

THESIS REPORT

Ph.D.

Dynamics and Stability of Spacecraft with Fluid-Filled Containers

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Abstract

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In this dissertation, we study the dynamics, stability and control of spacecraft with fluid-filled containers. The spacecraft with fluid-filled containers is modeled as a rigid body containing perfect fluid. A general model for the system is obtained by using a Lagrangian approach where the configuration manifold is the cartesian product of the rotation group and the group of volume preserving diffeomorphisms. The dynamical equations are interpreted as a non-canonical Hamiltonian system on an infinite dimensional Poisson space. The geometry of the model is explicitly given by identifying its Lie-Poisson and Euler-Poincare structure. The equilibria of the system are investigated. Based on the developed model, three control problems are studied for spacecraft with fluid-filled containers. These problems are the stability of rigid rotations equilibria, the stabilization of rigid rotations by means of torque control and the attitude control problem. All stability and control problems are studied in an infinite dimensional nonlinear setting without resorting to approximations. A key feature of this dissertation is the exploitation of the mechanical and geometric structure of the system to address the stability and control problems.

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by

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1994

Dedication

dedem
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için

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Table of Contents

<u>Section</u>	<u>Page</u>
1 Introduction	1
2 Geometry and Mechanics	9
2.1 Differentiation Concepts	9
2.2 Tensors and Differential Forms	11
2.3 Lie Groups and Lie Algebras	14
2.4 Lagrangian Mechanics	19
2.5 Poisson Mechanics	22
2.6 Divergence, Gradient, Etc.	27
2.7 Some Useful Equations	31
2.8 Some Infinite Dimensional Systems	32
2.9 Basic Equations of Incompressible Fluid Mechanics	39
3 Rigid Bodies Containing Fluids	42
3.1 Introduction	42
3.2 Lagrangian Formulation	43
3.2.1 Kinematics of the Configuration Manifold	45
3.2.2 Euler-Lagrange Equations	49

3.3	Constants of Motion	64
3.4	Equilibria of the System	69
3.5	Geometric Interpretations of the Model	73
3.5.1	Hamiltonian Structure	73
3.5.2	Generalized Bernoulli's Equation	78
3.5.3	Lie-Poisson Structure	82
3.5.4	Euler-Poincare Structure	89
3.6	Rigid Bodies Containing Viscous Fluids	94
3.6.1	Equilibria of the Dissipative Model	99
4	Stability and Control	101
4.1	Stability Notions	102
4.2	Stability Methods for Mechanical Systems	108
4.3	Control of Mechanical Systems	110
4.3.1	Static Dissipative Controllers	111
4.3.2	Dynamic Dissipative Controllers	117
4.4	A Modified Energy-Casimir Method	124
4.5	Stability of Rigid Bodies Containing Fluid	136
4.5.1	Effect of Viscosity on Stability	145
4.6	Velocity Control Problem	147
4.6.1	Effect of Viscosity	158
4.7	Attitude Control Problem	159
4.7.1	Effect of Viscosity	169
5	Conclusions and Future Directions	174
A	Pseudo Models for Rigid Bodies Containing Fluid	179

Chapter 1

Introduction

An embarrassing episode of the history of space studies is the story of an early U.S. satellite: *Explorer I* [48]. The satellite had been designed to spin about its long axis (the one corresponding to the minimum moment of inertia of the satellite). The attitude dynamics of the satellite in the orbit was modelled by using the celebrated equations of Euler for rigid bodies. The steady spin of a rigid body about the axes corresponding to the minimum or maximum moment of inertia is a stable motion. Indeed, *Explorer I* was almost a rigid body, it consisted of a long slender rigid body and light elastic booms attached to the main body of the satellite. The mass of the booms was small compared to that of the main body, so they could not exert large reaction torques on the body, hence the satellite could be modeled as a perfect rigid body neglecting the dynamics of the elastic booms. Indeed, the small torque assumption of JPL engineers was correct. The fatal mistake was the expectation that these negligible effects would produce negligible results. This is not the correct way of thinking in the realm of conservative mechanical systems. Because of conservative nature of mechanical systems, even infinitesimally small perturbations in the models might turn stable

motions into unstable ones. Indeed, this was what happened to *Explorer I*; the satellite began to tumble over within the first ninety minutes of its mission in orbit [38], [48]. The failure was due to the energy dissipation associated with the “negligible” dynamics of the elastic booms which turned the formally stable equilibrium of rotation along the long axis into an unstable equilibrium. The U.S. aerospace community learned the lesson of *Explorer I* well and developed successful solutions for satellite control technology since then.

Mechanical systems enjoy a special place among all dynamical systems. Indeed, conservative mechanical systems can be thought of as bifurcation points in the set of all possible systems [52]. This is why they should be treated with care as far as their modeling is concerned. In the context of conservative mechanical systems, all physically meaningful perturbations of a nominal model should be given equal treatment in order to understand their effects on the nominal model. The best way, of course, is to incorporate the perturbative dynamics as an integral part of the models. With this in mind, we will approach the main object of investigation of this dissertation: spacecraft with fluid-filled containers.

Satellites and spacecraft are composed of interconnected rigid and non-rigid parts. The antennas and booms form the major flexible appendages to the rigid bodies of the space structures and their interactive dynamics and control have been the subject of numerous studies. On the other hand, the rigid body-fluid interactions which take place in spacecraft has received far less attention. The liquid propellant fuel (which in some communication satellites constitutes half of the satellite mass), cooling liquids and mercury ring dampers are parts of spacecraft in which a rigid body-fluid interface appears. The effect of the rigid body-fluid interaction on the spacecraft is twofold. One is the liquid slosh phe-

nomenon which changes the mass distribution of the spacecraft. The second aspect is the energy dissipation that should be associated with the fluid motion w.r.t. spacecraft, since any real fluid would have non-zero viscosity and would behave as an energy sink. It is this dissipation that is believed to be a major source of instability in spacecraft [4]. The liquid slosh problem can be dealt with effectively by using variable structure fuel tanks. Indeed bellows, pistons and diaphragms are being used in spacecraft to change the volume and shape of the fuel tanks as the fuel is consumed. These type of devices are known to have a good center of mass control performance [83], [82]. Although by using these methods, it is possible to constrain the fluids into fully-filled cavities, this is no easy solution to cancel their effects on the rigid bodies of the spacecraft. This is because of the internal degrees of freedom associated with the fluid motion. The infinite dimensionality of the fluid dynamics makes the problem a little more difficult and interesting. While there have been some studies of the rigid body-fluid interaction, the problem has not been solved in a form suitable for incorporation into attitude dynamics modeling of spacecraft [38]. In contemporary satellite control technology, the dynamics of spacecraft-fluid interaction is only addressed as a perturbation to the attitude control systems. Although, by experimental and computational studies [4], [35] it is generally believed that the fluid in spacecraft is a cause of instability for attitude dynamics of the spacecraft, satisfactory analytical results are hard to find but see [65].

Before discussing our approach to the rigid body-fluid interaction problem, we would like to mention briefly some of the previous studies related to this area. The studies of the dynamics of body-fluid interactions can be seen to fall into three groups. The first group is the historic work about various aspects

of rotating fluids. It is known that Stokes was the first scientist who studied the problem. The names of Helmholtz, Lubeck, Lamb, Kelvin, Greenhill, Zhukowsky, Sludsky, Gaf and Poincare are also cited in this regard [65]. The second set of studies is the work of the Soviet applied mathematics community during the fifties and sixties. The basic motivation behind this group of work was the Soviet space studies. Various aspects of these studies are covered in a book by Moiseyev and Rumyantsev [65]. Although, mechanical nature of the rigid body-fluid interaction was well-formulated in this book, the stability results were not of an exact nature either due to the approximate models or due to an “approximate” stability concept they used. Yet, they considered various models including both fully-filled and partially-filled cavities in rigid bodies. The last group of studies are due to the U.S. aerospace circles. Numerous reports and papers mostly about the technological aspects of the rigid body-fluid interaction in spacecraft have been published since the sixties, see [4], [35]. Experimental and computational studies dominate in this group whereas the Soviet studies were more on the theoretical side. Overall, it could be said that despite various studies about the rigid body-fluid interaction problem, a general framework in which it is possible to address the problems of stability and control of spacecraft containing fluids is not available yet [38]. We aim to develop such a framework in this dissertation and to work out the related stability and control problems.

