

## ABSTRACT

Title of Dissertation:       RETENTION OF CONCEPTS AND SKILLS IN  
TRADITIONAL AND REFORMED APPLIED  
CALCULUS

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A fundamental question is currently being asked throughout the collegiate mathematics education community: "How can we help students understand and remember calculus better?" There seems to be general dissatisfaction with the knowledge and abilities of students who have completed calculus courses.

Reformers in the calculus arena are striving to change instruction to help students understand better and remember longer what they have learned. The Calculus Consortium based at Harvard (CCH) recently published new textbooks for applied calculus which embody a major switch in the philosophy of calculus teaching. The CCH texts, in which applications are the central motivation and not coincidental afterthoughts, emphasize concepts more than symbol manipulation and encourage student-driven discovery of fundamental ideas.

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Is this reformed way of teaching applied calculus more effective than the traditional method? Which method leads to better long-term understanding and ability? The purpose of this study was to shed light on these questions by characterizing and comparing the skills and conceptual understandings of students of traditional and reformed methods several months after they completed their applied calculus course.

A sample of 108 students of applied calculus (57 reformed, 51 traditional) who completed their course in April of 1997 were given a written test in November of 1997 to assess their conceptual understandings and computational skills. Sixteen of these students (8 traditional, 8 reformed) were interviewed to ascertain more about their conceptual understandings as well as their motivation, commitment and attitudes with respect to their applied calculus courses. Test results indicate that although there was no significant difference in overall performance between the two groups, students of the reformed method performed better on conceptual problems, while students of the traditional method performed better on computational problems. Interview results indicate that of the two groups, reformed course students were more confident in their ability to explain derivatives. Reformed course students mentioned graphs and applications more, and they also were more inclined to use estimation techniques than traditional course students. The traditional course students had a clearer idea of the connection between the derivative and the integral.

RETENTION OF CONCEPTS AND SKILLS IN TRADITIONAL AND REFORMED  
APPLIED CALCULUS

by

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## DEDICATION

To my wife, Carrie, who contributed to this work in more ways than anyone would guess; and to my children, who will be seeing a lot more of Daddy now that his “book” is finished.



## TABLE OF CONTENTS

|  |    |
|--|----|
| LIST OF TABLES.....                        | v  |
| LIST OF FIGURES.....                       | vi |
| CHAPTER 1: RATIONALE .....                 | 1  |
| Introduction/Background.....               | 1  |
| Goals.....                                 | 3  |
| Research Questions.....                    | 4  |
| Limitations.....                           | 7  |
| Significance .....                         | 7  |
| Definitions .....                          | 9  |
| CHAPTER 2: LITERATURE REVIEW .....         | 10 |
| Mathematics Learning.....                  | 10 |
| Cognitive Science.....                     | 11 |
| Retention .....                            | 13 |
| Social Constructivism.....                 | 17 |
| Conceptual and Procedural Knowledge .....  | 18 |
| Mathematics Teaching .....                 | 19 |
| Procedures vs. Concepts.....               | 19 |
| Teacher-Directed vs. Student-Directed..... | 21 |
| College-level Difficulties.....            | 23 |
| Technology .....                           | 25 |
| Calculus Reform .....                      | 26 |
| History.....                               | 26 |
| Recent Studies .....                       | 28 |
| CHAPTER 3: METHODOLOGY .....               | 32 |
| The Texts .....                            | 33 |
| Study Site.....                            | 34 |
| Instruments.....                           | 36 |
| Target Population .....                    | 53 |
| Data Collection.....                       | 54 |
| Data Analysis.....                         | 58 |

|   |     |
|---|-----|
| CHAPTER 4: RESULTS .....  | 61  |
| Tests.....  | 61  |
| Interviews: Concepts Portion .....  | 78  |
| Interviews: Motivation/Affect Portion and Instructor Insights .....                     | 97  |
| CHAPTER 5: CONCLUSIONS .....  | 105 |
| Implementation of Control and Reform Instruction.....                                   | 105 |
| Interpretation of Results .....   | 108 |
| Future Research .....   | 111 |
| Summary.....  | 113 |
| APPENDIX A: CONTENT ANALYSIS.....   | 116 |
| APPENDIX B: WRITTEN TEST .....  | 119 |
| APPENDIX C: STUDENT INTERVIEW.....  | 125 |
| APPENDIX D: CORRESPONDENCE BETWEEN RESEARCH QUESTIONS AND<br>TEST/INTERVIEW ITEMS ..... | 130 |
| APPENDIX E: TEST GRADING KEY .....  | 131 |
| REFERENCES.....   | 137 |

## LIST OF TABLES

|   |    |
|---|----|
| Table 3.1 Categorization of Test Items .....                      | 46 |
| Table 3.2 Sample Grade Distribution .....                         | 57 |
| Table 4.1 Total Score Distributions Summary.....                  | 61 |
| Table 4.2 Total Scores ANOVA, Reformed vs. Traditional.....       | 62 |
| Table 4.3 Categorization of Test Items .....                      | 63 |
| Table 4.4 Conceptual Score Distributions Summary.....             | 63 |
| Table 4.5 Procedural Score Distributions Summary .....            | 63 |
| Table 4.6 Conceptual Scores ANOVA, Reformed vs. Traditional ..... | 64 |
| Table 4.7 Procedural Scores ANOVA, Reformed vs. Traditional ..... | 65 |
| Table 4.8 Test Scores Summary .....                               | 76 |

## LIST OF FIGURES

|   |    |
|---|----|
| Figure 3.1 Graph for Test Item 1 .....                            | 39 |
| Figure 3.2 Graph for Test Item 7 .....                            | 43 |
| Figure 3.3 Graph and Function Table for Test Item 8.....          | 44 |
| Figure 3.4 Function Graph for Interview Conceptual Item 3 .....   | 49 |
| Figure 3.5 Derivative Graphs for Interview Conceptual Item 3..... | 50 |
| Figure 3.6 Graph for Interview Conceptual Item 4.....             | 51 |
| Figure 3.7 Graph for Interview Conceptual Item 7.....             | 52 |
| Figure 3.8 Definition for Interview Conceptual Item 8 .....       | 52 |
| Figure 4.1 Graph for Test Item 1 .....                            | 66 |
| Figure 4.2 Graph for Test Item 7 .....                            | 71 |
| Figure 4.3 Graph and Function Table for Test Item 8.....          | 72 |
| Figure 4.4 Function Graph for Interview Conceptual Item 3 .....   | 88 |
| Figure 4.5 Derivative Graphs for Interview Conceptual Item 3..... | 89 |
| Figure 4.6 Graph for Interview Conceptual Item 4.....             | 91 |



## CHAPTER 1: RATIONALE

### **Introduction/Background**

Over the past decade there has been much change in secondary school and collegiate calculus teaching. Many calculus instructors have drastically changed their materials and methods. There have been a great number of calculus reform projects nationwide. The National Science Foundation Calculus Initiative, from 1988 to 1994, granted \$22.2 million to 127 different calculus reform projects (Tucker & Leitzel, 1995). Growing emphasis has been placed on calculus reform at meetings of the national mathematics societies and associations, and a conference including a major focus on the teaching of calculus has been held annually beginning in 1991. It seems that each year, more and more colleges and universities begin reforming the way their faculty teach calculus.

Why is this reform going on? Persistent high dropout and failure rates in traditional courses and mounting evidence that even apparently “successful” students acquire limited understanding of calculus concepts have convinced many professors that there must be better curriculum materials and teaching strategies (Selden, Selden & Mason, 1994, p. 19). Recent calculus reform initiatives have sought ways to make the subject more meaningful for students and to lead them to better understanding of key ideas and techniques. They have tried to present critical calculus concepts in more depth with less emphasis on memorization of algorithms and symbol manipulation (Schoenfeld,



1997, p. 2). In this respect, the calculus reform is much the same as the concurrent mathematics education reform from kindergarten through twelfth grade.

Most of the calculus reform efforts have focused on the courses offered for mathematics, physical science, and engineering majors (henceforth referred to as “theoretical calculus”). At the same time, little reform work has been done in the courses for business, social science, and life science majors (henceforth referred to as “applied calculus”) (Narasimhan, 1993, p. 254 ). Certain reform developments have occurred over the last few years, but the movement is still small compared to that dealing with theoretical calculus.

As calculus curricula and pedagogical methods are developed which strive to bring about changes in the mathematical understanding of students, research is needed to help determine how successful the reforms are (Kaput & Dubinsky, 1994, p. vii; Becker & Pence, 1994, p. 6). Testimonials about better student understanding and achievement are commonplace, but evidential research is sparse. Such research will not only help determine the success of new curricula; it can also serve to guide their evolution.

Some research has begun into the effectiveness of calculus reforms. Most of this evaluative work has focused on the theoretical calculus. Some studies have compared students of reformed courses with their counterparts from traditional courses, but few (if any) have sought to establish whether the reformed courses have any lasting effect on their students. With the large amount of effort being expended on reforming calculus curricula, it behooves us to try to establish their value in terms of enhancement to the future capabilities of calculus students. Perhaps students of the reformed calculus

understand calculus concepts and procedures better than their counterparts at first, upon completion of the courses--but if there is really no difference between the knowledge and abilities of these same students a year or two later, then it could be that the reforms are of little practical value. On the other hand, if one reform effort can be shown to have a positive effect on how long calculus knowledge remains useful to its students, then it is certainly worth noting. This characteristic of the present study--the search for "residual" knowledge of calculus after the passage of time--is in some sense the heart of the whole matter: is reformed calculus making a difference? What better way to find out than to assess the understandings of calculus students some time after completion of their course?

## **Goals**

In this study, we sought to gain insight into the effectiveness of one popular reformed applied calculus curriculum. The Calculus Consortium based at Harvard (CCH), long active in the theoretical calculus reform, recently published a new text for applied calculus entitled "Applied Calculus for Business, Social Sciences, and Life Sciences." Since CCH is often looked to as the leader in calculus reform, it seems natural that its applied calculus should be among the first to be evaluated. Although fairly new, this textbook is currently in use at 100 different colleges and universities across the United States.

A great deal of time and effort went into the production of the CCH Applied Calculus--did it pay off? There are many ways one could try to answer this question, but we are most interested in the knowledge and abilities of students of the CCH Applied

Calculus several months after they complete the course. How do they compare with students of a more traditional program? The presumed goal of any applied calculus course is to help students achieve the best possible understanding of calculus and its applications to business, life sciences and social sciences. How much difference does the particular offering of applied calculus make in students' future capabilities?

A good "traditional" text with which to compare the CCH text is "Calculus and its Applications," written by Goldstein, Lay and Schneider. These authors put much effort into their book, thinking of what would make sense to students as they wrote it (Goldstein, Lay & Schneider, 1996, p. ix). This book (which will henceforth be referred to as the GLS text) continues to be widely used and is now in its seventh edition. Its success and popularity make it a good choice for comparison with reformed texts.

### **Research Questions**

For these questions the "key concepts" and "key procedures" of applied calculus refer to those concepts and procedures distilled from a content analysis of the CCH and GLS texts. In particular, the concepts and procedures include the following: description of graphs of functions (both with and without using derivatives), differentiation of functions both graphically and analytically, interpretation of the derivative (mostly in economic situations), integration of functions both graphically and analytically, interpretation of the definite integral, and the relationship between the derivative and the definite integral (the fundamental theorem of calculus).



1. How well do students of the CCH and GLS applied calculus courses understand key concepts of applied calculus seven months after taking their courses?
2. How well do students of the CCH and GLS applied calculus courses perform key procedures of applied calculus seven months after taking their courses?
3. How do students of the CCH and GLS applied calculus courses compare to each other in terms of their understanding of key concepts and performance of key procedures of applied calculus seven months after taking their courses?

An explanation of how we went about answering the foregoing questions is detailed in chapter 3; we mention the process here only briefly. First we analyzed and compared the CCH and the GLS applied calculus texts with several questions in mind: How are they different? Is the content the same? How does teaching differ between the two? Next, we distilled from these two courses a small but representative cross-section of essential concepts and skills common to both and put forward a pair of instruments designed to assess the learning of applied calculus. Then we used these instruments with two groups of students who completed the applied calculus course within the past year: one group from a reformed (CCH) course, and the other from a traditional (GLS) course. We used the results to describe the abilities of each group, to characterize the way they think about calculus, and to compare the two groups.

While most studies of this sort use students who are enrolled in the appropriate math course at the time the study is conducted, this study used students who completed

the course seven months prior to their selection as subjects. Finding out the effects that courses really have on their students in the future is very valuable; after all, the whole reason mathematics courses are taught is presumably so that the knowledge gained thereby can be put to good use in the future. Instructors need to pay more attention to the matter of retention, for the sad truth is that “the state of most students’ conceptual knowledge of mathematics after they have taken calculus is abysmal” (Epp, 1986, p. 48).

It is important to point out that we will focus not on what should be learned in the study of applied calculus but rather on what these particular students learned and how they compare to each other. There is some degree of disagreement about what students should learn and what they will be required to use later in their professional lives--in this study we do not attempt to answer this question explicitly. We simply note that educators remain committed to including applied calculus in the list of required courses for many students. Whatever the reasons, the educational community would like to increase the number of students who study applied calculus successfully.

It is hoped that this study will contribute to answering the call for more research in the teaching of collegiate mathematics issued by Kaput and Dubinsky in the introduction of “Research Issues in Undergraduate Mathematics Learning” (Kaput & Dubinsky, 1994). A great deal of study has gone on in mathematics education at primary levels up through high school, but clearly math education does not end there. Many questions remain to be answered about college mathematics.



## **Limitations**

There are, of course, limitations. How effective is the reform of applied calculus in actuality? Clearly this study cannot answer this question definitively. Being exploratory in nature and small in scale, this study cannot measure the success or failure of a national applied calculus reform. It can describe in part how we might go about measuring and evaluating college math curricula and instruction.

Truly determining the success or failure of applied calculus courses requires knowing the standard with which to compare; this study also cannot define the appropriate standard. It can only describe two prominent views of that standard and measure effects of curricula that reflect those views.

A study describing how students think is necessarily limited in terms of the number of subjects that can be involved, and because of this differences in student attitudes and performance may be affected by variables difficult to control (such as instructor, site, etc.).

How technology is used in the two types of applied calculus courses is presumed not to be a confounding variable, yet the two courses differed in the way technology was emphasized and used.

## **Significance**

This study has several possible positive consequences. First, it sheds light on questions about how students think about difficult math concepts. In order to teach better, math faculty need to understand better how students think, why they think that way, and how their thinking can be enhanced and encouraged. This study is not just an

evaluation of the Harvard Consortium's new applied calculus text, and it is not an evaluation of the GLS text; it contributes to our long-term understanding of mathematics learning.

Second, this study helps answer questions about the effect of math education reforms on long-term learning. If the philosophy of emphasizing concepts turns out to have an advantage in helping students retain what they've learned, this study and others like it may lend strength to the cause of the reformers. On the other hand, if the philosophy of emphasizing concepts does not appear to significantly affect how well students retain what they've learned, reformers may need to give more careful consideration to what actually is being accomplished through their efforts.

Third, better communication between mathematics departments and "client departments" (those requiring applied calculus of their students) may result from this study. There is a need for these departments to communicate more with each other (Narasimhan, 1993, p. 261). This dialogue can only help interdepartmental relations on the nation's campuses; it can also help produce college graduates who are better prepared for their careers.

Fourth, calculus instructors will be enabled to consider whether or not to participate in reform projects from a more informed vantage point in light of this study. Knowing how much and in what ways student understanding and retention was affected in the case of this particular implementation of a reformed curriculum will give instructors more information upon which to draw in making their own decisions about what and how to teach. Perhaps some faculty are waiting for more success to be shown

before they switch to new texts (Armstrong, Garner, & Wynn, 1994, p. 303); the results of this study may give them more confidence in turning to CCH and like curricula. Whatever the influence this study will have on its readers, it will lead them to more informed decisions.

Fifth, if indeed calculus instructors learn to change their teaching for the better, then not only will students do better with applied calculus than in the past, but the client disciplines will also become healthier.

### **Definitions**

theoretical calculus: a course in calculus offered at most universities and colleges for majors in mathematics, engineering or physical sciences. This course traditionally places emphasis on theoretical results and formulae.

applied calculus: A survey course in calculus offered at most universities and colleges for majors in fields other than mathematics, engineering or physical sciences. Emphasis is on practical application of concepts rather than theoretical results.

client disciplines: Those college/university departments or areas of study requiring that their students take an applied calculus course (e.g., business, social sciences, life sciences).

traditional applied calculus: the most common curriculum/methods used to teach applied calculus over the last several decades.

reformed business calculus: curriculum/methods employed by the Calculus Consortium based at Harvard or other similar projects (including use of technology, student discovery and real-world problems).



## CHAPTER 2: LITERATURE REVIEW

The purpose of this chapter is to give background information about how this study fits into prior research in mathematics education and to explain some of the reasons for the way the study was conducted.

The research literature in the area of mathematics education reform is growing daily. Whereas the majority of research still focuses on primary school issues, research on questions about college mathematics education has made substantial gains over the last decade. Since college mathematics education reform over this time period began with calculus reform (Schoenfeld, 1997, p. 2), most of the available literature on college math reform deals with calculus. Given the purpose of this study, such literature is of primary interest. We turn first, however, to two areas of research which build the foundation for discussing applied calculus reform: mathematics learning and mathematics teaching.

### **Mathematics Learning**

We will begin by considering prominent theories of mathematical learning. Much has been written in this area, particularly in reference to primary-level education. We give outlines of major relevant positions on memory and mental processes--issues at the heart of this investigation.

## Cognitive Science

One of the most widely accepted theories about how mathematics is learned is based on ideas and research results from the field of cognitive science. The two main goals of cognitive science research are understanding mechanisms of human memory and information processing (Silver, 1987). For purposes of the current study we focus primarily on memory.

It is generally theorized that human memory is made up of three main components (with various other ingredients attached): sensory buffers, working memory, and long-term memory (Silver, 1987; Byrnes, 1996). Sensory buffers take in information from the world around us, these observations are processed in working memory, and some permanent record is stored in long-term memory. Long-term memory is where mathematical knowledge, be it conceptual or procedural, is stored. Working memory is where everything students learn is processed; where information about a problem can interact with information from long-term memory and with metacognitive processes such as monitoring and planning (Silver, 1987). According to Hiebert and Carpenter (1992), long-term memory records can include representations of mathematical ideas as well as connections between the representations; thus mathematical knowledge can be described as a network (pp. 66-67). As these connections become more numerous, stronger, and well organized, mathematical understanding and the ability to remember mathematical knowledge increase (pp. 69, 75). Since internal memory connections between mathematical ideas are assumed to be “influenced by external activity” (p. 66), what and how mathematics is taught become



very important. Current reforms in mathematics education, including those involving calculus, aim to help students construct well-connected mathematical knowledge, leading to better retention and application of that knowledge.

In addition to basic processes of observation, information processing, and memory, research in cognitive science has highlighted the importance of higher-level mental activities referred to as metacognition. Schoenfeld defines metacognition as having three components: knowledge about one's own thought processes, self-regulation in thinking, and beliefs and intuitions (Schoenfeld, 1987, p. 190). Silver points out that although the processes of planning, monitoring, and evaluation during problem solving are considered to be important, very little time is given to developing these skills in actual instruction (Silver, 1987, p. 55)--yet we expect students to become good problem solvers. Silver goes on to note that teachers must also be aware that students come to class with their own unique experiences and beliefs, and that these help shape how and what students learn (p. 57).

It makes sense for teachers to think more about how to develop better metacognitive skills in their students, as these skills (or the lack thereof) dictate how people solve problems. Yet they must be careful, for metacognitive thinking changes depending upon context. As Pollak (1987) pointed out, the very fact that a problem appears in a textbook may change the kinds of strategies useful in solving it; for example, we might guess that the problem was contrived to have a simple answer (p. 254). The kinds of problems students will eventually need to solve on the job won't come from a

textbook and may not have one crisp answer--these are the kinds of problems we should prepare most students for (Pollak, 1987, p. 255).

Schoenfeld (1987) recommends four ways of fostering the growth of metacognitive skills in mathematics classes. The first of these is using videotape during class. As teacher and students watch themselves solve mathematical problems, they can see more clearly where errors occurred and discuss ideas for increasing future success. The second way is the modeling of good metacognitive behavior. For example, in working a problem on the board, the teacher might point out not only the mathematical ideas involved in the problem, but also the pertinent self-regulatory thoughts and monitoring processes which lead to success. The third technique for increasing metacognitive skills is to involve the whole class in discussions; these discussions are viewed as a way of creating a microcosm of mathematics culture in the classroom (i.e., a small version of the kind of collaboration in which professional mathematicians participate). The fourth suggestion is to have students work together in small groups, where they can help each other solve problems effectively. Cooperative work in small groups also has led to better achievement and higher self-confidence in learning (Davidson, 1990a, pp. 7-8). Since the CCH calculus is a proponent of group work, theory suggests that its students should acquire greater metacognitive skills and higher self-confidence than students in more traditional lecture-oriented courses.

### Retention

How does the passage of time affect retention of mathematical knowledge? This question is of central interest in this study. In this section we briefly discuss two views

of remembering and forgetting--two views which at first seem contradictory, yet after careful consideration appear to support each other.

The first widely accepted view considers the processes of human memory to be similar to those of a computer; the processor (working memory) calls for a record from disk storage (long term memory), which is then available for use. The human process is more elaborate, however, with three kinds of record retrieval: recall, whereby cues call up some stored record with which they are associated; recognition, whereby something observed is known to match with some record in long-term memory; and inferential reconstruction, wherein cues fetch fragments of records, and one must "build up a plausible story" around them in a reconstructive effort (Byrnes, 1996, p. 43). Those who support this view of remembering say that forgetting is caused either by decay (a weakening of memory records or association of cues with records through lack of accessing those records over time) or by interference (learning new things slightly different from or conflicting with existing memory records). Students' loss of memory of calculus concepts is most likely due to infrequency of their use of those "calculus records" once the course is completed. It seems less likely that subsequent learning interferes with calculus knowledge.

