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DECONVOLUTION FOR THE CASE OF MULTIPLE CHARACTERISTIC FUNCTIONS OF CUBES IN R^n

by

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Acknowledgments

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Introduction: Deconvolution and Machine Vision

Our interest in deconvolution is in part a consequence of a point of view in machine vision [1] that we have been developing. In this introduction we shall indicate this point of view and we shall also indicate certain constraints to deconvolution that arise in machine vision.

The deconvolution problems that are of interest here are of the type: on $\mathbb{R}^n$, given N distributions of compact support $\mu_1, \mu_2, \ldots, \mu_N$ (called convolutors), determine the existence, support, and construction of N distributions $\nu_1, \nu_2, \ldots, \nu_N$ (called deconvolutors) such that

$$\sum_{i=1}^{N} \mu_i * \nu_i = \delta ,$$

where δ is the Dirac distribution.

For machine vision the interest is in $\mathbb{R}^2$ or $\mathbb{R}^3$. Existence of the deconvolutors depends on the μ_i : e.g., the μ_i cannot all be smooth (C^∞) functions. A condition can be placed on the μ_i, called strong co-primeness, such that the desired ν_i exist and have compact support [2]. The cases for which the μ_i are characteristic functions of a) two intervals on $\mathbb{R}$ and b) two discs on $\mathbb{R}^2$ have been examined by Berenstein, Taylor, et al [3],[4],[5]. For these cases deconvolutors with compact support exist when, for example, the interval lengths or the disc diameters have the ratio $\sqrt{2}$. Explicit formulas for the deconvolutors in cases a) are reported in [6].

Let us consider a role for deconvolution, or signal reconstruction, in machine vision. In machine vision one seeks information about objects by means of one or more images. Let us consider objects that can be modelled as a finite union $\bigcup_j M_j$ of C^1 2-manifolds M_j in $\mathbb{R}^3$. An emitted or reflected radiation can be associated with an object by defining a density F on the sphere bundle of $\mathbb{R}^3$ restricted to $\bigcup_j M_j$, $S\mathbb{R}^3|\bigcup_j M_j$, where the density is with respect to a choice of volume form for $S\mathbb{R}^3|\bigcup_j M_j$. To include the variable time we consider the product space $S\mathbb{R}^3|\bigcup_j M_j \times \mathbb{R}$. Let $\mathcal{M}$ denote a subset of the set of such densities along with their support

$$\mathcal{M} \subset \{ F : S\mathbb{R}^3|\bigcup_j M_j \times \mathbb{R} \longrightarrow \mathbb{R} \}.$$

Let E_2 denote a subset of $\mathbb{R}^2$. This subset will represent what is typically referred to as the 'image plane'. Let $\mathcal{F}$ denote a subset of the set of time varying image intensities

$$\mathcal{F} \subset \{ f : E_2 \times \mathbb{R} \longrightarrow \mathbb{R} \}.$$

A basic problem in machine vision is the definition and construction of a suitable left inverse $\mathfrak{p}$ of an image forming map p,

$$\mathcal{M} \underset{\mathfrak{p}}{\overset{p}{\rightleftarrows}} \mathcal{F} .$$

Additionally, and most importantly, appropriate topologies are sought so that $\mathfrak{p}$ is continuous. For example, if $\mathcal{M}$ is a finite set with the discrete topology then $\mathcal{F}$ should consist of disconnected components, each containing at most one point from $p(\mathcal{M})$, and on each component $\mathfrak{p}$ is constant. If $p(m)$ is in component C then the convolution $\phi * p(m)$ of a given function ϕ with $p(m)$ may not be in C. In this example, the role for deconvolution is to map $\phi * p(m)$ back into C. Since we require only that the deconvolution yield a point in a neighborhood of $p(m)$, we use the term approximate deconvolution

We shall leave further mathematical details on this point of view to a future paper, but we will discuss the motivation. The motivation is that we wish to consider separately the questions of image quality and the questions of machine vision, and then to join these questions through continuity of vision on an appropriate space of images. We separately consider these questions because it seems ill advised to address the issue of vision over some neighborhood of an image (which might include the image plus some additive noise, convolutions of the image, or non-linear sensor degradation of the image) when the issue of vision at the idealized, perfect image remains an open question. With this separation, we consider the idealized, perfect image (e.g., $p(m)$ for $m \in \mathcal{M}$) as a limit point in an appropriate function space $\mathcal{F}$, and we shall require that any well defined vision algorithm $\mathfrak{a}$ be continuous on this space. (An example of a topology for $\mathcal{M}$ is the smallest topology such that p is continuous.)

We now turn to the specific issues in image quality and convolution-deconvolution that are the subject of this paper. For any image $f \in \mathcal{F}$ we never know f: we measure, for example, $\int_Q f$, where Q is a neighborhood of $(0,0) \in E_2 \times \mathbb{R}$, instead of $f(0,0)$. To use our continuous vision algorithm, if we cannot know f then we would like to be sufficiently close to f. Let us consider an example of what we can know about f. The set Q could be $(-\frac{a}{2} , \frac{a}{2}) \times (-\frac{a}{2} , \frac{a}{2}) \times (-T , 0)$. That is, Q models a square detector of side length $a > 0$ centered at $0 \in E_2 \subset \mathbb{R}^2$ and which integrates over the time interval $(-T , 0)$, $T > 0$. Let

$$A = (-\frac{a}{2} , \frac{a}{2}) \times (-\frac{a}{2} , \frac{a}{2}) \times (0 , T) = \{ x : -x \in Q \} \equiv -Q ,$$

and let χ_A be the characteristic function of A. Then

$$\int_Q f = \int f\chi_Q = (\chi_A * f)(0,0).$$

Let us model a *staring* array with a simple integration time response. A set of non-overlapping subsets which covers $E_2 \times \mathbb{R}$ (up to Lebesgue measure zero) is

$$\{ Q_{p,q} = (-\tfrac{a}{2}+p_1a , \tfrac{a}{2}+p_1a) \times (-\tfrac{a}{2}+p_2a , \tfrac{a}{2}+p_2a) \times ((q-1)T , qT) : p = (p_1,p_2) \in \mathbb{Z}^2 , q \in \mathbb{Z} \}$$

Let each $Q_{p,q}$ model a square detector of side length a centered at $pa = (p_1a,p_2a) \in E_2$ which integrates over the time interval $((q-1)T , qT)$. Let $(\aleph_A)_{[(u,s)]}$ be the shift of $\aleph_A$ by (u,s) ,

$$(\aleph_A)_{[(u,s)]}(x,t) = \aleph_A(x-u,t-s) .$$

With this $(\aleph_Q)_{[(pa,qT)]} = \aleph_{Q_{p,q}}$, and

$$\int(\aleph_Q)_{[(pa,qT)]}f = (\aleph_A * f)(pa,qT).$$

For the staring array we do not measure $f \in \mathcal{F}$ but rather

$$\{ (\aleph_A * f)(pa,qT) : p \in \mathbb{Z}^2, q \in \mathbb{Z} , (pa,qT) \in E \}$$

where E is some bounded subset of $E_2 \times \mathbb{R}$.

An answer to the question of what can be said about f based on the measured data is that for f in a suitable choice of normed function space, these measured values can be used to approximate $\aleph_E(\aleph_A * f)$ by the interpolation

$$\aleph_E \sum_{p,q} (\aleph_A * f)(pa,qT)\psi_{p,q} ,$$

where $\psi_{p,q}$ is a choice of interpolating function (e.g., $\psi_{p,q} = \aleph_Q(x-pa,t-qT)$). Moreover, for suitable normed spaces, $\aleph_E(\aleph_A * f)$ approximates $\aleph_E f$. For a choice of A let $\mathcal{I}$ denote the set of all interpolations

$$\mathcal{I} = \{ \aleph_E \sum_{p,q} (\aleph_A * f)(pa,qT)\psi_{p,q} : f \in \mathcal{F} \} .$$

Let $\mathcal{G}^N$ denote the direct product of N such sets, in each of which a different characteristic function is used,

$$\mathcal{G}^N = \mathop{\times}_{i=1}^{N} \{ \chi_E \sum_{p,q} (\chi_{A_i} * f)(pa_i, qT)\Psi_{p,q,i} : f \in \mathcal{F} \} .$$

Thus, what is known about f is a set of approximating interpolations of approximating convolutions.

We summarize the above by the diagram

$$\begin{tikzcd} \mathfrak{M} \arrow[r, shift left, "p"] & \mathcal{F} \arrow[l, shift left, "\wp"] \arrow[d, shift right, "r"'] \\ & \mathcal{G}^N \arrow[u, shift right, "\mathfrak{r}"'] \end{tikzcd} .$$

The relations $\wp$ and p are between objects and images in the sense we have modelled them above. The map r is the composition (interpolation $\circ$ sample $\circ$ convolution) just discussed.

The map $\mathfrak{r}$ is the subject of this paper. For a given choice of norm on $\mathcal{F}$, $\mathfrak{r}$ is a map from $r(f)$, $f \in \mathcal{F}$, to an approximate reconstruction of f. This may be viewed as a numerical implementation of the deconvolution from $\{\chi_{A_i} * f\}_{i=1,\dots,N}$ to f, for the reconstruction is based on a finite set of values from the convolutions. The existence and continuity of the operator which deconvolves $\{\chi_{A_i} * f\}_{i=1,2,\dots N}$ is discussed later. Given this operator, its continuity permits us to discuss approximate deconvolution based on approximations of $\chi_{A_i} * f$ by interpolation.

We now turn to a second item, certain physical constraints on deconvolution. Let A, Q, $A_{p,q}$, E, and $f : E_2 \times \mathbb{R} \longrightarrow \mathbb{R}$ be as above.

It has already been suggested that the set $\{A_{p,q}\}$ is to be a cover of $E_2 \times \mathbb{R}$ by non-overlapping sets. Recall

$$(\aleph_A * f)(pa,qT) = \int \aleph_{Q_{p,q}} f \ .$$

The physical constraint is that $Q_{p,q} \cap Q_{p',q'} = \emptyset$ for $(p,q) \neq (p',q')$. This is because two detectors cannot occupy the same space similtaneously. This constraint can be modified (e.g., using beam splitters) such that the constraint is

$$\sum_{p,q} c(p,q)\aleph_{Q_{p,q}}(x,t) = 1$$

where $c : \mathbb{Z}^2 \times \mathbb{Z} \longrightarrow [0,1]$. For the staring array example $c \equiv 1$. (Here we do not include detector efficiency in our discussion.) This constraint will determine in part the 'observation points', that is, points at which $\aleph_A * f$ can be evaluated. For example, the set

$$\{ \aleph_A * f(p\beta,t_o) : p \in \mathbb{Z}^2 \}$$

is not physically realizable for $\beta < a$.

In addition to constraints on the points at which $\aleph_E(\aleph_A * f)$ is measured, we also have bounds on the measured values. From Hölder's inequality

$$\| \aleph_E(\aleph_A * f) \|_\infty \leq \| \aleph_A \|_p \| f \|_{p'} \quad , \frac{1}{p} + \frac{1}{p'} = 1 \quad , 1 \leq p \leq \infty \ .$$

Let $|A|$ be the three dimensional Lebesgue measure of A, i.e., $|A| = \| \aleph_A \|_1 = a^2 T$. Thus

$$\| \aleph_E(\aleph_A * f) \|_\infty \leq |A|^{1/p} \|f\|_{p'} \ ,$$

and for $p < \infty$, $f \in L_{p'}(\mathbb{R}^n)$,

$$\| \aleph_E(\aleph_A * f) \|_\infty \xrightarrow[|A| \to 0]{} 0 \ .$$

(For $p = \infty$, $f \in L_1(\mathbb{R}^n)$, we get pointwise convergence by the Dominated Convergence Theorem, $[\aleph_E(\aleph_A * f)](x,t) \xrightarrow[|A| \to 0]{} 0$.)

In the case where noise or errors for each measurement do not decrease as $|A|^{1/p}$, and for f with unit $L^{p'}$ norm, we have a lower bound α_0 on $|A|$ imposed as an additional constraint.

For A as above, the simplest pattern of observation points in $\mathbb{R}^2 \times \mathbb{R}$ is the staring array with simple integrator,

$$\{ (pa,qT) : p \in \mathbb{Z}^2 , q \in \mathbb{Z} \} .$$

See Figure 1.

With $|A| = \alpha_0$, we may rescale a and T,

$$A(s) = (-s \frac{a}{2} , s \frac{a}{2}) \times (-s \frac{a}{2} , s \frac{a}{2}) \times (0 , T/_{s}2) , \quad s > 0 ,$$

so that $|A(s)| = |A| = \alpha_0$. See Figure 2a and 2b. In other words, the detector size can be reduced if the integration time is appropriately increased, and visa versa, without altering the upper bound due to Hölder.

A second simple observation scheme is a *continuous scanning* pattern. Let v be a unit vector in $\mathbb{R}^2$ and define

$$B = \{ (x,t) \in \mathbb{R}^2 \times [0,T] : x - vt \in (-\frac{a}{2} , \frac{a}{2}) \times (-\frac{a}{2} , \frac{a}{2}) \subset \mathbb{R}^2 .$$

See Figure 3. Note that $|B| = |A|$.

A third scheme is an alternative to the continuous scan, the *shift scanning* pattern of Figure 4.

In all of these cases, the number of observation points in a fixed set $E \subset E_2 \times \mathbb{R}$ is approximately $|E|/\alpha_0$. We have here the 'mesh size' or sampling interval bounded below due to α_0 .

Let us examine the consequences of f being independent of time. Let $f(x,t) = g(x)$, and let $P(1/n) = (-\frac{a}{2n},\frac{a}{2n}) \times (-\frac{a}{2n},\frac{a}{2n})$, $P = P(1)$. For the rescaling case of the staring array

$$(\chi_{A(1/n)} * f)(x,t) = ((\chi_{P(1/n)} \cdot \chi_{[(0,n^2T)]}) * f)(x,t)$$
$$= \int_{[t-n^2T,t]} \int \chi_{P(1/n)}(x-y)\, g(y)\, dy\, ds$$
$$= n^2 T(\chi_{P(1/n)} * g)(x)$$
$$= a^2 T\left(\frac{\chi_{P(1/n)}}{\|\chi_{P(1/n)}\|_1} * g \right)(x) .$$

The observation points are $(p\frac{a}{n}, qn^2T)$, $p \in \mathbb{Z}^2$, $q \in \mathbb{Z}$, and

$$(\chi_{A(1/n)} * f)(p\tfrac{a}{n}, qn^2T) = a^2 T \left(\frac{\chi_{P(1/n)}}{\|\chi_{P(1/n)}\|_1} * g \right)(p\tfrac{a}{n}) .$$

We conclude

Remark 1 For rescaling of A to $A(1/n)$, and for $g \in L^p(\mathbb{R}^2)$, $1 \le p < \infty$,

a) The underlying convolution which is sampled converges in L^p to g as n increases:

$$\alpha_o \left(\frac{\chi_{P(1/n)}}{\|\chi_{P(1/n)}\|_1} * g \right) \xrightarrow[n \to \infty]{} \alpha_o g \quad \text{in } L^p(\mathbb{R}^2) ;$$

b) the number of observation points on any set in E_2 increases as n^2;

c) the time interval associated with each observation point increases in length as n^2.

For shifted scanning, a choice for an observation point set is

$$\{(p_1,p_2)a + j(\tfrac{1}{n},\tfrac{1}{n^2})a \;,\; jT) : (p_1,p_2) \in \mathbb{Z}^2 \;,\; j \in \mathbb{Z} \} ,$$

while the set A remains the same for any n. See Figure 5. Moreover,

$$\{(\chi_A * f)(pa + ja(\tfrac{1}{n},(\tfrac{1}{n})^2) \;,\; jT) : p \in \mathbb{Z}^2 \;,\; j \in \mathbb{Z} \} =$$
$$\{ T(\chi_P * g)(pa + ja(\tfrac{1}{n},(\tfrac{1}{n})^2) : p \in \mathbb{Z}^2 \;,\; j \in \mathbb{Z} \} =$$
$$\{T(\chi_P * g)(k(0,1)\tfrac{a}{n} + j(1,\tfrac{1}{n})\tfrac{a}{n}) : (j,k) \in \mathbb{Z}^2 \} .$$

We conclude

Remark 2 For shifted scanning according to

$$\{(pa + j(\tfrac{1}{n},(\tfrac{1}{n})^2)a \ , \ jT) : p \in \mathbb{Z}^2 \ , \ j \in \mathbb{Z} \}$$

a) the underlying convolution remains

$$\alpha_o \frac{\aleph_p * g}{\|\aleph_p\|_1} \ ;$$

b) the number of observation points on any set increases as n^2 ;

c) the time required to acquire a full observation set increases as n^2 .

Let us compare rescaling (Remark 1) and scanning (Remark 2). For α_o fixed and for f independent of time, to reduce the mesh size of the observation points projected onto E_2 , we can use smaller detectors and observe over a longer time interval, or we can use (shifted) scanning and a sequence of time intervals. In both cases $|A| = \alpha_o$ and the total observation time to get all observation points is the same.

Therefore,

1. Rescaling and scanning are equivalent in terms of observation time required.

However,

2. Rescaling and scanning differ in that

   a) rescaling uses decreasing detector size to approach the desired function g ,

   b) scanning uses a fixed detector size to approach $\aleph_p * g$.

Our interest is in the scanning case. In particular, we examine the case of

$$\bigtimes_{i=1}^{3} \{ (\chi_{p_i} * g)(x_j) \ : \ j \in \mathbb{Z} \} ,$$

that is, more than one detector of fixed but appropriate sizes and a sequence of observation points whose mesh size can be as fine as required. For such a case, the desired approximate reconstruction of g can be given.

Construction of an approximate deconvolution on $\mathbb{R}^n$

While we shall address in detail the case in which the convolutors are the characteristc functions of cubes in $\mathbb{R}^n$ (e.g., the cubes P_1, P_2, and P_3 in $\mathbb{R}^2$ mentioned just above), we may begin somewhat more generally. Specifically, we shall assume we are given N convolutors μ_i , i=1,2,...,N , and each μ_i is in $L^\infty(\mathbb{R}^n)$ and has compact support. Let f be in $L^1 \cap L^\infty(\mathbb{R}^n)$. We wish to approximately reconstruct f from $r(f) \in \mathcal{S}^N$, where r and $\mathcal{S}^N$ are as in the Introduction. Approximate will mean any of the L^p norms, $1 \le p < \infty$ (and $p = \infty$ with some additional qualifications).

For approximate reconstruction of f it suffices that the reconstruction approximate, for sufficiently large $\tau > 0$, $\varphi_\tau * f$, where $\varphi \in L^1(\mathbb{R}^n)$ and $\varphi_\tau(x) = \tau^n \varphi(\tau x)$ for $x \in \mathbb{R}^n$, since

$$\varphi_\tau * f \xrightarrow[\tau \to \infty]{} f \quad \text{in} \quad L^p(\mathbb{R}^n) \ , \ 1 \le p < \infty \ .$$

In this case we seek N deconvolutors $\nu_1, \nu_2, \ldots, \nu_N$ such that

$$\sum_{i=1}^{N} \mu_i * (\nu_i * \varphi_\tau) * f = \varphi_\tau * f \ .$$

The ingredients for a solution $\{\nu_i * \varphi_\tau\}_{i=1,2,\ldots,N}$ were noted by Berenstein and Taylor [ ]. Let ^ denote the Fourier transform. The Fourier transform of distributions in the equation $\sum_{i=1}^{N} \mu_i * \nu_i = \delta$ results in the Bezout equation $\sum_{i=1}^{N} \hat{\mu}_i \ \hat{\nu}_i = 1$. A necessary condition on $\{\mu_i\}_{i=1,\ldots,N}$ for the existence of a solution is then $\sum_{i=1}^{N} |\hat{\mu}_i|^2(\omega) > 0$ for all $\omega \in \mathbb{R}^n$. For such

μ_i a solution of $\sum_{i=1}^{N} \hat{\mu}_i D_i = 1$ is

$$D_i = \frac{\overline{\hat{\mu}}_i}{\sum_{i=1}^{N} |\hat{\mu}_i|^2} ,$$

where $^{-}$ denotes complex conjugation. However, the D_i are not the solutions $\hat{\nu}_i$ if each ν_i is to be a distribution with compact support, for the D_i are not analytic. On the other hand we have the following. For $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ let

$$\|\omega\|_\infty = \max_{j=1,2,\ldots,n} \{|\omega_j|\} .$$

The growth of D_i as $\|\omega\|_\infty$ gets large is known once a lower bound is established for $\sum_{i=1}^{N} |\hat{\mu}_i|^2(\omega)$. For the μ_i of interest we shall exhibit such a bound as well as a choice of φ_τ such that $D_i\hat{\varphi}_\tau \in L^2(\mathbb{R}^n)$. In this case there exists $h_i \in L^2(\mathbb{R}^n)$ such that $\hat{h}_i = D_i\hat{\varphi}_\tau$ and

$$\sum_{i=1}^{N} h_i * \mu_i * f = \varphi_\tau * f ,$$

this last equation easily seen by taking Fourier transforms. (We have assumed $f \in L^1(\mathbb{R}^n)$ so that $\mu_i * f \in L^1(\mathbb{R}^n)$ and the left hand side is in $L^2(\mathbb{R}^n)$.)

