Reorganizing Nothingness
-- the dynamics of empty black holes

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At a cocktail party a longtime friend (and retired lawyer) asked me "What does this excitement about *detecting gravitational waves* have to do with me?" where 'me' clearly meant the non-physicists among us. Forced to come up with a quick and brief reply I said (approximately) "It could change our views of the aims and achievements of physical science." To this I added, until other topics intervened, a few sentences about the philosophical attitudes of physicists, which I expand at some length below.

[In place of citations, this note contains italicized keyword phrases that will lead to citations via an internet search.]

What has been seen?

The source for this inquiry was the widely reported *LIGO "We did it!"* news conference from the LIGO/VIRGO Scientific Collaboration. There it was announced that (1) gravitational waves had been directly detected, (2) evidence for the strong field details of real black holes (as deduced from Einstein's equations) was obtained, and (3) a merger of unexpectedly large black holes was found to have occurred. In addition this first observation from LIGO (Laser Interferometric Gravitational Observatory) demonstrated that a new window on the Universe – views using gravitational waves – was now open for future study. This essay explores a few important aspects of this new science.

The first observational data set in itself looks meager and needs explanation. Figure 1 displays these basic data:

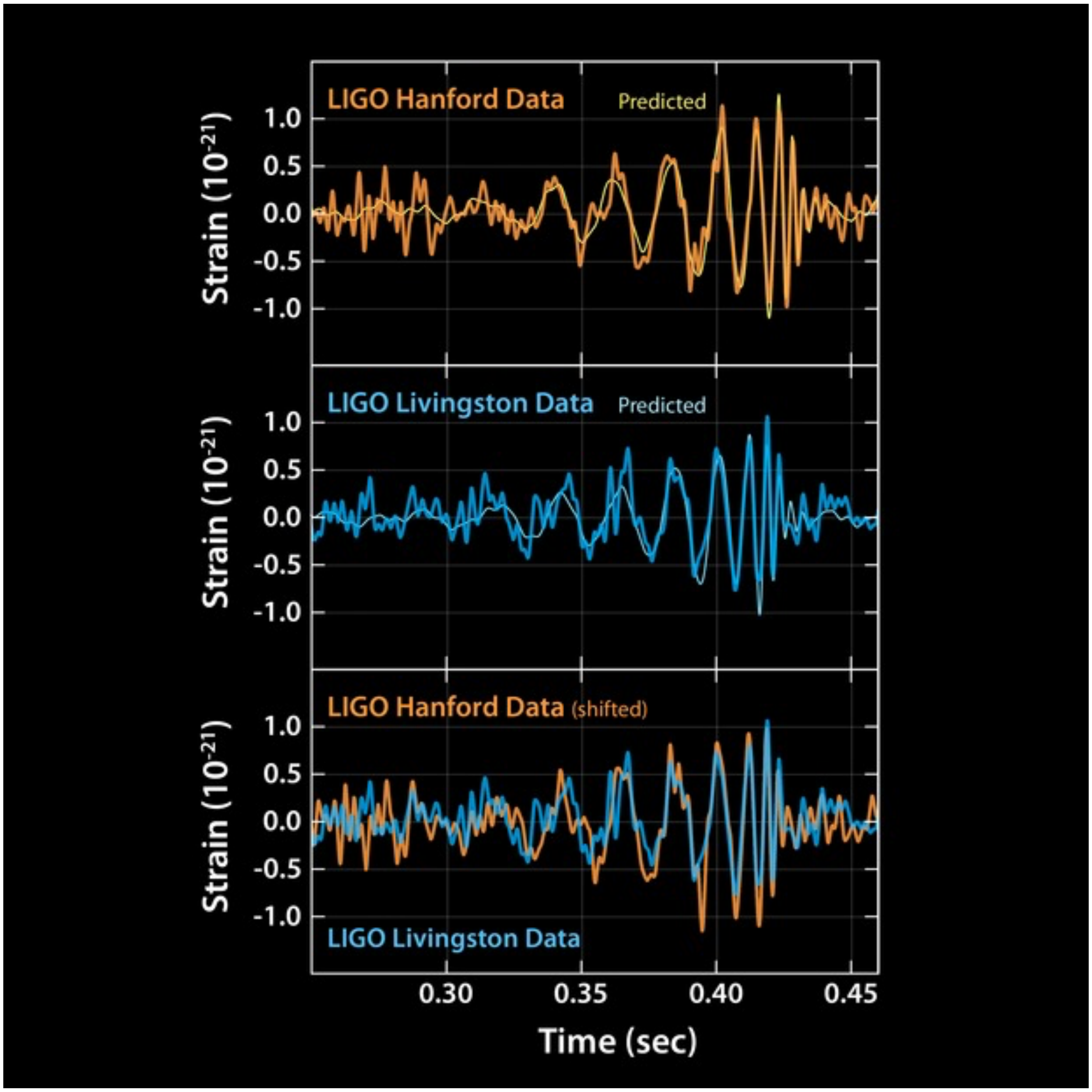


Figure 1. Data showing the fractional difference in length between the two interferometer arms at each facility (Hanford and Livingston) for the interesting fifth of a second on September 11, 2015.

The steps needed to get from these data to a video showing the appearance and motion of the two merging black holes are very complex and technical, so are merely suggested below. But comparison to an older data interpretation milestone may make these omitted steps seem more acceptable. Thus let us compare the LIGO observational data as evidence for black hole videos presented below to Tycho Brahe's observations of Mars' positions in the sky as evidence for the now familiar picture of the solar system of planets orbiting the Sun. Brahe's data have been put into a convenient form shown in Figure 2.