Our approach to the problem of rigid body-fluid interaction in spacecraft will be a geometric one which we will pay special attention to the conservative nature of the problem. The concepts and tools of geometric mechanics such as Poisson manifolds, Lie groups and algebras, Lie-Poisson systems, geometric reduction, energy-Casimir method etc. have proved to be useful not only in the

geometrization of mechanics but also in the control and stability of various mechanical phenomena [14], [43], [79]. We believe that the geometric framework for mechanics is powerful enough to address the stability and control problems of rigid bodies containing fluids without resorting to ad hoc approximation methods. This dissertation is organized as follows.

In chapter 2, we compiled some geometric and mechanical concepts and tools which we use in the later chapters to address the dynamics of a spacecraft containing fluid. Among other things, we introduce the notions of tensors, differential forms, Lie groups and algebras, Poisson manifolds, Hamiltonian systems on Poisson manifolds, Poisson reduction, Lagrangian mechanics as well as various examples of the infinite dimensional systems which can be interpreted as conservative (Lagrangian or Hamiltonian) systems. We also give different formulations of the dynamics of incompressible fluids in this chapter.

Chapter 3 is devoted to the development and analysis of dynamical models for spacecraft containing incompressible fluids. Along the lines of [8], [29] we choose the configuration manifold of incompressible fluid flow as the group of volume preserving diffeomorphisms. The configuration manifold of the rigid body-fluid system is taken as the cartesian product of this infinite dimensional group and the rotation group $SO(3)$. The coupled dynamics of the system is obtained from the Euler-Lagrange equations (section 2). In this derivation the Helmholtz decomposition of vector fields [21] and a simple decomposition of the Lie algebra $gl(3)$ proved useful. The resulting model gives a general and complete characterization of the interaction between the rigid body of the spacecraft and the fluid in the cavity. The dynamical model is quite general and its form does not depend on the particular shape or location of containers in the rigid body.

We derive four basic equations of the rigid body mechanics and fluid mechanics from the general rigid body-fluid model as special cases. These equations are: the Euler's equation for a rigid body, gyrostat (a rigid body containing internal momentum wheels) equation, the Euler's equation for ideal incompressible fluid flow and the Euler's equation for incompressible flow in rotating reference frames. By using a Legendre transformation, we represent the model in terms of momentum variables as well as velocity variables (section 2). The constants of motion for the rigid body-fluid system are determined (section 3). By using both the momentum and velocity space representations of the model, we identify and classify the equilibria of the system (section 4). As a by product of this analysis, we show that Beltrami flows are equilibrium solutions for Euler's equation for an incompressible ideal fluid. Various geometric interpretations of the rigid body-fluid system are studied in section 5. In particular, we identify the model as a non-canonical Hamiltonian system on an infinite dimensional Poisson space which can be interpreted as in duality with the cartesian product of Lie algebra $so(3)$ and the Lie algebra of incompressible vector fields. A direct result of the Hamiltonian nature of the system is the conservation of the energy and the magnitude of the total angular momentum of the rigid body-fluid system. Inspired by the form of the Poisson structure of the dynamical model, we generalize Bernoulli's equation for incompressible flow to a general Riemannian manifold setting. Furthermore, we explicitly identify the Lie-Poisson structure of the model expressed in momentum variables and the Euler-Poincare structure of the model expressed in terms of velocity variables. The Euler-Poincare interpretation of the model also reveals the symmetries of the system and shows that our derivation of the model from the Euler-Lagrange equations is indeed a

Lagrangian reduction process [57], [58]. In the last section of this chapter, we modify the dynamical equation of the rigid body-fluid model in order to incorporate the viscosity of the fluid, and investigate the asymptotics of the resulting dynamical system. The main contribution of this chapter is the development of a general model for rigid body-fluid interaction in spacecraft and the identification of various geometric structures of the model.

In chapter 4, we used the models developed in chapter 3 to address some stability and control problems after presenting some generalities for the stability and control of infinite dimensional mechanical systems. In section 1, following a brief review of some stability concepts we give a stability theorem in a Banach space setting which is essentially a reformulation of V. Arnold's "convexity conditions" [37] for the stability of infinite dimensional systems. In section 3, we formally develop two dissipative control methods for the stabilization of mechanical systems expressed in Lagrangian form. Here, we show that the vibrations of a Lagrangian system (under an observability condition) can be damped by using any passive controller satisfying certain conditions which generalize the state-space conditions for a positive-real system [6]. In section 4, we specialize the energy-Casimir stability method to mechanical systems with quadratic energy and casimirs. Here, we apply this modified test to various examples some of which are novel. The stability of the rigid rotation equilibria of a spacecraft with fluid is studied in section 5, where we apply the energy-Casimir method to obtain a stability result. Based on the results of this section, in section 6 we study the velocity control of a rigid body containing fluid. Here, we develop a (conservative) control method which stabilizes any given rigid rotation of the system. The proposed controller is simple and can be implemented by using internal or

external torque actuators. An important aspect is that the controller uses only the angular velocity information to manipulate the torque input; it is a finite dimensional controller which stabilizes an equilibrium of an infinite dimensional nonlinear dynamical system. In section 7, we address the attitude control problem for the rigid body-fluid system. By using the Euler parametrization of the rotation group $SO(3)$, we develop a controller which drives the orientation of the system to a desired orientation asymptotically in time. For all the stability and control problems studied in this chapter, the solutions are developed by using the conservative nature of the equations of a rigid body containing ideal incompressible fluid. The effect of viscosity is also investigated for each problem. The main contributions of this chapter are the stability and control results obtained in sections 5,6 and 7. By exploiting the mechanical and geometric structure of the models we are able to address these problems in an infinite dimensional nonlinear setting without resorting to approximations.

In the appendix, we develop some finite dimensional models which approximate the dynamics of rigid bodies containing fluids in a qualitative sense. These models, which we call pseudo models, by construction retain the essential conservative and geometric structure of the infinite dimensional model hence they are of some merit to understand the qualitative dynamics of the rigid body-fluid systems.

Chapter 2

Geometry and Mechanics

In this chapter, we present some basic concepts from geometry and geometric mechanics. Basic references we follow are [1], [2], [56], [21], [8].

2.1 Differentiation Concepts

The concepts of Frechet and Gateaux differentiation are generalizations of differentiation on $\mathbb{R}^n$ to normed spaces. For detailed treatments of differential calculus in normed spaces see [13], [10], [2] where we extracted the concepts given in this section.

Definition 2.1 *Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two Banach spaces, and $\mathcal{O} \subset E_1$ be an open subset. Then a function $f : \mathcal{O} \rightarrow E_2$ is said to be Frechet differentiable at $x_0 \in \mathcal{O}$ if there exists a (bounded) linear map $Df(x_0) \in \mathcal{B}(E_1, E_2)$ such that*

$$\lim_{\|x\|_1 \rightarrow 0} \frac{\|f(x_0 + x) - f(x_0) - Df(x_0)x\|_2}{\|x\|_1} = 0.$$

The operator $Df(x_0)$ is called the Frechet derivative of f at x_0 .

If f is differentiable at all points $x_0 \in \mathcal{O}$ then f is said to be differentiable on $\mathcal{O}$ and the mapping $Df : \mathcal{O} \rightarrow \mathcal{B}(E_1, E_2)$ is called the derivative of f . The second

derivative $D^2f : \mathcal{O} \rightarrow \mathcal{B}(E_1, \mathcal{B}(E_1, E_2))$ is defined as $D^2f = D(Df)$. Higher order derivatives are defined similarly. A slightly more general differentiation concept for functions defined on normed spaces is Gateaux derivative.