The $\{h_i\}_{i=1,\ldots,N}$ are the desired approximate deconvolutors. However, they do not have compact support. On the other hand, they can be explicitly exhibited using only the knowledge of the Fourier transforms of the convolutors μ_i . Because of this simplicity and potential utility, in the following we conduct an error analysis for a digital implementation of this approximate deconvolution for the special case of convolutors which are characteristic functions of cubes in $\mathbb{R}^n$. In addition the cases described in the introduction provide two further restrictions on the problem and these we adopt.

First, it will suffice to approximately reconstruct f on some compact set E. For example, it suffices to choose τ such that

$$\varepsilon_1 = \| \aleph_E [f - \varphi_\tau * f] \|_p$$

is sufficiently small.

Second, the measurements consist of a discrete set in $\mathbb{R}^n$ on which a set of convolutions is evaluated. Let $\{x_q\}_{q\in Q}$ denote the discrete set of points, with $x_q \in \mathbb{R}^n$ and with Q a finite index set. The convolution values are

$$\left\{(\mu_i * f)(x_q) : q \in Q \ , \quad i=1,2,\ldots,N\right\} \ .$$

We seek to use these values to approximate f on E by constructing an interpolation. In particular we seek functions $\psi_j : \mathbb{R}^n \longrightarrow \mathbb{R}$, $j \in J \subset Q$, and we seek a map $\tilde{G} : \{x_j\}_{j\in J} \longrightarrow \mathbb{R}$ such that $\aleph_E = \aleph_E \sum_{j\in J} \psi_j$ and $\aleph_E f$ is approximated in L^p by $\aleph_E \sum_{j\in J} \tilde{G}(x_j)\psi_j$. That is, we seek to make the error ε,

$$\varepsilon = \| \aleph_E [f - \sum_{j\in J} \tilde{G}(x_j)\psi_j] \|_p \quad ,$$

sufficiently small.

For brevity let F denote $\varphi_\tau * f$. From above we have

$$F \equiv \varphi_\tau * f = \sum_{i=1}^{N} h_i * \mu_i * f \ .$$

Let $\check{}$ denote the inverse Fourier transform, let $\aleph_\lambda$ be the characteristic function of the set $\{\omega\in\mathbb{R}^n : \|\omega\|_\infty \leq \lambda\}$, and let $\mathfrak{B}$ be a compact set in $\mathbb{R}^n$ with characteristic function $\aleph_{\mathfrak{B}}$. Then define

$$G = \sum_{i=1}^{N} ([\aleph_\lambda \hat{h}_i]^{\vee} \aleph_{\mathfrak{B}}) * (\mu_i * f) \ .$$

We shall seek to choose λ and $\mathfrak{B}$ such that

$$\varepsilon_3 = \| \chi_E \sum_{j\in J} [F(x_j) - G(x_j)] \psi_j \|_p$$

is sufficiently small.

The triangle inequality now indicates the additional two terms needed to have a bound for ε. One term is

$$\varepsilon_2 = \| \chi_E [F - \sum_{j\in J} F(x_j)\psi_j] \|_p \quad ,$$

and the second term is

$$\varepsilon_4 = \| \chi_E \sum_{j\in J} [G(x_j) - \tilde{G}(x_j)]\psi_j \|_p \quad .$$

The defining expression for G above suggests the consideration of $\tilde{G}(x_j)$ of the form

$$\tilde{G}(x_j) = \sum_{i=1}^{N} \sum_{q\in Q} \tilde{H}_i(x_j - x_q) [\mu_i * f](x_q) \quad ,$$

where $\tilde{H}_i : \{x_q\}_{q\in Q} \longrightarrow \mathbb{R}$. The $\tilde{H}_i$ then are the deconvolutors which we shall implement. An L^p error bound for this approximation is thus

$$\varepsilon = \| \chi_E [f - \sum_{j\in J} \tilde{G}(x_j)\psi_j] \|_p \leq \varepsilon_1+\varepsilon_2+\varepsilon_3+\varepsilon_4 \ .$$

In the following we develop bounds for each of the four error terms.

The lower bound $C(\omega)$: $\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2 \geq C(\omega) > 0$

For a choice of n+1 positive numbers $a_0, a_1, \ldots, a_n$ let A_i be the cube of side length a_i in $\mathbb{R}^n$. Let χ_{A_i} be the characteristic function of A_i and let $\mu_i = \frac{\chi_{A_i}}{(a_i)^n}$. For this specific case of convolutor we shall prove the following

Theorem. (Berenstein) Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be a choice of n+1 positive integers such that the collection is pairwise relatively prime and none is a perfect square. Let $a_i = \sqrt{\alpha_i}$, and let $M = \max_{i=0,1,\ldots,n} \{a_i\}$. Let μ_i be the characteristic function of the cube in $\mathbb{R}^n$ with side length a_i normalized to unit L^1 norm as above. Then

$$\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2 \geq \left[\frac{8}{5M^4}\right]^{2n} \prod_{j=1}^{n} \frac{1}{\max\{(\frac{\pi}{M})^4, |\omega_j|^4\}}, \quad \omega \in \mathbb{R}^n .$$

The proof will follow from several lemmas. We begin with

Lemma 1. Let p and q both be positive integers. If p and q are relatively prime then $\sqrt{p/q}$ is rational if and only if both p and q are perfect squares.

Proof: Let p and q be relatively prime.

($\Leftarrow$) If $p = p_1^2$ and $q = q_1^2$, then $\sqrt{p/q} = p_1/q_1$ is rational.

($\Rightarrow$) Since $\sqrt{p/q}$ is rational it equals r/s for some pair of integers r and s, which may be chosen to be relatively prime. Hence,

$$qr^2 = ps^2 . \qquad (*)$$

Let $q = q_1^{m_1} \cdots q_\ell^{m_\ell}$, where each q_i is prime. Since p and q are relatively prime and since each q_i is prime, from (*) each q_i divides s. Hence,

$$s^2 = q_1^{2n_1} q_2^{2n_2} \cdots q_\ell^{2n_\ell} (s')^2 \ , \ 2n_i \ge m_i \ \text{ for } i=1,2,\dots,\ell \ .$$

If q is not a perfect square then, reordering factors, m_1 is odd, $2n_1 - m_1 \ge 1$, hence, from (*), q_1 divides r^2 as well as s^2. This contradicts r and s being relatively prime. This along with a similar argument for p shows that both p and q are perfect squares.

□

To proceed it will be necessary to define some maps. Let $\mathcal{A} = \{a_0, a_1, \dots, a_n\}$, $m = \min_{a_i \in \mathcal{A}} \{a_i\}$. Let $x \in \mathbb{R}$. Define the maps

$$d : \mathbb{R} \times \mathcal{A} \longrightarrow \mathbb{Z} \ , \qquad r : \mathbb{R} \times \mathcal{A} \longrightarrow [-\tfrac{\pi}{2m}, \tfrac{\pi}{2m}]$$

by

$$x = d(x,a_i)\,\frac{\pi}{a_i} + r(x,a_i) \ , \qquad \operatorname{sgn}(x)\, r(x,a_i) \in [-\frac{\pi}{2a_i}, \frac{\pi}{2a_i}) \ .$$

For fixed x we have the maps defined by restriction

$$d_x : \mathcal{A} \longrightarrow \mathbb{Z} \ , \qquad r_x : \mathcal{A} \longrightarrow [-\tfrac{\pi}{2m}, \tfrac{\pi}{2m}]$$

$$d_x(a_i) = d(x,a_i) \qquad r_x(a_i) = r(x,a_i) \ .$$

For each fixed $x \in \mathbb{R}$ we also define the subset $\Gamma_x \subset \mathcal{A}$ by

$$\Gamma_x = \left\{ a \in \mathcal{A} : |r_x(a)| = \min_{a_i \in \mathcal{A}} \{ |r_x(a_i)| \} \right\} \ .$$

A choice from Γ_x will be denoted γ_x. (The set Γ_x may consist of more than one element. Any element γ_x may be interpreted as an element of $\mathcal{A}$ for which some integer multiple of π/γ_x is as near x as any element from the set $\{ z\frac{\pi}{a_i} : z \in \mathbb{Z} \ , \ a_i \in \mathcal{A} \}$.)

The following simple observation will be used.

Lemma 2. For every $Q \geq 0$

$$|r_\varkappa(\gamma_\varkappa)| \leq Q$$

or

$$\text{for every } a_i \in \mathcal{A} \quad |r_\varkappa(a_i)| > Q$$

Proof: The definition of $\gamma_\varkappa$.

∎

Lemma 3. For every $a_i, a_j \in \mathcal{A}$, $a_i \neq a_j$, and for every $d \in \mathbb{Z} - \{0\}$

$$\left|\sin\left(\frac{a_i}{a_j} d\pi\right)\right| \geq \frac{4}{\left(4\frac{a_i}{a_j}|d| + 1\right)a_j^{\,2}} .$$

Proof: There exists a nonnegative integer n and $\varepsilon \in [-1/2, 1/2]$ such that

$$\frac{a_i}{a_j}|d|\pi = n\pi + \varepsilon\pi . \qquad (*)$$

From this and the properties of the sine function

$$\left|\sin\left(\frac{a_i}{a_j} d\pi\right)\right| = \left|\sin\left(\frac{a_i}{a_j}|d|\pi\right)\right| = |\sin(\varepsilon\pi)| \geq \frac{2}{\pi}|\varepsilon\pi| . \qquad (**)$$

From $(*)$

$$|\varepsilon\pi| = \left|\frac{a_i}{a_j}|d| - n\right|\pi = \frac{\left|\left(\frac{a_i}{a_j}\right)^2 d^2 - n^2\right| \pi^2}{\left|\frac{a_i}{a_j}\ |d| + n\right| \pi} . \qquad (***)$$

This cannot vanish, for

$$\left(\frac{a_i}{a_j}\right)^2 d^2 - n^2 = 0 \Longleftrightarrow \frac{a_i}{a_j} = \frac{n}{|d|} ,$$

hence $\frac{a_i}{a_j}$ is rational, which by Lemma 1 contradicts the initial assumptions for $\mathcal{A}$. This nonvanishing along with the fact that $a_i^{\,2}$ and $a_j^{\,2}$ are both integers implies $|a_i^2 d^2 - n^2 a_j^2| \geq 1$. This with (***) implies

$$|\varepsilon\pi| \geq \frac{\pi}{a_j^2\left|\frac{a_i}{a_j}|d| + n\right|}.$$

However, from (*) $n \leq \frac{a_i}{a_j}|d| + \frac{1}{2}$, hence

$$\left|\frac{a_i}{a_j}|d| + n\right| \leq \frac{a_i}{a_j}|d| + \frac{a_i}{a_j}|d| + \frac{1}{2},$$

consequently

$$|\varepsilon\pi| \geq \frac{\pi}{a_j^2\left|2\frac{a_i}{a_j}|d| + \frac{1}{2}\right|},$$

which, when substituted in (**), yields the desired inequality.

□

Recall $M = \max_{a\in A} \{a\}$.

Lemma 4. Let $x \in \mathbb{R}$ be fixed. For every $\gamma_x \in \Gamma_x$ if

$$|d_x(\gamma_x)| \geq 1$$

and

$$|r_x(\gamma_x)| \leq \frac{2}{M}\frac{1}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2},$$

then for every $a_i \neq \gamma_x$

$$|\sin(a_i x)| \geq \frac{2}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2} \geq \frac{2}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2}.$$

Proof: Since $x = d_x(\gamma_x)\frac{\pi}{\gamma_x} + r_x(\gamma_x)$, by the Mean Value Theorem there exists $\xi \in \mathbb{R}$ such that

$$\sin(a_i x) = \sin\left[\frac{a_i}{\gamma_x}d_x(\gamma_x)\pi + a_i r_x(\gamma_x)\right] = \sin\left[\frac{a_i}{\gamma_x}d_x(\gamma_x)\pi\right] + a_i r_x(\gamma_x)\cos\xi.$$

Thus

$$|\sin(a_i x)| \ge \left|\sin\left[\frac{a_i}{\gamma_x} d_x(\gamma_x)\pi\right]\right| - |a_i r_x(\gamma_x)|$$

(apply Lemma 3)

$$\ge \frac{4}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2} - a_i|r_x(\gamma_x)|$$

(apply the hypothesis)

$$\ge \frac{4}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2} - \frac{2a_i}{M}\frac{1}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2}$$

$$\ge \frac{2}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2} \ge \frac{2}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2}.$$

□

The next two lemmas will address the case $|x| \ge \pi/2M$. For this case the condition $|d_x(\gamma_x)| \ge 1$ is not vacuous.

Lemma 5. If $|x| \ge \frac{\pi}{2M}$ then there exists $\gamma_x \in \Gamma_x$ such that

$$|d_x(\gamma_x)| \ge 1 \quad .$$

Proof: Case $x = \frac{\pi}{2M}$.

Since $x = \frac{\pi}{M} - \frac{\pi}{2M}$ then $d_x(M) = 1$ and $r_x(M) = -\frac{\pi}{2M}$. Moreover, for all $a_i \ne M$, $x < \frac{\pi}{2a_i}$, hence, $d_x(a_i) = 0$ and $r_x(a_i) = \frac{\pi}{2M}$, therefore $|r_x(M)| \le |r_x(a_i)|$, or $M \in \Gamma_x$.

Case $x = -\frac{\pi}{2M}$.

Use an analogous argument with $d_x(M) = -1$, $r_x(M) = \frac{\pi}{2M}$, and for $a_i \ne M$ $r_x(a_i) = -\frac{\pi}{2M}$.

Case $|x| > \frac{\pi}{2M}$.

Consider any $\gamma_x \in \Gamma_x$. Since $x = d_x(\gamma_x)\frac{\pi}{\gamma_x} + r_x(\gamma_x)$ while $|r_x(\gamma_x)| \le |r_x(M)| \le \frac{\pi}{2M} < |x|$, then $|d_x(\gamma_x)| \ge 1$.

□

Lemma 6. If $|x| \geq \frac{\pi}{2M}$ then there exists $\gamma_x \in A$ such that for every $a_i \in A - \{\gamma_x\}$

$$\frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{5|x|^2 M^4} .$$

Proof: Fix $x \in \mathbb{R}$, $|x| \geq \frac{\pi}{2M}$. Lemma 5 above provides a $\gamma_x \in \Gamma_x$ such that

$$x = d_x(\gamma_x)\frac{\pi}{\gamma_x} + r_x(\gamma_x) , \qquad |d_x(\gamma_x)| \geq 1 .$$

Since $|r_x(\gamma_x)| \leq \frac{\pi}{2\gamma_x}$, $|x| \geq \left[|d_x(\gamma_x)| - \frac{1}{2}\right]\frac{\pi}{\gamma_x}$. Therefore,

$$\frac{1}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]} \geq \frac{1}{|x|}\frac{\left[|d_x(\gamma_x)| - \frac{1}{2}\right]\frac{\pi}{\gamma_x}}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]} .$$

The right hand expression is increasing in $|d_x(\gamma_x)|$ for $|d_x(\gamma_x)| \geq 1$, so we may use $|d_x(\gamma_x)| = 1$ to get a lower bound:

$$\frac{1}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]} \geq \frac{1}{|x|}\frac{\frac{1}{2}\frac{\pi}{\gamma_x}}{4\frac{a_i}{\gamma_x} + 1} \geq \frac{1}{|x|}\frac{\pi/2}{4M + \gamma_x} \geq \frac{1}{|x|}\frac{\pi/2}{5M} .$$

By the first proposition it suffices to consider two cases.

Case 1: $|r_x(\gamma_x)| \leq \frac{2}{M}\frac{1}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2}$.

In this case, if $a_i \neq \gamma_x$ then by Lemma 4

$$|\sin(a_i x)| \geq \frac{2}{\left[4\frac{a_i}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^2} ,$$

hence,

$$|\sin(a_i x)| \geq \frac{2}{\gamma_x^2}\frac{1}{|x|}\frac{\pi/2}{5M} \geq \frac{\pi}{|x|5M^3} \geq \frac{2}{|x|5M^3} ,$$

and

$$\frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{x^2 5 a_i M^3} \geq \frac{2}{x^2 5 M^4} .$$

Case 2: For every $a_i \in \mathcal{A}$ $\quad |r_x(a_i)| > \frac{2}{M} \frac{1}{\left[4\frac{M}{\gamma_x}|d_x(\gamma_x)| + 1\right]\gamma_x^{\,2}}$.

Here, $x = d_x(a_i)\frac{\pi}{a_i} + r_x(a_i)$, $|r_x(a_i)| \leq \frac{\pi}{2a_i}$, hence

$$|\sin(a_i x)| = |\sin(a_i r_x(a_i))| \geq \frac{2}{\pi}\, |r_x(a_i)|\, a_i$$

$$\geq \frac{2a_i}{\pi} \frac{2}{M} \frac{1}{\left[4\frac{M}{\gamma}|d(\gamma)| + 1\right]\gamma^2} \geq \frac{2a_i}{\pi} \frac{2}{M} \frac{1}{|x|} \frac{\pi/2}{5\ M} \frac{1}{\gamma^2} \geq \frac{2a_i}{5|x|M^4} ,$$

and

$$\frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{5|x|^2 M^4} .$$

$\square$

Remark: In Case 2 $a_i x$ is never an integer multiple of π . In Case 1 $a_i \neq \gamma_x$ is used.

The case $|x| \leq \frac{\pi}{2M}$ is addressed next. As usual $\frac{\sin(x)}{x}$ is defined to be 1 for $x = 0$.

Lemma 7. If $|x| \leq \frac{\pi}{2M}$, then $\frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{\pi}$ for every $a_i \in \mathcal{A}$.

Proof: $|x| \leq \frac{\pi}{2M} \implies$ for every $a_i \in \mathcal{A}$, $|x| \leq \frac{\pi}{2a_i}$

$$\implies \frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{\pi} .$$

$\square$

These last two results can be combined:

Lemma 8. For every $x \in \mathbb{R}$, there exists $\gamma_x \in \mathcal{A}$ such that for every $a_i \in \mathcal{A} - \{\gamma_x\}$

$$\frac{|\sin(a_i x)|}{|a_i x|} \geq \frac{2}{5M^4} \frac{1}{\left[\max\left\{\frac{\pi}{2M}, |x|\right\}\right]^2} .$$

Proof: Since $M > 1$,

$$\frac{2}{\pi} \geq \frac{2}{\pi} \frac{4}{5\pi M^2} = \frac{2}{5M^4} \frac{1}{\left[\frac{\pi}{2M}\right]^2} .$$

With this we consider separately $|x| \leq \frac{\pi}{2M}$ and $|x| \geq \frac{\pi}{2M}$ and apply the preceding lemmas.