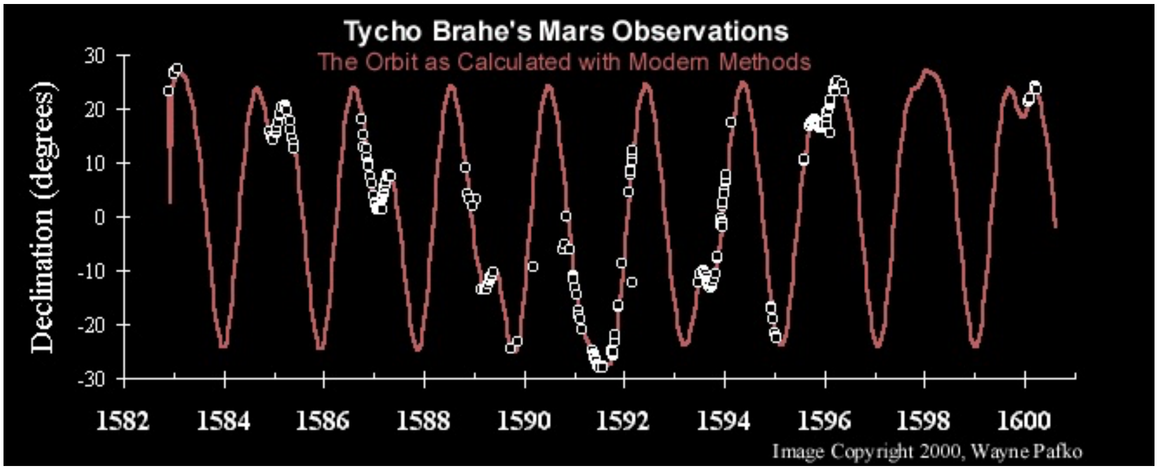


Fig. 2 Tycho Brahe's 20 years of observation of Mars, taken from his notes and compared by Wayne Pafko (http://www.pafko.com/tycho/home.html) to Newtonian theory.

The black hole merger pictures we provide below are based on the Fig. 1 data collections that are comparable in size to the Fig. 2 data Kepler had when he promoted the Copernican view of the solar system. There are further parallels between these two milestones. First note that Kepler's interpretation of Brahe's data was a step toward a deeper theory of the solar system accomplished by Newton, while the interpretation of the LIGO data is based on Einstein's 1915 gravitation theory. (Philosophers often feel that a prediction – theory first – is stronger support than a construction – observation first – of a theory to fit the data.) But understanding a theory's conclusions is often dependent upon visualization – the human oriented organization of the theory's statements. In the solar system case this may well have been Kepler's use of an analog computer to display his three laws. He could point to the Galilean moons of Jupiter whose motions made visible a scaled simulation of what a solar system satisfying his three laws might look like from a suitable viewpoint. In the black hole case we use the digital simulations of the result of applying Einstein's equations to the case of merging black holes, as provided by SXS (a LIGO related collaboration).

Philosophical Themes

Certain philosophical attitudes are unenforceable laws about what is treated as an important advance in science, or clues about where to look for improved understanding of Nature. Thus they are important for anyone wanting to judge where science fits into public life and into their personal views of Nature. *Gerald Holton (Thematic Origins ...)* has called them 'themata'.

The currently best known such theme is radical reductionism, which holds the *Higgs boson* as a recent success and the *'Theory of Everything'* as a current goal. A less widely known (but widely practiced) theme is 'emergence' whose icon is Philip Anderson's paper "*More is Different*". The reductionist view is that to understand Nature one must discover the ultimate micro-constituents of natural objects, and the laws by which these constituents interact. The emergent view is that major importance must be given to theories (which can often later be derived from the reductionists' theories by some approximation or limitation) giving languages and sets of laws that provide essential insight into the structure of natural things. Examples of emergent theories are hydrodynamics, Newtonian dynamics, thermodynamics, Schrödinger quantum mechanics, classical electricity and magnetism, etc. Each of these is "certified"-- known to be confirmed in some limited domain by a better theory (e.g., a not yet *falsified theory*) that may have a very different set of basic concepts and laws that often were understood only recently. A discussion of how essential these emergent theories are to understanding nature could begin by asking to what extent a molecular geneticist finds *Quantum ChromoDynamics* (the theory of quarks and gluons) useful.

Phase change of nothingness

Back to the question about detecting gravitational waves: On this occasion (*LIGO GW150914*) the detection confirmed that the black holes found in Einstein's theory truly exist in our Universe. **These black holes are a dramatic example of an object that cannot be reduced to its micro-constituents**. This shows that the very successful reductionist approach to understanding Nature is not adequate everywhere. The preceding sentence is the main point of this long essay.

To repeat: Nature has supplied us with black holes --- impressive, powerful, large objects that are not composed of any micro-constituents. Although the fact is little appreciated, black holes are made entirely of nothingness. Nothingness in this context is another word for pure empty spacetime containing no particles or matter fields but described by the spacetime metric that Einstein used to equate gravity with the curvature of spacetime. You will see that the SXS video (below) does not make any use of the matter (collapsed supersized stars?) that may have helped the two 30 solar mass black holes form – something that presumably preceded their merger by hundreds of million years or more. So the video shows two once-independent configurations of nothingness merging into one.

Why do I say black holes are made of nothing but nothingness? Aren't black holes made out of the particles in the stars that collapsed to form them? Answer: No. The particles, and the stars they formed, are not being rearranged or converted to other things. They are acting as an *enzyme* (to draw from a biological term) that brings the spacetime around them into a difficult shape which can then, via a *phase transition*, change into a stable configuration (the black hole) which no longer needs their help and can discard them.