The second prominent view of memory is based not on storage and retrieval, but on imagination. According to Bolles (1988), "...remembering is a much more active process than the old storage metaphor supposed. To construct our memories we need arousal and understanding" (p. 180). "Emotions, perceptions, and reminders all stir the imagination, and imagination, not storage, is the basis of memory" (p. 181). The claim is



that things such as “desire, continuing attention,...interpretation...and ambitions” aid in the building of memory. According to this view, forgetting happens simply because people don’t actively remember--for one reason or another, they choose not to use the memory power available to them (Bolles, 1988, p. 180). Hence we could conjecture that students don’t remember their calculus because they didn’t care enough about it or give it sufficient attention. They may not be good at remembering, which can be viewed as a learned skill rather than an innate mechanical ability. White (1997) agrees with these ideas when he points out that “Lack of intent to remember and a poor encoding process...” (p. 142) are responsible for forgetting. This view is consistent with the idea, discussed by Craik and Lockhart (1972), that the “level” or depth of processing information receives from the learner helps determine how well the learner retains that information (p. 676).

In reality both views described above must be right. Surely storage must occur, for we cannot remember something we have never seen or heard before; and very probably the more attention and imagination we apply to some idea, the better we remember it. Instructional implications from the “storage-retrieval” view include recommendations to teachers to view their practice as building long-term records in students’ memories and to use multiple methods of teaching. Yet following these recommendations would lead one to put more imagination, elaboration, and even emotion into teaching--all of which enhance remembering in the “memory-is-imagination” view.

In light of the foregoing discussion, it is quite plausible that the CCH applied calculus course--with its emphasis on more and varied student attention to concepts through applications and the use of technology--should lead to better remembering of calculus concepts on the part of its students.

Another factor which may lead students of reformed calculus to better recall of concepts and/or procedures is the "generation effect." Researchers have observed in some situations that when students generate for themselves part of the content they learn, they remember it better than those who were simply given the necessary information (McDaniel, Waddill & Einstein, 1988). Hence it may be that courses using CCH methods, with emphasis on having students construct their own understandings of central calculus concepts rather than merely accepting the instructor's understanding, may lead to better retention because of the generation effect. McDaniel, Waddill & Einstein (1988) point out, however, that studying with intent to learn may have just as strong an effect on students' recall ability (p. 532). This suggests that perhaps calculus students can learn as much as possible in any calculus course if they have a strong desire to learn. We note also that the results cited above were obtained immediately after the learning occurred.

Other researchers have investigated the rate at which forgetting occurs. Anderson (1995) explains that forgetting occurs very quickly at first and then slows over time (p. 234). Increased practice with the target material increases retention but does not change the rate of forgetting (p. 237).



### Social Constructivism

A second main theory about how mathematics is learned is called social constructivism. This theory deals with the way knowledge develops depending on social factors in one's environment. The claim of this and related theories is that successful learning depends more on when, where, and with whom the knowledge is gained than on how that knowledge is constructed specifically within the mind. As Hiebert and Carpenter put it,

A major goal of these theories, with their emphasis on situated knowledge and social cognition, is to explain the apparent understanding that comes with learning in everyday contexts and the lack of such understanding that accompanies learning in formal school settings. (Hiebert & Carpenter, 1992, p. 65)

Jean Piaget is generally considered to be the founding father of constructivism. He maintained that a child must develop his/her own understandings independent of a teacher who would "transmit" information; one must be placed in a situation (usually with other learners) where learning is possible and then be given ample time (and perhaps encouragement) to discover important things on one's own. This theory has been described with the "brick wall" analogy; teachers toss "bricks" to the students who try to build a wall with them (Byrnes, 1996). What the students do with these bricks should not be dictated; indeed, some have concluded that we should not tell students how to think (Speiser & Walter, 1996).

Efforts like the CCH reformed calculus texts are based in part upon ideas from social constructivism. It is for this reason that they encourage letting students work in small groups and promote the learning of calculus ideas through consideration of problems from real life.

### Conceptual and Procedural Knowledge

There are two types of knowledge which generally are thought to make up mathematical knowledge: procedural knowledge and conceptual knowledge. Procedural knowledge includes things such as performing algorithmic calculations, while conceptual knowledge would include understanding the reasons why the algorithms work or when they should be applied.

As Hiebert and Lefevre (1986) define it, conceptual knowledge is “rich in relationships....In fact, a unit of conceptual knowledge cannot be an isolated piece of information; by definition it is a part of conceptual knowledge only if the holder recognizes its relationship to other pieces of information” (pp. 3-4). Procedural knowledge is defined to have two parts, symbolic language and rules for completing mathematical tasks (p. 6).

Although some researchers focus on conceptual learning and others focus on procedural learning, both are important to competency in mathematics, and linking the two helps to create the kind of networked information that is retained better (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). On choosing which of the two types should come first, theory generally recommends concepts before procedures (Hiebert & Carpenter, 1992, p. 78)--yet, as Hiebert and Lefevre (1986) note, “Formal mathematics

instruction seems to do a better job of teaching procedures than concepts or relationships between them” (p. 22). Tempering this emphasis on procedures with deeper conceptual understanding is one of the central goals of curricular reforms such as the CCH applied calculus.

## **Mathematics Teaching**

Having discussed major views on the learning of mathematics, we turn now to research about the teaching of mathematics. There are three areas in what we know about the teaching of mathematics that are of primary interest: procedures vs. concepts, methods, and technology.

### Procedures vs. Concepts

If we accept the notion that mathematical knowledge is made up of procedures and concepts, then it seems clear that the teaching of mathematics should focus on both types of knowledge. Yet historically, math teachers have tended to focus mainly on the procedures. “Indeed, a study of college calculus syllabi and final exams from 1945 to 1990 found that 65 to 75 percent of the first semester calculus exam items could be categorized as symbol manipulations and calculations requiring little deep thought.... (Dubinsky and Ralston, 1992)” (Schoenfeld, 1997, p. 14). We should not be surprised, therefore, that deep conceptual understanding of mathematics is lacking in students. As Francis concluded,

...a variety of starting points, both procedural and conceptual, appears to provide the path to greater student learning. Therefore, textbooks and teachers need to



employ multiple problem representations on a regular basis. Further research is needed to uncover the conditions under which a particular format will prove most effective for learning. (Francis, 1992, p. 86)

It may well be that no particular format can answer this call, but that multiple methods will always be required.

Part of the trouble experienced by students of mathematics stems from the various ways in which teachers of mathematics react to the gap they find between their understanding and that of their students. Calculus professors generally react to this gap in one of two ways: either they ignore it, stating definitions and proving theorems, expecting students to fill in the gap and understand them; or, they

expose the basic concepts of calculus at a moderately high level, emphasizing intuition, but focus primarily on skills, and only test students on their ability to perform certain mechanical computations in response to certain verbal cues.....never asking students to do anything on an exam that requires genuine knowledge of concepts. (Epp, 1986, pp. 47-48)

Epp goes on to state that examples of poor student performance given by college departments serviced by the calculus courses involve poor thinking, not just inadequate computational skills (Epp, 1986, p. 48). Perhaps this is due to the emphasis instructors place on knowing the right algorithm to apply to a given problem rather than on how to formulate the problem; students tend to wade through all the “why” information looking for the “how to” information, which they are generally required to know for exams (Crosswhite, 1987, pp. 268-269). The CCH text represents a move towards the “why.”



Some argue that teaching of concepts is actually more important than teaching procedures. This seems to be the point of view of the CCH authors as well, and this idea seems to make even more sense in relation to applied calculus than to theoretical calculus. In speaking of biology students who take calculus, van der Vaart stated: "Memorization of a large number of derivatives and indefinite integrals will not do them nearly as much good as the actual derivation (not necessarily a rigorous proof, but an argument that illustrates the concepts) of a few well-chosen examples. Use of graphical methods is strongly recommended" (van der Vaart, 1986, p. 223).

Meanwhile, the argument against teaching strictly procedural information seems clear: "...the mere accumulation of additional factual knowledge is unlikely to improve nonroutine problem-solving ability" (Selden, Selden & Mason, 1994, p. 25). Rodi (1986) also said it nicely when he stated that the study of calculus should lead its students to comprehension of limits, derivatives, and integrals, but it should also lead to knowledge of how these ideas are "pertinent to the solution of a whole range of important problems. Understanding, not regurgitation, is the goal" (Rodi, 1986, p. 121).

Finally, as will be seen in the technology section, the current trends toward use of graphing calculators and computers in calculus classes implies a greater need for conceptual understanding as procedures are left more and more to technology (Epp, 1986).

#### Teacher-Directed vs. Student-Directed

Perhaps the toughest battle being fought in the arena of teaching methods is that of teacher-directed learning vs. student-directed learning. In the former, the teacher is in

control of the classroom, directing everything that is done, supposedly imparting knowledge to the avid students. The typical scenario that springs to mind is the traditional lecture, with the professor standing at the blackboard writing theorems out of a textbook which the students frantically copy into their notebooks. In the latter, the teacher relinquishes control to the students, who, often working with each other in small groups, discover ideas on their own.

Schoenfeld gives three main reasons for endorsing group work: professionally, students will probably work with others rather than alone; group work allows the tackling of more difficult problems; and group work can lead to individual learning (Schoenfeld, 1997, p. 31). Using small group work in calculus courses, Davidson (1990b) found that "...students were successful with limited guidance in making conjectures, proving the main theorems of calculus, developing techniques for solving various classes of problems, and in coming up with problem solutions and proofs not previously known to the teacher" (pp. 348-349). Students themselves have reacted positively in general to working in small groups (Rosenzweig, 1994).

Perhaps somewhere between the two extremes of lecture and group work is the best ground on which instructors should stand, especially if different students learn best in different ways. Here, as in the procedures vs. concepts debate, it may be that a good mixture is the best way to proceed:

Recent results in intellectual development seem to point to different styles with which instructors should be familiar so that they can design learning experiences

which match, or mismatch, students' styles, depending on the instructor's intentions. (Becker & Pence, 1994, p. 7)

Becker and Pence go on to say that studies have shown that the kind of teaching which works best is clear, sensitive to students, and involves cooperative learning (Becker & Pence, 1994, p. 11).

Epp makes a strong argument for group or student-directed learning when she notes that attending lectures and completing exercises in grammar are not sufficient to teach one a foreign language. Students who would learn a foreign language well "...have to make fools of themselves by talking out loud. The same goes for learning the language of mathematics" (Epp, 1986, pp. 56-57). She notes also that instructors would do well to let their students discuss questions with each other sometimes rather than jumping in with a quick answer.

It also appears that the way teaching is done is at least as important and often more important than the actual content that is being taught (Renz, 1986b; Ganter, 1998; MAA, 1998, Part 2, p. 1).

### College-level Difficulties

The teaching of college-level mathematics is made more difficult by the nature of the subject matter itself. Calculus in particular, with its dependence on functions, limits, and infinity, poses cognitive difficulties for students (Tall, 1992). Cognitive obstacles in calculus for students are primarily linguistic and intuitive (such as pointy graphs being thought non-functions) (Norman & Prichard, 1994).



We must not forget that calculus is a difficult subject. As Douglas points out, “the calculus student is attempting to understand the culminations of humankind’s grappling with the notions of limit and of the continuum. This stretches and enlarges the students’ physical and geometrical intuition” (Douglas, 1986a, p. 4).

One of the reasons that learning mathematics tends to be difficult for students is that they bring with them their own intuitions and misconceptions, from basic concepts to limits and proofs (Becker & Pence, 1994, pp. 8-9).

Selden, Selden and Mason (1994) selected 19 calculus students to whom they gave two tests: one which required the nontrivial application and combination of calculus concepts and skills, and another which contained problems requiring simple, separate use of the same concepts and skills. They found that students who had the knowledge necessary to solve the nontrivial problems often could not because they couldn’t apply their knowledge creatively. They conclude that traditional calculus teaching does not give students sufficient preparation in this area, partly because traditional courses don’t include many problems of a “cognitively nontrivial” nature (p. 19).

Ferrini-Mundy and Graham (1994) interviewed a group of calculus students and discovered that they held competing views of ideas from calculus in several different areas. For example, there is not harmony between algebraic and graphical representations, nor is there agreement in general between computational procedures and the concepts behind them (in other words, facility with an algorithm is not equivalent to an understanding of why the algorithm works). They also observed conflicting

conclusions about calculus, some of them formal and some of them “idiosyncratic” or “personal.” This happens because students build their own knowledge upon the foundation of their own prior experience, transforming it into incorrect notions.

### Technology

Many involved with the reforming of calculus instruction believe that technology is here to stay; in other words, students would do well to learn how to use technology correctly because they will likely always have access to calculators and computers (Schoenfeld, 1997, p. 10). But in order to use technology effectively, students must come to understand the concepts behind the calculations:

...the main requirement to use calculus packages effectively is firm conceptual understanding of the subject matter. With computers to take care of mechanical details, the premium is on the abilities to abstract, to infer, and to translate back and forth between formal mathematics and real world problems. (Epp, 1986, p. 49)

Francis (1992) also concluded that greater conceptual understanding and hence better teaching of concepts are needed in the wake of technological advances (p. 92). In addition, although technology may never completely replace pencil and paper, using technology is “an indispensable companion” for those investigating numerical methods (Lax, 1986, p. 71). Technology also changes course content since it enables students to do more (Renz, 1986a, pp. 104-105).

One apparent advantage of accompanying the teaching of calculus with technology use is that student misconceptions can be more readily discerned (and hence

better corrected) by the instructor. Judson (1992) explains that when student computations go awry, students often blame their own mistakes for the unexpected answers, whereas unexpected answers from computers lead students to make bizarre explanations which point out misconceptions they have about calculus ideas (p. 94).

More research needs to be done on using computers in classes, but results to date point to better conceptual understanding on the part of students (Becker & Pence, 1994, p. 9; Cooley, 1996; Heid, 1988; Judson, 1988).

### **Calculus Reform**

To speak of mathematics education reform at the collegiate level is practically synonymous with speaking of calculus reform. Although subjects other than calculus are being taught in new ways, the majority of reform efforts involve calculus. As time goes on, the calculus reform effort is becoming more widespread. Not only are calculus courses the most common kind of reformed courses, but reformed calculus courses comprise an ever-larger percentage of all calculus courses. Tucker and Leitzel estimated in 1995 that 150,000 students or more, which represents about one-third of all calculus students, were in reformed classes of some sort (Tucker and Leitzel, 1995, p. 5). To understand why this is so, we must look at the history of calculus reform.

### History

The emphasis on calculus in college-level math education reforms arose historically out of general dissatisfaction with the way calculus was being taught as well as an effort on the part of some to remove calculus from its place as the core of



introductory college mathematics and replace it with discrete math courses (Schoenfeld, 1997, p. 2). Calculus classes were generally small before the late 1950's, when Russian technological advances spurred a huge growth in calculus enrollment--which, in turn, led to large lecture sections and growing dissatisfaction with calculus teaching in the 1960's (Teles, 1992). Some reform efforts ensued which were similar to other modern reforms in math education in that they "emphasized thinking; conceptual understanding; interpersonal communication; and active, collaborative, or mastery learning....In contrast...to many of the reform calculus courses of today..., content of these courses between 1958 and 1987 varied little from that of traditional courses" (Teles, 1992, p. 225).

Current calculus reform can trace its roots to the Tulane Conference of 1986, where a group of mathematicians gathered under the direction of Ronald Douglas to chart a course for national calculus reform (Douglas, 1986b). This conference included workshops on what calculus content should be taught, the methods that should be used to teach it, and how all these changes should be implemented across the country. The content working group even came up with sample syllabi for this new "lean and lively" calculus. Participants in the Tulane conference reached a general consensus that calculus needed fewer topics and that it should emphasize concepts more (Douglas, 1986b, p. v). The methods workshop reported that motivation to change could be gained by noting the following common complaints: students were being driven out of mathematical career paths by traditional teaching practices; treatment of the subject matter was superficial; expectations of both teachers and students were too narrow; tenured faculty members

paid the teaching of calculus too little attention; departments other than mathematics served by the calculus courses were dissatisfied; and, finally, nearly half of the students taking calculus would never need to use the kind of calculus they were being taught (Davis, et al., 1986, p. xv).

The Tulane conference generated great interest across the country in calculus reform. The National Science Foundation began funding calculus reform projects in 1988; by 1994 the number of these projects had reached 127 (Tucker & Leitzel, 1995). Susan Ganter (1998) reported that “More than 500 mathematics departments at postsecondary institutions nationwide are currently implementing some level of calculus reform” (Ganter, 1998, p. 1). Ganter points out the need for evaluation of reform projects, stating that growing uncertainty about their effectiveness may lead to their abandonment. Ganter herself conducted a study to document the “impact of these efforts on student learning, faculty and student attitudes, and the general environment at undergraduate institutions” (Ganter, 1998, p. 3). Her findings are summarized along with the findings of others in the next section.

### Recent Studies

Much has happened in calculus reform since 1986. Yet the number of published research studies investigating the effectiveness of reformed curricula is relatively small in comparison to the number of testimonials recommending use of reformed calculus curricula. The research that has been done has been mainly in the form of doctoral dissertations and master’s theses (Ganter, 1998, p. 2). Some of these have sought to



conceptual and procedural measures, although the curriculum was not fully implemented as intended. Hadfield (1996) found that CCH students understood concepts better and procedures as well as students of a traditional method. In contrast, Brunett (1996) showed that a group of students who completed a traditional calculus course performed significantly better on a problem solving test than a group of students who completed a CCH course.

Taken together, what do these results mean? It appears that the results are mixed. Ganter (1998), in her analysis of all 127 NSF-funded calculus reform projects, concluded that reforms have had an overall positive effect on student achievement, with some negative results. Technology use, a part of 90 percent of the projects, was the most common element of reform (p. 4). In general, those projects involving technology led students to increased conceptual understanding in comparison with traditional curricula; levels of procedural understanding achieved in reformed courses were usually the same as or slightly lower than those achieved in traditional courses (pp. 6-7). Ganter reported further that the students most likely to do well in reformed calculus courses using long-term projects or group work were above average mathematics students, those who do poorly on traditional tests, and engineering majors (p. 7); this agrees in part with the results of Ratay (1993), who found that weaker students benefited most from reformed calculus. In summarizing student achievement results, Ganter says,

Clearly, student achievement has been affected by the calculus reform efforts.

What is perhaps less clear is the degree to which achievement has been affected and the appropriate 'mixture' of reform ideas that should be implemented at



various institution types to achieve the greatest positive effect. (Ganter, 1998, p. 8)

In light of this, it is reasonable (although certainly not guaranteed) that some positive effect will be observed in studies such as the present one.

It is interesting to note that students generally either love or hate reformed calculus, with the students most successful in previous math courses hating it the most and not many opinions falling between these two extremes (Ganter, 1998, p. 8).

Because student and faculty attitudes both improve when modifications are made to reformed courses on the basis of student input, Ganter concludes that “the instructor and the instructional methods--not reform textbooks--are the real crux of reform” (p. 10).

This implies that the impact which a textbook (such as the CCH applied calculus text) can have on the success of calculus reform lies in the degree to which it can induce positive changes in how instructors teach.

### CHAPTER 3: METHODOLOGY

The goal of this study was to gain insight into the effectiveness of one popular reformed applied calculus curriculum. The particular curriculum chosen was the textbook “Applied Calculus for Business, Social Sciences, and Life Sciences” published by the Calculus Consortium based at Harvard (CCH). To learn about its effectiveness, we compared the CCH applied calculus course with a traditional applied calculus course. The traditional course used the textbook “Calculus and Its Applications” by Goldstein, Lay and Schneider (GLS).

There are several ways in one might try to determine the success of particular mathematics curricula. But before this can be done, one must define the meaning of success for the curricula in question. The natural way to do this is to analyze the content of the particular courses and define success to be good understanding and ability in those areas. To define success for students of applied calculus, it also makes sense to look to the departments of the client disciplines. There are differences of opinion, however, on the value of calculus in business and social sciences. No matter how carefully a curriculum is tailored, the fact remains that on many campuses, the client disciplines do not often require their students to use specific concepts or techniques of calculus. For example, van der Vaart noted that “...biology students rarely see an application of calculus in their undergraduate biology courses” (van der Vaart, 1986, p. 213). Disciplines that require study of applied calculus see the course more as a way of giving students mathematical maturity. In other words, they don’t care about specific calculus

skills to be learned per se, but they do want their students to have the mathematical maturity that taking a course in applied calculus presumably brings about. For example, the authors of the CCH Applied Calculus note the following in their book:

We began work on this book by talking to faculty in business, economics, biology, and a wide range of other fields as well as to many mathematicians who teach applied calculus. As a result of these discussions we included some new topics...and omitted some traditional topics... (Hughes-Hallett, et al. 1996, p. viii)

Yet at the university where this study was conducted, math department faculty once asked the client disciplines about how calculus concepts and skills are used in their coursework and got the general response that they don't really use calculus at all. What, then, will define success for applied calculus curricula for this study? To answer this question, we must turn to the chosen texts.