□

We can now conclude the

Proof of the Theorem: Let $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in \mathbb{R}^n$. It is readily checked that

$$\hat{\mu}_i(\omega) = \prod_{j=1}^{n} \frac{\sin(a_i\omega_j/2)}{a_i\omega_j/2} .$$

For simplicity define $S(a_i,\omega_j) = \dfrac{\sin(a_i\omega_j/2)}{a_i\omega_j/2}$.

Let $\omega \in \mathbb{R}^n$ be fixed. For each coordinate ω_j of ω let γ_{ω_j} be an element of $\mathcal{A}$ provided by Lemma 8. Since there are at most n distinct γ_{ω_j} .and since there are $n+1$ distinct elements in $\mathcal{A}$, there exists an element $a(\omega) \in \mathcal{A}$ such that $a(\omega) \neq \gamma_{\omega_j}$, $j=1,2,\ldots,n$. To complete the proof let $\mu_{a_i} = \mu_i$ hence

$$\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2 = \sum_{i=0}^{n} |\hat{\mu}_{a_i}(\omega)|^2 \geq |\hat{\mu}_{a(\omega)}(\omega)|^2 = \prod_{j=1}^{n} |S(a(\omega),\omega_j)|^2$$

(apply Lemma 8)

$$\geq \left[\frac{8}{5M^4}\right]^{2n} \prod_{j=1}^{n} \frac{1}{\max\left\{\frac{\pi}{M}, |\omega_j|\right\}^4} \quad .$$

□

Piecewise Polynomial Approximate Identities

Our choice for φ in $\varphi_\tau * f$ is a piecewise polynomial, for φ can have

i. compact support,

ii. nonnegative values everywhere,

iii. an analytic representation in digital simulations,

iv. a predetermined number of continuous derivatives,

v. a tractable Fourier transform.

Let R be the characteristic function of the unit cube in $\mathbb{R}^n$ centered at the origin. We use following notation: for any function $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, for $a \in \mathbb{R}^n$, for $s > 0$, and for $x \in \mathbb{R}^n$,

$$g_s(x) = \frac{1}{s^n}\, g(\tfrac{x}{s}) \quad ,$$

$$g_{[a]}(x) = g(x-a) \quad .$$

Our choice for φ is denoted $\varphi_{\langle k\rangle}$, $k \in \mathbb{N}$,

$$\varphi_{\langle k\rangle} = (\overbrace{R * R * \cdots * R}^{k+1 \text{ times}})_{1/(k+1)} \quad .$$

It has the following readily checked properties:

i. The support of $\varphi_{\langle k\rangle}$ is the centered unit cube in $\mathbb{R}^n$;

the support of $\varphi_{\langle k\rangle s}$ is the centered cube of side length s.

$\| \varphi_{\langle k\rangle s} \|_1 = 1$.

ii. $\varphi_{\langle k\rangle}(x) \geq 0$, $x \in \mathbb{R}^n$.

iii. $\varphi_{\langle k\rangle}$ is a piecewise polynomial of degree k .

iv. $\varphi_{\langle k\rangle}$ has $k-1$ continuous derivatives.

v. $\hat{\varphi}_{\langle k\rangle s}(\omega) = \left[\hat{R}\left[\frac{s}{k+1}\,\omega\right] \right]^{k+1}$, $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$,

$$\hat{R}(\omega) = \prod_{j=1}^{n} \frac{\sin(\omega_j/2)}{\omega_j/2} \quad \text{(for } t=0 \ \ \frac{\sin(t)}{t} = 1 \text{ by definition).}$$

As usual we have for the first error term ε_1 , for $f \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$,

$$\varepsilon_1 = \| \aleph_E [f - \varphi_{\langle k \rangle s} * f] \|_p \xrightarrow[s \to 0]{} 0 \quad , \ 1 \le p < \infty \quad ,$$

and for $p = \infty$, for x a point of continuity of f ,

$$| f(x) - (\varphi_{\langle k \rangle s} * f)(x) | \xrightarrow[s \to 0]{} 0 \quad .$$

For any f of interest we can choose a suitable s , but the convergence is not uniform (e.g., f a square wave on $D \subset \mathbb{R}^1$, $D = \mathrm{supp}(\varphi_{\langle k \rangle s} * \aleph_E)$, with unit amplitude and period L , then for any fixed s , $p \ne \infty$, ε_1 approaches $\left[\|\aleph_E\|_1 \right]^{1/p}$ as L approaches 0). Consequently, we have no more to say about any upper bound for ε_1 .

We note, however, that for a fixed choice of k , the set

$$\{ \varphi_{\langle k \rangle s} * f \ : \ s > 0 \}$$

is a one parameter subset of $L^p(\mathbb{R}^n)$, and each $\varphi_{\langle k \rangle}$ is a piecewise polynomial with compact support. These properties make it practical to evaluate by simulation the appropriate size of s for the vision task at hand. Such a choice for s determines ε_1 which in turn suggests an upper bound for ε_2 , ε_3 , and ε_4 . For $\varepsilon_2 + \varepsilon_3 + \varepsilon_4$ defines the radius of a neighborhood about $\varphi_{\langle k \rangle s} * f$. The approach will be to make this radius as small as desired for a fixed s , hence for a fixed ε_1 . A conservative guide would be that the radius should be small compared to ε_1 . Beyond these remarks any additional significance for the size of the neighborhood depends on additional problem structure such as that discussed in the Introduction. Our interest hereafter is solely how to achieve an error bound radius of a predetermined size.

Interpolation in $L^p(E)$

With the choice of $\varphi_{\langle k\rangle s}$ above we turn to the error ε_2 . We shall use frequently the facts that for g and h functions on $\mathbb{R}^n$ such that $g*h$ is defined, for $s > 0$ and for $a \in \mathbb{R}^n$,

$$(g*h)_s = g_s*h_s \quad ,$$

$$(g*h)_{[a]} = g*h_{[a]} \quad ,$$

$$\|g_s\|_1 = \|g\|_1 \quad \text{for } g \in L^1(\mathbb{R}^n) \quad \text{, and}$$

$$\|g*h\|_p \le \|g\|_q\|h\|_r \quad \text{for } \frac{1}{p} = \frac{1}{q} + \frac{1}{r} - 1 \ , \quad 1 \le p,q,r \le \infty \ , \quad 0 \equiv \frac{1}{\infty}$$

(Young's inequality) .

We shall also need

Lemma 1. For $k \ge 1$, for $y = (y_1,y_2,\ldots,y_n)$, and for $1 \le p \le \infty$

$$\| (\varphi_{\langle k\rangle s})_{[y]} - \varphi_{\langle k\rangle s} \|_p$$
$$\le \left[\frac{k+1}{s}\right]^{n(1-\frac{1}{p})} \min\left\{\frac{2(k+1)}{s}\sum_{i=1}^{n}|y_i| \ , \ \left[\frac{2(k+1)}{s}\sum_{i=1}^{n}|y_i|\right]^{1/p}, \ 2^{1/p}\right\}$$

with the convention $\frac{1}{\infty} = 0$.

Proof: Define

$$\mathfrak{R} = (\overbrace{R * R * \cdots * R}^{k-1 \text{ times}})_{s/(k+1)} \quad .$$

To establish the first term in the minimum use

$$\| (\varphi_{\langle k\rangle s})_{[y]} - \varphi_{\langle k\rangle s} \|_p = \| \mathfrak{R} * R_{s/(k+1)} * \left[(R_{s/(k+1)})_{[y]} - R_{s/(k+1)}\right] \|_p$$
$$\le \| \mathfrak{R} \|_1 \ \| R_{s/(k+1)}\|_p \ \| (R_{s/(k+1)})_{[y]} - R_{s/(k+1)} \|_1 \ ,$$

with the obvious modifications if $k=1$.

We have, with $\frac{1}{\infty} = 0$,

$$\| \mathfrak{R} \|_1 = 1 \qquad \text{and} \qquad \| R_{s/(k+1)} \|_p = \left[\frac{k+1}{s}\right]^{n-\frac{n}{p}} \quad ,$$

while $\| (R_{s/(k+1)})_{[y]} - R_{s/(k+1)} \|_1$ is $\left[\frac{k+1}{s}\right]^n$ times the Lebesgue measure of

$$S = \left[[y_1-\frac{s/2}{k+1}\,,\ y_1+\frac{s/2}{k+1}] \times \cdots \times [y_n-\frac{s/2}{k+1}\,,\ y_n+\frac{s/2}{k+1}]\right] \triangle \left[[-\frac{s/2}{k+1}\,,\ \frac{s/2}{k+1}] \times \cdots \times [-\frac{s/2}{k+1}\,,\ \frac{s/2}{k+1}]\right] ,$$

where for sets A and B , $A \triangle B = (A - B) \cup (B - A)$. Let

$$I = [-\frac{s/2}{k+1}\,,\ \frac{s/2}{k+1}] \quad , \qquad I_{y_i} = [y_i-\frac{s/2}{k+1}\,,\ y_i+\frac{s/2}{k+1}] \quad .$$

Then

$$S \subset \left[\bigcup_{i=1}^{n} I_{y_1} \times\cdots\times I_{y_{i-1}} \times (I_{y_i} - I) \times I_{y_{i+1}} \times\cdots\times I_{y_n}\right] \cup \left[\bigcup_{i=1}^{n} I \times\cdots\times I \times (I - I_{y_i}) \times I \times\cdots\times I\right] ,$$

so that, with $\|S\|$ denoting the measure of S ,

$$\|S\| \le 2\sum_{i=1}^{n} |y_i| \left[\frac{s}{k+1}\right]^{n-1} \quad .$$

Hence,

$$\| \, (R_{s/(k+1)})_{[y]} - R_{s/(k+1)} \, \|_1 \le 2\,\frac{k+1}{s}\sum_{i=1}^{n} |y_i| \quad ,$$

which completes the proof of the first term in the minimum.

To establish the second and third terms in the minimum use

$$\| \, (\varphi_{\langle k\rangle s})_{[y]} - \varphi_{\langle k\rangle s} \, \|_p \le \| \, \Re \, \|_1 \, \| \, R_{s/(k+1)} \|_1 \, \| \, (R_{s/(k+1)})_{[y]} - R_{s/(k+1)} \, \|_p \ .$$

For the case $p < \infty$ the second term follows from

$$\| \, (R_{s/(k+1)})_{[y]} - R_{s/(k+1)} \, \|_p \le \left[\frac{k+1}{s}\right]^n \|S\|^{1/p} \ ,$$

while the third term follows from $\|S\| \le 2\left[\frac{s}{k+1}\right]^n$.

For the case $p = \infty$ it suffices for both the second and third terms to note that $\varphi_{\langle k\rangle s}$ is nonnegative, hence

$$\|(\varphi_{\langle k\rangle s})_{[y]} - \varphi_{\langle k\rangle s}\|_\infty \le \|\varphi_{\langle k\rangle s}\|_\infty \le \|\Re\|_1 \|R_{s/(k+1)}\|_1 \|R_{s/(k+1)}\|_\infty \le \left[\frac{k+1}{s}\right]^n \ .$$

$\square$

The following lemma indicates that we have many choices for an interpolating function.

Lemma 2. Let $g \in L^1(\mathbb{R}^n)$, $g \geq 0$, and $\operatorname{supp} g \subset B(0,r)$, the ball of radius r in $\mathbb{R}^n$. Let $\tau > 0$, $N \in \mathbb{N}$, and $\bar{N} = (2N+1)^n$. Let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $\mathbb{R}^n$, and let $\{x_j\}_{j=1}^{\bar{N}}$ denote the set of points in $\mathbb{R}^n$

$$\left\{ \sum_{i=1}^{n} p_i \tau e_i \;:\; p_i \in \mathbb{Z} \;,\; |p_i| \leq N \right\} \;.$$

Let $R_{\{\tau\}}(x) = R(\frac{1}{\tau} x)$, $x \in \mathbb{R}^n$. Define $\Psi = g * R_{\{\tau\}}$ and $\Psi_j = \Psi_{[x_j]}$. For $E \subset \mathbb{R}^n$ such that $\operatorname{supp}(\chi_E * \chi_{B(0,r)}) \subset \overbrace{[-N\tau, N\tau] \times \cdots \times [-N\tau, N\tau]}^{n \text{ times}} = D$, then

$$\|g\|_1 \, \chi_E = \chi_E \sum_{j=1}^{\bar{N}} \Psi_j \quad .$$

Proof: From the definitions

$$\chi_E(x) \sum_{j=1}^{\bar{N}} \Psi_j(x) = \chi_E(x) \int_{B(x,r)} g(x-t) \sum_{j=1}^{\bar{N}} \left[R_{\{\tau\}} \right]_{[x_j]}(t) \, dt \quad .$$

It suffices to check that for $x \in E$ then $B(x,r) \subset D$, and

$$\sum_{j=1}^{\bar{N}} \left[R_{\{\tau\}} \right]_{[x_j]} = 1 \quad \text{on} \quad D \quad .$$

□

Since $R_{\{\tau\}} = \tau^n R_\tau$ by definition,

Corollary. For $\operatorname{supp}(\chi_E * \chi_{B(0,\sqrt{n}\,\ell\tau)}) \subset D$, for

$$\Psi = \tau^n (\overbrace{R_\tau * \cdots * R_\tau}^{\ell \text{ times}}) \;,$$

then $\sum_{j=1}^{\bar{N}} \Psi_j = 1$ on E.

With this we can now establish an upper bound for ε_2 . We choose $\psi = \tau^n(\overbrace{R_\tau * \cdots * R_\tau}^{\ell \text{ times}})$. Let $\{x_j\}$ and ψ_j be as in Lemma 2, and let N be sufficiently large to satisfy the condition in Lemma 2.

Theorem. Let $\varphi = \varphi_{\langle k \rangle s}$, $1 \le q, q' \le \infty$, $\frac{1}{q} + \frac{1}{q'} = 1$ $(\frac{1}{\infty} \equiv 0)$, $f \in L^q(\mathbb{R}^n)$, and let $h = \ell\tau$. For $\varepsilon_2 = \| \aleph_E [\varphi * f - \sum_{j=1}^{\overline{N}} (\varphi * f)(x_j) \psi_j] \|_p$, $1 \le p \le \infty$,

$$\varepsilon_2 \le \|f\|_q \left[\frac{k+1}{s}\right]^{n/q} \|\aleph_E\|_p \min\left\{\frac{k+1}{s} nh , \left[\frac{k+1}{s} nh\right]^{1/q'} , 2^{1/q'}\right\} .$$

Proof: We have

$$\left|\aleph_E(x) \sum_{j=1}^{\overline{N}} \int [\varphi(x-t) - \varphi(x_j - t)] f(t) \, dt \, \psi_j(x)\right|$$

$$\le \aleph_E(x) \sum_{j=1}^{\overline{N}} \|\varphi_{[x-x_j]} - \varphi\|_{q'} \|f\|_q \psi_j(x)$$

$$\le \|f\|_q \left[\frac{k+1}{s}\right]^{n(1-\frac{1}{q'})} \min\left\{\frac{k+1}{s} nh , \left[\frac{k+1}{s} nh\right]^{1/q'} , 2^{1/q'}\right\} \aleph_E(x) \sum_{j=1}^{\overline{N}} \psi_j(x) ,$$

where the last inequality follows from Lemma 1 and from

$$\operatorname{supp}\psi_j \subset \{\|x - x_j\|_\infty \le \tfrac{h}{2}\} .$$

□

We conclude this section with some remarks. First, we have required that $f \in L^\infty(\mathbb{R}^n)$ because only for $q = \infty$ does ε_2 depend on s and h according to h/s . This is the simplest case for applications. As we shall see, we will obtain $\|\aleph_E\|_p$ as a factor in the bounds for ε_3 and ε_4 as well.

A further remark is that for h/s sufficiently small the minimum has the value of $\frac{k+1}{s} nh$.

A final observation is that the smallest bound is obtained for the choice of $\psi = R_{\{\tau\}}$, that is, $\ell = 1$.

Approximate reconstruction of $\varphi * f$ from $\{\varphi * \mu_i * f\}_{i=0}^{n}$

In this section we shall determine an explicit upper bound for the third error ε_3 . We use the notation and definitions of the Construction section and we use the specific convolutors $\{\mu_i\}_{i=0}^{n}$ of the Lower Bound section. This bound requires more work than any of the others. The first task is to determine the values of k in $\varphi_{\langle k\rangle s}$ for which $\hat{h}_i = (\varphi_{\langle k\rangle s})^{\wedge} D_i$ is in $L^2(\mathbb{R}^n)$. To use effectively the lower bound $C(\omega)$ we shall need the following lemmas.

Lemma 1. For a , b , p , q , and x all nonnegative real numbers, for $p - q \geq 0$, and for $b \neq 0$

$$\frac{(\max\{a,x\})^p}{(\max\{b,x\})^q} \leq \max\left\{\frac{a^p}{b^q} , x^{p-q}\right\} .$$

Proof: It suffices to show that the left hand side of the inequality is bounded by some member of the set on the right hand side for each of the cases: $x \leq a, b$; $a \leq x \leq b$; $b \leq x \leq a$; $a, b \leq x$.

$\Box$

Lemma 2. For a , b , p , q , and x all nonnegative real numbers, for $b \geq a$, $p - q \leq 0$, and for $x \neq 0$, $b \neq 0$

$$\frac{(\max\{a,x\})^p}{(\max\{b,x\})^q} \leq \min\left\{\frac{(\max\{a,x\})^p}{b^q} , x^{p-q}\right\} \leq \min\{b^{p-q} , x^{p-q}\} .$$

Proof: For the first inequality it suffices to show that the left hand side of the inequality is bounded by each term in the set on the right hand side for each of the cases: $x \leq a$, b ; $a \leq x \leq b$; $b \leq x$.

For the second inequality, since $b \geq a$, it suffices to check the cases $x \geq b$ and $x \leq b$.

$\Box$

We can now prove

Proposition 1 For $(k-2)p > 1$, $\hat{h}_i = (\varphi_{\langle k\rangle s})^{\wedge} D_i \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$. In particular $k = 3$ is sufficient for $\hat{h}_i \in L^2(\mathbb{R}^n)$.