The matter (including electromagnetic and other fields) that is discarded has enabled a phase change in a sector of spacetime from flat Minkowski spacetime to a curved black hole configuration. But what has happened to that enzymatic matter? Where has it gone? The answer given by classical (non-quantum) general relativity is "Not Here!"

As is generally known, nothing that falls into a black hole can send out a report; see *Finkelstein's 'unidirectional membrane'*, now properly called the event horizon. So, once the black-hole-enabling matter has disappeared into the black hole, its future light cone (events that could receive a signal from the collapsed matter) does not include anything/anyone outside the black hole. I prefer to say that the matter that collapsed in forming a black hole is not just hiding beyond our reach inside the horizon, but has actually been banished from our Universe, as it can have no further effect on us, nor we on it.

Views of Nothingness

We can return later to the question of what happened to the matter that helped the black hole form, but first I propose to let you understand in more detail what the merger video showed. Seeing the shape of nothingness is somewhat like the problem of photographing a spirit; and it is very like it if you take spirit to mean an invisible object such as clear air turbulence.

The problem of making a video record of colliding black holes has been solved by the *SXS Project* with great skill and good financial support. Their approach is similar to one that has been used to visualize the motion of dry air blowing past airfoils in a wind tunnel. The air, like empty space, does not emit, reflect or absorb light, so it can't be photographed when pure. But by adding some reflective marker such as smoke or oil or fog, in an amount so small that it won't change the airflow, one can get photographs showing the airflow patterns. See the example in the Figure below.

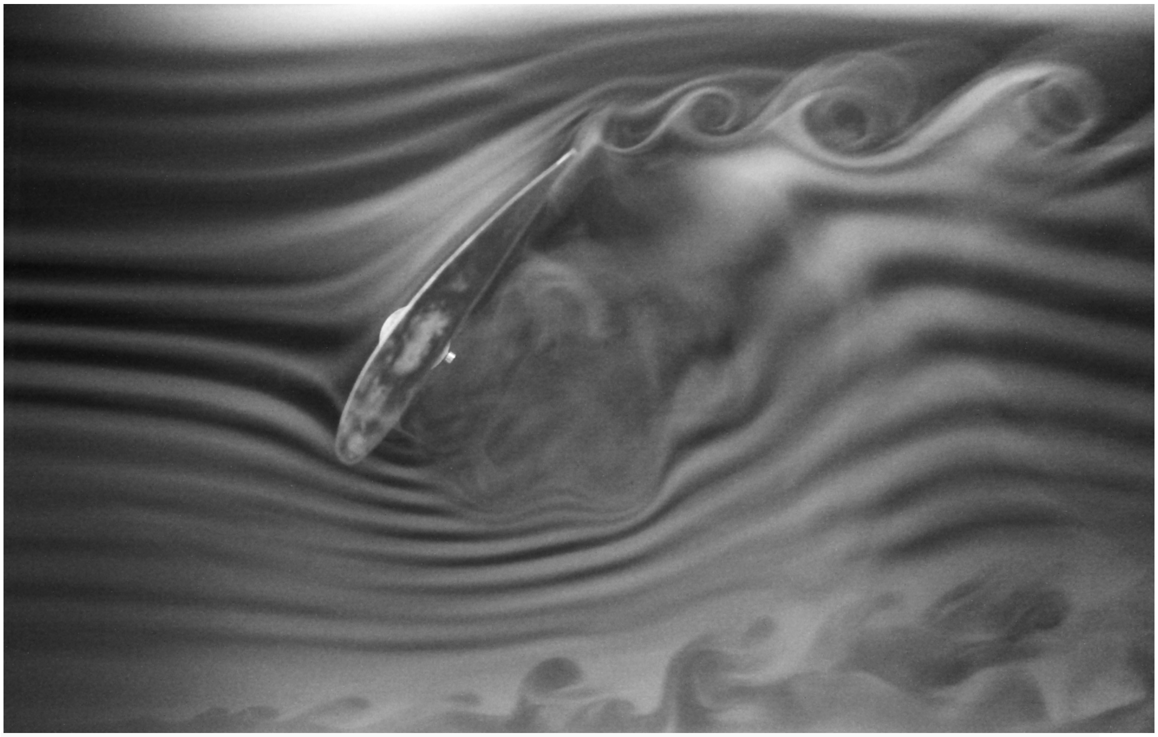
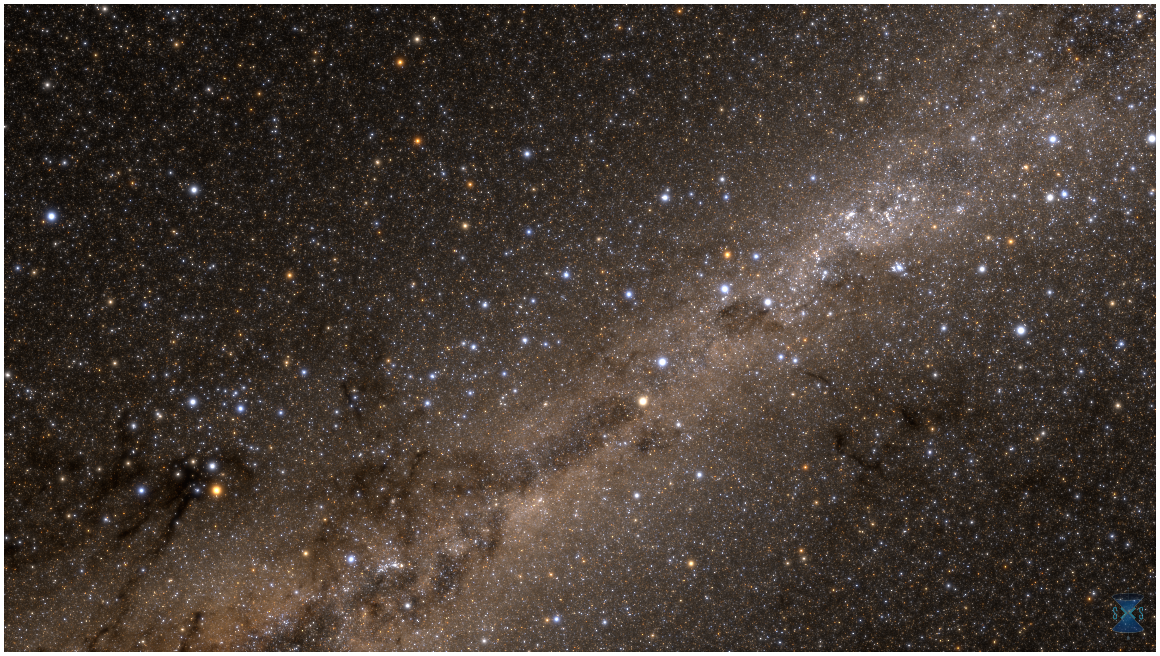


