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The Closed-Form Control Laws of the Constrained Model Predictive Control Algorithm

by H-W. Chiou and E. Zafiriou

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Hung-Wen Chiou and Evanghelos Zafiriou*

Department of Chemical Engineering and Institute for Systems Research
University of Maryland, College Park, MD 20742

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Abstract

The analysis of quadratic stability and strongly H_∞ performance of Model Predictive Control (MPC) with hard constraints (or called Constrained Model Predictive Control (CMPC)) can be accomplished by reformulating the hard constraints of CMPC. From the CMPC algorithm, each term of the closed-form of CMPC control law corresponding to an active constraint situation can be decomposed to have an uncertainty block, which is time varying over the control period. The control law also contains a bias from the bounds of the constraints which cause difficulty in stability and performance analysis. An alternative way to avoid this difficulty is to reformulate the hard constraints to adjustable constraints with time varying adjustable weights on the adjustable variables added to the on-line objective function. The time varying weights in the adjustable constraint control law make the control action just the same as the hard constrained control. Theoretical derivatives and examples are given. The same reformulation is applied to the softened constraint cases.

On the analysis of the quadratic stability and strongly H_∞ performance, the control system for hard constraint control law without bias satisfies the stability and performance criteria if and only if the control system for adjustable constraint control law with time varying adjustable weights satisfies the same criteria. The details will be shown in the technical reports on quadratic stability and strongly H_∞ performance analysis, which are in preparation.

1 Introduction

The effects of using an on-line optimization formulation in the feedback loop of a control system were analyzed and the on-line optimization techniques were applied to the control system to obtain a suboptimal control [Sznaier, 1989]. The objective function of this suboptimal control was subject to the constraints of the control variables and the system states. In addition, a norm bounded system state was set as an extra constraint of the objective function proved to guarantee the asymptotical

*Author to whom correspondence should be addressed. E-mail: zafiriou@src.umd.edu

stability of the control system. Under the assumptions of the objective function bounded by certain defined functions, asymptotical stability of the optimal infinite-horizon and moving-horizon control of a general class of constrained discrete-time systems was discussed [Keerthi and Gilbert, 1988].

A framework on the receding horizon control has been developed by Rawlings and Muske (1993). The end constraint used in the moving horizon on-line optimization was proposed by Genceli and Nikolaou (1992). A method to soften the hard constraint by adding a linear weighting function to the on-line objective function was given by de Oliveira and Biegler (1992).

Contrasting with the above research work, Constrained Model Predictive Control (CMPC) has tunable prediction horizon and control horizon in the quadratic objective function and no extra norm bounded state or end constraint. Its control law, similar to the variable structure control law, is a piecewise linear function which contains a sequence of linear control laws to handle different active constraint situations at the optimal condition.

In this paper, based on the state space model, the control law of CMPC is formulated. The closed-form control law of CMPC with uncertainty blocks which are time varying corresponding to active constraint situation at the optimum can be represented by the adjustable constraint control law with the uncertainty blocks containing adjustable weights. Applying the adjustable constraint control law shows the feasibility on analyzing the quadratic stability and strongly H_∞ performance of CMPC.

2 Hard Constrained MPC

The on-line objective function is:

$$\min_{u(k),\cdots,u(k+M-1)} \sum_{l=1}^{P}[e^T(k+l)\Gamma^2 e(k+l) + u^T(k+l-1)B^2u(k+l-1) + \Delta u^T(k+l-1)D^2\Delta u(k+l-1)] \tag{1}$$

subject to

$$\Delta\underline{u}(k+i) \le \Delta u(k+i) \le \Delta\bar{u}(k+i), \quad i = 0 \cdots M-1$$

and/or

$$\underline{u}(k+i) \le u(k+i) \le \bar{u}(k+i), \quad i = 0 \cdots M-1$$

and/or

$$\underline{y}(k+j) \le \hat{y}(k+j) \le \bar{y}(k+j), \quad w_b \le j \le w_e$$

where M : control horizon; P : prediction horizon; e : the predicted error (the difference between predicted output and reference input); u : manipulated variable; Δu : the change rate of manipulated variable ($\Delta u(k+i) = u(k+i) - u(k+i-1)$); Γ : the diagonal weighting matrix of predicted error; B : the diagonal weighting matrix of manipulated variable; D : the diagonal weighting matrix of Δu; $\hat{y}$: the predicted output; $\underline{u}$, $\bar{u}$ are the lower and upper bounds of u respectively; $\Delta\underline{u}$, $\Delta\bar{u}$ are the lower and upper bounds of Δu respectively; $\underline{y}$, $\bar{y}$ are the lower and upper bounds of $\hat{y}$ respectively; w_b, w_e are the beginning and ending points of predicted output constraint window respectively; k : is the time index.

The process model is described as:

$$\begin{aligned} x(k+1) &= \phi x(k) + \Theta u(k) \\ y(k) &= Cx(k) + d(k) \end{aligned} \tag{2}$$

where $x(k)$: the state variable; $d(k)$: the disturbance; $y(k)$: the output measurement; ϕ, Θ, C : the coefficients of the state space model. Based on the model to predict the future output (predicted output), the quadratic optimization problem (1) can be written as a standard Quadratic Programming problem:

$$\min_v q(v) = \frac{1}{2} v^T G v + g^T v \tag{3}$$

subject to

$$A^T v \geq b$$

where

$$v = [\ u^T(k) \ \ \ldots \ \ u^T(k+M-1)\ ]^T \tag{4}$$

the matrices G, A, and vectors g, b are functions of the tuning parameters (weights, horizons P, M), and some bounds of constraints. The vectors g, b are also linear functions of state, disturbance, and/or $u(k-1)$.

For the optimal solution v^* we have [Fletcher, 1981]:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{b} \end{bmatrix} \tag{5}$$

where $\hat{A}^T$, $\hat{b}$ consist of the rows of A^T, b that correspond to the constraints that are active at the optimum and λ^* is the vector of the Lagrange multipliers corresponding to the constraints. The optimal $u(k)$ corresponds to the first m elements of the v^* that solves (5), where m is the dimension of u.

The special form of the LHS matrix in (5) allows the numerically efficient computation of its inverse in a partitioned form [Fletcher, 1981]:

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & 0 \end{bmatrix}^{-1} = \begin{bmatrix} H & -T \\ -T^T & U_L \end{bmatrix} \tag{6}$$

Then

$$v^* = -Hg + T\hat{b} \tag{7}$$

$$\lambda^* = T^T g - U_L \hat{b} \tag{8}$$

The general control law for the optimization problem (1) or (3) is:

$$\begin{aligned} u(k) &= [\, I\, 0 \ldots 0\,](-Hg + T\hat{b}) \\ &= -[\, I\, 0 \ldots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}]S^T \Gamma^T \Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \end{aligned}$$

$$+ \quad [\, I\, 0 \ldots 0\,] G^{-1} \hat{A} (\hat{A}^T G^{-1} \hat{A})^{-1} \varpi^T \begin{bmatrix} \Delta\tilde{u}(k) + u(k-1) \\ \Delta\tilde{u}(k+1) \\ \vdots \\ \Delta\tilde{u}(k+M-1) \\ \tilde{u}(k) \\ \vdots \\ \tilde{u}(k+M-1) \\ \bar{\eta} x(k) + \bar{\alpha} d(k) + \tilde{y} \end{bmatrix}$$

$$+ \quad [\, I\, 0 \ldots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}] \Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{9}$$

where $R(k+1)$ is the reference input; ϖ^T is a extraction matrix, and it extracts elements from its right hand side matrix corresponding to an active constraint situation at the optimum; $\tilde{y} = [\ \tilde{y}^T(k+w_b) \ \cdots \ \tilde{y}^T(k+w_e)\ ]^T$; $\bar{\eta}, \bar{\alpha}$ are the matrices consisting of the rows from the η, α that correspond to the predicted output constraint window; $\tilde{y}(k+j)$,$\Delta\tilde{u}(k+i)$,$\tilde{u}(k+i)$ are the upper or lower bounds of the $\hat{y}(k+j)$, $\Delta u(k+i)$, $u(k+i)$ respectively; $\eta = [\ (C\phi)^T\ (C\phi^2)^T \ \cdots \ (C\phi^P)^T\ ]^T$ and $\alpha = [\ I_1 \ \cdots \ I_P\ ]^T$ (I_i : is an identity matrix);

$$\Pi = \begin{bmatrix} I & 0 & \ldots & 0 \\ -I & I & 0 & 0 \\ 0 & \ddots & \ddots & 0 \\ 0 & \ldots & -I & I \end{bmatrix}$$

When the control system gets into the unconstraint control region, the term $\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T = 0$ in (9), then the control law becomes unconstraint control law.