Proof: It is straightforward that for $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$

$$(\varphi_{\langle k\rangle s})^{\wedge}(\omega) = \prod_{j=1}^{n} \left[\frac{\sin\left[\frac{\omega_j s}{2(k+1)}\right]}{\frac{\omega_j s}{2(k+1)}}\right]^{k+1} .$$

Since $|\sin(x)| \le \min\{|x|, 1\}$,

$$\begin{aligned} |(\varphi_{\langle k\rangle s})^{\wedge}(\omega)| &\le \prod_{j=1}^{n} \min\left\{1 \,,\, \frac{2(k+1)}{|\omega_j| s}\right\}^{k+1} \\ &= \prod_{j=1}^{n} \frac{1}{\max\left\{1 \,,\, \frac{|\omega_j| s}{2(k+1)}\right\}^{k+1}} \\ &= \left[\frac{2(k+1)}{s}\right]^{n(k+1)} \prod_{j=1}^{n} \frac{1}{\max\left\{\frac{2(k+1)}{s} \,,\, |\omega_j|\right\}^{k+1}} . \end{aligned}$$

From the definitions, $\mu_i = \varphi_{\langle \emptyset\rangle a_i}$. With this and the theorem from the Lower Bounds section

$$|\hat{h}_i(\omega)| = \left| (\varphi_{\langle k\rangle s})^{\wedge}(\omega) \, \frac{\overline{\hat{\mu}}_i(\omega)}{\sum_{\ell=\emptyset}^{n} |\hat{\mu}_\ell(\omega)|^2} \right|$$

$$\le \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2(k+1)}{s}\right]^{n(k+1)} \left[\frac{2}{a_i}\right]^{n} \prod_{j=1}^{n} \frac{\max\left\{\frac{\pi}{M} \,,\, |\omega_j|\right\}^4}{\max\left\{\frac{2}{a_i} \,,\, |\omega_j|\right\} \max\left\{\frac{2(k+1)}{s} \,,\, |\omega_j|\right\}^{k+1}} .$$

Note that a_i may be replaced by $m = \min_{a_i \in A} \{a_i\}$. With this and Lemma 1

$$|\hat{h}_i(\omega)| \le \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^{n} \left[\frac{2(k+1)}{s}\right]^{n(k+1)} \prod_{j=1}^{n} \frac{\max\left\{\frac{m}{2}\left[\frac{\pi}{M}\right]^4 \,,\, |\omega_j|^3\right\}}{\max\left\{\frac{2(k+1)}{s} \,,\, |\omega_j|\right\}^{k+1}} .$$

Note that $\frac{2(k+1)}{s}$ may be replaced by $K = \max\left\{ \frac{2(k+1)}{s} , \left[\frac{m}{2}\left[\frac{\pi}{M}\right]^4 \right]^{\frac{1}{3}} \right\}$. Then by Lemma 2

$$|\hat{h}_i(\omega)| \le \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n K^{n(k+1)} \prod_{j=1}^{n} \min\left\{ \frac{\max\left\{\frac{m}{2}\left[\frac{\pi}{M}\right]^4 , |\omega_j|^3\right\}}{K^{k+1}} , |\omega_j|^{2-k} \right\} .$$

(The case $\frac{2(k+1)}{s} \ge \left[\frac{m}{2}\left[\frac{\pi}{M}\right]^4 \right]^{\frac{1}{3}}$ is typically the case of interest here.) Hence, $\hat{h}_i \in L^p(\mathbb{R}^n)$, $1 \le p < \infty$, whenever $(k-2)p > 1$.

$\square$

The next result is a well known tool. Our notation for some standard items is: ∂U for the boundary of the set U ; i for the imaginary element in $\mathbb{C}$; $\omega \cdot x$ for the usual scalar product of ω and x in $\mathbb{R}^n$; $d|\omega|$ for the standard volume form for $\mathbb{R}^n$ represented by $d\omega_1 \wedge d\omega_2 \wedge \cdots \wedge d\omega_n$ in the coordinates $(\omega_1,\ldots,\omega_n)$, where $\wedge$ is the wedge product of differential forms; and $\cdots \wedge \widehat{d\omega_i} \wedge \cdots$ for the deletion of the factor $d\omega_i$ in a wedge product.

Lemma 3. Let $g : \mathbb{R}^n \longrightarrow \mathbb{R}$, $g \in C^k(\mathbb{R}^n)$ (i.e., g has k continuous derivatives). Let $\partial_j g$ denote the partial derivative $\frac{\partial g}{\partial \omega_j}$. Let U be an open set in $\mathbb{R}^n$ with compact closure $\bar{U}$. Let $\bar{U}$ have a triangulation consisting of differentiable singular n-simplexes in $\mathbb{R}^n$. Then

$$\int_U (\partial_{j_1}\partial_{j_2}\cdots\partial_{j_k} g)\, e^{i\omega\cdot x}\, d|\omega|$$

$$= \sum_{\ell=2}^{k-1}\left[\prod_{r=1}^{\ell-1}(-ix_{j_r})\right](-1)^{j_k+1}\int_{\partial U}(\partial_{j_{\ell+1}}\cdots\partial_{j_k} g)\, e^{i\omega\cdot x}\, d\omega_1\wedge\cdots\wedge\widehat{d\omega}_{j_\ell}\wedge\cdots\wedge d\omega_n$$

$$+ (-1)^{j_1+1}\int_{\partial U}(\partial_{j_2}\cdots\partial_{j_k} g)\, e^{i\omega\cdot x}\, d\omega_1\wedge\cdots\wedge\widehat{d\omega}_{j_1}\wedge\cdots\wedge d\omega_n$$

$$+ \left[\prod_{r=1}^{k-1}(-ix_{j_r})\right](-1)^{j_k+1}\int_{\partial U} g\, e^{i\omega\cdot x}\, d\omega_1\wedge\cdots\wedge\widehat{d\omega}_{j_k}\wedge\cdots\wedge d\omega_n$$

$$+ \left[\prod_{r=1}^{k}(-ix_{j_r})\right]\int_U g\, e^{i\omega\cdot x}\, d|\omega| \quad .$$

Proof: Stokes theorem and induction on k.

∎

Corollary Let $\partial_{j_1}\partial_{j_2}\cdots\partial_{j_\ell} g = 0$ on ∂U for $0 \le \ell \le k-2$ and for any indices $j_1, j_2, \ldots, j_\ell$. Then, for $\|x\|$ any norm of $x \in \mathbb{R}^n$,

$$\int_U g\, e^{i\omega\cdot x}\, d|\omega| = O\left[\|x\|^{-k}\right] \quad \text{as} \quad \|x\| \longrightarrow \infty \quad .$$

Proof: From the lemma

$$\int_U (\partial_{j_1}\partial_{j_2}\cdots\partial_{j_k} g)e^{i\omega\cdot x}d|\omega| - (-1)^{j_1+1}\int_{\partial U}(\partial_{j_2}\partial_{j_3}\cdots\partial_{j_k} g)e^{i\omega\cdot x}d\omega_1\wedge\cdots\wedge\widehat{d\omega}_{j_1}\wedge\cdots\wedge d\omega_n$$

$$= \left[\prod_{r=1}^{k}(-ix_{j_r})\right]\int_U g\, e^{i\omega\cdot x}\, d|\omega| \ .$$

Consequently, by letting $j_1 = 1, 2, \ldots, n$ and taking the sum,

$$\int_U \sum_{t=1}^{n}(x_t\partial_t\partial_{j_2}\cdots\partial_{j_k} g)e^{i\omega\cdot x}d|\omega| - \int_{\partial U}(\partial_{j_2}\partial_{j_3}\cdots\partial_{j_k} g)e^{i\omega\cdot x}(x \lrcorner d|\omega|)$$

$$= \left[\sum_{t=1}^{n} -ix_t^2\right]\left[\prod_{r=2}^{k}(-ix_{j_r})\right]\int_U g\, e^{i\omega\cdot x}\, d|\omega| \quad ,$$

where $x \lrcorner d|\omega| = \sum_{j=1}^{n} x_j\, d\omega_1\wedge\cdots\wedge\widehat{d\omega}_j\wedge\cdots\wedge d\omega_n$.

Repeat this for the remaining indices $j_2, j_3, \ldots, j_k$ and normalize by dividing by $|x|^k$, $|x|$ the Euclidean norm of x. Let $\partial_{x/|x|} = \sum_{j=1}^{n} \frac{x_j}{|x|} \partial_j$.

Thus

$$\int_U (\partial_{x/|x|}^{k} g)\, e^{i\omega\cdot x}\, d|\omega| - \int_{\partial U} (\partial_{x/|x|}^{k-1} g)\, e^{i\omega\cdot x} \frac{x}{|x|} d|\omega| = (-i)^k |x|^k \int_U g\, e^{i\omega\cdot x} d|\omega| \quad .$$

$\square$

We now begin the comparison of $\sum_j (\varphi * f)(x_j)\psi_j$ with an approximation $\sum_j G(x_j)\psi_j$. We consider first $(\varphi * f)(0)$. Some required notation is:

Let $\mathcal{F} = \operatorname{supp} \varphi_{\langle k\rangle s}$, $\mathcal{M} \supset \bigcup_{i=0}^{n} \operatorname{supp} \mu_i$, let $\chi_{\mathcal{F}}$ and $\chi_{\mathcal{M}}$ denote the characteristic functions of $\mathcal{F}$ and $\mathcal{M}$ respectively, and define

$$\mathcal{F} + \mathcal{M} = \operatorname{supp} \chi_{\mathcal{F}} * \chi_{\mathcal{M}} \quad ,$$

$$\mathcal{F} + p\mathcal{M} = \operatorname{supp} \chi_{\mathcal{F}} * (\overbrace{\chi_{\mathcal{M}} * \cdots * \chi_{\mathcal{M}}}^{p \text{ times}}) \quad .$$

We have $-\mathcal{F} = \mathcal{F}$ and we shall require $-\mathcal{M} = \mathcal{M}$, where $-\{x \in \mathcal{M}\} = \{x : -x \in \mathcal{M}\}$. As usual, we abbreviate $\varphi_{\langle k\rangle s}$ by φ. Recall that we have the relation

$$\sum_{i=0}^{n} h_i * \mu_i * f = \varphi * f \quad .$$

Lemma 4.

$$(\varphi * f)(0) = \left[\sum_{i=0}^{n} h_i * [(\mu_i * f)\chi_{\mathcal{F}+\mathcal{M}}]\right](0)$$

$$+ \left[\sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\chi_{\mathcal{F}+2\mathcal{M}}(1-\chi_{\mathcal{F}})]\right]\chi_{\mathcal{F}+3\mathcal{M}}(1-\chi_{\mathcal{F}+\mathcal{M}})\right]\right](0) \quad .$$

Proof:

$$\sum_{i=0}^{n} h_i * [(\mu_i * f)\aleph_{\mathfrak{f}+\mathfrak{m}}] = \sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(\aleph_{\mathfrak{f}}+1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+\mathfrak{m}}\right]$$

$$= \sum_{i=0}^{n} h_i * \left[\mu_i * [f\aleph_{\mathfrak{f}}]\right] + \sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+\mathfrak{m}}\right] .$$

The first term in this sum evaluated at $\emptyset$ is

$$\left[\varphi * [f\aleph_{\mathfrak{f}}]\right](\emptyset) = (\varphi * f)(\emptyset) .$$

The second term evaluated at $\emptyset$, after adding and subtracting

$$\left[\sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]\right](\emptyset) ,$$

is $$\left[\sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+3\mathfrak{m}}\right]\right](\emptyset)$$

$$- \left[\sum_{i=0}^{n} h_i * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]\right](\emptyset) ,$$

and the first term in this sum is

$$\left[\sum_{i=0}^{n} h_i * \left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\right](\emptyset) = \left[\varphi * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right](\emptyset) = \emptyset .$$

□

Now we further decompose this expression for $(\varphi * f)(\emptyset)$,

$$(\varphi * f)(\emptyset) = \left[\sum_{i=0}^{n} (\hat{h}_i\aleph_\lambda)^\vee * [(\mu_i * f)\aleph_{\mathfrak{f}+\mathfrak{m}}]\right](\emptyset)$$

$$+ \left[\sum_{i=0}^{n} (\hat{h}_i\aleph_\lambda)^\vee * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]\right](\emptyset)$$

$$+ \left[\sum_{i=0}^{n} [\hat{h}_i(1-\aleph_\lambda)]^\vee * [(\mu_i * f)\aleph_{\mathfrak{f}+\mathfrak{m}}]\right](\emptyset)$$

$$+ \left[\sum_{i=0}^{n} [\hat{h}_i(1-\aleph_\lambda)]^\vee * \left[\left[\mu_i * [f\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right]\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]\right](\emptyset) .$$

This same procedure can be carried out at x_j :

Corollary.

$$(\varphi * f)(x_j) = \left[\sum_{i=\emptyset}^{n} (\hat{h}_i \aleph_\lambda)^\vee * [(\mu_i * f)(\aleph_{\mathfrak{f}+\mathfrak{m}})_{[x_j]}]\right](x_j)$$

$$+ \left[\sum_{i=\emptyset}^{n} (\hat{h}_i \aleph_\lambda)^\vee * \left[\left[\mu_i * [f(\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}}))_{[x_j]}]\right]\left[\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]_{[x_j]}\right]\right](x_j)$$

$$+ \left[\sum_{i=\emptyset}^{n} [\hat{h}_i (1-\aleph_1)]^\vee * [(\mu_i * f)(\aleph_{\mathfrak{f}+\mathfrak{m}})_{[x_j]}]\right](x_j)$$

$$+ \left[\sum_{i=\emptyset}^{n} [\hat{h}_i (1-\aleph_\lambda)]^\vee * \left[\left[\mu_i * [f(\aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}}))_{[x_j]}]\right]\left[\aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}})\right]_{[x_j]}\right]\right](x_j) \quad .$$

Proof: Apply $(a*b)(x_j) = (a*b_{[-x_j]})(\emptyset)$, e.g.

$$(\varphi * f)(x_j) = (\varphi * f_{[-x_j]})(\emptyset) \quad \text{and}$$

$$(a*\{f_{[-x_j]}b\})(\emptyset) = (a*\{fb_{[x_j]}\})(x_j) \quad .$$

□

The convolutions above can be replaced by the scalar product. Define

$$\text{for} \quad a,\, b : \mathbb{R}^n \longrightarrow \mathbb{R} \quad , \quad \langle a,b \rangle = \int_{\mathbb{R}^n} ab \quad .$$

Whenever $b(t) = b(-t)$, $(a*b)(x_j) = \langle a, b_{[x_j]} \rangle = \langle a_{[-x_j]}, b \rangle$. Therefore

Corollary.

$$(\varphi * f)(x_j) = \sum_{i=\emptyset}^{n} \left\langle \left[(\hat{h}_i \aleph_\lambda)^\vee\right]_{[x_j]} , (\mu_i * f)(\aleph_{\mathfrak{f}+\mathfrak{m}})_{[x_j]} \right\rangle$$

$$+ \sum_{i=\emptyset}^{n} \left\langle (\hat{h}_i \aleph_\lambda)^\vee , \left[\mu_i * [f_{[-x_j]} \aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right] \aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}}) \right\rangle$$

$$+ \sum_{i=\emptyset}^{n} \left\langle [\hat{h}_i (1-\aleph_\lambda)]^\vee , (\mu_i * f_{[-x_j]}) \aleph_{\mathfrak{f}+\mathfrak{m}} + \left[\mu_i * [f_{[-x_j]} \aleph_{\mathfrak{f}+2\mathfrak{m}}(1-\aleph_{\mathfrak{f}})]\right] \aleph_{\mathfrak{f}+3\mathfrak{m}}(1-\aleph_{\mathfrak{f}+\mathfrak{m}}) \right\rangle .$$

Let $\eta_1(x_j)$ and $\eta_2(x_j)$ be the second and third terms respectively of the right hand side above. Then

$$\begin{aligned}\varepsilon_3 = \Big\|\aleph_E \Big[\sum_{j\in J}(\varphi * f)(x_j)\psi_j \; - \; \sum_{j\in J}\sum_{i=0}^{n}\langle[(\hat{h}_i\aleph_\lambda)^{\vee}]_{[x_j]}\,,\,(\aleph_{\mathcal{J}+\mathbb{M}})_{[x_j]}[\mu_i * f]\rangle\psi_j\Big]\Big\|_p \\ \le \|\aleph_E \sum_{j\in J}\eta_1(x_j)\psi_j\|_p + \|\aleph_E \sum_{j\in J}\eta_2(x_j)\psi_j\|_p \\ \le \max\{|\eta_1(x_j)|\}\|\aleph_E\|_p + \max\{|\eta_2(x_j)|\}\|\aleph_E\|_p\end{aligned}$$

We turn then to determining bounds for $\eta_1(x_j)$ and $\eta_2(x_j)$. We address $\eta_2(x_j)$ first, this case being easier.

First, $\eta_2(x_j)$ is the sum of two terms, each of the form

$$\sum_{i=0}^{n}\langle\; [\hat{h}_i(1-\aleph_\lambda)]^{\vee}\;,\;\aleph_S[\mu_i*(f_{[-x_j]}\aleph_T)]\;\rangle\;,\text{ which is bounded by}$$

$$\begin{aligned}&\sum_{i=0}^{n}\|[\hat{h}_i(1-\aleph_\lambda)]^{\vee}\|_2\;\|\aleph_S[\mu_i*(f_{[-x_j]}\aleph_T)]\|_2 \\ &\le \sum_{i=0}^{n}(2\pi)^{-n/2}\;\|\hat{h}_i(1-\aleph_\lambda)\|_2\;\|\mu_i*(f_{[x_j]}\aleph_T)\|_2 \\ &\le \sum_{i=0}^{n}(2\pi)^{-n/2}\;\|\hat{h}_i(1-\aleph_\lambda)\|_2\;\|\mu_i\|_1\;\|f\|_2\quad .\end{aligned}$$

It is easy to see that this also bounds $\eta_2(x_j)$. Recall that $\|\mu_i\|_1 = 1$.

We use the proof of Proposition 1 to bound $\|\hat{h}_i(1-\aleph_\lambda)\|_p$, $p>1$. If $s \le M$ and $k \ge 3$ then $\frac{2(k+1)}{s} \ge \left[\frac{m}{2}\left[\frac{\pi}{M}\right]^4\right]^{\frac{1}{3}}$, and this is the case we shall assume. Let

$$a = \left[\frac{m}{2}\left[\frac{\pi}{M}\right]^4\right]^{\frac{1}{3}}\quad,\quad b = \frac{2(k+1)}{s}$$

so that by the proof of Proposition 1, with C a constant,

$$|h_i(\omega)| \le C\prod_{j=1}^{n}\min\left\{\frac{\max\left\{a^3\;,\;|\omega_j|^3\right\}}{b^{k+1}}\;,\;|\omega_j|^{2-k}\right\}\quad .$$

By definition, $1-\aleph_\lambda$ is the characteristic function of the set $\{\|\omega\|_\infty > \lambda\}$, where $\|\omega\|_\infty = \max\{|\omega_i|, i=1,2,\dots,n\}$. Note that the zero set of $(\varphi_{\langle k\rangle s})^\wedge$ contains $\{\|\omega\|_\infty = \ell\pi \frac{2(k+1)}{s}, \ell \in \mathbb{N}\}$. Consequently, because of Lemma 3 and its corollary, we shall later choose $\lambda = \ell\pi \frac{2(k+1)}{s} = \ell\pi b$, and this is convenient here also. Finally observe that

$$\{\|\omega\|_\infty > \lambda\} = \bigcup_{i=1}^{n} \{\|\omega\|_\infty > \lambda\} \cap \{|\omega_i| = \|\omega\|_\infty\} .$$

We outline the integration.

$$C^{-p}\|\hat{h}_i(1-\aleph_\lambda)\|_p^p \le \sum_{i=1}^{n} \int_{\{|\omega_i|=\|\omega\|_\infty\}\cap\{\|\omega\|_\infty>\lambda\}} \prod_{j=1}^{n} \min\{\cdots\}^p \le n \int_{\{|\omega_n|=\|\omega\|_\infty\}\cap\{\|\omega\|_\infty>\lambda\}} \prod_{j=1}^{n} \min\{\cdots\}^p$$

$$= n2^n \int_\lambda^\infty y^{p(2-k)} \left[\int_0^a \left[\frac{a^3}{b^{k+1}}\right]^p dx + \int_a^b \left[\frac{x^3}{b^{k+1}}\right]^p dx + \int_b^y x^{p(2-k)} dx \right]^{n-1}$$

(use $a \le b$ and the inequality $|v^n - (v+u)^n| \le n|u|\left[|v|+|u|\right]^{n-1}$)

$$\le n2^n \left[b^{1+p(2-k)}\right]^n \left[1 + \frac{1+(\ell\pi)^{1+p(2-k)}}{p(k-2)-1}\right]^{n-1} \frac{(\ell\pi)^{1+p(2-k)}}{p(k-2)-1} .$$

With this and with $p = 2$,

$$\max_j\{|\eta_2(x_j)|\}$$

$$\le (n+1)\|f\|_2 \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n \left[\frac{2(k+1)}{s}\right]^{\frac{7n}{2}} \left[\frac{n}{\pi^n}\right]^{\frac{1}{2}} \left[1+\frac{1+(\ell\pi)^{1+2(2-k)}}{2(k-2)-1}\right]^{\frac{n-1}{2}} \left[\frac{(\ell\pi)^{1+2(2-k)}}{2(k-2)-1}\right]^{\frac{1}{2}} .$$

We now turn to $\eta_1(x_j)$. Whereas we used λ to control the size of $\eta_2(x_j)$, we shall depend on $\mathfrak{M}$ to control the size of

$$\eta_1(x_j) = \sum_{i=0}^{n} \langle (\hat{h}_i\aleph_\lambda)^\vee \aleph_{\mathcal{Y}+3\mathfrak{M}}(1-\aleph_{\mathcal{Y}+\mathfrak{M}}) , \left[\mu_i * [f_{[-x_j]}\aleph_{\mathcal{Y}+2\mathfrak{M}}(1-\aleph_{\mathcal{Y}})] \right] \rangle .$$

How this is done is indicated by Lemma 5 below. First some notation. As previously noted, we use $\aleph_\lambda$ to denote the characteristic function of $\{\|\omega\|_\infty \le \lambda\}$ and we choose $\lambda = \ell\pi\frac{2(k+1)}{s}$, where ℓ is some positive integer.