Fig.3 Fog (water particle) wind tunnel visualization of a NACA 4412 airfoil at a low speed flow (Re=20.000). The image is released to the public domain courtesy of Smart Blade GmbH (www.smart-blade.com)
By Georgepehli (Own work) [CC BY-SA 3.0 (http://creativecommons.org/licenses/by-sa/3.0)], via Wikimedia Commons https://en.wikipedia.org/wiki/Wind_tunnel#/media/File:Fog_visualization.jpg .

The added marker used by the SXS Project is light rays. The SXS crew wanted to recreate the coalescing black holes of the famous *GW150914* gravitational wave event, but to place them on a video studio stage with a camera to view the action. Their model is a pair of cyber black holes which we have reason to believe is a much more accurate scale model of the GW150914 black holes than aeronautical engineers can typically obtain for a wind tunnel aircraft model. These model black holes are "illuminated" in the video studio by hanging, upstage, a cyber model of a section of the Milky Way Galaxy with a few million accurately placed stars, seen here:



https://www.black-holes.org/lensing/GW150914-FlatStill.png
Fig. 4 Backdrop for the SXS stage where a black hole pair will be placed and videotaped.

This backdrop provides the illumination for the video and in the process emits the light rays that can be bent when traversing curved spacetime. Thus the stage director places the model black holes between this backdrop and the video camera, and proceeds with the video performance where an early frame is

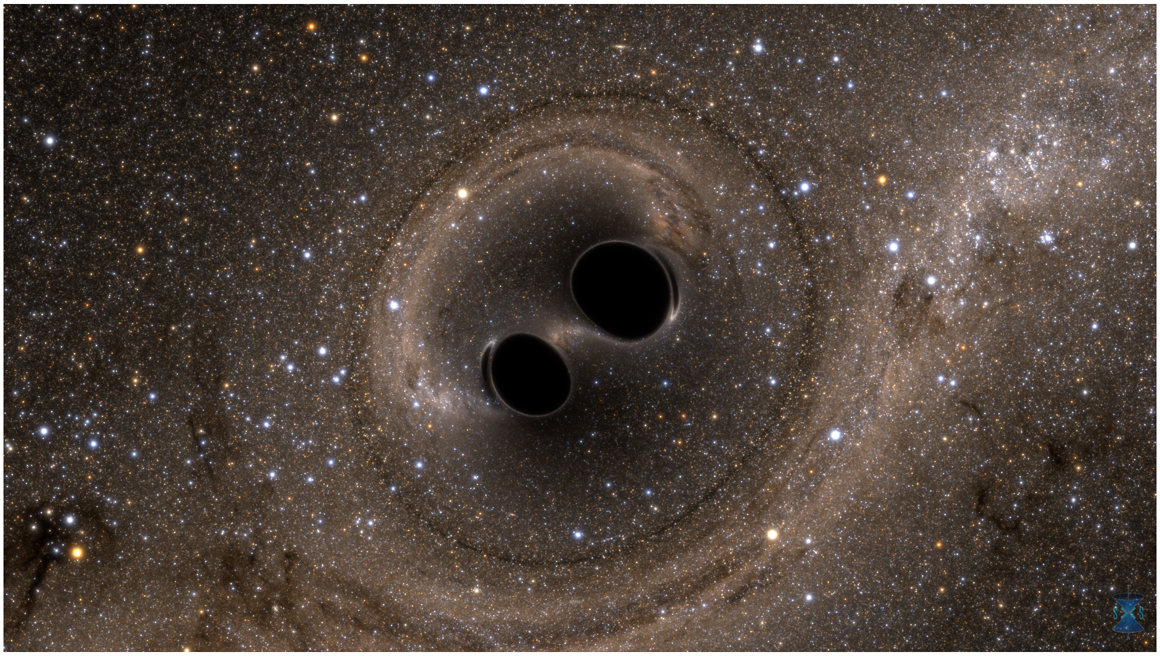


Fig. 5 A frame from the SXS video of coalescing black holes.
For the complete performance see:
https://www.youtube.com/watch?v=Zt8Z_uzG71o&feature=youtu.be

One has a temptation to treat this image as an artist's rendering of what scientists are thinking when they talk of black holes. But an artist's brush (or Photoshop control) has had no input to this display. Every pixel of the image has been placed there by a computer that has been programmed to obey Einstein's equations for the dynamics of the spacetime curvature and for the paths of light rays emitted by the backdrop. There is some artistry involved, but it is comparable to the artistry seen in the photographer *Karsh's portrait of Winston Churchill*, which also did not need Photoshop. The artistry was to pick a rhetorically effective backdrop and to place the backdrop, photo model, and camera in a way to get an insightful result. Karsh used three material components; SXS has three cyber equivalents---the backdrop, model, and camera---but both work with outstanding artistry (but no Photoshop) to provide a memorable souvenir of reality.