3 Adjustable Constrained MPC

The on-line objective function is:

$$\min_{u(k),\cdots,u(k+M-1),\bar{\epsilon}} \sum_{l=1}^{P} [e^T(k+l)\Gamma^2 e(k+l) + u^T(k+l-1)B^2 u(k+l-1)$$

$$+ \Delta u^T(k+l-1) D^2 \Delta u(k+l-1)] + \bar{\epsilon}^T W_t^2 \bar{\epsilon} \tag{10}$$

subject to

$$-\bar{\epsilon}_{\Delta u}(k+i) \leq \Delta u(k+i) \leq \bar{\epsilon}_{\Delta u}(k+i), \quad i = 0 \ \cdots \ M-1$$

and/or

$$-\bar{\epsilon}_{u}(k+i) \leq u(k+i) \leq \bar{\epsilon}_{u}(k+i), \quad i = 0 \ \cdots \ M-1$$

and/or

$$-\bar{\epsilon}_{\hat{y}}(k+j) \leq \hat{y}(k+j) \leq \bar{\epsilon}_{\hat{y}}(k+j), \quad w_b \leq j \leq w_e$$

and

$$\bar{\epsilon} = [\ \bar{\epsilon}_{\Delta u}^{T}(k)\ \cdots\ \bar{\epsilon}_{\Delta u}^{T}(k+M-1)\ \bar{\epsilon}_{u}^{T}(k)\ \cdots\ \bar{\epsilon}_{u}^{T}(k+M-1)\ \bar{\epsilon}_{\hat{y}}^{T}(k+w_b)\ \cdots\ \bar{\epsilon}_{\hat{y}}^{T}(k+w_e)\ ]^T \geq 0$$

where

$$W_t = \begin{bmatrix} W_{t\Delta u}(k) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_{t\Delta u}(k+M-1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_{tu}(k) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_{tu}(k+M-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_{t\hat{y}}(k+w_b) & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_{t\hat{y}}(k+w_e) \end{bmatrix}$$

$\bar{\epsilon}$ is the adjustable variable and the W_t is the adjustable weight; $\bar{\epsilon}_u(k+i)$, $\bar{\epsilon}_{\Delta u}(k+i)$, $\bar{\epsilon}_{\hat{y}}(k+j)$ are the adjustable variables for $u(k+i)$, $\Delta u(k+i)$, $\hat{y}(k+j)$ respectively. $W_{tu}(k+i)$, $W_{t\Delta u}(k+i)$, $W_{t\hat{y}}(k+j)$ are the adjustable diagonal weighting matrices for $\bar{\epsilon}_u(k+i)$, $\bar{\epsilon}_{\Delta u}(k+i)$, $\bar{\epsilon}_{\hat{y}}(k+j)$ respectively. Based on the model (2) to predict the future output (predicted output), the quadratic optimization problem (10) can be written as a standard Quadratic Programming problem:

$$\begin{aligned} \min_{v,\bar{\epsilon}} q(v) &= \frac{1}{2}v^T G v + g^T v + \frac{1}{2}\bar{\epsilon}^T W_t^2 \bar{\epsilon} \\ &= \frac{1}{2}\bar{v}^T \begin{bmatrix} G & 0 \\ 0 & W_t^2 \end{bmatrix} \bar{v} + [g^T\ \ 0]\bar{v} \end{aligned} \tag{11}$$

subject to

$$\bar{A}^T \bar{v} \geq \bar{b}$$

where $\bar{b}$ is the b with zero bounds of constraints ($\Delta\tilde{u}(k+i) = 0$ and/or $\tilde{u}(k+i) = 0$ and/or $\tilde{y}(k+j) = 0$).

$$\bar{A}^T = [\,A^T\ \ I\,],\ \ \bar{v} = [\ u^T(k)\ \ \cdots\ \ u^T(k+M-1)\ \ \bar{\epsilon}^T\ ]^T$$

where I is an identity matrix.

For the optimal solution v^* we have [Fletcher, 1981]:

$$\begin{bmatrix} G & 0 & -\hat{A} \\ 0 & W_t^2 & -\hat{I} \\ -\hat{A}^T & -\hat{I}^T & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \bar{\epsilon}^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ 0 \\ \hat{\bar{b}} \end{bmatrix} \tag{12}$$

where $\hat{I}^T\hat{I} = I$. Then

$$\bar{\epsilon}^* = W_t^{-2}\hat{I}\lambda^*$$

and

$$\begin{bmatrix} G & -\hat{A} \\ -\hat{A}^T & -\hat{W}_t^{-2} \end{bmatrix} \begin{bmatrix} v^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} g \\ \hat{\bar{b}} \end{bmatrix} \tag{13}$$

where $\hat{W}_t^{-2} = \hat{I}^T W_t^{-2} \hat{I}$.

Following the similar method in section 2 to solve (13), we have the hard constraint control law:

$$
\begin{aligned}
u(k) &= [\,I\,0\ldots0\,](-Hg+T\hat{b})\\
&= -[\,I\,0\ldots0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^TG^{-1}\hat{A}+\hat{W}_t^{-2})^{-1}\hat{A}^TG^{-1}]S^T\Gamma^T\Gamma(\eta x(k)+\alpha d(k)-R(k+1))\\
&+ [\,I\,0\ldots0\,]G^{-1}\hat{A}(\hat{A}^TG^{-1}\hat{A}+\hat{W}_t^{-2})^{-1}\varpi^T\begin{bmatrix} u(k-1)\\ 0\\ \vdots\\ 0\\ 0\\ \vdots\\ 0\\ \bar{\eta}x(k)+\bar{\alpha}d(k)\end{bmatrix}\\
&+ [\,I\,0\ldots0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^TG^{-1}\hat{A}+\hat{W}_t^{-2})^{-1}\hat{A}^TG^{-1}]\Pi^TD^TD\begin{bmatrix} u(k-1)\\ 0\\ \vdots\\ 0\end{bmatrix}
\end{aligned}
\tag{14}
$$

Remark I

1. When $W_t \to \infty$, the control law (14) is the same as the hard constraint control law (9) with zero bounds of constraints ($\Delta\tilde{u}(k+i)=0$ and/or $\tilde{u}(k+i)=0$ and/or $\tilde{y}(k+j)=0$).
2. When $W_t \to 0$, the control law (14) with $\hat{A}(\hat{A}^TG^{-1}\hat{A}+\hat{W}_t^{-2})^{-1}\hat{A}^T=0$ becomes unconstraint control law which is the same as the control law (9) with the term $\hat{A}(\hat{A}^TG^{-1}\hat{A})^{-1}\hat{A}^T=0$.
3. The control laws (9) and (14) are the same to work on controlling the process with constraints. The adjustable time varying W_t can make the adjustable constraint control system work the same as the hard constraint control.
4. The adjustable constraint control law can represent the hard constraint control law with nonzero bounds of constraints ($\Delta\tilde{u}(k+i)\neq0$ and/or $\tilde{u}(k+i)\neq0$ and/or $\tilde{y}(k+j)\neq0$).
5. If the adjustable constraint control law can make the control system satisfy the quadratic stability and/or strongly H_∞ performance criteria, then the corresponding hard constraint control law also can make the control system satisfy the same stability and/or performance criteria.
6. The advantage to take the adjustable constraint control law for the stability and performance analysis is that there are no any bias terms from bounds of constraints ($\Delta\tilde{u}(k+i)\neq0$ and/or $\tilde{u}(k+i)\neq0$ and/or $\tilde{y}(k+j)\neq0$). Because of this advantage, it makes the stability and performance analysis of CMPC be feasible.
7. The W_t^2 can be solved by the following equation for a specified $\bar{\epsilon}^*$:

$$W_t^2\bar{\epsilon}^* = \hat{I}(\hat{A}^TG^{-1}\hat{A})^{-1}[(\hat{A}G^{-1}g+\hat{b})-\hat{I}^T\bar{\epsilon}^*] \tag{15}$$

Closed-Form Control Law with Uncertainty Blocks

The control law (9), (14) can be reformulated as closed-form control law with uncertainty blocks. To do so, let

$$\hat{A}^T = \varpi^T \bar{s}$$

where $\bar{s}$ is a full rank submatrix of A^T. Then, several facts should be followed:

1. $(\bar{s}G^{-1}\bar{s}^T)^{-1} \geq \varpi(\hat{A}^T G^{-1}\hat{A})^{-1}\varpi^T \geq \varpi(\hat{A}^T G^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\varpi^T$ (Proof is in Appendix A),
2. $(\bar{s}G^{-1}\bar{s}^T)^{-1} = E^{-1}E^{-T}$, where E is a nonsingular matrix,
3. $\varpi(\hat{A}^T G^{-1}\hat{A})^{-1}\varpi^T = E^{-1}U_h{}^T\Delta_c U_h E^{-T}$ and $\varpi(\hat{A}^T G^{-1}\hat{A}+\hat{W}_t^{-2})^{-1}\varpi^T = E^{-1}U^T\Delta_{sc}UE^{-T}$, where Δ_c, Δ_{sc} are diagonal matrices with value of each entry element between 0 and 1,
4. The unitary matrices $U_h,\ U$ are variably dependent of the active constraint situation.

