ABSTRACT

Title of dissertation: MOTIVIC COHOMOLOGY OF GROUPS OF ORDER p^3

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In this thesis we compute the motivic cohomology ring (also known as Bloch's higher Chow groups) with finite coefficients for the two nonabelian groups of order 27, thought of as affine algebraic groups over $\mathbb{C}$. Specifically, letting τ denote a generator of the motivic cohomology group $H^{0,1}(BG, \mathbb{Z}/3) \cong \mathbb{Z}/3$ where G is one of these groups, we show that the motivic cohomology ring contains no τ-torsion, and so can be computed as a weight filtration on the ordinary group cohomology. In the case of a prime $p > 3$, there are also two nonabelian groups of order p^3. We make progress toward computing the motivic cohomology in the general case as well by reducing the question to understanding the τ-torsion on the motivic cohomology of a p-dimensional variety; we also compute the motivic cohomology of BG for general p modulo the τ-torsion classes.

MOTIVIC COHOMOLOGY OF GROUPS OF ORDER p^3

by

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Dedication

For Adam, for the set of all primes not discussed in this thesis.

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I am eternally thankful to everyone who helped support me through the demoralizing, lonely, isolating existential purgatory that is grad school. Okay perhaps that's a bit hyperbolic; but more seriously, no thesis ever gets completed without the help of a solid support network, and I am extremely fortunate to have one of the best.

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[1]...but do you know what a comathematician is?

me find the energy to keep working. (Okay some of that might have just been the caffeine, but still.)

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Chapter 1: Introduction

For p an odd prime, there are two nonabelian groups of order p^3, both arising as central extensions of $\mathbb{Z}/p \times \mathbb{Z}/p$ by $\mathbb{Z}/p$. The geometry of the classifying stacks of these groups provides an interesting example for computations in algebraic geometry, because they are situated right at the edge of what current theories are able to compute. While the group cohomology with both integral ([12]) and finite ([10], [4]) coefficients has been known for a long time, significant recent progress has been made on computing other invariants of these groups: The Chow groups were computed by Yagita in 2012 ([18]); the stable cohomology was computed by Bogomolov and Böhning in 2014 ([3]); in 2015 Pădurariu showed that the motives of these groups are mixed Tate ([14]).

In this paper we contribute to the endeavor of computing algebraic invariants of these groups by computing the motivic cohomology rings with coefficients in $\mathbb{Z}/3$ of the nonabelian groups of order 27. The motivic cohomology ring of a scheme X is a bigraded ring $H^{*,*'}(X, A)$ that specializes for certain degrees and coefficients to other important classical invariants, such as the Chow groups, the Milnor K-theory, and the étale cohomology with finite coefficients, among other examples. As Voevodsky describes in the introduction to [13], the power of motivic cohomology comes from applying results about the structural properties of the motivic cohomology ring to the specific cases of these classical invariants, in order to uncover new results about these invariants. In this paper we exploit exactly that power, combining seemingly disparate results about the Chow groups and the stable cohomology of these groups of order 27 in order to deduce the structure of the motivic cohomology ring. We also compute for general odd p the motivic cohomology modulo the kernel of a map to the étale cohomology.

The narrative goes as follows: Given an algebraic group G, considered as an affine group scheme over $\mathbb{C}$, we can define the Chow groups of G (as defined by Totaro in [15]), the ring of cohomological invariants of G, and the étale cohomology of BG with finite coefficients. These three invariants are related to each other by the spectral sequence defined by Bloch and Ogus in [2]. The motivic cohomology ring of BG with finite coefficients is related by a long exact sequence to the page two terms of that spectral sequence; there is also a map

$$\times\tau^{*-*'} : H^{*,*'}(X, \mathbb{Z}/p) \to H^*_{\text{et}}(X, \mathbb{Z}/p)$$

from the motivic cohomology ring to the étale cohomology, given by multiplying by a generator τ of $H^{0,1}(X, \mathbb{Z}/p) \cong \mathbb{Z}/p$. For the nonabelian groups of order p^3, Yagita proved in [18] that the map from the Chow groups to the étale cohomology is injective. We use that result together with the connection between the Chow groups and the cohomological invariants given by the Bloch-Ogus spectral sequence to show that these groups have no nontrivial cohomological invariants of degree three. (In fact we prove a slightly more general result about the vanishing of degree three invariants.) The connection between the cohomological invariants and the motivic cohomology then allows us to prove that for the case $p = 3$, a certain variety X that 'approximates' the classifying space BG has no τ-torsion in its motivic cohomology. Armed with that fact, we can then leverage structural properties of the motivic cohomology ring—specifically the existence of a localization sequence relating the motivic cohomology of a scheme to that of its subschemes—to prove that BG itself has no τ-torsion.

For $p > 3$ there are too many unknown terms on page two of the Bloch-Ogus spectral sequence to use the same method of proof. However, we still get the result that G has no nontrivial degree three cohomological invariants, and that turns out to be the key piece of the puzzle in determining the structure of $H^{*,*'}(BG, \mathbb{Z}/p)/(\ker(\times\tau^{*-*'}))$; so although we only prove that this kernel is trivial in the case $p = 3$, we're still able to make progress in the computation for general p.

In chapter 2 we review background on the definition and properties of Chow groups of a scheme, including the equivariant Chow groups for a scheme equipped with an action by a group G; this leads to Totaro's definition of the Chow ring of an algebraic group. Chapter 3 defines the motivic cohomology ring as Bloch's higher Chow groups, and surveys some basic properties. We also describe in more detail the $\times\tau$ map on motivic cohomology and its relationship to the étale cohomology and the Bloch-Ogus spectral sequence. In chapter 4 we describe the two nonabelian groups of order p^3 in more detail and give the known results about their ordinary cohomology and representation theory. Chapter 5 switches gears a bit to define the ring of cohomological invariants and discuss its connection to the Bloch-Ogus spectral sequence; here we prove a theorem on sufficient conditions for the vanishing of degree three cohomological invariants. These conditions are satisfied by our groups of order p^3. In chapter 6 we put the pieces together to show that in the case $p = 3$ there is no τ-torsion in the motivic cohomology of either group. Finally in chapter 7 we describe the ring $H^{*,*'}(BG, \mathbb{Z}/p)/(\ker(\times\tau^{*-*'}))$ for each of the two groups for general odd p.

Unless otherwise specified, p is always an odd prime, and finite groups are considered as affine algebraic groups defined over $\mathbb{C}$.

Chapter 2: Chow groups

Before moving up in life to higher Chow groups and motivic cohomology, we should quickly review the classic Chow groups of a smooth scheme X, following the development in Fulton [6]. Let $Z_i(X)$ be the group of dimension i *algebraic cycles* on X, that is, finite $\mathbb{Z}$-linear combinations of dimension i subvarieties of X. For W a dimension $i+1$ subvariety and f a rational function, define the *divisor* of f as

$$(f) = \sum_D \mathrm{ord}_D(f) D,$$

where the sum is taken over all codimension 1 subvarities of W, and $\mathrm{ord}_D(f)$ is the order of vanishing of f along D (see [6] section 1.2). The subgroup of $Z_i(X)$ generated by the divisors of functions as W ranges over all $(i+1)$-dimensional subvarieties is the subgroup of cycles *rationally equivalent to zero*. Define the Chow group $CH_i(X)$ to be the group $Z_i(X)$ modulo this equivalence.

2.1 Properties of Chow groups

We'll briefly survey some of the properties of the Chow groups that will ultimately generalize to the motivic cohomology ring (higher Chow groups). For more details and proofs of these properties see for example [6].

- **Functoriality.** There are two ways in which maps between varieties lead to maps on the corresponding Chow rings. For a proper map $f : X \to Y$ there is a *proper pushforward* $f_* : CH_i(X) \to CH_i(Y)$, which can be thought of as essentially mapping a dimension i cycle to its image under f. If $f : X \to Y$ is a flat morphism with fibers of dimension d, then there is a pullback map $f^* : CH_i(Y) \to CH_{i+d}(X)$, which can be thought of as essentially taking the preimage of cycles of Y. (Again see [6] for all the technical details.)

- **Localization sequence.** One specific example of the pullback and pushforward maps is the case where we have a closed subvariety $Z \subset X$. Then the inclusion map $i : Z \to X$ is proper, and the map $j : X - Z \to X$ is flat. In this case the pushforward and pullback fit into an exact sequence

$$CH_i(Z) \to CH_i(X) \to CH_i(X - Z) \to 0$$

called the *localization sequence.*

- **Ring structure.** In the case that X is smooth, the intersection product of cycles (see [6]) gives a ring structure on the Chow groups graded by codimension. In other words, letting $CH^i(X)$ be the group generated by codimension i cycles up to rational equivalence, there is a product map

$$CH^i(X) \times CH^j(X) \to CH^{i+j}(X)$$

that makes the Chow groups into a graded commutative ring $CH^*(X)$.

- **Cycle map.** For X a smooth variety over $\mathbb{C}$, there is a map

$$CH^*(X) \to H^{2*}(X, \mathbb{Z})$$

called the *cycle map.*

2.2 Equivariant Chow groups and $CH(BG)$

Let G be an affine group scheme of finite type over a field k. (For our purposes, we will typically be taking G to be a finite group thought of as a scheme over the complex numbers.) Let X be a smooth k-scheme with an action by G. For $i \geq 0$, pick a representation V of G over k such that there is a closed G-invariant subset $S \subset V$ with $\operatorname{codim}_V S > i$ such that G acts freely on $V - S$ (see discussion in Totaro [16] for why this is possible for all i). Suppose that the quotient $(X \times_k (V - S))/G$ exists as a scheme. With this set-up, we can define the *equivariant Chow group* $CH^i_G X$ as

$$CH^i_G X = CH^i(X \times_k (V - S)/G).$$

Of course, we have to check that this is well-defined and does not depend on the choices of V and S, and that we can find appropriate representations for any codimension i. See for example [16] (section 2.3) or [15] for these details. Fortunately it turns out that this definition gives rise to a well-defined equivariant Chow ring $CH^*_G X$. That these equivariant Chow rings satisfy the same properties of the Chow ring discussed above, such as functoriality and the existence of a cycle map, follows from the fact that they're defined as the Chow groups of quotient varieties.

The special case we're most interested in is when the variety X is just a point.

Definition 2.2.1. For G an affine group scheme of finite type over a field k and $i > 0$, let V be a representation of G over k, and $S \subset V$ a G-invariant closed subset with $\operatorname{codim}(S) > i$ such that G acts freely on $V - S$. Then define $CH^iBG = CH^i(V-S)/G$.

Chapter 3: Higher Chow groups

3.1 Definition

In this section we define the higher Chow groups of a scheme, following Bloch's construction in [1]. One way to motivate this construction is to think of it as a way to extend the localization sequence for Chow groups to the left.

For a positive integer n, define the standard n-simplex as the affine space

$$\Delta^n := \operatorname{Spec}\left(k[x_0, \dots, x_n] / \left(\left(\sum x_i\right) - 1\right)\right) \cong \mathbb{A}^n.$$

The *faces* of Δ^n are the subvarieties given by setting some subset of the variables equal to zero. For $i = 0, \dots, n$, let $f_i : \Delta^{n-1} \to \Delta^n$ denote the face map that sends x_i to zero.

