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# The Hamiltonian Structure of Nonlinear Elasticity: The Material and Convective Representations of Solids, Rods, and Plates

by

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# §1. *Introduction*

It is our belief that a thorough understanding of the mathematical underpinnings of elasticity is crucial to its analytical and numerical implementation. For example, in the analysis of rotating structures, the coupling of the equations for geometrically inexact models, obtained by linearization or other approximations, with those for rotating rigid bodies can easily lead to misleading artificial "softening" effects that can significantly alter numerical results; see Simo & Vu-Quoc [1986c] (especially equations (3) and (5)). In this paper, we consider fully nonlinear geometrically exact models for rods, plates (and shells) which take into account shear and torsion as well as the usual bending effects in traditional rod and plate models. These models can be obtained either from the three-dimensional theory by a systematic use of projection methods; see e.g., Antman [1972] and Naghdi [1972], or by a direct approach within the context of Coserrat continuum. Remarkably, the two approaches lead to essentially the same form of the governing field equations. In the present context, we have chosen as a model problem a particular rod model which may be regarded as an extension of the classical Kirchhoff Love model (see Love [1944]) to include shear deformation, as in Reissner [1973,1981], Antman [1974], Antman & Jordan [1975], and Simo [1985]. The counterpart of this model for plates is also considered. Our objective is a systematic development of the Hamiltonian structure underlying the dynamics of these geometrically exact models in the material and convective representations. The convective representation, which is often referred to as the body representation in the context of rigid-body mechanics, is a useful counterpart to the more familiar material and spatial representations. Understanding the relation between these is useful for computational purposes and for coupling of these models to the dynamics of rigid body motion, as in Krishnaprasad & Marsden [1987]. We confine ourselves to the material and convective representations only for simplicity; one can also treat the spatial representation and in fact we shall use this in a follow up work on stability of rigid bodies with geometrically exact flexible attatchments (see Krishnaprasad, Marsden, Posb~~u~~ergh, and Simo [1987]).

One of the topics that is of importance in the foundations of elasticity is a geometric formulation of the equations in Hamiltonian form. This form is useful in the dynamical analysis of systems; for example in the study of nonlinear stability (see Holm, Marsden, Ratiu, & Weinstein [1985], Krishnaprasad [1985], and Lewis, Marsden, & Ratiu [1986a]), of bifurcation theory (see Golubitsky & Stewart [1986] and Lewis, Marsden & Ratiu [1986b]) and of chaotic solutions (see Holmes & Marsden [1983] and Guckenheimer & Holmes [1983]). Our own motivation is to provide additional insight for work on rotating structures using geometrically exact models (see Krishnaprasad and Marsden [1987] and Krishnaprasad, Marsden, Posb~~u~~ergh and Simo [1987]). Of course, independent of this motivation is the fact that these Hamiltonian structures are of intrinsic interest for the mathematical foundations of elasticity theory.

The Hamiltonian structure for the material (or Lagrangian) representation of elasticity is given in terms of canonically conjugate variables - namely the placement field and its conjugate momentum, the momentum density. This standard result is well known and is indicated in, for example, Marsden & Hughes [1983, Chapter 5]. The relation beween this and other structures in spatial and body representations is an important result that goes back to Arnold [1966] and was developed by Marsden & Weinstein [1974, 1982], and others. A noncanonical Hamiltonian structure for elasticity that is partially a spatial representation is given in Holm and Kuperschmit [1983], and a Hamiltonian structure for isotropic elasticity in spatial representation is given in Marsden, Ratiu, & Weinstein [1984a,b]. We deal with these as well as the *convective representation* and also develop a Hamiltonian formalism for rods and plates. A geometric setting that is useful for understanding the general relation between the material, inverse material, spatial, and convective representations is given in Holm, Marsden & Ratiu [1986].

The noncanonical brackets found in this paper are obtained by the general methods of reduction from the canonical structure in material representation, as in Arnold [1966] and Marsden and Weinstein [1982]. When these procedures are done for fixed boundary problems, one obtains Lie-Poisson brackets associated with the dual of a Lie algebra of a semi-direct product. (See Marsden, Weinstein, Ratiu, Schmit & Spencer [1983] for a general introduction to this geometric theory.) These sorts of brackets appear for example in the equations for compressible fluids and magnetohydrodynamics (see Marsden [1982], Holm and Kuperschmit [1983] and Marsden, Ratiu and Weinstein [1984a,b]). (The geometric reason for this appearance is that if a configuration space is a group G, then the reduction of the phase space $T^*G$ by the isotropy subgroup $G_a$ for a representation of G on a vector space V is essentially the dual of the Lie algebra of the semi-direct product G Ⓢ V. This result, due to Ratiu, Guillemin and Sternberg, is proved in a sharpened version in Marsden, Ratiu & Weinstein [1984a] (to which we refer for the original references). When free boundaries are present, however, the brackets are only partially of the Lie-Poisson type. The geometric setting for these is the "gauged Lie-Poisson" context of Montgomery, Marsden & Ratiu [1984]. This was applied to free boundary fluid problems in Lewis, Marsden, Montgomery & Ratiu [1986]. In this paper, we shall not require the fairly sophisticated context of the gauged Lie-Poisson structures, but rather we shall obtain the results by a direct calculation. We do note, however, that when no boundaries are present, the Poisson brackets we get for three dimensional elasticity do reduce to Lie-Poisson brackets for a semidirect product. For rods and plates, the brackets also reduce to the Lie-Poisson type in the cases that the configuration space reduces to a group; for example, this happens for the torsional motion of a rod.

The geometric point of view adopted in this paper has proven particularly useful in the numerical solution of initial boundary value problems. For the geometrically exact rod model, for instance, *exact update procedures* for the configuration, stress resultants and stress couples can be developed by employing discrete algorithmic counterparts of the exponential map and parallel transport (see Simo & Vu-Quoc [1986a,b,1987]). These ideas also play a central role in the numerical analysis of geometrically exact shell models, as in Simo & Fox [1987]. This methodology results in algorithms that *exactly preserve* the fundamental physical requirement of material frame indifference. Similarly, for three-dimensional nonlinear viscoelastic solids, by exploiting the convective representation, one can develop unconditionally stable algorithms, accurate to second order, which exactly preserve covariance of the continuum formulation (see Simo [1986]). Thus, these algorithms go beyond the notion of incremental objectivity, as proposed by Hughes & Winget [1980]. Finally, we believe that the Hamiltonian structures developed in this paper will play a central role in the future development, design, and stability analysis of time-stepping integration algorithms for nonlinear elastodynamics, which ensure not only conservation of energy, (as in Chorin et. al. [1978] or Hughes, Liu & Caughy [1978]), but exactly preserve other fundamental integrals of motion such as global angular momentum.

## §2. *Covariant Three-Dimensional Elasticity*

We summarize the notation to be used in the description of three dimensional elastodynamics, following to a large extent the usage of Marsden & Hughes [1983]. Emphasis is placed on a covariant formulation of the field equations independent of the choice of coordinate charts.

### *The Configuration Space*

Let $(\mathcal{B}, G)$ and $(\mathcal{S}, g)$ be two smooth Riemannian manifolds carrying metrics $G$ and $g$ respectively. Typically we have $\mathcal{B} \subset \mathcal{S}$, where $\mathcal{S} = \mathbb{R}^3$ is the Euclidean three-space with the standard Euclidean metric. We refer to $\mathcal{B}$ as the ***reference configuration*** with points denoted by $X \in \mathcal{B}$, and we speak of $\mathcal{S}$ as the ***ambient space*** in which the body $\mathcal{B}$ moves. Points in $\mathcal{S}$ are denoted $x \in \mathcal{S}$. We shall consider coordinate charts $\hat{X}^A : \mathcal{B} \to \mathbb{R}$ and $\hat{x}^a : \mathcal{S} \to \mathbb{R}$ so that the local coordinates of the points X and x are denoted

$$X^A = \hat{X}^A(X) \quad \text{for } X \in \mathcal{B} \qquad \text{and} \qquad x^a = \hat{x}^a(x) \quad \text{for } x \in \mathcal{S}. \tag{2.1}$$

The configuration space $\mathcal{C}$ is the set of (orientation-preserving) embeddings $\varphi : \mathcal{B} \to \mathcal{S}$; we write

$$\mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S}), \tag{2.2}$$

and call the set $\varphi(\mathcal{B})$ the ***current configuration.*** It is known that, when suitably topologized, $\mathcal{C}$ is a smooth infinite-dimensional manifold (see Abraham, Marsden & Ratiu [1983], Ebin & Marsden [1970] and references therein).

To construct the tangent space to $\mathcal{C}$ at a configuration $\varphi \in \mathcal{C}$, consider a smooth curve $\varepsilon \mapsto \varphi_\varepsilon$ such that $\varphi_\varepsilon \big|_{\varepsilon=0} = \varphi$. By definition, $d\varphi_\varepsilon/d\varepsilon \big|_{\varepsilon=0}$ is tangent to $\mathcal{C}$ at $\varphi$. Let $X \in \mathcal{B}$; then :

$$\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_\varepsilon(X) \in T_{\varphi(X)}\mathcal{S}, \tag{2.3}$$

where $T_{\varphi(X)}\mathcal{S}$ is the tangent space to $\mathcal{S}$ at $\varphi(X)$. Consequently, the map:

$$X \in \mathcal{B} \mapsto \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_\varepsilon(X) \in T_{\varphi(X)}\mathcal{S} \tag{2.4}$$

is a vector field over $\varphi : \mathcal{B} \to \mathcal{S}$ . Hence, we define the tangent space $T_\varphi \mathcal{C}$ as:

$$T_\varphi \mathcal{C} = \left\{ \mathbf{V}_\varphi : \mathcal{B} \to T\mathcal{S} \;\middle|\; \mathbf{V}_\varphi(X) \in T_{\varphi(X)}\mathcal{S} \text{ for all } \mathbf{X} \in \mathcal{B} \right\}. \tag{2.5}$$

In local coordinates relative to the chart $\{X^A\}$ we have

$$\mathbf{V}_\varphi(X) = V^i(X) \frac{\partial}{\partial x^i} \in T_{\varphi(X)}\mathcal{S}. \tag{2.6}$$

We often use the notation $x=\varphi(X)$.

## *Kinematics*

A ***motion*** is a curve of configurations; we let $\varphi_t$ be the configuration at time $t$ and write $\varphi_t(X) = \varphi(X,t)$. Given a motion $\varphi_t$ , we define the following quantities:

*(i)* ***material velocity*** : $\mathbf{V}_t \in T_{\varphi_t} \mathcal{C}$ given by

$$\mathbf{V}_t(X) := \frac{\partial}{\partial t} \varphi(X, t), \tag{2.7a}$$

*(ii)* ***spatial velocity*** : $\mathbf{v}_t \in \mathfrak{X}(\varphi_t(\mathcal{B}))$ [the space of vector fields on $\varphi_t(\mathcal{B})$] is defined by

$$\mathbf{v}_t = \mathbf{V}_t \circ \varphi_t^{-1}, \tag{2.7b}$$

*(iii)* ***convective velocity*** : $\boldsymbol{\mathcal{V}}_t \in \mathfrak{X}(\mathcal{B})$ is defined by

$$\boldsymbol{\mathcal{V}}_t = \varphi_t^*(v_t) := T\varphi_t^{-1} \circ v_t \circ \varphi_t = T\varphi_t^{-1} \circ V_t \tag{2.7c}$$

The ***deformation gradient,*** denoted $F_t$ , is defined to be the tangent map of $\varphi_t$; we write $F_t = T\varphi_t$. In coordinates,

$$\mathbf{F}_t = F^a{}_A \frac{\partial}{\partial x^a} \otimes dX^A; \qquad F^a{}_A = \frac{\partial \varphi^a}{\partial X^A}, \tag{2.8}$$

where $dX^A$ is the dual basis to $(\partial/\partial X^A)$.

The following diagram illustrates these concepts.

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "\varphi"] \arrow[d, "\boldsymbol{\mathcal{V}}_t"'] \arrow[dr, "\mathbf{V}_t"] & \mathcal{S} \arrow[d, "\mathbf{v}_t"] \\
T\mathcal{B} \arrow[r, "T\varphi"'] & T\mathcal{S}
\end{tikzcd}$$

**Proposition 2.1.** *The convected velocity is the negative of the spatial velocity of the inverse motion* $\varphi_t^{-1} : \mathcal{S} \longrightarrow \mathcal{B}$ ; ***i.e.,***

$$\boldsymbol{\mathcal{V}}_t = - \frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t. \tag{2.9}$$

**Proof.** Applying the chain rule to the identity $X = \varphi^{-1}(\varphi(X, t), t)$ gives

$$\frac{\partial \varphi_t^{-1}}{\partial t} \circ \varphi_t + T\varphi_t^{-1} \circ V_t = 0$$

and so the result follows by noting that $\boldsymbol{\mathcal{V}}_t = T\varphi_t^{-1} \circ v_t \circ \varphi_t$. ■

## *The Metric and Convected Metric Tensors. Convected Lie Derivative*

We define the convected metric tensor by the pull-back relation:

$$C_t = \varphi_t^*(g); \quad \text{i.e.,} \qquad C_{AB} = F^a{}_A F^b{}_B g_{ab} \circ \varphi. \tag{2.10}$$

$C_t$ is called the ***right Cauchy-Green*** tensor.

Let $\nabla$ be the Levi-Civita connection associated with the spatial Riemannian metric $g$. The corresponding Christoffel symbols are given by the standard relation:

$$\gamma^{d}{}_{ab} = \frac{1}{2} g^{dc}\left[\frac{\partial g_{ac}}{\partial x^{b}} + \frac{\partial g_{bc}}{\partial x^{a}} - \frac{\partial g_{ab}}{\partial x^{c}}\right]. \tag{2.11}$$

Associated with the convected metric **C**, we define a Riemannian connection $\nabla$ by the pull-back relation:

$$\nabla_{\mathcal{W}}\mathcal{V} := \varphi^{*}[\nabla_{\varphi_{*}(\mathcal{W})}\, \varphi_{*}(\mathcal{V})], \tag{2.12}$$

for any convected vector fields $\mathcal{V}$ and $\mathcal{W} \in \mathfrak{X}(\mathcal{B})$.

Using the properties of pull-backs and covariant differentiation, it follows from (2.12) that

$$\Gamma^{C}{}_{AB} := \frac{\partial^{2}\varphi^{a}}{\partial X^{A}\partial X^{B}}\,(F^{-1})^{C}{}_{a} + (F^{-1})^{C}{}_{c}F^{b}{}_{B}F^{a}{}_{A}\gamma^{c}{}_{ab}. \tag{2.13}$$

It can be readily shown that the connection $\nabla$ is the Levi-Civita connection for **C** with the Christoffel symbols given by the standard formula:

$$\Gamma^{D}{}_{AB} = \frac{1}{2} C^{DC}\left[\frac{\partial C_{AC}}{\partial X^{B}} + \frac{\partial C_{BC}}{\partial X^{A}} - \frac{\partial C_{AB}}{\partial X^{C}}\right]. \tag{2.14}$$

Let $\mathcal{W} \in \mathfrak{X}(\mathcal{B})$ be a convected vector field. We define the convected Lie derivative of $\mathcal{W}$, denoted by $\mathcal{L}_{\mathcal{V}}\mathcal{W}$, as the lie derivative relative to the convected velocity field $\mathcal{V}_t$. Consequently $\mathcal{L}_{\mathcal{V}}\mathcal{W}$ is given by the pull-back relation:

$$\mathcal{L}_{\mathcal{V}_t}\mathcal{W} := \varphi_t^{*}[\mathcal{L}_{v_t}(\varphi_t^{*}\mathcal{W})]. \tag{2.15}$$

Here, $\mathcal{L}_{v}w$ is the (spatial) Lie derivative of vector fields $w \in \mathfrak{X}(\varphi(\mathcal{B}))$ defined by the formula (see Marsden & Hughes [1983], §1.6)

$$\mathcal{L}_{v_t} w := \varphi_{t*}\frac{\partial}{\partial t}\varphi_t^{*} w. \tag{2.16}$$

## *Acceleration Vector Fields*

We define the ***material acceleration*** $A_t : \mathcal{B} \to T\mathcal{S}$ and the ***spatial acceleration*** $a_t : \mathcal{S} \to T\mathcal{S}$ associated with the motion $\varphi_t$ by the expressions

$$A_t = \partial^2\varphi_t/\partial t^2 = \partial V_t/\partial t \quad , \quad a_t = A_t \circ \varphi^{-1}. \tag{2.17}$$

The *convected acceleration* $\mathcal{A}_t : \mathcal{B} \to T\mathcal{B}$ is defined by the pull-back relation

$$\mathcal{A}_t = \varphi_t{}^*(a_t). \tag{2.18}$$

**Proposition 2.2.** *The convected velocity and acceleration are related by the formula*

$$\mathcal{A}_t = \frac{\partial \boldsymbol{\mathcal{V}}_t}{\partial t} + \nabla_{\boldsymbol{\mathcal{V}}_t} \boldsymbol{\mathcal{V}}_t \,. \tag{2.19a}$$

*In coordinates,* $\mathcal{A}$ *is given by*

$$\mathcal{A}^A = \partial \mathcal{V}^A/\partial t + \mathcal{V}^C \mathcal{V}^A{}_{,C} + \Gamma_{CD}{}^A \mathcal{V}^C \mathcal{V}^D . \tag{2.19b}$$

**Proof.** Recall (Marsden & Hughes [1983] p. 33) that the spatial velocity and accelerations are related by

$$a_t = \frac{\partial v_t}{\partial t} + \nabla_{v_t} v_t \,. \tag{2.20}$$

Now pull back the relation (2.21) by $\varphi_t$ to get

$$\varphi_t^*(\nabla_v v_t) = \nabla_{\varphi^* v} \varphi^* v_t = \nabla_{\boldsymbol{\mathcal{V}}} \boldsymbol{\mathcal{V}}_t \tag{2.21}$$

The Lie derivative formula (2.16) then gives

$$\frac{\partial \boldsymbol{\mathcal{V}}_t}{\partial t} = \frac{\partial}{\partial t}(\varphi_t^* v_t) = \varphi_t^*(L_{v_t} v_t) + \varphi_t^*\left(\frac{\partial v_t}{\partial t}\right) = \varphi_t^*\left(\frac{\partial v_t}{\partial t}\right) \,. \tag{2.22}$$

Adding (2.21) and (2.22) gives (2.19). ■

Next, we record some formlae useful for our subsequent developement of the convected equations of motion. Let $T^*\mathcal{B}$ be the cotangent bundle of $\mathcal{B}$, and denote by $\flat : T\mathcal{B} \to T^*\mathcal{B}$ the standard *index-lowering* action (see Abraham Marsden & Ratiu [1983]) induced by the convected metric C. Given any vector field $\mathcal{W} \in \mathfrak{X}(\mathcal{B})$ there is a unique one form $\mathcal{W}^\flat$ defined by the relation

$$\mathcal{W}_A(X)\,\mathcal{U}^A(X) = C_{AB}(X)\,\mathcal{W}^A(X)\,\mathcal{U}^B(X)\,, \tag{2.23}$$

for any $\mathcal{U} \in \mathfrak{X}(\mathcal{B})$ and $X \in \mathcal{B}$; consequently, in coordinates, $\mathcal{W}^\flat$ is defined as :

$$\mathcal{W}^\flat(X) := C_{AB}(X)\;\mathcal{W}^B(X)\;dX^A. \tag{2.24a}$$

Similarly, following standard notation, we denote by $\# : T^*\mathcal{B} \to T\mathcal{B}$ the index-raising action induced by $C$. Associated with any $\mathcal{M} : \mathcal{B} \to T^*\mathcal{B}$ there is a unique vector field $\mathcal{M}^\# \in \mathfrak{X}(\mathcal{B})$, with coordinate expression

$$\mathcal{M}^\#(X) = C^{AB}(X)\;\mathcal{M}_B(X)\;\frac{\partial}{\partial X^A}\,, \tag{2.24b}$$

relative to a chart $\{X^A\}$ on $\mathcal{B}$. With this notation at hand we have

**Lemma 2.1.** *For the convected velocity field, the following relations hold:*

i.
$$\mathcal{L}_\mathcal{V}\mathcal{V}^\flat = \frac{1}{2}\,d\,|\mathcal{V}|^2_C + (\nabla_\mathcal{V}\mathcal{V})^\flat, \tag{2.25}$$

*in coordinates:*

$$(\mathcal{L}_\mathcal{V}\mathcal{V}^\flat)_A = \frac{1}{2}\,\frac{\partial}{\partial X^A}\,[\mathcal{V}_B C^{BC}\mathcal{V}_C] + \mathcal{V}_{A|B}\mathcal{V}^B. \tag{2.26}$$

ii.
$$\dot{C} = \mathcal{L}_\mathcal{V} C; \qquad \text{i.e.,} \qquad \dot{C}_{AB} = \mathcal{V}_{A|B} + \mathcal{V}_{B|A} \tag{2.27}$$

**Proof:** In a coordinate chart $\{X^A\}$ we have:

$$(\mathcal{L}_{\mathcal{V}}\mathcal{V}^{\flat})_{A} = \frac{1}{2}\ \mathcal{V}_{A|B}\mathcal{V}^{B} + \mathcal{V}_{A}\mathcal{V}^{B}|_{C}$$

$$= \frac{1}{2}\ \frac{\partial}{\partial X^{A}}\,[\mathcal{V}_{B}\mathcal{V}^{B}] + \mathcal{V}_{A|B}\mathcal{V}^{B}$$

$$= \frac{1}{2}\ \frac{\partial}{\partial X^{A}}\,[\mathcal{V}_{B}C^{BC}\mathcal{V}_{C}] + \mathcal{V}_{A|B}\mathcal{V}^{B}, \tag{2.28a}$$

where a vertical bar denotes covariant differentiation relative to the convected correction. This proves formulae **i.** As for formulae **ii**, since the Lie derivative is natural with respect to pull-backs, from (2.16) we have:

$$\mathcal{L}_{\mathcal{V}_t}C_t = \varphi_t^{*}L_{v_t}(\varphi_{t_*}C_t) = \varphi_t^{*}\mathcal{L}_{v}g = 2\varphi^{*}\mathbf{d}, \tag{2.28b}$$

where $d := [v_{a|b} + v_{b|a}]dx^{a} \otimes dx^{b}$ is the (spatial) ***rate of deformation*** tensor. Formula **ii** then follows by virtue of the well-known relation $\dot{C} = 2\varphi_t^{*}(d)$. ■

In view of formula **ii** one says that $C$ is ***Lie-dragged*** by the flow generated by the convected velocity field.

## *The Convected Volume Element and One Form Densities*

Next, we turn our attention to the definition of volume elements. Let $d^3X := dX^1 \wedge dX^2 \wedge dX^3$ and $d^3x := dx^1 \wedge dx^2 \wedge dx^3$ be the standard volume elements in $\mathcal{B}$ and $\mathcal{S}$ respectively, and let $\Lambda^3{}_x(\mathcal{S})$ and $\Lambda^3{}_X(\mathcal{B})$ denote the space of 3-forms at $x \in \mathcal{S}$ and $X \in \mathcal{B}$, so that $d^3X \in \Lambda^3{}_X(\mathcal{B})$ and $d^3x \in \Lambda^3{}_x(\mathcal{S})$. Associated with $g$ and $G$ there are unique *volume densities*, denoted by $\mu(g)$ and $\mu(G)$, respectively, and defined locally as

$$\mu(g) := \{\det[g]\}^{1/2}\, d^3x, \qquad \mu(G) := \{\det[G]\}^{1/2}\, d^3X. \tag{2.29}$$

We use the notations $\mu(G) \in |\Lambda^3(\mathcal{B})|$ and $\mu(g) \in |\Lambda^3(\mathcal{S})|$ to designate volume densities.