### **The Texts**

Although the texts chosen for this study contain essentially the same material, they do differ in philosophy. The CCH book includes a long preface in which fairly specific goals for student learning are listed, whereas the GLS preface is more general in its discussion of the topics. A content analysis (see Appendix A) shows that both books deal largely with the same procedures and concepts, yet their approaches differ. The CCH authors tend to use applications to introduce and motivate a concept or skill, treating the application as the reason for calculus as well as the end result of using calculus (Hughes-Hallett, et al. 1996, p. vii). The GLS text often presents facts or theoretical ideas first and embodies them in applications thereafter: "As we shall see



later, the concept of a derivative occurs frequently in applications” (Goldstein, Lay & Schneider, 1996, p. 92). The CCH text seems to lead students along gradually, spending more time on the development of the reasoning behind calculus; the GLS text develops the calculus concepts and techniques more quickly, allowing less time for student intuition and exploration to develop. The CCH text is designed to cover fewer topics in greater depth.

Another difference between the two approaches to calculus is the number of approaches to each topic offered to students. The CCH book expressly states its goal of treating topics in four different ways (“the rule of four”): geometrically, numerically, algebraically, and verbally (Hughes-Hallett, et al. 1996, p. vii). The GLS text also uses all four of these ways at various points in time, but in a much less focused way. Finally, the CCH text is aimed more at student activity than the GLS text. In traditional courses, large lecture sections with little interaction between teachers and students are commonplace (Douglas, 1986a, p. 5). The CCH authors, however, encourage teacher/student interaction and interaction between students (Garner and Hughes-Hallett, 1996, pp. 5-6).

### **Study Site**

The subjects chosen for this study were students at a large private university located in the western U.S. Although one reason for this choice was convenience and accessibility, it also made sense for several other reasons. The first of these was that the mathematics department was experimenting with applied calculus reform and had used the CCH Applied Calculus textbook. The traditional textbook in use at the university

was the GLS text. As mentioned in chapter one, these two texts are good choices for comparison. The fact that students of both methods were found at one university greatly simplified the sampling of students. It allowed the sampling to be completed at just one university rather than two. This also meant that the two groups of students would be more similar in other ways than groups from two different universities would be; thus any observed differences between the groups could be attributed more readily to the different applied calculus courses they took.

The university research site also had a large population of students of reformed and traditional methods (approximately 1700 students who completed their applied calculus course during the 1996-1997 school year<sup>1</sup>); this offered good chances for obtaining a reasonably large sample. The fact that this large population of students took applied calculus not just days but rather a few months prior to the time the study was to be conducted offered an opportunity to compare the traditional and reformed methods on the critical measure of long-term retention.

All students at the university research site have been given computer accounts for electronic mail, so they were easily and inexpensively contacted. The university also has a testing center where most exams at the university are administered. This allows students to complete required tests from their courses on their own time and makes more class time available for instructors. This testing center was utilized in the collection of

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<sup>1</sup> Roughly half the students took reformed and half traditional sections; this is based on the class schedule for the Winter (second) semester of the 1996-1997 school year, in which four sections of traditional and 4 sections of reformed applied calculus were listed. The listing simply stated which professor would be teaching each section, and students were asked to call the department office for information about whether specific calculators would be required.

data; it automatically provided a way of testing students under controlled circumstances without bringing them together all at one time.

### **Instruments**

The natural question which arises in the mind of anyone learning about an educational reform is “Does the reformed curriculum work better?” If reforms are to be considered worthwhile, they must be shown to be valuable; they must offer something the old ways did not. They must be shown effective in order to merit widespread adoption. Thus one thrust of this study was to perform some quantitative comparison between the understanding and performance on the part of students in the reformed course and that of their counterparts in the traditional course. This quantitative comparison came via a written test with 10 constructed response questions and problems.

But quantitative comparison alone cannot answer the research questions adequately. In order to learn about how well students understand mathematical ideas, one must consider more than test results. Insight was needed into how students were thinking and why they thought the way they did. As Schoenfeld notes, “Interviews...with students are a source of rich information about their understandings and feelings about calculus....[and] can also be used for research purposes” (Schoenfeld, 1997, p. 18). Hence another thrust of this study was qualitative investigation of student conceptions. What students say about calculus and how they learn and use it can help us better understand their performance on written tests. Taken together, quantitative and



qualitative data give us a clearer picture of the mathematical understandings and abilities than either kind of data could give alone.

Subsequent to the content analysis of the two textbooks, two measurement instruments were designed; the written test and the interview can be found in Appendices B and C, respectively. These were designed to cover the main thrusts of both applied calculus courses under study. Both texts were also checked to insure that each included problems similar to those on the written test.

The written assessment instrument was designed to include items in three categories: primarily conceptual, primarily procedural, or both conceptual and procedural. Items were structured in this fashion because the interplay of conceptual and procedural knowledge was of primary interest in the study; more and deeper conceptual understanding is the main thrust of the CCH reform.

The test and the interview together cover the description of graphs of functions, differentiation of functions both graphically and analytically, interpretation of the derivative (mostly in economic situations), integration of functions both graphically and analytically, interpretation of the definite integral, and the relationship between the derivative and the definite integral (the fundamental theorem of calculus). The interview also includes questions about motivation and satisfaction with the applied calculus course.

Both the test and the interview were piloted with University of Maryland students (math students for the test, math education students who had taken at least one semester of calculus for the interview) in an effort to refine their clarity and to determine

whether they assessed the desired concepts and procedures. After the piloting, slight modifications were made in the wording of questions and the appearance of graphs.

### Test Item Selection

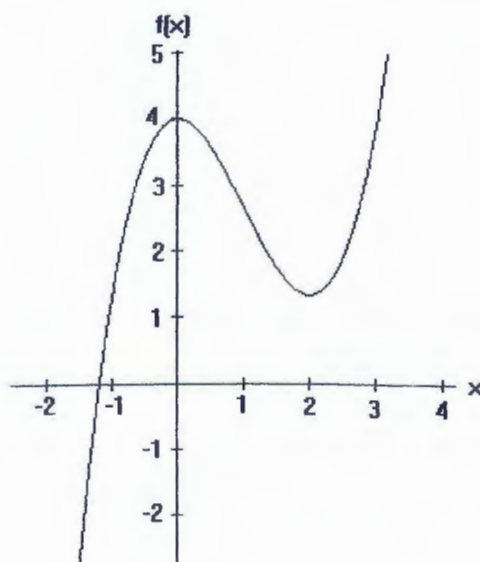
To give the reader a better understanding of the composition of the written test, we next present an item-by-item explanation of how and why each test item was chosen. Most of the items were constructed without particular reference to any example or exercise found in the textbooks, but at the same time they were designed to be both conceptually and procedurally like those in the textbooks.

In light of the focus of this study on retention and the fact that the test was to be given seven months after the students completed their applied calculus course, an effort was made to keep many of the test items simple. A test on which everyone scored poorly would leave little room for differentiating between groups of students.

*Item 1.*

Given below is the graph of a function.

- a. On what intervals is  $f(x)$  increasing? Decreasing?
- b. Estimate any points at which a local maximum or a local minimum occurs.



**Figure 3.1 Graph for Test Item 1**

The first item asks students to identify on a graph intervals where a function is increasing and intervals where it is decreasing as well as where local maxima and minima are located (see Figure 3.1). This kind of analysis of graphs is a major topic in any calculus course. Time is spent analyzing function graphs before using derivatives in association with the graphs so that students will understand what information the derivatives are going to provide; then function graphs are analyzed again with derivatives



so that the students can see that calculus does indeed supply desired information about graphs and can even help one create graphs. Item 1 was designed to assess the graph-analyzing capabilities of the students without derivatives. It constitutes a good first question in that it takes students back into the realm of calculus in a gradual way, hopefully acclimating them to the test and preparing them for what is to come. The particular function whose graph appears in item 1 ( $f(x) = \frac{2}{3}x^3 - 2x^2 + 4$ ) was chosen such that the demarcations between the increasing and decreasing intervals would fall at integer values of  $x$  so that the correct answers could be clearly viewed and easily obtained.

*Item 2.*

Consider the function:  $f(x) = \frac{2}{3}x^3 - 2x^2 + 4$ .

- a. Using calculus, determine whether the above function is increasing or decreasing at  $x = 1$ .
- b. Also using calculus, determine any values at which a local maximum or a local minimum occurs.

Now that students have analyzed a graph, the next task is to let calculus answer the same questions asked in item 1. Part (a) is slightly more focused than part (a) of item 1, in that students need to determine the sign of the slope only at one point on the graph. It is conceptually more challenging in that the right application of the derivative must be made. Part (b) asks students to use calculus to find places where local maxima and minima occur. This part too is conceptually more difficult than its counterpart in item 1, yet these applications of the derivative are perhaps the most common (and likely best remembered). The function given is the same function used in item 1, which (if

recognized) gave students a way to check their work against the graph. The fact that the function's local extrema occur at integer values of  $x$  makes the arithmetic involved in differentiating simple, hopefully reducing student errors.

*Item 3.*

Find the first derivatives of the following functions:

a.  $y = (3x^2 + 1)^2$

b.  $y = xe^{2x}$

Item 3 is a procedural question, asking students to perform some simple differentiation. The function in part (a) requires use of the chain rule, and part (b) requires the product rule. Once again, the functions were chosen to be fairly simple in the hope that many students would remember how to differentiate them.

*Item 4.*

A candy company knows that the revenue,  $R$ , from sales of a certain product is a function of the selling price,  $p$ , and is given by  $R = -500p^2 + 4500p$ . What is the maximum revenue? ( $R$  and  $p$  are in dollars).

The inclusion of a word problem about the maximization of a function was essential. Again a fairly simple function was chosen. The goal of this question was to require students to follow through with necessary steps to obtain a correct answer after computing the derivative. In other words, it's not just finding the critical value that is of interest; students need to be able to understand how the critical value is important to the original problem and how to use that value to answer another question.

*Item 5.*

A publishing company knows that the cost (in dollars) of printing a book depends on the number of copies printed. The cost of printing  $N$  copies is  $C = f(N)$ . Suppose that when  $N = 500$ , the derivative of the function is 2 (that is,  $f'(500) = 2$ ). What conclusions can you draw?

The goal of this item was to get students to explain that the derivative was marginal cost: the approximate cost of printing the 501st book is \$2. Marginal cost was one of the main business applications found in both the GLS text and the CCH text. This item is very open and conceptual, requiring students to describe their understanding about the meaning of the derivative. The particular values employed here, although not very important in answering the question, were chosen to be reasonable for the situation described in the problem.

*Item 6.*

A compact car in motion on the freeway uses  $f(x)$  gallons of gas in driving  $x$  miles.

- a. For what values of  $x$ , if any, is  $f'(x)$  positive?  
For what values of  $x$ , if any, is  $f'(x)$  negative?
- b. A motorhome uses  $g(x)$  gallons of gas in driving  $x$  miles. How would you expect the graphs of  $f(x)$  and  $g(x)$  to compare?

This problem was adapted from one found in the GLS text (see Chapter 1 Supplementary Exercise number 73 of Goldstein, Lay & Schneider, 1996). Conceptual in nature, it requires students to think about the sign and magnitude of the derivative of a function from every-day life. The goal was twofold. First, students hopefully would realize that the slope of  $f(x)$  was positive for all positive  $x$  (indeed, the function is only

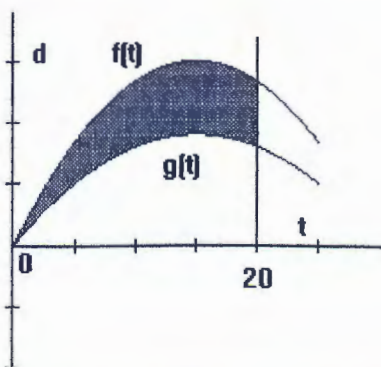


defined for positive  $x$ ). Second, we hoped students would make the inference that the motorhome used more gas, and thus its graph would be above the other graph.

*Item 7.*

Suppose that on a typical day the rate at which the electric company accumulates revenue (in dollars per minute) is given for any time in the day by the function  $f(t)$ . The rate at which the company's operating costs accumulate (in dollars per minute) is given for any time in the day by the function  $g(t)$ . These functions are graphed below.

- How could you use calculus to find the company's revenue and cost functions?
- How could you use calculus to compute the shaded area?
- What does the shaded area represent economically?



**Figure 3.2 Graph for Test Item 7**

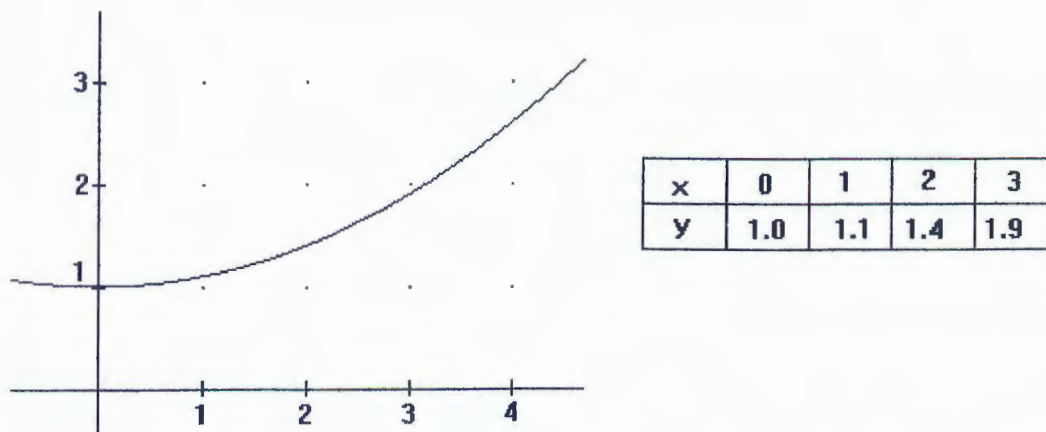
This item was perhaps the most difficult one on the test. The goal in designing it was threefold: to require students to explain how antidifferentiation can transform one useful function into another; to find out if they remembered how definite integrals can be used to compute area; and to require them to give the meaning of the area of a shaded portion of a graph in economical terms. All these are conceptual ideas important in applied calculus. The thing that made this problem more difficult is that it is somewhat

nonconventional in presenting the graphs of rate of revenue and rate of cost functions (as opposed to the graphs of revenue and cost functions).

*Item 8.*

Given the following graph and table of values for a function,  $f(x)$ ,

- Approximate the derivative of the function at  $x = 2$ .
- Approximate the integral of the function from  $x = 1$  to  $x = 3$ .



**Figure 3.3 Graph and Function Table for Test Item 8**

In this item the student's ability to estimate derivatives and integrals was tested. The problem gives a graph and a table of  $x$  and  $y$  values, but no function rule. For part (a), the hope was that they would use the formula for the slope of the line joining the point (2, 1.4) to either (1, 1.1) or (3, 1.9) to get an estimate for the derivative at  $x = 2$ . For part (b), the hope was that they would add up areas approximating the area under the function curve.

*Item 9.*

Compute the following definite integrals:

a.  $\int_1^2 4x^3 dx$

b.  $\int_0^{1/2} e^{2x} dx$

This item, requiring computation of simple definite integrals, was the counterpart to item 3, which required computation of simple derivatives. The functions chosen ( $4x^3$  and  $e^{2x}$ ) called upon knowledge of integrating with the power rule and exponentials, both very common in either brand of applied calculus.

*Item 10.*

There is a theorem, called the Fundamental Theorem of Calculus, which is a statement about the relationship between the derivative and the definite integral. What is this theorem?

The connection between the derivative and the integral was the subject of this test item. We hoped students would remember something related to the fundamental theorem of calculus, since that is essentially what enables us to integrate functions easily. It was postulated that not many students from either course would be able to state this theorem entirely correctly, since most applied calculus classes focus on it only briefly. Yet it would be enlightening to see what students remembered about the relationship of the derivative and the definite integral, the two central ideas of calculus, after the passage of time.

A few of the items on the written test were of a type more heavily emphasized by one course or the other, but each type was covered in both texts. This is no cause for



alarm, as it is of interest to find out whether the problems more heavily emphasized in a particular course actually were completed more successfully by those students in that course.

For purposes of later analysis, the questions were also grouped into three categories: primarily conceptual, primarily procedural, and both conceptual and procedural. The breakdown of the items into these three categories is shown in Table 3.1 below.

|              | Conceptual     | Procedural | Both Conceptual and Procedural |
|--------------|----------------|------------|--------------------------------|
| Item Numbers | 1, 5, 6, 7, 10 | 2, 3, 9    | 4, 8                           |

**Table 3.1 Categorization of Test Items**

### Interview Question Selection

In this section we offer reasons for the selection of the main types of questions included in the interviews. Two groups of questions were chosen for the student interviews; the first group included questions about the students' motivation and attitudes, and the second group was comprised of primarily conceptual questions about calculus itself. For individual questions in each group, follow-on questions were devised which might spark a stalled conversation or trigger a student's memory. Our goal was to "...provide sufficient cues in [our] questions to maximize the chances that students [would] retrieve the [desired] information" (Byrnes, 1996, p. 56).

The first group of questions (subtitled the motivation/affect portion of the interview) asked about the classroom experience in each course (see the list below).

*Affect/motivation portion.*

1. What was a typical class like? Follow-up: Was there enough time for questions?
2. How much homework was given?
3. Why did you take this particular section of applied calculus? Follow-up: Did you have a choice?
4. Did you like your applied calculus class? Follow-up: What did you like best?
5. What did you think of the way the course was taught?
6. How would you rate the instructor? Follow-up: What made the instructor “good?” Any improvements needed? Would you say that your instructor taught in a way that matched the goals of the textbook? Did the teaching style agree with your learning style?
7. How does this course compare to other math classes you've had? Follow-up: Did you work as hard for this class as for other math classes you've had?
8. What would you change about your applied calculus class? (Possible follow-on points: The teaching? The homework? The book? Etc.)
9. Would you recommend your applied calculus class to a friend? Follow-up: Would you take it again? Is there anything else you'd like to say?

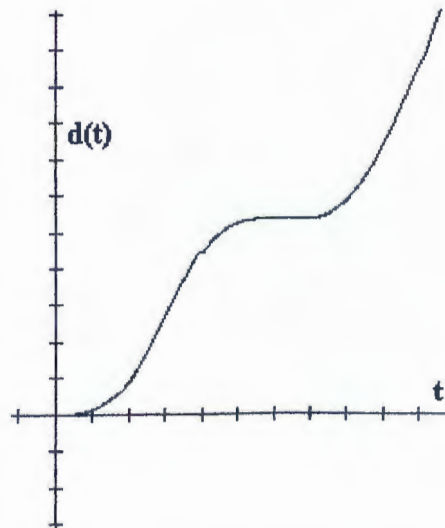
There were several reasons for including this group of questions. First, they offered a way of “breaking the ice” between the student and the interviewer; for this reason this group of questions came before the calculus questions. They allowed the researcher and the student to become comfortable in talking with each other, and they allowed the researcher to begin to understand how the student approached his or her applied calculus course. Second, they supplied valuable information about the sample of students (for example, whether they knew whether they would be using a reformed text). This information helped in painting a more accurate picture of the characteristics of the traditional and reformed groups of students. It allowed the researcher to describe better

how the two groups compared before they took their applied calculus course, so that possible differences in test and interview results could be explained better and properly attributed to the most likely causes. Third, these questions provided insight into the way the two different applied calculus courses were administered and taught; this also gave valuable comparison information. Not being able to choose the instructors for the courses, we wanted to find out as much as possible about the teaching of the courses in hopes that discovered differences would not cloud the interpretation of results from the tests and interviews.

The second group of questions allowed insight into the views and understandings of calculus held by the students. These included questions such as “How would you explain what derivatives/integrals are to your brother?” Students also were asked specific questions about graphs of various functions and their derivatives. These questions allowed discussion and probing into student interpretations and conceptions of derivatives and integrals and their applications that no written test could provide. In addition to offering direct answers to our research questions, student answers to these questions would provide a back-drop for understanding the results of the written test. A complete listing of the conceptual interview items follows (beginning on the next page) and is also included in Appendix C. Here the graphs accompany the questions; in the actual interviewing, the students saw only the graphs (on separate pages) while the researcher asked the questions verbally.

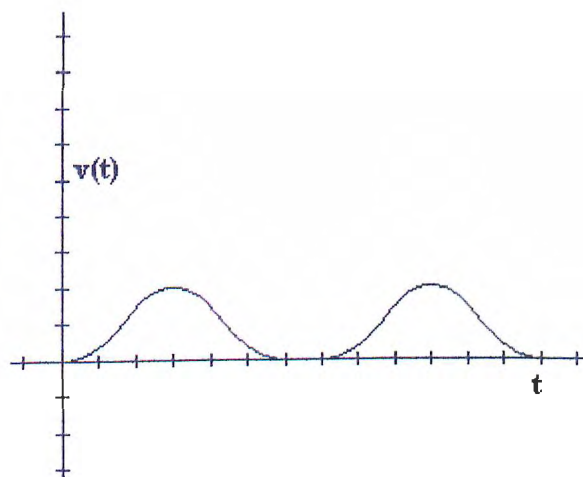


1. Suppose your brother understands what a function is, but has never heard of “the derivative.” How would you explain it to him? Possible follow-up cues: What is a derivative? Do you remember computing derivatives? If so, how do you do it? Do you remember what derivatives are were used for? Suppose I tell you that a derivative describes how something changes....
2. Suppose your roommate understands what a function is, but has never heard of “the integral.” How would you explain it to him/her? Possible follow-up cues: What is a definite integral? How is it used? What is an indefinite integral?
3. Below is a graph showing a car's distance traveled as a function of time. Choose a graph (of the three on the following page) which could represent the car's velocity over time and tell why. Possible follow-up cues: How is velocity related to distance traveled? Is there something you could say involving derivatives and/or integrals which would relate graphs a, b, and/or c to the first one?