My preference for this particular view of black hole dynamics is due to the fact that it is independent of the choice made for the coordinate system used in the computer formulation for solving the Einstein equations. There are no coordinate grid lines needed in the presentation.

Fig 6. Karsh's portrait of Winston Churchill
[public domain:
Yousuf Karsh. Library and Archives Canada, e010751643]

Conclusions

The example above of the LIGO event GW150914 gives a strong case for the view that can be described as Reorganizing Nothingness. The computer program that modeled this merger of two black holes did not contain any reference to matter nor to any non-gravitational fields. So what the videos presented was two tangles of nothingness (the two original black holes) interacting: spacetime geometry was being reorganized into a final stationary tangle of nothingness, the resultant spinning black hole.

But some readers may find this view too limited. If the final black hole has a strong gravitational field capable of bending passing light rays or holding stars or plasma clouds in orbit at a safe distance around it, must there not be somewhere a mass producing that Newtonian-like field? The brief answer to this question is that the mass/energy that is now responsible for this attractive distant gravitational field is in the tangle of unfamiliar Einstein gravitational fields that make up the black hole.

The next sections go searching for whatever matter may have helped the black holes to form originally, but this is just to extend the idea that "the original collapsing matter is gone since we can never see its remnant again" to the stronger version "It is gone since we can not even influence it a short time after the collapse".

But even without studying how far ('out of our Universe') some early enzymatic matter may have been dispatched, a summary of the Einstein gravitational fields may be illuminating. Einstein was forced by finding that just making the Newtonian gravitational potential a modifier of the velocity of light didn't make a good theory. But he learned, in the context of Minkowski spacetime, that he needed to use Riemannian geometry. There the basic gravitational potential is a 4 by 4 symmetric matrix called the metric tensor, which has 10 different components. Of these ten gravitational potentials, four control how the coordinate system is laid out, another two propagate waves at the speed of light and can carry unrestricted information. The last four potentials include one that survives in the Newtonian limit and is sufficient for most solar system relativistic effects such as the redshift, bending of light, and the correction to the orbit of Mercury. The remaining three potentials are tied to motion and are sometimes called gravito-magnetic potentials. They are important for the dragging of inertial frames near rotating masses. Although barely measurable in satellite orbits around a rotating Earth, these are immensely important in the case of spinning black holes. They, together with the Newton-like potential, give the structure of a quiescent black hole, and interact with each other to give a stable structure for the nothingness (empty spacetime) that constitutes the black hole. They allow a spinning black hole to be something like a battery. A spinning BH stores huge amounts of energy (with an upper limit of 29% of its mass) that can be released to spectacular effects if properly discharged. But for our purposes in this essay, they just provide an answer to "What is responsible for the gravitational fields outside a black hole?"

Addendum: Disposing of Matter Inside Black Holes

Why should we also want the picture that the Einstein equations provide of the unobservable spacetime inside a black hole? My motivation is to support the claim that the matter that helped form the black hole has been not just piled out of sight, but has been effectively ejected from our Universe. It is to support that claim that I push on to describe what seems believable about the behavior of spacetime beyond the Finkelstein horizon.

The most important change in our scientific viewpoint is to treat the black hole horizon as a three-dimensional object, rather than as simple two-dimensional surface dividing inside from outside. Since nothing is happening outside the black hole once its formation event has calmed down, picturing the horizon as two-dimensional would seem adequate. But this obscures the nature of that horizon, as it separated the stationary outside region from the dynamically collapsing vacuum inside. We then tend to ignore the difference between light cones tilting from just narrowing. Outside an isolated black hole spacetime is stationary, whether we treat the Schwarzschild metric or the more realistic Kerr metric (for a rotating BH). That means that if one finds a description of something falling into a BH (e.g. the last scraps of a collapsing star that helped the BH to form), the same trajectory should serve for a stray test particle a few days or a few hundred million years later by changing only the initial t-coordinate. That time-step, outside the BH, compares motions that differ in physical time. Inside the BH the light cones have tilted over; there the hidden portion of the second trajectory lies a physically <u>spatial</u> distance from the first. (The two trajectory world lines are spacelike related, not timelike related, inside the BH horizon.) The three-dimensional horizon has topologically spherical cross sections but becomes a tube when the third (t-coordinate) dimension is included. Outside the BH horizon a tube with r constant has a -++ metric signature, but inside the horizon it will importantly be +++, i.e. all directions are spacelike so any two points there will be outside each other's future light cone. A sketch of a spacetime diagram for the spherical BH is given in Fig. 7. For $r > M$ the difference between coordinate lengths and physical lengths is only factors of 2 or 3. Thus if we take a 60 solar mass BH as the object, the distance (time or space) M in gravitational units is about 90 km or 300 light-microseconds. For the largest well-measured super BH (NGC 1600) whose mass is about 20 billion solar masses, the distance M is about 85,000 seconds or 24 light-hours. [Note: the Sun-Earth distance is 500 light-sec or 8 minutes; the distance to Pluto is about 5.5 light-hours.]