Definition 2.2 Let $(E_1, \|\cdot\|_1)$ and $(E_2, \|\cdot\|_2)$ be two normed spaces, and $f : E_1 \rightarrow E_2$ be a mapping. The map f is called Gateaux differentiable at $x_0 \in E_1$, if there exist $\delta f(x_0, \cdot) : E_1 \rightarrow E_2$ such that

$$\lim_{t \rightarrow 0} \left\| \frac{1}{t} (f(x_0 + tx) - f(x_0)) - \delta f(x_0, x) \right\|_2 = 0.$$

Then, one calls $\delta f(x_0, x)$ the Gateaux derivative of f at x_0 in the direction x .

Gateaux derivative $\delta f(x_0, x)$ have to be neither continuous nor linear in x . An important fact [13] is that if f is Frechet differentiable at x_0 , then it is also Gateaux differentiable and both derivatives are equal. Based on this, if f is Frechet differentiable at x_0 , then $Df(x_0)$ can be calculated as

$$Df(x_0)x = \delta f(x_0, x) = \lim_{t \rightarrow 0} \frac{1}{t} (f(x_0 + tx) - f(x_0)).$$

We note that this is not equivalent to the definition of the Frechet derivative.

Example: Consider a bounded linear operator $\mathcal{A} : \mathcal{X} \rightarrow \mathcal{X}$, where $\mathcal{X}$ is a real Hilbert space. Let $f : \mathcal{X} \rightarrow \mathbb{R}$ be given by $f(x) = \frac{1}{2} \langle x, \mathcal{A}x \rangle$. Then, the Frechet derivative calculated at a point $x_0 \in \mathcal{X}$ is given by

$$Df(x_0)x = \frac{1}{2} \langle x, \mathcal{A}x_0 \rangle + \frac{1}{2} \langle \mathcal{A}x, x_0 \rangle.$$

If $\mathcal{A}$ is a symmetric operator, then $Df(x_0) = \mathcal{A}x_0$ and we can write $Df = \mathcal{A}$. If $\mathcal{A}$ is not bounded then, $Df = \mathcal{A}$ is only the Gateaux derivative of f .

If the domain space E_1 of a map $f : E_1 \rightarrow \mathfrak{R}$ is a function space where the norm on E_1 is given by the standard L^2 inner product, then we have

$$Df(u)h = \int \frac{\delta f}{\delta u} h dz$$

and $\frac{\delta f}{\delta u} \in \mathcal{B}(E_1, \mathfrak{R}) = E_1^*$ is called the *functional variation* of f .

These differentiation notions are for functions defined on linear spaces. However, if f is a mapping between manifolds instead of linear spaces, then we can use the local coordinate patches to determine the derivatives.

The tangent and cotangent lifts of the functions defined on manifolds are defined as follows [1].

Definition 2.3 *Let f be a function from manifold M to manifold N . Then, the tangent lift $Tf : TM \rightarrow TN$ is defined as*

$$Tf(x, v_x) = (f(x), Df(x)v_x)$$

where $v_x \in T_x M$ and $Df(x) : T_x M \rightarrow T_{f(x)} N$ is the derivative of f at x .

Definition 2.4 *Let $f : M \rightarrow N$ be a diffeomorphism. Then, the cotangent lift $T^*f : T^*N \rightarrow T^*M$ is defined by*

$$T^*f(y, p_y) = (x, D^*f(x)p_y)$$

where $(y, p_y) \in T^*N$, $x \in M$ and $y = f(x) \in N$. Here, $D^*f(x) : T_{f(x)}^*M \rightarrow T_x^*N$ is the dual map of $Df(x)$ and defined by $\langle D^*f(x)p_y, v_x \rangle = \langle p_y, Df(x)v_x \rangle$ for all $v_x \in T_x N$ and $p_y \in T_y^*M$.

2.2 Tensors and Differential Forms

Tensors and differential forms are important tools for formulating mechanics in a geometric setting. Some references are [1], [2], [56], [30]. The following brief

introduction closely follows [56]. Let E be a linear space, and E^* be its dual. Then a multilinear function f of the form

$$f : \underbrace{E^* \times E^* \times \dots \times E^*}_{r \text{ times}} \times \underbrace{E \times E \times \dots \times E}_{s \text{ times}} \rightarrow \mathfrak{R}$$

is called a *tensor of contravariant order r , and covariant order s* (or an (r,s) tensor). Let M be a manifold. An (r,s) *tensor field* on M is a function which assigns an (r,s) tensor to each point $x \in M$ on the manifold such that $E = T_x M$ and $E^* = T_x^* M$. A vector field can be interpreted as a contravariant tensor field of order 1 or a $(1,0)$ tensor field. Covector fields are covariant tensor fields of order 1 or $(0,1)$ tensor fields and a Riemannian metric on a manifold defines a $(0,2)$ tensor field. Let $v_i, g_i \in T_x M$ and $p_i, f_i \in T_x^* M$. Let α be a (k,l) tensor and β be an (m,n) tensor. Then the $(k+m, l+n)$ tensor $\alpha \otimes \beta$ will be called the *tensor product* of α and β and is given by

$$\begin{aligned} (\alpha \otimes \beta)(x)(p_1, \dots, p_k, f_1, \dots, f_m, v_1, \dots, v_l, g_1, \dots, g_n) = \\ \alpha(x)(p_1, \dots, p_k, v_1, \dots, v_l) \beta(x)(f_1, \dots, f_m, g_1, \dots, g_n). \end{aligned}$$

Let $\{e_i\}$ be a basis for E and $\{e^i\}$ be a dual basis for E^* . Then, in terms of these bases, we calculate the components of an (r,s) tensor α as

$$\alpha_{j_1, \dots, j_s}^{i_1, \dots, i_r} = \alpha(e^{i_1}, \dots, e^{i_r}, e_{j_1}, \dots, e_{j_s}).$$

A skew-symmetric $(0,k)$ tensor field is called a *differential k -form* or a *k -form*. The set of k -forms on a manifold M is denoted by $\Lambda^k(M)$. The tensor multiplication of two differential forms does not necessarily give another differential form. In order to define a closed multiplication between differential forms we first define a skew-symmetrization operator $\mathbf{A}$, which when applied to a $(0,k)$

tensor α produces a k -form $\mathbf{A}(\alpha)$ which is given by

$$\mathbf{A}(v_1, \dots, v_k) = \frac{1}{k!} \sum_{\pi \in S_p} \text{sgn}(\pi) \alpha(v_{m(1)}, \dots, v_{m(k)}). \quad (2.1)$$

Here, $\text{sgn}(\pi)$ is the sign of permutation π (i.e. 1 for even permutations, -1 for odd permutations) S_p is the group of permutation of numbers $1, \dots, k$, and $m(i) \in \{1, \dots, k\}$. Then, the *wedge product* $\alpha \wedge \beta$ of a k -form α and an l -form β is defined as the $(k+l)$ -form given as

$$\alpha \wedge \beta = \frac{(k+l)!}{k! l!} \mathbf{A}(\alpha \otimes \beta). \quad (2.2)$$

We can represent a k -form as

$$\alpha = \alpha_{i_1, \dots, i_k} dx^{i_1} \wedge \dots \wedge dx^{i_k}.$$

by using the basis differential forms. Then, the *exterior derivative* $\mathbf{d} : \Lambda^k \rightarrow \Lambda^{k+1}$ is defined as

$$\mathbf{d}\alpha = \frac{\partial \alpha_{i_1, \dots, i_k}}{\partial x^j} dx^j \wedge dx^{i_1} \wedge \dots \wedge dx^{i_k}. \quad (2.3)$$

In particular, if α is a 0-form (a function), then $\mathbf{d}\alpha$ is just the derivative of α , i.e.;

$$(\mathbf{d}\alpha)_i = \frac{\partial \alpha}{\partial x^i}$$

and we sometimes use the notation $\mathbf{d}\alpha = d\alpha$ for this case. If α is a 1-form, then we have:

$$(\mathbf{d}\alpha)_{ij} = \frac{\partial \alpha_j}{\partial x^i} - \frac{\partial \alpha_i}{\partial x^j}.$$

The exterior derivative is a linear operation and it commutes with the pull back operation, i.e., $\varphi^* \mathbf{d}\alpha = \mathbf{d}\varphi^* \alpha$. In other words it commutes with coordinate changes on k -forms. Another important property is that $\mathbf{d}^2 \alpha = 0$ for any differential form α . This is known as *cocycle property* and it is closely related to

the identity $\text{curl}(\text{grad}\psi) = 0$ for a function ψ . A k -form α is said to be *closed* if $\mathbf{d}\alpha = 0$. The Poincare lemma states that any closed k -form α can be written locally as $\alpha = \mathbf{d}\beta$ for a $(k-1)$ -form β . An analogous fact is that if a vector field $\mathbf{v}$ satisfies $\text{curl}(\mathbf{v}) = 0$ then it can be written as $\mathbf{v} = \nabla\psi$ for some function ψ .