For simplicity, let $\Lambda = \{\|\omega\|_\infty \le \lambda\}$. Then

$$\Lambda = \bigcap_{j=1}^{n} \{ |\omega_j| \le \lambda \} ,$$

$$\partial\Lambda \subset \bigcup_{j=1}^{n} \{ |\omega_j| = \lambda \} .$$

A second item of notation is the multi-index $\alpha = (\alpha_1,\alpha_2,\ldots,\alpha_n) \in \mathbb{N}^n$. Define $|\alpha| = \sum_{j=1}^{n} \alpha_j$, $\alpha! = \alpha_1!\alpha_2!\cdots\alpha_n!$, and $\partial^\alpha = \partial_1^{\alpha_1}\partial_2^{\alpha_2}\cdots\partial_n^{\alpha_n}$.

Lemma 5. For $|\alpha| \le k$, $\partial^\alpha \hat{h}_i(\omega) = 0$ for $\omega \in \partial\Lambda$. Consequently, for all $r \le k+1$, $x \in \mathbb{R}^n$, with $|x|$ the Euclidean norm of x ,

$$|(\hat{h}_i \aleph_\lambda)^\vee(x)| \le \frac{1}{(2\pi)^n} \frac{1}{|x|^r} \| \partial_{x/|x|}^{\,r} \hat{h}_i \|_{1,\Lambda} .$$

Proof: The first statement follows from property v of $\varphi_{\langle k\rangle s}$ in the Piecewise Polynomial section , the product formulas for derivatives, and the definition of h_i . The second statement follows from the proof of the Corollary to Lemma 3.

□

Several lemmas will be required to bound $\|\partial_v^{\,r} \hat{h}_i(\omega)\|_{1,\Lambda}$, where $v \in \mathbb{R}^n$, $|v| = 1$. Since $\hat{h}_i(\omega) = (\varphi_{\langle k\rangle s})^\wedge(\omega)\, D_i(\omega)$, it suffices to bound $\partial^\alpha\left[(\varphi_{\langle k\rangle s})^\wedge\right](\omega)$ and $\partial_v^{\,r} D_i(\omega)$, to apply Leibnitz's rule, and finally to integrate. Recall Leibnitz's rule for $\mathbb{C}$ valued functions f and g on $\mathbb{R}^n$, with α , β , γ multi-indices:

$$\partial^\alpha(fg) = \sum_{\substack{\beta,\gamma \\ \beta+\gamma=\alpha}} \frac{\alpha!}{\beta!\gamma!} (\partial^\beta f)(\partial^\gamma g) .$$

(In the particular cases examined here all Fourier transforms have values only on the real axis.)

It will be convenient to have a bound for $\partial^\alpha\left[(\varphi_{\langle k\rangle s})^\wedge\right](\omega)$ which depends only on $|\alpha|$.

Lemma 6.

$$|\partial^{\alpha}\left[(\varphi_{\langle k\rangle s})^{\wedge}\right](\omega)| \le \left[\frac{s}{2(k+1)}\right]^{|\alpha|} \frac{(k+|\alpha|)!}{k!} 2^{|\alpha|} \prod_{j=1}^{n} \min\left\{ 1 , \frac{1}{\frac{|\omega_j| s}{2(k+1)}} \right\}^{k+1} .$$

($\min\{1 , 1/x\}$ is understood to be 1 for $x = 0$.)

Proof: Let $\varphi'_{\langle k\rangle s}$ denote $\varphi_{\langle k\rangle s}$ constructed for $\mathbb{R}^1$. From the definitions, for $\omega = (\omega_1,\omega_2,\ldots,\omega_n) \in \mathbb{R}^n$,

$$(\varphi_{\langle k\rangle s})^{\wedge}(\omega) = \prod_{j=1}^{n} (\varphi'_{\langle k\rangle s})^{\wedge}(\omega_j) .$$

Claim : For $\nu \in \mathbb{R}$, for $r \in \mathbb{N}$,

$$|\partial^{r}\left[(\varphi'_{\langle k\rangle s})^{\wedge}\right](\nu)| \le \left[\frac{s}{2}\right]^{r} \min\left\{ 1 , \frac{(k+r)!}{k!\ (k+1)^{r}} \left[1 + \frac{2(k+1)}{|\nu|s}\right]^{r} \left[\frac{2(k+1)}{|\nu|s}\right]^{k+1} \right\} .$$

Proof of Claim: The first element in the minimized set is established by

$$|\partial^{r}\left[(\varphi'_{\langle k\rangle s})^{\wedge}\right](\nu)| \le \int_{[-\frac{s}{2},\frac{s}{2}]} |\varphi'_{\langle k\rangle s}(t)|\,|t|^{r}dt \le \left[\frac{s}{2}\right]^{r} \|\varphi'_{\langle k\rangle s}\|_{1} .$$

The second element is established by induction on k . Recall

$$(\varphi'_{\langle k\rangle s})^{\wedge}(\nu) = \left[\mathrm{sinc}\left[\frac{\nu s}{2(k+1)}\right]\right]^{k+1} , \quad \mathrm{sinc}(\nu) = \frac{\sin(\nu)}{\nu} .$$

By Leibnitz's rule

$$|\partial^{r}\left[(\varphi'_{\langle 0\rangle s})^{\wedge}\right](\nu)| \le \sum_{\ell=0}^{r} \left[\begin{matrix} r \\ \ell \end{matrix}\right] \left[\frac{s}{2}\right]^{r-\ell} \frac{\ell!}{|\nu|^{\ell+1}} \frac{1}{s/2} \le r! \left[\frac{s}{2}\right]^{r} \left[1 + \frac{2}{|\nu|s}\right]^{r} \left[\frac{2}{|\nu|s}\right] ,$$

hence the result for $k = 0$. Recall the notation $g_{\{s\}}(\nu) = g(\nu/s)$ for $s > 0$. Consequently $(\partial^{r} g_{\{s\}})(\nu) = (1/s)^{r}(\partial^{r}g)(\nu/s)$. With this notation

$$(\varphi'_{\langle k\rangle s})^{\wedge} = \left[(\varphi'_{\langle k-1\rangle s})^{\wedge}\right]_{\{\frac{k+1}{k}\}} \cdot \left[(\varphi'_{\langle 0\rangle s})^{\wedge}\right]_{\{k+1\}} .$$

From Leibnitz's rule, from the result for $k = 0$, and from the induction hypothesis

$$|\partial^{r}\left[(\varphi'_{\langle k\rangle s})^{\wedge}\right](\nu)| \le \left[\frac{s}{2}\right]^{r} \frac{1}{(k+1)^{r}} \left[1 + \frac{2(k+1)}{|\nu|s}\right]^{r} \left[\frac{2(k+1)}{|\nu|s}\right]^{k+1} \sum_{\ell=0}^{r} \left[\begin{matrix} r \\ \ell \end{matrix}\right] \frac{(\ell+k-1)!}{(k-1)!} (r-\ell)! ,$$

and the final sum is checked by induction to be $\frac{(k+r)!}{k!}$.

□ claim

Observe that $\frac{k!\,(k+1)^r}{(k+r)!} \le 1$ and that for $y > 0$

$$\min\{1\ ,\ (1+y)^r y^{k+1}\} \le \min\{1\ ,\ 2^r y^{k+1}\} \le 2^r \min\{1\ ,\ y^{k+1}\}\ .$$

With these observations

$$|\partial^r\left[(\varphi'_{\langle k\rangle s})^\wedge\right](\nu)| \le \left[\frac{s}{2(k+1)}\right]^r \frac{(k+r)!}{k!}\, 2^r \prod_{j=1}^{n} \min\left\{1\ ,\ \frac{1}{\frac{|\nu|s}{2(k+1)}}\right\}^{k+1}\ .$$

Since $$\partial^\alpha\left[(\varphi_{\langle k\rangle s})^\wedge\right](\omega) = \prod_{j=1}^{n} \partial^{\alpha_j}\left[(\varphi'_{\langle k\rangle s})^\wedge\right](\omega_j)\ ,$$

it only remains to check by induction on n that for a multi-index α

$$\prod_{j=1}^{n} \frac{(k+\alpha_j)!}{k!} \le \frac{(k+|\alpha|)!}{k!}\ .$$

□

We next bound derivatives of $D_i = \bar{\hat{\mu}}_i \left[\sum_{i=0}^{n} |\hat{\mu}_i|^2\right]^{-1}$. From Leibnitz's rule and since $\hat{\mu}_i = (\varphi_{\langle 0\rangle a_i})^\wedge$, it suffices to consider derivatives of the second factor. A formula for higher derivatives of compositions of functions will be needed. Let $s(\ell)$ denote a multi-index with ℓ coordinates

$$s(\ell) = (s_1, s_2, \ldots, s_\ell)\ .$$

For a function f of one variable let $f^{(s_i)}$ denote the derivative of order s_i.

Lemma 7. For $f\ ,\ g \in C^r(\mathbb{R})$, $r \ge 1$

$$(f\circ g)^{(r)} = \sum_{\ell=1}^{r} \sum_{\substack{|s(\ell)|=r \\ s_i \ge 1}} \begin{bmatrix} s_1+\cdots+s_\ell-1 \\ s_1-1 \end{bmatrix} \begin{bmatrix} s_2+\cdots+s_\ell-1 \\ s_2-1 \end{bmatrix} \cdots \begin{bmatrix} s_\ell-1 \\ s_\ell-1 \end{bmatrix}$$

$$\times \left[f^{(\ell)}\circ g\right]\left[g^{(s_1)} g^{(s_2)} \cdots g^{(s_\ell)}\right]\ .$$

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Deconvolution

for the case of

Multiple Characteristic Functions of Cubes in $\mathbb{R}^n$

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Abstract

Explicit error bounds are exhibited for a case of deconvolution with elementary convolutors on $\mathbb{R}^n$. The convolutors studied are a set of $n+1$ characteristic functions of cubes (e.g., with side length $\sqrt{j}$, $j=1,2,\ldots,n+1$) which operate by convolution on $L^1 \cap L^2(\mathbb{R}^n)$. For a suitable choice of approximate identity, a set of $n+1$ functions (deconvolutors) in $L^2(\mathbb{R}^n)$ are exhibited which restore $L^1 \cap L^2(\mathbb{R}^n)$, up to convolution with the approximate identity, from the $n+1$ convolutions. For the case of the convolutors operating on $L^1 \cap L^2 \cap L^p(\mathbb{R}^n)$, $1 \leq p < \infty$, explicit bounds for the restoration error in the norm $L^p(E)$, E compact, are exhibited; that is, error bounds for restoration restricted to a compact subset. The motivation for this study is the digital implementation of this deconvolution for the application to signal detectors which act by integrating over cubic regions. This motivation is discussed along with remarks on the significance of the topology for signals that is implied by the notion of restoration or deconvolution.

Proof: The proof is by induction on r . With the convention $0! = 1$ the case $r = 1$ is clear. Assume the result for $r - 1$.

$$(f\circ g)^{(r)} = \left[(f\circ g)^{(1)}\right]^{(r-1)} = \left[[f^{(1)}\circ g]\cdot g^{(1)}\right]^{(r-1)}$$

(apply Leibnitz's rule and the induction hypothesis)

$$= \sum_{s=1}^{r-1} \begin{bmatrix} r-1 \\ s-1 \end{bmatrix} \sum_{\ell=1}^{r-s} \sum_{\substack{|s(\ell)|=r-s \\ s_i\geq 1}} \begin{bmatrix} s_1+\cdots+s_\ell-1 \\ s_1-1 \end{bmatrix} \begin{bmatrix} s_2+\cdots+s_\ell-1 \\ s_2-1 \end{bmatrix} \cdots \begin{bmatrix} s_\ell-1 \\ s_\ell-1 \end{bmatrix}$$

$$\times \left[f^{(\ell+1)}\circ g\right]\left[g^{(s_1)} g^{(s_2)} \cdots g^{(s_\ell)} g^{(s)}\right] + [f^{(1)}\circ g]\cdot g^{(r)}$$

Observe

$$\sum_{s=1}^{r-1} \sum_{\ell=1}^{r-s} \sum_{\substack{|s(\ell)|=r-s \\ s_i\geq 1}} \begin{bmatrix} r-1 \\ s-1 \end{bmatrix}\left[\cdots\right] = \sum_{\ell=1}^{r-1} \sum_{s=1}^{r-\ell} \sum_{\substack{|s(\ell)|=r-s \\ s_i\geq 1}} \begin{bmatrix} r-1 \\ s-1 \end{bmatrix}\left[\cdots\right]$$

$$= \sum_{\ell=1}^{r-1} \sum_{\substack{s_0+s_1+s_2+\cdots+s_\ell=r \\ s_0, s_i\geq 1}} \begin{bmatrix} r-1 \\ s_0-1 \end{bmatrix}\left[\cdots\right] .$$

The term $[f^{(1)}\circ g]\cdot g^{(r)}$ corresponds to an additional $\ell = 0$ term in the last formulation of the summation. By renaming these r values for the index ℓ (add 1) the desired form is obtained.

□

Some miscellaneous results that will be needed are collected in

Lemma 8. Let ∂_v be the directinal derivative in direction $v \in \mathbb{R}^n$.

i. Lemma 7 holds for $f \in C(\mathbb{R})$, $g \in C(\mathbb{R}^n)$ if ∂^r is replaced by ∂_v^r .

ii. $$\sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \begin{bmatrix} s_1+\cdots+s_\ell-1\\ s_1-1\end{bmatrix}\begin{bmatrix} s_2+\cdots+s_\ell-1\\ s_2-1\end{bmatrix}\cdots\begin{bmatrix} s_\ell-1\\ s_\ell-1\end{bmatrix}$$

$$= \sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \begin{bmatrix} r-1\\ s_1-1\end{bmatrix}\begin{bmatrix} r-1-s_1\\ s_2-1\end{bmatrix}\cdots\begin{bmatrix} r-1-s_1-\cdots-s_{\ell-1}\\ s_\ell-1\end{bmatrix}$$

$$\le r^r \ .$$

iii. For $\varphi \in C^k(\mathbb{R}^n)$ and for $r = |\alpha| \le k$, if $M(|\alpha|)$ is a bound for $\partial^\alpha\varphi$ which depends only on $|\alpha|$, then , for $|v|$ the Euclidean norm of v ,

$$|\partial_v{}^r\varphi| \le (\sqrt{n})^r|v|^rM(r) \ .$$

Proof: For *i*, if $f \in C(\mathbb{R}^n)$, x , $v \in \mathbb{R}^n$, and if $p_v : \mathbb{R}^n \longrightarrow \mathbb{R}$, $p_v(t) = x + vt$, then $\left[\partial_v{}^r f\right](x) = \left[\partial^r(f\circ p_v)\right](0)$.

For *ii*, the first relation uses $s_1+s_2+\cdots+s_\ell = r$. The inequality follows from

$$\sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \frac{r}{r}\,\frac{(r-1)!}{(s_1-1)!(r-s_1)!}\,\frac{(r-s_1-1)!}{(s_2-1)!(r-s_1-s_2)!}\,\frac{(r-s_1-s_2-1)!}{(s_3-1)!(r-s_1-s_2-s_3)!}\cdots$$

$$\times\ \frac{(r-s_1-s_2-\cdots-s_{\ell-1}-1)!}{(s_\ell-1)!\ 0!}$$

$$= \sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \frac{r!}{[r(r-s_1)(r-s_1-s_2)\cdots(r-s_1-\cdots s_{\ell-1})][(s_1-1)!(s_2-1)!\cdots(s_\ell-1)!]}$$

$$= \sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \frac{r!}{[r(s_1-1)!][(r-s_1)(s_2-1)!]\cdots[(r-s_1-\cdots-s_{\ell-1})(s_\ell-1)!]}$$

$$\le \sum_{\ell=1}^{r}\sum_{\substack{|s(\ell)|=r\\ s_i\ge 1}} \frac{r!}{s_1!s_2!\cdots s_\ell!} \le \sum_{|\alpha|=r}\frac{r!}{\alpha!}$$, with this last summation in

multi-index notation ($\alpha \in \mathbb{N}^r$), while for $m \in \mathbb{N}$, $a_i \in \mathbb{R}$, $i=1,2,\dots,r$

$$\left[\sum_{i=1}^{r} a_i\right]^m = \sum_{|\alpha|=m} \frac{m!}{\alpha!} a_1^{\alpha_1} a_2^{\alpha_2} \cdots a_r^{\alpha_r} .$$

For *iii*, use

$$\sum_{i=1}^{n} |v_i| \le \sqrt{n}|v| , \quad \partial_v \varphi = \sum_{i=1}^{n} v_i \partial_i \varphi ,$$

hence

$$|\partial_v{}^r \varphi| = |\sum_{t_1=1}^{n} v_{t_1} \partial_{t_1} \cdots \sum_{t_r=1}^{n} v_{t_r} \partial_{t_r} \varphi| \le \sum_{t_1=1}^{n} \cdots \sum_{t_r=1}^{n} |v_{t_1} \cdots v_{t_r}| M(r)$$
$$\le (\sqrt{n}|v|)^r M(r) .$$

$\square$

Now we can complete the bound for $|\partial_v{}^r D_i(\omega)|$.