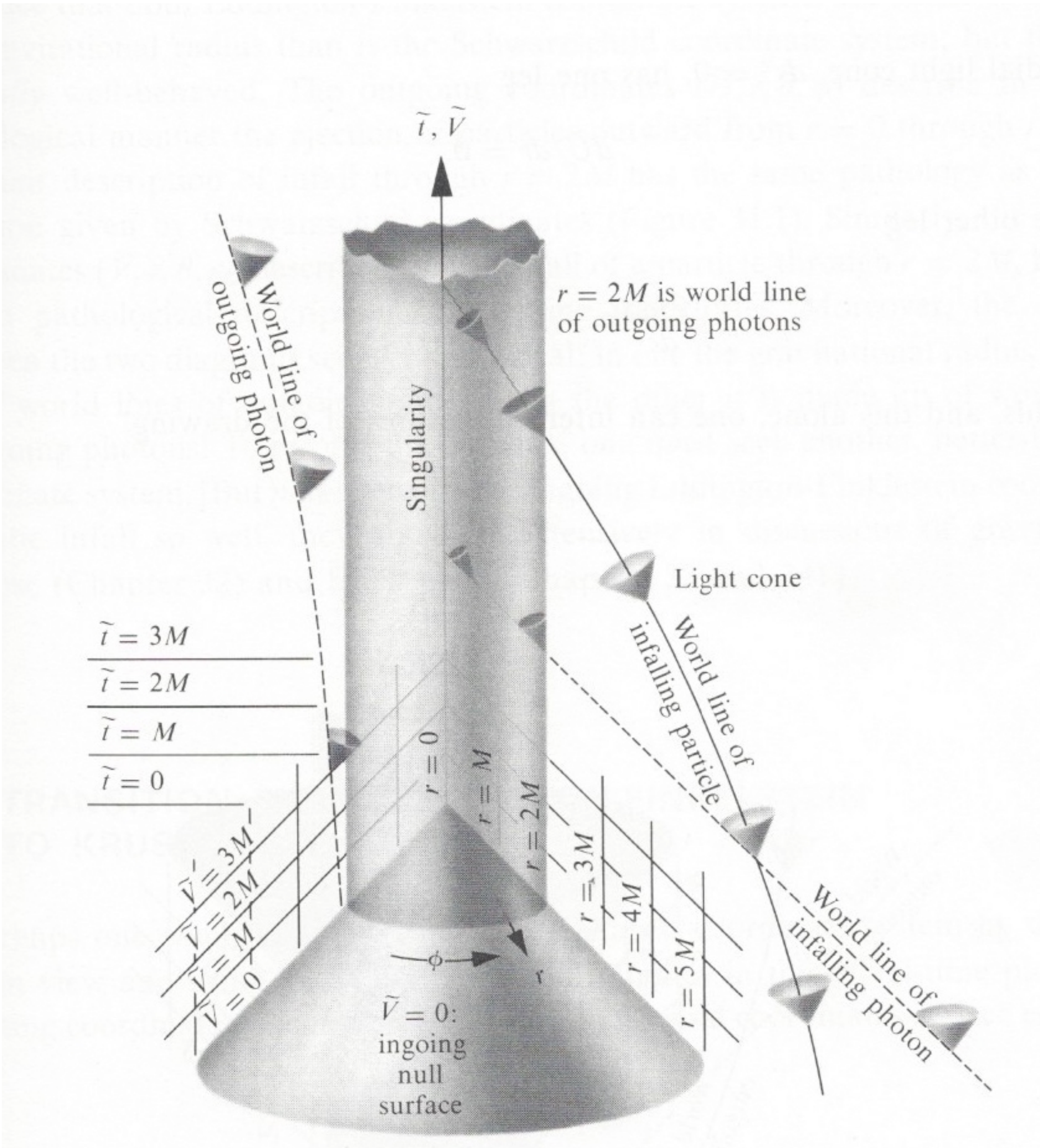


Fig. 7 Spacetime diagram for the Finkelstein BH. The implied horizontal coordinates are x and y, (the z-direction has been omitted) while horizontal planes represent the 3-dimesional hypersurfaces of constant $\tilde{t}$. Coordinate values of $\tilde{V} < 0$

should sketch a different metric describing the formation of this black hole. Note that the hypersurfaces (tubes) of constant r for $r < 2M$ and $\tilde{V} \geq 0$ are spacelike as they contain no directions inside the light cones.

This is totally out of line with intuitive ideas of what might be going on beyond a black hole horizon. Who could imagine a spacelike tube a billion light-years length in size existing inside an horizon that appears only a couple hundred kilometers in diameter? But that is what one gets from Einstein's equations when they lead to tumbling light cones as in Fig. 7 where the axis of coordinate time should be extended to great lengths upward that don't fit on the page. (Downwards the spacetime diagram needs to be modified to show the dynamics and matter as the BH is formed.) Einstein's gravity can make outgoing light rays fall backward – the extreme case of the gravitational bending of light rays. And those equations have been found to work accurately even at the levels where one finds the near velocity-of-light relative motion between two black holes as in the merger producing the *LIGO GW150914* gravitational wave pulse. This conundrum leads us to study an old mystery under a new name.

Autonomic Spacetime Creation

The flow of time has always seemed mysterious and did get attention, but little insight, from ancient and recent philosophers. I propose the name 'Autonomic Spacetime Creation' for this conundrum although that name does not explain anything. But it can serve to remind us that this mysterious creation of future spacetime from presently existing or previously archived spacetime is not just one overwhelming feature of nature, but exists in at least three variants. The first is the continual creation of spacetime that we experience everyday as the present evolves into the future, which previously, although anticipated, did not yet exist. The second is the continual new creation of spacetime inside every black hole as it prepares a place to put anything that might fall into the black hole. The third version of autonomic spacetime creation is that by which the expanding universe manages, on the largest scales where "dark energy" is noticed, to allow a region of space filled with dark energy to expand at unchanged density. The horizon of a black hole is a place from which the black hole (visible curved empty spacetime outside the horizon) "sheds" new spacetime continually into the (future) spacetime just inside the horizon. [The word 'autonomic' is chosen by analogy to the '*autonomic nervous system*' in physiology where it denotes the actions (such as heartbeat and digestion) that occur without any need or ability for us to consciously control them.]