Definition 2.5 *Let α be a k -form and $\mathbf{X}$ be a vector field on M . Then, the interior product $\mathbf{i}_{\mathbf{X}}\alpha$ of α with $\mathbf{X}$ is the $(k-1)$ -form defined by*

$$(\mathbf{i}_{\mathbf{X}}\alpha)(x)(v_2, \dots, v_k) = \alpha(x)(\mathbf{X}(x), v_2, \dots, v_k).$$

In terms of the exterior derivative and the interior product, the *Lie derivative* of a k -form α along the vector field $\mathbf{X}$ is given by

$$\mathcal{L}_{\mathbf{X}}\alpha = \mathbf{d}\mathbf{i}_{\mathbf{X}}\alpha + \mathbf{i}_{\mathbf{X}}\mathbf{d}\alpha. \quad (2.4)$$

This equality is known as the *Cartan's formula*.

2.3 Lie Groups and Lie Algebras

In the context of mechanical systems, Lie groups and Lie algebras play an important role as the configuration manifolds of some important mechanical systems and also as tools to characterize the dynamical symmetries. We present some basic notions extracted from [1], [56].

Definition 2.6 *A Lie group is a group G which is also a manifold such that the group multiplication $(g, h) \rightarrow gh$ and the group inverse $g \rightarrow g^{-1}$ are smooth maps.*

The *left translation* on G is defined by the map $L_g : G \rightarrow G$ which is given as $L_g(h) = gh$ where $g, h \in G$. Similarly, the *right translation* $R_g : G \rightarrow G$ is

defined as $R_g(h) = hg$.

Example: The set of all $n \times n$ invertible matrices $GL(n)$ forms a Lie group with matrix multiplication and matrix inversion taken as the group multiplication and the group inverse respectively:

$$L_A(B) = AB, \quad R_A(B) = BA, \quad (A)^{-1} = A^{-1}.$$

$GL(n)$ is an n^2 dimensional Lie group and all finite dimensional matrix groups can be interpreted as subgroups of $GL(n)$.

The concept of Lie algebra is closely related to the notion of Lie groups.

Definition 2.7 *A Lie algebra is a vector space $\mathcal{G}$ equipped with a bilinear map $[\cdot, \cdot] : \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$ satisfying the conditions:*

1. *Skew-Symmetry:* $[a, b] = -[b, a]$
2. *Jacobi Identity:* $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

for every $a, b, c \in \mathcal{G}$. This bilinear map is called Lie bracket.

Example: The set of all $n \times n$ matrices forms a Lie algebra with the matrix commutation as the Lie bracket:

$$[A, B] = AB - BA.$$

This Lie algebra is denoted by $gl(n)$.

Example: Let M be a manifold and let $\mathcal{X}(M)$ denote the set of smooth vector fields defined on M . The linear space $\mathcal{X}(M)$ is a Lie algebra with the bracket:

$$[f, g]_J = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g$$

for $f, g \in \mathcal{X}(M)$. This bracket on vector fields is known as the *Jacobi-Lie bracket*.

Let $\{a_i\}$ be a basis for an n -dimensional Lie algebra $\mathcal{G}$. Then, the (2,1) tensor c which is given by

$$[a_i, a_j] = \sum_k c_{ij}^k a_k \quad i, j : 1, 2, \dots, n \quad (2.5)$$

determines the bracketing operation on $\mathcal{G}$ in terms of the coordinates w.r.t. basis $\{a_i\}$. The constants c_{ij}^k are called the *structure constants* of the Lie algebra $\mathcal{G}$. Due to the skew-symmetry and Jacobi identity conditions on Lie brackets, the structure constants satisfy the following conditions:

$$c_{ij}^k = -c_{ji}^k \quad i, j, k : 1, 2, \dots, n \quad (2.6)$$

$$\sum_k c_{ij}^k c_{kl}^m + c_{li}^k c_{kj}^m + c_{jl}^k c_{ki}^m = 0 \quad i, j, l, m : 1, 2, \dots, n. \quad (2.7)$$

Any (2,1) tensor satisfying the above identities defines a Lie algebra on an n dimensional vector space.

Definition 2.8 *The adjoint map $ad_a : \mathcal{G} \rightarrow \mathcal{G}$ on a Lie algebra $\mathcal{G}$ is defined as:*

$$ad_a b = [a, b]$$

where $a, b \in \mathcal{G}$. Let $\mathcal{G}^*$ denote the dual space of $\mathcal{G}$. Then, the coadjoint map $ad_a^* : \mathcal{G}^* \rightarrow \mathcal{G}^*$ is defined as the dual of the adjoint map, i.e.;

$$\langle ad_a^* c, b \rangle = \langle c, ad_a b \rangle$$

where $a, b \in \mathcal{G}$, $c \in \mathcal{G}^*$ and $\langle \cdot, \cdot \rangle$ is a pairing between $\mathcal{G}^*$ and $\mathcal{G}$.

A Lie algebra $\mathcal{G}$ can be associated with a Lie group G in a particular way. To do this, we need an invariance notion. A vector field $\mathbf{X}$ on a Lie group G is called *left invariant* if $(T_h L_g) \mathbf{X}(h) = \mathbf{X}(gh)$ for every $g, h \in G$. Similarly, a

vector field $\mathbf{X}$ is called *right invariant* if $(T_h R_g)\mathbf{X}(h) = \mathbf{X}(hg)$. Let $\mathcal{X}_L$ denote the set of all left invariant vector fields on G . $\mathcal{X}_L$ is a linear vector space and it is isomorphic to the tangent space $T_e G$ of G at the identity element e . Let $\xi \in T_e G$, then $\mathbf{X}_\xi(g) = T_e L_g(\xi)$ is a left invariant vector field on G , and $\xi \rightarrow \mathbf{X}_\xi$ defines an isomorphism between $T_e G$ and $\mathcal{X}_L$. Let $[\cdot, \cdot]_J$ denote the Jacobi-Lie bracket on vector fields, then

$$[\xi, \eta] = [\mathbf{X}_\xi, \mathbf{X}_\eta]_J(e)$$

where $\xi, \eta \in T_e G = \mathcal{G}$ defines a Lie bracket on $T_e G$. The tangent space $\mathcal{G}$ equipped with this Lie bracket is called the *Lie algebra of G* . A similar construction is possible via right invariant vector fields on G , then the induced Lie bracket on $\mathcal{G}$ is obtained as the negative of the one given above.

Example: The Lie algebra $gl(n)$ is the Lie algebra of $GL(n)$.

Example: The smooth vector fields $\mathcal{X}(M)$ on a manifold M can be interpreted as the Lie algebra of the diffeomorphism group of manifold M [29]. The Lie bracket on $\mathcal{X}(M)$ is given by the negative of the Jacobi-Lie bracket [56].