Locally, by the change of variable formula, we have $d^3x = \det[F]\, d^3X$. Since the Jacobian $J_\varphi$ is defined as

$$\mu(g) \circ \varphi =: J_\varphi\, \mu(G), \tag{2.30}$$

we obtain the expression

$$J_\varphi := \det[F]\,\{\det[g]/\det[G]\}^{1/2}. \tag{2.31}$$

In the convected description, associated with the metric C we have the C-volume density locally defined as

$$\mu(G) := \{\det[G]\}^{1/2}\, d^3X. \tag{2.32}$$

Since $\det[C] = \det^2[F]\,\det[g]$, from (2.30), (2.31) and (2.32) we obtain

$$\mu(C) := J_\varphi\ \mu(G). \tag{2.33}$$

A ***one-form convected density*** is a mapping $\overline{\mathcal{M}} : \mathcal{B} \to T^*\mathcal{B} \otimes |\Lambda^3(\mathcal{B})|$ obtained by tensor product of a convected one-form $\mathcal{M} : \mathcal{B} \to T^*\mathcal{B}$ with the convected volume element $\mu(C)$. Accordingly, we have

$$\overline{\mathcal{M}} = \mathcal{M} \otimes \mu(C) = J_\varphi \mathcal{M} \otimes \mu(G). \tag{2.34}$$

Next we define the ***divergence*** operator which is needed in the formulation of the equations of motion. By recalling the covariant definition of the divergence (see, e.g., Abraham, Marsden, & Ratiu [1983] page 389), we have the following expressions in the convected and spatial descriptions.

$$(\mathcal{D}iv_C \mathcal{W})\mu(C) := \mathcal{L}_{\mathcal{W}_t}\mu(C); \quad (\mathrm{div}_g \mathrm{w})\mu(g) = \mathcal{L}_{\mathrm{w}_t}\mu(g), \tag{2.35}$$

for all vector fields $\mathcal{W} \in \mathfrak{X}(\mathcal{B})$ and $\mathrm{w} \in \mathfrak{X}(\varphi(\mathcal{B}))$. Vector fields, volume force densities and Lie derivatives are related in a simple manner. For the convected vector field density, one has the useful formula:

$$\mathcal{L}_{\mathcal{V}}\overline{\mathcal{W}} = L_{\mathcal{V}}\mathcal{W} \otimes \mu(C) + (\mathcal{D}iv_C \mathcal{W}) \otimes \mu(C) \tag{2.36}$$

which follows at once from (2.35) and standard properties of the Lie-derivative.

## *The Stress Tensor and Covariance*

We assume the existence of a stored energy function $W : \mathcal{M}_{\mathcal{S}} \times \mathcal{C} \times \mathcal{M}_{\mathcal{B}} \longrightarrow \mathbb{R}$, where $\mathcal{M}_{\mathcal{S}}$ is the space of Riemannian metrics on $\mathcal{S}$ and $\mathcal{M}_{\mathcal{B}}$ is the space of Riemannian metrics on $\mathcal{B}$, of the form

$$W = \overline{W}(g, F, G), \tag{2.37}$$

where $\overline{W}$ depends only on the point values of $g$, $F$, and $G$. This dependence is in keeping with the classical assumption that the stored energy function in an elastic material depends on the configuration $\varphi$ only locally through the point values of the deformation gradient $F$ (see Marsden & Hughes [1983], §§3.2 and 3.3). The dependence of the stored energy $\overline{W}$ on the metric tensor $g$ is essential to introduce the notion of covariance which embodies objectivity (or material-frame-indifference) as a particular case. Covariance is the statement of (left) invariance relative to the group of spatial diffeomorphisms, whereas objectivity merely requires (left) invariance relative to the group of spatial isometrics. Clearly, the former notion implies the latter but not conversely. We recall that relativity is a covariant theory.

To introduce the notion of covariance we begin by summarizing a few facts concerning the actions of the groups of spatial and material diffeomorphism the configuration space $\mathcal{C} = \mathrm{Emb}(\mathcal{B}, \mathcal{S})$ for elasticity. For further details, including functional analysis issues not addressed here, we refer to Ebin & Marsden [1973].

**i.** *The groups of spatial and material diffeomorphisms..* We denote by $\mathrm{Diff}(\mathcal{B})$ and $\mathrm{Diff}(\mathcal{S})$ the groups of diffemorphisms in $\mathcal{B}$ and $\mathcal{S}$, respectively. Let $\varphi \in \mathcal{C}$. Define the *left* and *right translations* of $\varphi \in \mathcal{C}$ as the mappings:

$$L_\eta : \mathrm{Diff}(\mathcal{S}) \times \mathcal{C} \to \mathcal{C}, \qquad R_\eta : \mathcal{C} \times \mathrm{Diff}(\mathcal{B}) \to \mathcal{C}, \tag{2.38}$$

given by

$$(\eta, \varphi) \mapsto L_\eta(\varphi) := \eta \circ \varphi, \quad (\varphi, \eta) \mapsto R_\eta(\varphi) := \varphi \circ \eta. \tag{2.39}$$

The tangent maps associated with these actions are obtained in the standard manner as follows. Consider a smooth curve $\varepsilon \to \varphi_\varepsilon \in \mathcal{C}$ such that $\varphi_\varepsilon\big|_{\varepsilon=0} = \varphi$, and $d\varphi_\varepsilon/d\varepsilon\big|_{\varepsilon=0} = V_\varphi$. Then, the tangent maps to $L_\eta$ and $R_\eta$ are computed as:

$$(T_\varphi L_\eta)(\mathbf{V}_\varphi(X)) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} L_\eta \varphi_\varepsilon(X) = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \eta(\varphi_\varepsilon(X) = D\eta(\varphi(X)) \cdot \mathbf{V}_\varphi(X). \tag{2.40}$$

Consequently, we have

$$T_\varphi L_\eta(\mathbf{V}_\varphi) = T_\eta \circ \mathbf{V}_\varphi, \tag{2.41}$$

for any $\eta \in \mathrm{Diff}(\mathcal{S})$. A similar calculation for $\eta \in \mathrm{Diff}(\mathcal{B})$ shows that

$$T_\varphi R_\eta(\mathbf{V}_\varphi) = \mathbf{V}_{\varphi \circ \eta^{-1}}. \tag{2.42}$$

The mechanical interpretation of these relations should be clear. We regard *left translations by* $\mathrm{Diff}(\mathcal{S})$ as superposed motions, *not necessarily rigid,* onto the current configuration. By virtue of (2.41), under any superposed $\eta \in \mathrm{Diff}(\mathcal{S})$ material vector fields at $\varphi(X)$ are transformed tensorially to vector fields at $\eta(\varphi(X))$. On the other hand, we regard *right translations* by $\mathrm{Diff}(\mathcal{B})$ as diffeomorphic changes of reference configuration $\mathcal{B}$ which, by virtue of (2.42), shift material points $X \in \mathcal{B}$ to $\eta^{\circ}(X)$, for any $\eta^{\circ} \in \mathrm{Diff}(\mathcal{B})$.

It can be readily shown that the action of $\mathrm{Diff}(\mathcal{S})$ on *spatial vector fields* is by push-forward; i.e.:

$$\mathbf{w} \in \mathfrak{X}(\varphi(\mathcal{B})) \mapsto \eta_*(\mathbf{w}) \in \mathfrak{X}(\varphi(\mathcal{B})). \tag{2.43}$$

**ii.** *Covariant stored energy functions..* We say that a stored energy function is *covariant* if it is *left invariant* under the action of $\mathrm{Diff}(\mathcal{S})$; accordingly, in view of (2.41) and (2.43), $W(g, F, G)$ is covariant if:

$$\eta \circ \overline{W}(\mathbf{g}, \mathbf{F}, \mathbf{G}) = \overline{W}(\eta_*\mathbf{g}, T\eta \circ \mathbf{F}, \mathbf{G}). \tag{2.44}$$

For any $\eta \in \mathrm{Diff}(\mathcal{S})$. The classical result that $W$ depends only on the point values of $C=\varphi^*(g)$ is an immediate consequence of the covariant assumption (2.44). To see this, consider the case $\mathcal{B} \subset \mathcal{S}$ so that $\mathrm{Diff}(B) \subset \mathrm{Diff}(\mathcal{B},\mathcal{S})$. By choosing $\eta=\varphi^{-1}$, (2.44) yields

$$\varphi^{-1} \circ \overline{W}(\mathbf{g}, \mathbf{F}, \mathbf{G}) = \overline{W}(\varphi^*\mathbf{g}, T\varphi^{-1} \circ \mathbf{F}, \mathbf{G}) = \overline{W}(\mathbf{C}, \mathbf{1}, \mathbf{G}); \tag{2.45}$$

where 1 is the identity. Consequently, there is a function $\overline{\overline{W}}$ of the point values of metrics on $\mathcal{B}$

such that

$$\overline{W}(g, F, G) = \overline{W}(C, G). \tag{2.46}$$

Let $\sigma$ be the ***Cauchy stress tensor*** and let $\Sigma$ be the ***convected stress tensor,*** which is defined by the pull-back relation

$$\Sigma := \varphi^*(\sigma). \tag{2.47}$$

Let $\rho(x,t)$ and $\rho_0(X,t)$ denote the ***density*** functions in the current and reference configurations, respectively. Define the **convected density** function as $\mathcal{R} := \varphi^*(\rho) = \rho \circ \varphi$. We then have the following constitutive equations

$$\sigma = 2\rho \frac{\partial \overline{W}(g, F, G)}{\partial g} \qquad \text{and} \qquad \Sigma = 2\mathcal{R} \frac{\partial \overline{W}(C, G)}{\partial C}, \tag{2.48}$$

Relation $(2.48_1)$ is referred to as the spatial ***Doyle-Ericksen formula*** (see Marsden & Hughes [1983] §3.3, and, for the material counterpart, Simo & Marsden [1984]). In terms of the ***Lagrangian strain tensor*** defined by $E = (C - G)/2$, and the ***Eulerian strain tensor*** defined by $e = \varphi_*(E) = (g - b^{-1})/2$, where $b^{-1} = \varphi_*(G)$ is the ***Finger deformation tensor,*** formulae (2.48) read

$$\sigma = \rho \frac{\partial \overline{W}(e, F, G)}{\partial e} \qquad \text{and} \qquad \Sigma = \mathcal{R} \frac{\partial \overline{W}(E, G)}{\partial E}. \tag{2.49}$$

Note that the dependence of $\overline{W}$ on the material metric tensor $G$ has been explicitly assumed in equation (2.46), but that $G$ is treated as a parameter as far as the covariance assumption is concerned.

Next, we consider the invariance group of $W$ on the right. According the (2.42), $\mathrm{Diff}(\mathcal{B})$ acts on the right on vector fields $V_\varphi$ by shifting the base points. On the other hand, it can be readily seen that the action of $\mathrm{Diff}(\mathcal{B})$ on convected vector fields $\mathcal{W} \in \mathfrak{X}(\mathcal{B})$ is by push-forward i.e.:

$$\mathcal{W} \in \mathfrak{X}(\mathcal{B}) \mapsto \eta^0{}_*\mathcal{W} \in \mathfrak{X}(\mathcal{B}). \tag{2.50}$$

[This is called the ***adjoint action***]. Then, the stored energy function $\overline{W}(C, G)$ is *right invariant* under $\mathrm{Diff}(\mathcal{B})$ if:

$$\overline{W}(C, G) \circ \eta^0 = \overline{W}(\eta^0{}_*C, \eta^0{}_*G), \tag{2.51}$$

for any $\eta^0 \in \mathrm{Diff}(\mathcal{B})$. That right invariance is equivalent to ***isotropy*** follows by considering the case $\mathcal{B} \subset \mathcal{S}$. Choosing $\eta^0 = \varphi$ we conclude from (2.51) that

$$\overline{W}(C, G) \circ \varphi = \overline{W}(g, b^{-1}), \tag{2.52}$$

which is the clasical expression for the stored energy function of an isotropic material. Thus, whereas left invariant under $\mathrm{Diff}(\mathcal{S})$, i.e. covariance is a fundamental physical requirement, right invariance under $\mathrm{Diff}(\mathcal{B})$ merely expresses a particular constitutive behavior; namely isotropy.

## *The Hamiltonian*

In the Hamiltonian formalism for the material description, the kinetic and potential energy are expressed in the variables on $T^*\mathcal{C}$; i.e., in the variables $\varphi \in \mathcal{C}$, and its conjugate momentum $M_\varphi$.

We start by defining the cotangent space $T_\varphi{}^*\mathcal{C}$ at $\varphi \in \mathcal{C}$ as a vector space in duality with $T_\varphi\mathcal{C}$ by means of a weakly non-degenerate pairing

$$\langle \cdot , \cdot \rangle : \; T_\varphi{}^*\mathcal{C} \times T_\varphi\mathcal{C} \to \mathbb{R}, \tag{2.53}$$

which is constructed as follows. Let $\overline{\alpha}_\varphi = \alpha_\varphi dV$ be a one-form density covering $\varphi \in \mathcal{C}$, where $dV = \mu(G)$. As in (2.34), $\overline{\alpha}_\varphi$ is given locally as $(\alpha_i(X) dV_X) dX^i$. The natural pairing between one-form densitites $\overline{\alpha}_\varphi(X) \in T^*{}_{\varphi(X)}\mathcal{S} \otimes |\Lambda_X(\mathcal{B})|$ at $\varphi(X)$ and vectors $V_\varphi(X) \in T_{\varphi(X)}\mathcal{S}$ is given, locally in coordinates, as

$$\langle \overline{\alpha}_\varphi, V_\varphi \rangle := \int_{\mathcal{B}} \alpha_i(X) V^i(X) \mu(G), \tag{2.54}$$

where $V_\varphi(X) = V^i(X)\,(\partial/\partial x^i)$. With this pairing, $T^*\mathcal{C}$ becomes:

$$T^*_\varphi\,\mathcal{C} := \{\, \overline{\alpha}_\varphi : \mathcal{B} \to T^*\mathcal{S} \otimes |\Lambda^3(\mathcal{B})| \;\big|\; \overline{\alpha}_\varphi(X) \in T^*_{\varphi(X)}\mathcal{S} \otimes |\Lambda^3{}_X(\mathcal{B})| \,\}. \tag{2.55}$$

We consider next the Hamiltonian for elasticity in the material, convected and spatial descriptions.

**i.** ***Material description.*** We derive the appropriate expression by starting with the expression of the Hamiltonian relative to $T\mathcal{C}$; i.e., in terms of configurations $\varphi$ and material velocites $V_t$, and performing a change of variables from $T\mathcal{C}$ to $T^*\mathcal{C}$. As a function of $T\mathcal{C}$, the Hamiltonian is given by

$$H(g; V_\varphi, G) := \int_{\mathcal{B}} \rho_0(X)\,|\,V_t(X)\,|_g^2\, \mu_X(G) + \int_{\mathcal{B}} \rho_0(X) W(g(\varphi(X)), F_t(X), G(X)) \mu_X(G) \quad (2.56)$$

where $|\cdot|_g$ denotes the length induced by the spatial metric $g$. Observe that the Hamiltonian (2.56) depend parametrically on the spatial metric $g$; i.e., $H: \mathcal{M}_{\mathcal{S}} \times T\mathcal{C} \times \mathcal{M}_{\mathcal{B}} \longrightarrow \mathbb{R}$. To transform - (2.56) to a function defined on $T^*\mathcal{C}$ we introduce the weak Riemannian metric on $\mathcal{C}$ defined by the kinetic energy; i.e.:

$$\langle\langle V_\varphi,\ W_\varphi \rangle\rangle_X := \int_{\mathcal{B}} V^i(X) W^j(X) g_{ij}(\varphi(X)) \rho_0(X) \mu_X(G). \quad (2.57)$$

In finite dimensions, a metric on a manifold induces a bundle metric on the cotangent bundle. In infinite dimensions, on the other hand, this need not be the case. In the present situation we give an explicit construction following Marsden, Ratiu & Weinstein [1980]. Let $\overline{\alpha}_\varphi = \alpha_\varphi dV$ and $\overline{\beta}_\varphi = \beta_\varphi dV \in T^*\mathcal{C} \otimes |\Lambda(\mathcal{B})|$, where $dV = \mu(G)$ is the volume element. Then $\overline{\alpha}_\varphi/(\rho_0 dV)$ and $\overline{\beta}_\varphi/(\rho_0 dV)$ are one-forms over $\varphi \in \mathcal{C}$ so that, evaluated at $\varphi(X)$ become elements of $T_{\varphi(X)}\mathcal{S}$. Now, $T_{\varphi(X)}\mathcal{S}$ is a finite dimensional Riemannian manifold with metric $g(\varphi(X))$. Consequently, we have the standard index-lowering and index-raising actions induced by $g(\varphi(X))$; explicitly, as in (2.24a,b), for any $V_\varphi(X) \in T_{\varphi(X)}\mathcal{S}$ and $\alpha_\varphi(X) \in T^*_{\varphi(X)}\mathcal{S}$ we define the associated one-form and vector field by

$$V^\flat_\varphi(X) = g_{ij}(\varphi(X)) V^j(X) dx^i; \quad \alpha^\sharp_\varphi(X) = g^{ij}(\varphi(X)) \alpha_j(X) \frac{\partial}{\partial x^i}. \quad (2.58)$$

This induces a bundle metric on $T^*\mathcal{C}$ by the expression

$$(\overline{\alpha}_\varphi,\ \overline{\beta}_\varphi) := \langle\langle V_\varphi, W_\varphi \rangle\rangle, \quad (2.59)$$

where $V_\varphi(X) = [\overline{\alpha}_\varphi/(\rho_0 dV)]^\sharp = [\alpha_\varphi/\rho_0]^\sharp$ and $W_\varphi(X) = [\overline{\beta}_\varphi/(\rho_0 dV)]^\sharp$. We shall denote by $\|\cdot\|$ the bundle norm defined by the metric (2.59).

With this notation, define the ***material momentum density*** as:

$$\overline{M}_\varphi := \rho_0 V^\flat_t\, \mu(G) \in T^*_\varphi \mathcal{C}. \quad (2.60)$$

In view of (2.57), (2.59) and (2.60), the kinetic energy term in (2.56) becomes $\|M_\varphi\|/2$ and the Hamiltonian takes the form:

$$H(g;\varphi,M_\varphi;G)=\frac{1}{2}\,\|M_\varphi\|^2+\int_{\mathcal{B}}\rho_0 W(g\circ\varphi,\ F\ G)\mu(G). \tag{2.61}$$

Again, we observe the $H: \mathcal{M}_S \times T^*\mathcal{C} \times \mathcal{M}_{\mathcal{B}} \longrightarrow \mathbb{R}$ depends parametrically on the metrics $g$ and $G$. In addition, $H$ subject to the covariance assumption.

**ii.** ***Convected description.*.** To express the Hamiltonian in the convected representation we first transform the kinetic energy term using relation (2.7c) between convected and material velocity fields. We have, in coordinates,

$$\begin{aligned}\|M_\varphi\|^2 &= \int_{\mathcal{B}}\rho_0(X)V^a(X)V^b(X)g_{ab}(\varphi(X))\mu_X(G)\\ &= \int_{\mathcal{B}}\rho_0(X)F^a{}_A(X)F^b{}_B(X)g_{ab}(\varphi(X))\mathcal{V}^A(X)\mathcal{V}^B(X)\mu_X(G)\\ &= \int_{\mathcal{B}}\rho_0(X)C_{AB}(X)\mathcal{V}^A(X)\mathcal{V}^B(X)\mu_X(G).\end{aligned} \tag{2.62}$$

Sometimes, and in what follows, we shall write $\mu_X(G)$ as just $\mu(G)$, suppressing the variable $X$. In view of the integrand in (2.62), we define the ***convected one-form mementum density*** to be $\overline{\mathcal{M}} = \mathcal{M} \otimes \mu(C)$, where $\mathcal{M}$ is a one-form on $\mathcal{B}$, and $\mu(C)$ is the convected volume element defined by (2.32), as

$$\begin{aligned}\overline{\mathcal{M}} &:= [\rho_0(X)C_{AB}(X)\mathcal{V}^B(X)]dX^A \otimes \mu(G)\\ &= [\mathcal{R}(X)C_{AB}(X)\mathcal{V}^B(X)]dX^A \otimes \mu(C).\end{aligned} \tag{2.63}$$

(Recall that $\mathcal{R}$ is the conveted density). Consequently, we have the equivalent expression

$$\overline{\mathcal{M}} := \rho_0\mathcal{V}^\flat{}_t \otimes \mu(G) = \mathcal{R}_t\mathcal{V}^\flat{}_t \otimes \mu(C), \tag{2.64}$$

where $\flat : T\mathcal{B} \longrightarrow T^*\mathcal{B}$ denotes the lowering-index action induced by $C$, as defined by (2.24a).

Again this construction induces a bundle metric given, locally, by the expression

$$\langle \overline{\mathcal{M}}, \overline{\mathcal{M}} \rangle := \int_{\mathcal{B}} \frac{1}{\rho_0(X)} (\mathcal{M}^{\#}, \mathcal{M}^{\#})_C \mu(G)\,, \tag{2.65a}$$

where

$$(\mathcal{M}^{\#}, \mathcal{M}^{\#})_C := (\rho_0(X)\mathcal{V}^A(X))(\rho_0(X)\mathcal{V}^B(X))C_{AB}(X) \tag{2.65b}$$

is the local inner product induced by $C$ on $T_X\mathcal{B}$. The Hamiltonian in the convected description then becomes

$$H(\overline{\mathcal{M}}, C, G) := \frac{1}{2} \langle \overline{\mathcal{M}}, \overline{\mathcal{M}} \rangle + \int_{\mathcal{B}} \rho_0 W(C, G)\mu(G). \tag{2.66}$$

Note that the kinetic energy is now a function of $\overline{\mathcal{M}}$ and $C$ alone. We also note that the kinetic energy is just one half the squared-length of the momentum density in the metric on the space of convective momentum densities that is induced by (2.65). We regard (2.66) as the energy induced on the original phase space of $\varphi$, $\overline{M}_{\varphi}$, $g$'s after factoring by the group of spatial diffeomorphisms $\mathrm{Diff}(\mathcal{S})$. (Again $G$ enters parametrically). This idea is central to the reduction procedure that will be explained in the next section.

**iii.** ***Spatial description.*** The expression for the kinetic energy in the spatial description readily follows by recalling relation (2.7b) between material and spatial velocity fields. We have

$$\begin{aligned} \frac{1}{2}\,\|M_{\varphi}\|^2 &= \frac{1}{2}\int_{\mathcal{B}} \rho_0(X)V^a(X)V^b(X)g_{ab}(\varphi(X))\mu_X(G) \\ &= \frac{1}{2}\int_{\varphi_t(\mathcal{B})} \rho_t(x)V^a(x)v^b(x)g_{ab}(x)\mu_x(\underset{\sim}{g}). \end{aligned} \tag{2.67}$$

Thus, we define the *spatial momentum density* $\overline{m}_t$ as

$$\overline{m}_t := \rho\, v_t^{\flat} \otimes \mu(g) \tag{2.68}$$

where $\flat$: $TS \to T^*S$ now denotes the *lowering-index* action induced by the spatial metric $g$. As in the convected description, this defines the bundle metric

$$\langle \bar{m}, \bar{m} \rangle := \int_{\varphi_t(\mathcal{B})} \frac{1}{\rho_t(x)} m_a(x) m_b(x) g^{ab}(x) \mu_x(g). \tag{2.69}$$

in terms of which, the kinetic energy in the spatial description becomes $\langle \bar{m}, \bar{m} \rangle / 2$.

The central issue in the Hamiltonian formalism in the spatial description concerns the appropriate formulation of the potential energy term in such a way that the assumed form of the stored energy function *does not preclude anisotropic response*. Early attempts, e.g., Marsden et al. [1983], have been restricted to isotropic response. In the present context we proceed as follows. Let

$$\mathbf{f}_t := -T\varphi_t^{-1}\ ; \quad \text{i.e.,} \quad \mathbf{f}_t(x) = -\frac{\partial(\varphi^{-1})^A}{\partial x^a} \frac{\partial}{\partial X^A} \otimes dx^a, \tag{2.70}$$

be minus the inverse deformation gradient. Assume a *covariant* stored energy function of the form

$$W = \tilde{w}(g, \mathbf{f}_t, \mathbf{G} \circ \varphi^{-1}), \tag{2.71}$$

where, as in the preceeding descriptions, we regard $\tilde{w}$ as depending parametrically on $G$. *Covariance* is now understood in the following sense:

$$\tilde{w}(g, \mathbf{f}_t, \mathbf{G} \circ \varphi^{-1}) \circ \eta = \tilde{w}(\eta_* g, \mathbf{f}_t \circ T\eta^{-1}, \mathbf{G} \circ \varphi^{-1} \circ \eta^{-1}). \tag{2.72}$$

That is, *right invariance* relative to the group $\mathrm{Diff}(\mathcal{S})$. Before proceeding further we remark on the interpretation of $f_t : T\mathcal{S} \longrightarrow T\mathcal{B}$. By performing a straight forward coordinate calculation we find:

$$\frac{\partial}{\partial t} f^A{}_a + \frac{\partial f^A{}_a}{\partial x^b} v^b + f^A{}_b \frac{\partial v^b}{\partial x^a} = 0. \tag{2.73}$$

It is apparent that (2.73) is the coordinate expression in a chart $\{x^a\}$ of a one-form on $\mathcal{S}$. This motivates the interpretation of $f_t$ as a collection of *one-forms*: $\pi^A : \mathcal{S} \longrightarrow T^*\mathcal{S}$, (i.e. a vector valued one form) rather than a two-point tensor, i.e.

$$\mathbf{f}_t = \frac{\partial}{\partial X^A} \otimes \pi_t^A; \quad \pi_t^A := f^A{}_a\, dx^a. \tag{2.74}$$

Expression (2.73) then reads:

$$\frac{\partial}{\partial t} \pi_t^A + L_{v_t} \pi_t^A = 0 \tag{2.75}$$

That is, $f_t$ is lie-dragged by the flow of the spatial velocity field. With these observations in mind, the Hamiltonian in the spatial description now reads:

$$H(g, \overline{m};\ \mathbf{f}_t, G) = \frac{1}{2} \langle \overline{m}, \overline{m} \rangle + \int_{\varphi_t(\mathcal{B})} \rho \tilde{w}(g, \mathbf{f}_t, G\ \ \varphi^{-1}) \mu(g). \tag{2.76}$$

## *Equations of Motion*

We conclude this section with a summary of the equations of motion in the spatial, material and convected descriptions.