Cautions

There is a huge difference between reading what Einstein's equations tell us about spacetime outside a black hole and what they can say about inside it. The difference is stability and robustness. For the region outside a black hole horizon the black hole solutions of Einstein's equations have been proven stable against small disturbances (*Vishveshwara, dissertation 1968*) and with unique configurations (*Israel, black hole uniqueness*) when reasonably distant from other important masses. There is a much more generic condition than an horizon, the *Penrose trapped surface*, for which it was importantly shown that some sort of singularity must occur inside that surface according to the Einstein equations. Numerical simulations, after much effort, eventually showed that two black holes would, according to Einstein's equations, interact somewhat like other heavy masses and could collide and merge if they once got sufficiently close to each other. These predictions of possible solutions of Einstein's equations were confirmed by finding such an interaction in nature as seen by LIGO, so we must take Einstein's equations under black hole conditions seriously.

Concerning what happens to matter that has fallen into a black hole, the Einstein equations become much less specific. They do say that we cannot get direct confirmation of any proposed behavior, as no reports can escape the black hole interior. [There is a slight hope that some conditions proposed for occurring inside a black hole might observably be found, with a reversed direction of time, in the very early stages of the big bang cosmology.] But it may be impossible to explore the Einstein equation behaviors far inside a black hole, as these equations strongly suggest that the simple solutions we know will prove very unstable so that every numerical solution of the equations will give a different answer from a calculation with almost identically posed input. These instabilities could set in under conditions far from the limiting case where quantum gravity (not yet available) would be needed to make the theory plausible. Therefore we want to look only at moderate depths into the BH interior, where the view provided by Einstein's equations can be taken quite seriously.

For a spinning black hole (most black holes in nature are expected to have some spin) the instabilities are expected to begin toward the Kerr causality horizon at a moderate fraction of the black hole size. A causality horizon is a point in a numerical solution of the Einstein equations where the program finds that, to perform the next time-step, it needs more data than can be supplied by the entire previous history. It may want information from a recently created a singularity where, by definition, no information is available. Thus, beyond this point, the Einstein equations effectively recuse themselves from participating in physics. Unfortunately, little progress can be expected from numerical relativity explorations of the causality horizon in the next decade or so as preparing to interpret the anticipated LIGO observations takes precedence for the available resources. (See *Berger in Living Reviews 2002* for some past work in this area.)

In a spinning black hole the Finkelstein horizon is located at a level slightly below the Schwarzschild (spherical BH) position $r = 2M$ at $r = M + \sqrt{M^2 - a^2}$ while the causality horizon (or 'troublesome horizon') is at the level $r = M - \sqrt{M^2 - a^2}$ where $a/M < 1$ is the fraction of the maximum spin plausible. A reasonable choice of $a/M = 0.6$ as in the final GW150914 BH puts these two horizons at $r_F = 1.8M$ and $r_t = 0.2M$. Between these two horizons every constant r hypersurface is spacelike. So we think in terms of the plausible spacetime near $r = M$. Describing this limit to the assertive capability of the Einstein equations is clearest for large black holes. The causality horizon found in the Kerr solution of Einstein's equations for the NGC1600 black hole would be moderate fraction of its 50 light-hour size, perhaps only ten percent of that size if its spin were about 60% of the maximum spin. The curvature of spacetime (i.e., the gravitational field gradient) there would be of the order of M/r^3 with r near M. At the horizon of the *NCG 1600* black hole the radius of curvature of spacetime (approximately $\sqrt{r^3/M}$) is about

25 light-hours. By comparison, the radius of curvature of spacetime just above the Earth (where astronauts have spent hours in low orbit) is about one third of a light-hour, so the tidal stresses the astronauts didn't even notice were nearly 100 times stronger than those at the horizon of the *NCG 1600* black hole. Those at the causality horizon might be only a few times weaker than those in low Earth orbit.

Oppenheimer and Snyder in 1939 gave the original description of what we now see as the formation of a (spherical) black hole. Their presentation did not include any description of the empty spacetime outside the collapsing matter but beyond the sight of external observers. Thanks to Finkelstein's exposition of that sector of empty spacetime, the Oppenheimer and Snyder collapse description could be completed (e.g., by *Beckedorff and Misner, DRUM, 1962*) to include that previously ignored sector. By looking at that sector of spacetime (the empty part inside the black hole horizon, but outside the collapsed matter) we find, in the iconic Oppenheimer-Snyder example, that while the matter may collapse to a single central point, the spacetime outside the matter also collapses, but to a long line, not to a point. (In a spinning BH the line singularity is replaced by a long tube with topologically spherical cross section, the causality horizon.) Stability was essential in arguing that something like the *Oppenheimer-Snyder collapse* might occur in the real world rather than being a zero probability fluke among solutions of the Einstein equations.

There is a lot of non-singular spacetime inside the black hole horizon that was produced by the *LIGO GW150914* event about a billion years ago. The observed part of that formation event took only about a fifth of a second. If a few bits of stray matter were to have fallen into that 60 solar mass black hole, each would take about 300 microseconds to fall to the causal horizon of the empty space inside. But a similar bit of matter that fell in on a parallel world line a billion years later would end up a billion light years in spacelike distance away from the first in this interior empty space. Thus the matter that probably helped the BHs to form is so far away (in a spacelike direction) that any attempt to offer it assistance (drop in a magic elixir to fight the squeeze?) will fail to reach that matter in the few milliseconds available before it too reaches a causality barrier. With the enzymatic BH-assisting matter that far out of reach, I regard it as having departed our Universe.