Definition 2.9 Let M be a manifold and G be a Lie group. A left action of G on M is a smooth mapping $\Phi : G \times M \rightarrow M$ such that

$$\Phi(e, x) = x \quad \forall x \in M$$

$$\Phi(g, \Phi(h, x)) = \Phi(gh, x) \quad \forall g, h \in G, x \in M.$$

A *right action* is a map $\Psi : M \times G \rightarrow M$ that satisfies $\Psi(x, e) = x$ and $\Psi(\Psi(x, g), h) = \Psi(x, gh)$. Let $g \in G$, and let $\Phi_g : M \rightarrow M$ be given by $\Phi_g(x) = \Phi(g, x)$. Then, a left action is characterized by $\Phi_e = id$ and $\Phi_{gh} = \Phi_g \Phi_h$.

If $M = V$ is a linear space and $\Phi_g : V \rightarrow V$ is a linear map then the action of G on V is called a *representation* of G on V . An action Φ is said to be

1. *transitive* if for any $x, y \in M$ there is a $g \in G$ s.t. $\Phi(g, x) = y$
2. *effective* if $\Phi_g = id$ implies $g = e$
3. *free* if $\Phi_g(x) = x$ for some $x \in M$ implies $g = e$.

Example: Every group acts on itself by translations. The left translation on a group is a left action of G on itself. Similarly, the right translation is a right action on the group. These actions are transitive, effective and free.

Example: Let the map $I_g : G \rightarrow G$ defined by $I_g(h) = ghg^{-1} = R_{g^{-1}}L_g(h)$ be called the *conjugation map*. This map defines a left action of G on G since $I_e = id$ and

$$I_g \circ I_h(x) = ghxh^{-1}g^{-1} = I_{gh}.$$

Example: By differentiating the conjugation map at $g = e$, we get the *adjoint action* of G on $\mathcal{G}$:

$$Ad : G \times \mathcal{G} \rightarrow \mathcal{G}$$

$$Ad_g(\xi) = (T_e I_g)\xi = T_e(R_g^{-1} \circ L_g)\xi.$$

This action is also called the *adjoint representation* of G on $\mathcal{G}$. For the matrix group $GL(n)$, the adjoint action is given by $Ad_g\xi = g\xi g^{-1}$ where $g \in GL(n)$ and $\xi \in gl(n)$. The adjoint map ad on $\mathcal{G}$ can also be characterized by linearizing the conjugation map at $g = e$, i.e:

$$ad_\xi\eta = T_e(Ad_g\eta)\xi = [\xi, \eta].$$

Let $Ad_g^* : \mathcal{G}^* \rightarrow \mathcal{G}^*$ be the dual of the adjoint action i.e.;

$$\langle Ad_g^* \alpha, \xi \rangle = \langle \alpha, Ad_g \xi \rangle$$

for all $\xi \in \mathcal{G}$ and $\alpha \in \mathcal{G}^*$. Then, the map $\Phi : G \times \mathcal{G}^* \rightarrow \mathcal{G}^*$ which is defined as

$$\Phi(g, \alpha) = Ad_{g^{-1}}^* \alpha$$

is a left action and is called the *coadjoint action* of G on $\mathcal{G}^*$. On the matrix group $GL(n)$ the coadjoint action is given by $Ad_g^* \xi = g^{-1} \xi g$.

Example: Let Φ be a left action of G on M . Then, the map $\tilde{\Phi} : G \times TM \rightarrow TM$ which is defined as:

$$\tilde{\Phi}(g, (x, v_x)) = (\Phi_g(x), (T_x \Phi_g) v_x)$$

is called the *tangent lifted action* of G on TM where $(x, v_x) \in T_x M$. Similarly, the mapping $\bar{\Phi} : G \times T^*M \rightarrow T^*M$ which is given by

$$\bar{\Phi}(g, (x, p_x)) = (\Phi_g(x), (T_x^* \Phi_{g^{-1}}) p_x)$$

is called the *cotangent lifted action* of G on T^*M where $(x, p_x) \in T_x^* M$.

2.4 Lagrangian Mechanics

Lagrangian mechanics is the formulation of conservative mechanical systems in terms of the velocity variables. A classical reference is [34]. For an excellent geometric treatment see [1] which we follow here. Let M be a manifold, $z \in M$ and TM be the tangent bundle. We denote points on TM by (z, v_z) . A mapping $L : TM \rightarrow \mathfrak{R}$ is called a *Lagrangian*. To model a physical system, the Lagrangian is taken as $L = K - V$ where K is the kinetic energy and V is the potential energy of the mechanical system we want to model.

The *principle of critical action* states that, if L is the Lagrangian of a conservative system, then the system evolves such that the condition

$$\delta \int L(z(t), v_z(t)) dt = 0 \quad (2.8)$$

holds where the variation is taken w.r.t fixed end-points solutions. This principle is equivalent to the *Euler-Lagrange equation*:

$$\frac{d}{dt} D_2 L(z, v_z) - D_1 L(z, v_z) = 0 \quad (2.9)$$

where $D_i L$ is the (Gateaux) derivative of L w.r.t. the i -th argument.

Let (z, p_z) denote a point in the cotangent bundle T^*M . Define the *Legendre transformation* $\mathbf{FL} : TM \rightarrow T^*M$ as

$$\mathbf{FL}(z, v_z) = (z, D_2 L). \quad (2.10)$$

That is we define the momentum variable $p_z = D_2 L \in T_z^*M$. Define the *energy* $E : TM \rightarrow \mathfrak{R}$ as

$$E(z, v_z) = \langle D_2 L, v_z \rangle - L(z, v_z). \quad (2.11)$$

Then, the Euler-Lagrange equations can be interpreted as a Hamiltonian system $\mathbf{X}_E$ on TM equipped with a symplectic structure Ω_L :

$$\Omega_L(z, v_z)(\mathbf{X}_E, \alpha) = \langle dE, \alpha \rangle \quad \forall \alpha \in T_z M. \quad (2.12)$$

The symplectic structure Ω_L is called the *Lagrangian symplectic structure* and is given by:

$$\Omega_L(z, v_z) = \begin{bmatrix} D_2 D_1 L - D_1 D_2 L & D_2 D_2 L \\ -D_2 D_2 L & 0 \end{bmatrix}.$$

If $\mathbf{FL}$ is a diffeomorphism, then the Lagrangian L is called *hyperregular*. For a hyperregular Lagrangian L we define the *Hamiltonian* $H : T^*M \rightarrow \mathfrak{R}$ by

$H \circ \mathbf{F}L = E$. That is:

$$H(z, p_z) = \langle p_z, \phi_z(p_z) \rangle - L(z, \phi_z(p_z)) \quad (2.13)$$

where $\phi_z(p_z) = v_z$. Note that if L is hyperregular then for any $z \in M$ there exist a unique map $\phi_z : T_z^*M \rightarrow T_zM$. With this definition of H , we get $v_z = D_2H \in T_zM$ and the Euler-Lagrange equations can be re-written as

$$\dot{z} = D_2H, \quad \dot{p}_z = -D_1H. \quad (2.14)$$

This is the Hamiltonian vector field of H on T^*M equipped with the standard Poisson structure.

Let u be a covector field on M which assigns a covector u_z to a point $z \in M$. This external force field is incorporated into the Euler-Lagrange equations as

$$\frac{d}{dt}D_2L(z, v_z) - D_1L(z, v_z) = u_z. \quad (2.15)$$

Under the effect of the external force u , the Hamiltonian H evolves as

$$\dot{H}(z, p_z) = \langle u_z, \dot{z} \rangle = \langle u_z, v_z \rangle$$

A force field u is called *dissipative* if $\langle u_z, v_z \rangle < 0, \forall z \in M$. A force field is called *conservative* if $\langle u_z, v_z \rangle = 0, \forall z \in M$. Let $R(z) : T_zM \rightarrow T_z^*M$ be a symmetric positive linear map, then the covector field u given as $u_z = -R(z)v_z$ is a dissipative force field. Let $G(z) : T_zM \rightarrow T_z^*M$ be a skew-symmetric linear map, then the covector field u which is given by $u_z = G(z)v_z$ is a conservative force field. In particular, if the (0,2) tensor field associated with G can be written as $G = \mathbf{d}\alpha$ for some covector field α , then $u_z = G(z)v_z$ is called a *gyroscopic* force and it can be incorporated into the Lagrangian formalism by modifying the Lagrangian L as $L \rightarrow L - \langle \alpha_z, v_z \rangle$. For an investigation of gyroscopic forces in the stability and control of mechanical systems see [79].