**i.** ***Conservation of mass.*** Let $\varphi_t \in \mathcal{C}$ be a motion, $\mathcal{U} \subset \mathcal{B}$ a compact set, and $\varphi(\mathcal{U}) \subset \mathcal{S}$. Conservation of mass requires that:

$$\int_{\mathcal{U}} \rho_0(X) \mu_X(G) = \int_{\varphi_t(\mathcal{U})} \rho_t(x) \mu_x(g). \tag{2.77}$$

By the change of variable formula, assuming enough smoothness, we have

$$\varphi_t^*(\rho_t(x)\mu_x(g)) = (\rho_t \circ \varphi_t)(X)\varphi^*(\mu_x(g)) = \rho_0(X)\mu_X(G). \tag{2.78}$$

Alternatively, since $\mu(g) \circ \varphi =: J_\varphi \mu(G)$ by (2.30), we have the equivalent expression

$$\rho_0(X) = (\rho_t \circ \varphi_t) J_\varphi(X). \tag{2.79}$$

**i.a**. *Spatial Description.*. From (2.78) it follows that

$$\varphi_{t*} \frac{\partial}{\partial t} \varphi_t^* (\rho_t \mu(g)) = 0\,. \tag{2.80}$$

The Lie derivative formula (2.16) then yields:

$$\mathcal{L}_{v_t}[\rho_t \otimes \mu(g)] = 0. \tag{2.81}$$

Alternatively, using the fact that $\mathcal{L}_v\mu(g)=(\mathrm{div}_g v)\mu(g)$ we recover from (2.81) the classical expression

$$\frac{\partial \rho_t}{\partial t} + \mathrm{div}_g(\rho_t v_t) = 0. \tag{2.82}$$

Note, however, that either expression (2.78) or expression (2.81) are more convenient than the classical relation (2.82). This observation is crucial in our subsequent development of the Poisson bracket.

**i.b.** *Convected Description.* By taking the pull-back of (2.78) and using standard properties of the Lie derivative we obtain:

$$0 = \varphi_t^*[\mathcal{L}_{v_t}(\rho_t \otimes \mu(g))] = \mathcal{L}_{\varphi_t^* v_t}(\varphi_t^* \rho_t \otimes \varphi_t^* \mu(g)). \tag{2.83}$$

Using (2.15), (2.30) and (2.33) we obtain

$$\mathcal{L}_{\mathcal{V}_t}(\mathcal{R}_t \otimes \mu(C)) \equiv \mathcal{L}_{\mathcal{V}_t}(\rho_0 \otimes \mu(G)) = 0. \tag{2.84}$$

Equivalently, by exploiting the connection between Lie derivatives and volume elements we have:

$$\frac{\partial \mathcal{R}_t}{\partial t} + \mathcal{D}iv_C(\mathcal{R}_t \mathcal{V}_t) = 0. \tag{2.85}$$

**i.c.** *Material Description.* From either (2.78) or (2.84) we obtain:

$$\frac{\partial \rho_0}{\partial t} = 0, \tag{2.86}$$

which constitutes the statement of conservation of mass in the material description.

**ii.** ***Conservation of Momentum.*** We summarzie below the *covariant* version of the classical forms of the local momentum equations in the spatial, convected and material descriptions.

**ii.a.** *Spatial Description.* Assuming for simplicity zero body forces, we have the classical equations

$$\operatorname{div}_{\mathbf{g}} \sigma = \rho_t\left[\frac{\partial \mathbf{v}_t}{\partial t} + \nabla_{\mathbf{v}_t}\mathbf{v}_t\right]. \tag{2.87}$$

To express (2.87) in *covariant form*; i.e., in a form independent of the choice of coordinates; and in terms of the spatial momentum density, we proceed as follows. Time differentiation of definition (2.68) for $\overline{m}_t$ and use of (2.87) yields

$$\frac{\partial \overline{\mathbf{m}}_t}{\partial t} = \frac{\partial \rho_t}{\partial t}\mathbf{v}_t^\flat \otimes \mu(\mathbf{g}) - [\rho_t(\nabla_{\mathbf{v}_t}\mathbf{v}_t)^\flat - (\operatorname{div}_{\mathbf{g}}\sigma)^\flat] \otimes \mu(\mathbf{g}). \tag{2.88}$$

On the other hand, by the Lie derivative formula:

$$\mathbf{L}_{\mathbf{v}_t}\overline{\mathbf{m}}_t = \rho_t(\mathbf{L}_{\mathbf{v}_t}\mathbf{v}_t^\flat) \otimes \mu(\mathbf{g}) + (\mathbf{d}\rho_t \cdot \mathbf{v}_t)\mathbf{v}_t^\flat \otimes \mu(\mathbf{g}) + \rho_t\mathbf{v}_t^\flat \otimes \mathbf{L}_{\mathbf{v}_t}\mu(\mathbf{g}). \tag{2.89}$$

Using again the fact that $\mathbf{L}_V(\mu(g)) = [\operatorname{div}_g V]\mu(g)$ along with the spatial counterpart of (2.26), (2.89) becomes

$$\mathbf{L}_{\mathbf{v}_t}\overline{\mathbf{m}}_t = \left[\rho_t(\nabla_{\mathbf{v}_t}\mathbf{v}_t)^\flat + \frac{1}{2}\rho_t\mathbf{d}\,|\mathbf{v}_t|_{\mathbf{g}}^2 + (\mathbf{d}\rho_t \cdot \mathbf{v}_t)\mathbf{v}_t^\flat + \rho_t(\operatorname{div}_{\mathbf{g}}\mathbf{v}_t)\mathbf{v}_t^\flat\right] \otimes \mu(\mathbf{g}). \tag{2.90}$$

Adding (2.88) and (2.90) and using the continuity equation (2.82) we finally obtain

$$\frac{\partial \overline{\mathbf{m}}_t}{\partial t} + \mathbf{L}_{\mathbf{v}_t}\overline{\mathbf{m}}_t = \left[(\operatorname{div}_{\mathbf{g}}\sigma)^\flat + \frac{\rho_t}{2}\mathbf{d}\,|\mathbf{v}_t|_{\mathbf{g}}^2\right] \otimes \mu(\mathbf{g}). \tag{2.91}$$

Which constitutes the desired expression.

**ii.b.** *Convective Expression.* By pull-back of the classical balance of momentum equations (2.87) one finds:

$$\mathfrak{Div}_C\, \boldsymbol{\Sigma} = \mathcal{R}\left[\frac{\partial \boldsymbol{\mathcal{V}}_t}{\partial t} + \nabla_{\boldsymbol{\mathcal{V}}_t}\boldsymbol{\mathcal{V}}_t\right]; \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma}^T . \tag{2.92}$$

The covariant version of (2.92) may be obtained by an identical argument to that leading to (2.91), now phrased in the convective description. Alternatively, since

$$\frac{\partial \mathbf{M}_t}{\partial t} = \frac{\partial}{\partial t}\varphi_t^* \bar{\bar{\mathbf{m}}}_t = \varphi_t^*\left[\frac{\partial \bar{\bar{\mathbf{m}}}_t}{\partial t} + \mathcal{L}_{\mathbf{V}_t}\bar{\bar{\mathbf{m}}}_t\right] \tag{2.93}$$

by pull back of (2.91), one finds

$$\frac{\partial \mathbf{M}_t}{\partial t} = \left[(\mathfrak{Div}_C\, \boldsymbol{\Sigma})^\flat + \frac{\mathcal{R}_t}{2}\, \mathbf{d}\,|\boldsymbol{\mathcal{V}}_t|_C^2\right] \otimes \mu(\mathbf{C}). \tag{2.94}$$

**ii.c.** *Material Representation.* For completeness we also record the form of the momentum equations in the material description. Making use of the Piola transformation (e.g. Marsden & Hughes [1983, Chap. 1]) from the spatial form (2.87) we obtain

$$\mathrm{DIV}_G\mathbf{P} = \rho_0\frac{\partial \mathbf{V}_t}{\partial t}; \quad \mathbf{P}\mathbf{F}_t^T = \mathbf{F}_t\mathbf{P}^T. \tag{2.95}$$

where $\mathsf{P} := \mathsf{J}_\varphi \boldsymbol{\sigma} \mathsf{F}_t^{-1}$ is the (non-symmetric) ***first Piola-Kirchhoff*** tensor.

In the next section, we show that the field equations in the convected description are Hamiltonian relative to a non-canonical Lie-Poisson structure on the material phase space reduced on the left by the group of spatial diffeomorphisms.

## §3. *The Hamiltonian Structure of Threee-Dimensional Elasticity in the Material and Convective Representations*

In this section we show that the equations of elastodynamics in the convective representation are Hamiltonian relative to a non-canonical Poisson structure on the space of pairs $(\mathfrak{M}, \mathbf{C})$, where $\mathfrak{M}$ is the convective momentum density and $\mathbf{C}$ is the Cauchy-Green tensor, as in the preceeding section. This means that the equations of elastodynamics are equivalent to the following condition: For any function $f(\mathfrak{M}, \mathbf{C})$,

$$\dot{f} = \{f, H\}, \tag{3.1}$$

where H is the Hamiltonian, given by equation (2.25) and the bracket $\{\ \}$ appearing in (3.1) satisfies the usual conditions for a Poisson bracket, including Jacobi's identity (see, for example, Marsden et al. [1983]).

This bracket is obtained by reducing the canonical bracket on $T^*\mathcal{C}$ by the group of spatial isometries of the metric $\mathbf{g}$ on $\mathcal{S}$. Equivalently, as in Marsden, Ratiu and Weinstein [1984a,b], we can add the metrics $\mathbf{g}$ and $\mathbf{G}$ as a parameter and then reduce the cotangent bundle $T^*(\mathcal{M}_\mathcal{S} \times \mathcal{C} \times \mathcal{M}_\mathcal{B})$ by the left action of $\mathcal{M}_\mathcal{S} \times \mathrm{Diff}(\mathcal{S}) \times \mathcal{M}_\mathcal{B}$. As is shown in this reference, the result is the same as reducing $\mathcal{M}_\mathcal{S} \times T^*\mathcal{C} \times \mathcal{M}_\mathcal{B}$ by the action of $\mathrm{Diff}(\mathcal{S})$. This reduction procedure will be explicitly explained by direct calculation in what follows. Before reading this section, the reader may find it helpful to first review the parallel case of rigid body dynamics in §4.

### *The Canonical Bracket on the Material Phase Space*

We start with the canonical phase space $T^*\mathcal{C}$; the space of configurations $\varphi \in \mathcal{C}$ and their canonically conjugate momenta $\overline{\mathbf{M}}_\varphi = \mathbf{M}_\varphi \otimes \mu(\mathbf{G})$, the material momentum densities. in addition, to accommodate the covariance assumption and the influence of the reference configuration (i.e., anisotropy), one introduces the spatial and material metrics, $\mathbf{g} \in \mathcal{M}_\mathcal{S}$ and $\mathbf{G} \in \mathcal{M}_\mathcal{B}$, as additional parameters. Consequently, we consider the material phase space.

$$\mathcal{P}_{\mathrm{can}} := \mathcal{M}_\mathcal{S} \times T^*\mathcal{C} \times \mathcal{M}_\mathcal{B}. \tag{3.2}$$

We write $f(\mathbf{g}; \varphi, \overline{\mathbf{M}}_\varphi, \mathbf{G})$ for functions $f : \mathcal{P}_{\mathrm{can}} \to \mathbb{R}$.

To define the canonical bracket on $\mathcal{P}_{\mathrm{can}}$ we start by introducing the notation of the *partial*

*Fréchet* and *functional derivatives* of a function $f : \mathcal{P}_{can} \to \mathbb{R}$. The *partial Fréchet* and *functional derivatives* of $f : \mathcal{P}_{can} \to \mathbb{R}$ relative to $\overline{\mathbf{M}}_\varphi$ is defined by considering a curve $\varepsilon \mapsto \overline{\mathbf{M}}_\varphi + \varepsilon\delta\overline{\mathbf{M}}_\varphi \in \mathfrak{X}^*(\mathcal{B})$ and setting

$$\mathbf{D}_{\bar{\mathbf{M}}_\varphi} f \cdot \delta\overline{\mathbf{M}}_\varphi := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(\mathbf{g}, \varphi, \overline{\mathbf{M}}_\varphi + \varepsilon\delta\overline{\mathbf{M}}_\varphi, \mathbf{G}), \tag{3.3}$$

where $\delta\overline{\mathbf{M}}_\varphi := \delta\mathbf{M}_\varphi \otimes \mu(\mathbf{G})$ is a *one-form density* covering $\varphi \in \mathcal{C}$.

The definition of the partial Fréchet derivative of $f : \mathcal{P}_{can} \to \mathbb{R}$ with respect to $\varphi \in \mathcal{C}$ requires some caution since $T^*\mathcal{C}$ is not simply a product space. Essentially, one needs to "fix" $\overline{\mathbf{M}}_\varphi$ while allowing $\varphi \in \mathcal{C}$ to vary. To formalize this intuitive notion, we proceed as in Lewis, Marsden, Montogmery and Ratiu [1987]. We identify $T^*_\varphi\mathcal{S}$ with $\varphi(\mathcal{B}) \times (\mathbb{R}^3)^*$ and denote by $\tilde{\mathbf{M}}_\varphi : \mathcal{B} \to (\mathbb{R}^3)^*$ the principal part of $\mathbf{M}_\varphi$; i.e., the projection of $\mathbf{M}_\varphi$ onto $(\mathbb{R}^3)^*$. Thus we regard $\mathbf{M}_\varphi$ as mapping the following spaces:

$$\mathbf{M}_\varphi = \varphi \times \tilde{\mathbf{M}}_\varphi : \mathcal{B} \to \varphi(\mathcal{B}) \times (\mathbb{R}^3)^* \equiv T^*_\varphi\mathcal{S}. \tag{3.4}$$

As usual, we set $\overline{\mathbf{M}}_\varphi := \mathbf{M}_\varphi \otimes \mu(\mathbf{G})$.

With the foregoing identifications, let $\varepsilon \mapsto \varphi_\varepsilon \in \mathcal{C}$ be a smooth curve such that

$$\delta\varphi := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \varphi_\varepsilon \in T_\varphi\mathcal{C}. \tag{3.5}$$

Then, set

$$\overline{\mathbf{M}}_{\varphi_\varepsilon} := \varphi_\varepsilon \times \tilde{\mathbf{M}}_\varphi \otimes \mu(\mathbf{G})$$

and define

$$\mathbf{D}_\varphi f \cdot \delta\varphi := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(\mathbf{g}; \varphi_\varepsilon, \overline{\mathbf{M}}_{\varphi_\varepsilon}; \mathbf{G}). \tag{3.6}$$

Next, we define partial functional derivatives. Since boundary conditions are involved, we consider two possible situations.

**i.** *Pure displacement boundary conditions*. The configuraitons $\varphi \in \mathcal{C}$ are restricted as follows:

$$\varphi\,|_{\partial\mathcal{B}} = \overline{\varphi} \quad \text{(prescribed)} \tag{3.7a}$$

Consequently, the admissible variations $\mathbf{V}_\varphi \in T_\varphi C$ must satisfy:

$$\mathbf{V}_\varphi(X) = \mathbf{O}, \text{ for } X \in \mathcal{B}, \tag{3.7b}$$

so that

$$\overline{\mathbf{M}}_\varphi(X) = \mathbf{O}, \text{ for } X \in \mathcal{B}. \tag{3.8}$$

We now define the ***partial functional derivative***, $\delta\,\overline{f}/\delta\varphi = (\delta f/\delta\varphi) \otimes \mu(\mathbf{G})$, as the ***one-form density*** covering $\varphi \in C$, which is given by the relation

$$\mathbf{D}_\varphi f \bullet \delta\varphi = \int_{\mathcal{B}} \frac{\overline{\delta f}}{\delta\varphi} \bullet \delta\varphi = \int_{\mathcal{B}} \frac{\delta f}{\delta\varphi} \bullet \delta\varphi\, \mu(\mathbf{G}). \tag{3.9}$$

Similarly, the ***partial functional derivative*** $(\delta f/\delta\overline{\mathbf{M}}_\varphi)$, is a **vector field** covering $\varphi \in C$ given by the expression:

$$\mathbf{D}_{\bar{\mathbf{M}}_\varphi} f \bullet \delta\overline{\mathbf{M}}_\varphi = \int_{\mathcal{B}} \frac{\delta f}{\delta\overline{\mathbf{M}}_\varphi} \bullet \delta\overline{\mathbf{M}}_\varphi = \int_{\mathcal{B}} \frac{\delta f}{\delta\overline{\mathbf{M}}_\varphi} \bullet \delta\mathbf{M}_\varphi\, \mu(\mathbf{G}) \tag{3.10}$$

with these definitions, the canonical poisson bracket $\{\,\cdot\,,\cdot\,\} : \mathcal{F}(\mathcal{P}_{\text{can}}) \times \mathcal{F}(\mathcal{P}_{\text{can}}) \to \mathbb{R}$ takes the form:

$$\{f, g\} = \int_{\mathcal{B}} \left[ \frac{\overline{\delta f}}{\delta\varphi} \cdot \frac{\delta g}{\delta\overline{\mathbf{M}}_\varphi} - \frac{\overline{\delta g}}{\delta\varphi} \cdot \frac{\delta f}{\delta\overline{\mathbf{M}}_\varphi} \right] \tag{3.11}$$

**ii**. *General Boundary conditions*. Next we consider general boundary conditions of mixed type. Assuming dead loading, we have

$$\varphi \,|_{\partial_\varphi \mathcal{B}} = \tilde{\varphi} \quad \text{(prescribed)} \tag{3.12a}$$

$$\mathbf{P}\hat{\mathbf{N}} \,|_{\partial_\sigma \mathcal{B}} = \tilde{t} \quad \text{(prescribed)} \tag{3.12b}$$

where $\mathbf{P}$ is the first Piola-Kirchhoff tensor referred in (2.9.5 ), and $\hat{\mathbf{N}}$ is the normal to $\partial_\sigma\mathcal{B}$ and

$$\text{closure}\,(\partial_\varphi\mathcal{B} \cup \partial_\sigma\mathcal{B}) = \text{closure}(\partial\mathcal{B}) \quad\text{and}\quad \partial_\varphi\mathcal{B} \cap \partial_\sigma\mathcal{B} = \varnothing. \tag{3.12c}$$

To account for the traction boundary conditions on $\partial_\sigma\mathcal{B}$, we modify our definition (3.9) of functional derivative as follows. We set

$$\mathbf{D}_\varphi f \cdot \delta\varphi = \int_{\mathcal{B}} \frac{\overline{\Delta f}}{\delta\varphi} \cdot \delta\varphi + \int_{\partial_\sigma\mathcal{B}} \frac{\overline{\partial f}}{\delta\varphi} \cdot \delta\varphi\,|_{\partial_\sigma\mathcal{B}}, \tag{3.13a}$$

for all variations $\delta\varphi \in T_\varphi C$ satisfying the essential boundary conditions; i.e.,

$$\delta\varphi\,|_{\partial_\varphi\mathcal{B}} = \mathbf{0}. \tag{3.13b}$$

Here, $\overline{\Delta f}/\delta\varphi$ is a one form density defined on points $\mathbf{X} \in \delta_\sigma\mathcal{B}$. For further elaboration on the possible alternative defintions of partial functional derivatives that account for the (natural) boundary conditions (3.12b) we refer to Lewis et al. [1987].

By defining

$$\frac{\delta f}{\delta\varphi} := \frac{\Delta f}{\delta\varphi} + \delta_{(\partial_\sigma\mathcal{B})}\frac{\partial f}{\delta\varphi}\,; \quad \frac{\delta f}{\delta \overline{M}_\varphi} \equiv \frac{\Delta f}{\delta M_\varphi}, \tag{3.14}$$

where $\delta_{(\partial_\sigma\mathcal{B})}$ is the Dirac delta measure on $\mathcal{B}$ which is concentrated on $\partial_\sigma\mathcal{B}$, the standard definition (3.11) of the canonical bracket now becomes

$$\{f, g\} = \int_{\mathcal{B}} \frac{\overline{\Delta f}}{\delta\varphi} \cdot \frac{\delta g}{\delta\overline{M}_\varphi} - \frac{\overline{\Delta g}}{\delta\varphi} \cdot \frac{\delta f}{\delta\overline{M}_\varphi}$$

$$+ \int_{\partial_\sigma\mathcal{B}} \left[ \frac{\partial f}{\delta\varphi} \cdot \frac{\delta g}{\delta\overline{M}_\varphi}\bigg|_{\partial_\sigma\mathcal{B}} - \frac{\partial g}{\delta\varphi} \cdot \frac{\delta f}{\delta\overline{M}_\varphi}\bigg|_{\partial_\sigma\mathcal{B}} \right] \gamma(\mathbf{G}) \tag{3.15}$$

where $\overline{\Delta f}/\delta\varphi := \Delta f/\delta\varphi \otimes \mu(\mathbf{G})$ is a one-form density covering $\varphi : \mathcal{B} \to \mathcal{S}$, and $\gamma(\mathbf{G})$ is the surface area element on $\partial_\sigma\mathcal{B}$ induced by $\mathbf{G}$.

**Proposition 3.1.** *The canonical Hamilton equations* $\dot{f} = \{f, H\}$ *on material phase space,* $\mathcal{P}_{\text{can}}$,

***yield the material balance laws.***

$$\left.\begin{aligned} &\frac{\partial\varphi}{\partial t} = \mathbf{V}_t \\ &\frac{\partial \overline{\mathbf{M}}_\varphi}{\partial t} = \mathrm{DIV}_G\,\mathbf{P}\otimes\mu(\mathbf{G})\,; \quad \mathbf{P} = \rho_0\,\frac{\partial W(g\circ\varphi,\, T\varphi,\, \mathbf{G})}{\partial \mathbf{F}} \end{aligned}\right\}\ \text{in } \mathcal{B} \tag{3.16}$$

and

$$\frac{\partial\varphi}{\partial t} = \mathbf{V}_t \ \text{ on } \partial_\sigma\mathcal{B}, \ \text{ and } \bar{\mathbf{t}} = \mathbf{P}\hat{\mathbf{N}} \ \text{ on } \partial_\sigma\mathcal{B}.$$

**Proof.** The Hamiltonian in the material (Lagrangian) description is given by (2.56), i.e.,

$$\mathrm{H}(g;\varphi,\overline{\mathbf{M}}_\varphi,\mathrm{G}) = \frac{1}{2}\int_{\mathcal{B}} \rho_0\,|\mathbf{V}_t|^2\,\mu(\mathrm{G}) + \int_{\mathcal{B}} \rho_0 W(g\circ\varphi,\, T\varphi,\, \mathrm{G})\,\mu(\mathrm{G})$$

$$-\int_{\partial_\sigma\mathcal{B}} \varphi\cdot\bar{\mathbf{t}}\,\gamma(\mathrm{G}). \tag{3.17}$$

where $\overline{\mathbf{M}}_\varphi = \mathbf{M}_\varphi\otimes\mu(\mathrm{G}) = \rho_0\mathbf{V}^\flat_t\otimes\mu(\mathrm{G})$. By considering a curve $\varepsilon\mapsto\varphi_\varepsilon\in\mathcal{C}$ with $\varphi_{\varepsilon|\varepsilon=0} = \varphi$ and $(d\varphi_\varepsilon/d\varepsilon)|_{\varepsilon=0} = \delta\varphi$, and keeping in mind the identifications elaborated upon above, we find

$$\mathrm{D}_\varphi\mathrm{H}\cdot\delta\varphi = \int_{\mathcal{B}} \rho_0\,\frac{\partial W}{\partial \mathbf{F}} : \mathrm{GRAD}\,\delta\varphi\otimes\mu(\mathrm{G}) - \int_{\partial_\sigma\mathcal{B}} \delta\varphi\cdot\bar{\mathbf{t}}\,\gamma(\mathrm{G})$$

$$= \int_{\mathcal{B}} -[\mathrm{DIV}_G\mathbf{P}\otimes\mu(\mathrm{G})]\cdot\delta\varphi + \int_{\partial_\sigma\mathcal{B}} (\mathbf{P}\hat{\mathbf{N}} - \bar{\mathbf{t}})\cdot\delta\varphi\,\gamma(\mathrm{G}). \tag{3.18}$$

It follows that

$$\frac{\overline{\Delta \mathrm{H}}}{\delta\varphi} = -\,\mathrm{DIV}_G[\mathbf{P}\otimes\mu(\mathrm{G})]; \quad \frac{\overline{\partial \mathrm{H}}}{\delta\varphi} = (\mathbf{P}\hat{\mathbf{N}} - \bar{\mathbf{t}})\otimes\gamma(\mathrm{G}). \tag{3.19}$$

Similarly one finds

$$\frac{\delta \mathrm{H}}{\delta\overline{\mathbf{M}}_\varphi} = \mathbf{V}_t. \tag{3.20}$$

The canonical bracket (3.15) then becomes:

$$\{f, H\} = \int_{\mathcal{B}} \frac{\overline{\Delta f}}{\delta\varphi} \cdot \mathbf{V}_\varphi + \frac{\delta f}{\delta \mathbf{M}_\varphi} \cdot [\mathrm{DIV}_{\mathbf{G}}(\mathbf{P} \otimes \mu(\mathbf{G})]$$

$$+ \int_{\partial_\sigma \mathcal{B}} \left[ \frac{\partial f}{\delta\varphi} \cdot \mathbf{V}_\varphi|_{\partial_\sigma \mathcal{B}} + (\bar{\mathbf{t}} - \mathbf{P}\hat{\mathbf{N}}) \cdot \frac{\delta f}{\delta \mathbf{M}_\varphi}\bigg|_{\partial_\sigma \mathcal{B}} \right] \gamma(\mathbf{G}). \tag{3.21}$$

On the other hand, for any $f : \mathcal{P}_{can} \to \mathbb{R}$ we have:

$$\dot{f} = \int_{\mathcal{B}} \frac{\overline{\Delta f}}{\delta\varphi} \cdot \frac{\partial\varphi}{\partial t} + \frac{\delta f}{\delta \mathbf{M}_\varphi} \cdot \frac{\partial \mathbf{M}_\varphi}{\partial t} + \int_{\partial_\sigma \mathcal{B}} \frac{\partial f}{\delta\varphi} \cdot \frac{\partial\varphi}{\partial t}\bigg|_{\partial_\sigma \mathcal{B}} \gamma(\mathbf{G}). \tag{3.22}$$

By comparing (3.21) and (3.22) the result follows. ■

## *The Reduced Convective Phase Space*

As remarked in the introduction, the reduced phase space in the convective representation is obtained by *left reduction* of the material phase space by the group of spatial diffeomorphisms $\mathrm{Diff}(\mathcal{S})$; i.e.,

$$\mathcal{P}_{conv} := \mathrm{Diff}(\mathcal{S})\backslash\mathcal{P}_{can}. \tag{3.23}$$

Our discussion in the preceeding section on the Hamiltonian structure of elasticity in the convective representation led to the functional form $H(\mathbf{C}, \mathcal{M}, \mathbf{G})$. Correspondingly, the left reduced phase space $\mathcal{P}_{conv}$ is:

$$\mathcal{P}_{conv} := S_2(\mathcal{B}) \times \mathfrak{X}^*(\mathcal{B}) \times \mathcal{M}_{\mathcal{B}} \tag{3.24}$$

where

$S_2(\mathcal{B})$ = symmetric, covariant rank-two tensors on $\mathcal{B}$.