The control law (14) can be rewritten as:

$$\begin{aligned}
u(k) &= -[\,I\,0\cdots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\hat{A}^T G^{-1}]S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\
&\quad +[\,I\,0\cdots 0\,]G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\varpi^T \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix} \\
&\quad +[\,I\,0\cdots 0\,][G^{-1} - G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\hat{A}^T G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\
&= -[\,I\,0\cdots 0\,][G^{-1} - G^{-1}\bar{s}^T E^{-1}U^T\Delta_{sc}UE^{-T}\bar{s}G^{-1}]S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\
&\quad +[\,I\,0\cdots 0\,]G^{-1}\bar{s}^T E^{-1}U^T\Delta_{sc}UE^{-T} \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix} \\
&\quad +[\,I\,0\cdots 0\,][G^{-1} - G^{-1}\bar{s}^T E^{-1}U^T\Delta_{sc}UE^{-T}\bar{s}G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\end{aligned} \tag{16}$$

Based on the closed-form control law with uncertainty blocks, the quadratic stability and strongly H_∞ performance of hard constrained MPC can be analyzed. A lemma in the following would be useful to simplify part of the closed-form control law to make the stability and performance analysis less conservative.

Lemma 1 *If the active constraint set includes some inputs reaching their active constraints of $\Delta u(k)$, then the term $u(k-1)$ arising from the bound $(\hat{b})$ in the hard constraint control law (9) does not affect the equivalent control law for quadratic stability and strongly H_∞ performance analysis.*
Proof: see appendix A.

4 Softening the hard constraints

The on-line objective function is:

$$\min_{u(k),\cdots,u(k+M-1),\epsilon} \sum_{l=1}^{P}[e^T(k+l)\Gamma^2 e(k+l) + u^T(k+l-1)B^2 u(k+l-1)$$

$$+ \Delta u^T(k+l-1)D^2\Delta u(k+l-1)] + \epsilon^T W^2 \epsilon \tag{17}$$

subject to

$$-\epsilon_{\Delta u}(k+i) + \Delta \underline{u}(k+i) \le \Delta u(k+i) \le \Delta\bar{u}(k+i) + \epsilon_{\Delta u}(k+i), \quad i = 0 \cdots M-1$$

and/or

$$-\epsilon_u(k+i) + \underline{u}(k+i) \le u(k+i) \le \bar{u}(k+i) + \epsilon_u(k+i), \quad i = 0 \cdots M-1$$

and/or

$$-\epsilon_{\hat{y}}(k+j) + \underline{y}(k+j) \le \hat{y}(k+j) \le \bar{y}(k+j) + \epsilon_{\hat{y}}(k+j), \quad w_b \le j \le w_e$$

and

$$\epsilon = [\ \epsilon_{\Delta u}{}^T(k)\ \cdots\ \epsilon_{\Delta u}{}^T(k+M-1)\ \epsilon_u{}^T(k)\ \cdots\ \epsilon_u{}^T(k+M-1)\ \epsilon_{\hat{y}}{}^T(k+w_b)\ \cdots\ \epsilon_{\hat{y}}{}^T(k+w_e)\ ]^T \ge 0$$

where

$$W = \begin{bmatrix} W_{\Delta u}(0) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & W_{\Delta u}(M-1) & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & W_u(0) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & W_u(M-1) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & W_{\hat{y}}(w_b) & 0 & 0 \\ 0 & & 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & W_{\hat{y}}(w_e) \end{bmatrix}$$

ϵ is the softening variable, and W is the softening weight. $\epsilon_u(k+i)$ $\epsilon_{\Delta u}(k+i)$ $\epsilon_{\hat{y}}(k+j)$ are the softening variables for $u(k+i)$, $\Delta u(k+i)$, $\hat{y}(k+j)$ respectively. $W_u(i)$, $W_{\Delta u}(i)$, $W_{\hat{y}}(j)$ are the time invariant softening diagonal weighting matrices for $\epsilon_u(k+i)$, $\epsilon_{\Delta u}(k+i)$, $\epsilon_{\hat{y}}(k+j)$ respectively.

Follow the same method in transforming the hard constrained MPC to adjustable constrained MPC in section 2, 3. The above optimization problem with time invariant W can be replaced by the following adjustable softened constraint optimization problem with time invariant W and time varying W_t:

$$\min_{u(k),\cdots,u(k+M-1),\epsilon,\bar{\epsilon}} \sum_{l=1}^{P}[e^T(k+l)\Gamma^2 e(k+l) + u^T(k+l-1)B^2 u(k+l-1)$$
$$+\Delta u^T(k+l-1)D^2\Delta u(k+l-1)] + \epsilon^T W^2 \epsilon + \bar{\epsilon}^T W_t{}^2 \bar{\epsilon} \tag{18}$$

subject to

$$-\epsilon_{\Delta u}(k+i) - \bar{\epsilon}_{\Delta u}(k+i) \leq \Delta u(k+i) \leq \epsilon_{\Delta u}(k+i) + \bar{\epsilon}_{\Delta u}(k+i), \quad i = 0 \cdots M-1$$

and/or

$$-\epsilon_u(k+i) - \bar{\epsilon}_u(k+i) \leq u(k+i) \leq \epsilon_u(k+i) + \bar{\epsilon}_u(k+i), \quad i = 0 \cdots M-1$$

and/or

$$-\epsilon_{\hat{y}}(k+j) - \bar{\epsilon}_{\hat{y}}(k+j) \leq \hat{y}(k+j) \leq \epsilon_{\hat{y}}(k+j) + \bar{\epsilon}_{\hat{y}}(k+j), \quad w_b \leq j \leq w_e$$

and

$$\epsilon = [\ \epsilon_{\Delta u}{}^T \ \ \epsilon_u{}^T \ \ \epsilon_{\hat{y}}{}^T\ ]^T \geq 0$$
$$\bar{\epsilon} = [\ \bar{\epsilon}_{\Delta u}^T \ \ \bar{\epsilon}_u^T \ \ \bar{\epsilon}_{\hat{y}}^T\ ]^T \geq 0$$

Based on the model (2) to predict the future output (predicted output), the quadratic optimization problem (18) can be written as a standard Quadratic Programming problem:

$$\begin{aligned}
\min_{v,,\epsilon,\bar{\epsilon}} q(v) &= \frac{1}{2}v^T G v + g^T v + \frac{1}{2}\epsilon^T W^2 \epsilon + \frac{1}{2}\bar{\epsilon}^T W_t^2 \bar{\epsilon} \\
&= \frac{1}{2}\bar{v}^T \begin{bmatrix} G & 0 & 0 \\ 0 & W^2 & 0 \\ 0 & 0 & W_t{}^2 \end{bmatrix} \bar{v} + [\ g^T \ \ 0 \ \ 0\ ]\bar{v}
\end{aligned} \tag{19}$$

subject to

$$\bar{A}^T \bar{v} \geq \bar{b} \tag{20}$$

where

$$\bar{A}^T = \begin{bmatrix} A_1{}^T & 0 & I_i^T \\ A_2{}^T & I_2 & I_{ii}{}^T \end{bmatrix}$$

$$\bar{v} = [\ u^T(k) \ \ \cdots \ \ u^T(k+M-1) \ \ \epsilon^T \ \ \bar{\epsilon}^T\ ]^T$$

where $A_1{}^T$, $A_2{}^T$ consist of the rows of A^T that correspond to the hard constraints without and with softening respectively; I_2, $[\ I_i \ \ I_{ii}\ ]^T = I$ are identity matrices and their dimension correspond to the row dimension of $A_2{}^T$, $\bar{A}^T$ respectively. For the optimal solution v^* we have [Fletcher, 1981]:

$$\begin{bmatrix} G & 0 & 0 & -\hat{A}_1 & -\hat{A}_2 \\ 0 & W^2 & 0 & 0 & -\hat{I}_2 \\ 0 & 0 & W_t{}^2 & -\hat{I}_i & -\hat{I}_{ii} \\ -\hat{A}_1^T & 0 & -\hat{I}_i^T & 0 & 0 \\ -\hat{A}_2^T & -\hat{I}_2^T & -\hat{I}_{ii}^T & 0 & 0 \end{bmatrix} \begin{bmatrix} v^* \\ \epsilon^* \\ \bar{\epsilon}^* \\ \lambda_1{}^* \\ \lambda_2{}^* \end{bmatrix} = - \begin{bmatrix} g \\ 0 \\ 0 \\ \hat{b}_1 \\ \hat{b}_2 \end{bmatrix} \tag{21}$$

where $\hat{I}_2^T\hat{I}_2 = I$; $\hat{I}_i^T\hat{I}_i = I$, $\hat{I}_{ii}^T\hat{I}_{ii} = I$; $[\ \lambda_1{}^{*T}\ \ \lambda_2{}^{*T}\ ]^T = \lambda^*$, $\hat{b} = [\ \hat{b}_1^T\ \ \hat{b}_2^T\ ]^T$; $\hat{A}_1^T$, $\hat{A}_2^T$, $\hat{b}_i$, $\hat{I}_2^T$, $\hat{I}_i^T$, $\hat{I}_{ii}^T$ consist of the rows of $A_1{}^T$, $A_2{}^T$, $\bar{b}^T$, I_2, $I_i{}^T$, $I_{ii}{}^T$ that correspond to the active constraints at the optimum. Let

$$\bar{G} = \begin{bmatrix} G & 0 \\ 0 & W^2 \end{bmatrix}$$

and

$$\hat{\mathcal{A}}^T = \begin{bmatrix} \hat{A}_1^T & 0 \\ \hat{A}_2^T & \hat{I}_2^T \end{bmatrix}$$