Mathematical Appendix

The Finkelstein Metric can be written

$$ds^2 = \left(dt_* + dr\right)\left[\left(\frac{2M}{r} - 1\right)dt_* + \left(\frac{2M}{r} + 1\right)dr\right] + r^2 d\Omega^2$$

$$t_* = u - r$$

Here $r^2 d\Omega^2 = dx^2 + dy^2 + dz^2 - dr^2$ shows the spherically symmetric metric in rectangular coordinates with $r^2 = x^2 + y^2 + z^2$ so that the smoothness of this metric can be verified for all points *t,x,y,z* with $r > 0$.
To find the more familiar form of this metric introduce, for *r > 2M*, a different time coordinate:

$$t = t_* - 2M \ln\left|\frac{r}{2M} - 1\right|$$

This t and r are the Schwarzschild coordinates that confused physicists for decades because the *t* coordinate is singular at $r = 2M$. Although others had used the better coordinate t^* before Finkelstein, none had recognized the geometry of the light cones at $r = 2M$, which Finkelstein called a "perfect unidirectional membrane" but is now called the Schwarzschild horizon. (Eddington did not even notice that in this form the spacetime is smooth, i.e., non-singular, at $r = 2M$). The two radial null vectors

$$\mathbf{k} = \frac{\partial}{\partial t^*} - \frac{\partial}{\partial r} \quad \text{and} \quad \mathbf{l} = \frac{\partial}{\partial t^*} + \frac{\left(1 - \frac{2M}{r}\right)}{\left(1 + \frac{2M}{r}\right)}\frac{\partial}{\partial r}$$

mark the limits of the light cones in the r, t^* plane. Note that for $r < 2M$ both of these null vectors point in the direction of decreasing r.

In order to better notice the behavior of the light cones when $r < 2M$ one can rename coordinates:

$$r = T,\ t_* = Z$$

$$ds^2 = \left(dZ + dT\right)\left[\left(\frac{2M}{T} - 1\right)dZ + \left(\frac{2M}{T} + 1\right)dT\right] + T^2 d\Omega^2$$

This way of writing the metric encourages us to notice its Z-independence which implies that a translation, Z –>Z + Z_{shift} is a symmetry. With T=const this metric gives a spacelike slice when $T < 2M$:

$$d\ell^2 = \left(\frac{2M}{T} - 1\right)dZ^2 + T^2 d\Omega^2$$

As the "count down time" T moves toward the (matter-free) singularity at T=0 any fixed coordinate part of this spacelike hypersurface approaches zero size in the transverse (angular) directions but expands in the Z-direction. This part of the spacetime is well visualized as in Fig. 7.

Kerr metric

In Kerr coordinates with $t_* = u - r$

the Kerr metric is

$$ds^2 = ds_0{}^2 + \frac{2Mr^3}{r^4 + a^2 z^2}\omega^2$$

where

$$ds_0{}^2 = -dt_*{}^2 + dx^2 + dy^2 + dz^2$$

and

$$\omega = dt_* + \frac{rx + ay}{r^2 + a^2}dx + \frac{ry - ax}{r^2 + a^2}dy + \frac{z}{r}dz$$

The function $r(x, y, z)$ is defined in an unusual way here by the equation

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

with the result that $r = 0$ can only be achieved when $z = 0$ in which case $r^2 = x^2 + y^2 - a^2$ so that $r = 0$ only on a ring. But r is nonzero when $x^2 + y^2 > a^2$ and is undefined when $z = 0$ and $x^2 + y^2 < a^2$.

Computations with the Kerr metric are complicated and are usually assisted by using other coordinates than the t_*, x, y, z above; but the $r(x, y, z)$ defined above is the one for which the two Kerr horizons (Finkelstein horizon and trouble horizon) are given by the simple formulae $r_\pm = M \pm \sqrt{M^2 - a^2}$ used above in the main text.

[from: The Kerr spacetime: A brief introduction   Matt Visser 35]

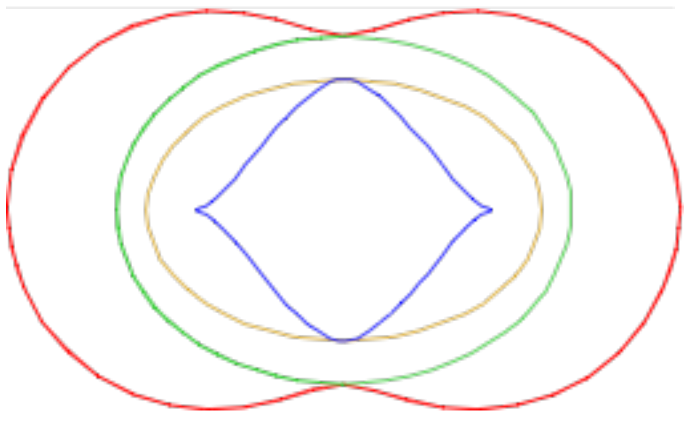


Figure 8: Polar slice through the Kerr spacetime in Cartesian Kerr–Schild coordinates. Location of the horizons, ergosurfaces, and curvature singularity is shown for $a = 0.99\ M$ and $M = 1$. Note that the inner and outer horizons are ellipses in these coordinates, while the inner and outer ergosurfaces are more complicated. The curvature singularity lies at the kink in the inner ergosurface.