$\mathfrak{X}^*(\mathcal{B})$ = convective one-form densities on $\mathcal{B}$.
$\mathcal{M}_{\mathcal{B}}(= S_2(\mathcal{B}))$ = Riemmanian metrics on $\mathcal{B}$.

Thus, by left reduction of the material phase space to the convective representation, a function $\hat{f}$: $\mathcal{P}_{can} \to \mathbf{R}$ is transformed to a function $f : \mathcal{P}_{conv} \to \mathbf{R}$ through the change of variables:

$$\hat{f}\,(g;\, \varphi,\, \overline{M}_\varphi,\, G) = f(\varphi^* g,\, (T\varphi)^T \circ \overline{M}_\varphi,\, G) \tag{3.25}$$

where $(T\varphi)^T = F^T : T\mathcal{S} \to T\mathcal{B}$ is the transpose of the deformation gradient, $F = T\varphi$, $\overline{\mathcal{M}} := (T\varphi)^T \circ M_\varphi \in \mathfrak{X}^*(\mathcal{B})$ is the convected momentum density, and $C = \varphi^* g \in S_2(\mathcal{B})$ the right Cauchy-Green tensor.

Conceptually, the procedure to derive the Poisson bracket for the convective representation is straight forward: One simply expands the canonical bracket (3.11) using the chain rule and the change of variables (3.25), to obtain the desired expression in terms of the convective variables $(C, \overline{\mathcal{M}}, G) \in \mathcal{P}_{conv}$.

Before proceeding with this calculation, we make some additional remarks. First, if $\mathcal{C}$ were a group, we would expect the reduced bracket to be of a special type, namely a Lie-Poisson bracket on the dual of the Lie algebra of a semi-direct product (see Marsden, Ratiu & Weinstein [1984a,b]). We shall see analogues of such a structure here. Because the boundary of $\mathcal{B}$ can move, $\mathcal{C}$ is not a group and indeed our bracket differs from a semi-direct product Lie-Poisson bracket only in boundary integral terms. For another bracket in *spatial* representation, see Lewis, Marsden, Montgomery & Ratiu [1986] and also Holm, Marsden & Ratiu [1986].) The boundary conditions here can either be displacement, traction, or a combination thereof. In the former case we restrict $\mathcal{C}$, which imposes corresponding restrictions on $\mathcal{V}$ and $\mathcal{M}$; specifically, if a portion of $\partial\mathcal{B}$ is fixed, $\mathcal{M}$ will vanish on that portion. For traction boundary conditions we add a corresponding boundary term to the Hamiltonian as illustrated above.

Second, we observe that given a dynamic solution $\mathcal{M}(X, t)$, $C(X, t)$, we can reconstruct the original motion $\varphi(X, t)$ by constructing $\mathcal{V}$ with (2.24a), and then solving the ordinary differential equation:

$$\frac{\partial\psi(x, t)}{\partial t} = \mathcal{V}(\psi(x, t), t), \tag{3.26a}$$

and finally letting

$$\varphi_t = \psi_t^{-1}. \tag{3.26b}$$

This is a special case of the general reconstruction procedure that is used in reduction theory. (See

Abraham & Marsden [1978]).]

## *The Poisson Bracket in the Convective Phase Space*

As in our development of the canonical Hamiltonian structure, we begin with the definition of partial Fréchet and functional derivatives relative to the convective variables $(\mathbf{M}, \mathbf{C}) \in \mathfrak{X}^*(\mathcal{B}) \times S_2(\mathcal{B})$.

Let $\varepsilon \mapsto \varphi_\varepsilon \in \mathcal{C}$ be a smooth curve with $\varphi_\varepsilon|_{\varepsilon=0} = \varphi$ and $d/d\varepsilon|_{\varepsilon=0}\, \varphi_\varepsilon = \delta\varphi \in T_\varphi \mathcal{C}$. Then, $\varepsilon \mapsto \mathbf{C}_\varepsilon := \varphi_\varepsilon^* \mathbf{g}$ is a curve in $S_2(\mathcal{B})$ with tangent vector given by

$$\delta \mathbf{C} := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \mathbf{C}_\varepsilon = \varphi^* \mathcal{L}_{(\delta\varphi \circ \varphi^{-1})} \mathbf{g} = \mathcal{L}_{(T\varphi^{-1} \circ \delta\varphi)} \mathbf{C} \tag{3.27}$$

Similarly, $\varepsilon \mapsto \overline{\mathbf{M}}_\varepsilon := \rho_0 (T\varphi)^T \circ \mathbf{V}^\flat{}_t \otimes \mu(\mathbf{G})$ is a curve in $\mathfrak{X}^*(\mathcal{B})$ whose tangent is computed as follows. Recall that

$$\mathrm{GRAD}\ \delta\varphi := \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} T\varphi_\varepsilon \tag{3.28a}$$

so that the following relation holds

$$\mathrm{GRAD}\ \delta\varphi = T\varphi \circ \nabla(\overset{T\varphi^{-1}\circ\,\delta\varphi}{\cancel{\varphi^{-1}\circ\delta\varphi}}), \tag{3.28b}$$

where $\nabla$ denotes covariant derivative relative to the convective metric $\mathbf{C} = \varphi^* \mathbf{g}$; i.e., $\nabla \mathcal{W} = \mathcal{W}^A|_B\, \partial/\partial X^A \otimes dX^B$, with $\mathcal{W}^A|_B = \partial \mathcal{W}^A / \partial X^B + \Gamma^A{}_{BC} \mathcal{W}^C$. Then, we have

$$\begin{aligned}
\delta \overline{\mathbf{M}} &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \overline{\mathbf{M}}_\varepsilon = [\mathrm{GRAD}\ \delta\varphi]^T\, \overline{\mathbf{M}}_\varphi \\
&= [\nabla(T\varphi^{-1} \circ \delta\varphi)]^T\, \overset{T\varphi}{\cancel{\varphi^{-1}}} \circ \overline{\mathbf{M}}_\varphi \\
&= [\nabla(T\varphi^{-1} \circ \delta\varphi)]^T \overline{\mathcal{M}}
\end{aligned} \tag{3.29}$$

Consider next functions $\hat{f} : \mathcal{P}_{\mathrm{can}} \to \mathbb{R}$ and $f : \mathcal{P}_{\mathrm{conv}} \to \mathbb{R}$ related by the change of variables (3.25). By the chain rule, we obtain

$$D_\varphi \hat{f} \cdot \delta\varphi = \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(C_\varepsilon, \overline{\mathcal{M}}_\varepsilon, G) = D_C f \cdot \delta C + D_{\overline{\mathcal{M}}} f \cdot \delta\overline{\mathcal{M}}, \tag{3.30}$$

where $\delta C$ and $\delta\overline{\mathcal{M}}$ are given by (3.27) and (3.29), respectively. We define the ***partial functional derivatives*** of $\hat{f} : \mathcal{P}_{conv} \to \mathbf{R}$ as:

$$\left.\begin{aligned} D_C \hat{f} \cdot \delta C &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(C_\varepsilon, \overline{\mathcal{M}}, G) = \int_{\mathcal{B}} \frac{\overline{\delta f}}{\delta C} : \delta C \\ D_{\overline{\mathcal{M}}} \hat{f} \cdot \delta\overline{\mathcal{M}} &= \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(C, \overline{\mathcal{M}}_\varepsilon, G) = \int_{\mathcal{B}} \frac{\delta f}{\delta\overline{\mathcal{M}}} \cdot \delta\overline{\mathcal{M}} \end{aligned}\right\} . \tag{3.31}$$

Thus, $\overline{\delta f/\delta C} = \delta f/\delta C \otimes \mu(C) \in S_2^*(\mathcal{B})$ and $\delta f/\delta\overline{\mathcal{M}} \in \mathfrak{X}(\mathcal{B})$ where $S_2^*(\mathcal{B}) = S^2(\mathcal{B}) \otimes |\Lambda^3(\mathcal{B})|$, the space of contravariant tensor densities, is the dual space to $S_2(\mathcal{B})$. We assume that no boundary term contributions are present in the definitions (3.31). From (3.29), (3.30) and (3.31) we obtain

$$\begin{aligned} D_\varphi \hat{f} \cdot \delta\varphi &= \int_{\mathcal{B}} \frac{\overline{\delta f}}{\delta C} : L_{(T\varphi^{-1} \circ\, \delta\varphi)} C + \frac{\delta f}{\delta\overline{\mathcal{M}}} \cdot [\nabla(T\varphi^{-1} \circ \delta\varphi)]^T \overline{\mathcal{M}} \\ &= \int_{\mathcal{B}} \frac{\overline{\delta f}}{\delta C} : L_{(T\varphi^{-1} \circ\, \delta\varphi)} C + \overline{\mathcal{M}} \cdot [\nabla(T\varphi^{-1} \circ \delta\varphi)] \frac{\delta f}{\delta\overline{\mathcal{M}}} . \end{aligned} \tag{3.32}$$

We now state the main result needed in the derivation of the Poisson bracket in the convective representation.

**Lemma 3.2.** *Assume mixed boundary conditions of the type (3.12) with zero traction vector; i.e.:*

$$\overline{\mathbf{M}}_\varphi = 0 \text{ on } \delta_\varphi\mathcal{B}, \quad \mathbf{P}\hat{\mathbf{N}} = \hat{\mathbf{t}} \equiv \mathbf{O} \text{ on } \partial_\sigma\mathcal{B}. \tag{3.33}$$

*Then:*

$$\frac{\overline{\Delta f}}{\delta\varphi} = -(T\varphi^{-T})\ \mathfrak{Div}_C(\mathfrak{E}_f)\otimes\mu(C)\ \text{ in }\ \mathcal{B},$$

$$\frac{\overline{\partial f}}{\delta\varphi} = (T\varphi^{-T})\circ \underset{\sim}{\overline{\mathcal{E}}_f}\, \underset{\sim}{\hat{N}}\quad \text{on } \partial_\sigma\mathcal{B}, \tag{3.34}$$

$$\frac{\delta f}{\delta\overline{M}_\varphi} = T\varphi \circ \frac{\delta f}{\delta\overline{\mathcal{M}}}$$

*where* $\overline{\mathfrak{E}}_f = \mathfrak{E}_f\otimes\mu(C)$ *is a mixed (covariant-contravariant) tensor density, defined as*

$$\mathfrak{E}_f := \left[\overline{\mathcal{M}}\otimes\frac{\delta f}{\delta\overline{\mathcal{M}}} + 2C\,\frac{\overline{\delta f}}{\delta\varphi}\right]. \tag{3.35}$$

**Proof.** Set $\mathcal{W} := T\varphi^{-1}\circ\delta\varphi \in \mathfrak{X}(\mathcal{B})$. A coordinate calculation shows that $(\mathfrak{L}_{\mathcal{W}}C)_{AB} = \mathcal{W}_{A|B} + \mathcal{W}_{B|A}$. Therefore, (3.32) becomes:

$$D_\varphi\hat{f}\cdot\delta\varphi = \int_{\mathcal{B}} \nabla\mathcal{W} : \left[2C\,\frac{\overline{\delta f}}{\delta C} + \overline{\mathcal{M}}\otimes\frac{\delta f}{\delta\overline{\mathcal{M}}}\right] = \int_{\mathcal{B}} \nabla\mathcal{W} : \overline{\mathfrak{E}_f}. \tag{3.36}$$

Use of the divergence theorem yields:

$$D_\varphi\hat{f}\bullet\delta f = \int_{\mathcal{B}} -\mathcal{W}\bullet(\mathfrak{Div}_C\,\mathfrak{E}_f)\otimes\mu(C) + \int_{\partial\mathcal{B}} \mathcal{W}^\flat\bullet\mathfrak{E}_f\hat{N}\otimes\gamma(C), \tag{3.37}$$

where $\gamma(C)$ is the surface element induced by $\mathbf{C}$ and $\hat{\mathbf{N}}$ the one-form normal to $\partial\mathcal{B}$. Since $\mathcal{W} = T\varphi^{-1}\circ\delta\varphi$ and $\delta_\varphi|\partial_\varphi\mathcal{B} = \mathbf{0}$ from (3.13a) and (3.37) we have

$$\int_{\mathcal{B}}\frac{\overline{\Delta\hat{f}}}{\delta\varphi}\bullet\delta\varphi + \int_{\partial_\sigma\mathcal{B}}\frac{\overline{\partial\hat{f}}}{\delta\varphi}\bullet\delta\varphi\,|_{\partial_\sigma\mathcal{B}} = -\int_{\mathcal{B}}\delta\varphi\bullet T\varphi^{-T}\mathfrak{Div}_C(\mathfrak{E}_f)\otimes\mu(C)$$

$$+\int_{\partial_\sigma\mathcal{B}}\delta\varphi\,|_{\partial_\sigma\mathcal{B}}\bullet T\varphi^{-T}\mathfrak{E}_f\otimes\gamma(C) \tag{3.38}$$

and result $(3.34)_{1,2}$ follows. Relation $(3.34)_3$ also follows by a similar calculation. ■

The Poisson bracket is now obtained by substituting relations (3.34) into expression (3.15) for the canonical bracket. This yields

$$\{f, g\} = \int_{\mathcal{B}} -\left[\frac{\delta g}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{Div}_C(\mathfrak{L}_f) - \frac{\delta f}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{Div}_C(\mathfrak{L}_g)\right] \otimes \mu(C)$$

$$+ \int_{\partial_\sigma \mathcal{B}} \left(\frac{\delta g}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{L}_f \hat{N} - \frac{\delta f}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{L}_g \hat{N}\right) \otimes \gamma(C). \tag{3.39}$$

Use of the divergence theorem leads to

$$\{f, g\} = \int_{\mathcal{B}} \left(\nabla \frac{\delta g}{\delta \overline{\mathcal{M}}}\right) : \overline{\mathfrak{L}}_f - \left(\nabla \frac{\delta f}{\delta \overline{\mathcal{M}}}\right) : \overline{\mathfrak{L}}_g$$

$$- \int_{\partial_\varphi \mathcal{B}} \left[\frac{\delta g}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{L}_f \hat{N} - \frac{\delta f}{\delta \overline{\mathcal{M}}} \cdot \mathfrak{L}_g \hat{N}\right] \gamma(C). \tag{3.40}$$

Since $\overline{\mathcal{M}}\,|\partial_\varphi \mathcal{B} = 0$, from (3.35) we have $\mathfrak{L}_f|\partial_\varphi \mathcal{B} = 2C\,\delta f/\delta C$. By recalling the definition of the bracket of vector fields in $\mathfrak{X}(\mathcal{B})$ and using (3.35), (3.40) reduces to:

$$\{f, g\} = \int_{\mathcal{B}} \overline{\mathcal{M}} \cdot \left[\frac{\delta f}{\delta \overline{\mathcal{M}}}, \frac{\delta g}{\delta \overline{\mathcal{M}}}\right] + \int_{\mathcal{B}} \left(\nabla \frac{\delta g}{\delta \overline{\mathcal{M}}}\right) C : 2\overline{\frac{\delta f}{\delta C}} - \left(\nabla \frac{\delta f}{\delta \overline{\mathcal{M}}}\right) C : 2\overline{\frac{\delta g}{\delta C}}$$

$$- \int_{\partial_\varphi \mathcal{B}} \left[\frac{\delta f}{\delta \mathcal{M}} : 2C \frac{\delta f}{\delta C} \hat{N} - \frac{\delta f}{\delta \mathcal{M}} : 2C \frac{\delta g}{\delta C}\right] \gamma(C). \tag{3.41}$$

The following theorem summarizes the various expressions for the Poisson bracket:

***Theorem 3.3.*** *The reduced poisson bracket on the convective phase space has the following equivalent expressions*

$$\text{i.)}\quad \{f,g\} = \int_{\mathcal{B}} \bar{\mathcal{M}} \cdot \left[\frac{\delta f}{\delta \bar{\mathcal{M}}}, \frac{\delta g}{\delta \bar{\mathcal{M}}}\right]$$

$$+ \int_{\mathcal{B}} \mathbf{C} : \left[\frac{\delta f}{\delta \bar{\mathcal{M}}} \otimes \mathrm{Div}_C\!\left(2\,\frac{\delta g}{\delta \mathbf{C}}\right) - \frac{\delta g}{\delta \bar{\mathcal{M}}} \otimes \mathrm{Div}_C\!\left(2\,\frac{\delta f}{\delta \mathbf{C}}\right)\right] \mu(\mathbf{C})$$

$$- \int_{\partial_\sigma \mathcal{B}} \mathbf{C} : \left[\frac{\delta f}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta g}{\delta \mathbf{C}}\,\hat{\mathbf{N}} - \frac{\delta g}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta f}{\delta \mathbf{C}}\,\hat{\mathbf{N}}\right] \gamma(\mathbf{C}) \qquad (3.42a)$$

$$\text{ii.)}\quad = \int_{\mathcal{B}} \bar{\mathcal{M}} \bullet \left[\frac{\delta f}{\delta \bar{\mathcal{M}}}, \frac{\delta f}{\delta \bar{\mathcal{M}}}\right] + \int_{\mathcal{B}} \left[\overline{\frac{\delta f}{\delta \mathbf{C}}} : \mathcal{L}_{\frac{\delta g}{\delta \bar{M}}}\,\mathbf{C} - \overline{\frac{\delta g}{\delta \mathbf{C}}} : \mathcal{L}_{\frac{\delta f}{\delta \bar{M}}}\,\mathbf{C}\right]$$

$$- \int_{\partial_\varphi \mathcal{B}} \mathbf{C} : \left[\frac{\delta g}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta f}{\delta \mathbf{C}}\,\hat{\mathbf{N}} - \frac{\delta f}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta g}{\delta \mathbf{C}}\,\hat{\mathbf{N}}\right] \gamma(\mathbf{C}) \qquad (3.42b)$$

$$\text{iii.)}\quad = \int_{\mathcal{B}} \bar{\mathcal{M}} \bullet \left[\frac{\delta f}{\delta \bar{\mathcal{M}}}, \frac{\delta g}{\delta \bar{\mathcal{M}}}\right] + \int_{\mathcal{B}} \mathbf{C} : \left[\mathcal{L}_{\frac{\delta g}{\delta \bar{\mathcal{M}}}}\,\overline{\frac{\delta f}{\delta \mathbf{C}}} - \mathcal{L}_{\frac{\delta f}{\delta \bar{\mathcal{M}}}}\,\overline{\frac{\delta g}{\delta \mathbf{C}}}\right]$$

$$- \int_{\partial_\varphi \mathcal{B}} \mathbf{C} : \left[\frac{\delta f}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta g}{\delta \mathbf{C}}\,\hat{\mathbf{N}} - \frac{\delta g}{\delta \bar{\mathcal{M}}} \otimes 2\,\frac{\delta f}{\delta \mathbf{C}}\,\hat{\mathbf{N}}\right] \gamma(\mathbf{C})$$

$$- \int_{\partial \mathcal{B}} \left[\left(\mathbf{C} : \frac{\delta g}{\delta \mathbf{C}}\right)\left(\frac{\delta f}{\delta \bar{\mathcal{M}}} \bullet \hat{\mathbf{N}}\right) - \left(\mathbf{C} : \frac{\delta f}{\delta \mathbf{C}}\right)\left(\frac{\delta f}{\delta \bar{\mathcal{M}}} \bullet \hat{\mathbf{N}}\right)\right] \gamma(\mathbf{C}). \qquad (3.42c)$$

**Proof**. Expression (3.42a) follows from (3.41) by the divergence theorem. Expression (3.42b) follows from (3.41) by the coordinate expression for the Lie derivative, $\mathcal{L}_{w}\mathbf{C}$, and the symmetry of $\overline{\delta f}/\delta \mathbf{C}$. Finally, expression (3.42c) follows form (3.41b) by integation by parts. ■

We observe that for pure displacement boundary conditions, the boundary term vanishes in expression (2.42a) since $\partial_\sigma \mathcal{B} = \varnothing$, whereas for traction free boundary conditions the boundary term in (3.42b) vanishes since $\partial_\varphi \mathcal{B} = \varnothing$.

**Corollary 3.1.** *Hamilton's equations* $\dot{f} = \{f, H\}$ *are equivalent to the convective equations of motion*

$$\frac{\partial \overline{\mathcal{M}}}{\partial t} = \left[\frac{1}{2}\,\mathrm{Rd}\,|\mathcal{V}|_C^2 + (\mathrm{Div}_C\,\Sigma)^\flat\right] \otimes \mu(C)$$

$$\frac{\partial C}{\partial t} = L_{\mathcal{V}} C \tag{3.43}$$

$$\Sigma \hat{N} = O \text{ on } \partial_\sigma \mathcal{B} \quad (\text{with } \overline{\mathcal{M}} = O \text{ on } \partial_{\underset{\sim}{\varphi}} \mathcal{B}).$$

This follows, as we remarked earlier, by the general theory of reduction. For completeness we include a direct verification of this statement.