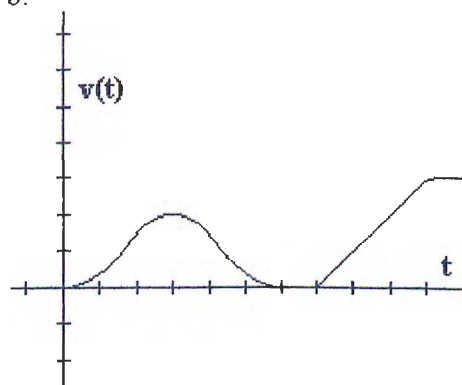


**Figure 3.4 Function Graph for Interview Conceptual Item 3**

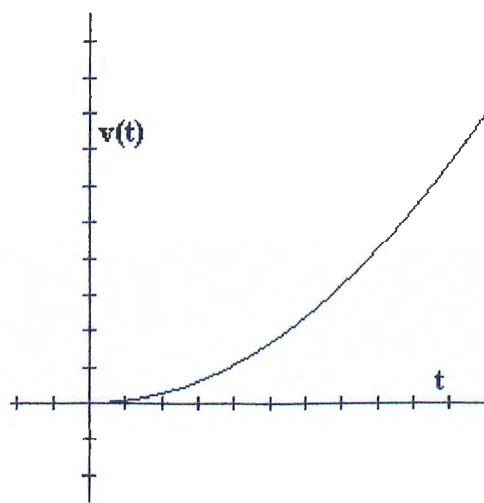
a.



b.

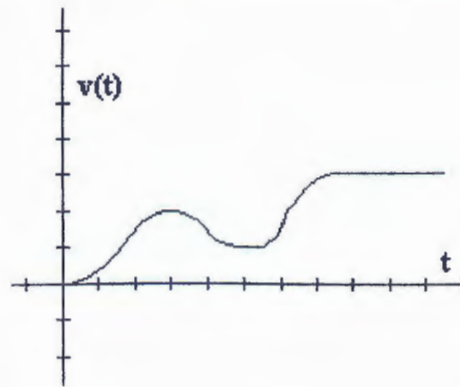


c.



**Figure 3.5 Derivative Graphs for Interview Conceptual Item 3**

4. Here is a graph showing the rate of travel of a car (i.e., velocity). What could I learn from the derivative? (Pause for response and discussion.) Follow-up cues: What could I learn from the derivative at a specific point? What could I learn from the general derivative of the function? What could I learn from the definite integral between two  $t$ -values? What could I learn from the integral in general?

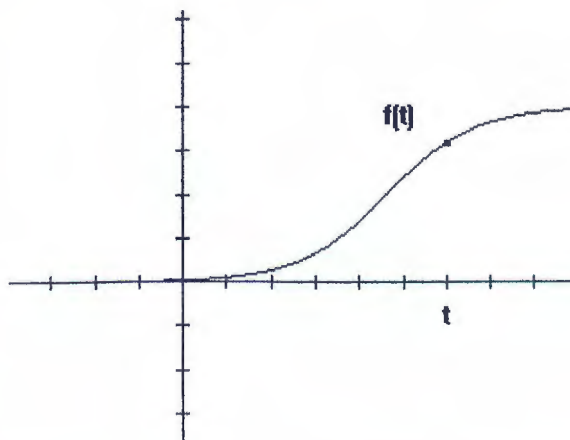


**Figure 3.6 Graph for Interview Conceptual Item 4**

5. How are integrals and derivatives related? Follow-up cues: Are they completely separate ideas, or is there something that connects them? Can you use both terms in a single statement?
6. Suppose I have a function of two variables,  $f(x,y)$ . What's the interpretation of the partial derivative of  $f$  with respect to  $x$ ? Possible follow-up cues: The partial derivative of  $f(x,y)$  with respect to  $x$  is written this way.... Do you remember computing partial derivatives?



7. Here is a graph showing how a population of wild animals varies over time. Suppose you know the rule for this function,  $y = f(t)$ , where  $f(t)$  is the number of animals at time  $t$ . How could you use the rule to determine rate of increase in the number of animals at a particular time  $t$ ? Follow-up cues: What (in calculus terms) would give you the rate of increase at a particular point?



**Figure 3.7 Graph for Interview Conceptual Item 7**

8. Here is the definition of something. Have you seen it? What does this tell us?

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

**Figure 3.8 Definition for Interview Conceptual Item 8**

9. For things you forgot, can you pinpoint a reason why you forgot them? Follow-up: the course, or your own study? Is there anything else you would like to say?

Interviews were also conducted with the three instructors whose students were involved in the study. The main purpose of these interviews was to help determine how the teaching was carried out in the applied calculus classes attended by the subjects. The assumption made in designing the study was that the instructors would be comfortable with and committed to a teaching style that was in harmony with the textbook they were using; this would lend credibility to study results. For example, in order for the CCH to be effective, it must be taught by someone who embraces the reformed pedagogy. Similarly, the traditional instructors needed to be committed to their text and its methods in order to be successful. The instructor interviews included questions about teaching style, the textbook, use of class time, technology use, etc. The instructors were also asked what they thought influenced the students in their choosing of an applied calculus course. In addition, the researcher observed each instructor teaching a class in order to more accurately categorize his teaching style. Results of the interviews (along with all other results) are given in Chapter 4.

### **Target Population**

Subjects were chosen from among those students who completed an applied calculus course at the university research site during the Winter semester (January through April) of 1997. This semester was one in which both the CCH text and the GLS text were used. Just students from this particular semester were chosen in order to insure greater comparability between students. Since applied calculus is usually taken early in one's collegiate career, the great majority of these students were still enrolled at the university for the Fall semester of 1997, when this study's data were collected.

Using students who had already completed the course had several advantages in addition to adding the retention aspect to the study, among which were the following: valuable class time was not be taken for data collection; no one knew this study was going to be done when they took their applied calculus course, so no one altered their studying of the material from what it normally would have been; and less lead time was necessary in coordinating the study with the instructors.

### **Data Collection**

#### Sampling

The careful sampling procedures which were developed for this study required the collaboration of a faculty member of the university mathematics department. Not only was that faculty member able to access the pertinent information needed to select the subjects, but he also completed the selection--concealing important characteristics of the subjects from the author, thus insuring a high degree of objectivity.

As mentioned earlier, it was determined that the easiest way to contact the greatest number of potential subjects for this study would be through the use of electronic mail. Using class rolls from the Winter semester of 1997, a list of potential subjects was assembled. Next an Internet directory containing e-mail addresses for university students was consulted. This process produced addresses for a total of 801 potential subjects. A message was sent to these students which briefly introduced them to the study and invited them to participate. This message also informed them that those selected to take the written test would receive \$10, and that those selected to participate



in both the test and the interview would receive \$30. After a few days, a second message similar to the first was sent in an effort to obtain more responses.

We desired to use only those students who had received a grade of A, B, or C in their applied calculus course so that we would be dealing with those who were actively involved and put forth a somewhat substantial effort. Among those responding to our solicitations were 58 students from the traditional course and 93 from the reformed course. Hypothesizing that the effect size of eventual differences found between the two groups would be medium (.25), we originally sought 64 students from each group to participate in the study. This is the number per group required to perform a one-way analysis of variance (ANOVA) with power = .80 and with  $\alpha = .05$  (Cohen, 1992, p. 158). Once we realized we would not be able to obtain 64 traditional students, we determined to invite all 58 responding traditional students to take the written test and to trim the sample of reformed students to 58 also. Having 50 students per group would still allow for power = .80 with  $\alpha = .10$  (Cohen, 1992, p. 158). For an exploratory study of this type, we decided that a .10 probability of Type I error is acceptable.

The grade distribution of the group of 58 traditional students who wanted to participate in our study was as follows: 29 A's, 16 B's, 13 C's. Of the 93 reformed students who wanted to participate, there were 29 A's, 49 B's, and 15 C's. The reformed B's were numbered in the order they were received, and a pseudo-random number generator (HP48G calculator) was used to randomly select 16 of the B's to include in the study. Similarly, the reformed C's were numbered and 2 of them were selected for exclusion from the study. This process resulted in a stratified sample with

comparable numbers of A's, B's, and C's (29, 16, and 13 respectively) in each of the reformed and traditional groups.

Based on available funds and estimated time required for interviewing, it was decided that 18 students (9 from each group) would be randomly selected from among the 116 students chosen for the study. In order to get the best results, only A and B students were chosen for interviews. Appropriate e-mail messages were sent to the various groups of students chosen; the messages told them when the test would be available and where, how to collect their money, and how to schedule an interview with the researcher.

Initially the written test was to be available to students over a six-day period at the testing center on campus. As testing and interviewing proceeded, some students dropped out. When possible, alternates were randomly selected from the groups of students previously excluded. Care was taken to choose alternates whose grades matched those of the students who dropped out. In only one case were we unable to substitute a student with a matching grade (a B student was used in place of an A student). In order to allow as many students as possible to take the test, it was kept available at the testing center for a total of 11 days, with the interviews being conducted during the last 3 days of the same time period.

The final sample of students who took the test was made up of 57 reformed course students and 51 traditional course students. The distribution by grade is shown in Table 3.2 below.

|                    | A  | B  | C  | Total |
|--------------------|----|----|----|-------|
| <b>Reformed</b>    | 27 | 16 | 13 | 57    |
| <b>Traditional</b> | 29 | 11 | 11 | 51    |

**Table 3.2 Sample Grade Distribution**

### Testing

Testing was accomplished at the testing center on the university campus. A cover sheet was placed on each test for recording a number and a student's name. The second page of each test (first page with problems) included the same number as on the first page. After the tests were completed and before they were graded, the cover sheet was removed so that the researcher could perform blind, unbiased grading. The testing center provided summaries containing name, test number, and elapsed time for each student who took the test. These summaries were used after the initial grading of the tests to match numbers with students.

### Interviewing

During the last three days the test was offered, the researcher interviewed a total of 16 students (8 traditional, 8 reformed). An empty office was made available in the math department for the interviews. One hour was allotted for each interview; most interviews lasted approximately thirty minutes. Students were allowed to use scratch paper if they desired; what they wrote (if anything) was kept along with a tape recording of each interview. Instructors were also interviewed for approximately 30 minutes each, and a tape recording of each interview was made.



As discussed earlier, the interviews served two main purposes: the obtaining of valuable sample control information and insight into the teaching and learning of applied calculus for the specific instructors and students involved in the study. In order to make the interviews easier to analyze, each one was transcribed.

## **Data Analysis**

### Grading

The grading of the written tests was accomplished in three phases. First, a grading key was prepared which gave in detail the answers expected and the number of points allotted for various responses. This key was used to grade all 108 tests. Each problem was graded in turn (all 108 solutions to problem 1, then all 108 solutions to problem 2, etc.). Then preliminary total scores were computed for each test.

The second phase of grading involved refining and clarifying the grading key. A subset of 10 tests was chosen (5 reformed, 5 traditional), the scores were concealed, and copies were distributed (along with the original key) to three other math education graduate students at the University of Maryland. Once their grading was complete, the author met separately with each one and compared scores, discussed any differences, and decided upon changes to be made in the key.

In phase three, the grading key was revised (see Appendix E for the final version) and all 108 tests were searched for places where scores would need to be changed due to the revisions in the key. Finally, scores were changed where it was deemed necessary and final scores for each test were recorded. The final scores reflect agreement rates of

82 percent, 88 percent, and 90 percent between the author and the other three graders. If scoring differences of one point or less are counted for agreement, the rates are 96 percent, 95 percent, and 99 percent.

### Statistics

Statistical comparisons were performed on the scores from the written tests in three separate ways. First, mean total test scores from the traditional and reformed groups were compared using standard one-way analysis of variance (ANOVA) procedures on the total scores of all students. For this and all other statistical inference testing, the SPSS/PC+ Studentware Plus 1.0 software was used.

Second, each of the ten test items was placed into one of three categories: primarily conceptual, primarily procedural (computational), or both conceptual and procedural. This classification (see Table 3.1, presented earlier) was accomplished based on a majority vote of five people: the author, the instructor of the reformed applied calculus course, and the three graduate students who served as supplementary graders. Each student was given three new scores: the first a total of all points earned on conceptual problems, the second a total of all points earned on the procedural problems, and the third a total of all points earned on problems in the “both” category. The ANOVA procedure was again performed, this time on the conceptual, procedural, and “both” averages from each group of students.

Third, a separate ANOVA was completed for the group means on each individual test item (a total of 18 separate tests, since most of the 10 problems had more than one part).

## Interviews

For the interview data, analysis was accomplished by categorizing responses for each question across interviews, and a summary sheet showing the frequencies of the most common responses for each was created. These summaries were then analyzed for trends, and the original transcripts were consulted again to verify these trends.

Particularly relevant or insightful quotes were also extracted.



## CHAPTER 4: RESULTS

The key questions of this study ask how well students of traditional and reformed courses understand concepts and procedures of applied calculus and retain that knowledge some time after the end of formal study. The written test and the interview employed in this study help to answer these questions. We turn first to the test results. We will present information about the scores, then we will discuss some particulars about student responses to the questions.

### Tests

Student scores on the written test are of interest on at least three different levels. The overall score is the first level; from this we can compute and compare distributions for the two groups of students. This will tell us if there is any difference between the groups, considering the whole content of the applied calculus courses (at least that portion covered by the test). Table 4.1 below summarizes the distribution of students' total test scores by group (a total of 67 points was possible).

|             | Mean    | Standard Deviation | Maximum | Minimum |
|-------------|---------|--------------------|---------|---------|
| Reformed    | 25.8860 | 12.3809            | 53      | 2       |
| Traditional | 25.1373 | 11.8030            | 50.5    | 5.5     |

**Table 4.1 Total Score Distributions Summary**

As one can tell from the table, there is not really much difference between the two groups when we consider overall scores. Indeed, the ANOVA analysis bears out the fact that there is not a significant difference of means. The effect size (difference between the means divided by the pooled estimate of the combined standard deviation) is .062 (six hundredths of one standard deviation). This effect size is so small that no conventional tables for computing desired sample size show this value. We can safely conclude that the number of students per sample required to find an existing difference of such small magnitude between the groups at a .10 confidence level is prohibitively large (aside from the fact that such a small difference is not interesting enough to pursue anyway). Table 4.2 below shows the results of the ANOVA analysis on the total scores from each group of students.

| <b>Source</b>         | <b>Sum of Squares</b> | <b>D. F.</b> | <b>Mean Square</b> | <b>F</b> | <b>Sig.</b> |
|-----------------------|-----------------------|--------------|--------------------|----------|-------------|
| <b>Between Groups</b> | 15.0886               | 1            | 15.0886            | .1029    | .7491       |
| <b>Within Groups</b>  | 15549.5480            | 106          | 146.6938           |          |             |
| <b>Total</b>          | 15564.6366            | 107          |                    |          |             |

**Table 4.2 Total Scores ANOVA, Reformed vs. Traditional**

The second level of interest in scores on the written test concerns a partitioning of the test items into three groups: conceptual, procedural, and a combination of both. As mentioned in chapter 3, a majority vote of five people (four graduate students and one professor) was used to place each question in its appropriate category. The result of

this categorization is shown in Table 4.3. Each student's total score was split into three subtotals, one for each of the categories of test items. This allows us to consider each subtotal as coming from a separate test. The two tables following 4.3 (4.4, 4.5) show the statistics describing the distributions by group of the conceptual and procedural subtotal scores, respectively (a total of 37 conceptual and 16 procedural points were possible). The score distributions for the portion of the test that was both procedural and conceptual are not presented; this category included only two items and no significant difference in the means occurred.

|                     | <b>Conceptual</b> | <b>Procedural</b> | <b>Both Conceptual and Procedural</b> |
|---------------------|-------------------|-------------------|---------------------------------------|
| <b>Item Numbers</b> | 1, 5, 6, 7, 10    | 2, 3, 9           | 4, 8                                  |

**Table 4.3 Categorization of Test Items**

|                    | <b>Mean</b> | <b>Standard Deviation</b> | <b>Maximum</b> | <b>Minimum</b> |
|--------------------|-------------|---------------------------|----------------|----------------|
| <b>Reformed</b>    | 15.1053     | 5.6693                    | 28             | 2              |
| <b>Traditional</b> | 12.6176     | 4.8452                    | 26.5           | 4.5            |

**Table 4.4 Conceptual Score Distributions Summary**

|                    | <b>Mean</b> | <b>Standard Deviation</b> | <b>Maximum</b> | <b>Minimum</b> |
|--------------------|-------------|---------------------------|----------------|----------------|
| <b>Reformed</b>    | 5.0526      | 4.7034                    | 16             | 0              |
| <b>Traditional</b> | 7.7255      | 5.3482                    | 16             | 0              |

**Table 4.5 Procedural Score Distributions Summary**



The most interesting results from statistical analysis of the test scores came from comparing the groups of students by their mean procedural and conceptual subtotals. We consider first the conceptual comparison, since the emphasis of the reformed curriculum was a deeper understanding of concepts. If there was any difference at all between our two groups of students, we expect it to manifest itself here. Looking at Table 4.4, we note that the reformed course students did score higher than the traditional course students; and, as shown in Table 4.6, this difference did turn out to be statistically significant. Recall that we tested with power .80 at the  $\alpha = .10$  level.

| Source         | Sum of Squares | D. F. | Mean Square | F      | Sig.  |
|----------------|----------------|-------|-------------|--------|-------|
| Between Groups | 166.5666       | 1     | 166.5666    | 5.9375 | .0165 |
| Within Groups  | 2973.6625      | 106   | 28.0534     |        |       |
| Total          | 3140.2291      | 107   |             |        |       |

**Table 4.6 Conceptual Scores ANOVA, Reformed vs. Traditional**

Similarly, we might expect any difference between the traditional course and reformed course students in computational performance to favor the traditional course. After all, the reformed curriculum claims to emphasize concepts, which suggests that it may not emphasize computation to the same extent as traditional instruction. Table 4.5 shows that the mean for the traditional course was higher; this difference also was shown to be statistically significant (see Table 4.7).

| Source         | Sum of Squares | D. F. | Mean Square | F      | Sig.  |
|----------------|----------------|-------|-------------|--------|-------|
| Between Groups | 192.2973       | 1     | 192.2973    | 7.6371 | .0067 |
| Within Groups  | 2668.9990      | 106   | 25.1792     |        |       |
| Total          | 2861.2963      | 107   |             |        |       |

**Table 4.7 Procedural Scores ANOVA, Reformed vs. Traditional**

The third level of interest in scores on the written test comes at the item level, i.e. when we consider differences between student groups on each individual item. As with the other tests for significance, one-way ANOVA procedures were used. As we shall see, the significant differences on conceptual and procedural subtotals was not due to significance on all the conceptual problems and all the procedural problems. Where differences are statistically significant, we give the value of the  $F$  statistic and its associated probability.<sup>1</sup> We present the results item by item, noting common mistakes students made as we go. We also note for each item the numbers of students from each group who received perfect scores on that item.

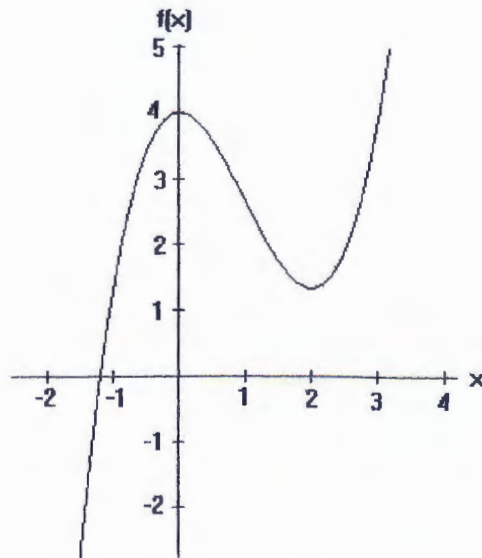
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<sup>1</sup> The reader is reminded that using multiple  $F$ -tests runs the risk of producing spurious significant results.

*Item 1.*

Given below is the graph of a function.

- a. On what intervals is  $f(x)$  increasing? Decreasing?
- b. Estimate any points at which a local maximum or a local minimum occurs.



**Figure 4.1 Graph for Test Item 1**

No significant difference was found on either part of the first item. There seemed to be general confusion about this problem; only 35 of 108 students (19 reformed, 16 traditional) received all points possible on part (a). In writing down the intervals on which the given function was increasing and decreasing, one common error students of both groups made was that of using  $y$ -values rather than  $x$ -values. For example, students wrote that  $f$  was increasing on the interval  $(-\infty, 4]$  when a local maximum existed at the



point (0, 4) and the response should have been the interval  $(-\infty, 0]$ . An equal number of students from each group (7) made this kind of error. For part (b) similar errors occurred, but students were generally more successful. A total of 73 students (37 reformed, 36 traditional) received all possible points.

*Item 2.*

Consider the function:  $f(x) = \frac{2}{3}x^3 - 2x^2 + 4$ .

- a. Using calculus, determine whether the above function is increasing or decreasing at  $x = 1$ .
- b. Also using calculus, determine any values at which a local maximum or a local minimum occurs.

This item was the same as item 1, except that the students were instructed to use calculus to find their answers. Part (a), which was the companion problem to part (a) of item 1, showed no statistical difference between the groups. The tally of students receiving full credit was 20 to 16 in favor of the traditional group. Statistical significance was evident on part (b), where the tally was 20 to 14 favoring the traditional group ( $F = 4.759, p = .031$ ). The difference occurred not because the reformed students couldn't find the extrema on the graph, but rather because they didn't always use calculus to do it. There were 10 students from the reformed group and just 1 from the traditional group who found both extrema by methods other than setting the derivative of the function equal to zero and solving for  $x$ . There were also 5 from the reformed group and 2 from the traditional group who found one of the two extrema without using calculus. If method didn't matter and these students had received full credit, the difference between

the groups may have vanished. The scores as they are show that 31 students from the traditional group earned full credit, whereas 17 from the reformed group earned full credit.