Let X be a scheme of finite type over a field k. Bloch's key insight for generalizing the Chow groups is the following equivalent characterization of rational equivalence of cycles, which is proved in [6] (as proposition 1.6). First, a bit of notation: If $V \subseteq X \times \mathbb{P}^1$ is a subvariety such that projection onto $\mathbb{P}^1$ induces a dominant map $f : V \to \mathbb{P}^1$, and $P \in \mathbb{P}^1$ is any point, then denote by $V(P)$ the projection of $f^{-1}(P) \subseteq X \times \{P\}$ onto X.

Proposition 3.1.1. *A cycle $\alpha \in Z_k(X)$ is rationally equivalent to zero if and only* *if there exist $(k+1)$-dimensional subvarieties $V_1, \ldots, V_n \subseteq X \times \mathbb{P}^1$ such that*

$$\alpha = \sum_{i=1}^{n} [V(0)] - [V(\infty)],$$

where $[Z]$ denotes the class of Z in $Z_k(X)$.

In other words, a cycle is rationally equivalent to zero essentially if it can be expressed in terms of differences of fibers of maps to $\mathbb{P}^1$. It's this conception that Bloch generalizes.

Let $z^i(X, j)$ be the free abelian group generated by codimension i cycles of the product $X \times_k \Delta^j$ that intersect each face $X \times_k \Delta^r$ in codimension i, where $\Delta^r \subseteq \Delta^j$ is the image of a face map as defined above. (In other words, we want to toss out any cycles of $X \times_k \Delta^j$ that contain faces of Δ^j; our interest is in cycles more native to X, in a sense.) For these cycles the pullback along the face maps is well defined: Let $\delta_m = f_m^* : z^i(X, j) \to z^i(X, j-1)$ for $m = 0, \ldots, j$.

We can now re-frame rational equivalence in terms of these face maps. By proposition 3.1.1, a cycle on X is rationally equivalent to zero precisely when it lies in the image of the map

$$\delta_0 - \delta_1 : z^i(X, 1) \to z^i(X, 0).$$

Here for a cycle $Z \in z^i(X, 1)$ we're thinking of $\delta_m(Z)$ as essentially the fiber over $m \in \Delta^1 \cong \mathbb{A}^1$ for $m = 0, 1$. Hence we can think of the Chow group $CH^i(X)$ as being cycles on X modulo the image of $\delta_0 - \delta_1$. This formulation now generalizes naturally:

Definition 3.1.2. Let X be a scheme of finite type over a field k and Δ^n and

$z^i(X,j)$ as above. The face maps δ_i give rise to a simplicial complex of abelian groups:

$$\cdots \to z^i(X,2) \substack{\to\\\to\\\to} z^i(X,1) \rightrightarrows z^i(X,0).$$

Define the *higher Chow groups* of X as the homology of this complex. Specifically,

$$CH^i(X,j) := \frac{\ker\left(\sum(-1)^m\delta_m : z^i(X,j) \to z^i(X,j-1)\right)}{\mathrm{im}\left(\sum(-1)^m\delta_m : z^i(X,j+1) \to z^i(X,j)\right)}.$$

It follows from the discussion above that $CH^i(X,0)$ is the normal Chow group $CH^i(X)$.

3.2 Properties of higher Chow groups

This section summarizes the main properties of the higher Chow groups that we will make use of. All of these are extensions of similar properties for the classic Chow groups. See Bloch's original paper [1] as well as the corrections to a couple of the proofs given by Levine [11] for more details.

- **Ring structure.** In the case that X is a smooth scheme of finite type over k, the higher Chow groups have the structure of a bigraded commutative ring (see [1]). The product map

  $$CH^p(X,q) \otimes CH^r(X,s) \to CH^{p+r}(X,q+s)$$

  comes from composing the natural map

  $$CH^p(X,q) \otimes CH^r(X,s) \to CH^{p+r}(X \times X,q+s)$$

with the pullback of the diagonal $\Delta : X \to X \times X$.

- **Localization sequence.** As mentioned above, one of the main useful things about higher Chow groups is that they provide a way to extend the localization sequence for Chow groups to the right. For a closed subvariety $Z \subset X$ of codimension d and $U = X - Z$, the localization sequence takes the form

$$\cdots \to CH^i(U, j+1) \to CH^{i-d}(Z, j) \to CH^i(X, j) \to CH^j(U, j-1) \to \cdots.$$

For a scheme X with an action of a group scheme G, we can define equivariant higher Chow groups analogously to how we defined the equivariant Chow ring above; see [5] for details.

3.3 Connection to motivic cohomology

For schemes X of finite type over a field, Bloch's higher Chow groups coincide with the motivic cohomology groups of X as defined by Voevodsy (see [17] and [13]), but with annoyingly different indexing. Specifically, the conversions are as follows:

$$CH^i(X, j) \cong H^{2i-j,i}(X, \mathbb{Z});$$

$$H^{m,n}(X, \mathbb{Z}) \cong CH^n(X, 2n-m).$$

For the remainder of this paper we will use the motivic cohomology indexing. Define the *weight* of an element of the motivic cohomology group of bidegree (m, n) to be $w = 2n - m$. It follows from the relationship to higher Chow groups that the weight zero motivic cohomology corresponds to the normal Chow groups. It further follows that the motivic cohomology groups vanish for $w < 0$, as well as for $n < 0$.

3.4 Connection to étale cohomology

In the case of finite coefficients, the Beilinson-Lichtenbaum conjecture, proven by Voevodsky in [17], provides an identification of certain motivic cohomology groups with the étale cohomology groups of X. For X a smooth scheme of finite type over k and n an integer invertible in k, there is a cycle map

$$H^{m,n}(X, \mathbb{Z}/n) \to H^m_{\mathrm{et}}(X, \mu_n^{\otimes n}).$$

The Beilinson-Lichentenbaum conjecture asserts that this map is in fact an isomorphism in a wide range of degrees.

Theorem 3.4.1 (Voevodsky)**.** *Let X be a smooth scheme of finite type over a field k, and suppose that n is invertible in k. Then the cycle map*

$$H^{m,n}(X, \mathbb{Z}/n) \to H^m_{et}(X, \mu_n^{\otimes n})$$

is an isomorphism for $m \leq n$, and is injective for $m = n + 1$.

Note that for $k = \mathbb{C}$, the étale cohomology is identified with the ordinary cohomology $H^m(X, \mathbb{Z}/n)$.

Since we know by the identification with higher Chow groups that motivic cohomology is trivial for $m > 2n$, in the case of finite coefficients this creates a wedge in which potentially interesting behavior can occur, namely bidegrees with $m/2 \leq n < m$.

3.5 The $\times\tau$ map

We are now ready to introduce the main protagonist (or antagonist?) of our story. For a field k of characteristic not p and containing a pth root of unity, we have (by theorem 3.4.1 or corollary 4.9 in [13]) that $H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/p) \cong \mu_p \cong \mathbb{Z}/p$. Fix a generator $\tau \in \mathbb{Z}/p$ of this group. For an irreducible variety X over k, we also have that $H^{0,1}(X, \mathbb{Z}/p) \cong H^0_{\text{et}}(X, \mathbb{Z}/p) \cong \mathbb{Z}/p$, and by slight abuse of notation we'll also write τ for the image of τ under the map $H^{0,1}(\operatorname{Spec} k, \mathbb{Z}/p) \to H^{0,1}(X, \mathbb{Z}/p)$ induced by the structure map $X \to \operatorname{Spec} k$.

By the product structure on the motivic cohomology, multiplication by τ gives a map of bidegree $(0, 1)$. It turns out that understanding this map is key to understanding the structure of the motivic cohomology ring with finite coefficients. We know by the Beilinson-Lichtenbaum conjecture that for fixed m, as n increases we eventually hit a point where $H^{m,n}(X, \mathbb{Z}/p)$ is isomorphic to the étale cohomology (specifically when $n \geq m$). In other words, given a nontrivial class $\alpha \in H^{m,n}(X, \mathbb{Z}/p)$ where $n < m$, we can identify $\tau^{m-n}\alpha$ with a class in $H^m_{\text{et}}(X, \mathbb{Z}/p)$. To the extent that there are interesting 'extra' classes in the motivic cohomology that are invisible in the étale cohomology ring, those classes must get sent to zero by some power of τ. Hence to study the information contained in the motivic cohomology ring that's absent from the étale cohomology, we want to zoom in and focus on the τ-torsion elements.

Yagita [18] makes this precise as follows, in the case that $k = \mathbb{C}$: In this case the étale cohomology is the same as the ordinary cohomology, and the $\times\tau^{m-n}$ map composed with the isomorphism from the Beilinson-Lichtenbaum conjecture can be identified with the cycle map $cl : H^{m,n}(X, \mathbb{Z}/p) \to H^m(X, \mathbb{Z}/p)$. Yagita defines the motivic filtration of $H^*(X, \mathbb{Z}/p)$ by $\operatorname{gr}^i H^*(X, \mathbb{Z}/p) = F_i^*/F_{i-1}^*$, where a class x is in F_i^* if it is equal to $\tau^r y$ for some class y with $w(y) \leq i$. (Recall that for $y \in H^{m,n}(X, \mathbb{Z}/p)$, the weight $w(y)$ is $w(y) = 2n - m$.) In other words, to find the grading of a class x, 'divide out' powers of τ as much as possible, and when you can't go any further take the weight. Defining the ring

$$h^{*,*'}(X, \mathbb{Z}/p) = H^{*,*'}(X, \mathbb{Z}/p)/(\ker(\times\tau^{*-*'}),$$

we get that

$$h^{m,n}(X, \mathbb{Z}/p) \cong \bigoplus_i \operatorname{gr}^{2(n-i)-m} H^m(X, \mathbb{Z}/p)\tau^i.$$

For example, $h^{2n,n}(X, \mathbb{Z}/p) \cong \operatorname{gr}^0 H^{2n}(X, \mathbb{Z}/p)$ is the image of the cycle map $H^{2n,n}(X, \mathbb{Z}/p) \cong CH^n(X) \otimes \mathbb{Z}/p \to H^{2n}(X, \mathbb{Z}/p)$ (in other words, the classes in the regular cohomology that come from weight zero classes, i.e. the Chow ring).

Computation of the motivic cohomology group, then, can be broken down into steps:

- Understand $\ker\left(\times\tau^{*-*'} : H^{*,*'}(X, \mathbb{Z}/p) \to H^{*,*}(X, \mathbb{Z}/p) \cong H^*_{\text{et}}(X, \mathbb{Z}/p)\right)$;
- Understand the 'motivic grading' on $H^*_{\text{et}}(X, \mathbb{Z}/p)$.

If we're lucky (or unlucky, depending on our goals) the answer to the first question will be that the $\times\tau$ map is in fact injective in all degrees, so that $h^{*,*'}(X, \mathbb{Z}/p) \cong H^{*,*'}(X, \mathbb{Z}/p)$, and we can focus on just understanding the motivic grading. For future convenience, we'll give this property the rather uninspired name of τ-injectivity.

Definition 3.5.1. Let X be a smooth scheme separated and of finite type over a field k of characteristic not equal to p and containing a pth root of unity. Call X τ-injective if the map $\times\tau : H^{m,n}(X, \mathbb{Z}/p) \to H^{m,n+1}(X, \mathbb{Z}/p)$ is injective in all degrees. Call an algebraic group over k τ-injective if BG is τ-injective.