2.5 Poisson Mechanics

Classical mechanics can be generalized to a geometric setting. One approach to geometrization of mechanics is to characterize the conservative mechanical systems on Poisson manifolds. Some references in this regard are [56], [53], [54], [80], [68].

Definition 2.10 *A Poisson bracket on a manifold M is an $\mathfrak{R}$ -bilinear operation $\{\cdot, \cdot\}$ on $C^\infty(M)$ satisfying*

1. *skew-symmetry:* $\{f, g\} = -\{g, f\}$
2. *Jacobi identity:* $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$
3. *Leibniz rule:* $\{fg, h\} = g\{f, h\} + f\{g, h\}$

for every $f, g, h \in C^\infty(M)$. A manifold endowed with a Poisson bracket is called a Poisson Manifold.

The third condition on the bracket implies that the value of $\{f, g\}$ at $z \in M$ depends on the functions f and g only via their differentials. This together with the skew-symmetry condition implies that there is a contravariant skew-symmetric (2,0) tensor $W(z) : T_z^*M \times T_z^*M \rightarrow \mathfrak{R}$ such that

$$W(z)(df, dg) = \{f, g\}(z). \quad (2.16)$$

This tensor W is called a *Poisson tensor* or a *Poisson structure*. Associated with the Poisson tensor W , there is a vector bundle map $W^\sharp(z) : T_z^*M \rightarrow T_zM$ which is characterized by

$$\langle \alpha, W^\sharp(z)\beta \rangle = W(z)(\alpha, \beta) \quad (2.17)$$

for $\alpha, \beta \in T_z^*M$. In this study we will use the symbol W to denote both the tensor and the associated bundle map; the meaning will be clear from the context.

If M is finite dimensional and $\{z_i\}$ are local coordinates, then the Poisson bracket W is given by

$$\{f, g\}(z) = W^{ij}(z) \frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j}. \quad (2.18)$$

The conditions on a Poisson bracket impose the following conditions on the Poisson tensor W :

$$W^{ij}(z) = -W^{ji}(z) \quad i, j : 1, \dots, n \quad (2.19)$$

$$\sum_l W^{li} \frac{\partial W^{jk}}{\partial z^l} + W^{lj} \frac{\partial W^{ki}}{\partial z^l} + W^{lk} \frac{\partial W^{ij}}{\partial z^l} = 0 \quad i, j, k : 1, \dots, n. \quad (2.20)$$

Example: Let V be an inner product space, and $\mathcal{W}$ be a skew-symmetric operator on V , i.e., $\langle \mathcal{W}z, y \rangle = \langle z, -\mathcal{W}y \rangle$. Then,

$$\{f, g\}(z) = \langle df(z), \mathcal{W}dg(z) \rangle$$

defines a Poisson bracket on V and $\mathcal{W} : V^* \rightarrow V$ is a Poisson structure. Let M be a manifold, T^*M be its cotangent bundle and $(z, p_z) \in T_z^*M$. Then,

$$\mathcal{W}(z, p_z) = \begin{pmatrix} 0 & \mathcal{I} \\ -\mathcal{I} & 0 \end{pmatrix}$$

defines a Poisson structure on T^*M . This constant structure on T^*M is known as the *canonical Poisson structure* on T^*M .

Example: Let $\mathcal{G}$ be a Lie algebra and $\mathcal{G}^*$ be its dual. Then, the brackets

$$\{f, g\}(\mu) = \pm \langle \mu, [df, dg] \rangle \quad (2.21)$$

define Poisson structures on $\mathcal{G}^*$. Such brackets are known as the *Lie-Poisson brackets*. We will denote $\mathcal{G}^*$ equipped with $+$ and $-$ Lie-Poisson structures by

$\mathcal{G}_+^*$ and $\mathcal{G}_-^*$ respectively. If $\mathcal{G}$ is a finite dimensional Lie algebra with the structure constants c_{ij}^k , then the $+$ Lie-Poisson structure $W(\mu) : \mathcal{G} \times \mathcal{G} \rightarrow \mathfrak{R}$ on $\mathcal{G}^*$ is given by

$$W_{ij}(\mu) = \sum_k c_{ij}^k \mu_k. \quad (2.22)$$

Furthermore, one can check that a structure of this form is a Poisson structure iff c_{ij}^k are structure constants of a Lie algebra.

Example: Assume W is an invertible Poisson structure on a manifold M . Let the 2-form Ω on M is defined by $\Omega = W^{-1}$. Any 2-form which can be obtained as the inverse of a Poisson tensor is called a *symplectic form*. A manifold equipped with a symplectic form is called a *symplectic manifold*. On a finite dimensional symplectic manifold, a symplectic form Ω satisfies

$$\Omega_{ij} = -\Omega_{ji} \quad i, j : 1, \dots, n \quad (2.23)$$

$$\frac{\partial \Omega_{ij}}{\partial z_k} + \frac{\partial \Omega_{jk}}{\partial z_i} + \frac{\partial \Omega_{ki}}{\partial z_j} = 0 \quad i, j, k : 1, \dots, n. \quad (2.24)$$

These conditions directly follow from (2.19) and (2.20) if $\Omega = W^{-1}$. We also note that (2.24) is equivalent to $\mathbf{d}\Omega = 0$, i.e., Ω is a closed form.

Example: Jacobi-Lie bracket $[\cdot, \cdot]_J$ can be interpreted as a Poisson bracket on the covector fields $\mathcal{X}^*(M)$ defined on a manifold M . Let $f, g \in C^\infty(\mathcal{X}^*(M))$, then we have

$$\{f, g\} = [df, dg]_J$$

where we interpret $df, dg \in \mathcal{X}$. Note that, Poisson nature of the Jacobi-Lie bracket comes from the fact that Jacobi-Lie bracket is a Lie algebra bracket on vector fields.

Given a smooth function H on a Poisson manifold M , we define a vector field $\mathbf{X}_H : M \rightarrow TM$ on M by

$$\mathbf{X}_H(z) = W(z)dH(z) \quad (2.25)$$

and call it the *Hamiltonian vector field of H* . Alternatively, the Hamiltonian vector field $\mathbf{X}_H$ satisfies the equality

$$\langle dF(z), \mathbf{X}_H(z) \rangle = \{F, H\}(z) \quad (2.26)$$

for any $F \in C^\infty(M)$. A key aspect of a Hamiltonian vector field is the conservation of the Hamiltonian H along $\mathbf{X}_H$:

$$\dot{H} = \langle dH, \mathbf{X}_H \rangle = \{H, H\} = 0.$$

Another important property is that the flow of a Hamiltonian system preserves the Poisson structure, i.e.;

$$\{F, G\} \circ \varphi_t = \{F \circ \varphi_t, G \circ \varphi_t\}$$

where φ_t denotes the flow of $\mathbf{X}_H$. A function $C \in C^\infty(M)$ is called a *Casimir* function if

$$\{C, F\} = 0$$

for any $F \in C^\infty(M)$. Equivalently, a Casimir is a function which has its differential lying in the kernel of the Poisson structure: $WdC = 0$.

A Hamiltonian vector field $\mathbf{X}_H$ of the Hamiltonian H on a symplectic manifold is defined as

$$\Omega(z)(\mathbf{X}_H(z), v) = \langle dH(z), v \rangle \quad (2.27)$$

for all $v \in T_z M$ and $z \in M$. If $\Omega = W^{-1}$ then the vector fields characterized by (2.26) and (2.27) become identical.