Lemma 9. For $v \in \mathbb{R}^n$, $|v| = 1$, for $r \ge 1$, and for $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{R}^n$,

$$|\partial_v{}^r D_i(\omega)| \le (M\sqrt{n}(n+1)r)^r ((r+1)!) \frac{\left[\frac{5M^4}{8}\right]^{2n(r+1)}}{\left[\frac{m}{2}\right]^{n(2r+1)}}$$
$$\times \prod_{j=1}^{n} \max\left\{ \left[\frac{n}{M}\right]^{4(r+1)} \left[\frac{m}{2}\right]^{2r+1} , |\omega_j|^{2r+3} \right\} .$$

Proof: From Leibnitz's rule and from Lemma 7 with $f(t) = t^{-1}$,

$g = \sum_{i=0}^{n} |\hat{\mu}_i|^2$,

$$\partial_v{}^r D_i = \sum_{p=0}^{r} \begin{bmatrix} r \\ p \end{bmatrix} \partial_v^{r-p} \bar{\hat{\mu}}_i \sum_{\ell=1}^{p} \sum_{\substack{|s(\ell)|=p \\ s_i \ge 1}} \begin{bmatrix} s_1+s_2+\cdots s_\ell-1 \\ s_1-1 \end{bmatrix} \cdots \begin{bmatrix} s_\ell-1 \\ s_\ell-1 \end{bmatrix} \frac{(-1)^\ell \ell!}{\left[\sum_{i=0}^{n} |\hat{\mu}_i|^2\right]^{\ell+1}}$$
$$\times \prod_{q=1}^{\ell} \sum_{j=0}^{n} \sum_{t_q=0}^{s_q} \begin{bmatrix} s_q \\ t_q \end{bmatrix} \left[\partial_v^{s_q - t_q} \hat{\mu}_j\right] \left[\partial_v^{t_q} \bar{\hat{\mu}}_j\right] .$$

From Lemma 6 with $\mu_i = \varphi_{\langle 0\rangle a_i}$, $m = \min_i\{a_i\}$, $M = \max_i\{a_i\}$, and from Lemma 8,

$$|\partial_v{}^r D_i(\omega)| \le \sum_{p=0}^{r} \begin{bmatrix} r \\ p \end{bmatrix} (\sqrt{n}M)^{r-p}(r-p)! \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}$$

$$\times \sum_{\ell=1}^{p} \sum_{\substack{|s(\ell)|=p \\ s_i\ge 1}} \begin{bmatrix} s_1+s_2+\cdots s_\ell-1 \\ s_1-1 \end{bmatrix} \cdots \begin{bmatrix} s_\ell-1 \\ s_\ell-1 \end{bmatrix} \frac{\ell!}{\left[\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2\right]^{\ell+1}}$$

$$\times \prod_{q=1}^{\ell} \sum_{j=0}^{n} \sum_{t_q=0}^{s_q} \begin{bmatrix} s_q \\ t_q \end{bmatrix} (\sqrt{n}M)^{s_q}(s_q-t_q)!\,t_q! \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^2 .$$

For the last factor,

$$\prod_{q=1}^{\ell} \sum_{j=0}^{n} \sum_{t_q=0}^{s_q} \left[\cdots\right] \le \prod_{q=1}^{\ell} (n+1)(s_q+1)!(\sqrt{n}M)^{s_q} \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^2$$

(using $\prod_{j=1}^{n} (\alpha_j+1)! \le (|\alpha|+1)!$ as in the proof of Lemma 6)

$$\le (n+1)^{\ell}(\sqrt{n}M)^{p} \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^{2\ell} (p+1)! .$$

Combining,

$$|\partial_v{}^r D_i(\omega)| \le r!(\sqrt{n}M)^{r} \frac{\prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}}{\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2} \sum_{p=0}^{r} (p+1)$$

$$\times \sum_{\ell=1}^{p} \sum_{\substack{|s(\ell)|=p \\ s_i\ge 1}} \begin{bmatrix} s_1+s_2+\cdots s_\ell-1 \\ s_1-1 \end{bmatrix} \cdots \begin{bmatrix} s_\ell-1 \\ s_\ell-1 \end{bmatrix} \ell! \left[\frac{(n+1) \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^2}{\sum_{i=0}^{n} |\hat{\mu}_i(\omega)|^2}\right]^{\ell} .$$

Since $|\hat{\mu}_i(\omega)|^2 \le \prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^2$,

$$|\partial_v{}^r D_i(\omega)| \le r!(\sqrt{n}M)^r \sum_{p=\emptyset}^{r} (p+1)!p^p(n+1)^p \frac{\prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^{2p+1}}{\left[\sum_{i=\emptyset}^{n} |\hat{\mu}_i(\omega)|^2\right]^{p+1}}$$

$$\le ((r+1)!)^2(\sqrt{n}Mr)^r(n+1)^r \frac{\prod_{j=1}^{n} \min\{1 , \frac{2}{|\omega_j|m}\}^{2r+1}}{\left[\sum_{i=\emptyset}^{n} |\hat{\mu}_i(\omega)|^2\right]^{r+1}} .$$

To complete the proof apply the Lower Bound Theorem, then apply Lemma 1.

∎

At last we bound $|\partial_v{}^r \hat{h}_i(\omega)|$.

Lemma 10. For $v \in \mathbb{R}^n$, $|v| = 1$, for $r \ge 1$, and for $\omega = (\omega_1, \omega_2, \ldots, \omega_n) \in \mathbb{R}^n$,

$$|\partial_v{}^r \hat{h}_i(\omega)| \le A(r,k,n,\mathcal{A}) \left[\frac{2(k+1)}{s}\right]^{n(k+1)} \frac{\prod_{j=1}^{n} \max\{ C , |\omega_j| \}^{2r+3}}{\prod_{j=1}^{n} \max\left\{\frac{2(k+1)}{s} , |\omega_j| \right\}^{k+1}} ,$$

where $C = \max_{\ell=\emptyset,1,\ldots,r} \left\{ \left[\left[\frac{n}{M}\right]^{4(\ell+1)} \left[\frac{m}{2}\right]^{2\ell+1}\right]^{1/(2\ell+3)} \right\}$, and where

$$A(r,k,n,\mathcal{A}) = ((r+1)!)^2 \begin{bmatrix} k+r \\ r \end{bmatrix} \frac{\left[\frac{5M^4}{8}\right]^{2n}}{\left[\frac{m}{2}\right]^n} \left[\frac{s}{(k+1)C^{2n}} + \sqrt{n}M(n+1)r \frac{\left[\frac{5M^4}{8}\right]^{2n}}{\left[\frac{m}{2}\right]^{2n}} \right]^r ,$$

with $\mathcal{A}$ the set $\{a_\emptyset, a_1, \ldots, a_n\}$ of which M and m are the max and min , respectively.

Proof: Apply Leibnitz's rule, Lemma 9, and Lemma 6, then use the inequalities

$$\max\left\{ \left[\frac{n}{M}\right]^{4(\ell+1)} \left[\frac{m}{2}\right]^{2\ell+1} , |\omega_j|^{2\ell+3} \right\} \le \max\left\{ C , |\omega_j| \right\}^{2\ell+3}$$
$$\le C^{2\ell+3} \max\left\{ 1 , \frac{|\omega_j|}{C} \right\}^{2r+3} , \text{ for } \ell \le r ,$$

and

$$((\ell+1)!)^2 \frac{(k+r-\ell)!}{k!} \le (\ell+1)!(\ell+1) \frac{(k+r)!}{k!} \le ((r+1)!)^2 \binom{k+r}{r} ,$$

and finally apply the binomial formula.

□

We return to our original goal, a bound for $|\eta_1(x_j)|$. Lemma 5 and Lemma 10 narrow the choices for r and k . By Hölder's inequality and by Young's inequality, along with $\|\mu_i\|_1 = 1$,

$$|\eta_1(x_j)| = \left| \sum_{i=0}^{n} \left\langle (\hat{h}_i \aleph_\lambda)^{\vee} \aleph_{\mathcal{F}+3\mathfrak{m}}(1-\aleph_{\mathcal{F}+\mathfrak{m}}) , \left[\mu_i * [f_{[-x_j]}\aleph_{\mathcal{F}+2\mathfrak{m}}(1-\aleph_{\mathcal{F}})] \right] \right\rangle \right|$$
$$\le \sum_{i=0}^{n} \|(\hat{h}_i \aleph_\lambda)^{\vee}(1-\aleph_{\mathfrak{m}})\|_2 \|f\|_2 .$$

We may assume that $\mathfrak{m}$ is a centered cube in $\mathbb{R}^n$ with side length β . Let $B_i(r,k,n,\lambda)$ bound $\|\partial^r_{x/|x|} \hat{h}_i\|_{1,\Lambda}$ uniformly in $x \in \mathbb{R}^n - \{0\}$. Then by Lemma 5,

$$\|(\hat{h}_i \aleph_\lambda)^{\vee}(1-\aleph_{\mathfrak{m}})\|_2 \le \frac{B_i(r,k,n,\lambda)}{(2\pi)^n} \left[\int_{\mathbb{R}^n - \mathfrak{m}} |x|^{-2r} \, d|x| \right]^{1/2} .$$

Consequently, we choose $2r \ge n + 1$. Since $\|x\|_\infty \le |x|$, we integrate over $\{\|x\|_\infty > \beta\}$ in the manner as that outlined in the discussion preceeding Lemma 5:

$$\|(\hat{h}_i \aleph_\lambda)^{\vee}(1-\aleph_{\mathfrak{m}})\|_2 \le \frac{B_i(r,k,n,\lambda)}{(2\pi)^n} \frac{\sqrt{n}}{\sqrt{2r-n}} 2^r \left[\frac{1}{\beta}\right]^{\frac{2r-n}{2}} .$$

From Lemma 10 $B_i(r,k,n,\lambda)$ can be determined.

Lemma 11. For υ , r , ω , $\mathcal{A}$, C , and A as in Lemma 10, for $k-2r-2 \neq 1$, for $\lambda = \ell\pi \frac{2(k+1)}{s}$, $\ell \in \mathbb{N}-\{0\}$, $\Lambda = \{\|\omega\|_\infty \le \lambda\}$,

$$\|\partial_\upsilon{}^r\hat{h}_i\|_{1,\Lambda} \le A(r,k,n,\mathcal{A})\ 2^n\ K_1{}^{2n(r+2)} \left\{\min\{1,\frac{\lambda}{K_1}\}^n + u\left[\left[\frac{N+(\ell\pi)^{N+1}}{N+1}\right]^n - 1\right]\right\}$$

where

$$N = 2r+2-k\ , \qquad u = \begin{cases} 0 & \text{if } \lambda \le K_1 \\ 1 & \text{if } \lambda > K_1 \end{cases}\ , \qquad K_1 = \max\left\{\frac{2(k+1)}{s}\ ,\ C\right\}.$$

Proof: Note that each occurrence of $\frac{2(k+1)}{s}$ in the first inequality of Lemma 10 may be replaced by K_1 . Thus, by Lemma 1 and Lemma 2 combined,

$$|\partial_\upsilon{}^r\hat{h}_i(\omega)| \le A(r,k,n,\mathcal{A})\ K_1{}^{n(k+1)} \prod_{j=1}^{n} \max\{\ K_1\ ,\ |\omega_j|\ \}^{2r+3-k-1}\ .$$

Integrate, treating separately $\|\omega\|_\infty \le K_1$ and $\|\omega\|_\infty > K_1$. For example, for $\lambda \ge K_1$ and using $\mathbb{R}^n = \bigcup_{i=1}^{n} \{|\omega_i| = \|\omega\|_\infty\}$

$$\|\partial_\upsilon{}^r\hat{h}_i\|_{1,\Lambda} = \|\chi_\lambda \partial_\upsilon{}^r\hat{h}_i\|_1 \le A(r,k,n,\mathcal{A})\ K_1{}^{n(k+1)} \left\{ n2^n \int_0^{K_1} K_1{}^{nN}\ y^{n-1} dy \right.$$
$$\left. +\ n2^n \int_{K_1}^{\lambda} y^N \prod_{j=1}^{n-1} \left[\int_0^{K_1} K_1{}^N dy_j + \int_{K_1}^{y} y_j{}^N dy_j \right] dy \right\}$$

$$\le A(r,k,n,\mathcal{A})\ K_1{}^{n(k+1)}\ n2^n\ K_1{}^{n(N+1)} \left\{ \frac{1}{n} + \frac{1}{n}\left[\left[\frac{N+(\lambda/K_1)^{N+1}}{N+1}\right]^n - 1\right]\right\}.$$

Note that, regardless of the sign of N , $\left[\frac{N+(\lambda/K_1)^{N+1}}{N+1}\right] \le \left[\frac{N+(\ell\pi)^{N+1}}{N+1}\right]$.

□

From Lemma 10 $B_i(r,k,n,\lambda)$ can be chosen to be independent of λ . For if $k+1 > 2r+3+1$ then $|\partial_\upsilon{}^r\hat{h}_i|$ is in $L^1(\mathbb{R}^n)$. Explicitly,

Corollary. For $k > 2r+3$, (i.e., $N < -1$)

$$\|\partial_\upsilon{}^r\hat{h}_i\|_{1,\Lambda} \le \|\partial_\upsilon{}^r\hat{h}_i\|_1 \le A(r,k,n,\mathcal{A})\ 2^n\ K_1{}^{2n(r+2)} \left[\frac{(-N)}{(-N)-1}\right]^n .$$

In Lemma 5 it was seen that r could be as large as $k+1$. However, r must be less than half of this value for the Corollary to apply. For example, the Corollary does not apply for $k \le 5$. For $k=9$, r can be no greater than 2. In this case, $\eta_2(x_j)$ decreases as $\lambda^{-13/2}$ while $\eta_1(x_j)$ decreases as β^{-1} for $n = 2$. To have $\eta_1(x_j)$ converge more rapidly we must choose between large values for k, and hence for K_1, and a bound for $\eta_1(x_j)$ which depends on λ.

We can finally state our bound for $|\eta_1(x_j)|$. We conclude this section by collecting the results in

Theorem.

For E a compact subset of $\mathbb{R}^n$, for $\varphi = \varphi_{\langle k\rangle s}$ as defined above, for $f \in L^1 \cap L^2(\mathbb{R}^n)$, for h_i, $\aleph_\lambda$, $\aleph_{\mathcal{F}+\mathcal{M}}$, ψ_j, $\{x_j\}_{j\in J}$, and μ_i as defined above, and for $k \ge 3$,

$$\varepsilon_3 = \Big\|\aleph_E \Big[\sum_{j\in J}(\varphi * f)(x_j)\psi_j - \sum_{j\in J}\sum_{i=\emptyset}^{n}\langle[(\hat{h}_i\aleph_\lambda)^{\vee}]_{[x_j]}, (\aleph_{\mathcal{F}+\mathcal{M}})_{[x_j]}[\mu_i * f]\rangle\psi_j\Big]\Big\|_p$$
$$\le \max_j\{|\eta_1(x_j)|\}\|\aleph_E\|_p + \max_j\{|\eta_2(x_j)|\}\|\aleph_E\|_p .$$

Let $C = \max\limits_{\ell=\emptyset,1,\cdots,r}\left\{ \left[\left[\frac{\pi}{M}\right]^{4(\ell+1)}\left[\frac{m}{2}\right]^{2\ell+1}\right]^{1/(2\ell+3)} \right\}$, and let

$$A(r,k,n,\mathcal{A}) = ((r+1)!)^2\begin{bmatrix}k+r\\ r\end{bmatrix}\frac{\left[\frac{5M^4}{8}\right]^{2n}}{\left[\frac{m}{2}\right]^{n}}\left[\frac{s}{(k+1)C^{2n}} + \sqrt{n}M(n+1)r\,\frac{\left[\frac{5M^4}{8}\right]^{2n}}{\left[\frac{m}{2}\right]^{2n}}\right]^r ,$$

with $\mathcal{A}$ the set $\{a_\emptyset, a_1, \ldots, a_n\}$ of which M and m are the max and min, respectively. Let $K_1 = \max\left\{\frac{2(k+1)}{s}, C\right\}$, $\lambda = \ell n\frac{2(k+1)}{s}$, with $\ell \in \mathbb{N}$.

For $2r \geq n+1$, $2r + 2 - k \neq -1$

$$\max_j\{ |\eta_1(x_j)| \} \leq (n+1)\|f\|_2 \frac{1}{(2\pi)^n} \frac{\sqrt{n}}{\sqrt{2r-n}} 2^{r+n} A(r,k,n,A)\ K_1^{2n(r+2)} \left[\frac{1}{\beta}\right]^{\frac{2r-n}{2}}$$

$$\times \left\{ 1 + u \left[\left[\frac{N + (\ell\pi)^{(N+1)}}{N+1}\right]^n - 1 \right] \right\}$$

and where $N = 2r + 2 - k$, $u = \begin{cases} 0 & \text{if } \lambda \leq K_1 \\ 1 & \text{if } \lambda > K_1 \end{cases}$.

Secondly, for the case $\frac{2(k+1)}{s} \geq \left[\left[\frac{\pi}{M}\right]^4 \frac{m}{2}\right]^{1/3}$,

$$\max_j\{ |\eta_2(x_j)| \}$$

$$\leq (n+1)\|f\|_2 \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n \left[\frac{2(k+1)}{s}\right]^{\frac{7n}{2}} \left[\frac{n}{\pi^n}\right]^{\frac{1}{2}} \left[1 + \frac{1+(\ell\pi)^{1+2(2-k)}}{2(k-2)-1}\right]^{\frac{n-1}{2}} \left[\frac{(\ell\pi)^{1+2(2-k)}}{2(k-2)-1}\right]^{\frac{1}{2}} .$$

A discrete implementation of approximate reconstruction

We have in this section the payoff for all of the preceding analysis: We can exhibit maps defined on discrete spaces which may be used in a digital implementation of the approximate reconstruction. For these maps we develop the final error term ε_4 .

To begin, recall the interpolating function Ψ used in the Construction section and in the Interpolation section: $\Psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ along with a discrete set of points $\mathfrak{J}$ in $\mathbb{R}^n$ with index set J , $\{x_j\}_{j\in J} = \mathfrak{J}$, such that, with $\Psi_j = \Psi_{[x_j]}$, $\chi_E \sum_{j\in J} \Psi_j = \chi_E$, where E is a subset of $\mathbb{R}^n$ with compact closure. To condense the notation of the previous section, let

$$H_i = (\chi_\lambda \hat{h}_i)^{\vee} : \mathbb{R}^n \longrightarrow \mathbb{R} \quad , \quad i=0,1,\ldots,n \quad ,$$

(recall $\hat{h}_i$ is symmetric) and let the set $\mathfrak{J}+\mathfrak{M}$ be denoted by $\mathfrak{B}$. As in the Construction section, let

$$G = \sum_{i=0}^{n} (H_i \chi_{\mathfrak{B}}) * \mu_i * f \quad .$$

In the preceding sections we have developed the manner in which $\sum_{j\in J} G(x_j)\Psi_j$ is an approximate reconstruction of f . The set $\mathfrak{J}$ may be viewed as the 'reconstruction set' in $\mathbb{R}^n$.

A second discrete subset of $\mathbb{R}^n$ is the 'data set' $\mathfrak{Q}$, the set on which the convolutions $\mu_i * f$, $i=0,1,\ldots,n$, are evaluated. As in the Construction section let Q be the index set for $\mathfrak{Q}$, $\mathfrak{Q} = \{x_q\}_{q\in Q}$. We shall require

$$\mathfrak{Q} \supset \mathfrak{J}$$

and

$$\text{for every } x_q \in \mathfrak{Q} \ , \ x_j \in \mathfrak{J} \ , \ q \in Q \ , \ j \in J : \ x_q = x_j \iff q = j \ .$$

With this notation the objective of this section is to exhibit a map

$$\tilde{H}_i : \mathbb{Q} \cap \mathbb{B} \longrightarrow \mathbb{R}$$

such that the discrete convolution

$$\tilde{G} : \mathbb{J} \longrightarrow \mathbb{R}$$

$$\tilde{G}(x_j) = \sum_{i=0}^{n} \sum_{q \in Q} (\tilde{H}_i \aleph_{\mathbb{B}})(x_j - x_q)\ [\mu_i * f](x_q)$$

approximates G in the sense that

$$\varepsilon_4 = \| \aleph_E \sum_{j \in J} \left[G(x_j) - \tilde{G}(x_j) \right] \psi_j \|_p \xrightarrow[|\mathbb{Q}| \longrightarrow 0]{} 0 \quad ,$$

where $|\mathbb{Q}|$ is a suitable measure of the 'mesh' of $\mathbb{Q}$. Here the irregular notation $\tilde{H}_i \aleph_{\mathbb{B}}$ is used in place of $\tilde{H}_i(\aleph_{\mathbb{B}}|_{\mathbb{Q}})$. Also $\tilde{G}$ depends on Q , but this dependence is suppressed in the notation. We have immediately

$$\varepsilon_4 \le \sum_{i=0}^{n} \max_{j \in J} \left\{ \left| ([H_i \aleph_{\mathbb{B}}] * \mu_i * f)(x_j) - \sum_{q \in Q} [\tilde{H}_i \aleph_{\mathbb{B}}](x_j - x_q)\ [\mu_i * f](x_q) \right| \right\} \|\aleph_E\|_p \ .$$

We require that the set Q have associated with it a set $S \subset \mathbb{R}^n$. With the notation $\aleph_{S_q}(x) = \aleph_S(x - x_q)$ for $q \in Q$, $x_q \in \mathbb{Q}$, $x \in \mathbb{R}^n$, $\aleph_S$ the characteristic function of S , the sets $\mathbb{Q}$ and S are to satisfy

i) $\aleph_E * \aleph_{\mathbb{B}} \sum_{q \in Q} \aleph_{S_q} = \aleph_E * \aleph_{\mathbb{B}}$ almost everywhere , and

$\aleph_{\mathbb{B}} = \sum_{x_q \in \mathbb{B} \cap \mathbb{Q}} (\aleph_S)_{[x_q]}$ almost everywhere

ii) for $x_j \in \mathbb{J}$, for x_q , $x_{q'} \in \mathbb{Q}$

$$(\aleph_{S_q} * \aleph_{S_{q'}})(x_j) = \begin{cases} \|\aleph_S\|_1 & \text{if } x_{q'} = x_j - x_q \\ 0 & \text{otherwise} \end{cases}$$

iii) $\aleph_S(-x) = \aleph_S(x)$, $x \in \mathbb{R}^n$.