In order for this definition to be meaningful, it would be nice to know that there exist cases in which τ-injectivity fails. Fortunately such examples have been computed; for instance, in [9] Kameko defines for each prime p a family of groups H, the smallest of which is order p^{p+3}, such that the cycle map

$$cl : CH^2(BH) \otimes \mathbb{Z}/p \to H^4(BH, \mathbb{Z}/p)$$

fails to be injective. (Here BH is thought of as an affine group scheme over $\mathbb{C}$.) In fact his proof is constructive, in the sense that his proof works by constructing a class as a Chern class of a specific so-called virtual complex representation and then proving that it is in the kernel of the mod p cycle class map. Since we can identify that map with the $\times\tau^2$ map on motivic cohomology, the class constructed in that paper must be τ-torsion. Kameko's construction generalizes a family of examples of larger order described by Totaro in [16], and Kameko conjectures that his examples of order p^{p+3} are in fact the smallest examples where the cycle map fails to be injective in degree 2.

3.6 Connection to the Bloch-Ogus spectral sequence

One computational tool for connecting the Chow ring of a variety to the cohomology with finite coefficients is the spectral sequence constructed by Bloch and

Ogus in [2]. For a variety X over an algebraically closed field k, the first page of this spectral sequence is given by

$$E_1^{rs} = \coprod_{x \in X^{(r)}} H^{s-r}(k(x), \mathbb{Z}/p),$$

where the sum is taken over points $x \in X$ such that $\overline{\{x\}}$ has codimension r, and $H^{s-r}(k(x), \mathbb{Z}/p)$ is the mod p Galois cohomology of the residue field of the stalk at x. Let $\mathcal{H}^d$ denote the sheafification of the Zariski presheaf given by $U \mapsto H^d_{\text{et}}(U, \mathbb{Z}/p)$. In their 1974 paper Bloch and Ogus showed that the rows

$$H^s(k(X), \mathbb{Z}/p) \to \coprod_{x \in X^{(1)}} H^{s-1}(k(x), \mathbb{Z}/p) \to \cdots \to \coprod_{x \in X^{(s)}} H^0(k(x), \mathbb{Z}/p) \to 0$$

are the global sections of a flasque resolution of the sheaf $\mathcal{H}^s$, with the differentials given by sums of residue maps. Therefore taking homology to turn the page amounts to computing the sheaf cohomology of the sheaves $\mathcal{H}^s$, and page two looks like

$$E_2^{rs} = H^r(X, \mathcal{H}^s).$$

It follows naturally from the construction that the diagonal entries on the second page are the mod p Chow groups $CH^r X \otimes \mathbb{Z}/p$, and Bloch and Ogus showed that this spectral sequence converges to the étale cohomology $H^*_{\text{et}}(X, \mathbb{Z}/p)$.

As we've seen, the motivic cohomology ring also provides a connection between the Chow ring and the étale cohomology ring, as it specializes to each in specific degrees. Therefore it's natural to expect some sort of close relationship between the motivic cohomology ring and the terms of this spectral sequence; that relationship is given by the following long exact sequence (see [18]):

$$\cdots \to H^{m,n-1}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} H^{m,n}(X,\mathbb{Z}/p) \to H^{m-n}(X,\mathcal{H}^n) \to$$
$$H^{m+1,n-1}(X,\mathbb{Z}/p) \xrightarrow{\times\tau} H^{4,3}(X,\mathbb{Z}/p) \to \cdots. \tag{3.1}$$

Hence the kernel of the $\times\tau$ map that we're interested in is intimately related to the page two terms of the Bloch-Ogus spectral sequence.

The following lemma follows immediately from this sequence and the fact that the Galois cohomology of the function field vanishes above $\dim(X)$, and is proved as lemma 5.4 in [18].

Lemma 3.6.1. *Let X be a smooth variety over $\mathbb{C}$ of dimension two. Then X is τ-injective; in other words, $H^{*,*'}(X,\mathbb{Z}/p) \cong h^{*,*'}(X,\mathbb{Z}/p)$.*

Chapter 4: Extraspecial groups of order p^3

For p an odd prime, there are precisely two nonabelian groups of order p^3, both arising as central extensions of $\mathbb{Z}/p \times \mathbb{Z}/p$ by $\mathbb{Z}/p$. Throughout, let G_i be the nonabelian group of order p^3 and exponent p^i, for $i = 1, 2$. (These groups go by various notations and names; my notation here is by no means standard, but hopefully the subscripts provide a bit of a mnemonic.) These groups have the presentations

$$G_i = \langle a, b, c : c^p = b^p = [a, c] = [b, c] = 1,\ [a, b] = c,\ a^p = c^{i-1}\rangle.$$

In other words, c generates the center, and a and b generate the two factors under the projection map to $\mathbb{Z}/p \times \mathbb{Z}/p$.

Both the integral and mod p cohomology of these groups have been computed (see for example [10] and [12]), as have the Chow groups [18]. In this section we summarize those results, since we will be building off of them to do computations of the motivic cohomology.

4.1 Mod p cohomology

In [10], Leary computes the mod p cohomology of G_1 using the method of first computing the cohomology of a group $\tilde{G}_1$ fitting into the exact sequence

$$0 \to G_1 \to \tilde{G}_1 \to S^1 \to 0,$$

where S^1 is the unit circle considered as a subgroup of $\mathbb{C}$. The following (with modified notation) is Theorem 6 of [10].

Theorem 4.1.1. *Let $p > 3$ be prime, and G_1 be the nonabelian group of order p^3 and exponent p. Then $H^*(BG_1, \mathbb{Z}/p)$ is generated by elements*

$$y, y', x, x', Y, Y', X, X', d_4, \ldots, d_p, c_4, \ldots, c_{p-1}, z;$$

writing y^*, x^* etc. to stand at once for y and y', x and x', and so on, we have

$$\deg y^* = 1, \quad \deg x^* = \deg Y^* = 2, \quad \deg X^* = 3,$$

$$\deg d_i = 2i - 1, \quad \deg c_i = 2i, \quad \deg z = 2p;$$

and these are subject to the following relations, organized by degree (here when *

appears in a relation it has a consistent value, e.g. y^*x^* means yx and $y'x'$):

Degree	*Relations*
2	$yy' = 0$
3	$xy' = x'y,\ \ yY' = y'Y,\ \ y^*Y^* = 0$
4	$(Y^*)^2 = YY' = 0,\ \ y^*X^* = x^*Y^*,\ \ Xy' = 2xY' + x'Y,\ \ X'y = 2x'Y + xY'$
5	$X^*Y^* = 0,\ \ XY' = -X'Y,\ \ xX' = -x'X$
6	$x^*(xY' + x'Y) = 0$
$2p-1$	$c_{p-1}y^* = -(x^*)^{p-1}y^*$
	$d_{p-1}x^* = -(x^*)^{p-1}y^*$
$2p$	$d_py = -x^{p-1}Y,\ \ d_py' = x'^{p-1}Y',\ \ c_{p-1}x^* = -(x^*)^p$
	$c_{p-1}Y^* = -(x^*)^{p-1}Y^*,\ \ d_{p-1}X^* = -(x^*)^{p-1}Y^*$
$2p+1$	$x^py' = x'^py,\ \ c_{p-1}X^* = -(x^*)^{p-1}X^*$
	$d_px = x^{p-1}x,\ \ d_px' = -x'^{p-1}X'$
$2p+2$	$x^px' = x'^px,\ \ x^pY' = -x'^pY$
$2p+3$	$x^pX' = -x'^pX$
$4p-5$	$d_{p-1}c_{p-1} = x^{2p-3}y + x'^{2p-3}y' - x^{p-1}x'^{p-2}y'$
$4p-4$	$c_{p-1}^2 = x^{2p-2} + x'^{2p-2} - x^{p-1}x'^{p-1}$
	$d_pd_{p-1} = x^{2p-3}Y - x'^{2p-3}Y' + x^{p-1}x'^{p-2}Y'$
$4p-3$	$d_pc_{p-1} = -x^{2p-3}X + x'^{2p-3}X' - x^{p-1}x'^{p-2}X'$

Additionally if not specified on this list c_i(anything) $= d_i$(anything) $= 0$ for 'anything' $\neq z$, and $(y^*)^2$ and d_p^2 are expressible in terms of the other generators.[1] *The Bockstein acts by $\beta(y^*) = x^*$, $\beta(Y^*) = X^*$, $\beta(d_i) = c_i$ for $i < p$, and $\beta(d_p) = 0$.*

[1] As far as I can tell, Leary's computation doesn't discuss relations for y^2, $(y')^2$, or d_p^2, but it's

For the case $p = 3$, which will be the focus of much of this thesis, the following appears as Theorem 7 in [10].

Theorem 4.1.2. *Let G be the nonabelian group of order 27 and exponent 3. Then $H^*(BG, \mathbb{Z}/p)$ is generated by elements $y, y', x, x', Y, Y', X, X'$, and z, with*

$$\deg y^* = 1, \quad \deg x^* = \deg Y^* = 2, \quad \deg X^* = 3, \quad \deg z = 6,$$

and relations

Degree	*Relations*
2	$yy' = 0$
3	$xy' = x'y, \;\; yY' = y'Y, \;\; y^*Y^* = xy'$
4	$YY' = xx', \;\; Y^2 = xY', \;\; Y'^2 = x'Y$
	$xx' = x^*Y^* - y^*X^*, \;\; Xy' = x'Y - xY', \;\; X'y = xY' - x'Y$
5	$XY = x'X, \;\; X'Y' = xX', \;\; XY' = -X'Y, \;\; xX' = -x'X$
6	$XX' = 0, \;\; x(xY' + x'Y) = -xx'^2, \;\; x'(xY' + x'Y) = -x'x^2$
7	$x^3y' = x'^3y$
8	$x^3x' = x'^3x, \;\; x^3Y' + x'^3Y = -x^2x'^2$
9	$x^3X' = -x'^3X$

As before we have that $\beta(y^*) = x^*$ and $\beta(Y^*) = X^*$.

In the case of the group G_2 of exponent p^2, the mod p cohomology was computed by Diethelm in [4], and is significantly simpler to write down than the exponent p case. The following (with modified notation) is theorem 2(a) in [4].

clear from counting dimensions and comparing to the integral cohomology that they must not be independent generators.

Theorem 4.1.3. *Let p be an odd prime and let G_2 be the nonabelian group of order p^3 and exponent p^2. Then $H^*(G_2, \mathbb{Z}/p)$ is generated by elements*

$$a_1, \ldots, a_{p-1}, b, y, v, w,$$

with

$$\deg a_i = 2i - 1, \quad \deg b = 1, \quad \deg y = 2, \quad \deg v = 2p - 1, \quad \deg w = 2p,$$

with relations

$$a_i a_j = a_i y = a_i v = b^2 = v^2 = 0.$$

4.2 Integral Cohomology

We will also make use of the structure of the integral cohomology rings of G_i, which were computed by Lewis in [12]. The following appeared as theorems 5.2 and 6.26 respectively in that paper.