The equation (21) can be rewritten as:

$$\begin{bmatrix} \bar{G} & 0 & -\hat{\mathcal{A}} \\ 0 & W_t^2 & -\hat{I} \\ -\hat{\mathcal{A}}^T & -\hat{I}^T & 0 \end{bmatrix} \begin{bmatrix} \nu_\epsilon{}^* \\ \bar{\epsilon}^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} \bar{g} \\ 0 \\ \hat{b} \end{bmatrix} \tag{22}$$

where $\hat{I} = [\ \hat{I}_i\ \ \hat{I}_{ii}\ ]$; $\nu_\epsilon{}^* = [\ \nu^{*T}\ \ \epsilon^{*T}\ ]^T$; $\bar{g} = [\ g^T\ \ 0\ ]^T$. Then

$$\bar{\epsilon}^* = W_t{}^{-2}\hat{I}\lambda^*$$

and

$$\begin{bmatrix} \bar{G} & -\hat{\mathcal{A}} \\ -\hat{\mathcal{A}}^T & -\hat{W}_t^{-2} \end{bmatrix} \begin{bmatrix} \nu^* \\ \lambda^* \end{bmatrix} = - \begin{bmatrix} \bar{g} \\ \hat{b} \end{bmatrix} \tag{23}$$

where $\hat{W}_t^{-2} = \hat{I}^T W_2{}^{-2}\hat{I}$.

Following the similar method in section 2 to solve (23), we have the adjustable softened constraint control law:

$$\begin{aligned} u(k) &= [\ I\,0\ldots0\ ](-Hg + T\hat{b}) \\ &= -[\ I\,0\ldots0\ ][\bar{G}^{-1} - \bar{G}^{-1}\hat{\mathcal{A}}(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}} + \hat{W}_t{}^{-2})^{-1}\hat{\mathcal{A}}^T\bar{G}^{-1}] \begin{bmatrix} S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\ 0 \end{bmatrix} \\ &+ [\ I\,0\ldots0\ ]\bar{G}^{-1}\hat{\mathcal{A}}(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}} + \hat{W}_t{}^{-2})^{-1}\varpi^T \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix} \\ &+ [\ I\,0\ldots0\ ][\bar{G}^{-1} - \bar{G}^{-1}\hat{\mathcal{A}}(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}} + \hat{W}_t{}^{-2})^{-1}\hat{\mathcal{A}}^T\bar{G}^{-1}] \begin{bmatrix} \Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \end{aligned} \tag{24}$$

Remark II

1. When $W_t \rightarrow \infty$, the control law (24) is the same as the softened constraint control law of the optimization problem (17) with zero bounds of constraints ($\Delta\tilde{u}(k+i) = 0$ and/or $\tilde{u}(k+i) = 0$ and/or $\tilde{y}(k+j) = 0$).
2. When $W_t \rightarrow 0$, the control law (24) with the term $\hat{\mathcal{A}}(\hat{\mathcal{A}}^T G^{-1}\hat{\mathcal{A}})^{-1}\hat{\mathcal{A}}^T = 0$ becomes unconstrained control law, which is the same as the control law (9) with $\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T = 0$.
3. The control laws of the optimization problem (17) and (18) are the same to work on controlling the process with softened constraints. The adjustable time varying W_t can make the adjustable softened constraint control system (18) work the same as the softened constraint control (17).
4. The adjustable softened constraint control law (24) can represent the softened constraint control law with nonzero bounds of constraints ($\Delta\tilde{u}(k+i) \neq 0$ and/or $\tilde{u}(k+i) \neq 0$ and/or $\tilde{y}(k+j) \neq 0$).
5. If the adjustable softened constraint control law (24) can make the control system satisfy the quadratic stability and/or Strongly H_∞ performance criteria, then the corresponding softened constraint control law from the optimization problem (17) should be able to make the control system satisfy the same stability and/or performance criteria.
6. The advantage to take the adjustable softened constraint control law for the stability and performance analysis is that there are no any bias terms from bounds of constraints ($\Delta\tilde{u}(k+i) \neq 0$ and/or $\tilde{u}(k+i) \neq 0$ and/or $\tilde{y}(k+j) \neq 0$). Because of this advantage, it makes the stability and performance analysis of CMPC be feasible.
7. The W_t^2 can be solved by the following equation for a specified $\bar{\epsilon}^*$:

$$W_t^2 \bar{\epsilon}^* = \hat{I}(\hat{\mathcal{A}}^T \bar{G}^{-1}\hat{\mathcal{A}})^{-1}[(\hat{\mathcal{A}}\bar{G}^{-1}\bar{g} + \hat{\bar{b}}) - \hat{I}^T \bar{\epsilon}^*] \tag{25}$$

Closed-Form Control Law with Uncertainty Blocks

The control law (24) can be reformulated as closed-form control law with uncertainty blocks. To do so, let

$$\hat{\mathcal{A}}^T = \varpi^T \bar{z}$$

where $\bar{z}$ is a full rank submatrix of $\mathcal{A}^T$ with row dimension not great than M times the number of manipulated variable. Then, several facts should be followed:

1. $(\bar{s}G^{-1}\bar{s}^T)^{-1} \geq (\bar{z}\bar{G}^{-1}\bar{z}^T)^{-1} \geq \varpi(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}})^{-1}\varpi^T \geq \varpi(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}} + \hat{W}_t^{-2})^{-1}\varpi^T$,
2. $(\bar{z}\bar{G}^{-1}\bar{z}^T)^{-1} = E^{-1}E^{-T}$, where E is a nonsingular matrix,
3. $\varpi(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}})^{-1}\varpi^T = E^{-1}U_{sh}^T\Delta_c U_{sh}E^{-T}$ and $\varpi(\hat{\mathcal{A}}^T\bar{G}^{-1}\hat{\mathcal{A}} + \hat{W}_t^{-2})^{-1}\varpi^T = E^{-1}U^T\Delta_{sc}UE^{-T}$, where Δ_c, Δ_{sc} are diagonal matrices with value of each entry element between 0 and 1,
4. The unitary matrices U_{sh}, U are variably dependent of the active constraint situation.

The control law (24) can be rewritten as:

$$
\begin{aligned}
u(k) &= -[\,I\,0\cdots0\,][\bar{G}^{-1} - \bar{G}^{-1}\hat{A}(\hat{A}^T\bar{G}^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\hat{A}^T\bar{G}^{-1}]\begin{bmatrix} S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\ 0 \end{bmatrix} \\
&\quad +[\,I\,0\cdots0\,]\bar{G}^{-1}\hat{A}(\hat{A}^T\bar{G}^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\varpi^T\begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix} \\
&\quad +[\,I\,0\cdots0\,][\bar{G}^{-1} - \bar{G}^{-1}\hat{A}(\hat{A}^T\bar{G}^{-1}\hat{A} + \hat{W}_t^{-2})^{-1}\hat{A}^T\bar{G}^{-1}]\begin{bmatrix} \Pi^TD^TD\begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \\
&= -[\,I\,0\cdots0\,][\bar{G}^{-1} - \bar{G}^{-1}\bar{z}^TE^{-1}U^T\Delta_{sc}UE^{-T}\bar{z}\bar{G}^{-1}]\begin{bmatrix} S^T\Gamma^T\Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\ 0 \end{bmatrix} \\
&\quad +[\,I\,0\cdots0\,]\bar{G}^{-1}\bar{z}^TE^{-1}U^T\Delta_{sc}UE^{-T}\begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix} \\
&\quad +[\,I\,0\cdots0\,][\bar{G}^{-1} - \bar{G}^{-1}\bar{z}^TE^{-1}U^T\Delta_{sc}UE^{-T}\bar{z}\bar{G}^{-1}]\begin{bmatrix} \Pi^TD^TD\begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \\ 0 \end{bmatrix} \qquad (26)
\end{aligned}
$$

Based on the closed-form control law with uncertainty blocks, the quadratic stability and strongly H_∞ performance of softened constraint MPC can be analyzed. A lemma in the following would be useful to simplify part of the closed-form control law to make the stability and performance analysis less conservative.

Lemma 2 *If the active constraint set includes some inputs reaching their active constraints of $\Delta u(k)$, then the term $u(k-1)$ arising from the bound $(\hat{b})$, which can be obtained by reformulating the constraints of optimization problem (17) to the constraint of a standard Quadratic Programming*

problem (please see the section 2) does not affect the equivalent control law for quadratic stability and strongly H_∞ performance analysis.
The approach followed in the proof is similar to that in lemma 1.