With these conditions on S and $\mathcal{Q}$ the difference in the expression for ε_4 splits:

$$|([H_i\chi_{\mathcal{B}}]*\mu_i*f)(x_j) - \sum_{q\in Q}[\tilde{H}_i\chi_{\mathcal{B}}](x_j-x_q)\ [\mu_i*f](x_q)| \qquad (*)$$

$$= \Big|\Big[\Big[\sum_{q'\in Q}\Big[H_i\chi_{\mathcal{B}} - \frac{1}{\|\chi_S\|_1}[\tilde{H}_i\chi_{\mathcal{B}}](x_{q'})\Big]\ \chi_{S_{q'}}\Big] *\mu_i*f\Big](x_j)$$

$$+ \Big[\Big[\sum_{q'\in Q}\frac{1}{\|\chi_S\|_1}[\tilde{H}_i\chi_{\mathcal{B}}](x_{q'})\ \chi_{S_{q'}}\Big] * \Big[\sum_{q\in Q}\Big[\mu_i*f - [\mu_i*f](x_q)\Big]\chi_{S_q}\Big]\Big](x_j)\Big| .$$

To define $\tilde{H}_i$ and to bound ε_4 we specify certain remaining choices. In particular, let the index set Q be a finite subset of $\mathbb{Z}^n$ and let $\hat{Q}$ be a second finite subset of $\mathbb{Z}^n$. Choose $\delta > 0$ and $\Delta > 0$ and let

$$\mathcal{Q} = \Big\{x_q = q\delta\ ,\ q\in Q\Big\}\ , \qquad S = \overbrace{[-\tfrac{\delta}{2},\tfrac{\delta}{2}] \times \cdots \times [-\tfrac{\delta}{2},\tfrac{\delta}{2}]}^{n\ \text{times}}\ ,$$

$$\hat{\mathcal{Q}} = \Big\{\nu_t = t\Delta\ ,\ t\in\hat{Q}\Big\}\ , \qquad \hat{S} = \overbrace{[-\tfrac{\Delta}{2},\tfrac{\Delta}{2}] \times \cdots \times [-\tfrac{\Delta}{2},\tfrac{\Delta}{2}]}^{n\ \text{times}}\ ,$$

where Δ and $\hat{Q}$ are chosen such that, with $\chi_{\hat{S}}(\omega - t\Delta) = \chi_{\hat{S}_t}(\omega)$, $\omega\in\mathbb{R}^n$,

$$\sum_{t\in\hat{Q}}\chi_{\hat{S}_t} = \chi_\lambda\ .$$

We now define

$$\frac{1}{\|\chi_S\|_1}(\tilde{H}_i\chi_{\mathcal{B}})(x_q) = \Big[\sum_{t\in\hat{Q}}(\chi_\lambda\hat{h}_i)(t\Delta)\ \chi_{\hat{S}_t}\Big]^{\vee}(x_q)\ \chi_{\mathcal{B}}(x_q)$$

$$= \sum_{t\in\hat{Q}}(\chi_\lambda\hat{h}_i)(t\Delta)\ e^{i\Delta\delta(t\cdot q)}\ (\chi_{\hat{S}})^{\vee}(q\delta)\ \chi_{\mathcal{B}}(q\delta)\ .$$

Theorem. Let μ_i*f , $i=0,1,\ldots,n$, be given on the set $\mathcal{Q}$, where δ and Q are such that condition $i)$ holds for given sets E and $\mathcal{B}$. Let $\tilde{H}_i$, Δ , $\hat{Q}$, and H_i be as above. Let $\mathcal{J}$ be a subset of $\mathcal{Q}$, with J the corredsponding subset of the index set Q , such that as above $\chi_E = \chi_E\sum_{j\in J}\psi_j$

Then

$$\varepsilon_4 = \| \aleph_E \sum_{j\in J} \sum_{i=0}^{n} \Big[\Big[[H_i \aleph_{\mathfrak{B}}] * \mu_i * f \Big](x_j) - \sum_{q\in Q} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_j - x_q)[\mu_i * f](x_q) \Big] \varphi_j \|_p$$

$$\le \|\aleph_E\|_p \, \|f\|_1 \left[\frac{\lambda}{\pi}\right]^n \times$$

$$\Bigg[\max_{\omega\in\Lambda} \left\{ A(1,k,n,\mathcal{A}) \left[\frac{2(k+1)}{s}\right]^{n(k+1)} \frac{\prod_{j=1}^{n} \max\{C \;,\; |\omega_j|\}^5}{\prod_{j=1}^{n} \max\left\{\frac{2(k+1)}{s},\; |\omega_j|\right\}^{k+1}} \right\} \frac{\sqrt{n}\Delta}{2}\left(1+\frac{n\lambda\delta}{2}\right)$$

$$+ \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n K^{3n} \left[\frac{n\lambda\delta}{2} + \|\aleph_{\mathfrak{B}}\|_1 \, m^{-n} \min\left\{\frac{n\delta}{m} \;,\; 1\right\} \right] \Bigg] \quad ,$$

where $\Lambda = \{\|\omega\|_\infty \langle\lambda\}$, $A(1,k,n,\mathcal{A})$ and C are as in Lemma 10 of Approximate reconstruction, and where K is as in the proof of Proposition 1 of Approximate reconstruction.

Proof: In the following let $1 \le r,\ r' \le \infty$, with $\frac{1}{r} + \frac{1}{r'} = 1$ and $\frac{1}{\infty} \equiv 0$. A bound for the first term in the splitting (*) is

$$\Big| \Big[\Big[\sum_{q'\in Q} \Big[H_i \aleph_{\mathfrak{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \Big] \aleph_{S_{q'}} \Big] * \mu_i * f \Big](x_j) \Big|$$

$$\le \Big\| \Big[\sum_{q'\in Q} \Big[H_i \aleph_{\mathfrak{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \Big] \aleph_{S_{q'}} \Big] * \mu_i * f \Big\|_\infty$$

$$\le \Big\| \sum_{q'\in Q} \Big[H_i \aleph_{\mathfrak{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \Big] \aleph_{S_{q'}} \Big\|_r \, \|\mu_i\|_1 \, \|f\|_{r'} \ .$$

A bound for the first factor of this bounding product is

$$\Big\| \sum_{q'\in Q} \Big[H_i \aleph_{\mathfrak{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \Big] \aleph_{S_{q'}} \Big\|_r$$

$$\le \max_{q'\in Q} \left\{ \Big\| \Big[H_i \aleph_{\mathfrak{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \Big] \aleph_{S_{q'}} \Big\|_\infty \right\} \|\aleph_{\mathfrak{B}}\|_r \quad .$$

We now apply the definition of $\tilde{H}_i$ to bound

$$\left\| \left[H_i \aleph_{\mathcal{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathcal{B}}](x_q) \right] \aleph_{S_q} \right\|_\infty$$

$$\le \sup_{x \in S_q} \left\{ \frac{1}{(2\pi)^n} \sum_{t \in \hat{Q}} \int_{\hat{S}_t} \left| (\aleph_\lambda \hat{h}_i)(\omega) e^{i\omega\cdot x} - (\aleph_\lambda \hat{h}_i)(t\triangle) e^{i\omega\cdot x_q} \right| d\omega \right\}$$

$$\le \sup_{x \in S_q} \left\{ \frac{1}{(2\pi)^n} \sum_{t \in \hat{Q}} \left[\int_{\hat{S}_t} \left| \aleph_\lambda \hat{h}_i - (\aleph_\lambda \hat{h}_i)(t\triangle) \right| d\omega \right.\right.$$

$$+ \int_{\hat{S}_t} \left| (\aleph_\lambda \hat{h}_i)(\omega) - (\aleph_\lambda \hat{h}_i)(t\triangle) \right| \left| e^{i\omega\cdot x} - e^{i\omega\cdot x_q} \right| d\omega$$

$$\left.\left. + \int_{\hat{S}_t} \left| (\aleph_\lambda \hat{h}_i)(\omega) \right| \left| e^{i\omega\cdot x} - e^{i\omega\cdot x_q} \right| d\omega \right] \right\} .$$

For $x \in S_q$ and $\|\omega\|_\infty < \lambda$,

$$\left| e^{i\omega\cdot x} - e^{i\omega\cdot x_q} \right| \le \left| \omega\cdot(x-x_q) \right| \le |\omega| \left| (x-x_q) \right| \le \frac{n\lambda\delta}{2} .$$

Combining these bounds

$$\max_{i,q} \left\{ \left\| \left[H_i \aleph_{\mathcal{B}} - \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathcal{B}}](x_q) \right] \aleph_{S_q} \right\|_\infty \right\}$$

$$\le \frac{1}{(2\pi)^n} \max_i \left\{ \max_{t \in \hat{Q}} \sup_{\omega \in \hat{S}_t} \left\{ \left| (\aleph_\lambda \hat{h}_i)(\omega) - (\aleph_\lambda \hat{h}_i)(t\triangle) \right| \right\} \|\aleph_\lambda\|_1 \left(1 + \frac{n\lambda\delta}{2}\right) \right.$$

$$\left. + \frac{n\lambda\delta}{2} \|\aleph_\lambda \hat{h}_i\|_1 \right\} .$$

We have $\|\aleph_\lambda\|_1 = (2\lambda)^n$, and from the proof of Proposition 1 in Approximate reconstruction

$$\|\aleph_\lambda \hat{h}_i\|_1 \le \|\aleph_\lambda\|_1 \, \|\hat{h}_i\|_\infty \le (2\lambda)^n \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n K^{n(k+1)} \, K^{(2-k)n} ,$$

where, as always, $k \ge 3$.

To bound $\sup_{\omega\in\hat{S}_t} \{|(\aleph_\lambda\hat{h}_i)(\omega) - (\aleph_\lambda\hat{h}_i)(t\Delta)|\}$ we observe that with $v = \omega - t\Delta$ there exists $\omega' \in \hat{S}_t$ such that

$$|(\aleph_\lambda\hat{h}_i)(\omega) - (\aleph_\lambda\hat{h}_i)(t\Delta)| \le |\partial_v\hat{h}_i(\omega')| \le \max_{\omega'\in\Lambda}\left\{|\partial_{v/|v|}\hat{h}_i(\omega')|\right\} |\omega - t\Delta| \le \max_{\omega'\in\Lambda}\left\{|\partial_{v/|v|}\hat{h}_i(\omega')|\right\} \sqrt{n}\frac{\Delta}{2} .$$

By Lemma 10 in Approximate reconstruction

$$\sup_{\omega\in\hat{S}_t} \{|(\aleph_\lambda\hat{h}_i)(\omega) - (\aleph_\lambda\hat{h}_i)(t\Delta)|\}$$

$$\le \max_{\omega\in\Lambda}\left\{A(1,k,n,A)\left[\frac{2(k+1)}{s}\right]^{n(k+1)} \frac{\prod_{j=1}^{n}\max\{C\ ,\ |\omega_j|\}^5}{\prod_{j=1}^{n}\max\left\{\frac{2(k+1)}{s},\ |\omega_j|\right\}^{k+1}}\right\} \frac{\sqrt{n}\Delta}{2} .$$

This completes the bound for the first term of the splitting (*) .

For the second term of the splitting (*) we first use

$$\left|\left[\left[\sum_{q'\in Q} \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i\aleph_{\mathfrak{B}}](x_{q'})\ \aleph_{S_{q'}}\right] * \left[\sum_{q\in Q}\left[\mu_i*f - [\mu_i*f](x_q)\right]\aleph_{S_q}\right]\right](x_j)\right|$$
$$\le \left\| \sum_{q'\in Q}\frac{1}{\|\aleph_S\|_1} [\tilde{H}_i\aleph_{\mathfrak{B}}](x_{q'})\ \aleph_{S_{q'}}\right\|_r \left\| \sum_{q\in Q}\left[\mu_i*f - [\mu_i*f](x_q)\right]\aleph_{S_q}\right\|_{r'} .$$

To bound $\| \sum_{q\in Q}\left[\mu_i*f - [\mu_i*f](x_q)\right]\aleph_{S_q}\|_{r'}$ note that, with $1 \le v,\ v' \le \infty$, $\frac{1}{v} + \frac{1}{v'} = 1$,

$$\left|\sum_{q\in Q}\left[(\mu_i*f)(x) - [\mu_i*f](x_q)\right]\aleph_{S_q}(x)\right|$$
$$\le \sum_{q\in Q}\left[\|(\mu_i)_{[x]} - (\mu_i)_{[x_q]}\|_v\ \|f\|_{v'}\ \aleph_{S_q}(x)\right.$$
$$\le \left[\frac{1}{a_i}\right]^{n(1-\frac{1}{v})}\min\left\{ \frac{n\delta}{a_i}\ ,\ \left[\frac{n\delta}{a_i}\right]^{1/v},\ 2^{1/v}\right\} \|f\|_{v'} \sum_{q\in Q}\aleph_{S_q}(x) ,$$

where $\frac{\delta}{2} = \max_{x\in S_q}\left\{\|x - x_q\|_\infty\right\}$. The last inequality follows from Lemma 1 of the

Interpolation section and from $\varphi_{\langle\emptyset\rangle a_i} = \mu_i$. As usual, all occurances of a_i in the last expression may be replaced by m .

To bound $\| \sum_{q' \in Q} \frac{1}{\|\aleph_S\|_1} [\tilde{H}_i \aleph_{\mathfrak{B}}](x_{q'}) \ \aleph_{S_{q'}} \|_r$ use

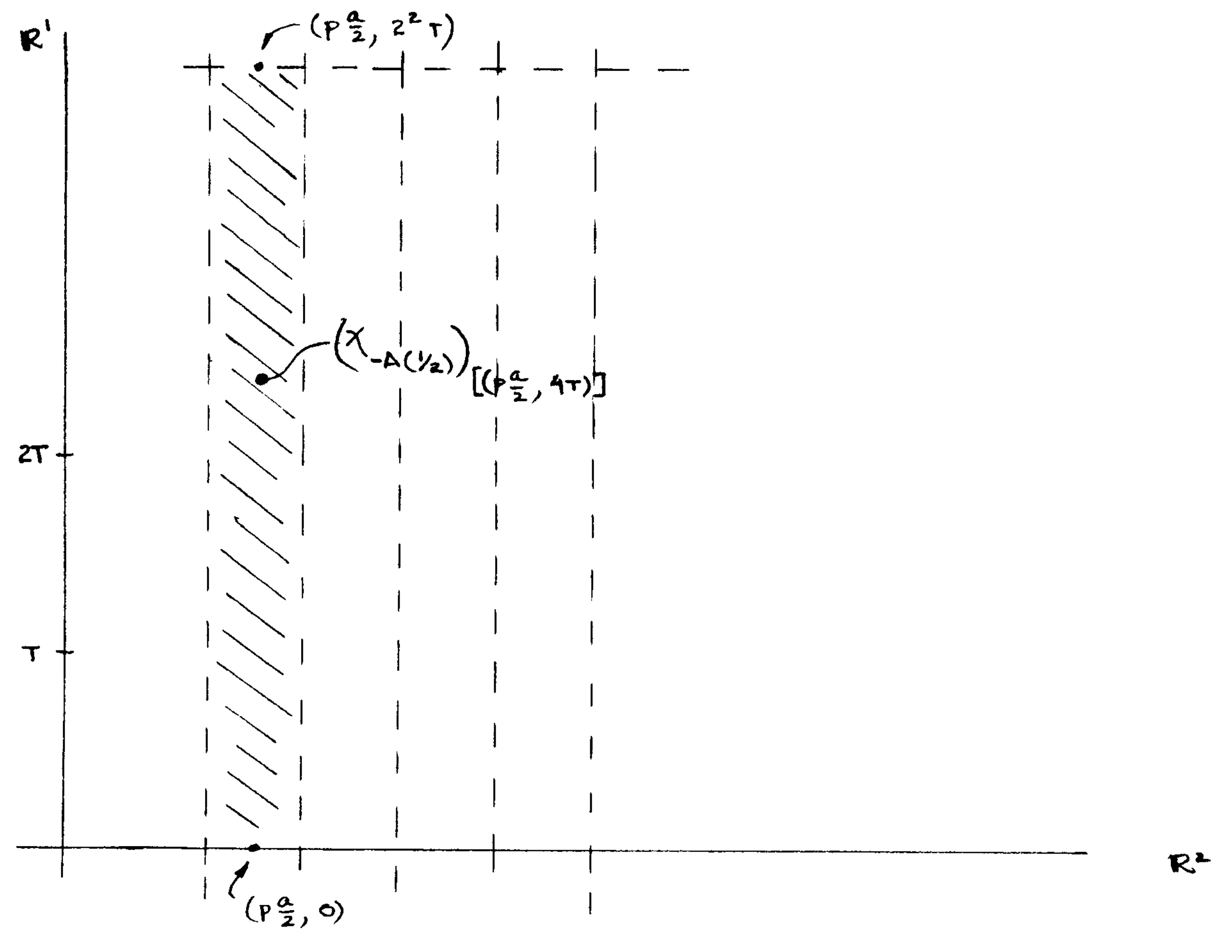
$$\max_{i,q} \left\{ \left| \frac{(\tilde{H}_i \aleph_{\mathfrak{B}})(x_q)}{\|\aleph_S\|_1} \right| \right\} \leq \max_i \left\{ \frac{1}{(2\pi)^n} \| \sum_{t \in \hat{Q}} (\aleph_\lambda \hat{h}_i)(t\triangle) \ \aleph_{\hat{S}_t} \|_1 \right\}$$

$$\leq \max_i \left\{ \|\aleph_\lambda \hat{h}_i\|_\infty \right\} \frac{\|\aleph_\lambda\|_1}{(2\pi)^n} \leq \left[\frac{\lambda}{\pi}\right]^n \|\hat{h}_i\|_\infty$$

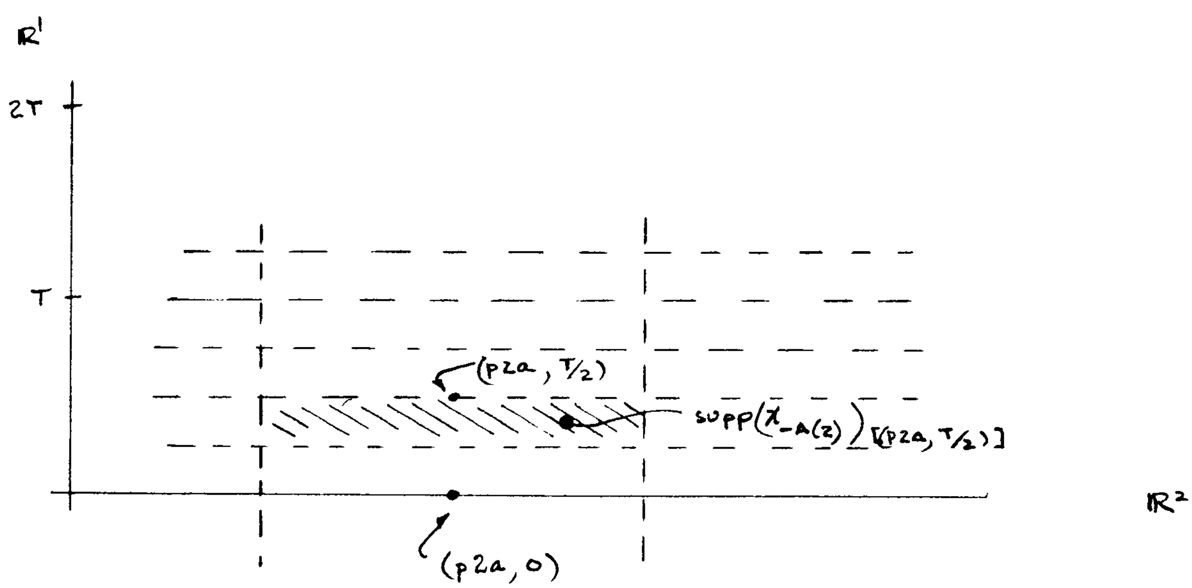
$$\leq \left[\frac{\lambda}{\pi}\right]^n \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n K^{3n} \quad .$$

To complete the proof, combine the above bounds and use $r' = 1$ in the bounds for the first term of the splitting (*) , and use $r = v' = 1$ in the bounds for the second term of the splitting (*) .