Theorem 4.2.1. *Let p be an odd prime and G_2 be the nonabelian group of order p^3 and exponent p^2. Then $H^*(G_2, \mathbb{Z})$ has generators $\alpha, \chi, \zeta, \beta_1, \ldots, \beta_{p-1}$, with*

$$\deg \alpha = 2, \quad \deg \beta_i = 2i, \quad \deg \zeta = 2p, \quad \deg \chi = 2p + 1,$$

and satisfying the following relations:

$$p^2 \zeta = p\alpha = p\chi = p\beta_i = 0;$$

$$\chi^2 = \beta_i \alpha = \beta_i \chi = \beta_i \beta_j = 0 \text{ for all } i, j.$$

As with the mod p cohomology, the integral cohomology of G_1 is more complicated to write down:

Theorem 4.2.2. *Let p be an odd prime and G_1 be the nonabelian group of order p^3 and exponent p. Then $H^*(G_1, \mathbb{Z})$ has generators $\alpha, \beta, \mu, \nu, \chi_2, \ldots, \chi_{p-2}, \gamma$, with*

$$\deg \alpha = \deg \beta = 2, \quad \deg \mu = \deg \nu = 3, \quad \deg \zeta = 2p, \quad \deg \chi_i = 2i, \quad \deg \gamma = 2p-2,$$

and

$$p^2\zeta = p\alpha = p\beta = p\nu = p\mu = p\chi_i = p\gamma = 0,$$

and satisfying relations

Degree	*Relations*
5	$\alpha\mu = \beta\nu$
6	$\mu^2 = \nu^2 = 0$
	$\chi_2 = d\mu\nu$ *(for $p > 3$)*
	$p\zeta = e\mu\nu$ *(for $p = 3$)*
$2p$	$\alpha\gamma = \alpha\beta^{p-1}, \;\; \beta\gamma = \beta\alpha^{p-1}$
$2p+1$	$\mu\gamma = \mu\alpha^{p-1}, \;\; \nu\gamma = \nu\beta^{p-1}$
$2p+2$	$\alpha\beta^p = \beta\alpha^p$
$2p+3$	$\alpha^p\mu = \beta^p\nu$
$4p-4$	$\gamma^2 = \alpha^{p-1}\beta^{p-1}$

where d, e denote some element of $\mathbb{Z}_p^*$; in addition for all i, j,

$$\chi_i\chi_j = \alpha\chi_i = \beta\chi_i = \mu\chi_i = \nu\chi_i = \gamma\chi_i = 0.$$

Our interest in the integral cohomology comes mainly from the fact that, as proven by Yagita in [18], for these two groups the Chow groups (and hence the weight zero motivic cohomology groups) are isomorphic to the even degree integral cohomology. The tool we'll use in the computation of the motivic filtration is the universal coefficients sequence for motivic cohomology: from the short exact sequence

$$0 \to \mathbb{Z} \xrightarrow{\times p} \mathbb{Z} \to \mathbb{Z}/p \to 0$$

we get a long exact sequence in motivic cohomology, which gives the following key fact:

Key Fact 4.2.3.

$$\dim H^{m,n}(X, \mathbb{Z}/p) = \dim(H^{m,n}(X, \mathbb{Z})/p) + \dim(p\text{-torsion}(H^{m+1,n}(X, \mathbb{Z})).$$

To that end, the information that we really care about is the dimension of the integral cohomology in each degree. For simplicity of bookkeeping, I'll record dimensions of the cohomology groups modulo the generator ζ of degree $2p$; note that with the exception of the case $G = G_1$ and $p = 3$, which we'll discuss in detail later, we have that

$$H^*(G, \mathbb{Z}) = R[\zeta]/(p^2\zeta),$$

where R is the ring given by the other generators and relations, so we can focus our attention on understanding that R.

Corollary 4.2.4. *For each n, let R_n be the degree n graded piece of $H^*(G_2, \mathbb{Z})/(\zeta)$. Then we have*

$$R_n \cong \begin{cases} 0 & n = 2i-1, 1 \le i \le p; \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & n = 2i, 1 \le i < p; \\ \mathbb{Z}/p & n \ge 2p \end{cases}$$

Proof. Since $\beta_i\alpha = 0$, below degree $2p$ each even degree has generators α^i and β_i, and clearly there are no odd degree generators below degree $2p+1$. Since we are working modulo powers of ζ, the only generator in degree $2p$ is α^p. In higher degree, again since $\beta_i\chi = \chi^2 = 0$, the only generators that aren't multiples of ζ are of the form $\alpha^i\chi^j$ for $j = 0, 1$, meaning there is exactly one generator in each degree. $\square$

Taking multiples of ζ into account, we get the following for each degree:

Corollary 4.2.5.

$$H^n(G_2, \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/p)^i \oplus \mathbb{Z}/p^2 & n = 2ip, i > 0; \\ (\mathbb{Z}/p)^{\lfloor \frac{n}{2p} \rfloor} & n \text{ odd;} \\ (\mathbb{Z}/p)^{\lfloor \frac{n}{2p} \rfloor + 2} & n \text{ even, } 2p \nmid n. \end{cases}$$

Proof. For degrees $n \geq 2p+1$, a set of generators is given by $\alpha^1\chi^j$ as discussed above along with ζ multiplied by each of the generators of degree $n - 2p$, which leads to the given pattern of degrees. $\square$

We can do the same thing in the case of G_1, first counting generators modulo multiples of the class ζ and then using that to find the dimensions of each degree of the cohomology.

Corollary 4.2.6. *For each n, let R_n be the degree n graded piece of $H^*(G_1, \mathbb{Z})/(\zeta)$.*

Then we have

$$R_n \cong \begin{cases} 0 & n = 1; \\ \mathbb{Z}/p \oplus \mathbb{Z}/p & n = 2; \\ (\mathbb{Z}/p)^{i+2} & n = 2i, 2 \leq i < p; \\ (\mathbb{Z}/p)^{i} & n = 2i-1, 2 \leq i \leq p; \\ (\mathbb{Z}/p)^{p+1} & n \geq 2p. \end{cases}$$

Proof. The dimensions for $n = 1, 2, 3$ are clear from the generators of the rings. We break the rest into cases:

(i) Even degrees $n = 2i, 2 \leq i < p$. In this case generators are given by $\alpha^j \beta^{i-j}$ for $0 \leq j \leq i$ and χ_i, for a total of $i + 2$ independent generators. For $p > 3$, the fact that $\mu\nu$ is a multiple of χ_2 means that it does not contribute an extra dimension.

(ii) Odd degree $n = 2i-1, 3 \leq i \leq p$. Since $\alpha\mu = \beta\nu$, we can consider all multiples of $\beta\nu$ redundant, and $\chi_{i-2}\mu = \chi_{i-2}\nu = 0$, generators are given by multiplying each generator of degree $n - 3$ other than χ_{i-2} by μ, and additionally $\nu\alpha^{i-2}$. Since there are i total generators of degree $2i - 4$ by the above, this gives a total of i generators of degree n.

(iii) Even degree $n = 2i, i \geq p$. Because of the relation $\alpha\beta^p = \beta\alpha^p$, we can reduce all generators that are multiples of β so that the power on α is less than p. Hence a generating set is given by $\beta^i, \alpha\beta^{i-1}, \ldots, \alpha^{p-1}\beta^{i-p+1}$ and α^i. Note that for $p = 3$ the generator $\mu\nu$ in degree 6 is a multiple of ζ so is trivial in R_6.

(iv) Odd degree $n = 2i + 1, i \geq p$. In this case there are $p + 1$ generators of degree

$n-3$, and as before generators of degree n are given by multiplying these by μ and ν and applying the relations $\alpha\mu = \beta\nu$ and $\alpha^p\mu = \beta^p\nu$. Hence a generating set is given by $\mu\beta^{i-1}, \mu\alpha\beta^{i-2}, \ldots, \mu\alpha^{p-1}\beta i - p, \nu\alpha^{i-1}$, for a total of $p+1$ generators.

□

As before we can use this to compute the total rank of $H^n(G_1, \mathbb{Z})$ for each n, which tells us the dimension of the Chow groups or equivalently the weight zero motivic cohomology.

Corollary 4.2.7. *For each $n \geq 1$, let $s_n = \lceil \frac{n}{2p} \rceil - 1$, let $i_n = n - 2ps_n$, and let d_n be the dimension of R_{i_n} from the previous corollary. Then*

$$H^n(G_1, \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/p)^{d_n + s_n(p+1)} \oplus \mathbb{Z}/p^2 & n = 2ip, i > 0; \\ (\mathbb{Z}/p)^{d_n + s_n(p+1)} & \textit{else.} \end{cases}$$

4.3 Representations of G_i

Write $\zeta = \exp(2\pi i/p^2)$ and $\omega = \exp(2\pi i/p)$. There are p-dimensional complex representations ρ_i of the groups G_i given explicitly as follows:

For G_1, the representation ρ_1 is given by sending

$$a \mapsto \operatorname{diag}(1, \omega, \omega^2, \ldots, \omega^{p-1}),$$

$$b \mapsto (\delta_{i,j-1}),$$

$$c \mapsto \operatorname{diag}(\omega, \ldots, \omega),$$

where $\delta_{i,j} = 1$ if $i \equiv j \pmod p$ and 0 else.

For G_2, the representation ρ_2 is given by sending

$$a \mapsto \operatorname{diag}(\zeta, \zeta^{1+p}, \zeta^{1+2p}, \dots, \zeta^{1+p(p-1)}),$$

$$b \mapsto (\delta_{i,j-1}),$$

$$c \mapsto \operatorname{diag}(\omega, \dots, \omega).$$

In both cases, the action is free outside of a finite number of lines. Thinking of G_i as a complex algebraic group, the representation ρ_i gives a free action of G_i on the variety $U = \mathbb{A}^p_{\mathbb{C}} - S$, where S is the set on which the action has nontrivial stabilizers.

Understanding the structure and the cohomology of this set S on which the action is not free will be key to understanding the cohomology of BG; specifically we will prove as a lemma that (for both groups G_i) the motivic cohomology of S is τ-injective. First we describe the structure of this set for each group of order p^3 separately, then we present Yagita's computation of the ordinary cohomology of these sets (spoiler: the cohomology is the same for both groups) and use that to compute the motivic cohomology.

4.3.1 The group G_1

Let $V = \mathbb{C}^p - \{0\}$ have the action of G_1 given by restricting the representation ρ_1. Denote by $e_i = (0, \dots, i, \dots, 0)$ the ith standard basis vector of $\mathbb{C}^p$. Let

$$S_0 = \coprod_i \mathbb{C}^*\{e_i\}$$

be the disjoint union of the span of each basis vector intersected with V (here $\mathbb{C}^* = \mathbb{C} - \{0\}$).

Proposition 4.3.1. *G_1 acts on S_0 via the restriction of ρ_1.*

Proof. This is clear since the generators a and c act via diagonal matrices and hence simply scale each e_i, and the generator b acts by permuting the e_i. $\square$

The stabilizer for any point $s \in S_0$ is the subgroup $\langle ac^i \rangle$ for some i; for example any point in $\mathbb{C}^*\{e_1\}$ is stabilized by $\langle a \rangle$.

For each $1 \leq j < p$ we can define a similar set S_j as follows: the matrix $\rho_1(a)$ is conjugate to the matrix $\rho_1(ab^j)$ for $1 \leq j < p$ and to $\rho_1(b)$; let $M_j \in GL_p(\mathbb{C})$ be such that $M_j^{-1}\rho_1(a)M_j = \rho_1(ab^i)$ for $j < p$ and $M_p^{-1}\rho_1(a)M_p = \rho_1(b)$. Then the set $S_j = M_j^{-1}S_0$ is G_1-equivariant, and the stabilizer of any point in S_j is the subgroup $\langle ab^jc^i \rangle$ for $j < p$ and $\langle bc^i \rangle$ for $j = p$. Write S for the disjoint union of the S_j. (Think of the sets $S_0, \ldots, S_p$ as essentially corresponding to the $p+1$ automorphisms of the quotient group $G/\langle c \rangle = \langle \overline{a}, \overline{b} \rangle \cong \mathbb{Z}/p \times \mathbb{Z}/p$.)