5 Illustrations

Example 1

A SISO multieffect evaporator process is given [Ricker *et al.*, 1989] :

$$\tilde{p}(s) = \frac{2.69(-6s+1)e^{-1.5s}}{100s^2+25s+1}$$

(I) Set constraints on Δu, u over the control horizon (M):

$$-0.1 \le \Delta u(k+i) \le 0.1, \ -0.5 \le u(k+i) \le 0.5, \ i = 1, M$$

Select tuning parameters:

$$P = 10, \ M = 2, \ B = 0, \ D = 5, \ \Gamma = 1, \ T_s = 3$$

where T_s is the sampling time. Set the disturbance $d(s) = 1.2/s$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 1), and we can off-line find the W_t^2 (in Table 1) by using equation (15). Applying adjustable constrained MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 1. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

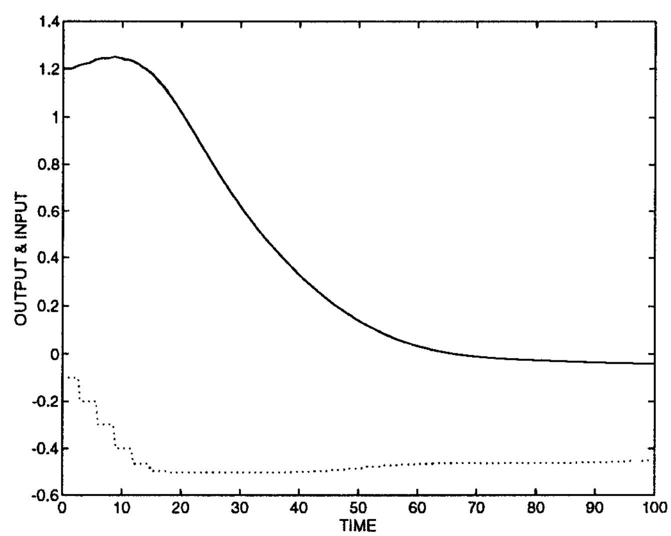

Figure 1: Simulation for example 1 (I); Solid line : output; Dotted line : input

(II) Set constraint on :

$$-0.5 \le \hat{y}(k+4) \le 0.5$$

k	Active Constraint Sets				Adjustable Weights			
	$\Delta u(k)$	$\Delta u(k+1)$	$u(k)$	$u(k+1)$	$(W_{t\Delta u}(k))^2$	$(W_{t\Delta u}(k+1))^2$	$(W_{tu}(k))^2$	$(W_{tu}(k+1))^2$
0	1	1	0	0	3.2904e+01	1.6552e+01	0	0
1	1	1	0	0	2.4086e+01	9.6007e+00	0	0
2	1	1	0	0	1.4231e+01	1.9543e+00	0	0
3	1	0	0	0	4.7794e+00	0	0	0
4	0	0	0	1	0	0	0	1.4085e+00
5	0	0	0	1	0	0	0	2.3070e+00
6	0	0	1	1	0	0	9.1895e-01	1.9499e+00
7	0	0	1	1	0	0	8.7099e-01	1.5026e+00
8	0	0	1	1	0	0	6.8672e-01	1.1203e+00
9	0	0	1	1	0	0	5.2925e-01	7.9285e-01
10	0	0	1	1	0	0	3.9435e-01	5.1186e-01
11	0	0	1	1	0	0	2.7859e-01	2.7051e-01
12	0	0	1	1	0	0	1.7915e-01	6.3060e-02
13	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
·	0	0	0	0	0	0	0	0

Table 1: The active constraint sets and adjustable weights of example 1 (I); where 1, 0 in the columns of u, Δu represent the corresponding constraint reaches its lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

Select tuning parameters:

$$P = 30,\ M = 1,\ B = 0,\ D = 0,\ \Gamma = 1,\ T_s = 3$$

Set the disturbance $d(s) = 2/s$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 2), and we can off-line find the $W_t{}^2$ (in Table 2) by using equation (15). Applying adjustable constrained MPC with $W_t{}^2$ to do the control simulation again, we see that these two simulations are exactly same shown on Figure 2. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

(III) Set constraint on :

$$-0.3 \le \Delta u(k) \le 0.3,\ \ -0.5 \le u(k) \le 0.5,\ \ -0.1 \le \hat{y}(k+7) \le 0.1$$

Select tuning parameters:

$$P = 10,\ M = 2,\ B = 0,\ D = 0,\ \Gamma = 1,\ T_s = 3$$

Set the disturbance $d(s) = 1/s$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 3), and we can off-line find the $W_t{}^2$ (in Table 3) by using equation (15). Applying adjustable constrained MPC with $W_t{}^2$ to do the control simulation again, we see that these two simulations are exactly same shown on Figure 3. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

k	Active Constraint Sets	Adjustable Weights
	$\hat{y}(k+4)$	$(W_{t\hat{y}}(k+4))^2$
0	2	1.0999e+04
1	1	1.2570e+02
2	1	1.1056e+03
3	0	0
$\vdots$	$\vdots$	$\vdots$
.	0	0

Table 2: The active constraint sets and adjustable weights of example 1 (II); where 2, 1, 0 in the columns of $\hat{y}$ represent the corresponding constraint reaches the upper, lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

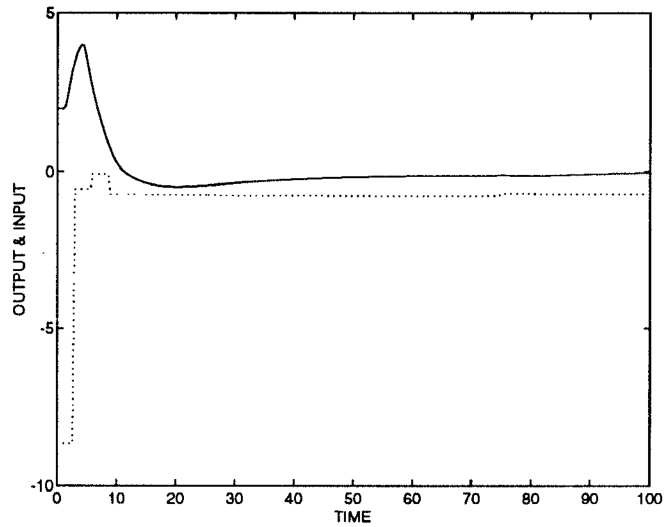

Figure 2: Simulation for example 1 (II); Solid line : output; Dotted line : input

k	Active Constraint Sets			Adjustable Weights		
	$\Delta u(k)$	$u(k)$	$\hat{y}(k+7)$	$(W_{t\Delta u}(k))^2$	$(W_{tu}(k))^2$	$(W_{t\hat{y}}(k+7))^2$
0	1	0	2	5.8738e+00	0	3.7205e+01
1	0	1	2	0	2.3633e+00	2.6568e+01
2	0	1	2	0	2.0248e+00	1.9203e+01
3	0	1	2	0	1.3823e+00	1.2526e+01
4	0	1	2	0	8.2551e-01	6.5942e+00
5	0	1	2	0	3.4428e-01	1.3871e+00
6	0	1	0	0	8.6829e-02	0
7	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
.	0	0	0	0	0	0

Table 3: The active constraint sets and adjustable weights of example 1 (III); where 2, 1, 0 in the columns of $\hat{y}$, u, Δu represent the corresponding constraint reaches its upper, lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

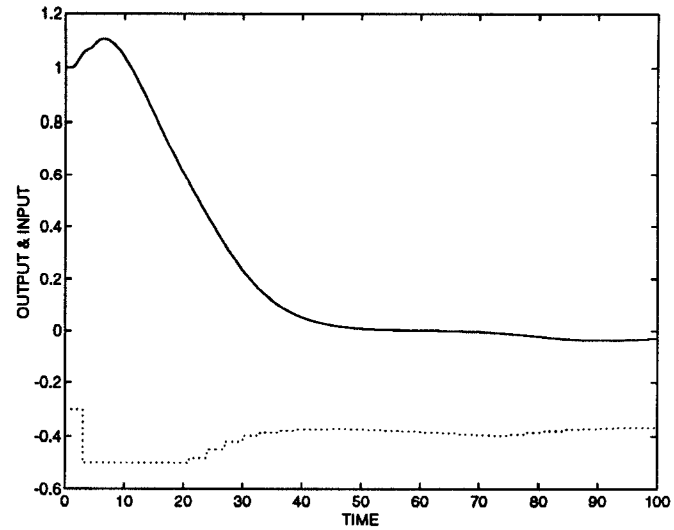

Figure 3: Simulation for example 1 (III); Solid line : output; Dotted line : input

Example 2

A 1×1 process model of a subsystem of the Shell Control Problem is given [Prett and Garcia, 1988]:

$$\tilde{p}(s) = \frac{4.05e^{-27s}}{50s+1}$$

Set constraints on $\hat{y}(k+7)$:

$$-0.5 \leq \hat{y}(k+7) \leq 0.5$$

Select tuning parameters:

$$P = 60, \; M = 1, \; B = 0, \; D = 0, \; W_{\hat{y}}(7) = 290, \; \Gamma = I, \; T_s = 4$$

Set the disturbance $d(s) = 1.1/s$ and apply softened hard constraint MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 4), and we can off-line find the W_t^2 (in Table 4) by using equation (25). Applying adjustable softened constraint MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 4. The result shows that the adjustable softened constraint control law (24) work just exactly same as the softened constraint control law from the optimization problem (17).