*Item 3.*

Find the first derivatives of the following functions:

a.  $y = (3x^2 + 1)^2$

b.  $y = xe^{2x}$

This item was purely computational. Part (a) required a simple application of the chain rule; part (b) required both the chain rule and the product rule. Students fared much better in general on part (a) than on part (b), although less than half the students received full credit on either part. Only 48 of 108 students (31 traditional, 17 reformed) correctly computed the derivative of  $y = (3x^2 + 1)^2$ , and only 19 of 108 students (13 traditional, 6 reformed) correctly computed the derivative of  $y = xe^{2x}$ . Obviously functions of the second type are more difficult for students to compute, and for neither type of function were the applicable differentiation rules remembered well. Differences between the groups' mean scores were statistically significant for both parts of this problem; in each case the traditional group performed better ( $F = 10.690$  and  $5.113$ ,  $p = .0015$  and  $.0258$ , respectively).

*Item 4.*

A candy company knows that the revenue,  $R$ , from sales of a certain product is a function of the selling price,  $p$ , and is given by  $R = -500p^2 + 4500p$ . What is the maximum revenue? ( $R$  and  $p$  are in dollars).

This item asked students to use calculus to maximize a revenue function. Only 37 students (21 reformed, 16 traditional) performed this correctly. The difference between the groups was not significant. However, as in part (b) of problem 2, there were some students from the reformed group (7) who found correct or nearly correct answers without using calculus. There was just one such student in the traditional group, so emphasis on correct answers rather than choice of method may have yielded a significant difference in favor of the reformed group of students. Common methods employed by those not receiving full credit involved simply plugging various  $x$ -values into the function or using a calculator.

*Item 5.*

A publishing company knows that the cost (in dollars) of printing a book depends on the number of copies printed. The cost of printing  $N$  copies is  $C = f(N)$ . Suppose that when  $N = 500$ , the derivative of the function is 2 (that is,  $f'(500) = 2$ ). What conclusions can you draw?

This item prompted students to explain the meaning of the numerical value of the derivative of a cost function at a certain point (the marginal cost). Few students answered this question correctly (10), but all of them were from the reformed group-- thus the difference between the groups was significant in that group's favor ( $F = 11.618$ ,  $p = .0009$ ; the most statistically significant of any difference found in this entire study). It is interesting to note that only 2 students in the traditional group received any points on this problem (13 from the reformed group received 3 or more of the 4 possible points). Some of the students' answers were hard to interpret, which was evidence that either they couldn't describe clearly or didn't understand completely the idea of marginal



cost. Most students had no insight at all, as evidenced by the fact that 91 of them received a score of zero on this problem. Needless to say, average scores on this item were lower than for any other item.

*Item 6.*

A compact car in motion on the freeway uses  $f(x)$  gallons of gas in driving  $x$  miles.

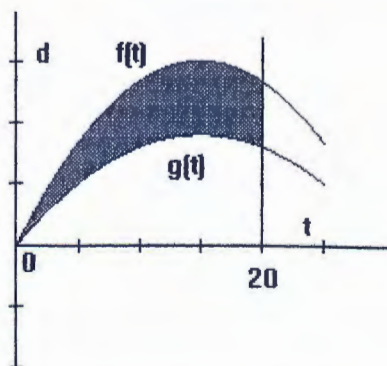
- a. For what values of  $x$ , if any, is  $f'(x)$  positive?  
For what values of  $x$ , if any, is  $f'(x)$  negative?
- b. A motorhome uses  $g(x)$  gallons of gas in driving  $x$  miles. How would you expect the graphs of  $f(x)$  and  $g(x)$  to compare?

This item required students to think about the sign and magnitude of a function from every-day life. No significant difference was manifest in part (a), where the traditional group had a slightly higher average score, or in part (b), where the reformed group average was slightly higher. Only 18 students (9 from each group) received all the points on the first part, and 37 (22 reformed, 15 traditional) received all possible points on the second part). The most common error on part (a) was that students said the derivative of the function was positive for “all” values of  $x$  without making it clear that the function was only defined for non-negative  $x$  (saying “all” doesn’t clearly rule out negative values). An interesting fact about the performance of students on part (a) was that although the problem was conceptual in nature, the traditional group of students actually scored higher on average than the reformed group.

*Item 7.*

Suppose that on a typical day the rate at which the electric company accumulates revenue (in dollars per minute) is given for any time in the day by the function  $f(t)$ . The rate at which the company's operating costs accumulate (in dollars per minute) is given for any time in the day by the function  $g(t)$ . These functions are graphed below.

- How could you use calculus to find the company's revenue and cost functions?
- How could you use calculus to compute the shaded area?
- What does the shaded area represent economically?



**Figure 4.2 Graph for Test Item 7**

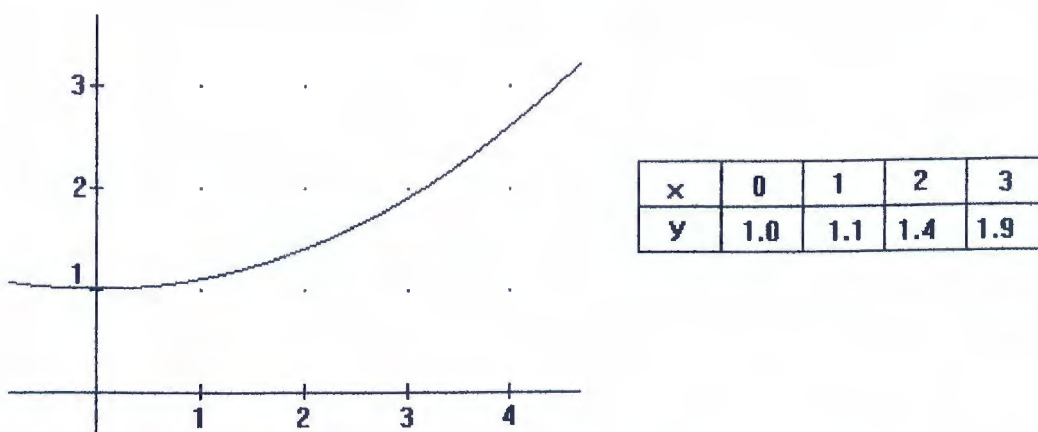
This item, which probed students' conceptual understandings of both indefinite and definite integration, proved to be one of the most difficult. Only 6 students (3 from each group) received all possible points on part (a). Students simply were not able to say that a cost (or revenue) function could be found by antidifferentiating a function for rate of change in cost (or rate of change in revenue). Students remembered more about how to find the area of a shaded region between the graphs of two functions (part (b)); 20 of them correctly identified how to use definite integration over the bounded interval (12 reformed, 8 traditional). This group of 20 still represents less than one fifth of all the students, however. Common mistakes included forgetting to mention the limits for the

definite integral and forgetting to subtract one function from the other in the integral. For part (c), where the only statistically significant difference between the groups was manifest ( $F = 8.626$ ,  $p = .0041$ ), 90 of the students knew that the shaded region of the given graph had something to do with profit (revenue minus cost). The counts were 52 to 38 in favor of the reformed group. It is interesting to note, however, that only 3 students, all from the reformed group, received full credit for this problem because everyone else forgot to mention the time interval portion of the correct answer. The shaded region of the graph represented profit for the first 20 minutes of the day, and those 3 students were the only ones who mentioned this fact.

*Item 8.*

Given the following graph and table of values for a function,  $f(x)$ ,

- Approximate the derivative of the function at  $x = 2$ .
- Approximate the integral of the function from  $x = 1$  to  $x = 3$ .



**Figure 4.3 Graph and Function Table for Test Item 8**



This problem tested students' techniques for estimating derivatives and integrals. The hope in part (a) was that students would use the slope of the line between the specified point and an "adjacent" point given in a table of function values to estimate the derivative. Full credit for part (a) was given to only 20 students (12 reformed, 8 traditional). Many others simply looked at the graph and guessed a reasonable value for the derivative at the given point. We note also that some of those receiving full credit actually did not use estimation; they correctly guessed the rule for the function and differentiated it to obtain the exact answer. No statistically significant difference was evident on part (a).

For part (b), the hope was that students would use Riemann sums, employing either the left-hand rule, the right-hand rule, the midpoint rule, or the trapezoidal rule. Before presenting the results for this item, we note that approximation techniques for integration were not covered specifically in the traditional course. Ironically, although the tally of students receiving all 5 of the possible points was small (5), it was 3 to 2 in favor of the traditional group. Yet the overall difference between the groups was statistically significant in favor of the reformed group ( $F = 3.190$ ,  $p = .077$ ). The reason for this is simply that the reformed group was more adept at "eyeballing" the correct answer. A group of 18 students received 3 of the 5 points possible, and 16 of these were from the reformed course. A score of 3 was earned either by giving a correct answer only (where method was not shown) or by using the correct method but making arithmetic errors. For this group of 18 students, most of them apparently just guessed a

correct answer. A total of 62 students (34 traditional, 28 reformed) scored zero on part (b).

*Item 9.*

Compute the following definite integrals:

a.  $\int_1^2 4x^3 dx$

b.  $\int_0^{1/2} e^{2x} dx$

The problems given in item 9 were, like those in item 3, purely procedural. Students were asked to compute two simple definite integrals. Whereas both of the differentiation problems in item 3 showed statistically significant differences between the group averages, only the difference in part (a) of item 9 was statistically significant. Once again, this procedural difference favored the traditional course ( $F = 6.557$ ,  $p = .012$ ). The tally of those receiving full credit for part (a) was 24 to 9 in favor of the traditional course. We point out, however, that this gap would have been much narrower if answers obtained by use of a calculator had received full credit (the tally would have been 25 to 17). For part (b), the desired answer was  $(e-1)/2$ , and only 1 of two possible points were awarded for numerical answers obviously obtained by calculator. The counts of those receiving full credit, which was 10 to 6 in favor of the traditional course, actually would have swung in favor of the reformed course (14 to 12) if calculator answers had been given 2 points.

*Item 10.*

There is a theorem, called the Fundamental Theorem of Calculus, which is a statement about the relationship between the derivative and the definite integral. What is this theorem?

As hypothesized, not many were successful in recalling the fundamental theorem of calculus; this item exhibited the second lowest mean scores of any item on the test.

No student received as much as 4 of the 5 possible points, and only one student received 3. Only 14 students received any points at all (8 reformed, 6 traditional). There was not a significant difference between the traditional course and the reformed course. Of note on item 10 was the fact that the traditional group of students had a higher average, although the problem was judged to be basically conceptual in nature.

Summary

Reviewing the results of the written test, we notice first that there was no overall difference between the traditional course and the reformed course. Yet this comparable performance by the two groups of students masked some real differences in terms of procedural and conceptual abilities of the students. The traditional course students were better at performing computations, whereas the reformed course students displayed deeper understandings of concepts. Finally, considering each item of the test separately, we discovered that the conceptual and procedural differences were not universal. The procedural difference was more marked, with 67 percent (4 of 6) of the purely computational items significantly favoring the traditional course, whereas just 22 percent (2 of 9) of the purely conceptual problems significantly favored the reformed course--and one of those was on approximations of integrals, which the traditional course instructors



did not teach. The one surprise was that item 2 part (b), judged to be conceptual in nature, showed a statistically significant difference favoring the traditional group. As noted earlier, this didn't mean the reformed course students were incompetent; it simply meant they used methods other than calculus to get their answers. For the problems that were both conceptual and procedural, 33 percent (1 of 3) showed statistical significance favoring the reformed course. The itemized results are summarized in Table 4.8.

| Item Number /Points possible |   | Reformed                        |                                  |               | Traditional        |                     |               | Sig. <sup>4</sup> |
|------------------------------|---|---------------------------------|----------------------------------|---------------|--------------------|---------------------|---------------|-------------------|
|                              |   | Pct. fully correct <sup>2</sup> | Pct. partly correct <sup>3</sup> | Average Score | Pct. fully correct | Pct. partly correct | Average Score |                   |
| <b>1a</b>                    | 3 | 33                              | 63                               | 2.114         | 31                 | 59                  | 1.971         | .414              |
| <b>1b</b>                    | 3 | 65                              | 35                               | 2.263         | 71                 | 25                  | 2.529         | .500              |
| <b>2a</b>                    | 4 | 28                              | 28                               | 1.737         | 39                 | 27                  | 2.216         | .156              |
| <b>2b</b>                    | 4 | 25                              | 18                               | 1.368         | 39                 | 22                  | 2.118         | <b>.031</b>       |
| <b>3a</b>                    | 2 | 30                              | 7                                | .667          | 61                 | 4                   | 1.255         | <b>.002</b>       |
| <b>3b</b>                    | 2 | 11                              | 12                               | .333          | 25                 | 16                  | .667          | <b>.026</b>       |
| <b>4</b>                     | 5 | 37                              | 33                               | 2.658         | 31                 | 45                  | 2.735         | .848              |
| <b>5</b>                     | 4 | 18                              | 9                                | .895          | 0                  | 4                   | .098          | <b>.001</b>       |
| <b>6a</b>                    | 3 | 16                              | 40                               | 1.333         | 18                 | 45                  | 1.441         | .654              |
| <b>6b</b>                    | 3 | 39                              | 32                               | 1.719         | 29                 | 29                  | 1.353         | .140              |
| <b>7a</b>                    | 4 | 5                               | 5                                | .298          | 6                  | 4                   | .294          | .983              |
| <b>7b</b>                    | 6 | 21                              | 74                               | 3.123         | 16                 | 59                  | 2.549         | .136              |
| <b>7c</b>                    | 5 | 5                               | 88                               | 2.790         | 0                  | 78                  | 2.167         | <b>.004</b>       |
| <b>8a</b>                    | 5 | 21                              | 37                               | 1.807         | 16                 | 33                  | 1.294         | .165              |
| <b>8b</b>                    | 5 | 4                               | 47                               | 1.263         | 6                  | 27                  | .765          | <b>.077</b>       |
| <b>9a</b>                    | 2 | 16                              | 21                               | .526          | 47                 | 2                   | .961          | <b>.012</b>       |
| <b>9b</b>                    | 2 | 11                              | 21                               | .421          | 20                 | 12                  | .510          | .537              |
| <b>10</b>                    | 5 | 0                               | 14                               | .211          | 0                  | 14                  | .216          | .965              |

**Table 4.8 Test Scores Summary**

<sup>2</sup> This means "the percentage of students who received full credit on this item."

<sup>3</sup> This means "the percentage of students who received partial credit on this item."

<sup>4</sup> This is the probability of the obtained *F* statistic in ANOVA testing. Significant results are in boldface.

Finally, we note that scores students earned on the written test were consistent with grades earned in the course. Students who received an A grade in their applied calculus course earned the highest mean scores (34.1 for the reformed group, 29.6 for the traditional group). Students who received a B grade earned the second highest mean scores (20.5 reformed, 24.6 traditional), and the lowest mean scores went to students who received a C grade (15.9 reformed, 13.8 traditional).

### **Interviews: Concepts Portion**

How do students think about calculus ideas? What is the depth of their understanding of the central concepts of calculus? The interviews were designed to provide insight into these questions and to help explain the various levels of student performance on the written test. Students' responses to the interview questions allow us a glimpse into their minds--a chance to see a small part of their thoughts about calculus.

We will consider the questions from the interviews in turn, analyzing each in three ways. First, we will describe the range of answers from all sixteen of the students by categorizing them, thus characterizing the main thrusts of the responses overall. Second, we will explore those categories where interesting differences appeared in the frequencies of responses between the reformed and traditional groups. Third and finally, we will summarize the results and speculate about their implications (remembering all the while that this small amount of qualitative data doesn't warrant decisive conclusions; rather, it indicates possible trends to investigate in future studies). We turn first to the questions about calculus concepts. For convenience in the discussion (and for brevity)

when referring to specific students, we use labels T1 through T8 for the traditional calculus students; we use labels R1 through R8 for the reformed calculus students.

Question One: What is a derivative?

The first calculus question in the interview asked students to describe how they would explain what a derivative is to someone who had never heard of it before. Several follow-on cues were used following question one during the interviews as a means of probing to see what cues students might need to recall important ideas. Different students might require different cues to enable them to retrieve pertinent information from their memories (Byrnes, 1996, p. 56). The cues used in the interview included these: “Do you remember computing derivatives?” “Do you remember what derivatives were used for?” As a last resort, this cue was used: “Suppose I tell you that the derivative has something to do with how something changes. Does that bring anything to mind?” The question about the use of derivatives proved most effective among the follow-on cues; for that reason we will present the responses to that question and responses to question one separately. We must remember, however, that these two sets of responses taken together will give us a clearer picture of students’ understanding of the definition of the derivative.

There were six main categories or groupings of elements included in student responses to the original question; the list below is by order of frequency from highest to lowest. Each of the sixteen students interviewed mentioned something that fits into one or more of the following (paraphrased) ways to explain the derivative (the categories



listed below are not mutually exclusive; any given student may be counted in several of them):

1. "I would show various rules for differentiating functions" or "I know the power rule."
2. "It has to do with change of a function or curve."
3. "It has to do with the slope of a line, the slope of a graph, or the slope of a line tangent to a graph."
4. "I don't know what it is" or "I know what it is but can't explain it."
5. "I would draw a graph and use it to explain derivatives."
6. "It has to do with finding the tangent to a curve."

Category 1 was by far the most popular. Most students, when asked to think about derivatives, thought first of the power rule. Thirteen of the sixteen students (eight traditional, five reformed) said something that fits into this category. Seven of the students showed a specific example of the power rule; all but one of these involved a quadratic monomial or binomial. The similarity of the examples was remarkable. The exhaustive list is as follows:  $3x + 2$ ,  $x^2$ ,  $3x^2$ ,  $3x^2 + 2x$ ,  $at^2 + bt$ .

The second-most common element among student responses as to what derivatives are were references to change. Nine of the sixteen students interviewed (three traditional, six reformed) said something about the change in a curve, a function, or a line associated with a function. It wasn't always clear, however, exactly what a particular student thought was changing. Student T8, for example, said that there was a line, which she called the slope of the graph, associated with a function. She said that

“saying how the line is changing is describing the function.” It seems that she was thinking of a line tangent to the graph of a function, and that the tangent line would have different slopes depending on where it was tangent to the function--yet the word “tangent” was not recalled. Student R7 had perhaps the same difficulty and referred to the “change of the slope” in a graph being like “the change of the line.” A somewhat more precise response came from student R2, who referred to a “rate of change at a point in time.” Student R6 mentioned that the derivative would show “the increase from one point to the next.”

The third most common idea among the responses to question one was that of slope. Seven students were counted in this category (two traditional, three reformed). Three students mentioned the slope of a line tangent to a curve, although one of these seemed slightly confused about what a tangent line is. Student T8 spoke of the derivative as being the slope of a line between two points on a graph. Students R5 and R8 spoke of “the slope of the line.” Student R8 went on to explain how “the line” was related to a graph, although the word tangent was not mentioned. R5 seemed not to have made a clear connection between “the line” and a graph. Student R7, who mentioned the “change of the slope” and “the change of the line,” seemed to have a correct understanding of how “the line” was related to the graph, although (as mentioned above) the word “tangent” was not used.

Fourth on the frequency list was the realization on the part of six students (five traditional, one reformed) that they could not explain what the derivative was. All of the students in this group said that although they could not explain derivatives, they either

knew what a derivative was or could compute, use, or graph them. Student T6 said that “...you focus so much on just doing the problems that I didn’t really think about what it meant.” Five of the six students did have some ability to describe the derivative, because they mentioned things in the other categories: three mentioned differentiation rules, and two mentioned change and/or slope.

The fifth category of response included those who said they would draw a graph to explain derivatives. All of these four students also fell into categories three and/or four (they mentioned change and/or slope).

The sixth category included two students (both traditional) who said that the derivative had something to do with finding the tangent to a curve. For one other student who was almost included in this category (T6, quoted above), the idea of a tangent was very vague, and the connection to the derivative was not strong at all: “...I get a picture of a circle, with a straight line, you know, and I know that it has something to do with that...” The two students who appeared to understand the notion of tangent were also included in category 1.

Several interesting differences appeared between the traditional and reformed groups in terms of their responses to the question about explaining derivatives. Four of the six categories showed differences of three or more between the number of responses from each group. We will consider the strongest differences first.

Of the six students who remarked that they didn’t know how to explain what derivatives are, five were from the traditional group. This is perhaps not surprising when we consider that one of the goals of the reformed curriculum was to focus more on the



careful development of concepts; yet given the relative similarity in performance on the written test, one wouldn't expect the contrast to be quite so strong. Still, the notion that one can understand and use derivatives without being able to explain them was clearly brought out in the interviews; this helps explain how the groups could differ in ability to explain and at the same time perform at similar levels on the test.

All four of the students who said they would use a graph came from the reformed group. Some other students (from both groups) did use graphs in their explanations or talked about graphs as they explained; yet none of them said specifically that they would use a graph in trying to explain derivatives. In other words, graphs or pictures would be used by traditional students to explain derivatives, but graphs might not be the main focus of their explanation. The reformed students were perhaps more comfortable with graphs as their primary vehicle for explanation.

Categories 2 and 3 each showed a difference of three between response frequencies. Category 2 included students who said the derivative had to do with the change in a curve or a function (six reformed, three traditional); category 3 included those who mentioned the slope of a line, a graph, or a line tangent to a graph (five reformed, two traditional). Both of these categories appear to favor slightly the reformed group of students: more of them thought of finding the change in or slope of a graph.

Category 1 (differentiation rules) included all eight traditional students and six of the eight reformed students. Perhaps the propensity to rely on differentiation rules is greater among students of the traditional calculus, but clearly both traditional and

reformed students still rely heavily on the rules. For many, these rules are the first thing called to mind when thinking of the notion of the derivative.