Since no point in V is stabilized by $\langle c \rangle$, this in fact exhausts the points with nontrivial stabilizers; the following is lemma 3.1 in [18]:

Proposition 4.3.2. *The group G_1 acts freely on $V - S$.*

4.3.2 The group G_2

Again let $V = \mathbb{C}^p - \{0\}$ and consider V as a G_2-space under the restriction of the action given by ρ_2. Let $v = (1, \ldots, 1) \in V$; clearly all multiples of v are stabilized by b, since b acts by permuting the basis vectors of $\mathbb{C}^p$. Define

$$v_i = \rho_2(a^i)v$$

for $0 \leq i \leq p-1$ (so $v = v_0$). Then

$$bc^{-i} \cdot v_i = bc^{-i}a^i \cdot v = a^i b \cdot v = a^i v = v_i,$$

since $a^i ba^{-i} = bc^{-i}$. So the span of v_i is stablized by $\langle bc^{-i} \rangle$ and G_2 acts on the disjoint union

$$H = \coprod \mathbb{C}^* \{v_i\}.$$

Since for $1 \leq i \leq p-1$ we have $(a^i b^j c^k)^p = c^i$, and no point is stabilized by $\langle c \rangle$, no point can be stabilized by $\langle a^i b^j c^k \rangle$. Hence we've found all the points with nontrivial stabilizers, and G_2 acts freely on $V - H$ (see [18]).

4.3.3 Equivariant motivic cohomology of S and H

Yagita computed the mod p equivariant cohomology of the varieties S and H defined above.

Theorem 4.3.3. *Let $S_0, \ldots, S_p, H$ be as defined above. Then we have the following:*

(a) $H^*_{G_1}(S_0, \mathbb{Z}/p) \cong H^*_{G_1}(S_i, \mathbb{Z}/p)$ for $i = 1, \ldots, p$;

(b) $H^*_{G_1}(S_0, \mathbb{Z}/p) \cong H^*_{G_2}(H, \mathbb{Z}/p) \cong \mathbb{Z}/p[y] \otimes \Lambda(x, z)$, where $\deg(y) = 2$ and $\deg(x) = \deg(z) = 1$.

Here in both cases the class y is the top Chern class of a one-dimensional representation $G \to \mathbb{C}^*$ taking a non-central generator of order p to the pth root of unity ω and the other two generators to 1. (Because of exactly how we've defined the sets H and S_0, in the former case this representation takes b to ω and in the latter case it takes a to ω, but the only importance of that is that it makes the notation slightly more confusing; the important information is that the element y is a Chern class.)

Theorem 4.3.4. *The $\times\tau$ map is injective on the equivariant motivic cohomology rings $H^{*,*'}_{G_1}(S_0, \mathbb{Z}/p)$ and $H^{*,*'}_{G_2}(H, \mathbb{Z}/p)$. This gives the isomorphisms*

$$H^{*,*'}_{G_1}(S_0, \mathbb{Z}/p) \cong H^{*,*'}_{G_2}(H, \mathbb{Z}/p) \cong \mathbb{Z}/p[\tau] \otimes \mathbb{Z}/p[y] \otimes \Lambda(x, z).$$

Proof. This argument follows the structure of the proof of lemma 6.4 in [18], which deals with the case $G = D_8$. We begin with the case of S_0. Consider the localization sequence in motivic cohomology induced by the inclusion $S_0 \times \{0\} \hookrightarrow S_0 \times \mathbb{C}$, where the G_1 action on $S_0 \times \mathbb{C}$ is given by

$$(a^i b^j c^k) \cdot (r, s) = (\rho_1(a^i b^j c^k) r, \omega^i s),$$

where again ω is a primitive pth root of unity. Importantly, since the stabilizer of every point in S_0 is $\langle ac^k \rangle$ for some k, this action is free on the complement $S_0 \times \mathbb{C}^*$. Therefore the localization sequence looks like

$$\cdots \to H^{m-2,n-1}_{G_1}(S_0 \times \{0\}, \mathbb{Z}/p) \xrightarrow{\times y} H^{m,n}_{G_1}(S_0 \times \mathbb{C}, \mathbb{Z}/p) \to$$
$$H^{m,n}\left((S_0 \times \mathbb{C}^*)/G_1, \mathbb{Z}/p\right) \to \cdots.$$

The variety $(S_0 \times \mathbb{C}^*)/G_1$ is not difficult to understand: the element $b \in G_1$ acts on the first factor by transitively permuting the p disjoint lines that make up S_0, so modding out by the action of b we may take a representative of each orbit on $\mathbb{C}^*\{e_1\} \times \mathbb{C}^*$ where e_1 is the first standard basis vector in $\mathbb{C}^p$ as defined earlier. Now

c acts as multiplication by ω on the first factor and trivially on the second, while a acts trivially on the first factor and by multiplication by ω on the second. In other words we have

$$(S_0 \times \mathbb{C}^*)/G_1 \cong \mathbb{C}^*/\langle c \rangle \times \mathbb{C}^*/\langle a \rangle.$$

The ordinary cohomology, then, is the cohomology of $S^1 \times S^1$:

$$H^*((S_0 \times \mathbb{C}^*)/G_1, \mathbb{Z}/p) \cong \Lambda(x, z),$$

where x, z are generators in degree 1 and Λ is the exterior algebra over $\mathbb{Z}/p$. Since the variety $(S_0 \times \mathbb{C}^*)/G_1$ is two-dimensional, by corollary 3.6.1 it is τ-injective; since both generators are degree 1, this means the only question about the motivic grading is whether the element $xz \in H^{2,2}((S_0 \times \mathbb{C}^*)/G_1, \mathbb{Z}/p)$ is a multiple of τ or not, a question which will turn out not to matter for the following computation.

We can now leverage this in the localization sequence to understand the equivariant motivic cohomology of S_0 itself. Of course up to homotopy $S_0 \times \{0\}$ and $S_0 \times \mathbb{C}$ are just S_0 in disguise, so we get the following diagram, writing X for $(S_0 \times \mathbb{C}^*)/G_1$:

$$\begin{tikzcd}
\cdots \arrow[r] & H^{m-1,n}(X,\mathbb{Z}/p) \arrow[r] \arrow[d, "\times\tau"] & H^{m-2,n-1}_{G_1}(S_0,\mathbb{Z}/p) \arrow[r, "\times y"] \arrow[d, "\times\tau"] & H^{m,n}_{G_1}(S_0,\mathbb{Z}/p) \arrow[r] \arrow[d, "\times\tau"] & H^{m,n}(X,\mathbb{Z}/p) \arrow[r] \arrow[d, "\times\tau"] & \cdots \\
\cdots \arrow[r] & H^{m-1,n+1}(X,\mathbb{Z}/p) \arrow[r] & H^{m-2,n}_{G_1}(S_0,\mathbb{Z}/p) \arrow[r, "\times y"] & H^{m,n+1}_{G_1}(S_0,\mathbb{Z}/p) \arrow[r] & H^{m,n+1}(X,\mathbb{Z}/p) \arrow[r] & \cdots
\end{tikzcd}$$

The plan of attack is to use the four lemma and induction on m. We first recall the relevant lemma:

Lemma 4.3.5. *Given the commutative diagram of groups*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "a"] & B \arrow[r] \arrow[d, "b"] & C \arrow[r] \arrow[d, "c"] & D \arrow[d, "d"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D'
\end{tikzcd}$$

suppose that the rows are exact and that the vertical map a is surjective and the maps b and d are injective. Then c is injective.

For any $m \leq 2$, let n be such that $2n - m \geq 0$; then the four lemma applies to the diagram above to give us that $\times\tau : H^{m,n}_{G_1}(S_0, \mathbb{Z}/p) \to H^{m,n+1}_{G_1}(S_0, \mathbb{Z}/p)$ is injective. Since the map is trivially injective if $2n - m < 0$ (because the domain is zero), this gives that S_0 is τ-injective for $m \leq 2$. Now for $m = 3$, the only cases in which the leftmost map might fail to be surjective are $(m, n) = (3, 0), (3, 1)$, which again are trivial. For all other values of n the four lemma again applies, meaning that S_0 is τ-injective for $m \leq 3$. Now induction fully kicks in, since the leftmost map is surjective in all degrees when $m > 3$. Hence the result for $m \leq 3$ implies the result for $m \leq 5$, which implies the result for $m \leq 7$, and so on.

The argument for H is essentially identical, with the action of G_2 on $H \times \mathbb{C}$ given by

$$(a^i b^j c^k) \cdot (r, s) = (\rho_2(a^i b^j c^k) r, \omega^j s).$$

Recall that the stabilizer of any point in H is the subgroup $\langle bc^k \rangle$ for some k, meaning that again this action is free on $H \times \mathbb{C}^*$. In this case by construction the element a acts to permute the p disjoint lines that make up H, and the rest of the argument goes through exactly as above.

$\square$

Chapter 5: Cohomological Invariants

In this chapter we define the notion of a cohomological invariant of a group—briefly, a natural transformation from G-torsors over a field extension K/k to abelian Galois cohomology of K. (See below or [7] for details.) As before we assume for simplicity that the base field k is algebraically closed. We are interested in the ring of cohomological invariants of G because its graded pieces appear on the second page of the Bloch-Ogus spectral sequence (defined in [2]), and the terms on that page slot into a long exact sequence with motivic cohomology groups. Therefore understanding the ring of cohomological invariants of G can be leveraged to gain information about the motivic cohomology.

The main result in this section that will be useful to us is theorem 5.3.1, which gives conditions under which a group G has no nontrivial degree three cohomological invariants. This result will serve as a lemma in our computation of the motivic cohomology of the nonabelian groups of order 27.

5.1 Definition and versal torsors

A cohomological invariant of G is a natural transformation of functors

$$\eta : H^1(-, G) \to H^*(-, \mathbb{Z}/p),$$

where $H^1(K, G)$ is the first nonabelian Galois cohomology set (which can be thought of as isomorphism classes of G-torsors over K), and $H^*(K, \mathbb{Z}/p)$ is the abelian Galois cohomology ring. For our purposes, however, this is not the most convenient way to think of cohomological invariants. Given a quotient variety $X = (V - S)/G$ as above, with $\operatorname{codim} S \geq 2$, the generic fiber T of the map $V - S \to X$ is a versal G-torsor, meaning that any given cohomological invariant is actually completely defined by its value on that specific torsor (see discussion in [8]). Since T is defined over $\operatorname{Spec} k(X)$, its image under an invariant η will lie in the Galois cohomology group $H^d(k(X), Z/p)$ for some degree d. Hence we can identify the group of degree d cohomological invariants of G with a certain subset of $H^d(k(X), \mathbb{Z}/p)$.

In fact, we can say much more about that certain subset: Given a point $x \in X$ with $\operatorname{codim} \overline{\{x\}} = 1$, we get a residue map

$$\nu_x : H^d(k(X), \mathbb{Z}/p) \to H^{d-1}(k(x), \mathbb{Z}/p),$$

where $k(x)$ is the stalk at x. If a class $\eta_T \in H^d(k(X), \mathbb{Z}/p)$ is the image of a versal torsor under an invariant, then $\nu_x(\eta_T) = 0$ for all such x; conversely, Totaro shows that if $\operatorname{codim} S \geq 2$, every class in the kernel of ν_x for all x does in fact define a cohomological invariant (letter to Serre, reprinted in [7]). Therefore we have the identification

$$\operatorname{Inv}^d G = \ker \left(H^d(k(X), \mathbb{Z}/p) \to \coprod_{x \in X^{(1)}} H^{d-1}(k(x), \mathbb{Z}/p) \right),$$

where $x \in X^{(1)}$ ranges over all codimension one points.