**Proof of Corollary 3.1.** From (2.66), the Hamiltonian in the convective description is

$$H(C, \overline{\mathcal{M}}, G) = \frac{1}{2}\int_{\mathcal{B}} R\,|\mathcal{V}|_C^2\,\mu(C) + \int_{\mathcal{B}} R\overline{W}(C, G)\mu(C) \tag{3.44}$$

where $\overline{\mathcal{M}} = \mathcal{M} \otimes \mu(C) = \mathcal{R}\mathcal{V}^\flat \otimes \mu(C) \equiv \rho_0 \mathcal{V}^\flat \otimes \mu(G)$. The partial functional derivatives are computed in the standard fashion by considering curves $\varepsilon \mapsto C_\varepsilon \in S_2(\mathcal{B})$ and $\varepsilon \mapsto \overline{\mathcal{M}}_\varepsilon \in \mathfrak{X}^*(\mathcal{B})$ and using the directional derivative formula. One finds

$$\frac{\delta \overline{H}}{\delta C} = \frac{1}{2}[-\mathcal{R}\mathcal{V} \otimes \mathcal{V} + \Sigma] \otimes \mu(C); \quad \frac{\delta H}{\delta \overline{\mathcal{M}}} = \mathcal{V}. \tag{3.45}$$

Using $(3.45)_2$ and the definition of the Lie bracket of vector fields in $\mathfrak{X}(\mathcal{B})$, the first term in (3.42b) becomes:

$$\int_{\mathcal{B}} \overline{\mathcal{M}} \cdot \left[\frac{\delta f}{\delta \overline{\mathcal{M}}}, \frac{\delta H}{\delta \overline{\mathcal{M}}}\right] = \int_{\mathcal{B}} \overline{\mathcal{M}} \cdot \left[\frac{\delta f}{\delta \overline{\mathcal{M}}}, \mathcal{V}\right] = \int_{\mathcal{B}} \overline{\mathcal{M}} \cdot L_{\frac{\delta f}{\delta \overline{\mathcal{M}}}} \mathcal{V}. \tag{3.46}$$

The second term in (3.42b) reduces to:

$$\int_{\mathcal{B}} \overline{\frac{\delta f}{\delta C}} : \mathcal{L}_{\frac{\delta H}{\delta \mathcal{M}}} C = \int_{\mathcal{B}} \frac{\delta f}{\delta C} : \mathcal{L}_{\nu} C. \tag{3.47}$$

The last term in (3.42b) requires a move elaborated computation. First, we use the derivation property of Lie derivatives to obtain:

$$-\int_{\mathcal{B}} \frac{\delta H}{\delta C} : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} C = -\int_{\mathcal{B}} \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}}\left(C : \frac{\delta H}{\delta C}\right) + \int_{\mathcal{B}} C : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} \frac{\delta H}{\delta C}. \tag{3.48}$$

Substitution of $(3.45)_1$ into (3.48) and use of standard additional properties of the Lie derivative yields

$$\begin{aligned}
-\int_{\mathcal{B}} \frac{\delta H}{\delta C} : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} C &= \int_{\mathcal{B}} \frac{1}{2} \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} [|\nu|_C^2 \otimes \mathcal{R}\mu(C) - (C : \Sigma) \otimes \mu(C)] \\
&\quad - \int_{\mathcal{B}} C : [(\mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} \nu) \otimes \nu \otimes \mathcal{R}\mu(C)] + \frac{|\nu|_C^2}{2} \otimes \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} [\mathcal{R} \otimes \mu(C)] \\
&\quad + \int_{\mathcal{B}} \frac{1}{2} C : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} [\Sigma \otimes \mu(C)] \\
&= \int_{\mathcal{B}} \left[\frac{\mathcal{R}}{2} d\, |\nu|_C^2\, \mu(C) - \overline{\mathcal{M}} \cdot \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} \nu\right] \\
&\quad + \frac{1}{2} \int_{\mathcal{B}} [C : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} \Sigma - \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} (C : \Sigma)] \otimes \mu(C).
\end{aligned} \tag{3.49}$$

By exploiting the symmetry of $\Sigma$ and the expression in local coordinates for the Lie derivative, the term $C : \mathcal{L}_{\delta f/\delta \mathcal{M}} \Sigma$ becomes:

$$\begin{aligned}
\frac{1}{2} C : \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} \Sigma &= \frac{1}{2} d(C : \Sigma) \cdot \frac{\delta f}{\delta \mathcal{M}} - \left(\nabla \frac{\delta f}{\delta \mathcal{M}}\right) : \Sigma \\
&= \frac{1}{2} \mathcal{L}_{\frac{\delta f}{\delta \mathcal{M}}} (C : \Sigma) - \left(\nabla \frac{\delta f}{\delta \mathcal{M}}\right) : \Sigma ,
\end{aligned} \tag{3.50}$$

where in coordinates, $(\nabla\, \delta f/\delta \overline{\mathcal{M}}) : \boldsymbol{\Sigma} = (\delta f/\delta \overline{\mathcal{M}}_A)_{|B}\, C_{AD}\, \boldsymbol{\Sigma}^{DB}$. Substitution of (3.50) into (3.49), use of integration by parts and Gauss theorem then yields

$$
\begin{aligned}
-\int_{\mathcal{B}} \frac{\delta H}{\delta C} : \mathbf{L}_{\frac{\delta f}{\delta \overline{\mathcal{M}}}} \mathbf{C} &= \int_{\mathcal{B}} \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet \left[ \frac{\mathcal{R}}{2}\, \mathrm{d} \left| \boldsymbol{\mathcal{V}} \right|_{C}^{2} + (\mathbf{Div}_C\, \boldsymbol{\Sigma})^{\flat} \right] \otimes \mu(C) \\
&\quad + \int_{\mathcal{B}} -\mathbf{Div}_C \left( \boldsymbol{\Sigma}\, \frac{\delta f}{\delta \overline{\mathcal{M}}} \right) \otimes \mu(C) - \overline{\mathcal{M}} \bullet \mathbf{L}_{\frac{\delta f}{\delta \overline{\mathcal{M}}}} \boldsymbol{\mathcal{V}} \\
&= \int_{\mathcal{B}} \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet \left[ \frac{\mathcal{R}}{2}\, \mathrm{d} \left| \boldsymbol{\mathcal{V}} \right|_{C}^{2} + (\mathbf{Div}_C\, \boldsymbol{\Sigma})^{\flat} \right] \otimes \mu(C) \\
&\quad - \int_{\partial \mathcal{B}} \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet (\boldsymbol{\Sigma} \hat{N})^{\flat}\, \gamma(C) - \int_{\mathcal{B}} \overline{\mathcal{M}} \bullet \mathbf{L}_{\frac{\delta f}{\delta \overline{\mathcal{M}}}} \boldsymbol{\mathcal{V}}.
\end{aligned}
\tag{3.51}
$$

Substituting (3.46), (3.47) and (3.51) into (3.42b) yields

$$
\begin{aligned}
\{f, H\} &= \int_{\mathcal{B}} \frac{\delta f}{\delta C} : \mathbf{L}_{\boldsymbol{\mathcal{V}}} C + \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet \left[ \frac{\mathcal{R}}{2}\, \mathrm{d} \left| \boldsymbol{\mathcal{V}} \right|_{C}^{2} + (\mathbf{Div}_C\, \boldsymbol{\Sigma})^{\flat} \right] \otimes \mu(C) \\
&\quad - \int_{\partial \mathcal{B}} \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet (\boldsymbol{\Sigma} \hat{N})^{\flat}\, \gamma(C) \\
&\quad - \int_{\partial_\varphi \mathcal{B}} \left[ \boldsymbol{\mathcal{V}} \bullet \left( 2 \frac{\delta f}{\delta C} \hat{N} \right)^{\flat} - \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet (\boldsymbol{\Sigma} \hat{N})^{\flat} - \mathcal{M}(\boldsymbol{\mathcal{V}} \cdot \hat{N}) \right] \gamma(C)\, .
\end{aligned}
\tag{3.52}
$$

Since $\boldsymbol{\mathcal{V}}\, |\partial_\varphi \mathcal{B} = \mathbf{O}$ the boundary term in (3.52) collapses to

$$
-\int_{\partial_\sigma \mathcal{B}} \frac{\delta f}{\delta \overline{\mathcal{M}}} \bullet (\boldsymbol{\Sigma} \hat{N})^{\flat}\, \gamma(C). \tag{3.53}
$$

On the other hand, for any $f : \mathcal{P}_{\text{conv}} \to \mathbb{R}$, we have

$$\dot{f} = \int_{\mathcal{B}} \frac{\delta f}{\delta C} : \dot{C} + \frac{\delta f}{\delta \mathcal{M}} \cdot \dot{\mathcal{M}} \tag{3.54}$$

Comparing (3.51) - (3.52) and (3.54), the desired result follows. ■

# §4. *The Rotation Group and Rigid Body Dynamics*

In what follows we summarize some basic notation and elementary properties of the rotation group needed for subsequent developments. For a more detailed account we refer the reader to standard textbooks, such as Abraham & Marsden [1982, §4.1] and Choquet, DeWitt-Morette & Dillard-Bleich [1984, pp. 181-194]. We then review the Hamiltonian structure of rigid body dynamics.

## *The Rotation Group and its Lie Algebra*

Following standard usage, we denote by SO(3) the Lie group of proper orthogonal transformations, i.e.,

$$\mathrm{SO}(3) := \left\{ \Lambda : \mathbb{R}^3 \to \mathbb{R}^3 \;\middle|\; \Lambda \text{ is linear}, \; \Lambda^T\Lambda = \mathbf{I}, \text{ and } \det \Lambda = \mathbf{1} \right\}. \qquad (4.1)$$

SO(3) is a compact subgroup of the general linear group GL(3). Its Lie algebra is $\mathbf{so}(3) := \{\hat{\Theta} : \mathbb{R}^3 \to \mathbb{R}^3 \mid \hat{\Theta}$ is linear and $\hat{\Theta} + \hat{\Theta}^T = 0\}$, the set of all skew-symmetric tensors. In coordinates, relative to an orthonormal basis $\{e_i\}$ in $\mathbb{R}^3$, we write $\hat{\Theta} = \hat{\Theta}^i{}_j e_i \otimes e^j$, $\Theta = \Theta^i e_i$, and, in matrix notation,

$$[\hat{\Theta}^i{}_j] = \begin{bmatrix} 0 & -\Theta^3 & \Theta^2 \\ \Theta^3 & 0 & -\Theta^1 \\ -\Theta^2 & \Theta^1 & 0 \end{bmatrix}, \quad \{\Theta^i\} = \begin{bmatrix} \Theta^1 \\ \Theta^2 \\ \Theta^3 \end{bmatrix}. \qquad (4.2)$$

Recall that $\mathbf{so}(3)$ and $\mathbb{R}^3$ are related through the isomorphism $\hat{\ } : \mathbb{R}^3 \to \mathbf{so}(3)$, defined by the relation $\hat{\theta} h = \theta \times h$, for any $h \in \mathbb{R}^3$. Here, $\theta \in \mathbb{R}^3$ is the axial vector of $\hat{\theta} \in \mathbf{so}(3)$, and $\times$ denotes the ordinary vector product. Physically, $\Lambda \in \mathrm{SO}(3)$ represents a finite rotation while infinitesimal rotations are linearized rotations about the identity. Geometrically, one speaks of $\mathbf{so}(3)$ as the tangent space of SO(3) at the identity $\mathbf{I} \in \mathrm{SO}(3)$, and employs the notation $\mathbf{so}(3) = T_{\mathbf{I}}\mathrm{SO}(3)$.

## *Left and Right Representations of the Tangent Space to SO(3)*

Given any $\Lambda \in SO(3)$, elements $\hat{\Theta}_\Lambda$ of the tangent space $T_\Lambda SO(3)$ at a point $\Lambda \in SO(3)$ are represented in two alternative forms:

(i) *Left invariant vector fields* defined by the relation $\hat{\Theta}_\Lambda := T_I L_\Lambda \hat{\Theta}$, where $\hat{\Theta} \in \mathbf{so}(3)$, and $L_\Lambda : SO(3) \to SO(3)$ denotes left translation; i.e., $L_\Lambda \Lambda_1 := \Lambda \Lambda_1$. A simple calculation yields $\hat{\Theta}_\Lambda = \Lambda \hat{\Theta}$. Accordingly, we have the following identification of $T_\Lambda SO(3)$:

$$T_\Lambda SO(3) = \{\hat{\Theta}_\Lambda := \Lambda \hat{\Theta} \mid \text{for any } \hat{\Theta} \in \mathbf{so}(3)\} . \tag{4.3a}$$

Geometrically, an element $\hat{\Theta}_\Lambda \in T_\Lambda SO(3)$ corresponds to a finite rotation superimposed onto an infinitesimal rotation $\hat{\Theta} \in \mathbf{so}(3)$.

(ii) *Right invariant vector fields*. The characterization is identical to that in (i), with left translations replaced by right translations $R_\Lambda : SO(3) \to SO(3)$ defined as $R_\Lambda \Lambda_1 = \Lambda_1 \Lambda$. This leads to the representation

$$T_\Lambda SO(3) := \{\hat{\theta}_\Lambda := \hat{\theta} \Lambda \mid \text{for any } \hat{\theta} \in \mathbf{so}(3)\} . \tag{4.3b}$$

Geometrically, an element $\hat{\theta}_\Lambda \in T_\Lambda SO(3)$ represents an infinitesimal rotation superimposed onto a finite rotation represented by $\Lambda$. Following the general conventions we use in elasticity, left invariant vector fields at a point $\Lambda$ are denoted by upper case letters; i.e., $\hat{\Theta}_\Lambda$. We think of $\hat{\Theta} \in \mathbf{so}(3)$, with axial vector $\Theta \in \mathbb{R}^3$, as a *material object* with coordinates $\Theta^I$ relative to a material basis $\{E_I\}$. In the context of rigid body dynamics one often speaks of *body representations* and *body coordinates* relative to the body frame $\{t_I\}$ defined by $t_I := \Lambda E_I$. On the other hand, right invariant vector fields at a point $\Lambda$ are denoted by lower case letters; i.e., $\hat{\theta}_\Lambda$. One thinks of $\hat{\theta} \in \mathbf{so}(3)$, with axial vector $\theta \in \mathbf{R}^3$, as a spatial object and coordinates $\theta^i$ relative to a spatially fixed basis $\{e_i\}$. In the context of rigid body mechanics one speaks of the *spatial representation*.

## *The Dual of the Lie Algebra*

We also recall that $\mathbf{so}(3)^*$, the cotangent space at the identity $I \in SO(3)$, is isomorphic to $[\mathbb{R}^3, \times]$ and to $\mathbf{so}(3)$ via the dot product; that is, we also identify elements of

$\mathbf{so}(3)^*$ with skew symmetric matrices and use the map $\hat{}$ and the pairing given by

$$\hat{\Pi}(\hat{W}) = \frac{1}{2}\hat{\Pi}:\hat{W} \equiv \Pi\cdot W \ , \tag{4.4}$$

where $\hat{\Pi}:\hat{W} = \mathrm{tr}[\Pi^T W]$, and we have used the notation $A:B = A^I{}_J B_I{}^J$ for any $A,B \in GL(3)$. Thus, the duality pairing

$$\langle\cdot\,,\cdot\rangle : T^*_\Lambda SO(3) \times T_\Lambda SO(3) \to \mathbb{R} \tag{4.5a}$$

is defined by

$$\langle\hat{\Pi}_\Lambda, \hat{W}_\Lambda\rangle := \frac{1}{2}\mathrm{tr}[\hat{\Pi}_\Lambda \hat{W}_\Lambda^T] \equiv \frac{1}{2}\hat{\Pi}_\Lambda:\hat{W}_\Lambda \ . \tag{4.5b}$$

Note that this duality pairing is *left -invariant* in the sense that

$$\langle\hat{\Pi}_\Lambda, \hat{W}_\Lambda\rangle := \frac{1}{2}\mathrm{tr}\left[\hat{\Pi}^T\Lambda^T\Lambda\hat{W}\right] \equiv \frac{1}{2}\hat{\Pi}:\hat{W} = \langle\hat{\Pi},\hat{W}\rangle. \tag{4.5c}$$

## *The Exponential Map*

Finally, we recall that the straight line $\varepsilon \mapsto \varepsilon\hat{\theta} \in T_I SO(3)$ , for $\varepsilon > 0$, is mapped by the *exponential map* onto the curve

$$\varepsilon \mapsto \exp[\varepsilon\hat{\theta}] = \Big[\sum_{k\geq 0} \frac{\varepsilon^k}{k!}\hat{\theta}^k\Big] \in SO(3)\ . \tag{4.6}$$

Note that for the case of $\hat{\theta} \in \mathbf{so}(3)$ one has the following explicit formula often credited to Rodrigues (Goldstein [1980, p. 165]):

$$\exp[\hat{\theta}] = I + \frac{\sin\|\theta\|}{\|\theta\|}\hat{\theta} + \frac{1}{2}\frac{\sin^2(\|\theta\|/2)}{(\|\theta\|/2)^2}\hat{\theta}^2 \tag{4.7}$$

This formula is of fundamental importance in the numerical solution of initial boundary value problems for geometrically exact rods (Simo & Vu-Quoc [1986a]).

Let us now recall the Hamiltonian structure of rigid body dynamics in the body (= convective) representation. This is done to enable the reader to see the parallel with the developments in the preceding and subsequent sections. It is also useful to keep this parallel in mind for the problem of coupled dynamics (Krishnaprasad and Marsden [1987] and Krishnaprasad, Marsden, Posburgh, & Simo [1987]). For further details on rigid body dynamics in this context, see Marsden, Ratiu & Weinstein [1984b].

The configuration space is $\mathcal{C} = SO(3)$ , and the Hamiltonian $H : T^*SO(3) \to \mathbb{R}$ is given by

$$H = \frac{1}{2} \Pi \cdot \mathbb{I}^{-1} \Pi \, . \tag{4.8}$$

Here, $\mathbb{I}$ is the *inertia dyadic* defined in terms of the Lagrangian coordinates $X = (X^1, X^2, X^3)$ as

$$\mathbb{I} := \int_\Omega \rho_0(X) \left[ \|X\|^2 1 - X \otimes X \right] d^3X \, , \tag{4.9}$$

and $\Pi$ is the *body momentum* defined as

$$\Pi := \mathbb{I} \, [\Lambda^T \dot{\Lambda}]^{\vee} . \tag{4.10}$$

We observe that $H$ is *invariant under spatial isometries* ; i.e., invariant under the left action of $SO(3)$ . Reduction by this symmetry amounts to considering functions $f : T^*SO(3) \to \mathbb{R}$ of the form

$$f(\Lambda, \hat{\Pi}_\Lambda) = \bar{f}(\Lambda^T \hat{\Pi}_\Lambda) \equiv \bar{f}(\hat{\Pi}) \, . \tag{4.11}$$

The canonical bracket on $T^*SO(3)$ is given in terms of the duality pairing $\langle \cdot , \cdot \rangle$ : $T^*_\Lambda SO(3) \times T_\Lambda SO(3) \to \mathbb{R}$ defined by (4.5) as

$$\{ f, g \} = \frac{1}{2} \left[ \left\langle \frac{\partial f}{\partial \Lambda} , \frac{\partial g}{\partial \hat{\Pi}_\Lambda} \right\rangle - \left\langle \frac{\partial g}{\partial \Lambda} , \frac{\partial f}{\partial \hat{\Pi}_\Lambda} \right\rangle \right] . \tag{4.12}$$

The *Lie-Poisson* bracket on the reduced space $\mathcal{P}$ = SO(3)\T*SO(3) is obtained with the aid of the results given below.

**Proposition 4.1.** *The following formulae hold*

$$\frac{\partial \bar{f}}{\partial \Lambda} = -\frac{1}{2}\,\Lambda\left[\Pi \times \frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge}, \tag{4.13}$$

$$\frac{\partial \bar{f}}{\partial \hat{\Pi}_{\Lambda}} = \Lambda\left[\frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge}. \tag{4.14}$$

where $\bar{f}(\hat{\Pi}) = \tilde{f}(\Pi)$.

**Proof.** To prove (4.13), observe that for any $\delta\Lambda \in T_{\Lambda}SO(3)$ we have the left representation $\delta\Lambda = \Lambda\delta\hat{\Theta}$, where $\delta\hat{\Theta} \in \mathbf{so}(3)$. Thus, by the chain rule and the left invariance of the duality pairing we have

$$\left\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \right\rangle = \left\langle \frac{\partial \bar{f}}{\partial \hat{\Pi}}, \delta\Lambda^{T}\hat{\Pi}_{\Lambda} \right\rangle$$

$$= \left\langle -\frac{\partial \bar{f}}{\partial \hat{\Pi}}, \delta\hat{\Theta}\,\hat{\Pi} \right\rangle$$

$$= \left\langle -\hat{\Pi}\,\frac{\partial f}{\partial \hat{\Pi}}, \delta\hat{\Theta} \right\rangle.$$

Since $\frac{\partial \bar{f}}{\partial \hat{\Pi}} \in \mathbf{so}(3)$, it follows that

$$\left\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \right\rangle = \left\langle -\frac{1}{2}\left[\hat{\Pi}, \frac{\partial \bar{f}}{\partial \hat{\Pi}}\right], \delta\hat{\Theta} \right\rangle,$$

so that, on setting $\bar{f}(\hat{\Pi}) = \tilde{f}(\Pi)$ and recalling the Lie bracket relation

$$\hat{A}\hat{B} - \hat{B}\hat{A} = [A \times B]^{\wedge},$$

we obtain

$$\langle \frac{\partial f}{\partial \Lambda}, \delta\Lambda \rangle = \langle -\frac{1}{2} \Lambda \left[ \Pi \times \frac{\partial \tilde{f}}{\partial \Pi} \right]^{\wedge}, \delta\Lambda \rangle,$$

and result (4.13) follows. An analogous calculation holds for the formula (4.14).■

Substitution of (4.13) and (4.14) into (4.12), and the use of standard vector product identities yield the result

$$\{f, g\} = -\Pi \cdot \left[ \frac{\partial \tilde{f}}{\partial \Pi} \times \frac{\partial \tilde{g}}{\partial \Pi} \right], \tag{4.16}$$

which is the standard Lie-Poisson bracket for rigid body dynamics. The equation $\dot{f} = \{f, H\}$ is then easily checked to be equivalent to Euler's equation for rigid body dynamics: $\dot{\Pi} = \Omega \times \Pi$, where $\hat{\Omega} = \Lambda^T \dot{\Lambda}$ is the body angular velocity.

# §5. *Geometrically Exact Rod Models*

The static version of the rod model summarized below goes back essentially to Reissner [1973] who modified the classical Kirchhoff - Love model (see Love [1944]) to account for shear deformation. An equivalent model, formulated as a constrained director theory - the so-called special theory of Cosserat rods - is due to Antman [1974] - see also Antman & Jordan [1975], Antman & Kenney [1981], and Antman [1984] for some applications. The dynamic version along with the parametrization discussed below is given in Simo [1985]. For completeness, a brief account is outlined next.

## *The Configuration Space*

From a physical standpoint, the configurations of a rod deforming in the ambient space $\mathbb{R}^3$ may be defined by specifying : (i) The position of its line of centroids by means of the map $\varphi : [0, L] \to \mathbb{R}^3$, and (ii) The orientation of cross sections at points $S \in [0, L]$. This can be done using the orientation of a moving basis $\{\mathbf{t}_I(S) \mid I = 1,2,3\}$ attached to the cross section relative to a fixed frame $\{\mathbf{E}_I \mid I = 1,2,3\}$, referred to as a *material frame* in what follows. The moving basis is described by means of an orthogonal transformation $\Lambda : [0, L] \to SO(3)$ such that $t_I(S) = \Lambda(S)E_I$.

If we view the rod as having a finite cross section given by a compact set $\Omega \subset \mathbb{R}^2$ with a smooth boundary $\partial\Omega$, then the placement of the rod in its reference configuration is determined by a map $\Phi_0(S) = (\varphi_0(S), \Lambda_0(S))$ in such a way that the corresponding set ocupied by the rod is given by

$$\mathcal{B} = \left\{ X \in \mathbb{R}^3 \;\middle|\; X = \varphi_0(S) + \sum_{\alpha=1}^{2} \xi^\alpha \Lambda_0(S) E_\alpha, \text{ where } (\xi^1, \xi^2, S) \in \Omega \times [0, L] \right\} \tag{5.1a}$$

Without loss of generality we can assume that $\|\varphi_0'(S)\| = 1$, so that $L$ is the length of the reference line of centroids. Typically, one chooses $\Lambda_0(S)$ so that $T_I(S) := \Lambda_0(S)E_I$ is the Frenet frame associated with the curve $\varphi_0$; that is,

$$T_3 := \varphi_0', \qquad T_1 := \varphi_0''/\|\varphi_0''\| \qquad \text{and} \qquad T_2 := T_3 \times T_1 \tag{5.1b}$$

Accordingly, we take for our configuration space the set

$$C = \{\Phi = (\varphi, \Lambda) \mid [0, L] \rightarrow \mathbf{R}^3 \times SO(3)\}. \tag{5.2}$$

For simplicity, we shall assume that $\varphi$ and $\Lambda$ satisfy pure displacement boundary conditions in what follows; i.e., $\varphi|_{S=0,L}$ and $\Lambda|_{S=0,L}$ are prescribed. More general boundary conditions are handled by a procedure similar to that explained in detail in Section 3. Hence, the tangent space at the identity configuration is given by

$$\begin{aligned} &T_{\text{identity}}C \\ &= \{(\delta\varphi, \delta\hat{\Theta}): [0, L] \rightarrow \mathbf{R}^3 \times \mathbf{so}(3) \mid \delta\varphi|_{S=0,L} = 0 \\ &\qquad \text{and } \delta\hat{\Theta}|_{S=0,L} = 0\}. \end{aligned} \tag{5.3}$$

Left and right invariant tangent vector fields at a configuration $\Phi \in C$ are defined below in the standard fashion by employing left and right translations. Finally, associated with any configuration $\Phi \in C$ is its *arc-length* defined by the mapping

$$S \in [0, L] \mapsto s = \tilde{s}(S) := \int_0^S \left\| \frac{\partial}{\partial \xi} \varphi(\xi) \right\| d\xi. \tag{5.3'}$$

The arc length may be used to parametrize points on the center line of the current configuration. For the convective description, this is not necessary, but it is convenient in the spatial description. We shall tacitly assume that the rod does not self-intersect; that is, that on the image of the above mapping, there is a well defined smooth inverse mapping $x \mapsto S$, where $x$ is a point in the image of $\Phi$. The image curve is parametrized by the arc length and so we will regard the role of $x$ as being played by $s$ in what follows.