Example: Let M be the space of the functions defined on the real line vanishing at infinity. Then, the dynamical equation $u_t = u_x$ can be interpreted as a Hamiltonian system on M , where the Poisson bracket is given by

$$\{f, g\}(u) = \int_{-\infty}^{\infty} \frac{\delta f}{\delta u} \frac{\partial}{\partial x} \frac{\delta g}{\delta u} dx$$

and the Hamiltonian H is given as

$$H(u) = \frac{1}{2} \int_{-\infty}^{\infty} u^2(x) dx.$$

The Poisson structure associated with this bracket is the differential operator ∂_x , which is a skew-symmetric linear operator on M . Then, one can calculate $dH = u$ and reconstruct the Hamiltonian equation as:

$$u_t = W dH = \partial_x(dH) = \partial_x(u) = u_x.$$

Let M_1 and M_2 be two Poisson manifolds with $\{\cdot, \cdot\}_1$ and $\{\cdot, \cdot\}_2$ as Poisson brackets. Then, a function $f : M_1 \rightarrow M_2$ is called *canonical* (or *Poisson*) if

$$\{F, K\}_2 \circ f = \{F \circ f, K \circ f\}_1 \quad (2.28)$$

for every $F, K \in C^\infty(M_2)$. The flow of a Hamiltonian vector field defines a one parameter family of canonical maps on the Poisson manifold itself. Let G be a Lie group with $\mathcal{G}$ as its Lie algebra. Then, a useful fact [56] is that the cotangent lifts of the left and the right actions on the group are canonical maps between T^*G equipped with the standard Poisson structure and $\mathcal{G}_-^*$ and $\mathcal{G}_+^*$ respectively.

A useful tool in the investigation of mechanics in a Poisson setting is Poisson reduction [56], [54], [80]. Consider a Poisson manifold M with the bracket $\{\cdot, \cdot\}_M$. Let G be a Lie group acting on M by Poisson maps. Assume that M/G is a smooth manifold which is equipped with the bracket $\{\cdot, \cdot\}_{M/G}$ which is given by

$$\{F \circ \pi, K \circ \pi\}_M = \{F, K\}_{M/G} \circ \pi \quad (2.29)$$

for all $F, K \in C^\infty(M/G)$. Here, $\pi : M \rightarrow M/G$ is the canonical projection which is by construction a Poisson map. Now, consider the Hamiltonian vector field $\mathbf{X}_H$ on M :

$$\langle dF, \mathbf{X}_H \rangle = \{F, H\}_M \quad \forall F \in C^\infty(M). \quad (2.30)$$

If the Hamiltonian $H : M \rightarrow \mathfrak{R}$ is G invariant, then $\mathbf{X}_H$ reduces to another Hamiltonian system $\mathbf{X}_{\tilde{H}}$ on M/G which is given by

$$\langle d\tilde{F}, \mathbf{X}_{\tilde{H}} \rangle = \{\tilde{F}, \tilde{H}\}_{M/G} \quad \forall \tilde{F} \in C^\infty(M/G) \quad (2.31)$$

where the reduced Hamiltonian $\tilde{H} : M/G \rightarrow \mathfrak{R}$ is defined by

$$\tilde{H} \circ \pi = H. \quad (2.32)$$

This process is known as *Poisson reduction* [54]. There are related reduction methods such as symplectic reduction [51], [1] cotangent bundle reduction [1], [56] and Lagrangian reduction [57], [58], [87] which can be interpreted as special cases of Poisson reduction. In particular, if $M = T^*G$ where G is a Lie group, then a left invariant Hamiltonian system on T^*G reduces to a Hamiltonian system on $\mathcal{G}_-^*$. Similarly, a right invariant Hamiltonian system on T^*G reduces to a Hamiltonian system on $\mathcal{G}_+^*$. This reduction is called *Lie-Poisson reduction* and has an important place in geometric mechanics [53], [56].

2.6 Divergence, Gradient, Etc.

We consider a finite dimensional Riemannian manifold M . Let G denote the Riemannian metric on M and $\ll \cdot, \cdot \gg$ denote the associated inner product, i.e.,

$$\ll a, b \gg = \sum_i \sum_j G_{ij}(x) a^i b^j$$

where $a, b \in T_x M$. We will also use the notation $a^\flat = G(x)a \in T_x^* M$ and $c^\sharp = G^{-1}(x)c \in T_x M$ for a vector $a \in T_x M$ and a covector $c \in T_x^* M$. If $M = \mathbb{R}^n$ with $G(x) = \mathbf{1}$ then it is called a Cartesian space. In this section we follow [1].

Definition 2.11 *Let $\mathbf{v}$ be a vector field on M . The divergence of $\mathbf{v}$ which we will denote by $\text{div} \mathbf{v}$ (or $\nabla \cdot \mathbf{v}$) is defined as the scalar field given by*

$$\nabla \cdot \mathbf{v} = \sum_i \frac{1}{\sqrt{\det(G(x))}} \frac{\partial}{\partial x^i} (\sqrt{\det(G(x))} v^i)$$

where v^i is the i -th component of $\mathbf{v}$.

In a Cartesian space, divergence is given by

$$\nabla \cdot \mathbf{v} = \sum_i \frac{\partial v^i}{\partial x^i}.$$

Definition 2.12 *Let f be a function on M . The gradient of f which we denote by $\text{grad } f$ (or ∇f) is the vector field defined as:*

$$\ll \nabla f, v_x \gg = \langle \mathbf{d}f, v_x \rangle \quad \forall v_x \in T_x M.$$

In coordinates, ∇f is given as:

$$\nabla f(x) = G^{-1}(x) \mathbf{d}f(x) = (\mathbf{d}f)^\sharp.$$

In a Cartesian space, this reduces to $\nabla f = \mathbf{d}f = df$.

The *Laplacian* Δf of a function f is defined as the scalar field given by

$$\Delta f = \text{div}(\text{grad } f) = \nabla \cdot \nabla f.$$

This operator is also called the *Laplace-Beltrami operator*. In coordinates, it is given as:

$$\Delta f = \frac{1}{\sqrt{\det(G(x))}} \sum_j \frac{\partial}{\partial x^j} (\sqrt{\det(G(x))} \sum_i (G^{-1}(x))^{ij} \frac{\partial f}{\partial x^i}).$$

It can be shown that $grad$ and $-div$ are adjoint w.r.t to the L^2 inner product. Hence, Δ acting on functions is symmetric and non-positive. In a Cartesian space, Δf is given by

$$\Delta f = \sum_i \frac{\partial^2 f}{\partial x^i \partial x^i}.$$

Analogous of all these vectorial operations can also be defined for differential forms.

Definition 2.13 *Let M be an n dimensional Riemannian manifold with the associated volume form $\mu \in \Lambda^n$. Then, there is a unique isomorphism $*$: $\Lambda^k \rightarrow \Lambda^{n-k}$ satisfying*

$$\alpha \wedge * \beta = \langle \alpha, \beta \rangle \mu$$

for all $\alpha, \beta \in \Lambda^k$. This linear operator is called the Hodge star operator.