5.2 Bloch-Ogus spectral sequence and stable cohomology

In their 1974 paper, Bloch and Ogus showed that the product of residue maps considered above is part of a flasque resolution of the sheaf $\mathcal{H}^d$ on X, defined as the sheafification of the Zariski presheaf $U \mapsto H^d_{\text{ét}}(U, \mathbb{Z}/p)$. Therefore we can actually think of the kernel as a sheaf cohomology group, and we get

$$\operatorname{Inv}^d G = H^0(X, \mathcal{H}^d).$$

This sheaf cohomology group appears as the $E_2^{0,d}$ term of the Bloch-Ogus spectral sequence for X, which converges to the étale cohomology $H^*_{\text{et}}(X, \mathbb{Z}/p)$. With our assumptions on X and the base field k, we can in fact identify these étale cohomology groups with the group cohomology $H^*(G, \mathbb{Z}/p)$ in low degree.

The diagonal entries $E_2^{r,r}$ are isomorphic to the mod p Chow groups $CH^r X \otimes \mathbb{Z}/p \cong CH^r G \otimes \mathbb{Z}/p$. Hence the differential $\delta : E_2^{0,3} \to E_2^{2,2}$ combined with the maps to and from the abutment give an exact sequence:

$$H^3(G, \mathbb{Z}/p) \to \operatorname{Inv}^3 G \xrightarrow{\delta} CH^2 G \otimes \mathbb{Z}/p \to H^4(G, \mathbb{Z}/p).$$

The kernel of the differential δ is composed of the classes that survive the turning of the page of the spectral sequence and hence appear in the étale cohomology; in other words, this kernel is the stable cohomology as discussed by Bogomolov and Böhning in [3]. In that paper they prove that the two groups we are interested in have stable cohomological dimension 2, meaning that no classes from $\operatorname{Inv}^3 G$ survive to the cohomology, so the kernel of δ is trivial. Combining this with the fact that the cycle map out of the Chow group is injective, which follows from [18], is enough to give that $\operatorname{Inv}^3 G_i = 0$ for our groups of order p^3. However, we can use the connection between the Bloch-Ogus spectral sequence and the motivic cohomology ring to prove a more general result that doesn't rely on the results of [3].

5.3 Vanishing of degree three invariants

In this section we prove the following theorem, which gives conditions on the integral cohomology and integral Chow groups that guarantees the vanishing of degree three invariants of G.

Theorem 5.3.1. *Let X be a variety over $\operatorname{Spec} \mathbb{C}$ satisfying the following two properties:*

(i) $CH^2 X \cong H^4(X, \mathbb{Z})$;

(ii) There is some power p^n with $p^n H^3(X, \mathbb{Z}) = 0$.

Then $H^0(X, \mathcal{H}^3) = 0$. In particular, if X is an approximation of the classifying stack BG for an algebraic group G such that the above two conditions hold, then $\operatorname{Inv}^3 G = 0$.

Proof. The group $H^0(X, \mathcal{H}^3)$ fits into the following long exact sequence:

$$\cdots \to H^{3,2}(X, \mathbb{Z}/p) \xrightarrow{\times\tau} H^{3,3}(X, \mathbb{Z}/p) \to H^0(X, \mathcal{H}^3) \to$$
$$H^{4,2}(X, \mathbb{Z}/p) \xrightarrow{\times\tau} H^{4,3}(X, \mathbb{Z}/p) \to \cdots.$$

Therefore, we get our result if we can show that

(a) $\times\tau : H^{4,2}(X, \mathbb{Z}/p) \to H^{4,3}(X, \mathbb{Z}/p)$ is injective, and

(b) $\times\tau : H^{3,2}(X, \mathbb{Z}/p) \to H^{3,3}(X, \mathbb{Z}/p)$ is surjective.

The injectivity is easier to show, so we will do that first. We know that $H^{4,2}(X, \mathbb{Z}/p) \cong CH^2X \otimes \mathbb{Z}/p$ is the mod p Chow group. The kernel of the map

$$c : CH^2X \to H^4(X, \mathbb{Z}/p)$$

that comes from composing the cycle map with the change of coefficients map $H^4(X, \mathbb{Z}) \to H^4(X, \mathbb{Z}/p)$ is exactly pCH^2X, since we are assuming the cycle map is an isomorphism. This means the mod p cycle map

$$c_p : CH^2X \otimes \mathbb{Z}/p \to H^4(X, \mathbb{Z}/p)$$

is injective. Since we can identify c with the map $\times\tau^2$ on motivic cohomology, we have shown (a).

For (b), denote by β the connecting homomorphism $\beta : H^3(X, \mathbb{Z}/p) \to H^4(X, \mathbb{Z})$; the plan of attack is first to show that

$$\ker(\beta) \subseteq \mathrm{im}(\tau) \subseteq H^{3,3}(X, \mathbb{Z}/p) \cong H^3(X, \mathbb{Z}/p),$$

and then to show that any class in $H^3(X, \mathbb{Z}/p)$ is equivalent to a class in $\ker(\beta)$ mod the image of τ.

The key to the first step is that, for any exponent n the short exact sequence

$$0 \to \mathbb{Z}/p \to \mathbb{Z}/p^{n+1} \to \mathbb{Z}/p^n \to 0$$

induces connecting maps on both étale cohomology and motivic cohomology:

$$\beta_{\mathrm{mot}} : H^{*,*}(X, \mathbb{Z}/p^n) \to H^{*+1,*}(X, \mathbb{Z}/p);$$

$$\beta_{\mathrm{et}} : H^*(X, \mathbb{Z}/p^n) \to H^{*+1}(X, \mathbb{Z}/p).$$

Under the isomorphisms from the Beilinson-Lichntenbaum conjecture, then, β_{et} maps from $H^{*,*}(X, \mathbb{Z}/p^n)$ to $H^{*+1,*+1}(X, \mathbb{Z}/p)$, and we have that

$$\beta_{\mathrm{et}} = \tau \circ \beta_{\mathrm{mot}}.$$

Therefore, crucially for us, $\mathrm{im}(\beta_{\mathrm{et}}) \subseteq \mathrm{im}(\tau)$.

Now let $x \in \ker(\beta) \subseteq H^3(X, \mathbb{Z}/p)$. Then we can pull x back to a class $\tilde{x} \in H^3(X, \mathbb{Z})$. By assumption, $H^3(X, \mathbb{Z})$ is p^n-torsion for some n, meaning that $\tilde{x}$ in turn comes from a class $\overline{x} \in H^2(X, \mathbb{Z}/p^n)$. Then we have $x = \beta_{\mathrm{et}}(\overline{x}) \in \mathrm{im}(\tau)$ as desired.

For the general case, we now assume that $\beta(x) \neq 0 \in H^4(X, \mathbb{Z})$. Recall that by assumption $H^4(X, \mathbb{Z}) \cong CH^2 X \cong H^{4,2}(X, \mathbb{Z})$; we write $y \in H^{4,2}(X, \mathbb{Z})$ for the image of $\beta(x)$ under this isomorphism. Since $py = 0$, we have $y = \beta(x')$ for some $x' \in H^{3,2}(X, \mathbb{Z}/p)$. Then $\beta(\tau x') = \beta(x) \in H^4(X, \mathbb{Z})$ (where we abuse notation a bit by conflating $\tau x'$ and its image under the isomorphism $H^{3,3}(X, \mathbb{Z}/p) \cong H^3(X, \mathbb{Z}/p)$). By the previous case, then, $\beta(x - \tau x') = 0$, so $x - \tau x' \in \mathrm{im}(\tau)$; therefore we also have $x \in \mathrm{im}(\tau)$ as desired.

$\square$

In the case that $X = (V - S)/G$ is an approximation to BG as described above, with $|G| = p^n$, we do automatically have that $H^3(X, \mathbb{Z}) \cong H^3(G, \mathbb{Z})$ is p^n-torsion, so the second condition of the theorem is automatically satisfied. Therefore we have shown that for finite p-groups G, if the degree two cycle class map is an isomorphism then G has no nontrivial degree three cohomological invariants. For example, Yagita proved that the cycle class map is an isomorphism in all degrees for the two nonabelian groups of order p^3 for odd primes p [18], meaning by our result these groups have no cohomological invariants of degree three. Since we'll use it in computations later on, we record that fact here as a corollary:

Corollary 5.3.2. *Let G be either of the nonabelian groups of order p^3. Then* $\mathrm{Inv}^3(G) = 0$*.*

Chapter 6: Motivic Cohomology of BG_i

In this chapter we prove the main result, which states that the motivic cohomology of the nonabelian groups of order 27 is τ-injective. The fact that enables this conclusion for $p = 3$ is that in that case the groups G_i have irreducible three-dimensional representations, letting us define a variety $X = (V - S)/G_i$ that serves as an approximation to BG with $\dim(X) = 3$. By the following lemma, that narrows the degrees in which the $\times\tau$ map on the motivic cohomology of X could fail to be injective down enough that we can use previous results to show that X is in fact τ-injective. In order to use the same method for $p > 3$ we would need to know more about the terms on page two of the Bloch-Ogus spectral sequence.

Lemma 6.0.1. *Let X be a complex variety with $\dim X = n$. Then the $\times\tau$ map on the motivic cohomology $H^{*,*'}(X, \mathbb{Z}/p)$ can only fail to be injective in bidegrees (i, j) satisfying $2 \leq j \leq n - 1$ and $j + 2 \leq i \leq 2j$.*

Proof. This follows directly from the long exact sequence 3.1:

$$\cdots \to H^{i-j-2}(X, \mathcal{H}^{j+1}) \to H^{i,j}(X, \mathbb{Z}/p) \to H^{i,j+1}(X, \mathbb{Z}/p) \to \cdots .$$

The sheaf cohomology group $H^{i-j-2}(X, \mathcal{H}^{j+1})$ is trivial if $j+1 > n$ or if $i-j-2 < 0$. Therefore in order for the $\times\tau$ map to have a nontrivial kernel, we must have $j \leq n-1$

and $j+2 \leq i$. If $i > 2j$ then the motivic cohomology group $H^{i,j}(X, \mathbb{Z}/p) = 0$ so the map is trivially injective. And finally, for $j < 2$, the set of i with $j+2 \leq i \leq 2j$ is empty. Therefore the given bidegrees are the only ones in which $\times\tau$ can fail to be injective. $\square$

In the case that $n = 3$, the only bidegree that we have to worry about is $(4, 2)$. This leads to the following corollary:

Corollary 6.0.2. *Let X be a complex variety with $\dim X = 3$, and assume that $H^0(X, \mathcal{H}^3) = 0$. Then X is τ-injective.*

6.1 The $p = 3$ case

Theorem 6.1.1. *Let G be a nonabelian group of order 27, thought of as an affine group over $\mathbb{C}$. Then the map*

$$\times\tau : H^{*,*'}(BG, \mathbb{Z}/p) \to H^{*,*'+1}(BG, \mathbb{Z}/p)$$

is injective.