## *Motions and Velocity Fields*

A ***motion*** is a curve of configurations $t \in [0,T] \mapsto \Phi_t = (\varphi_t, \Lambda_t) \in C$, for some time interval $[0,T]$. Associated with a motion, is the ***material velocity field*** $V_\Phi(S, t)$ defined by

$$V_\Phi(S, t) := \frac{\partial}{\partial t}\Phi(S,t) := \left(\dot{\varphi}_t(S, t), \dot{\Lambda}_t(S, t)\right) \tag{5.4}$$

Thus at any time t, the material velocity is an element of $T_\Phi C$, the tangent space to $C$ at the configuration $\Phi$ at time $t$. The ***spatial velocity field*** $v_\Phi(s,t)$ is defined by

$$v_\Phi(s,t) := (\dot{\varphi}(S,t), \dot{\Lambda}(S,t)), \tag{5.5b}$$

where, as above, $s = \tilde{s}(S,t)$ is the arc length in the current configuration. Finally, the ***convected velocity field*** $\boldsymbol{\mathcal{V}}_\Phi(S,t)$ is defined by the expression

$$\boldsymbol{\mathcal{V}}_\Phi(S,t) := \left([\Lambda(S,t)]^T\dot{\varphi}(S,t),\ [\Lambda(S,t)]^T\dot{\Lambda}(S,t)\right). \tag{5.5c}$$

Since $\Lambda(S,t) \in SO(3)$, we can write

$$\dot{\Lambda}(S,t) = \Lambda(S,t)\,\hat{W}(S,t) = \hat{w}(s,t)\,\Lambda(S,t) \quad , \tag{5.6}$$

where $\hat{W}(S,t) \in \mathbf{so}(3)$ and $\hat{w}(s,t) \in \mathbf{so}(3)$ . Accordingly, the convected and spatial velocity fields can be expressed as

$$\begin{aligned} \boldsymbol{\mathcal{V}}_\Phi(S,t) &= \left([\Lambda(S,t)]^T\,\dot{\varphi}(S,t),\ \hat{W}(S,t)\right), \\ v_\Phi(s,t) &= (\dot{\varphi}(S,t),\ \hat{w}(s,t))\,. \end{aligned} \tag{5.7}$$

**Remark.** One can easily check that the components of the spatial velocity in the moving frame $\{t_I(S)\}$ are equal to the components of the convected velocity in the inertial frame $\{E_I\}$. We also remark that the material velocity field $V_\Phi$ may be viewed either as a left extension of the convected velocity field $\boldsymbol{\mathcal{V}}_\Phi$ or as a right extension of the spatial velocity field $v_\Phi$. ■

## *Strain Measures.*

As in the three-dimensional theory, we define the (one-dimensional) ***deformation gradient*** as

$$\Phi'(S,t) := \left(\varphi'(S,t), \Lambda'(S,t)\right), \quad \text{where} \quad (\cdot)' := \partial/\partial S \quad . \tag{5.8}$$

Similarly, in parallel with the definition of velocity fields, starting from the (Lagrangian) deformation gradient $\Phi'(S,t)$ we define ***convected and spatial strains*** according to

| *convected* | *material* | *spatial* |
|---|---|---|
| $\Gamma(S,t) := [\Lambda(S,t)]^T \cdot \varphi'(S,t)$ | $\varphi'(S,t)$ | $\gamma(s,t) := \partial\varphi(S,t)/\partial s$ |
| $\hat{\Omega}(S,t) := [\Lambda(S,t)]^T \cdot \Lambda'(S,t)$ | $\Lambda'(S,t)$ | $\hat{\omega}(s,t) := [\partial\Lambda(S,t)/\partial s] \cdot \Lambda^T(S,t),$ |

(5.9)

where $s = \sim s(S,t)$. We again note that the components of the convected strains in the material frame $\{E_I\}$ coincide with the components of the spatial strains in the moving frame $\{\mathbf{t}_I(S)\}$ up to a factor (because of the arc length parametrization that may be used in the spatial description.) The above expressions can be derived from the three-dimensional theory by a duality argument employing the formula for the stress power given in the remark below.

### *The Equations of Motion in Spatial Description*

We associate with the motion $t \mapsto \Phi_t \in \mathcal{C}$ smooth vector fields $n(s,t)$, $m(s,t)$, and a scalar field $\rho(s,t)$ interpreted respectively as the contact resultant force, contact resultant couple, and density per unit of current length. These fields satisfy the following *spatial local form* of the equations of motion:

$$\dot{\rho} + (\dot{J}J^{-1})\frac{\partial \rho}{\partial s} = 0,$$

$$\rho A v = \frac{\partial n}{\partial s} + \bar{n}, \tag{5.10a}$$

$$\rho[\mathbb{j}\dot{w} + w \times \mathbb{j}w] = \frac{\partial m}{\partial s} + \gamma \times n + \bar{m}.$$

In these equations,

$$\cdot = \frac{\partial}{\partial t} + \frac{\partial}{\partial s}\frac{\partial \tilde{s}}{\partial t} \tag{5.10b}$$

denotes the material time-derivative, and $\bar{n}$, $\bar{m}$ are the body force and torque per unit of current length $s = \tilde{s}(S, t)$. The role of the Jacobian is played by $J = \partial\tilde{s}/\partial S$; $\mathbb{J}(s, t)$ is the time dependent inertia dyadic of the cross section relative to the moving frame $\{t_I\}$ given by

$$\mathbb{J} := J^{\alpha\beta}\, t_\alpha \otimes t_\beta + J^{33}\, t_3 \otimes t_3 \tag{5.11}$$

where $J^{\alpha\beta}$ and $J_{33}$ in (5.11) and $A$ in (5.10) are inertia constants associated with the reference cross section. (Typically $J^{\alpha\beta} := \int_\Omega \xi^\alpha \xi^\beta \, d\Omega$, $\alpha,\beta = 1,2$, and $A := \int_\Omega d\Omega$, where $\xi_1$, $\xi_2$ are coordinates in the reference configuration defined by (5.1), and $J^{33}$ is the torsion modulus.)

Equations (5.10) give the local statement in the spatial description of balance of mass, balance of linear momentum and balance of angular momentum, respectively. We refer to Antman [1972],[1976] for a derivation of analogous equations from the three-dimensional theory.

## *Equations of Motion in the Convective Description*

In the convective description, the balance laws are expressed directly in the inertial frame $\{E_I\}$. To this end we define vector fields $N(S, t)$, $M(S, t)$, and $\rho_0(S)$ as pull-backs of their spatial counterparts by means of the relations

$$\begin{aligned} N(S, t) &:= [\Lambda(S, t)]^T \cdot n(s, t), \\ M(S, t) &:= [\Lambda(S, t)]^T \cdot m(s, t), \\ \rho_0(S) &:= J(S, t)\rho(s, t), \end{aligned} \tag{5.12}$$

where $s = \tilde{s}(S, t)$ and $J(S, t) = \partial\tilde{s}(S, t)/\partial S$. We use the relation

$$\frac{\partial m}{\partial s} = J^{-1}\Lambda[M' + \Omega \times M] \tag{5.13}$$

and the analogous expression connecting $n$ and $N$ to obtain readily the following statements of balance of mass, linear and angular momentum:

$$\frac{\partial \rho_0}{\partial t} = 0,$$

$$A\rho_0[\dot{\boldsymbol{\nu}} + W \times \boldsymbol{\nu}] = \frac{\partial N}{\partial S} + \Omega \times N + \overline{N},$$
$$\rho_0[J\dot{W} + W \times JW] = \frac{\partial M}{\partial S} + \Omega \times M + \Gamma \times N + \overline{M}, \tag{5.14}$$

where $\boldsymbol{\nu} = \Lambda^T\dot{\varphi}$ is the convective velocity and $J := \Lambda^T j \Lambda$ is the *time independent* inertia dyadic.

**Remark:** Within the context of the three-dimensional theory, $n$ and $m$ are defined as the resultant force and the resultant torque relative to the line of centroids of the distribution of stress acting on a cross section. The definition of $\gamma$ and $\omega$ is unique in the sense that the stress power is given by

$$\int_{\Omega\times[0,L]} P:\dot{F}\ d\Gamma\ dS = \int_{\varphi([0,L])}[n\cdot\gamma^{\nabla} + m\cdot\omega^{\nabla}]\ ds = \int_{[0,L]} N\cdot\dot{\Gamma} + M\cdot\dot{\Omega}]\ dS \tag{5.15a}$$

Here, $P$ is the first Piola-Kirchhoff stress tensor, $F$ is the deformation gradient of the configuration given by

$$\Phi = \varphi(S,t) + \sum_{\alpha=1}^{2} \xi^\alpha t_\alpha(S,t), \qquad \text{and} \qquad {}^{\nabla} = \dot{} - w\ \times \tag{5.15b}$$

is the co-rotated rate measuring the rate of change relative to the moving frame.

Appropriate stress measures conjugate to the Lagrangian strains $(\varphi', \Lambda')$ may also be obtained by either (i) left extension of the spatial stress measure $(n, m)$ or (ii) right extension of the convected stress measures $(N, M)$. For instance, the latter extension takes the form

$$\hat{N}_\Lambda := \Lambda\hat{N} \quad \text{and} \quad \hat{M}_\Lambda := \Lambda\hat{M}. \tag{5.16}$$

Obviously, such an extension preserves the stress power. ■

## *Constitutive Equations*

In view of the expression (5.15) for the stress power, we characterize (isothermal) hyperelastic response by the existence of a free energy function $\psi(S, \gamma, \omega, \Lambda)$ such that

$$\mathrm{n} = \frac{\partial\psi(S,\gamma,\omega,\Lambda)}{\partial\gamma},$$

$$\mathrm{m} = \frac{\partial\psi(S,\gamma,\omega,\Lambda)}{\partial\omega}. \tag{5.17}$$

By postulating that these equations are frame indifferent; i.e.,covariant under the left action of the Euclidean group of spatial isometries, a further reduction is obtained as follows. Let $t \mapsto Q(t) \in SO(3)$ be an arbitrary superposed rigid body rotation, and let $\Phi_t^+ := (\, Q(t)\varphi,\ Q(t)\Lambda_t) \in C$. Denoting by $(\cdot)^+$ objects associated with $\Phi_t^+$, we have the relations

$$\begin{aligned} \gamma^+ &= Q(t)\gamma, \quad n^+ = Q(t)n, \\ \omega^+ &= Q(t)\omega, \quad m^+ = Q(t)m, \\ \psi^+ &= \psi, \end{aligned} \tag{5.18}$$

Since $\psi^+(S,\ Q\gamma,\ Q\omega,\ Q\Lambda) = \psi(S,\ \gamma,\ \omega,\ \Lambda)$, we choose $Q = \Lambda^T$ to show that the only possible form compatible with frame-indifference and locality (see, for example, Marsden & Hughes [1983]) is given by

$$N = \frac{\partial\Psi(S,\Gamma,\Omega)}{\partial\Gamma}, \qquad M = \frac{\partial\Psi(S,\Gamma,\Omega)}{\partial\Omega}. \tag{5.19}$$

**Example.** An example of a constitutive equation consistent with the above invariance requirements and useful in computation (see for example, Simo and Vu-Quoc [1986b]) is furnished by the ***uncoupled linear systems :***

$$N = C_N(\Gamma - \Gamma^0)\,, \qquad M = C_M(\Omega - \Omega^0)\,, \tag{5.20}$$

where $C_N$ and $C_M$ are symmetric positive definite and constant, $\Gamma^0 = \Lambda_0{}^T\varphi_0{}'$ and $\Omega^0 = \Lambda_0{}^T\Lambda_0{}'$. This ensures that the reference configuration $\Phi_0(S) := (\varphi_0(S),\ \Lambda_0(S))$ is stress-free. Linear constitutive models of the type (5.20) are analogous to the Saint Venant-Kirchhoff model of three dimensional elasticity and are typically restricted to small strains due to the behavior of $\psi$. We refer to Ciarlet [1986, Chapter 4] for a summary of appropriate growth conditions of $\psi$.

### *The Hamiltonian in the convective description*

We define convected linear and angular momenta by

$$\mathcal{M} := \rho_0 A \mathcal{V} \qquad \text{and} \qquad \Pi := \rho_0 \mathbb{J} \mathcal{W}. \tag{5.21}$$

If, for simplicity the body forces and couples pure displacement boundary conditions hold, then the Hamiltonian is given by

$$H((\Gamma, \Omega); (\mathcal{M}, \Pi)) = \int_0^L \left((A\,\rho_0)^{-1}\|\mathcal{M}\|^2 + \Pi\cdot(\rho_0\mathbb{J})^{-1}\Pi\right)dS$$

$$+ \int_0^L \psi(S, \Gamma, \Omega)\,dS \tag{5.22}$$

In the next section we shall consider the Hamiltonian structure of this rod model on the reduced space $\mathcal{P} = SO(3)\backslash T^*\mathcal{C}$ obtained by the *left reduction* of the canonical phase space $T^*\mathcal{C}$ by the special orthogonal group $SO(3)$.

## §6. *The Hamiltonian Structure for Rods in the Convective Material and Representation*

In this section we shall derive the Poisson structure which makes the equations for our geometrically exact rod model Hamiltonian. To do this we proceed in a way that is similar to that for three-dimensional elasticity and for the free rigid body, namely, we reduce the canonical bracket in material representation by means of spatial isometries. To carry this out, we begin by recalling that the configuration space for the rod model is given by

$$\mathcal{C} = \left\{ \Phi \;\middle|\; \Phi = (\varphi, \Lambda) : [0,L] \longrightarrow \mathbb{R}^3 \times SO(3) \right\}. \tag{6.1}$$

The tangent space to $\mathcal{C}$ at a configuration $\Phi$ is given by

$$T_\Phi\mathcal{C} = \left\{ V_\Phi \;\middle|\; V_\Phi = (V_\varphi, \hat{W}_\Lambda) : [0,L] \longrightarrow \mathbb{R}^3 \times T_\Lambda SO(3) \right\} \tag{6.2}$$

and the cotangent space is given by

$$T^*_\Phi\mathcal{C} = \left\{ M_\Phi \;\middle|\; M_\Phi = (M_\varphi, \hat{\Pi}_\Lambda) : [0,L] \longrightarrow \mathbb{R}^3 \times T^*_\Lambda SO(3) \right\}. \tag{6.3}$$

For simplicity, we shall restrict our attention to pure displacement boundary conditions; i.e., $V_\Phi = (0, 0)$ for $S = 0$ and $S = L$. As remarked in the preceding sections, more general boundary conditions are handled by procedures analogous to those considered in detail in Section 3. Next, we introduce the duality pairing

$$\langle \cdot , \cdot \rangle : T_\Phi\mathcal{C} \times T^*_\Phi\mathcal{C} \longrightarrow \mathbb{R}$$

defined by

$$\langle V_\Phi, M_\Phi \rangle = \int_0^L \left[ M_\varphi \cdot V_\varphi + \frac{1}{2} \hat{\Pi}_\Lambda : \hat{W}_\Lambda \right] dS. \tag{6.4}$$

Recall that the reason for the factor 1/2 is that (4.4) and (4.5) imply that

$$\hat{\Pi}_\Lambda : \hat{W}_\Lambda = 2\, \Pi \cdot W. \tag{6.5}$$

Next we turn our attention to the formulation of the canonical Hamiltonian structure on $T^*\mathcal{C}$.

### *The Canonical Bracket*

We consider functions $f,\ g : T_{\Phi}{}^{*}\mathcal{C} \longrightarrow \mathbf{R}$ so that the canonical Poisson bracket is given by

$$\{f, g\} \ := \left\langle \frac{\delta f}{\delta \Phi}, \frac{\delta g}{\delta M_{\Phi}} \right\rangle - \left\langle \frac{\delta g}{\delta \Phi}, \frac{\delta f}{\delta M_{\Phi}} \right\rangle . \tag{6.6}$$

$$= \int_0^L \left[ \frac{\delta f}{\delta \varphi} \cdot \frac{\delta g}{\delta M_{\varphi}} - \frac{\delta g}{\delta \varphi} \cdot \frac{\delta f}{\delta M_{\varphi}} \right] dS$$

$$+ \frac{1}{2} \int_0^L \left[ \frac{\delta f}{\delta \Lambda} : \frac{\delta g}{\delta \Pi_{\Lambda}} - \frac{\delta g}{\delta \Lambda} : \frac{\delta f}{\delta \Pi_{\Lambda}} \right] dS . \tag{6.7}$$

As in section 3, we remark that the Poisson bracket (6.7) is written as if the variables $(\varphi, M_{\varphi})$ and $(\Lambda,\ \Pi_{\Lambda})$ were both independent. This however, requires some caution since $T^{*}\mathcal{C}$ is not simply a product space. The formal justification for this convention relies on a construction analogous to that explained in detail in Section 3 (see equations (3.4)-(3.6)).

### *The Reduced (Poisson) Bracket*

In the present context, left reduction of $T^{*}\mathcal{C}$ by the group $SO(3)$ amounts to considering functions $f : T^{*}\mathcal{C} \longrightarrow \mathbb{R}$ which are $SO(3)$ invariant; these have the following form:

$$f(\Phi;\ M_{\Phi}) \ \equiv \ f\big((\varphi,\ \Lambda);\ (M_{\varphi},\ \Pi_{\Lambda})\big)$$

$$= \ \bar{f}\big((\Lambda^T\varphi',\ \Lambda^T\Lambda');\ (\Lambda^T M_{\varphi},\ \Lambda^T \Pi_{\Lambda})\big) , \tag{6.8}$$

where $(\cdot)' \equiv d(\cdot)/dS$. Introducing the notation

$$\Gamma := \Lambda^T\varphi' , \quad \Omega := \Lambda^T\Lambda',$$

$$\mathcal{M} := \Lambda^T M_{\varphi} , \quad \Pi := \Lambda^T \Pi_{\Lambda}, \tag{6.9}$$

which is consistent with the left representation of $T_{\Phi}\mathcal{C}$, we may rewrite equation (6.8) as

$$f((\varphi,\Lambda);(M_\varphi,\hat{\Pi}_\Lambda)) = \bar{f}((\Gamma,\hat{\Omega});(\mathcal{M},\hat{\Pi}))$$
$$= \tilde{f}((\Gamma,\Omega);(\mathcal{M},\Pi)) \ . \tag{6.10}$$

We now obtain a *reduced* bracket in terms of the variables $((\Gamma,\ \Omega);\ (\mathcal{M},\ \hat{\Pi}))$ by transforming the canonical bracket with the aid of the chain rule. The key result is contained in the following:

**Proposition 6.1.** *The following formulae hold:*

$$\frac{\delta f}{\delta \hat{\Pi}_\Lambda} = \Lambda\left[\frac{\partial \tilde{f}}{\partial \Pi}\right]^{\wedge} , \tag{6.11a}$$

$$\frac{\delta f}{\delta \varphi} = -\Lambda\left[\left(\frac{\partial \tilde{f}}{\partial \Gamma}\right)' + \Omega \times \frac{\partial \tilde{f}}{\partial \Gamma}\right], \tag{6.11b}$$

$$\frac{\delta f}{\delta M_\varphi} = \Lambda\,\frac{\partial \tilde{f}}{\delta \mathcal{M}} , \tag{6.11c}$$

$$\frac{\delta f}{\delta \Lambda} = -\Lambda\left[\Gamma \times \frac{\partial \tilde{f}}{\partial \Gamma} + \mathcal{M} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \frac{1}{2}\Pi \times \frac{\partial \tilde{f}}{\partial \Pi} + \left(\frac{\partial \tilde{f}}{\partial \Omega}\right)' + \Omega \times \frac{\partial \tilde{f}}{\partial \Omega}\right]^{\wedge} . \tag{6.11d}$$

**Proof.** The first three formulae above follow directly from the chain rule. To prove the last one, we make use of the chain rule, the skew-symmetry of $\partial\bar{f}/\partial\hat{\Omega}$, integration by parts, and relation $\Lambda' = \Lambda\,\hat{\Omega}$, to obtain

$$\left\langle \frac{\delta f}{\delta \Lambda}, \delta\Lambda \right\rangle = \int_0^L \left\{ \frac{\partial \bar{f}}{\partial \Gamma}\cdot \delta\Lambda^T\varphi' + \frac{1}{2}\,\mathrm{tr}\left[\frac{\partial \bar{f}}{\partial \hat{\Omega}}\left(\delta\Lambda^T\Lambda' + \Lambda^T\delta\Lambda'\right)\right]\right.$$
$$\left. + \frac{\partial \bar{f}}{\partial \mathcal{M}}\cdot \delta\Lambda^T\varphi' + \frac{1}{2}\,\mathrm{tr}\left[\frac{\partial \bar{f}}{\partial \hat{\Pi}}\,\delta\Lambda^T\Pi_\Lambda\right.\right\} dS \tag{6.12}$$

$$= \int_0^L \Lambda^T\delta\Lambda : \left[\Gamma \otimes \frac{\partial\bar{f}}{\partial\Gamma} + \mathcal{M} \otimes \frac{\partial\bar{f}}{\partial\mathcal{M}} - \frac{1}{2}\hat{\Omega}\,\frac{\partial\bar{f}}{\partial\hat{\Omega}}\right] ds$$

$$- \int_0^L \frac{1}{2}\,\delta\Lambda : \left[\left(\Lambda\frac{\partial\bar{f}}{\partial\hat{\Omega}}\right)' + \Lambda\hat{\Pi}\,\frac{\partial\bar{f}}{\partial\hat{\Pi}}\right] ds\,, \qquad (6.13)$$

Since $\delta\Lambda \in T_\Lambda SO(3)$ we have the left-representation $\delta\Lambda = \Lambda\delta\hat{\Theta}$, where $\delta\hat{\Theta} \in \mathbf{so}(3)$, so we may rewrite (6.13) as

$$\left\langle \frac{\delta f}{\delta\Lambda}, \delta\Lambda \right\rangle = \int_0^L \delta\hat{\Theta} : \left[\Gamma \otimes \frac{\partial\bar{f}}{\partial\Gamma} + \mathcal{M} \otimes \frac{\partial\bar{f}}{\partial\mathcal{M}} - \hat{\Omega}\,\frac{\partial\bar{f}}{\partial\hat{\Omega}}\right.$$
$$\left. - \frac{1}{2}\hat{\Pi}\,\frac{\partial f}{\partial\hat{\Pi}} - \frac{1}{2}\left(\frac{\partial\bar{f}}{\partial\hat{\Omega}}\right)'\right] ds\,. \qquad (6.14)$$

Since $\delta\hat{\Theta} \in \mathbf{so}(3)$, it follows that

$$\delta\hat{\Theta} : A \equiv \delta\hat{\Theta} : \left[\frac{A - A^T}{2}\right]\,, \qquad (6.15)$$

for any $A \in GL(\mathbb{R}^3)$. Since $\partial\bar{f}/\partial\hat{\Omega} = [\partial\tilde{f}/\partial\Omega]^\wedge$, the identities

$$[a \otimes b - b \otimes a] = -[a \times b]^\wedge\,, \quad a, b \in \mathbb{R}^3\,, \qquad (6.16)$$

$$[\hat{A}\hat{B} - \hat{B}\hat{A}] = [A \times B]^\wedge\,, \quad \hat{A}, \hat{B} \in \mathbf{so}(3)\,, \qquad (6.17)$$

yield

$$\left\langle \frac{\delta f}{\delta\Lambda}, \delta\Lambda \right\rangle = -\int_0^L \delta\hat{\Theta} : \frac{1}{2}\left[\Gamma \times \frac{\partial\tilde{f}}{\partial\Gamma} + \mathcal{M} \times \frac{\partial\tilde{f}}{\partial\mathcal{M}} + \Omega \times \frac{\partial\tilde{f}}{\partial\Omega}\right.$$
$$\left. + \frac{1}{2}\Pi \times \frac{\partial\tilde{f}}{\partial\Pi} + \left(\frac{\partial\tilde{f}}{\partial\Omega}\right)'\right]^\wedge ds\,. \qquad (6.18)$$

The result follows from the replacement of $\delta\hat{\Theta}$ by $\Lambda^T\delta\Lambda$. ■