Example 3

A 2×2 process model of a subsystem of the Shell Control Problem is given [Prett and Garcia, 1988]:

$$\tilde{p}(s) = \begin{bmatrix} \frac{4.05e^{-27s}}{50s+1} & \frac{1.77e^{-28s}}{60s+1} \\ \frac{5.39e^{-18s}}{50s+1} & \frac{5.72e^{-14s}}{60s+1} \end{bmatrix}$$

(I) Set constraints on Δu, u over the control horizon (M).

$$-0.3 \leq \Delta u(k+i) \leq 0.3, \; -0.5 \leq u(k+i) \leq 0.5, \; i = 1, M$$

k	Active Constraint Sets	Adjustable Weights
	$\hat{y}(k+7)$	$(W_{r\hat{y}}(k+7))^2$
0	2	4.9986e+04
1	0	0
$\vdots$	$\vdots$	$\vdots$
.	0	0

Table 4: The active constraint sets and adjustable weights of example 2; where 2, 0 in the columns of $\hat{y}$ represent the corresponding constraint reaches its upper (active constraint), and stays between bounds (inactive constraint) respectively.

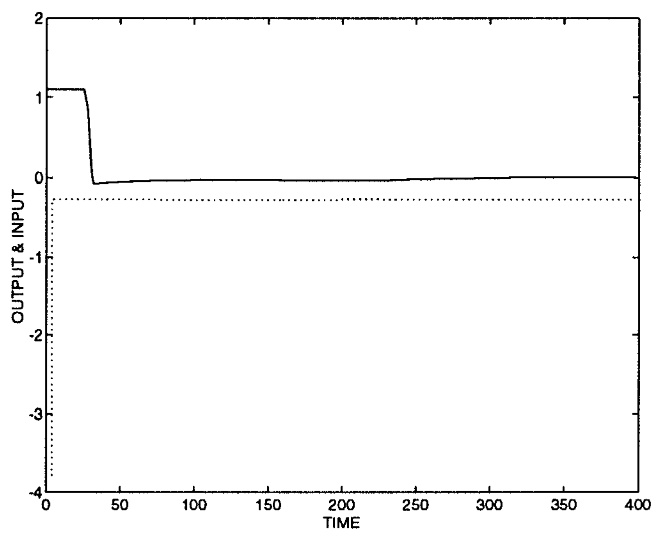

Figure 4: Simulation for example 2; Solid line : output; Dotted line : input

k	Active Constraint Sets									
	$\Delta u_1(k)$	$\Delta u_2(k)$	$u_1(k)$	$u_2(k)$	$u_1(k+1)$	$u_2(k+1)$	$u_1(k+2)$	$u_2(k+2)$	$u_2(k+3)$	$u_1(k+4)$
0	1	1	0	0	1	1	1	1	1	0
1	0	0	1	1	1	1	0	0	0	0
2	0	0	1	1	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0
8	0	0	1	0	1	0	0	0	0	0
9	0	0	1	0	1	0	1	0	0	0
10	0	0	1	0	1	0	1	0	0	0
11	0	0	1	0	1	0	1	0	0	0
12	0	0	1	0	1	0	1	0	0	0
13	0	0	1	0	1	0	1	0	0	0
14	0	0	0	0	1	0	0	0	0	1
15	0	0	0	0	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
·	0	0	0	0	0	0	0	0	0	0

Table 5: The active constraint sets of example 3 (I); where 1, 0 in the columns of u, Δu represent the corresponding constraint reaches its lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

Select tuning parameters:

$$P = 6,\ M = 5,\ B = 0,\ D = 1.5I,\ \Gamma = I,\ T_s = 6$$

Set the disturbance $d(s) = [\ 1.5/s\ 1.7/s\ ]^T$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 5), and we can off-line find the W_t^2 (in Tables 6 and 7) by using equation (15). Applying adjustable constrained MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 5. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

(II) Set constraint on :

$$-0.1 \le \hat{y}_1(k+6) \le 0.1,\ \ -0.1 \le \hat{y}_2(k+4) \le 0.1$$

Select tuning parameters:

$$P = 7,\ M = 2,\ B = 0,\ D = 0,\ \Gamma = I,\ T_s = 6$$

Set the disturbance $d(s) = [\ 5/s\ 5/s\ ]^T$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 8), and we can off-line find the W_t^2 (in Table 8) by using equation (15). Applying adjustable constrained MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 6. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

k	Adjustable Weights					
	$(W_{t\Delta u_1}(k))^2$	$(W_{t\Delta u_2}(k))^2$	$(W_{tu_1}(k))^2$	$(W_{tu_2}(k))^2$	$(W_{tu_1}(k+1))^2$	$(W_{tu_2}(k+1))^2$
0	5.93e+00	5.38e+00	0	0	6.55e-01	9.47e-01
1	0	0	1.28e+00	1.11e+00	2.81e-01	4.89e-01
2	0	0	6.84e-01	2.89e-01	0	0
3	0	0	0	0	0	0
4	0	0	0	0	0	0
5	0	0	0	0	0	0
6	0	0	0	0	0	0
7	0	0	0	0	0	0
8	0	0	8.68e-03	0	4.22e-02	0
9	0	0	1.35e-01	0	5.67e-02	0
10	0	0	1.25e-01	0	5.62e-02	0
11	0	0	1.01e-01	0	4.58e-02	0
12	0	0	6.92e-02	0	3.12e-02	0
13	0	0	3.47e-02	0	1.63e-02	0
14	0	0	0	0	7.97e-03	0
15	0	0	0	0	0	0
⋮	⋮	⋮	⋮	⋮	⋮	⋮
.	0	0	0	0	0	0

Table 6: The adjustable weights of example 3 (I)

k	Adjustable Weights			
	$(W_{tu_1}(k+2))^2$	$(W_{tu_2}(k+2))^2$	$(W_{tu_2}(k+3))^2$	$(W_{tu_1}(k+4))^2$
0	2.28e-01	6.74e-01	1.38e-01	0
1	0	0	0	0
2	0	0	0	0
3	0	0	0	0
4	0	0	0	0
5	0	0	0	0
6	0	0	0	0
7	0	0	0	0
8	0	0	0	0
9	1.47e-02	0	0	0
10	1.74e-02	0	0	0
11	1.54e-02	0	0	0
12	1.19e-02	0	0	0
13	8.86e-03	0	0	0
14	0	0	0	2.18e-03
15	0	0	0	0
⋮	⋮	⋮	⋮	⋮
.	0	0	0	0

Table 7: The adjustable weights of example 3 (I)

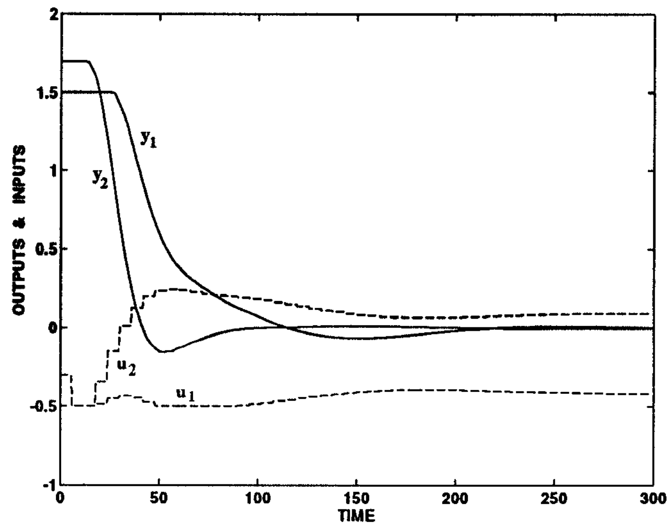

Figure 5: Simulation for example 3 (I); Solid line : output; Dashed line : input

k	Active Constraint Sets		Adjustable Weights	
	$\hat{y}_2(k+4)$	$\hat{y}_1(k+6)$	$(W_{t\hat{y}_1}(k+6))^2$	$(W_{t\hat{y}_2}(k+4))^2$
0	1	2	1.0550e+01	1.4174e+01
1	1	2	3.9173e+00	2.4911e+00
2	0	0	0	0
⋮	⋮	⋮	⋮	⋮
.	0	0	0	0

Table 8: The active constraint sets and adjustable weights of example 3 (II); where 2, 1, 0 in the columns of $\hat{y}$ represent the corresponding constraint reaches its upper, lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