Proof. We've already done much of the work of the proof; what remains is to put the pieces together. Let G act on the space $V = \mathbb{C}^3 - \{0\}$ via the faithful three-dimensional representation of G as described in section , and let $S \subset V$ be the closed subset on which the stabilizers of the G-action are nontrivial. By theorem 4.3.3, we know the motivic cohomology ring of S and that it is τ-injective. Let $X = (V - S)/G$ be the quotient variety. Since X is three-dimensional, by corollaries 5.3.2 and 6.0.2 X is also τ-injective.

The structure of the argument is to use the localization sequence in equivariant motivic cohomology induced by the inclusion $S \subset V$, combined with what we know about S and X, to obtain the result for BG. The first step is to prove τ-injectivity for $H_G^{*,*'}(V, \mathbb{Z}/p)$; the second step is to use this result and the localization sequence coming from the inclusion $\{0\} \subset \mathbb{C}^3$ to get τ-injectivity for $H_G^{*,*'}(\mathbb{C}^3, \mathbb{Z}/p) = H^{*,*'}(BG, \mathbb{Z}/p)$.

Step 1. We first focus on the following commutative diagram of exact sequences, coming from the localization sequence (3.1) from $S \subset V$:

$$\begin{array}{ccccccccc}
\cdots \to & H^{m+3,n+2}(X,\mathbb{Z}/p) & \longrightarrow & H_G^{m,n}(S,\mathbb{Z}/p) & \longrightarrow & H_G^{m+4,n+2}(V,\mathbb{Z}/p) & \to & H^{m+4,n+2}(X,\mathbb{Z}/p) & \to \cdots \\
& \downarrow{\scriptstyle \times\tau} & & \downarrow{\scriptstyle \times\tau} & & \downarrow{\scriptstyle \times\tau} & & \downarrow{\scriptstyle \times\tau} & \\
\cdots \to & H^{m+3,n+3}(X,\mathbb{Z}/p) & \to & H_G^{m,n+1}(S,\mathbb{Z}/p) & \to & H_G^{m+4,n+3}(V,\mathbb{Z}/p) & \to & H^{m+4,n+3}(X,\mathbb{Z}/p) & \to \cdots
\end{array}$$

As in the four lemma above, refer to the vertical maps from left to right as a, b, c, and d. Since the action of G on $V - S$ is free, the equivariant cohomology $H_G^{*,*'}(V - S, \mathbb{Z}/p)$ is the cohomology of the quotient variety X. We know that the vertical maps b and d are both injective. What we need in order for the four lemma to kick in and prove that c is also injective is to show that the map a is surjective. This is not true for general m and n. Going back to our trusty sequence 3.1, we can see that the map $a = \times\tau : H^{m,n-1}(X, \mathbb{Z}/p) \to H^{m,n}(X, \mathbb{Z}/p)$ is surjective if and only if the map $H^{m-n}(X, \mathcal{H}^n) \to H^{m+1,n-1}(X, \mathbb{Z}/p)$ is injective. Hence we only have to worry about bidegrees in which $H^{m-n}(X, \mathcal{H}^n)$ is nonzero. Since X is three-dimensional, this is a finite list. For each degree (m, n), the map $a : H^{m,n-1}(X, \mathbb{Z}/p) \to H^{m,n}(X, \mathbb{Z}/p)$ corresponds in the diagram above to the map $c : H_G^{m+1,n-1}(V, \mathbb{Z}/p) \to H_G^{m+1,n}(V, \mathbb{Z}/p)$. For each case that a might fail to be surjective, we can check that in fact the domain of c is trivial and therefore its injectivity is ensured:

(m,n)	$(m+1,n-1)$	$2(n-1)-(m+1)\geq 0$?
$(0,0)$	$(1,-1)$	No
$(1,1)$	$(2,0)$	No
$(2,2)$	$(3,1)$	No
$(3,1)$	$(4,0)$	No
$(3,2)$	$(4,1)$	No
$(4,2)$	$(5,1)$	No
$(4,3)$	$(5,2)$	No
$(5,3)$	$(6,2)$	No
$(6,3)$	$(7,2)$	No

Well, that's convenient. Since these are the only cases in which the injectivity isn't guaranteed by the four lemma, this shows that the map $c = \times\tau : H_G^{m,n}(V,\mathbb{Z}/p) \to H_G^{m,n+1}(V,\mathbb{Z}/p)$ is injective in all degrees, as desired.

Note that the result that $H^0(X,\mathcal{H}^3) = 0$ is a key corollary here: Without knowing this, it would be possible that $a : H_G^{4,2}(V,\mathbb{Z}/p) \to H_G^{4,3}(V,\mathbb{Z}/p)$ could fail to be injective.

We now use a similar diagram coming from the localization sequence for the inclusion $\{0\} \subset \mathbb{C}^3$, following the structure of the argument in [18]. Since the single point $\{0\}$ and the space $\mathbb{C}^3$ are both homotopic to a point, the equivariant cohomology of both is by definition $H^{*,*'}(BG,\mathbb{Z}/p)$. In this case we circumvent the issue of surjectivity of the first vertical map in the diagram a different way: Instead of considering the $\times\tau$ map itself, we take the vertical maps to be quotienting out by the kernel of $\times\tau^{*-*'}$ (a la the method in [18]). We can do this since in this case the map $H^*_G(\{0\}, \mathbb{Z}/p) \to H^{*+6}_G(\mathbb{C}^3, \mathbb{Z}/p)$ is given by multiplication by the Chern class z of a three-dimensional representation (see [10] or theorems 4.1.2 and 4.1.3 above), which comes from the Chow ring and is therefore weight zero, so preserves the grading. This map is injective, so we get a diagram

$$\begin{array}{ccccccc}
\longrightarrow H^{m,n}(BG, \mathbb{Z}/p) & \longrightarrow & H^{m+6,n+3}(BG, \mathbb{Z}/p) & \longrightarrow & H^{m+6,n+3}_G(V, \mathbb{Z}/p) \longrightarrow \\
\downarrow & & \downarrow & & \downarrow \\
\longrightarrow h^{m,n}(BG, \mathbb{Z}/p) & \longrightarrow & h^{m+6,n+3}(BG, \mathbb{Z}/p) & \longrightarrow & h^{m+6,n+3}_G(V, \mathbb{Z}/p) \longrightarrow
\end{array}$$

We proceed by induction on m, applying the five lemma: For $m < 0$, the $\times\tau$ map is injective since the domain is trivial; since we know the first and last vertical arrows in the above diagram are isomorphisms, by induction we can apply the five lemma to the above diagram to get that the $\times\tau$ map is injective for $m < 6$. Proceeding inductively, this gives injectivity for $m < 12$, and so on.

□

Chapter 7: Computing $h^{*,*'}(BG,\mathbb{Z}/p)$

Now that we know that there are no motivic cohomology classes that are killed by τ and hence 'hidden' from the étale cohomology ring, the work that remains is to understand the ring $h^{*,*'}(BG,\mathbb{Z}/p)$ with

$$h^{m,n}(X,\mathbb{Z}/p) \cong \bigoplus_i \operatorname{gr}^{2(n-i)-m} H^m(X,\mathbb{Z}/p)\tau^i.$$

Our main tools will be the universal coefficients sequence that links the mod p cohomology with the integral cohomology and the fact that we know the Chow groups of G agree with the integral cohomology.

The computation proceeds in steps. First, for each generator of the ordinary cohomology ring, we determine which graded piece of $\operatorname{gr}^* H^{*'}(BG,\mathbb{Z}/p)$ it appears in (in other words, how many powers of τ we can divide out by). Second, we use the knowledge of the integral cohomology and the Chow groups to determine what rank over $\mathbb{Z}/p$ we expect each bidegree to be. Finally we piece the information together to give generators of the motivic cohomology and their images in the ordinary cohomology.

Theorem 7.1. *Let G_2 be the nonabelian group of order p^3 and exponent p^2. Then the ring $h^{*,*'}(G_2,\mathbb{Z}/p)$ is generated by elements $\tau, a_1,\ldots,a_{p-1},u_1,\ldots,u_p,b,y,v,$ and*

w, with bidegrees

$$\deg \tau = (0,1), \deg a_i = (2i-1, i), \deg u_i = (2i, i), \deg b = (1,1),$$

$$\deg y = (2,1), \deg v = (2p-1, p), \deg w = (2p, p),$$

and relations

$$\tau u_i = a_i b$$

in addition to the relations corresponding to those of the ordinary cohomology.

Proof. By theorem 4.1.3, the ordinary cohomology of G_2 is generated by $a_1, \ldots, a_{p-1}$, b, y, v, and w. For ease (or extra confusion?) of notation we will sometimes below use the notation $v = a_p$. The relations give that in degrees $i < 2p$ there are two independent generators:

$$H^i(G_2, \mathbb{Z}/p) \cong \begin{cases} a_j\mathbb{Z}/p \oplus y^{j-1}b\mathbb{Z}/p & \text{for } i = 2j-1; \\ a_jb\mathbb{Z}/p \oplus y^j\mathbb{Z}/p & \text{for } i = 2j. \end{cases}$$

By the integral cohomology, which is isomorphic to the (integral) Chow groups, we have that

$$H^{2n,n}(G_2, \mathbb{Z}/p) \cong CH^n(G_2) \otimes \mathbb{Z}/p \cong H^{2n}(G_2, \mathbb{Z}) \otimes \mathbb{Z}/p,$$

which we know from corollary 4.2.4 has dimension two over $\mathbb{Z}/p$ for $n < p$. Since $\times\tau^n$ maps $H^{2n,n}(G_2, \mathbb{Z}/p)$ injectively to $H^{2n}(G_2, \mathbb{Z}/p)$, which is also dimension two, the $\times\tau$ map must be an isomorphism at every stage of that map. In other words, for $n < p$,

$$H^{2n}(G_2, \mathbb{Z}/p) \cong \operatorname{gr}^0 H^{2n}(G_2, \mathbb{Z}/p).$$

In particular this implies that we can take the generator y to have bidegree $(2,1)$ in the motivic cohomology.

Turning to the odd degrees less than $2p$, by our Key Fact we have that

$$\dim(H^{2n-1,n}(G_2,\mathbb{Z}/p)) = \dim(H^{2n-1,n}(G_2,\mathbb{Z})/p) + \dim(p\text{-torsion}(H^{2n,n}(X,\mathbb{Z})) \geq 2.$$

(Note that for $n = p$ here the Chow group $CH^p(G_2,\mathbb{Z})$ is generated by α^p and χ, so is rank two.)

Since for $n \leq p$, $\dim H^{2n-1}(G_2,\mathbb{Z}/p) = 2$, again we have that $\times\tau$ is an isomorphism at every step starting at degree $(2n-1,n)$, in other words

$$H^{2n-1}(G_2,\mathbb{Z}/p) \cong \operatorname{gr}^1 H^{2n}(G_2,\mathbb{Z}/p).$$

Specifically, we can take the generators a_i, b and v to have weight one. The elements $y^j b$ then naturally have weight one as well. By these dimensional arguments, there must be for each $i < p$ an element $u_i \in H^{2i,i}(G_2,\mathbb{Z}/p)$ with $\tau u_i = a_i b \in H^{2i,i+1}(G_2,\mathbb{Z}/p)$.