Using the formulae in Proposition 6.1 and employing the notation $\{f, g\} = \{f, g\}_\varphi + \{f, g\}_\Lambda$ where

$$\{f, g\}_\Lambda := \frac{1}{2}\int_0^L \left(\frac{\partial f}{\partial \Lambda} : \frac{\partial g}{\partial \hat{\Pi}_\Lambda} - \frac{\partial g}{\partial \Lambda} : \frac{\partial f}{\partial \hat{\Pi}_\Lambda}\right) dS \ , \tag{6.19a}$$

$$\{f, g\}_\varphi := \int_0^L \left(\frac{\partial f}{\partial \varphi} \cdot \frac{\partial g}{\partial M_\varphi} - \frac{\partial f}{\partial M_\varphi} \cdot \frac{\partial g}{\partial \varphi}\right) dS \ , \tag{6.19b}$$

we have

$$\{f, g\}_\varphi = -\int_0^L \left[\left(\frac{\partial \tilde{f}}{\partial \Gamma}\right)' \cdot \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \left(\frac{\partial \tilde{g}}{\partial \Gamma}\right)' \cdot \frac{\partial \tilde{f}}{\partial \mathcal{M}}\right] dS$$

$$-\int_0^L \Omega \cdot \left[\frac{\partial \tilde{f}}{\partial \Gamma} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \frac{\partial \tilde{g}}{\partial \Gamma} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}}\right] dS \ .$$

Since $\hat{A} : \hat{B}/2 \equiv A \cdot B$, we obtain

$\{f, g\}_\Lambda :=$

$$-\int_0^L \left(\frac{1}{2}\Pi \times \frac{\partial \tilde{f}}{\partial \Pi} + \mathcal{M} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \Gamma \times \frac{\partial \tilde{f}}{\partial \Gamma} + \Omega \times \frac{\partial \tilde{f}}{\partial \Omega} + \left(\frac{\partial \tilde{f}}{\partial \Omega}\right)'\right] \cdot \frac{\partial \tilde{g}}{\partial \Pi}\, dS$$

$$-\int_0^L \left(\frac{1}{2}\Pi \times \frac{\partial \tilde{g}}{\partial \Pi} + \mathcal{M} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} + \Gamma \times \frac{\partial \tilde{g}}{\partial \Gamma} + \Omega \times \frac{\partial \tilde{g}}{\partial \Omega} + \left(\frac{\partial \tilde{g}}{\partial \Omega}\right)'\right] \cdot \frac{\partial \tilde{f}}{\partial \Pi}\, dS, \tag{6.21}$$

so that

$$\{f, g\}_\Lambda = -\int_0^L \Big[\Pi \cdot \left(\frac{\partial \tilde{f}}{\partial \Pi} \times \frac{\partial \tilde{g}}{\partial \Pi}\right) + \left(\frac{\partial \tilde{f}}{\partial \Omega}\right)' \cdot \frac{\partial \tilde{g}}{\partial \Pi} - \left(\frac{\partial \tilde{g}}{\partial \Omega}\right)' \cdot \frac{\partial \tilde{f}}{\partial \Pi}$$

$$+ \mathcal{M} \cdot \left(\frac{\partial \tilde{f}}{\partial \mathcal{M}} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \mathcal{M}} \times \frac{\partial \tilde{f}}{\partial \Pi}\right) \tag{6.22}$$

$$+ \Omega \cdot \left( \frac{\partial \tilde{f}}{\partial \Omega} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \Omega} \times \frac{\partial \tilde{f}}{\partial \Pi} \right)$$

$$+ \Gamma \cdot \left( \frac{\partial \tilde{f}}{\partial \Gamma} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \Gamma} \times \frac{\partial \tilde{f}}{\partial \Pi} \right) \Big] \, dS.$$

Thus, we have proved the following

**Theorem 6.2.** *The reduced Poisson bracket on* $\mathcal{P} = SO(3)\backslash T^*C$ *is given by*

$$\{\tilde{f}, \tilde{g}\} =$$

*(canonical)*

$$- \int_0^L \Big\{ \left( \frac{\partial \tilde{f}}{\partial \Gamma} \right)' \cdot \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \left( \frac{\partial \tilde{g}}{\partial \Gamma} \right)' \cdot \frac{\partial \tilde{f}}{\partial \mathcal{M}} + \left( \frac{\partial \tilde{f}}{\partial \Omega} \right)' \cdot \frac{\delta \tilde{g}}{\partial \Pi} - \left( \frac{\partial \tilde{g}}{\delta \Omega} \right)' \cdot \frac{\partial \tilde{f}}{\partial \Pi} \Big\} \, dS$$

*(interaction)*

$$- \int_0^L \Big\{ \Omega \cdot \Big[ \frac{\partial \tilde{f}}{\partial \Gamma} \times \frac{\partial \tilde{g}}{\partial \mathcal{M}} - \frac{\partial \tilde{g}}{\partial \Gamma} \times \frac{\partial \tilde{f}}{\partial \mathcal{M}} \Big] \Big\} \, dS$$

*(Lie-Poisson for a semi-direct product)*

$$- \int_0^L \Pi \cdot \Big[ \frac{\partial \tilde{f}}{\partial \Pi} \times \frac{\partial \tilde{g}}{\partial \Pi} \Big] \, dS$$

$$+ \int_0^L \Big\{ \Omega \cdot \Big[ \frac{\partial \tilde{f}}{\partial \Omega} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \Omega} \times \frac{\partial \tilde{f}}{\partial \Pi} \Big] + \Gamma \cdot \Big[ \frac{\partial \tilde{f}}{\partial \Gamma} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \Gamma} \times \frac{\partial \tilde{f}}{\partial \Pi} \Big] + \mathcal{M} \cdot \Big[ \frac{\partial \tilde{f}}{\partial \mathcal{M}} \times \frac{\partial \tilde{g}}{\partial \Pi} - \frac{\partial \tilde{g}}{\partial \mathcal{M}} \times \frac{\partial \tilde{f}}{\partial \Pi} \Big] \Big\} \, dS \qquad (6.23)$$

*Note that the first integral gives the canonical term in the variables* $[(\Gamma, \Omega); (\mathcal{M}, \Pi)]$, *the second integral gives interaction terms, and the last two integrals are the Lie-Poisson terms.* ■

**Corollary 6.3.** *Hamilton's equations* $\dot{f} = \{ f, H \}$ *with Hamiltonian given by equation* (5.22) *and the Poisson bracket given by* (6.23) *are equivalent to the following convected equations of motion*

$$\begin{aligned} \dot{\mathcal{M}} &= N' + \Omega \times N - W \times \mathcal{M}, \\ \dot{\Pi} &= M' + \Omega \times M + \Gamma \times N - W \times \Pi, \\ \dot{\Gamma} &= \mathcal{V}' + \Omega \times \mathcal{V} - W \times \Gamma, \\ \dot{\Omega} &= W' + \Omega \times W, \end{aligned} \tag{6.24}$$

*where*

$$N = \frac{\partial \psi}{\partial \Gamma}, \qquad M = \frac{\partial \psi}{\partial \Omega}. \qquad \blacksquare$$

The last two equations in (6.24) can be checked by a direct calculation from the kinematic relations in section 5. The first two equations in (6.24) coincide with (5.14).

In summary, we have found the reduced space $\mathcal{P} = SO(3)\backslash T^*C$ to be the space of the convected variables $((\Gamma, \Omega); (\mathcal{M}, \Pi))$ with the Poisson bracket given by (6.23). This reduced bracket has been obtained from the canonical bracket in material representation by reduction. As with three-dimensional elasticity, if the motion on the reduced space is known, then the original motion in the material description can be reconstructed by solving the following system

$$\frac{\partial \Lambda}{\partial t} = \Lambda W, \qquad \frac{\partial \varphi}{\partial t} = \Lambda \mathcal{V}. \tag{6.25}$$

**Remark.** The Hamiltonian formulation can be of assistance in the establishment of conservation laws, in addition to being useful for stability and bifurcation studies. For example, suppose that the rod has an isotropic cross section, so that the inertia dyadic $\mathbb{J}$ has two equal eigenvalues, just as in the case of the Lagrange top. Then there is a material symmetry group acting: it is the group $S^1$ acting on the *right*. Namely, let the symmetry group $S^1$ consist of rotations about the symmetry axis, say $u$. The group action is then given by the following action of $R \in S^1$: $\varphi$, and $M_\varphi$ unchanged, and $(\Lambda, \Pi_\Lambda) \mapsto (\Lambda R, \Pi_\Lambda R)$. Here the momentum map corresponding to this action gives the conserved quantity

$$j = \int_0^L \Pi \cdot u \; dS. \qquad \blacksquare \tag{6.26}$$

# §7. *Geometrically Exact Plate Models*

In this section we summarize the basic equations governing a geometrically exact plate model. First, we introduce some necessary notation.

## *Notation*

In the geometric description of the configurations of a plate, the unit sphere

$$S^2 := \{t \in \mathbb{R}^3 \mid \|t\| = 1\} \ , \qquad (7.1a)$$

plays a central role. For any $t \in S^2$, the tangent space at $t$ is

$$T_t S^2 = \{v_t \in \mathbb{R}^3 \mid v_t \cdot t = 0\} \ . \qquad (7.1b)$$

Thus, $T_t S^2$ can be identified with $\mathbb{R}^2$. We now introduce a subset of $SO(3)$, which enables us to establish a link between the plate and rod theories.
Let $E \in \mathbb{R}^3$ be a fixed but otherwise arbitrary vector. Define $S_E$ to be the set of rotations $\Lambda \in SO(3)$ whose rotation axis $\psi \in \mathbb{R}^3$ is *normal* to $E$; i.e.,

$$S_E := \{\Lambda \in SO(3) \mid \psi \in \mathbb{R}^3 \text{ satisfies } \Lambda\psi = \psi \text{ and } \psi \cdot E = 0\}. \qquad (7.2)$$

The set $S_E$ is closely related to the quotient space $SO(3)/S^1 \cong S^2$, which realizes $SO(3)$ as a nontrivial $S^1$ bundle over $S^2$. This bundle is, when $SO(3)$ is replaced by $SU(2)$, the classical Hopf fibration. This idea is made explicit in the next proposition.

**Proposition 7.1.** *Given any* $t \in S^2$, *with* $t \neq -E$, *there is one and only one element* $\Lambda_t \in S_E$ *such that*

$$t = \Lambda_t E \ . \qquad (7.3a)$$

*In fact,* $S^2$ *and* $S_E$ *are diffeomorphic, and* $\Lambda_t$ *admits the following explicit characterization*

$$\Lambda_t = (E \cdot t)\mathbf{1} + [E \times t]^{\wedge} + (1 + E \cdot t)^{-1} [E \times t] \otimes [E \times t]. \qquad (7.3b)$$

**Proof.** Expression (7.3b) can be derived by suitably restricting the exponential map in $SO(3)$

given by Rodrigues formula (4.7) to $S_E$, see Simo & Fox [1987]. The result can be easily checked as follows. First, it is clear that $\Lambda_t E = t$. Second, a direct calculation readily shows that $\Lambda_t = \Lambda_t^T$. Finally, $\Lambda_t (E \times t) = (E \times t)$. Hence, $\psi = E \times t$ in (7.2) and $\Lambda_t$ is in $S_E$. Uniqueness follows at once from expresion (7.3b). ■

The geometric interpretation of this proposition should be clear. It constructs $\Lambda_t$ by rotating $E$ to $t$ in the plane they span, through the angle $\Theta := \cos^{-1}(E \cdot t)$. Formula (7.3b) plays a central role in the numerical analysis and implementation of geoemtrically exact shell models.

The tangent space to $S_E$ at the identity is given by

$$T_1 S_E := \{\hat{\theta} \in \mathbf{so}(3) \mid \theta \cdot E = 0\} . \tag{7.3c}$$

Finally, the tangent space $T_\Lambda S_E$ at $\Lambda \in S_E$ is obtained by using either left or right translations in $SO(3)$. For instance, for left-invariant vector fields we have

$$T_\Lambda S_E = \{\hat{\Theta}_\Lambda := \Lambda \hat{\Theta} \mid \text{for any } \hat{\Theta} \in T_1 S_E, \text{ and } \Lambda \in S_E\}. \tag{7.3d}$$

An analogous characterization holds for right invariant fields.

## *Kinematic description of the plate*

We consider the kinematic description of a plate with thickness $h > 0$. Essentially, no conceptual modification is required for the more general case of a shell. Let $\{E_I(X^0),\ I = 1,2,3\}$ be a fixed *orthonormal* basis for $\mathbb{R}^3$, with $E_3 \equiv E$. Any point $X \in \mathbb{R}^3$ may be expressed as

$$X = X^0 + \xi E, \quad \text{where} \quad X^0 = X^\alpha E_\alpha,\ \xi \in \mathbb{R} . \tag{7.4a}$$

From the point of view of the three-dimensional theory, the reference configuration of the plate is the set $\mathcal{B} \subset \mathbb{R}^3$ given by

$$\mathcal{B} := \left\{X = X^0 + \xi E \mid X^0 \in \Omega \text{ and } \xi \in \left[-\frac{h}{2}, \frac{h}{2}\right]\right\} , \tag{7.4b}$$

where $\Omega \subset \mathbb{R}^2$ is a given region in the plane. We refer to $\{X^0 + \xi E \mid X^0 \in \partial\Omega$ and $\xi$

$\in [-h/2, h/2]\}$ as the *edge* of the plate, and to $\Omega$ as its *mid-plane*. The basic kinematic assumption is that any *admissible configuration* $\tilde{\Phi} : \mathcal{B} \longrightarrow \mathbb{R}^3$ of the body is characterized as

$$x = \tilde{\Phi}(X) := \varphi(X^0) + \xi t(X^0) \quad , \quad \xi \in \left[-\frac{h}{2}, \frac{h}{2}\right] , \tag{7.5a}$$

where $\varphi : \Omega \longrightarrow \mathbb{R}^3$ maps the mid-plane onto $\varphi(\Omega) \in \mathbb{R}^3$, and $t$ is a *unit vector* attached to the point $\varphi(X^0) \in \mathbb{R}^3$, not necessarily normal to $\varphi(\Omega)$, which is referred to as *director* or *fiber direction*. The fact that $t$ need not be normal allows *shear* deformations.

Note that the mapping $X^0 \in \Omega \longrightarrow t(X^0) \in S^2$ assigns to points $X^0$, vectors $t$ in the *unit sphere* (at $\varphi(X^0)$). By Proposition 7.1, the unit vector $t$ at $\varphi(X^0)$ can be obtained from $E$ *uniquely* through a rotation $\Lambda(X^0) \in S_E$ with rotation axis normal to $E$.

## *Configuration space*

The kinematic assumption (7.3) embodies two essential ingredients: (i) Points $X^0 \in \Omega$ in the midplane are mapped onto points $x \in \mathbb{R}^3$ through the mapping $\varphi : \Omega \longrightarrow \mathbb{R}^3$; and (ii) *unit* vectors $E \in S^2$ attached to points $X^0 \in \Omega$ are mapped into *unit vectors* $t(X^0) \in S^2$ attached to points $\varphi(X^0) \in \mathbb{R}^3$ by rotations $\Lambda : \Omega \longrightarrow S_E$ with rotation axis normal to $E$. According to this view, two abstract characterizations of the set $C$ of possible configurations of the plate are possible:

(a) *Director point of view:*

$$C := \left\{ \overline{\Phi} \equiv (\varphi, t) \mid \varphi : \Omega \longrightarrow \mathbb{R}^3 \text{ and } t : \Omega \longrightarrow S^2 \right\} . \tag{7.6a}$$

The tangent space $T_{\overline{\Phi}} C$ at a configuration $\overline{\Phi} \in C$ is defined as

$$T_{\overline{\Phi}} C := \left\{ \delta\overline{\Phi} \equiv (\delta\varphi, \delta t) \mid \delta\varphi : \Omega \longrightarrow \mathbb{R}^3 \text{ and } \delta t : \Omega \longrightarrow T_t S^2 \right\} . \tag{7.6b}$$

(b) *Constrained-frame point of view.* Equivalently, as a result of Proposition 7.1, the configuration space $C$ may be defined as

$$C := \left\{ \Phi \equiv (\varphi, \Lambda) \mid \varphi : \Omega \longrightarrow \mathbb{R}^3 \text{ and } \Lambda : \Omega \longrightarrow S_E \right\} . \tag{7.7a}$$

According to this view, the tangent space is

$$T_{\Phi}C := \{\delta\overline{\Phi} \equiv (\delta\varphi, \delta\Lambda) \mid \delta\varphi : \Omega \longrightarrow \mathbb{R}^3 \text{ and } \delta\Lambda : \Omega \longrightarrow T_{\Lambda}S_E\}. \quad (7.7b)$$

Points of view (a) and (b), although equivalent, lead to a different parametrization of the equations of motion. Starting with the classical paper of Ericksen & Truesdell [1958], a substantial body of contemporary work, see, e.g., Antman [1976,1978], Cohen & DaSilva [1966], Green, Naghdi, & Wainwright [1965], and Naghdi [1972, 1980] has been typically concerned with the director point of view (a). Here, on the other hand, we take an alternative approach and adopt the constrained frame point of view (b) as a starting point. Our motivation for this lies in the strucure of the equations of motion and the Poisson bracket in approach (b) which are essentially identical to that of the geometrically exact rod model considered in Sections 5 and 6. The form of the Poisson bracket corresponding to approach (b), in the convected description, is obtained by a reduction process that amounts to enforcing explicitly the additional (symmetry) condition that the rotation fields must lie in the subset $S_E \subset SO(3)$ (i.e., a further reduction by $S^1$).

## *Motion and velocity fields*

A ***motion*** is a curve of configurations $t \in [0, T] \longrightarrow \Phi_t = (\varphi_t, \Lambda_t) \in C$, for some time interval $[0, T] \subset \mathbb{R}_+$. Associated with the motion, is the ***material velocity field*** $V_{\Phi}(X^0, t)$ defined as usual as

$$V_{\Phi}(X^0, t) := \frac{\partial}{\partial t}\Phi(X^0, t) \equiv \left(\dot{\varphi}_t(X^0), \dot{\Lambda}_t(X^0)\right). \quad (7.8a)$$

Thus the material velocity is a mapping $t \mapsto V_{\Phi_t} \in T_{\Phi}C$, where $T_{\Phi}C$ is the tangent space at configuration $\Phi$. We define the ***convected velocity*** field $\boldsymbol{\mathcal{V}}_{\Phi}(X^0, t) \in \boldsymbol{\mathfrak{X}}(\Omega)$ as in section 5, by the expression

$$\boldsymbol{\mathcal{V}}_{\Phi}(X^0, t) := \left([\Lambda(X^0, t)]^T\dot{\varphi}_t(X^0, t), [\Lambda(X^0, t)]^T\dot{\Lambda}(X^0, t)\right). \quad (7.8b)$$

Here, $\boldsymbol{\mathfrak{X}}(\Omega)$ denotes the space of smooth fields $\boldsymbol{\mathcal{V}}_{\Phi} : \Omega \longrightarrow \mathbb{R}^3 \times T_I S_E$. Since $\Lambda(X^0, t) \in S_E$, we have $\dot{\Lambda}(X^0, t) = \Lambda(X^0, t)\hat{W}(X^0, t)$ where $\hat{W}(X^0, t) \in T_I S_E$. Thus, the convected velocity field can be expressed as

$$\boldsymbol{\mathcal{V}}_{\Phi}(X^0, t) = (\boldsymbol{\mathcal{V}}, \hat{W})(X^0, t) \quad (7.8c)$$

where

$$\boldsymbol{\mathcal{V}}(X^0, t) := [\Lambda(X^0, t)]^T V(X^0, t) \ , \qquad V(X^0, t) := \dot{\phi}(X^0, t) \ .$$

Note that the material velocity field $V_\Phi$ is the *left extension* of the convected velocity field $\boldsymbol{\mathcal{V}}_\Phi$. Similarly, we could define a spatial field $v_\Phi$ whose second factor $w(X^0, t)$ is defined as

$$\hat{w}(X^0, t) := \dot{\Lambda}(X^0, t)[\Lambda(X^0, t)]^T \ . \tag{7.8d}$$

Then $\hat{w} : \Omega \times [0, T] \longrightarrow T_I S_t$ is interpreted as a ***spatial angular velocity***, where $T_I S_t := \{\hat{\theta} \in \mathbf{so}(3) \mid \mathbf{t} \cdot \boldsymbol{\theta} = 0\}$ is the tangent space at $\mathbf{t} = \Lambda E$.

## *Strain measures*

As in the geometrically exact rod model outlined in Section 5, starting from the Lagrangian deformation gradient $\Phi_{,\alpha}(X^0, t)$, $(\alpha = 1, 2)$, we define *convected* strains according to the expressions

| *Convected* | *Material* | *Spatial* |
|---|---|---|
| $\Gamma_\alpha(X^0,t) := \Lambda^T\varphi_{,\alpha}(X^0, t)$ | $\varphi_{,\alpha}(X^0, t)$ | $\gamma_\alpha = \varphi_{,\alpha}$ |
| $\Omega_\alpha(X^0, t) = \left[\Lambda^T\Lambda_{,\alpha}(X^0, t)\right]^\vee$ | $\Lambda_{,\alpha}(X^0, t)$ | $\omega_\alpha = \left[\Lambda_{,\alpha}\Lambda^T\right]^\vee$ |

We note that the above expressions can also be derived from the three-dimensional theory by a duality argument employing equivalence of the stress power. This is discussed briefly in a remark below (see equation (7.15)).

## *Stress resultants and stress couples; equations of motion in spatial description*

Associated with the motion $t \mapsto \Phi_t \in \mathcal{C}$, are vector fields $n_\alpha(X^0, t)$, $m_\alpha(X^0, t)$, $(\alpha = 1, 2)$, and $\rho_0(X^0, t)$, assumed to be smooth and interpreted as internal resultant force, internal resultant torque, and density per unit of area. These fields satisfy the following *spatial local form* of the equations of motion

$$
\begin{aligned}
n_{\alpha,\alpha} + \bar{n} &= \rho_0 h \dot{v} , \\
m_{\alpha,\alpha} + \varphi_{,\alpha} \times n_\alpha + \bar{m} &= \rho_0 \mathbb{K} \dot{w} ,
\end{aligned}
\tag{7.10}
$$

where $\bar{n}$, $\bar{m}$ are the applied body force and torque, and $\rho_0 h$, $\rho_0 \mathbb{K}$ are intertia coefficients. Typically, for plates of constant thickness, $\rho_0 \mathbb{K} = \rho_0 h^3/12$. Note that $w \cdot t = 0$, that is $\hat{w} \in T_I S_t$ for each $X^0 \in \Omega$ and $t \in [0, T]$. Equations (7.10) are the local statement in the spatial description of balance of linear momentum and balance of angular momentum, respectively. We refer to Green & Zerna [1968], Naghdi [1972], or Libai and Simmonds [1983], for a derivation of these equations from the three dimensional theory.

**Remark.** From the point of view of the three-dimensional theory, the right-hand side of (7.10) agrees with the standard definition of linear and angular momentum per unit of reference surface relative to the mid-plane. To see this, consider configurations $\tilde{\varphi} : \mathcal{B} \longrightarrow \mathbb{R}^3$ of the form $\tilde{\varphi} := \varphi + \xi \Lambda E$, where $\xi \in [-h/2, h/2]$. The angular momentum $\pi$ relative to the mid-plane is then given by

$$
\begin{aligned}
\pi &:= \int_{-h/2}^{h/2} (\tilde{\varphi} - \varphi) \times \rho_0 \dot{\tilde{\varphi}} \, d\xi \\
&= \int_{-h/2}^{h/2} \rho_0 \xi \Lambda E \times (\dot{\varphi} + \dot{\Lambda} E) \, d\xi \\
&= \frac{\rho_0 h^3}{12} \Lambda \left[ E \times \Lambda^T \dot{\Lambda} E \right] \\
&= \frac{\rho_0 h^3}{12} \Lambda [E \times \hat{W} E] \;=\; \frac{\rho_0 h^3}{12} \Lambda [E \times (W \times E)] \\
&= \frac{\rho_0 h^3}{12} \Lambda W \;=\; \frac{\rho_0 h^3}{12} w ,
\end{aligned}
\tag{7.11}
$$

where we have used the fact that $\hat{W} \in T_I S_E$ so that $W \cdot E = 0$. ■

## *Convective equations of motion*

An alternative form of the balance laws is obtained from the convective description. As far as we are aware, this form of the equations has not been stated explicitly in the literature. As in the rod model discussed previously, the basic idea is to pull-back the spatial balance laws (7.10) with the orthogonal transformation $\Lambda : \Omega \longrightarrow S_E$. Accordingly, one defines vector fields $N(X^0, t)$, $M(X^0, t)$ by the expressions

$$N(X^0, t) := \Lambda^T n(X^0, t) , \qquad M(X^0, t) := \Lambda^T m(X^0, t) . \tag{7.12}$$

The relation

$$m_{\alpha,\beta} = \Lambda[M_{\alpha,\beta} + \Omega_\beta \times M_\alpha] , \tag{7.13}$$

and the analogous expression relating $n_\alpha$ and $N_\alpha$, supports a straightforward calculation yielding the following statements of linear and angular momentum:

$$\rho_0 h[\dot{\boldsymbol{\nu}} + W \times \boldsymbol{\nu}] = N_{\alpha,\alpha} + \Omega_\alpha \times N_\alpha + \bar{N} . \tag{7.14a}$$

$$\rho_0 \mathbb{K}\dot{W} = M_{\alpha,\alpha} + \Omega_\alpha \times M_\alpha + \Gamma_\alpha \times N_\alpha + \bar{M} . \tag{7.14b}$$

**Remark** . Within the context of the three-dimensional theory, $n_\alpha$ and $m_\alpha$ represent resultant forces and resultant torques relative to the mid-surface of the distribution of stress acting on sections $S_\alpha$. As in the rod model discussed above, $\gamma_\alpha$, $\omega_\alpha$, and $\Gamma_\alpha$, $\Omega_\alpha$ are uniquely determined in the sense that the stress power is given by

$$\begin{aligned} \int_{\Omega\times\left[-\frac{h}{2},\frac{h}{2}\right]} P : \dot{F}\, d\Omega\, d\xi &= \int_\Omega \left[n_\alpha \cdot \overset{\nabla}{\gamma}_\alpha + m_\alpha \cdot \overset{\nabla}{\omega}_\alpha\right] d\Omega \\ &= \int_\Omega \left[N_\alpha \cdot \dot{\Gamma}_\alpha + M_\alpha \cdot \dot{\Omega}_\alpha\right] d\Omega \ , \end{aligned} \tag{7.15}$$

where $P$ is the first Piola-Kirchhoff stress tensor, $F$ the deformation gradient of the configuration $\varphi := \varphi(X^0, t) + \xi t(X^0, t)$, and $(\cdot)^\nabla = (\cdot)' - W \times (\cdot)$ is a ***co-rotated rate.*** ■

## *Further reduction of the convective equations of motion*

In contrast with the rod model considered in Section 5, the convective equations of motion (7.14) are amenable to further reduction by $S^1$. The reason for this additional reduction -- which can be carried out either in the spatial or in the convective descriptions -- is that only $SO(3)/S^1 \equiv S_E$ enters in the configuration space of the plate, not the entire $SO(3)$ as in the rod model.