In summary, what do we see in explanations of the derivative by traditional and reformed students? We see that although both groups rely on rules of differentiation and take some (if not all) of their definition of the derivative from them; the reformed students tend to focus more on change, slope and graphs; the reformed students also feel more confident in explaining derivatives than their counterparts.

Question one follow-on cue: What are derivatives used for?

The responses to this question shed some additional light on what students think about derivatives, and the responses to question one may have been clearer if this cue had been asked first. For example, student T1, who said he couldn't explain derivatives other than by showing the differentiation rules, also said the derivative was used to find rates of change, maxima and minima, slopes, and acceleration. Student T5, whose responses to question one matched those of T1, also said that derivatives were used to find maxima and minima as well as whether a function was increasing or decreasing.

Following are the four main categories of responses as to the uses of the derivative. Once again, the responses are listed in order by frequency:

1. Economic functions (marginal revenue, cost, profit, etc.)
2. Finding whether a function is increasing or decreasing; slopes
3. Finding change or rates of change in a function
4. Finding the maximum and minimum

Other less frequent responses included finding inflection points, velocity, acceleration, and “estimating.”

Ten students (four traditional, six reformed) said derivatives applied to economic functions. The range of functions mentioned included “marginal revenue and cost,” “profit,” “how much you should produce,” “rate of return,” and “profit margin.” One student, R7, mentioned using the second derivative: “...If you take the second derivative then you can find the change in the change so you can find out if you’re, like, gaining profit, you can find out if that gain is increasing or decreasing.” Some of the students just mentioned the economic applications, whereas others, such as student R6, explained what they meant: “...We used [derivatives] with, like, cost and revenues--it did the marginal cost and marginal revenue. Like, the marginal cost, the derivative would be the cost at that unit to make the next one...”

Seven students (four traditional, three reformed) said derivatives were used to find whether a function is increasing or decreasing or to find slopes; four students (two from each group) mentioned finding “rates of change” or “the change in a function;” three (all traditional) mentioned finding “maximums and minimums of functions” or “max and min.”

The differences between the traditional and reformed groups of students were not so strong as the differences found on the first question. None of the categories mentioned above showed a difference as large as three in the number of responses. The only interesting thing to note is that a few of the reformed students talked about



applications that none of the traditional students mentioned, such as “estimating,” “projectiles,” and inflection points.

Taken together, what do question one and its follow-on cue tell us about students’ understanding of the derivative? If we combine the responses to these two questions from those categories dealing with “slope,” “change,” “rate of change,” and “increasing/decreasing,” we find that only four of the sixteen students—one reformed, three traditional—are not accounted for. Of those four students, three (the traditional ones) did know that the derivative had something to do with the “tangent.” It appears that overall, the main idea about the derivative seems to have been remembered.

#### Question Two: What is an integral?

Very similar to question one, question two asked students how they would explain “the integral” to someone who knew about functions but had not heard about integrals. There were really only two responses which could be categorized as typical; other than these two responses there were a range of varied comments, each one coming only from one or two students. These other responses sometimes involved some specific application of integration, such as “economics” or “half-life.”

The first common response was that the integral is an anti-derivative, or it is the opposite of the derivative. Seven students mentioned this idea (five traditional, two reformed). The second common response was that the integral had to do with the area under a curve; nine students (six reformed, 3 traditional) said something to this effect.

Most of the seven students who defined the integral as the anti-derivative or the opposite of the derivative mentioned that if you have the derivative of a function, then

the integral takes you back to the original function. It is interesting to note that the three students from the traditional course who did not mention the idea of “opposites” at this point in the interview did mention something about it later during the discussion following question five (which asked about the relationship between the derivative and the integral). For example, student T6 said, “Well, every integral that I remember doing had a derivative in it. Like, we had to find the derivative and then work the integral, and I’m sure that’s just part of it...” Student T7 said, “...if you took the derivative, it would take it down a power, it took it down. Integral took it up a power. I don’t know the relation, but I know it’s related.” Student T4 said “Well integral, you do, take the anti-derivative, so they’re connected in that respect.” In general, it seemed that by the time the interview was finished, all the traditional course students had mentioned that the derivative and the integral were “opposites.”

For reformed course students, fewer of the ones who didn’t mention the “opposites” idea mentioned it later. In total, only four of the eight students in the reformed group connected derivatives and integrals this way in their minds. One who did make the connection later (student R1) really made it well, however; she ended up writing the symbolic form of the fundamental theorem of calculus on her scratch paper! Another student (R6), who knew the ideas were connected but didn’t quite define them as opposites, said that “...they’re definitely connected through the fundamental theorem of calculus....if you know one you can find the other and vice versa through that theorem.” Why didn’t the reformed course students understand the connection between

derivatives and integrals better? It could be that the reformed course stressed applications so much that students lost sight of the connections.

The responses about the integral giving the area under a curve (or something similar to that idea) favored the reformed course students. It seemed that more of them thought of area immediately when the integral was mentioned, almost as though “area” was their personal definition of what integration means. This could have stemmed from the approximation technique of Riemann sums that these students studied (and which the traditional course students did not study).

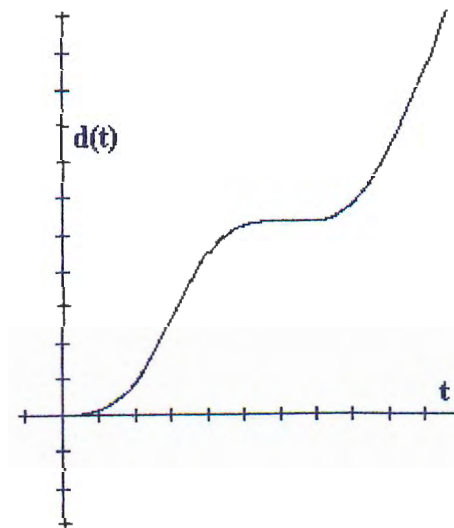
In sum, then, it seemed that students came away from the traditional applied calculus course thinking that integration meant antidifferentiation, whereas students in the reformed course came away thinking more about applications of integration to finding area.

### Question Three: Graphical Derivatives

This question challenged students to choose the graph of a function’s derivative from among three possibilities. The graph of the original function showed a car’s position over time, so the choices for the derivative represented the car’s velocity for the same time period. Problems of this type were not commonplace for either the traditional or the reformed course. This problem was included to probe whether students could reason from what they knew and connect the graph of a function with the graph of its derivative. Question three is included on the next two pages for reference.

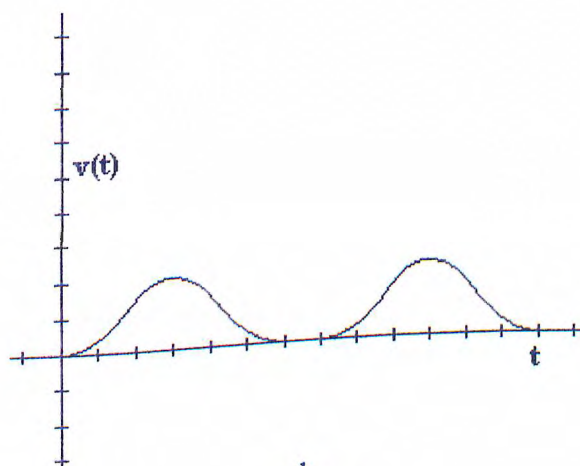


3. Below is a graph showing a car's distance traveled as a function of time. Choose a graph (of the three on the following page) which could represent the car's velocity over time and tell why. Possible follow-up cues: How is velocity related to distance traveled? Is there something you could say involving derivatives and/or integrals which would relate graphs a, b, and/or c to the first one?

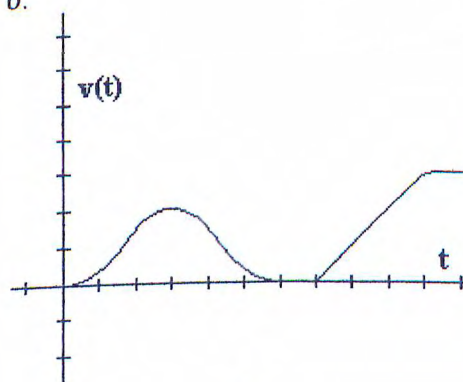


**Figure 4.4 Function Graph for Interview Conceptual Item 3**

a.



b.



c.

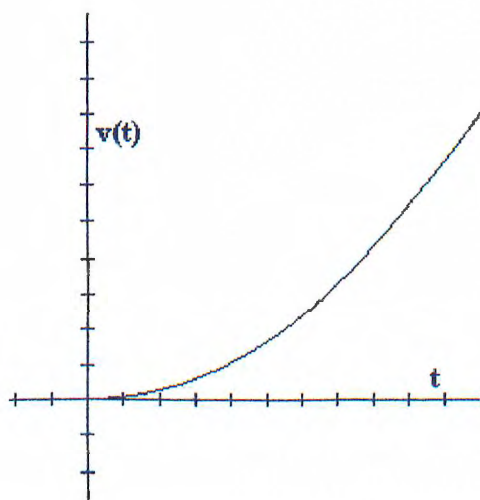


Figure 4.5 Derivative Graphs for Interview Conceptual Item 3

Choices (a) and (b) were designed to be very similar ((b) is the correct one), and choice (c) was designed to be obviously wrong (refer back to Figure 3.5). All but one of the students talked their way to a solution (correct or incorrect). Their thinking aloud generally involved either comments about the shape of the original graph compared to the shape of the derivative choices or comments about what the car may have been doing at various times along the graph. Only one student was unable to get a monologue going and resorted to saying “I don’t know.”

Of the fifteen students who talked their way to an answer, only six (three traditional, three reformed) chose the correct answer initially. These six were in large part successful because they were able to rule out choice (c) first because the original graph wasn’t strictly increasing; then they were able to eliminate choice (a) because the original graph didn’t “level off” a second time (velocity didn’t reach zero again). Six other students (two traditional, four reformed) picked (a) initially; the remaining three chose (c). Of the nine students who made an incorrect first choice, five later changed their answer to the correct choice (some were prodded to keep thinking aloud).

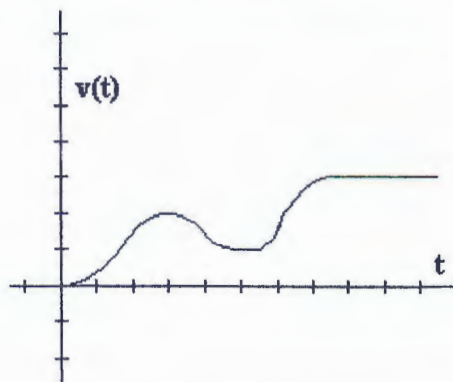
It was interesting that four of the five students who were able to correct their wrong choices were from the reformed course. Perhaps they had more experience in talking through problems or more confidence in working with graphs.

#### Question Four: Relationship Between Distance, Velocity and Acceleration

Students were shown the graph of a function giving the velocity of a car over time; they were then asked what meanings the derivative and integral of that function would have.



4. Here is a graph showing the rate of travel of a car (i.e., velocity). What could I learn from the derivative? (Pause for response and discussion.) Follow-up cues: What could I learn from the derivative at a specific point? What could I learn from the general derivative of the function? What could I learn from the definite integral between two  $t$ -values? What could I learn from the integral in general?



**Figure 4.6 Graph for Interview Conceptual Item 4**

In general, just over half of the students knew that the derivative of velocity would be acceleration and the integral of velocity would be distance traveled. There were a few students who thought mostly in terms of the graph only and not the car, explaining that the derivative could help us find maxima and minima as well as where the function was increasing and decreasing. When speaking of the integral of velocity, some mentioned that it would tell us average velocity over a time period, and one spoke of the area under the curve without interpreting what that area represented.

The only trend evident is that the idea of acceleration seemed to be best remembered by the students from the reformed course. That the definite integral between two values would yield the total distance traveled on that interval was

remembered by five students from each group, whereas reformed course students remembering acceleration outnumbered their traditional course counterparts six to three.

Question Five: How are Integrals and Derivatives Related?

This question asked students to use the words “integral” and “derivative” in one statement. The goal behind this question was to determine how the two notions were related to each other in the minds of the students.

This item was discussed in part earlier in connection with question two. In general, all the students believed that derivatives and integrals must be connected somehow, even though some of them couldn’t describe the connection very well. Some common responses included “the integral is the anti-derivative,” “they are opposites of each other or reverse each other,” and “the derivative of the integral is the original function.”

Only one student (R1) accomplished the original goal of this question and actually wrote down a symbolic form of the fundamental theorem of calculus. Another reformed course student tried to write it down but did so incorrectly. It seemed that more of the reformed course students knew that there was a fundamental theorem and mentioned its name, whereas the traditional course students were more familiar with the word “anti-derivative.”

It seemed clear from responses to this question that the traditional group of students had more ready answers and better general ideas about what the connection was between derivatives and integrals. The reformed group knew they had connected the ideas in class, but the memory was vague. From the traditional group, perhaps the

clearest explanation came from student T5 (who admittedly had studied calculus before taking the applied calculus course): “I think [the derivative and the integral] are connected, cause to find the integral, like, without using a calculator, then you have to do the anti-derivative. So you kind of have to know what a derivative is to know what the anti-derivative is.”

#### Question Six: Partial Differentiation

When asked to explain the meaning of the partial derivative with respect to  $x$  of a function of two variables ( $x$  and  $y$ ), most students had no clear answer. There seemed to be only four (two from each group) who had more than a shallow understanding of partial derivatives. Barely more than a majority of the students (four reformed and five traditional) knew that to compute the partial derivative with respect to  $x$  required treating  $y$  as a constant.

The only noticeable trend seemed to be that a few of the reformed course students said something that was close to being correct yet still somewhat unclear. For example, student R5 said, “It would show you what would happen to... $x$ ...in regards to  $y$  or...I don’t know.” Student R2 said the partial derivative would give the “...change in  $x$  with respect to....” and left the thought unfinished. In contrast, the traditional course students seemed to understand the principle very clearly or not at all.

#### Question Seven: The Derivative Revisited

Whereas question one asked students to explain what a derivative is, question seven was aimed at finding out if students could recognize a simple application of the



derivative. Shown the graph of a population function over time, students were asked how they might use the rule of the function to find the rate of increase at a certain point.

Eight of the students (six from the traditional course) responded with the correct answer, namely, “compute the derivative and plug in  $t$ .” Another four students (all from the reformed course) referred to “taking the derivative,” and it was unclear whether plugging in a particular value for  $t$  was implied or not. Still four others (3 from the reformed course) spoke of estimating the answer, and this usually meant finding the slope between two points on the graph.

One trend appeared to be that the traditional group knew better that a particular value of  $t$  was called for. This could mean either that the reformed course students assumed that their answers implied this necessity of a particular value (i.e., they weren’t careful in their explanation yet really knew what was going on), or it could mean they didn’t remember that a particular value was needed.

Another slight trend was that more of the reformed course students (three to two) talked about estimating the derivative using, for example, the slope between two points on the graph.

#### Question Eight: Recognizing the Definition of the Derivative

For this item, students were shown the formula for the definition of the derivative. They were then asked if they recognized it and could identify it. Most students either correctly identified the formula or they didn’t know what it was. A few gave errant answers such as “the central limit theorem” and “an object falling” and “the fundamental theorem of calculus.”

Just two of the students from the traditional course gave the right answer. One other student from this group mumbled something about the “central theorem,” then said later that he had known that was the derivative. One of the two who gave the right answer (T2) wasn’t completely certain. She did remember that it was something they had used early in the semester before finding a short cut method. In contrast, five of the reformed course students remembered correctly what the formula was or that it “has to do with derivatives.” It could be that the difference was the amount of emphasis placed on the definition in class; one of the reformed course students (R3) mentioned that “It could be like, there’s some formula, he just kept pounding it into our heads every time. It’d be on every quiz.”

Question Nine: Why did you forget?

Students were asked if they could pinpoint some reason (other than just the passage of time or the lack of review and practice) why they may have forgotten some things. Most agreed that not using calculus since they took the course was the main factor in their forgetting; some mentioned the fact that they didn’t need any more math, so they simply didn’t think about it anymore. Some felt that a little bit of reviewing would have helped a lot.

A few other interesting points were brought up in answer to question nine. Three students (two traditional) said that the material wasn’t ingrained enough in their minds at the time they learned it. One student from the reformed course said that the material was covered too quickly, and one from the traditional course said that some things were presented once in class and then never revisited. One student from the

reformed course said that the things best remembered were those that were used and developed more in other classes following calculus. Another student from the reformed course mentioned that the things he learned to do on his calculator were the easiest to remember. He apparently liked using the calculator, so it stands to reason that he probably used it often and learned to use it well--thus his memory of those ideas was more clear.

The most interesting comments in regard to how the course could have been changed to enhance memory came from two students, one in each group. They both talked about the importance of applications of calculus. The traditional course student (T8), was disappointed that her instructor

...just taught formulas and stuff....I've heard the other classes did more of the practical real life. That would probably help you remember more, cause then you could set different avenues of thought, 'cause if you're based solely on the formulas, it seems like then once you forget the formulas, you're dead. So it seems better....my mind works best if I learn these, then I can figure out the other stuff.

Student R7 said that she could see how some of the problems applied to other students' interests but not to hers, and that's why she forgot them. The implication was that if all the applications had been in the area of her particular interest, she would have remembered the calculus better.



One general note about all the student responses to calculus questions was that they seemed to be weak in their understanding of the vocabulary of calculus; they lacked the words they needed to clearly explain things.

### **Interviews: Motivation/Affect Portion and Instructor Insights**

The main purpose for this section of the student interview and for the instructor interviews was to find out more about the traditional and reformed courses so that we could get a clearer picture of how they compared to each other. If the present study is to make reasonable conclusions, we must understand how the traditional and reformed courses related to each other. Were they taught by instructors who adhered to the traditional and reformed methods? Were there any characteristics of the groups of students that made them different from each other? If observed differences in knowledge of calculus are to be attributed to the fact that one course was traditional and the other was reformed, then hopefully the reformed course group of students was as similar as possible to the traditional course group of students when they began their applied calculus course, and hopefully the instructors believed in and used styles commensurate with their respective courses. This portion of the student interview and the instructor interviews helped clarify these two aspects of the study. We will discuss first the questions which shed light on the comparability of the two groups of students.

#### Group Comparability

There were two main questions which helped clarify the comparability of the groups. The first of these asked students to tell why they chose the particular applied calculus course they took. There were basically three reasons given: (1) it fit their

schedule best; (2) the teacher had been recommended to them; (3) the traditional course required no calculator. The third reason was only mentioned by students from the traditional course; one didn't want to purchase a calculator just for one class, and another described the calculators as "huge" and did not want to learn how to use one (although this same student mentioned it probably would have been a good thing to learn). There were five students from the reformed course who mentioned that the teacher had been recommended, but in only one case was this given as the main reason for the choice of section. There were three students from the traditional course who mentioned that the teacher had been recommended, and again in just one of those cases was that the main factor in the choice. The main reason for fourteen of the sixteen students was simply that the section they chose fit their schedule better. Student R2 said that he had heard the night class (traditional) was easier, but he had still chosen the daytime one (reformed) because of his schedule.

In light of the fact that eight students had their instructor recommended to them, it is interesting to note that students were later asked to rate their instructor, and all but one of the eight (indeed, thirteen of the sixteen students in all) gave their instructors very high marks for their teaching.

The second question that helped in comparing the groups dealt with the amount of effort students had put into their calculus course. The groups were similar in this regard; all eight from the reformed group and six of seven from the traditional group said they worked harder or much harder in this course than they had in other math classes (one traditional student was not asked). A few mentioned that they had had some



calculus before taking the course, but this question was not uniformly asked of the students. In general, the students liked their applied calculus course and said they would recommend it to a friend.

The instructors of the traditional course both mentioned that they had students in their classes who had dropped out of the reformed course because they either didn't want to buy a graphing calculator, or they were uncomfortable with learning to use one. It was unclear how many students this applied to; the sections for both courses were large (over 200 students in each). The instructor of the reformed course noted that most students probably signed up by telephone, with nothing to influence their choice but the class schedule. The class schedule listed only the instructor for each section, not the book, and a note informed students to call the math office to learn whether a calculator would be required for their chosen section. One student interviewed mentioned that when he learned a calculator was required for his class, he simply went and bought one. Students could have learned about which book was to be used in their section by visiting the campus bookstore soon before classes started. Given the large numbers of students taking applied calculus, it seems most reasonable that scheduling was the largest single factor in most students' class choice.

### The Teaching

Several questions in the student interview dealt with the way the courses were taught. The first of these invited students to describe what a typical class period was like. There was little deviation from the following combination: time for questions on previous material, covering new material (lecture), and a quiz. The quiz came first in the



reformed course, last in the traditional one (instructors' responses clarified this). The only variation from this pattern came in the reformed course, where three students mentioned working in groups or doing homework in class. Students agreed in general that there was usually enough time in class to get questions answered.

On the subject of homework, students from the reformed course said that they had more of it. Not that they had a greater number of problems to do than their counterparts in the traditional course, but that the problems were tougher in nature. Two students said they were all "word problems," which probably meant they were descriptions of real-life applications such as were emphasized in the CCH textbook. Traditional course students said things such as "the homework wasn't too strenuous" and it took "maybe an hour a night," whereas one or two of the reformed course students remarked that they spent 2 or 3 hours each night. One student (R5) described the homework this way: "It wasn't doing the problems over and over again, it was each problem I had to think harder about."