The generator w is the top chern class of a dimension p representation (see the computations in [4]) and hence lies in the Chow ring; it is the image of the generator $\zeta \in CH^p(G_2,\mathbb{Z})$. Ignoring multiples of w for the moment, each degree for $n \geq 2p$ of the mod p cohomology has two generators: for $n = 2i$ these generators are y^i (weight zero) and $y^{i-p}vb$ (weight two); for $n = 2i+1$ they are $y^i b$ (weight one) and $y^{i-p+1}v$ (weight one). The only mystery about these generators, then, is whether $y^{i-p}vb$ is a multiple of τ. The answer is no: we know from the integral cohomology that the dimension of $H^{2n,n}(G_2,\mathbb{Z}/p)$ is $\lceil \frac{n}{p} \rceil + 1$, and that is precisely how many independent generators are accounted for by y^n and each generator of $H^{2(n-p),n-p}(G_2, \mathbb{Z}/p)$ multiplied by w.

To summarize, we have the following:

degree	dim	generators
$(0,1)$	1	τ
$(1,1)$	2	a_1, b
$(2,1)$	2	y, u_1
$(2,2)$	2	$\tau y, \tau u_1 = a_1 b$
$(3,2)$	2	a_2, yb
$\vdots$	$\vdots$	$\vdots$
$(2p,p)$	2	y^3, w
$(2p,p+1)$	3	$\tau y^3, \tau w, vb$
$(2p+1,p+1)$	4	$wa_1, wb, y^p b, yv$
$(2p+2,p+1)$	3	wu_1, wy, y^{p+1}
$(2p+2,p+2)$	4	$\tau w u_1, \tau w y, \tau y^{p+1}, ybv$
$\vdots$	$\vdots$	$\vdots$

Continuing these patterns shows that all odd degree classes are generated in weight one, and even degree classes are generated in either weight zero or two; no additional generators are needed other than the ones we've mentioned. Since the cycle maps are injective, the relations from ordinary cohomology hold true for the corresponding generators in motivic cohomology as well, where applicable. $\square$

For G_1, the relations in the cohomology look slightly different for the $p = 3$

and $p > 3$ cases, so we treat those separately.

Theorem 7.2. *Let $p > 3$ be an odd prime and let G_1 be the nonabelian group of order p^3 and exponent p. Then the ring $h^{*,*'}(G,\mathbb{Z}/p)$ is generated by elements*

$$\tau, y, y', x, x', Y, Y', X, X', c_2, \ldots, c_{p-1}, d_3, \ldots, d_p, z,$$

with bidegrees

$$\deg\tau = (0,1), \deg y^* = (1,1), \deg x^* = (2,1), \deg Y^* = (2,2),$$

$$\deg X^* = (3,2), \deg c_i = (2i,i), \deg d_i = (2i-1,i), \deg z = (2p,p),$$

and relations

$$\tau c_2 = xY' + x'Y$$

$$\tau d_3 = XY'$$

$$\tau c_3 = XX'$$

in addition to the relations corresponding to those of the ordinary cohomology.

Proof. As before, the plan is to use the knowledge of the integral cohomology and Chow groups in order to determine the weight zero motivic cohomology. The relations of the cohomology ring are much messier than in the case of G_2, so we start by recording what we know about generators of the ordinary cohomology in each degree less than $2p$. This comes from a careful examination of the relations. The dimension of $H^{2n,n}(G_1,\mathbb{Z}/p)$, or equivalently the zeroth graded piece of $H^{2n}(G_1,\mathbb{Z}/p)$, comes from the isomorphism to the mod p Chow group, which we know from the integral cohomology.

degree	dim	generators	$\dim(\mathrm{gr}^0)$
1	2	y, y'	
2	4	x, x', Y, Y'	2
3	6	$X, X', xy, xy', x'y', y'Y$	
4	7	$x^2, xx', (x')^2, xY, xY', x'Y, x'Y'$	4
5	8	$X'Y, xX', xX, x'X',$ $x^2y, x^2y', (x')^2y, (x')^2y'$	
6	9	$XX', x^3, x^2x', x(x')^2, (x')^3,$ $x^2Y, x^2Y', (x')^2Y, (x')^2Y'$	5
$\vdots$	$\vdots$	$\vdots$	
$2i-1$	$2i+2$	$x^{i-1}y, x^{i-2}X, (x')^{i-1}y', x(x')^{i-2}y', \ldots,$ $x^{i-1}y', (x')^{i-2}X', \ldots, x^{i-2}X', d_i$	
$2i$	$2i+3$	$x^i, x^{i-1}x', \ldots, (x')^i,$ $x^{i-1}Y', \ldots, (x')^{i-1}Y', x^{i-1}Y, c_i$	$i+2$

For degrees $2p$ and above, the generators that are multiples of the generators in degree three or less (in other words, not multiples of z) are given by

$$x^{i-1}y, x^{i-2}X, (x')^{i-1}y', x(x')^{i-2}y', \ldots, x^{p-1}(x')^{i-p}y',$$
$$(x')^{i-2}X', \ldots, x^{p-1}(x')^{i-p-1}X' \quad \text{for } n = 2i-1, i > p;$$

$$x^i, (x')^i, x(x')^{i-1}, \dots, x^{p-1}(x')^{i-p+1},$$

$$(x')^{i-1}Y', \dots, x^{p-1}(x')^{p-i}Y', x^{i-1}Y \quad \text{for } n = 2i, i \geq p.$$

The key to the whole computation is the fact that $H^0(G_1, \mathcal{H}^3) = 0$, since that implies that $\times\tau : H^{3,2}(G_1, \mathbb{Z}/p) \to H^{3,3}(G_1, \mathbb{Z}/p)$ is surjective and hence an isomorphism. This allows us to conclude that

$$H^3(G_1, \mathbb{Z}/p) \cong \mathrm{gr}^1 H^3(G_1, \mathbb{Z}/p) \cong h^{3,2}(G_1, \mathbb{Z}/p),$$

so specifically we may take the generators X, X' to be weight one. In lower degrees, clearly y, y' have no choice but to be weight one; since by [10] we have $x^* = \beta y^*$ we may take x, x' to be the two generators of $h^{2,1}(G_1, \mathbb{Z}/p)$, meaning we must have $Y, Y' \in \mathrm{gr}^2 H^2(G_1, \mathbb{Z}/p)$.

From the computations in [10], the classes c_i are in the Chow ring and therefore weight zero. Since $\beta(d_i) = c_i$ we may take the classes d_i to be in weight one. The lists of generators above, combined with the fact that z, being a Chern class, can be taken as weight zero, show that the even degree cohomology is completely generated in weights zero and two, and the odd degree cohomology is completely generated in weight one except possibly for the class $X'Y$.

In even degrees greater than six, the number of weight zero generators modulo z that we can enumerate from the given lists matches what we expect to see from the dimension of the integral cohomology. The degrees that we have to think a bit harder about are four and six. Again turning to the details of Leary's computation for guidance, he shows that the classes $xY' + x'Y$ in degree four and XX' in degree six are in fact the images of Chern classes, so these give the missing weight zero generators.

In other words, we may take generators $c_2 \in h^{4,2}(G_1, \mathbb{Z}/p)$ and $c_3 \in h^{6,3}(G_1, \mathbb{Z}/p)$ with $\tau c_2 = xY' + x'Y \in h^{4,3}(G_1, \mathbb{Z}/p)$ and $\tau c_3 = XX' \in h^{6,4}(G_1, \mathbb{Z}/p)$. This completes the picture we expect from the dimensions of the integral cohomology in even degrees.

The existence of c_3 with $\tau c_3 = XX'$ allows us to solve the remaining mystery, namely the weight of the element $X'Y$. Since $\beta(Y) = X$ and $\beta(X') = 0$, we have that $\beta(X'Y)$ is a multiple of XX'. The class c_3 should be in the image of β, since $\beta(c_3) = 0$ and we know $H^{6,3}(G_1, \mathbb{Z})$ is exponent p, but is not hit by any combination of the other generators of degree five. Therefore we get that there must be a $d_3 \in h^{5,3}(G_1, \mathbb{Z}/p)$ with $\beta(d_3) = c_3$ and τd_3 a multiple of $X'Y$. □

Finally, we can prove a similar result for G_1 in the case that $p = 3$. In this case we also know by theorem 6.1.1 that $H^{*,*'}(BG_1, \mathbb{Z}/p) \cong h^{*,*'}(BG_1, \mathbb{Z}/p)$.

Theorem 7.3. *Let G be the nonabelian group of order 27 and exponent 3. Then the motivic cohomology ring $H^{*,*'}(G, \mathbb{Z}/p)$ is generated by elements*

$$\tau, y, y', x, x', Y, Y', X, X', c_2, d_3, z,$$

with bidegrees

$$\deg \tau = (0,1), \deg y^* = (1,1), \deg x^* = (2,1), \deg Y^* = (2,2),$$

$$\deg X^* = (3,2), \deg c_2 = (4,2), \deg d_3 = (5,3), \deg z = (6,3),$$

and relations

$$\tau(c_2 - x^2 - (x')^2) = xY' + x'Y$$

$$\tau d_3 = XY'$$

in addition to the relations corresponding to those of the ordinary cohomology.

Proof. The proof follows the same plan as the $p > 3$ case. Up to degree five generating sets for the ordinary cohomology look the same as above, and in degree six the difference is that there is an extra generator z and we have the relation $XX' = 0$. Above degree six, as per the computations in [10] we again have that modulo z everything is generated by terms of the form

$$x^{i-1}y, (x')^{i-1}y', x(x')^{i-2}y', x^2(x')^{i-3}y',$$
$$x^{i-2}X, (x')^{i-2}X', x(x')^{i-3}X', x^2(x')^{i-4}X' \quad \text{for } n = 2i-1;$$

$$x^{i}, (x')^{i}, x(x')^{i-1}, x^2(x')^{i-2},$$
$$x^{i-1}Y, (x')^{i-1}Y', x(x')^{i-2}Y', x^2(x')^{i-3}Y' \quad \text{for } n = 2i.$$

Again these line up with the dimensions we are expecting in weights zero, one, and two from the integral cohomology. Similar to the $p > 3$ case, Leary's computations tell us that the class $xY' + x'Y + x^2 + (x')^2$ is the image of a Chern class, so we may take a generator $c_2 \in h^{4,2}(G, \mathbb{Z}/p)$ with $\tau(c_2 - x^2 - (x')^2) = xY' + x'Y \in h^{4,3}(G, \mathbb{Z}/p)$.

The final piece of the puzzle is the weight of the class XY'. Unlike the $p > 3$ case, here we have that $\beta(XY') = XX' = 0 \in H^6(G, \mathbb{Z}/p)$, so the reasoning that we employed in the previous proof doesn't work. This is where the fact that in the integral cohomology we have the relation

$$\mu\nu = 3\zeta \in H^6(G, \mathbb{Z})$$

comes into play. What's going on here is that at the level of ordinary cohomology, the universal coefficients sequence takes the class XY' to $\mu\nu = 3\zeta$. Changing coefficients back to $\mathbb{Z}/p$, that class vanishes, giving $\beta(XY') = XX' = 3z = 0$. (That relation that $XX' = 3z \in H^6(G, \mathbb{Z}/p)$ is trivially true since both are equal to zero, but is somehow more meaningful than that: it's the reflection in the mod p cohomology of the equivalent relation in integral cohomology.) Since we do have that $H^{6,3}(G, \mathbb{Z}) \cong CH^3(G) \cong H^6(G, \mathbb{Z})$, the class $3\zeta \in H^{6,3}(G, \mathbb{Z})$ must be in the image of the connecting map from $H^{5,3}(G, \mathbb{Z}/p)$, so let d_3 be a preimage; then τd_3 will be a multiple of XY'.

$\square$

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