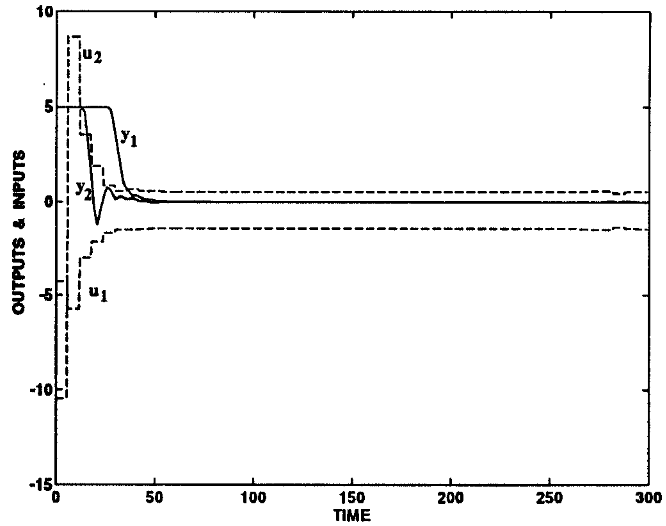

Figure 6: Simulation for example 3 (II); Solid line : output; Dashed line : input

(III) Set constraints on

$$-0.3 \leq \Delta u(k) \leq 0.3, \ -0.5 \leq u(k) \leq 0.5,$$

$$-0.1 \leq \hat{y}_1(k+8) \leq 0.1, \ -0.1 \leq \hat{y}_2(k+7) \leq 0.1$$

Select tuning parameters:

$$P = 10, \ M = 2, \ B = 0, \ D = 0, \ \Gamma = I, \ T_s = 6$$

Set the disturbance $d(s) = [\ 1.5/s\ 1.5/s\ ]^T$ and apply hard constrained MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control (in Table 9), and we can off-line find the W_t^2 (in Table 10) by using equation (15). Applying adjustable constrained MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 7. The result shows that the adjustable constraint control law (14) work just exactly same as the hard constraint control law (9).

(IV) Set constraint on :

$$-0.3 \leq \Delta u_1(k) \leq 0.3, \ -1 \leq u_1(k) \leq 1, \ -0.1 \leq \hat{y}_1(k+5) \leq 0.1$$

Select tuning parameters:

$$P = 73, \ M = 1, \ B = 0, \ D = 0, \ W_{\hat{y}_1}(5) = 10.0835, \ \Gamma = I, \ T_s = 6$$

Set the disturbance $d(s) = [\ 2/s\ 1/s\ ]^T$ and apply softened hard constraint MPC to simulate controlling this process. From the control simulation, we obtain a sequence of active constraint sets over control period (in Table 11), and we can off-line find the W_t^2 (in Table 11) by using equation (25). Applying adjustable softened constraint MPC with W_t^2 to do the control simulation again, we see that these two simulations are exactly same shown on Figure 8. The result shows that the adjustable softened constraint control law (24) work just exactly same as the softened constraint control law from the optimization problem (17).

k	Active Constraint Sets					
	$\Delta u_1(k)$	$\Delta u_2(k)$	$u_1(k)$	$u_2(k)$	$\hat{y}_2(k+7)$	$\hat{y}_1(k+8)$
0	1	1	0	0	2	2
1	0	0	1	1	2	2
2	0	0	1	1	0	2
3	0	0	1	1	0	0
4	0	2	0	0	0	0
5	0	2	0	0	0	0
6	0	2	1	0	0	0
7	0	0	1	0	0	0
8	0	0	1	0	0	0
9	0	0	1	0	0	0
10	0	0	1	0	0	0
11	0	0	1	0	0	0
12	0	0	1	0	0	0
13	0	0	1	0	0	0
14	0	0	1	0	0	0
15	0	0	1	0	0	0
16	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\cdot$	0	0	0	0	0	0

Table 9: The active constraint sets of example 3 (III); where 1, 2, 0 in the columns Of u, Δu, $\hat{y}$ represent the corresponding constraint reaches its lower, upper bound (active constraint), and stays between bounds (inactive constraint) respectively.

k	Adjustable Weights					
	$(W_{t\Delta u_1}(k))^2$	$(W_{t\Delta u_2}(k))^2$	$(W_{tu_1}(k))^2$	$(W_{tu_2}(k))^2$	$(W_{t\hat{y}_1}(k+8))^2$	$(W_{t\hat{y}_2}(k+7))^2$
0	9.6648e+00	7.4413e+00	0	0	1.9646e+01	7.8742e+00
1	0	0	3.8619e+00	2.9184e+00	1.1657e+01	1.6999e+00
2	0	0	2.1734e+00	1.4682e+00	4.7211e+00	0
3	0	0	7.4089e-01	2.2133e-01	0	0
4	0	7.6351e-01	0	0	0	0
5	0	8.4561e-01	0	0	0	0
6	0	4.7112e-01	1.2524e-01	0	0	0
7	0	0	3.5815e-01	0	0	0
8	0	0	2.7604e-01	0	0	0
9	0	0	2.2802e-01	0	0	0
10	0	0	1.8231e-01	0	0	0
11	0	0	1.4216e-01	0	0	0
12	0	0	1.0651e-01	0	0	0
13	0	0	7.4887e-02	0	0	0
14	0	0	4.6844e-02	0	0	0
15	0	0	2.1973e-02	0	0	0
16	0	0	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\cdot$	0	0	0	0	0	0

Table 10: The adjustable weights of example 3 (III)

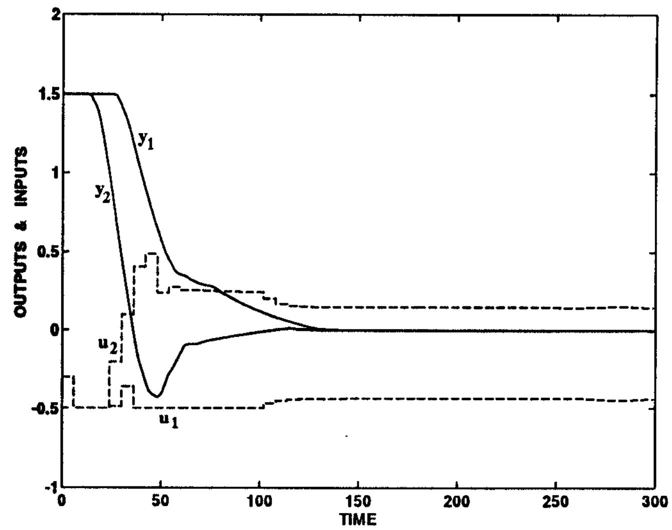

Figure 7: Simulation for example 3 (III); Solid line : output; Dashed line : input

k	Active Constraint Sets		Adjustable Weights	
	$\Delta u_1(k)$	$\hat{y}_1(k+5)$	$(W_{t\Delta u_1}(k))^2$	$(W_{t\hat{y}_1}(k+5))^2$
0	1	2	5.6633e+02	1.8622e+03
1	1	2	2.5508e+02	1.6842e+03
2	0	2	0	1.4537e+03
3	0	2	0	1.2292e+03
4	0	2	0	1.0371e+03
5	0	2	0	8.7271e+02
6	0	2	0	7.3206e+02
7	0	2	0	6.1175e+02
8	0	2	0	5.0885e+02
9	0	2	0	4.2087e+02
1	0	2	0	3.4565e+02
11	0	2	0	2.8136e+02
12	0	2	0	2.2642e+02
13	0	2	0	1.7949e+02
14	0	2	0	1.3941e+02
15	0	2	0	1.0519e+02
16	0	2	0	7.5980e+01
17	0	2	0	5.1055e+01
18	0	2	0	2.9795e+01
19	0	2	0	1.1667e+01
20	0	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
·	0	0	0	0

Table 11: The active constraint sets and adjustable weights of example 3 (IV); where 2, 1, 0 in the columns of $\hat{y}_1$, Δu_1 represent the corresponding constraint reaches its upper, lower bound (active constraint), and stays between bounds (inactive constraint) respectively.

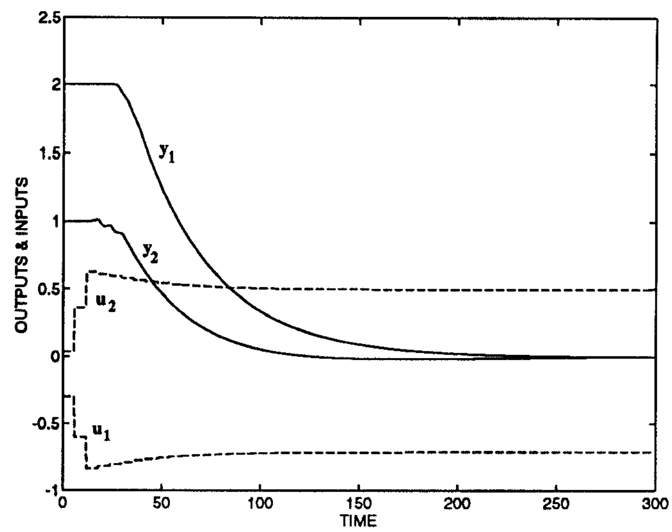

Figure 8: Simulation for example 3 (IV); Solid line : output; Dashed line : input

6 Concluding Remarks

This paper shows that the control law of CMPC can be reformulated to closed-form with uncertainty blocks inside. The control law for a set of specified constraints contains constant bias terms which cause difficulty in analyzing the quadratic stability and strongly H_∞ performance of CMPC. To avoid this difficulty, we can change the hard constraints to adjustable constraints with time varying weights. The control law of this adjustable constrained system with no constant bias, and the upper bound of its uncertainty block is found to be adequate to analyze the qualitative (quadratic stability) and quantitative (strongly H_∞ performance) properties of CMPC. The same methods also can be applied to cases with softened hard constraints.