Another question about how the class was taught concerned agreement between the teaching style and the students' learning styles. When asked about this agreement, all eight of the traditional course students replied that the teaching agreed with the way they liked to learn. In contrast, only four of the reformed course students replied in like manner--and two of them specifically said that the teaching disagreed with their preferred learning style. This is evidence that the reformed course was taught in a non-traditional style, at least to some degree. Students who have normally been taught in a traditional lecturing style may naturally be uncomfortable with new emphasis on working in groups

or being left more of the discovery to do on their own. As student R3 put it, "I'm more of a numbers person as opposed to, like, a story type person." Student R8 said it this way: "...he always said this isn't a plug and chug class, and I'm like, 'but that's what I'm used to, that's what I'm good at!'" Students generally agreed that the teaching styles they encountered in class matched their respective textbooks.

The instructors' comments about their own teaching help illuminate what happened during class time. The reformed instructor talked about trying "to get the students engaged in the process," both together as a whole class and in smaller groups. He mentioned that smaller groups had to be employed more than a whole-class group because of the large class size. He also said he "...had the classroom noisy a good part of the time" and that he tried to wander around and interact with the groups. When asked about the amount of individual interaction he got with students, his first response was "quite a bit."

The traditional course instructors were, indeed, more traditional. They rarely made efforts to let the students work in groups during class time, although they both said they tried to encourage group work outside of class. Each of them encouraged questions, but they also said that a lot of interaction was difficult and the class had to be run more on a lecture format because of class size. On the other hand, the reformed course instructor mentioned that because the class was so large, he had to let the students work more in groups than usual. One of the traditional course instructors described his covering of new material as lecturing, the other called it "developing." One of them said he tries to run "a real interactive classroom." When asked about the



opportunity for individual interaction with students, his first response (and that of the other traditional instructor) was “not much.” He thought that his teaching should be enough to bring students to understanding, as evidenced by this remark: “...if they’re not getting it from what I’m saying in class and the interactive dialogue, I’m just not saying the right things. And so I think they’re better off hearing it from another person.” This leaves room for wanting students to work with others, yet it is a very traditional “you can get it from my lecture” attitude.

#### Student Likes and Dislikes

Two of the questions in the student interview were aimed at determining what parts of each type of course are agreeable to students, as well as what turns them off. The first of these two questions was “What did you like best about your calculus course?” Various answers were given, but the most prevalent answer from the traditional course students was that they liked their teacher or his teaching style. For the reformed course students, the favorite response was that they liked the applications of calculus. Both of these responses seem appropriate, given the context from which they came: students in the traditional course, where the lecture seemingly could “make or break” students’ understandings, liked the teacher best; and students in the reformed course, which heavily emphasized applications, liked the applications.

The second question in this area was “What would you have changed about your calculus course?” This led to a wide range of responses. The most common one was that the class was too large; students would have preferred a smaller section (none was available). Of the six students who mentioned this, five were from the reformed course.



One student from the traditional course wanted more applications and one wanted more advanced material, whereas two students from the reformed course said their textbook was a bit unclear. Two students (T3 and T6) said they wanted more reviewing or a slower pace in class, and three students (T5, R6 and R8) said that the instructor should have covered all the steps of problems done in class.

#### Other Instructor Insights

The reformed instructor seemed definitely from the reformed school of thought, trying to involve students in the process of mathematics rather than just having them learn a body of knowledge. The two traditional instructors felt that the reformed methods could be good, but for some reason they weren't currently the best choice. For one of them, reformed teaching was too slow, not allowing enough time to cover the required amount of material. For the other, using applications to teach the material would be more beneficial if the students could be separated by major so that they would see only those applications in which they were interested (one student expressed a similar sentiment). This instructor also wanted students to have much better proficiency with differentiation rules before applications were introduced (again, a very traditional attitude).

#### Summary

Reviewing the results from the student interviews, we can make a few general statements in summary. First, students from the reformed course seemed more confident in their ability to explain derivatives. These students also mentioned graphs more in reference to derivatives than did their counterparts from the traditional course. In

general, integration meant antidifferentiation to the students from the traditional course. For the students from the reformed course, integration had more to do with finding area in various applications. Traditional course students seemed to have a clearer idea of the connection between the derivative and the integral than the reformed course students. The traditional course students also seemed to know the quick way of finding the slope of a graph at a particular point (differentiate and plug in the  $x$ -value), whereas the reformed course students seemed more inclined to use estimation techniques. The reformed course students also recognized the formula for the derivative more readily than did the traditional course students. Finally, many of the students from both groups gave imprecise descriptions of mathematical ideas. Sometimes this seemed to be due to confusion on their part about calculus concepts; other times they seemed to lack the vocabulary they needed to describe adequately what they were thinking.

## CHAPTER 5: CONCLUSIONS

We began this study with the goal of characterizing and comparing student retention of concepts and procedures studied in reformed and traditional applied calculus courses. It is time to review, interpret, and examine the findings and their implications. But we must consider also the validity of our findings; therefore we turn first to a brief discussion of issues related to the comparability of our student samples.

### Implementation of Control and Reform Instruction

In order for results of a curriculum comparison to be meaningful, the data under scrutiny need to have come from comparable samples of students. It was our hope in conducting this study that the two groups of students who participated were as much alike as possible in every way except for the curriculum (i.e., the textbook and the teaching) they experienced. Clearly it is impossible to have perfection in this endeavor; the uniqueness of individuals alone is enough to guarantee that. But we want to insure as much similarity as we can between the samples in characteristics other than the curricular ones.

The first question of comparability concerns the randomness of the student samples. Participants were chosen from among those responding to an invitation via electronic mail, so the group of respondents itself does not represent a random sample from the entire group of students who completed an applied calculus course during the Winter semester of 1997. Sampling procedures from this point on, however, guaranteed stratified random samples from among the volunteers. Thus, although not perfectly



random, the samples seem acceptably random--indeed, the fact that subjects must consent to participation may preclude the possibility of obtaining better samples.

The second comparability question concerns how the students were assigned to the traditional and reformed applied calculus classes. Student enrollment in all sections was by self-selection, which certainly is not random--yet students had a limited amount of information available to them in choosing their course. The course listing for the first semester of 1997 included only the time classes met and the name of the instructor. No information on textbooks was listed, and the only information about calculators was a note that students could phone the mathematics department to find out which sections required calculators. The textbooks for each course were available in the campus bookstore prior to the start of classes. Interviews revealed that students did indeed choose a class based on who was to teach it; yet it appeared that each of the three instructors involved in the study were equally preferred. There is also evidence that some students chose a course specifically because it did not require the purchase of a calculator. It is unclear how significant this factor was in course choice, but it seems that if having to use a calculator or not was important to a given student then that student would be likely to perform better in his/her chosen environment than in the other. But the sheer size of the enrollments in both course types renders it unlikely that the calculator issue was critical for a large percentage of the students. It is similarly unlikely that students chose their course based on the textbook. Visiting the bookstore before registering would have required fairly late registration, and it appears that getting a course that fit a particular schedule was high on most students' priority list. In

consideration of all these factors, it seems that assignment of students to reformed and traditional classes was not biased in any major way.

The third issue concerning comparability of results is the teaching of the two applied calculus courses. There are really two pertinent questions here: Did each instructor's teaching match the recommendations of the curriculum being used? And, was the quality of teaching comparable across the courses? In answer to the former question, it appears that the instructor of the reformed course was more "reformed" than his traditional counterparts. Although each instructor commented that class size made the teaching fairly traditional (lecture-oriented), the reformed instructor said a minimum of 20 percent of class time was spent with the students working in groups, whereas only one of the traditional instructors mentioned letting students work in groups during class, and that was only for about 5 percent of class time and involved quiz problems, not homework problems. The traditional instructors seemed concerned about getting their students to understand what they needed to know about the mathematics, whereas the reformed instructor tried to get students "engaged in the process" of mathematics. As to the quality of teaching, all three instructors seemed to be conscientious and caring, as evidenced by student comments and comments from the instructors themselves. Each instructor also seemed thorough and articulate, an impression confirmed by classroom observations by the researcher. Based on all the available evidence, it seems unlikely that results of this study were greatly biased by anything the instructors did other than the curricula they followed.



### Interpretation of Results

Having established that our results can reasonably be attributed to curriculum variables, we can discuss the meaning of student performance on the written and interview tasks. In overall comparison, there was no significant difference in terms of mean scores on the written test between the reformed course (mean = 25.9) and the traditional course (mean = 25.1); the probability of the observed  $F$  statistic was 0.75. In other words, if we consider only in broad terms how well students remembered what they learned in applied calculus seven months after taking the course, there really was no difference between the reformed course and the traditional course. This suggests that if an instructor or an institution is interested only in the total overall learning and retention obtained by its applied calculus students, it might not pay to invest in the change to a reformed curriculum (at least not the CCH applied calculus curriculum).

But the knowledge of calculus retained by students in the reformed course was clearly different from that of students in the traditional course. The significant difference between the groups in mean conceptual and procedural scores is evidence of this fact (the probabilities of the observed  $F$  statistics were .02 for conceptual scores and .007 for procedural scores). It appears that students in the reformed course gained in conceptual understanding of calculus when compared to students in the traditional course. But they did not acquire or retain the same procedural ability. What does this trade-off between concepts and procedures mean? It means that the choice of curricula made by applied calculus instructors should depend upon the goals they envision for students. If the desire is for students to gain increased conceptual understanding, it might well be



advisable to choose some type of reformed curriculum. If, on the other hand, facility with calculus computations is desired, then there is reason to choose a more traditional curriculum. In other words, the question “What curriculum should I use?” is best answered by “It depends on what you want students to learn!” And now we have arrived at the heart of the matter, for the debate about what we want students to learn is on-going. The debate really is a discussion of the relative merits of procedural and conceptual knowledge in mathematics.

Proponents of calculus reform argue that being able to get the right answers to procedural questions in calculus without understanding how or why the algorithms work or without being able to correctly interpret the methods and answers in authentic problem contexts is of little value. They also point out that, in light of modern trends in the use of technology, it seems clear that people will generally use calculators or computers to perform mathematical calculations. Therefore we should focus more on conceptual teaching than we have in the past, so that people will be able to use technology appropriately and effectively. As Harvey Keynes (a supporter of calculus reform) said in speaking of his own institution, “The main aspect of the reform here is to try to have students become much more actively involved and encourage group learning with technology” (Wilson, 1997, p. A13). The clear intention of such involvement is to enhance student construction of deep conceptual understanding.

Opponents of calculus reform argue that one cannot fully understand or appreciate the concepts until they gain proficiency in skills. They believe that knowing the algorithms and being able to use them is the most important part of mathematics, and

that reformers are trying to take a short-cut that leads nowhere. As George Andrews (one critic of reform) put it, "The reformers believe that they will get around the roadblocks of basic arithmetic so students can get to higher-order skills. But to learn piano, you must learn scales and chords before you move to the 'Moonlight Sonata'" (Wilson, 1997, p. A13).

This study has shown that the "scales before Beethoven" analogy is not necessarily true. In fact, it has shown that it is possible to understand concepts better than procedures or vice versa.

This study suggests that the differential effects of the CCH and traditional applied calculus curricula are not transient, although it does seem that a lot of forgetting occurred.<sup>1</sup> Most studies of this type occur soon after or during the time the courses are taught, so that differences between the conceptual understandings of students that have been observed are concurrent with the teaching. This study showed an effect that was present after some time passed. In other words, the differences in conceptual and procedural knowledge between the two groups of students were present seven months after the courses were completed.

Interviews with students in this study generally supported the results of the written test in that the reformed course students seemed to be somewhat more conceptually oriented than the traditional course students. They (the reformed group) seemed more confident in trying to explain things, talked more about applications of calculus and used graphical explanations more than did the traditional group. This

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<sup>1</sup> For example, it is discouraging to note that only 33 of 108 students remembered how to integrate the function  $4x^3$ ; even more discouraging is the fact that only 16 correctly integrated the function  $e^{2x}$ .



agrees with the research of Selden, Selden and Mason (1994), in which they found that students of traditional calculus have weaker graphical knowledge than analytical knowledge. They also found that these students knew the skills but couldn't apply them to solve problems. Not much can be concluded about computational skills of the students in the current study based on the interviews, as the interviews really were focused on concepts.

### Future Research

Clearly a study such as this cannot give a definitive answer about the value of applied calculus reform. But this study does add to existing knowledge about calculus learning and retention, and it also lends motivation to specific directions of future research.

First, it was discouraging to learn that for both traditional and reformed applied calculus students, retention rates seemed very low. If test scores indicated how much students remembered of all course content, then the average applied calculus student remembered only 38 percent of what he or she learned seven months earlier. This was perhaps not unexpected, yet actually observing it pointed out that we really are not very effective in how we teach calculus if our goal is long-term understanding and retention. Many instructors do focus on how to best help students learn calculus procedures and concepts, yet all our testing occurs during or just after the course and nobody seems to care much about what will be remembered at some point in the future. The way we currently teach is not geared toward fostering long-term retention of the subject matter. Thus one important question for future research is "How should one teach in order to



improve retention?” The answer is probably not simply to teach last what you want students to remember most; one student involved in this study (R7) said, “You know, it’s really strange that we did derivatives first, and I remember them better than I do the integral, which we did at the end of the semester.” Perhaps, as Bolles (1988) pointed out, we need more imagination in our teaching.

There is another important question related to retention: exactly what are students required to use from their applied calculus course at a later date? We mentioned in the introduction that finding out what the client disciplines want from calculus students was beyond the scope of this study, but the answer to the question would be very useful. If students remember so little, what should instructors focus on in order to maximize students’ success in future encounters with calculus?

Technology seems to occupy an important position in the debate about calculus reform. Certainly technology is a major component of most calculus reform projects (Ganter, 1997). This study did not deal with technology use in a major way, yet it is clear that technological issues were of great importance to some students (such as the student who spent so much time using his calculator or the students who may have switched to the traditional course to avoid the calculator required in the reformed course). Also, the use of calculators led some students (particularly reformed course students) to alternative strategies for solving problems (such as evaluating a parabolic function at many points near the apex to find a local maximum). Are there other effects of technology use on student performance and understanding? What is the best way to use calculators in an applied calculus class? Another whole vista of research questions

opens to our view when we consider the rapid changing of technology as well--what about using the TI-92 and other such symbol-manipulating machines? Obviously there is more research to do in this area.

There are also improvements or modifications that could be made in this study to increase the validity of its results. For example, the grading of the written test could have been modified to get more at calculus concepts; sometimes points were perhaps unduly focused on arithmetic (still, it is interesting that traditional students fared better with the current grading key). The test questions themselves were somewhat limited in their probing of computational skills; only three of the ten questions dealt exclusively with procedures (although they were the most basic procedures).

It would also be wise to perform a similar study using students who experience calculus instruction with smaller class sizes. The CCH reform (and other reforms) are perhaps best carried out in smaller, more interactive classes, so the particular implementation of applied calculus reform we studied was perhaps not in its native environment.

### Summary

After all is said and done, is reformed applied calculus better than traditional? We don't know yet. It does appear that the CCH applied calculus, at least, is achieving its goal of increasing the conceptual understanding of its students. Although losses in computational facility may not have been desired by the CCH authors, they likely realized that some loss was inevitable, and they were willing to accept that price because of what it would buy--namely, deeper conceptual understanding.

One thing that alarms reformers is the tendency on the part of some educators to discard current reform efforts in calculus. Glancing quickly at studies such as this, they might see no big gains and decide it's not worth the effort to make any big changes in the way they teach. But we must remember that current reform efforts are still relatively young; final judgment should be withheld until sufficient time for maturation has passed. There are still too many unanswered questions. In other words, reform hasn't failed yet and we need to keep working on and evaluating it. Many of the unknowns can be better understood as a result of research efforts such as the current study; indeed,

The lack of studies to indicate that [reform] efforts are having a positive impact on students, together with the increase in workload brought on by reform, is creating an environment of uncertainty that could result in the withdrawal of support for such activities by funding agencies, institutions, faculty and students.  
(Ganter, 1997, p. 3)

Not only does such research need to be done, it needs to be brought to the attention of those making decisions about teaching and curricula.

Hiebert & Lefevre (1986, p. 8) point out that procedural knowledge can be learned by rote (i.e., without meaning). At least one student interviewed in this study said something which exemplified this idea of rote learning. When asked for any final comments, he remarked:

Well, it occurred to me that I can still remember how to do some things...but I don't know what exactly they are....I know how to get the derivative of a function, I know how to get the integral and find areas between two curves at



certain points...but I just can't remember what it means--how to use it or...how  
[to] describe a derivative to a friend.

It is hoped that future research on the teaching of calculus will have as one of its primary  
motivations the desire to bring about deeper understanding--both conceptual and  
procedural--in the minds of students.

## APPENDIX A: CONTENT ANALYSIS

Below is a topical mapping between the two textbooks. This chart shows section numbers from both books covering the same topic. The left column is for Applied Calculus, Preliminary Edition (Hughes-Hallett, et al., 1996); the right column is for Calculus and its Applications (Goldstein, Lay & Schneider, 1996). On the page following the chart is a listing of the main headings from the table of contents of each book.

### CCH

### GLS

|  |   |
|--|---|
| 1.1 (includes supply and demand curves)                  | 0.1 (no supply and demand curves until p. 234)      |
| 1.2  | 0.2   |
| 1.3  | 0.6 (part)  |
| 1.4  | 4.1   |
| 1.5 (power functions dominated by exponential functions) | 0.2, 0.5  |
| 1.6  | 5.2, 4.2  |
| 1.7  | 4.4, 4.6  |
| 1.8  | 5.1   |
| 1.9  | 0.3   |
| 1.10   | 0.2   |
| 1.11   | 8.1, 8.2, 8.4                                       |
| 1.12   | 7.5   |
| 2.1  | 1.8   |
| 2.2  | 1.8   |
| 2.3 (limit; rate of change at point)                     | 1.2, 1.3 (slope), 1.4 (limit)                       |
| 2.4  | 1.3   |
| 2.5  | 1.7, 1.8  |
| 2.6  | 1.7, 1.8, 2.2                                       |
| 2.7  | 1.8, 2.7  |
| 3.1  | 6.2, 9.4 (approximation)                            |
| 3.2  | 6.3 (plus Fundamental Theorem), 9.4 (approximation) |

|   |   |
|---|---|
| 3.3   | 6.2, 6.3, 6.4 (two curves)  |
| 3.4   | 6.5   |
| 3.5 (notation and total change)                             | 6.3 (notation only)   |
| 3.6   | 6.3   |
| 4.1   | 1.3, 3.2 (chain rule), 1.6 (rules)  |
| 4.2   | 1.3, 3.2, 1.8, 1.6  |
| 4.3   | 4.3, 4.5, 8.3, 8.4  |
| 4.4   | 3.2   |
| 4.5   | 3.1   |
| 5.1   | 2.2   |
| 5.2   | 2.3, 2.4  |
| 5.3   | not found   |
| 5.4 (maximum profit, minimum average cost, maximum revenue) | 2.7 (maximum profit, maximum revenue), 3.1 (minimum average cost)   |
| 5.5   | 5.3   |
| 6.1   | not found   |
| 6.2   | 9.5 (some--present and future value, demographics), 5.4 (drug)  |
| 6.3   | 6.5, 9.5  |
| 6.4   | 6.5   |
| 6.5, 6.6  | 11.1-11.4   |
| 7.1 (definition and examples)                               | 10.1, 5.4 (learning curve)  |
| 7.2 (slope fields)  | not found   |
| 7.3   | 10.5 (and p. 650)   |
| 7.4 (modeling)  | 10.1 (Newton's Law of Cooling p. 604)   |
| not found   | Separation of variables and numerical methods for differential equations  |
| 8.1   | 7.1   |
| 8.2 (tour of 3-D space)                                     | not found   |
| 8.3 (better development; teaching to do it)                 | 7.1   |
| 8.4   | 7.1   |
| 8.5 (linear functions)                                      | not found   |
| 8.6   | 7.1-7.4   |
| 9.1, 9.2  | 7.2   |
| 9.3   | 7.3   |
| 9.4   | not much  |
| 9.5   | 7.4   |
| not found   | Many other topics, including: integration methods (substitution, parts); double integrals; changing limits of integration, etc. |



Table of Contents listings from both textbooks:

GLS

Functions  
The derivative  
applications of the derivative  
Techniques of differentiation  
The exponential and natural logarithm functions  
Applications of the exponential and  
natural logarithm functions  
The definite integral  
Functions of several variables

The trigonometric functions  
Techniques of integration  
Differential equations  
Probability and calculus  
Taylor polynomials and infinite series

CCH

A library of functions  
Key concept: The derivative  
Key concept: the definite integral  
Short-cuts to differentiation  
Using the derivative  
Using the definite integral  
Differential equations  
Functions of many variables  
Calculus for functions of many  
variables

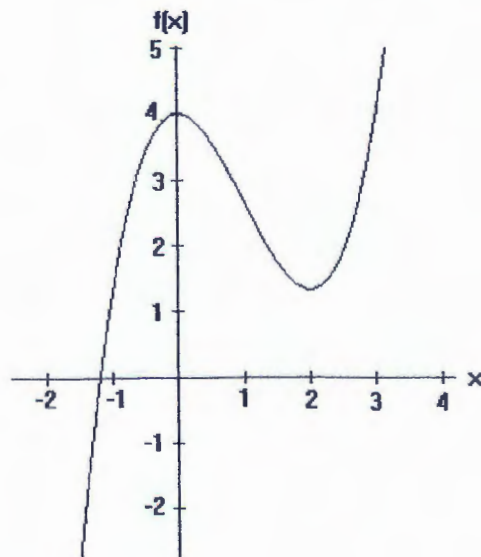
## APPENDIX B: WRITTEN TEST

1. Given below is the graph of a function.

Test # \_\_\_\_\_

a. On what intervals is  $f(x)$  increasing? Decreasing?

b. Estimate any points at which a local maximum or a local minimum occurs.