Using the convected description to carry out the reduction of equations (7.14) we first note that since $\hat{\Omega}_\alpha, \hat{M}_\alpha \in T_I S_E$, it follows that $\Omega_\alpha \cdot E = M_\alpha \cdot E \equiv 0$ and so $\Omega_\alpha \times M_\alpha$ is parallel to $E$. We exploit this fact by introducing the decomposition

$$N_\alpha \;=:\; N_\alpha^0 + Q_\alpha E \;,\quad \Gamma_\alpha \;=:\; \Gamma_\alpha^0 + \Xi_\alpha E \;. \tag{7.16}$$

A straightforward calculation then yields

$$\begin{aligned} \Gamma_\alpha \times N_\alpha &= \Gamma_\alpha^0 \times N_\alpha^0 + (\Gamma_\alpha^0 \times E)Q_\alpha - (N_\alpha^0 \times E)\Xi_\alpha \;, \\ \Omega_\alpha \times N_\alpha &= \Omega_\alpha \times N_\alpha^0 + (\Omega_\alpha \times E)Q_\alpha \;. \end{aligned} \tag{7.17}$$

The decomposition (7.16) and (7.17) reduce the balance of angular momentum (7.14b) to

$$\Omega_\alpha \times M_\alpha + \Gamma_\alpha^0 \times N_\alpha^0 = 0 \;, \tag{7.18a}$$

$$M_{\alpha,\alpha} + \left(\Gamma_\alpha^0 \times E\right)Q_\alpha - \left(N_\alpha^0 \times E\right)\Xi_\alpha + \overline{M}^{\,0} \;=\; \rho_0 \mathbb{K}\dot{W} \;. \tag{7.18b}$$

The structure of equations (7.18) suggests the introduction of the following notation:

$$\begin{aligned} \Omega_\alpha^0 &:= \Omega_\alpha \times E & & & \Omega_\alpha &= E \times \Omega_\alpha^0 \;, \\ W^0 &:= W \times E & &\text{or} & W &= E \times W^0 \;, \\ M_\alpha^0 &:= M_\alpha \times E & & & M_\alpha &= E \times M_\alpha^0 \;. \end{aligned} \tag{7.19}$$

In addition, we denote by $\mathbb{P}_E := I - E \otimes E$ the orthogal projection parallel to $E$. Since $W \times \boldsymbol{\mathcal{V}} = (\,E \times W^0\,) \times \boldsymbol{\mathcal{V}}$, the convected acceleration may be expressed as

$$\Lambda^T \ddot{\varphi} \;=\; \left[\dot{\boldsymbol{\mathcal{V}}}^0 + \mathcal{V} W^0\right] + \left[\,\dot{\mathcal{V}} - W^0 \cdot \boldsymbol{\mathcal{V}}^0\right]E \;, \tag{7.20a}$$

where

$$\boldsymbol{\nu}^0 := \mathbb{P}_E \boldsymbol{\nu}\ , \quad \text{and} \quad \mathcal{V} := E \cdot \boldsymbol{\nu}\ . \tag{7.20b}$$

We then have the following five equations of balance of momentum in the convected representation:

$$\begin{aligned} N^0_{\alpha,\alpha} + \Omega^0_\alpha Q_\alpha + \overline{N}^0 &= \rho_0 h\left[\dot{\boldsymbol{\nu}}^0 + \mathcal{V} W^0\right]\ , \\ Q_{\alpha,\alpha} - \Omega^0_\alpha \cdot N^0_\alpha + \overline{Q} &= \rho_0 h\left[\dot{\mathcal{V}} - \dot{W}^0 \cdot \boldsymbol{\nu}^0\right]\ , \\ M^0_{\alpha,\alpha} - \Gamma^0_\alpha Q_\alpha + N^0_\alpha \Xi_\alpha + \overline{M}^0 &= \rho_0 \mathbb{K} \dot{W}^0\ . \end{aligned} \tag{7.21}$$

**Remark.** *Interpretation of* $W^0$ *and* $\Omega^0_\alpha$. The equations of motion (7.21) along with conditions (7.18a) are in fact the convected form of the equations of motion corresponding to a *director description* of the plate, as in (7.6a,b); see Simo & Fox [1987] for further details. The vector field $W^0$ is the ***convected director velocity*** as the following identity shows:

$$\Lambda^T \dot{t} = \Lambda^T \dot{\Lambda} E = \hat{W} E \equiv W \times E =: W^0\ .$$

An entirely analogous interpretation holds for $\Omega^0_\alpha$. ■

## *Constitutive equations*

In view of expression (7.15) for the stress power, one characterizes (isothermal) hyperelastic response by assuming the existence of a free energy function of the form

$$\Psi(X^0,\ \Gamma^0_\alpha,\ \Xi_\alpha,\ \Omega^0_\alpha).$$

As in the rod model considered in Section 5, this form of the free energy in the convective description arises naturally by postulating invariance under left action of the Euclidean group; i.e., material frame-indifference. We then have the equations

$$N_\alpha^0 = \frac{\partial\Psi\left(X^0, \Gamma_\alpha^0, \Xi_\alpha, \Omega_\alpha^0\right)}{\partial\Gamma_\alpha^0}, \tag{7.22a}$$

$$M_\alpha^0 = \frac{\partial\Psi\left(X^0, \Gamma_\alpha^0, \Xi_\alpha, \Omega_\alpha^0\right)}{\partial\Omega_\alpha^0}, \tag{7.22b}$$

$$Q_\alpha = \frac{\partial\Psi\left(X^0, \Gamma_\alpha^0, \Xi_\alpha, \Omega_\alpha^0\right)}{\partial\Xi_\alpha}. \tag{7.22c}$$

We note that equation (7.18a), arising from balance of angular momentum about $E$, can be expressed as

$$\Omega_\alpha^0 \times M_\alpha^0 + \Gamma_\alpha^0 \times N_\alpha^0 = 0.$$

From this condition we obtain the following restriction on the constitutive equations (7.22):

$$\Omega_\alpha^0 \times \frac{\partial\Psi\left(X^0, \Gamma_\alpha^0, \Xi_\alpha, \Omega_\alpha^0\right)}{\partial\Omega_\alpha^0} + \Gamma_\alpha^0 \times \frac{\partial\Psi\left(X^0, \Gamma_\alpha^0, \Xi_\alpha, \Omega_\alpha^0\right)}{\partial\Gamma_\alpha^0} = 0. \tag{7.23}$$

Condition (7.23) is analogous to the symmetry of the Cauchy and second Piola-Kirchhoff stress tensor in three-dimensional continuum mechanics, which in turn results from balance of angular momentum.

### *The Hamiltonian in the convective description*

We define the *convected linear and angular momenta* by the expressions

$$\begin{aligned} \mathcal{M} &= \mathcal{M}^0 + \mathcal{M}E := \rho_0 h[\mathcal{V}^0 + \mathcal{V}E], \\ \Pi &:= \rho_0 \mathbb{K} W, \\ \Pi^0 &:= \Pi \times E \equiv \rho_0 \mathbb{K} W^0 . \end{aligned} \tag{7.24}$$

If, for simplicity the body forces and homogeneous Dirichlet boundary conditions are imposed, then the Hamiltonian is given by

$$H := \frac{1}{2}\int_\Omega \left\{ (\rho_0 h)^{-1}\left[\|\mathcal{M}^0\|^2 + \mathcal{M}^2\right] + (\rho_0\mathbb{K})^{-1}\|\Pi^0\|^2 \right\} d\Omega + \int_\Omega \Psi\left(X^0, \Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha\right) d\Omega \ . \qquad (7.25)$$

## *Convective Equations of Motion in Director Notation*

For completeness we record below the field equations governing our plate model in director notation. As in Section 7.2, let $\{E_I\}$ be the standard basis in $\mathbb{R}^3$, with $E_3 \equiv E$, $\{X_\alpha\}$ covering $\Omega \subset \mathbb{R}^2$, and $X_3 \equiv \xi \in [-h/2, h/2]$. Define

$$\begin{aligned} \mathfrak{M}^0 &:= M^0_\alpha \otimes E_\alpha , & \mathfrak{N}^0 &:= N^0_\alpha \otimes E_\alpha , & \mathfrak{Q} &:= Q_\alpha E_\alpha , \\ \mathfrak{W}^0 &:= \Omega^0_\alpha \otimes E_\alpha , & \mathfrak{G}^0 &:= \Gamma^0_\alpha \otimes E_\alpha , & \mathfrak{X} &:= \Xi_\alpha E_\alpha , \end{aligned} \qquad (7.26)$$

The momentum equations (7.21) may then be recast in the following form:

$$\begin{aligned} \mathcal{DIV}\mathfrak{N}^0 + \mathfrak{W}^0\mathfrak{Q} + \overline{N}^0 &= \rho_0 h[\dot{\boldsymbol{\mathcal{V}}}^0 + \mathcal{V}W^0] , \\ \mathcal{DIV}\mathfrak{Q} - \mathfrak{W}^0 : \mathfrak{N}^0 + \overline{Q} &= \rho_0 h[\dot{\mathcal{V}} - W^0 \cdot \boldsymbol{\mathcal{V}}^0] , \\ \mathcal{DIV}\mathfrak{M}^0 - \mathfrak{G}^0\mathfrak{Q} + \mathfrak{N}^0\mathfrak{X} + \overline{M}^0 &= \rho_0\mathbb{K}\dot{W}^0 . \end{aligned} \qquad (7.27)$$

This concludes our development of the present geometrically exact plate model.

# §8. *The Hamiltonian Structure for Plates in the Material and Convective Representation*

In this section we develop the Hamiltonian structure for the plate model in the convective representation with governing equations summarized in the previous section . We show that these equations are Hamiltonian relative to a non-canonical Poisson structure in the space $\mathcal{P}$ of convected variables

$$\{(\Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha),\ (\mathcal{M}^0, \mathcal{M}, \Pi^0)\}.$$

## *The First Reduced Bracket*

Our derivation of the corresponding Poisson bracket follows the same reduction scheme employed both in three-dimendional elasticity and in our treatment of geometrically exact rods, and can be outlined as follows. We start with the canonical Hamiltonian structure on the cotangent bundle of the configuration space, $T^*\mathcal{C}$ and the corresponding canonical bracket. From this bracket we obtain by left reduction by SO(3) a Poisson bracket for the plate model, which is the counterpart of the Poisson bracket derived for the rod model. An important difference, however, is that one can further reduce this bracket by enforcing the condition that only a part of SO(3), namely $SO(3)/S^1 \equiv S_E$, enters in the configuration space. This additional reduction is the result of an extra symmetry of the Hamiltonian, which is now invariant with respect to rotations about the axis $\mathbf{t}$. Physically , this symmetry corresponds to the fact that no inertia and no stiffness is associated with rotations about the director $\mathbf{t}$.

The bracket in terms of the variables $\{(\Gamma_\alpha, \Omega_\alpha), (\mathcal{M}, \Pi)\}$ is derived from the canonical bracket in material variables in the same way as for the rod model. This bracket is as follows:

$$\{\tilde{f}, \tilde{g}\} = \begin{cases} \text{(canonical)} & -\int_\Omega \Big\{\Big(\dfrac{\partial \tilde{f}}{\partial \Gamma_\alpha}\Big)_{,\alpha} \cdot \dfrac{\partial \tilde{g}}{\partial \mathcal{M}} - \Big(\dfrac{\partial \tilde{g}}{\partial \Gamma_\alpha}\Big)_{,\alpha} \cdot \dfrac{\partial \tilde{f}}{\partial \mathcal{M}} \\ & \quad + \Big(\dfrac{\partial \tilde{f}}{\partial \Omega_\alpha}\Big)_{,\alpha} \cdot \dfrac{\delta \tilde{g}}{\partial \Pi} - \Big(\dfrac{\partial \tilde{g}}{\partial \Omega_\alpha}\Big)_{,\alpha} \cdot \dfrac{\partial \tilde{f}}{\partial \Pi}\Big\}\, d\Omega \qquad (8.1a) \\ \textit{(interaction)} & -\int_\Omega \Big\{\Omega_\alpha \cdot \Big[\dfrac{\partial \tilde{f}}{\partial \Gamma_\alpha} \times \dfrac{\partial \tilde{g}}{\partial \mathcal{M}} - \dfrac{\partial \tilde{g}}{\partial \Gamma_\alpha} \times \dfrac{\partial \tilde{f}}{\partial \mathcal{M}}\Big]\Big\}\, d\Omega \qquad (8.1b) \end{cases}$$

*(Lie-Poisson for a semi-direct product)*

$$\left\{\begin{aligned} &-\int_\Omega \Big\{\Omega_\alpha\cdot\Big[\frac{\partial\tilde f}{\partial\Omega_\alpha}\times\frac{\partial\tilde g}{\partial\Pi}-\frac{\partial\tilde g}{\partial\Omega_\alpha}\times\frac{\partial\tilde f}{\partial\Pi}\Big] \\ &\qquad+\Gamma_\alpha\cdot\Big[\frac{\partial\tilde f}{\partial\Gamma_\alpha}\times\frac{\partial\tilde g}{\partial\Pi}-\frac{\partial\tilde g}{\partial\Gamma_\alpha}\times\frac{\partial\tilde f}{\partial\Pi}\Big] \\ &\qquad+\mathcal{M}\cdot\Big[\frac{\partial\tilde f}{\partial\mathcal{M}}\times\frac{\partial\tilde g}{\partial\Pi}-\frac{\partial\tilde g}{\partial\mathcal{M}}\times\frac{\partial\tilde f}{\partial\Pi}\Big]\Big\}\,d\Omega \\ &-\int_\Omega \Pi\cdot\Big[\frac{\partial\tilde f}{\partial\Pi}\times\frac{\partial\tilde g}{\partial\Pi}\Big]\,d\Omega. \end{aligned}\right. \tag{8.1c}$$

## *The Reduced Bracket*

The further reduction is acomplished by using the change of variables in (7.19) given by

$$\Gamma_\alpha = \Gamma^0{}_\alpha + \Xi_\alpha E\,, \qquad \mathcal{M} = \mathcal{M}^0 + \mathscr{M}E, \tag{8.2a}$$

$$\Omega_\alpha = E\times\Omega^0{}_\alpha\,, \qquad \Pi = E\times\Pi^0. \tag{8.2b}$$

Since $\hat\Omega_\alpha \in T_1S_E$ and $\hat\Pi \in T_1S_E$, we have the constraints

$$\Pi\cdot E = 0\,, \qquad \Omega_\alpha\cdot E = 0, \tag{8.3}$$

and therefore

$$\Pi\cdot\Big(\frac{\partial\tilde f}{\partial\Pi}\times\frac{\partial\tilde g}{\partial\Pi}\Big) = 0, \tag{8.4}$$

and

$$\Omega_\alpha\cdot\Big(\frac{\partial\tilde f}{\partial\Omega_\alpha}\times\frac{\partial\tilde g}{\partial\Pi}\Big) = 0\,. \tag{8.5}$$

From (8.2a) and the chain rule we get

$$\frac{\partial \tilde{f}}{\partial \Gamma_\alpha} = \frac{\partial f}{\partial \Gamma^0_\alpha} + E \frac{\partial f}{\partial \Xi_\alpha}, \tag{8.6}$$

where $\tilde{f}$ is a function of $\{(\Gamma_\alpha, \Omega_\alpha), (\mathcal{M}, \Pi)\}$, as in the bracket above, and f is a function of the variables $\{(\Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha), (\mathcal{M}^0, \mathcal{M}, \Pi^0)\}$, which are related to the preceding variables through equations (8.2). Similarly we have

$$\frac{\partial g}{\partial \Pi^0} = \frac{\partial \tilde{g}}{\partial \Pi} \times E, \tag{8.7}$$

and

$$\frac{\partial \tilde{f}}{\partial \mathcal{M}} = \frac{\partial f}{\partial \mathcal{M}^0} + E \frac{\partial f}{\partial \mathcal{M}}. \tag{8.8}$$

Substitution of (8.4) - (8.8) into (8.1) gives

**Theorem 8.1.** *The reduced bracket on the space* $\mathcal{P}$ *of the variables*

$$\{(\Gamma^0_\alpha, \Xi_\alpha, \Omega^0_\alpha), (\mathcal{M}^0, \mathcal{M}, \Pi^0)\}$$

*is given by :*

$$\{f, g\} =$$

*(interaction)*

$$\left\{\begin{aligned} &\int_\Omega \Big\{ \Omega^0_\alpha \cdot \Big[ \frac{\partial f}{\partial \mathcal{M}^0} \frac{\partial g}{\partial \Xi_\alpha} - \frac{\partial g}{\partial \mathcal{M}^0} \frac{\partial f}{\partial \Xi_\alpha} \Big] \\ &+ \Omega^0_\alpha \cdot \Big[ \frac{\partial f}{\partial \Gamma^0_\alpha} \frac{\partial g}{\partial \mathcal{M}} - \frac{\partial g}{\partial \Gamma^0_\alpha} \frac{\partial f}{\partial \mathcal{M}} \Big] \\ &+ \Gamma^0_\alpha \cdot \Big[ \frac{\partial f}{\partial \Xi_\alpha} \frac{\partial g}{\partial \Pi^0} - \frac{\partial g}{\partial \Xi_\alpha} \frac{\partial f}{\partial \Pi^0} \Big] \\ &+ \Xi_\alpha \cdot \Big[ \frac{\partial f}{\partial \Pi^0} \cdot \frac{\partial g}{\partial \Gamma^0_\alpha} - \frac{\partial g}{\partial \Pi^0} \cdot \frac{\partial f}{\partial \Gamma^0_\alpha} \Big] \Big\} \, d\Omega \end{aligned}\right.$$

*(Lie Poisson for a semidirect product)*

$$
+ \int_\Omega \Big\{ \mathfrak{M}^0 \cdot \Big[ \frac{\partial f}{\partial \mathcal{M}} \frac{\partial g}{\partial \Pi^0} - \frac{\partial g}{\partial \mathcal{M}^0} \frac{\partial f}{\partial \Pi^0} \Big] + \mathcal{M} \cdot \Big[ \frac{\partial f}{\partial \Pi^0} \cdot \frac{\partial g}{\partial \mathfrak{M}^0} - \frac{\partial g}{\partial \Pi^0} \cdot \frac{\partial f}{\partial \mathfrak{M}^0} \Big] \Big\}\, d\Omega
$$

*(canonical)*

$$
\begin{aligned}
- \int \Big\{ & \Big(\frac{\partial f}{\partial \Omega^0{}_\alpha}\Big)_{,\alpha} \cdot \frac{\partial g}{\partial \Pi^0} - \Big(\frac{\partial g}{\partial \Omega^0{}_\alpha}\Big)_{,\alpha} \cdot \frac{\partial f}{\partial \Pi^0} \\
& + \Big(\frac{\partial f}{\partial \Gamma^0{}_\alpha}\Big)_{,\alpha} \cdot \frac{\partial g}{\partial \mathfrak{M}^0} - \Big(\frac{\partial g}{\partial \Gamma^0{}_\alpha}\Big)_{,\alpha} \cdot \frac{\partial f}{\partial \mathfrak{M}^0} \\
& + \Big(\frac{\partial f}{\partial \Xi_\alpha}\Big)_{,\alpha} \cdot \frac{\partial g}{\partial \mathcal{M}} - \Big(\frac{\partial g}{\partial \Xi_\alpha}\Big)_{,\alpha} \cdot \frac{\partial f}{\partial \mathcal{M}} \Big\}\, d\Omega. \ \blacksquare
\end{aligned}
\tag{8.9}
$$

As for the rod model discussed in Sections 5 and 6, we obtain from the expression (7.25) for the Hamiltonian and from the bracket (8.9) the following:

**Corollary 8.2.** *Hamilton's equations in the form* $\dot{f} = \{ f, H \}$, *where* $f$ *is an arbitrary function on the phase space given by theorem 8.1, with the bracket given by (8.9) and Hamiltonian given by (7.25) are equivalent to the following convective equations of motion:*

$$
\left.
\begin{aligned}
\dot{\mathcal{M}} &= Q_{\alpha,\alpha} - \Omega^0{}_\alpha \cdot N^0{}_\alpha + W^0 \cdot \mathfrak{M}^0, \\
\dot{\mathfrak{M}}^0 &= N^0{}_{\alpha,\alpha} + \Omega^0{}_\alpha Q_\alpha - \mathcal{M} \mathcal{W}^0, \\
\dot{\Pi}^0 &= \mathfrak{M}^0{}_{\alpha,\alpha} + \Xi_\alpha N^0{}_\alpha - \Gamma^0{}_\alpha Q_\alpha, \\
\dot{\Xi}_\alpha &= \mathcal{V}_\alpha + W^0 \cdot \Gamma_\alpha - \Omega^0{}_\alpha \cdot \boldsymbol{\nu}^0, \\
\dot{\Gamma}^0{}_\alpha &= \boldsymbol{\nu}^0{}_{,\alpha} - W^0 \Xi_\alpha + \Omega^0{}_\alpha \mathcal{V}, \\
\dot{\Omega}^0{}_\alpha &= W^0{}_{,\alpha},
\end{aligned}
\right\}
\tag{8.10}
$$

*where* $N^0_\alpha = \partial\psi/\partial\Gamma^0_\alpha$, $Q_\alpha = \partial\psi/\partial\Xi_\alpha$ *and* $M^0_\alpha = \partial\psi/\partial\Omega^0_\alpha$. ■

The first three of these equations of motion have been derived already (see 7.21). The second group of three equations can be directly checked using the kinematic relations given in section 7. The corollary then follows by the general principles of reduction, but may also be verified by a direct computation.

## *Conclusions*

In this paper we have presented a systematic development of the Hamiltonian structure for geometrically exact nonlinear elasticity, including solids, rods, and plates. We have emphasized the convective representation since it is numerically convenient and it is useful for the coupling with rigid body dynamics to obtain models for rigid bodies with flexible attatchments. In particular, our formulation of the geometrically exact plate model in the convected representation constitutes the natural counterpart of the classical Kirchhoff-Love and Reissner-Antman models for rods. The derivation of the Poisson structure follows the same lines as earlier works of Marsden, Ratiu, Weinstein and their coworkers, namely it is obtained by reduction from canonical brackets in the material (or Lagrangian) representation by reduction; i.e., by the elimination of rotational symmetry (material frame-ndifference), and by introducing the Cauchy-Green tensor as a dynamical variable. This approach is consistent with the covariance investigations of nonlinear elasticity by Hughes, Marsden, & Simo. In a future work we shall use the Hamiltonian structures for geometrically exact rods and plates to study the nonlinear stability of rigid bodies with flexible attachments, following the ideas of Krishnaprasad and Marsden.

It is clear from the literature and related work that these methods have more significance and applicability than may be suggested from the preceding developments. For example, the attention to the proper geometry and the nonlinear context that is typified by the present investigation is of benefit for numerical work, as has been demonstrated by Simo & VuQuoc. Also, other models can be investigated; for example, it is clear from the literature (Iwinski and Turski [1976], Marsden and Weinstein [1982]) that one can also include electromagnetic effects into the same formalism.

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