$\square$



2a



2b

FIGURE 2. Two rescalings of a and T.
$|A(1/2)| = |A(2)|$

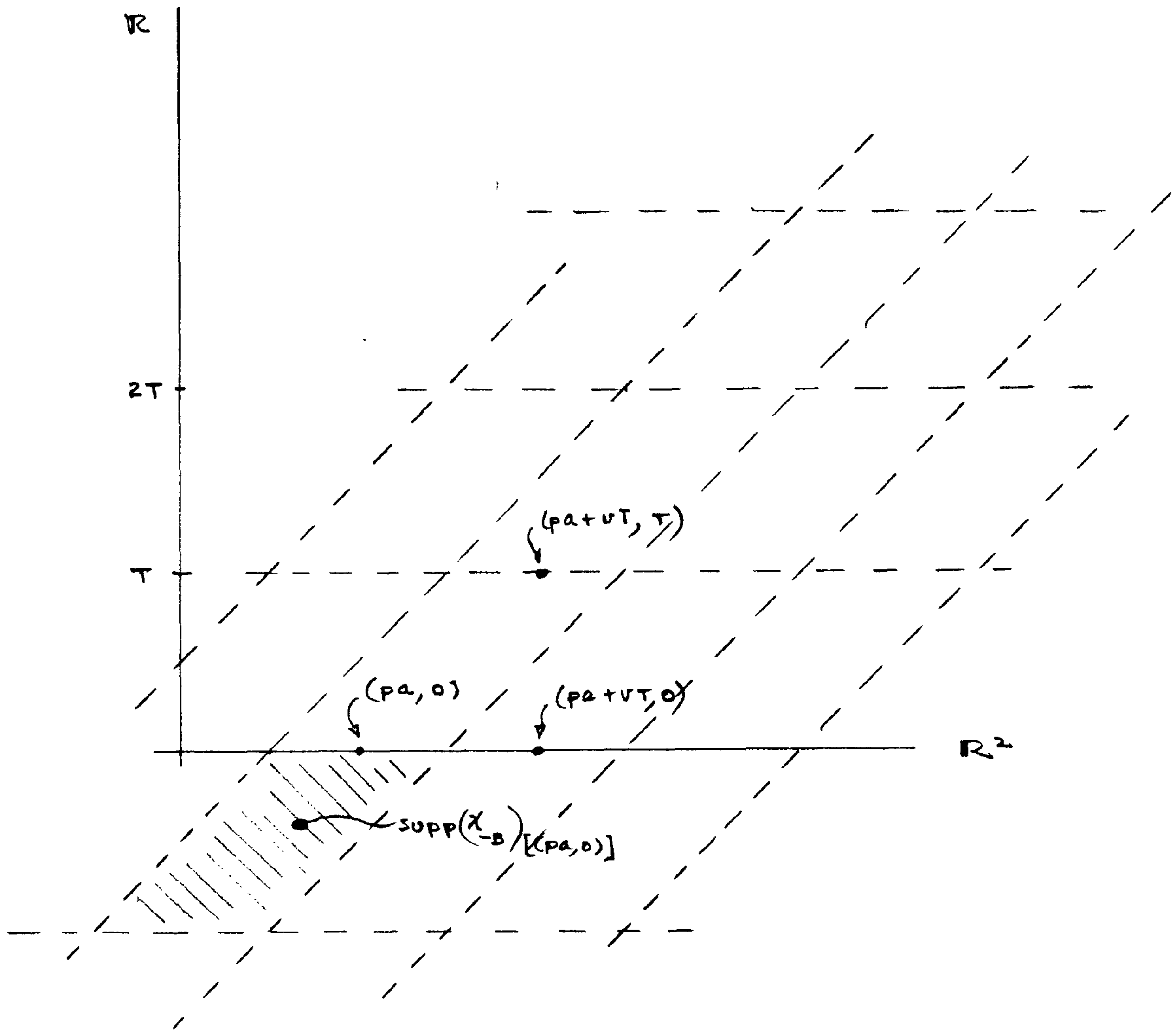


FIGURE 3. Representation of continuous scanning with scan velocity $v \in \mathbb{R}^2$
In this representation
$$\{pa + vT : p \in \mathbb{Z}^2\} = \{pa : p \in \mathbb{Z}^2\}$$

Discussion

A primary motivation for exhibiting an explicit error bound was to determine if a 'pracitical' support for the deconvolutors $(\hat{h}_i \aleph_\lambda)^{\vee} \aleph_{\mathcal{B}}$ could be established. A 'practical' support would be one for which the side length β of $\mathcal{B}$ differed from the side length m of the smallest convolutor by a factor of several tens. Such an support would be useful for applications.

The bounds established here do not satisfy our 'practical' criterion. Let us examine the error ε for a specific case. Consider

$$n = 2 , \qquad \mathcal{A} = \{1, \sqrt{2}, \sqrt{3}\} , \qquad k \geq 3 , \quad \text{and} \qquad s \leq 1 .$$

Then

$$m = 1 , \qquad M = \sqrt{3} , \qquad \frac{2(k+1)}{s} \geq 8 , \quad \text{and}$$

(see the Theorem in Approximate reconstruction for definitions)

$$C = \left[\left[\frac{n}{M}\right]^4 \frac{m}{2}\right]^{1/3} \cong 1.76 , \qquad K_1 = \frac{2(k+1)}{s} .$$

Since the side length $m = 1$ of the smallest convolutor is our unit in $\mathbb{R}^1$, it is easy to select a function f and a set E such that

$$\|f\|_p \leq 1 , \quad 1 \leq p \leq \infty , \quad \text{and} \quad \|\aleph_E\|_1 = 2^n$$

(e.g., a simple function with support in E). For such a case the error ε should certainly be no more than 1 .

Consider ε_3 of the Approximate reconstruction section. In the bound for ε_3 given in that section $|\eta_1(x_j)|$ and $|\eta_2(x_j)|$ have the common factor

$$L = (n+1)\sqrt{n} \left[\frac{5M^4}{8}\right]^{2n} \left[\frac{2}{m}\right]^n \cong 1.70 \times 10^4 .$$

This term also appears in $A(r,k,n,A)$ so that

$$|\eta_1(x_j)| \le \|f\|_2 \frac{2^{r+n}}{(2\pi)^n\sqrt{2r-n}} ((r+1)!)^2 \begin{bmatrix} k+r \\ r \end{bmatrix} \left[1+rM2^nL\right]^r L\, K_1^{4n+2nr} \left[\frac{1}{\beta}\right]^{\frac{2r-n}{2}} ,$$

$$|\eta_2(x_j)| \le \|f\|_2 \left[\frac{1}{\pi}\right]^{n/2} 3^{\frac{n-1}{2}} L\, K_1^{3n+\frac{n}{2}} \left[\frac{1}{\ell n}\right]^{k-3+\frac{1}{2}} .$$

If $k = 3$ then for $\varepsilon \le \|f\|_2$ it is necessary that $|\eta_2(x_j)| \le \|f\|_2$ which requires that

$$\ell n \ge K_1^{7n} \ge 8^{14} = 2^{42} .$$

For $|\eta_1(x_j)|$ to not exceed $|\eta_2(x_j)|$ it is necessary that

$$\beta^{r-\frac{n}{2}} \ge (\ell n)^{1/2} K_1^{n(2r+\frac{1}{2})} ,$$

whence, for $r = 2$,

$$\beta \ge (\ell n)^{1/2}\, 8^9 \ge 2^{21+27} = 2^{48} .$$

Clearly, such estimates are not 'practical'. Similar relations hold for Δ and δ that appear in the bound for ε_4.

References

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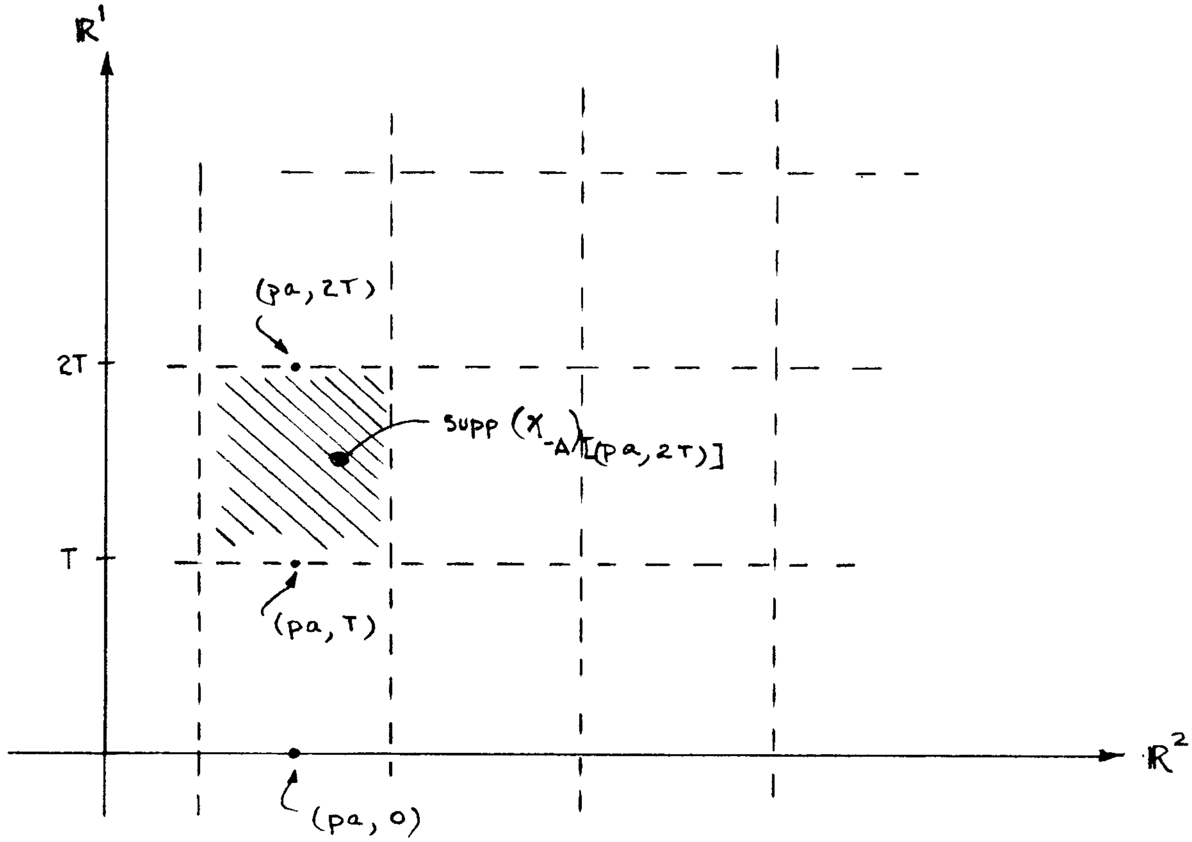

FIGURE 1. Representation of a staring array with a simple integration time response.

$$-A = \varphi = \left(-\tfrac{a}{2}, \tfrac{a}{2}\right) \times \left(-\tfrac{a}{2}, \tfrac{a}{2}\right) \times (-T, 0) \subset \mathbb{R}^3$$

$$p = (p_1, p_2) \in \mathbb{Z}^2$$

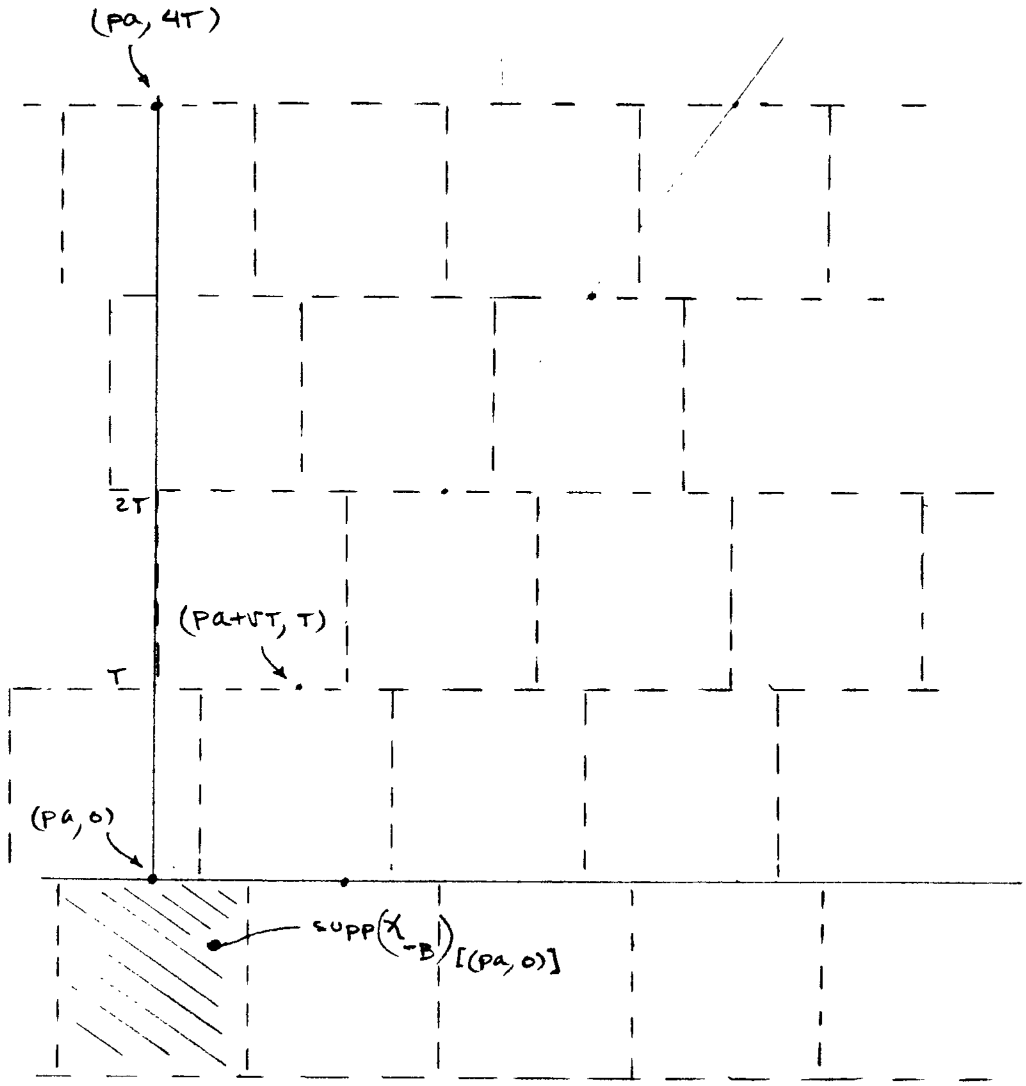

FIGURE 4. Representation of shift scanning. After each interval T the set B is shifted by vT, $v \in \mathbb{R}^2$. Here, for each $n \in \mathbb{Z}$

$$\{pa + nvT : p \in \mathbb{Z}^2\} = \{pa + (n+4)vT : p \in \mathbb{Z}^2\}$$

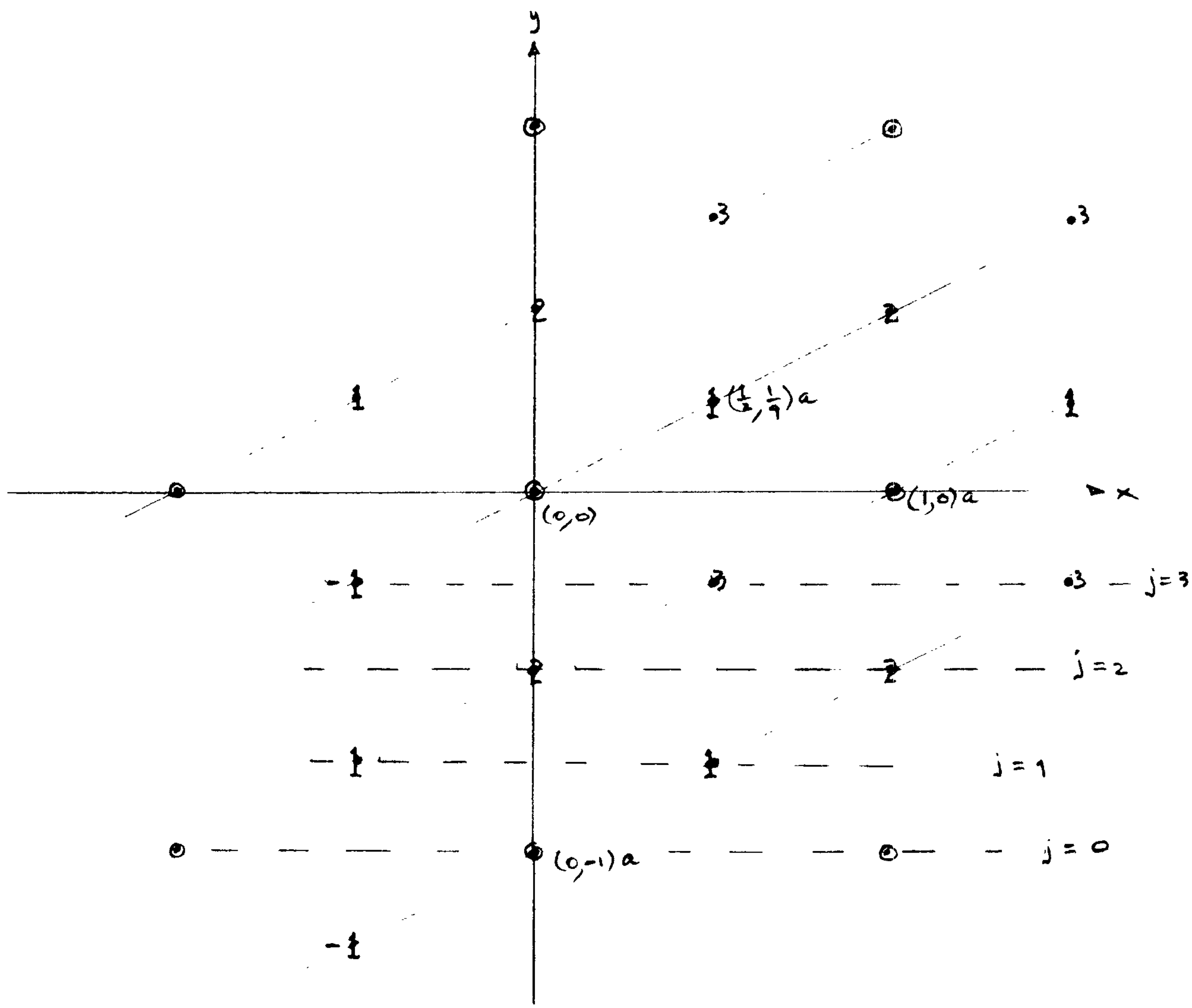

Figure 5. Scan in $\mathbb{R}^2$ for $(p_1,p_2) + j(\frac{1}{2}, \frac{1}{2}z)$, $(p_1,p_2) \in \mathbb{Z}^2$, $j \in \mathbb{Z}$

Legend: e.g., •2 means point sampled for $j=2$.