2. Consider the function:

$$f(x) = \frac{2}{3}x^3 - 2x^2 + 4$$

- a. Using calculus, determine whether the above function is increasing or decreasing at  $x = 1$ .
- b. Also using calculus, determine any values at which a local maximum or a local minimum occurs.

3. Find the first derivatives of the following functions:

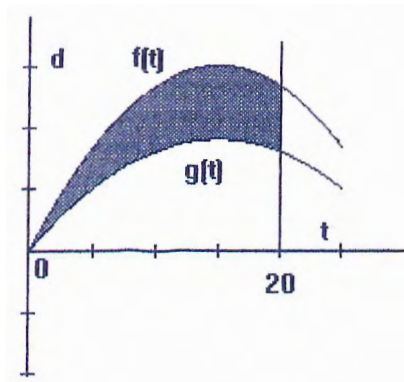
- a.  $y = (3x^2 + 1)^3$

- b.  $y = xe^{2x}$



4. A candy company knows that the revenue,  $R$ , from sales of a certain product is a function of the selling price,  $p$ , and is given by  $R = -500p^2 + 4500p$ . What is the maximum revenue? ( $R$  and  $p$  are in dollars).
5. A publishing company knows that the cost (in dollars) of printing a book depends on the number of copies printed. The cost of printing  $N$  copies is  $C = f(N)$ . Suppose that when  $N = 500$ , the derivative of the function is 2 (that is,  $f'(500) = 2$ ). What conclusions can you draw?
6. A compact car in motion on the freeway uses  $f(x)$  gallons of gas in driving  $x$  miles.
- For what values of  $x$ , if any, is  $f'(x)$  positive?  
For what values of  $x$ , if any, is  $f'(x)$  negative?
  - A motorhome uses  $g(x)$  gallons of gas in driving  $x$  miles. How would you expect the graphs of  $f(x)$  and  $g(x)$  to compare?

7. Suppose that on a typical day the rate at which the electric company accumulates revenue (in dollars per minute) is given for any time in the day by the function  $f(t)$ . The rate at which the company's operating costs accumulate (in dollars per minute) is given for any time in the day by the function  $g(t)$ . These functions are graphed below.



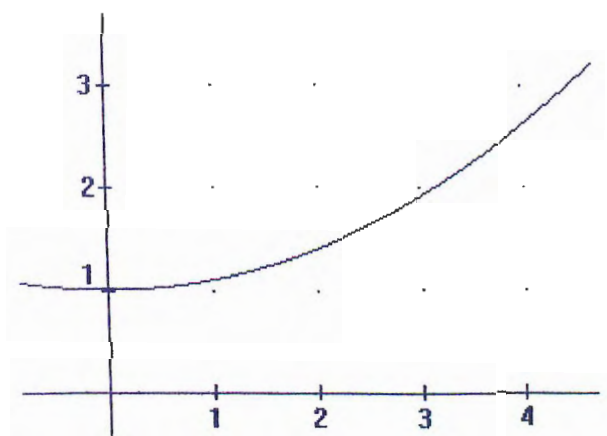
a. How could you use calculus to find the company's revenue and cost functions?

b. How could you use calculus to compute the shaded area?

c. What does the shaded area represent economically?

8. Given the following graph and table of values for a function,  $f(x)$ ,

a. Approximate the derivative of the function at  $x = 2$ .



| x | 0   | 1   | 2   | 3   |
|---|-----|-----|-----|-----|
| y | 1.0 | 1.1 | 1.4 | 1.9 |

b. Approximate the integral of the function from  $x = 1$  to  $x = 3$ .



9. Compute the following definite integrals:

a.  $\int_1^2 4x^3 dx$

b.  $\int_0^{1/2} e^{2x} dx$

10. There is a theorem, called the Fundamental Theorem of Calculus, which is a statement about the relationship between the derivative and the definite integral. What is this theorem?

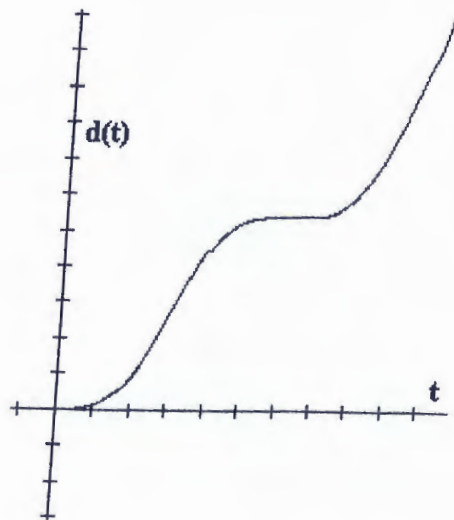
## APPENDIX C: STUDENT INTERVIEW

Affect/motivation portion.

1. What was a typical class like? Follow-up: Was there enough time for questions?
2. How much homework was given?
3. Why did you take this particular section of applied calculus? Follow-up: Did you have a choice?
4. Did you like your applied calculus class? Follow-up: What did you like best?
5. What did you think of the way the course was taught?
6. How would you rate the instructor? Follow-up: What made the instructor “good?”  
Any improvements needed? Would you say that your instructor taught in a way that matched the goals of the textbook? Did the teaching style agree with your learning style?
7. How does this course compare to other math classes you've had? Follow-up: Did you work as hard for this class as for other math classes you've had?
8. What would you change about your applied calculus class? (Possible follow-on points: The teaching? The homework? The book? Etc.)
9. Would you recommend your applied calculus class to a friend? Follow-up: Would you take it again?

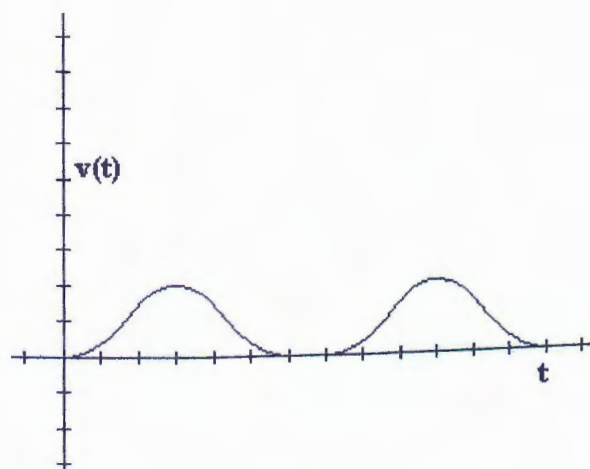
Concepts portion: note that the graphs accompanying the interview questions below were shown on separate pages to the students during the interviews.

1. Suppose your brother understands what a function is, but has never heard of "the derivative." How would you explain it to him? Possible follow-up cues: What is a derivative? Do you remember computing derivatives? If so, how do you do it? Do you remember what derivatives are were used for? Suppose I tell you that a derivative describes how something changes....
2. Suppose your roommate understands what a function is, but has never heard of "the integral." How would you explain it to him/her? Possible follow-up cues: What is a definite integral? How is it used? What is an indefinite integral?
3. Below is a graph showing a car's distance traveled as a function of time. Choose a graph (of the three on the following page) which could represent the car's velocity over time and tell why. Possible follow-up cues: How is velocity related to distance traveled? Is there something you could say involving derivatives and/or integrals which would relate graphs a, b, and/or c to the first one?

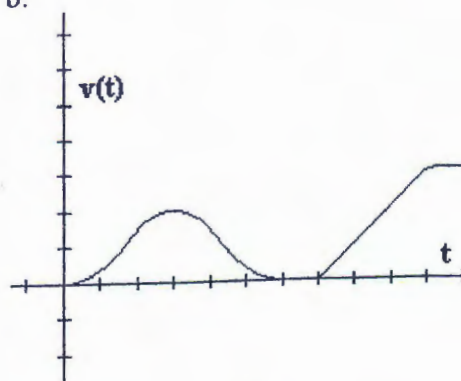




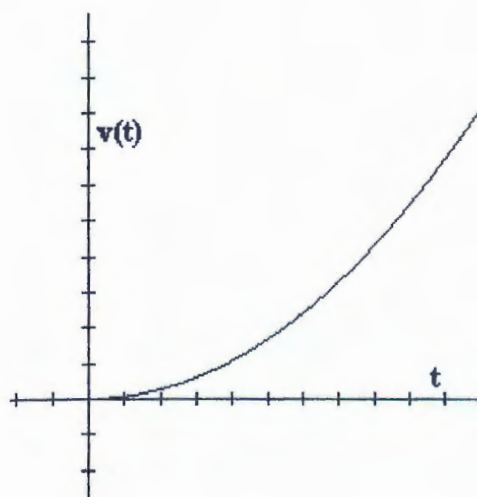
a.



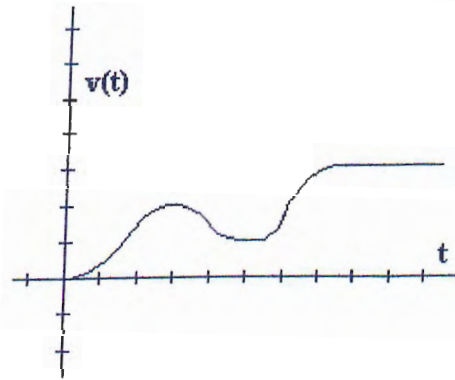
b.



c.

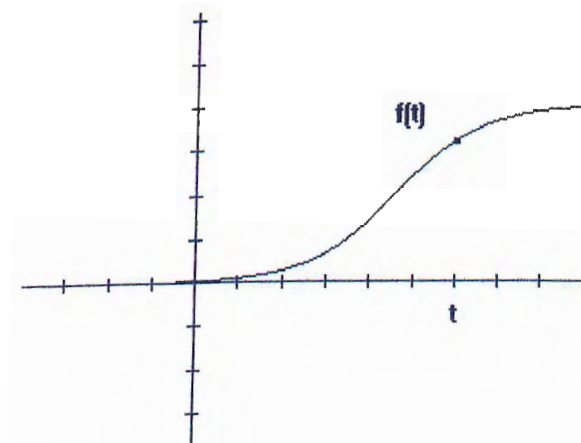


4. Here is a graph showing the rate of travel of a car (i.e., velocity). What could I learn from the derivative? (Pause for response and discussion.) Follow-up cues: What could I learn from the derivative at a specific point? What could I learn from the general derivative of the function? What could I learn from the definite integral between two  $t$ -values? What could I learn from the integral in general?



5. How are integrals and derivatives related? Follow-up cues: Are they completely separate ideas, or is there something that connects them? Can you use both terms in a single statement?
6. Suppose I have a function of two variables,  $f(x,y)$ . What's the interpretation of the partial derivative of  $f$  with respect to  $x$ ? Possible follow-up cues: The partial derivative of  $f(x,y)$  with respect to  $x$  is written this way.... Do you remember computing partial derivatives?

7. Here is a graph showing how a population of wild animals varies over time. Suppose you know the rule for this function,  $y = f(t)$ , where  $f(t)$  is the number of animals at time  $t$ . How could you use the rule to determine rate of increase in the number of animals at a particular time  $t$ ? Follow-up cues: What (in calculus terms) would give you the rate of increase at a particular point?



8. Here is the definition of something. Have you seen it? What does this tell us?

$$\lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

9. For things you forgot, can you pinpoint a reason why you forgot them? Follow-up: the course, or your own study? Is there anything else you would like to say?



## APPENDIX D: CORRESPONDENCE BETWEEN RESEARCH QUESTIONS AND

### TEST/INTERVIEW ITEMS

The six main areas from calculus targeted by the research questions for this study are found in the left column of the first table below, and the corresponding items from the data collection instruments are found in the right column.

|   |   |
|---|---|
| Description of graphs of functions                            | Test item 1 (without calculus)<br>Test item 2 (with calculus)   |
| Differentiation of functions                                  | Test item 3 (analytically)<br>Test item 4 (analytically)<br>Interview item 3 (graphically)  |
| Interpretation of the derivative                              | Test item 4      Interview item 1<br>Test item 5      Interview item 3<br>Test item 6      Interview item 4<br>Interview item 6<br>Interview item 7 |
| Integration of functions                                      | Test item 9 (analytically)<br>Interview item 4 (graphically)  |
| Interpretation of the definite integral                       | Test item 7      Interview item 2<br>Interview item 4   |
| Relationship between the derivative and the definite integral | Test item 10      Interview item 5  |

## APPENDIX E: TEST GRADING KEY

1. a. 3 points total

Desired answer:

$(-\infty, 0]$  increasing (1 point;  $(-\infty, 0)$  or  $x < 0$  or  $x \leq 0$  also acceptable)  
 $[0, 2]$  decreasing (1 point;  $(0, 2)$  or  $0 \leq x \leq 2$  or  $0 < x < 2$  also acceptable)  
 $[2, \infty]$  increasing (1 point;  $(2, \infty)$  or  $x \geq 2$  or  $x > 2$  also acceptable)

Notes:

1/2 point per endpoint of each interval (including  $\infty, -\infty$ );

1/2 point off per interval for flipping inequality signs;

1/2 point off per extra endpoint (using something other than  $-\infty$  in the first interval or something other than  $\infty$  in the last).

1. b. 3 points total

Desired answer:

$x = 0$  (1 point; the pair  $(0, 4)$  also acceptable)

$x = 2$  (1 point; the pair  $(2, y)$  also acceptable for  $1 \leq y \leq 1.5$ )

Classification (1/2 point for correctly stating that a local max occurs at  $x = 0$ ;  
 1/2 point for correctly stating that a local min occurs at  $x = 2$ )

Notes:

2 points for drawing correctly and labeling the graph ("max" and "min") without explicitly saying  $x = 0, x = 2$ ;

1 point for drawing with no labeling (student was looking at the right points);  
 $x$  values must be correct ( $x = 4$  gets no credit);

listing the correct values in the correct order but not explicitly stating which is a local max and which is a local min gets no point for classification;

correct classification point given if the  $x$  value is correct with correct classification and the other  $x$  value is incorrect but its classification is correct (for example, "local max at  $x = 0$ , local min at  $x = 3$ " gets 2 points, as does " $x = 4$  max,  $x = 2$  min");

1/2 point off for each extra endpoint included as a max or min such as  $(-1.2, -2.5)$  or  $(3, 5)$ .

2. a. 4 points total

Desired answer:

$f'(x) = 2x^2 - 4x$  (2 points for correctly computing the derivative)

$f'(1) = -2$  (1 point for correct computation)

decreasing (1 point)

Notes:

2 points for finding correct derivative in part b, even if not written in part a;

Arriving at the correct answer without using calculus (i.e., other than the above steps) gets no points.

2. b. 4 points total

Desired answer:

$$2x^2 - 4x = 0$$

$$x = 0$$

$$x = 2$$

(2 points for setting the derivative equal to 0)

(1 point for first correct value)

(1 point for second correct value)

Notes:

Don't deduct any points for using an incorrect derivative obtained in part a; not points extra for max, min classification--also no deduction for incorrect classification;

Arriving at the correct answer without using calculus (i.e., other than the above steps) gets no points.

3. a. 2 points total

Desired answer:

$$3(3x^2 + 1)^2 (6x) = 18x(3x^2 + 1)^2 \quad (2 \text{ points for correct final answer})$$

Notes:

1 point if chain rule is applied correctly and an arithmetic error occurs thereafter;

0 points if chain rule is applied incorrectly.

3. b. 2 points total

Desired answer:

$$x(2e^{2x}) + e^{2x} = e^{2x}(2x + 1) \quad (2 \text{ points for correct final answer})$$

Notes:

1 point if product rule is applied correctly and an arithmetic error occurs thereafter;

0 points if product rule is applied incorrectly.

4. 5 points total

Desired answer:

$$R' = -1000p + 4500 \quad (2 \text{ points for correct derivative; 1 point if half correct})$$

$$\text{Set } R' = 0$$

(1 point for knowing to set the derivative equal to 0)

$$p = 4.5 \text{ dollars}$$

(1 point for finding correct critical value)

$$R(4.5) = \$10,125$$

(1 point for correct answer--1/2 point for knowing to plug 4.5 into original equation, 1/2 point for correct calculation)

Notes:

No credit for guessing (without using calculus) the correct critical value and plugging it in--and also no point for the correct answer from a correctly guessed critical value.



5. 4 points total  
Desired answer: "Marginal cost at 500 books printed is \$2; i.e., 501st book costs \$2 to make."  
Notes:  
Only 3 points given for writing "each" in place of "501st;"  
2 points given for saying "each book costs \$2" without mentioning "at 500;"  
1 point given for saying that the slope = 2 at  $N = 500$ ;  
0 points given for saying "cost for 500 books=\$2."
6. a. 3 points total  
Desired answers:  
 $x > 0$  (1 1/2 points)  
none (1 1/2 points)  
Notes:  
"0" does not mean "none" (i.e., no credit for "0" on second part);  
"All" on first part and "none" on second part gets 2 1/2 points.
6. b. 3 points total  
Desired answer:  $g(x) > f(x)$  (3 points)  
Notes:  
"g used more gas" also gets 3 points; "Slope greater for g" gets 2 points.
7. a. 4 points total  
Desired answer: "Integrate  $f(t)$  to get the revenue function; integrate  $g(t)$  to get the cost function."  
Notes:  
Saying "find the antiderivative" rather than "integrate" is correct (4 points if used in reference to both functions);  
Saying "Integrate both" without reference to which is revenue and which is cost still receives 4 points; "Has to do with integrals" gets 2 points.
7. b. 6 points total  
Desired answer:  
 $\int_0^{20} [f(t) - g(t)] dt$  (2 points for knowing to integrate; 2 points for correct limits of integration; 2 points for  $f(t) - g(t)$ ).  
Notes:  
1 point for each of the correct limits, even if not associated with an integral (e.g., "plug in 20 for x" gets 1 point);  
" $g(t) - f(t)$ " or " $g - f$ " gets 0 points rather than 2 points;  
"subtract lower from higher" gets 2 points for that portion; so does  $f - g$ , even if not associated with an integral;  
Saying "integrate" gets the 2 points for knowing to integrate.

7. c. 5 points total

Desired answer:

"Total profit over the first 20 minutes of the day." (3 points for "total profit;"  
2 points for the rest of the  
desired answer)

Notes:

For the 3 point portion, the following answers are equivalent to "total profit:"  
"profit," "net income;"

For the 3 point portion, the following answers receive 2 points: "income,"  
"profit margin," "profit per minute."

8. a. 5 points total

Desired answer:

A good method

(3 points; acceptable answers involve using the slope

between  $x = 1$  and  $x = 2$  ( $\frac{1.4 - 1.1}{2 - 1} = .3$ ); or using the slope

between  $x = 2$  and  $x = 3$  ( $\frac{1.9 - 1.4}{3 - 2} = .5$ ); or using the slope

between  $x = 1$  and  $x = 3$  ( $\frac{1.9 - 1.1}{3 - 1} = .4$ ), or the average of  
the first two acceptable slopes (also .4))

Correct calculation (2 points)

Notes:

Estimation from the graph with an almost correct "eyeballed" answer  $a$ ,

$.2 \leq a \leq .6$  gets 2 points;

A correct answer only (.3, .4, or .5) gets 3 points;

Correctly guessing the function  $y = 1 + .1x^2$  and computing  $f'(2) = .4$  gets 5  
points;

Using a correct method but without 2 as an endpoint (i.e., using 1 as the right  
endpoint) gets 1 point;

Trying to differentiate any function (whether a good guess or not) gets 1 point;

Trying to find a slope or writing a slope formula gets 1 point.

8. b. 5 points total

Desired answer:

A good method

(3 points; all of the following methods are acceptable:  
left-hand rule ( $1.1+1.4 = 2.5$ ), right-hand rule ( $1.4+1.9 = 3.3$ ), midpoint rule ( $2 \times 1.4 = 2.8$ ), trapezoidal rule  
( $\frac{1.1+1.4}{2} + \frac{1.4+1.9}{2} = 2.9$ ), trapezoid between  $x = 1$  and  
 $x = 3$  ( $\frac{1.1+1.9}{2} = 3.0$ ))

Correct calculation (2 points)

Notes:

Estimation from the graph with an almost correct "eyeballed" answer  $a$ ,  
 $2.4 \leq a \leq 3.4$  gets 2 points;

A correct answer only (2.5, 2.8, 2.9, 3.0, 3.3) gets 3 points;

Wrong method, happenstance right answer gets 0 points;

Trying to integrate any function gets 1 point;

Knowing it's area, nothing more gets 1 point;

Correctly guessing the function  $y = 1+.1x^2$  and computing  $\int_1^3 (1+.1x^2)dx = 2.867$   
gets 5 points.

9. a. 2 points total

Desired answer:

$$x^4 \Big|_1^2 = 15 \quad (2 \text{ points})$$

Notes:

Arithmetic error after applying correct integration rule gets 1 point;

Incorrect integration rule gets 0 points;

Answers admittedly obtained by calculator get 1 point.

9. b. 2 points total

Desired answer:

$$\frac{e^{2x}}{2} \Big|_0^{1/2} = \frac{e-1}{2} \quad (2 \text{ points})$$

Notes:

Arithmetic error after applying correct integration rule gets 1 point;

Incorrect integration rule gets 0 points;

Answers admittedly obtained by calculator get 1 point.



10. 5 points total  
 Desired answer:  
 Correct statement of linkage (2 points; “undo each other” or “inverses” or “an integral is an anti-derivative” or an equivalent statement)  
 $\int_a^b F'(x)dx = F(b) - F(a)$  (2 points)  
 Stating other conditions (1 point; the conditions are that  $f(x)$  is continuous on the interval  $a \leq x \leq b$  and  $F(x)$  is an antiderivative of  $f(x)$ )
- Notes:  
 For the first portion, saying “opposites” gets 1 point.  
 For the portion where the actual theorem is written symbolically, the answer  $\int f'(x) = f(x) + C$ , with or without the  $C$ , gets 1 point.

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