Acknowledgments

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Appendix A: The Proof of fact 1 in 3 and Lemma 1

(1) Proof of the fact 1 in section 3

The inequality as following is obviously held:

$$\varpi(\hat{A}^T G^{-1} \hat{A})^{-1} \varpi^T \geq \varpi(\hat{A}^T G^{-1} \hat{A} + \hat{W}_t^{-2})^{-1} \varpi^T$$

We are going to prove:

$$(\bar{s} G^{-1} \bar{s}^T)^{-1} \geq \varpi(\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} \varpi^T$$

and the extraction matrix has the following property

$$\varpi^T \varpi = I$$

From the properties of positive definite matrix [Horn and Johnson, 1990], we know:

$$\begin{aligned}
& \varpi^T (\bar{s} G^{-1} \bar{s}^T)^{-1} \varpi \geq (\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} > 0 \\
\Longrightarrow\ & \varpi^T (\bar{s} G^{-1} \bar{s}^T)^{-1} \varpi \geq \varpi^T \varpi (\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} \varpi^T \varpi > 0 \\
\Longrightarrow\ & \varpi^T [\, (\bar{s} G^{-1} \bar{s}^T)^{-1} - \varpi(\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} \varpi^T \,] \varpi \geq 0 \\
\Longrightarrow\ & (\bar{s} G^{-1} \bar{s}^T)^{-1} \geq \varpi(\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} \varpi^T
\end{aligned} \tag{27}$$

or

$$(\bar{s} G^{-1} \bar{s}^T)^{-1} > 0, \ (\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi) > 0$$

the inequality (27) implies

$$\begin{bmatrix} \varpi^T & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} (\bar{s} G^{-1} \bar{s}^T)^{-1} & \varpi \\ \varpi^T & (\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi) \end{bmatrix} \begin{bmatrix} \varpi & 0 \\ 0 & I \end{bmatrix} \geq 0$$

It also implies

$$\begin{bmatrix} (\bar{s} G^{-1} \bar{s}^T)^{-1} & \varpi \\ \varpi^T & (\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi) \end{bmatrix} \geq 0$$

Hence, $(\bar{s} G^{-1} \bar{s}^T)^{-1} \geq \varpi(\varpi^T \bar{s} G^{-1} \bar{s}^T \varpi)^{-1} \varpi^T$ □

(2) Proof of Lemma 1

The control law of hard constrained MPC (9) with zeros bounds of constraints ($\Delta\tilde{u}(k+i) = 0$, $\tilde{u}(k+i) = 0$, $\tilde{y}(k+j) = 0$) can be reformulated as the following with uncertainty blocks (please see the "Closed-Form Control Law with Uncertainty Blocks" in section 3):

$$\begin{aligned}
u(k) \;=\; & -[\, I\, 0 \cdots 0 \,][G^{-1} - G^{-1} \bar{s}^T E^{-1} U_h{}^T \Delta_c U_h E^{-T} \bar{s} G^{-1}] S^T \Gamma^T \Gamma(\eta x(k) + \alpha d(k) - R(k+1)) \\
& + [\, I\, 0 \cdots 0 \,] G^{-1} \bar{s}^T E^{-1} U_h{}^T \Delta_c U_h E^{-T} \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta} x(k) + \bar{\alpha} d(k) \end{bmatrix}
\end{aligned}$$

$$+[\,I\,0\cdots 0\,][G^{-1}-G^{-1}\bar{s}^T E^{-1} U_h{}^T \Delta_c U_h E^{-T}\bar{s} G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (28)$$

and the control law of adjustable constrained MPC is:

$$\begin{aligned} u(k) \;=\; & -[\,I\,0\cdots 0\,][G^{-1}-G^{-1}\bar{s}^T E^{-1} U^T \Delta_{sc} U E^{-T}\bar{s} G^{-1}]S^T\Gamma^T\Gamma(\eta x(k)+\alpha d(k)-R(k+1)) \\ & +[\,I\,0\cdots 0\,]G^{-1}\bar{s}^T E^{-1} U^T \Delta_{sc} U E^{-T} \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k)+\bar{\alpha}d(k) \end{bmatrix} \\ & +[\,I\,0\cdots 0\,][G^{-1}-G^{-1}\bar{s}^T E^{-1} U^T \Delta_{sc} U E^{-T}\bar{s} G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (29) \end{aligned}$$

Both of them have the same bound on uncertainty block:

$$U_h{}^T \Delta_c{}^2 U_h \le I, \;\; U^T \Delta_{sc}{}^2 U \le I$$

It is obvious that these two control laws (28) and (29) are the same for working on analyzing the quadratic stability and strongly H_∞ performance (the details of analyzing quadratic stability and strongly H_∞ performance will be shown in the technical reports). Hence, we can conclude that the control law (28) can make the control system satisfy the the quadratic stability and strongly H_∞ performance criteria if and only if the control law (29) can make the control system satisfy the same criteria.

For a system with m inputs system, assume that ℓ constraints of Δu are active with $\ell \le m$. Without loss of generality assume these are the first ℓ elements of u. Then, the corresponding control law from (28) can be written as:

$$\begin{aligned} u(k) \;=\; & -[\,I\,0\;\cdots\;0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}]S^T\Gamma^T\Gamma(\eta x(k)+d(k)-R(k+1))+ \\ & [\,I\,0\;\cdots\;0\,]G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{b}+ \\ & [\,I\,0\;\cdots\;0\,][G^{-1}-G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T G^{-1}]\Pi^T D^T D \begin{bmatrix} u(k-1) \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (30) \end{aligned}$$

because

$$\bar{s}^T E^{-1} U_h{}^T \Delta_c U_h E^{-T}\bar{s} = \hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\hat{A}^T$$

and

$$\hat{b} = \varpi^T \begin{bmatrix} u_1(k-1) \\ \vdots \\ u_m(k-1) \\ 0 \\ \vdots \\ 0 \\ \bar{\eta}x(k) + \bar{\alpha}d(k) \end{bmatrix}$$

By the matrix operation, we have:

$$[\, I\, 0 \cdots 0\,]G^{-1}\hat{A}(\hat{A}^T G^{-1}\hat{A})^{-1}\mathcal{U}(k-1) = \begin{bmatrix} I_\ell & 0 \\ \bar{x}_5 & 0 \end{bmatrix} \mathcal{U}(k-1)$$

and the control law (30) can be rewritten as:

$$\begin{aligned} u(k) &= -\begin{bmatrix} I_\ell - q^{-1}I_\ell & 0 \\ -\bar{x}_5 q^{-1} & I \end{bmatrix}^{-1} \{ \begin{bmatrix} 0 & 0 \\ \bar{x}_1 & \bar{x}_2 \end{bmatrix} x(k) + \begin{bmatrix} 0 & 0 \\ \bar{x}_3 & \bar{x}_4 \end{bmatrix} d(k) - \begin{bmatrix} 0 & 0 \\ \bar{x}_6 & \bar{x}_7 \end{bmatrix} qR(k) - \\ &\quad \begin{bmatrix} 0 & 0 \\ \bar{x}_8 & \bar{x}_9 \end{bmatrix} q^{-1}u(k)\} \\ &= -\begin{bmatrix} 0 & 0 \\ \bar{x}_1 & \bar{x}_2 \end{bmatrix} x(k) - \begin{bmatrix} 0 & 0 \\ \bar{x}_3 & \bar{x}_4 \end{bmatrix} d(k) + \begin{bmatrix} 0 & 0 \\ \bar{x}_6 & \bar{x}_7 \end{bmatrix} R(k+1) + \begin{bmatrix} 0 & 0 \\ \bar{x}_8 & \bar{x}_9 \end{bmatrix} u(k-1) \end{aligned}$$

where $\mathcal{U}(k-1) = [\, u_1(k-1) \cdots u_\ell(k-1)\, 0 \cdots 0\,]^T$; $\bar{x}_i$ correspond to term which may not be zero in the matrix operations; q is a time shifting operator. From the above equation, since I_ℓ and $\bar{x}_5$ have been eliminated from the final expression, the term $u(k-1)$ arising from $\hat{b}$ owing to ℓ active constraints of Δu can be eliminated from the control law (28). Therefore, we can conclude that the control law (28) without or with the term $u(k-1)$ arising from $\hat{b}$ is the same as the control law (29) for working on analyzing the quadratic stability and strongly H_∞ performance. The statement of lemma follows. □

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