The *codifferential* operator $\delta : \Lambda^k \rightarrow \Lambda^{k-1}$ is defined by

$$\delta = (-1)^{n(k+1)+1} * \mathbf{d} *.$$

The operator δ satisfies the equality $\delta^2 = 0$. The exterior derivative $\mathbf{d}$ and δ are adjoint w.r.t. the inner product

$$(\alpha, \beta) = \int \alpha \wedge * \beta.$$

Definition 2.14 *The Laplace-deRham operator $\Delta : \Lambda^k \rightarrow \Lambda^k$ on differential forms is defined by*

$$\Delta = \mathbf{d}\delta + \delta\mathbf{d}.$$

Since $\mathbf{d}$ and δ are adjoint, Δ acting on forms is symmetric and non-negative:

$$(\Delta\alpha, \beta) = (\alpha, \Delta\beta) \quad , \quad (\Delta\alpha, \alpha) \geq 0.$$

Acting on a function (a 0-form), the Laplace-deRham operator gives the negative of the Laplace-Beltrami operator acting on functions. By using the same symbol

Δ , we define the *Laplacian of a vector field* $\mathbf{v}$ as the negative of its image under the Laplace-deRham operator:

$$\Delta \mathbf{v} = -(\Delta(\mathbf{v}^\flat))^\sharp.$$

In 3-d Cartesian space, $\Delta \mathbf{v}$ reduces to the usual definition of the Laplacian of a vector field:

$$\Delta \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v})$$

where the curl operator $\nabla \times$ is given as

$$\nabla \times \mathbf{v} = \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \right).$$

Geometrically, we have $\nabla \times \mathbf{v} = (*(\mathbf{d}\mathbf{v}^\flat))^\sharp$. A comparison of Laplace-deRham operator on differential forms, and Laplacian on vector fields points out that $\mathbf{d}$ is analogous to curl and δ is analogous to negative of the divergence. Let $\mathbf{m}$ be a k -form. Based on this analogy, the curl of a k -form $\mathbf{m}$ is defined as the $(k+1)$ form $\mathbf{d}\mathbf{m}$. Similarly, the divergence of $\mathbf{m}$ is defined as the $(k-1)$ form $-\delta\mathbf{m}$. The only missing concept is the curl of a vector field $\mathbf{v}$ which is defined on a Riemannian manifold. In [69], $\text{curl}\mathbf{v}$ is defined as the $(0,2)$ tensor field given by

$$\text{curl}\mathbf{v}(X, Y) = \langle \nabla_X \mathbf{v}, Y \rangle - \langle \nabla_Y \mathbf{v}, X \rangle$$

where X and Y vector fields on M and ∇_X is the covariant differentiation along vector field X . In coordinates, the components of $\text{curl}\mathbf{v}$ are given as

$$(\text{curl}\mathbf{v})_{ij} = \frac{\partial v^j}{\partial x_i} - \frac{\partial v^i}{\partial x_j}.$$

Note that $\text{curl}\mathbf{v}$ is a skew-symmetric object. Hence, it is not only a $(0,2)$ tensor, but also a 2-form. Therefore, $\text{curl}\mathbf{v}$ is nothing but $\mathbf{d}\mathbf{v}$ if we interpret $\mathbf{v}$ as a

covector field. This simply implies that, the curl operation is more “natural” for covector fields than it is for vector fields. In 3-d Cartesian space we have

$$\text{curl} \mathbf{v} = -\widehat{\nabla \times \mathbf{v}}$$

where $\hat{\cdot}$ of a vector $a = [a_1, a_2, a_3]^T$ is defined as

$$\hat{a} = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}.$$

We note that, here the $\hat{\cdot}$ operation acts as the negative of the Hodge star operator $*$. It is unfortunate that the curl of a vector field is generally represented as another vector field, since this classical convention hides the more natural (and more general) interpretation of the curl operator as the exterior derivative on covector fields. For more on the curl operation see [62], [89], [88].

2.7 Some Useful Equations

Stokes Theorem: Let M be a compact, oriented n -dimensional manifold with boundary ∂M . Let α be a smooth $(n-1)$ -form. Then

$$\int_M \mathbf{d}\alpha = \int_{\partial M} \alpha.$$

A useful consequence of the Stokes theorem is the *divergence theorem*:

$$\int_M \nabla \cdot \mathbf{v} dV = \int_{\partial M} \mathbf{v} \cdot \mathbf{n} dA$$

where $\mathbf{v}$ is a vector field on $M \subset \mathbb{R}^3$, and $\mathbf{n}$ is the unit normal to boundary ∂M .

An equality related to the divergence theorem is [9]

$$\int_M \mathbf{v} \cdot \nabla \psi dV = - \int_M (\nabla \cdot \mathbf{v}) \psi dV + \int_{\partial M} \mathbf{v} \cdot \mathbf{n} \psi dA$$

where ψ is a function on $M \subset \mathbb{R}^3$. A direct result of this equality is that the gradient operator is adjoint to the negative of the divergence operator if $\mathbf{v}$ vanishes at the boundary. Another boundary condition for which these operators are adjoint is that $\mathbf{v}$ is parallel to the boundary ∂M .

We will also make use of the following identity in this study [9]:

$$\int_M \mathbf{v} \cdot (\nabla \times \mathbf{s}) dV = \int_M (\nabla \times \mathbf{v}) \cdot \mathbf{s} dV + \int_{\partial M} \mathbf{v} \cdot (\mathbf{n} \times \mathbf{s}) dA.$$

Here both $\mathbf{v}$ and $\mathbf{s}$ are vector fields on $M \subset \mathbb{R}^3$. A direct result is that curl is a symmetric operator on smooth vector fields provided either $\mathbf{v}$ vanishes on the boundary or $\mathbf{v}$ is normal to the boundary. We note that parallelness of $\mathbf{v}$ to the boundary is not a boundary condition under which curl is a symmetric operator.

By combining the results that curl is a symmetric operator and the divergence and the negative of the gradient operator are adjoint under vanishing boundary conditions, we conclude that the Laplacian operator Δ on vector fields is symmetric and non-positive with vanishing boundary conditions.

2.8 Some Infinite Dimensional Systems

In this section, we give some examples of infinite dimensional systems which can be interpreted as conservative (Lagrangian or Hamiltonian) systems. We would like to emphasize that some of these examples (such as the generalization of electromagnetism and the Hamiltonian structure of an infinite transmission line) are not well-known.

Example: Maxwell's Equations

The dynamics of an electromagnetic field in 3-d Cartesian space is given by the

celebrated equations of Maxwell:

$$\frac{\partial E}{\partial t} = \nabla \times B \quad (2.33)$$

$$\frac{\partial B}{\partial t} = -\nabla \times E \quad (2.34)$$

$$\nabla \cdot E = \rho \quad (2.35)$$

$$\nabla \cdot B = 0 \quad (2.36)$$

where we have set all physical constants as unity for simplicity. Here, E is the electrical field, B is the magnetic field and ρ is the time invariant electrical charge density. Letting $z = (E, B) \in \mathcal{X}(\mathbb{R}^3) \times \mathcal{X}(\mathbb{R}^3)$, Maxwell's equations can be represented as a Hamiltonian system $z_t = W(z)dH$ where W is a Poisson structure given by

$$W(E, B) = \begin{pmatrix} 0 & \nabla \times \\ -\nabla \times & 0 \end{pmatrix}$$

and the Hamiltonian H is given as

$$H(E, B) = \frac{1}{2} \int E^T E dx + \frac{1}{2} \int B^T B dx.$$

The structure W does not depend on (E, B) , therefore skew-symmetry of W suffices to imply its Poisson nature. And, the skew-symmetry of W follows from the symmetry of the curl operator [89]. The Poisson bracket associated with W is known as the Born-Infeld-Pauli bracket [56]. It is interesting to note that the conditions given by (2.35), (2.36) are satisfied automatically by the solutions of the dynamical equations given by (2.33), (2.34). By using the vector identity $\nabla \cdot \nabla \times = 0$ we get:

$$(\nabla \cdot B)_t = \nabla \cdot (B_t) = -\nabla \cdot (\nabla \times E) = 0$$

$$\rho_t = (\nabla \cdot E)_t = \nabla \cdot (E_t) = \nabla \cdot (\nabla \times B) = 0.$$

Therefore, the manifold determined by the conditions (2.35), (2.36) is invariant under Maxwell's equations. The given Poisson structure W can be obtained from a canonical structure by reduction [56] to this invariant manifold. Maxwell's equations are equivalent to the (electrical) wave equation:

$$E_{tt} - \Delta E = -\nabla \rho \quad (2.37)$$

where Δ is the Laplacian operation on the vector fields. The left side of this second order equation can be interpreted as a Lagrangian system associated with the Lagrangian

$$L(E, E_t) = \frac{1}{2} \int E_t^T E_t dx + \frac{1}{2} \int E^T \Delta E dx.$$

The right hand side is the external electrical field induced by the charge density.

Example: Generalized Electromagnetism

Let M be a Riemannian manifold of dimension $n \geq 3$. Let

$$\rho \in \Lambda^0(M) \quad , \quad E \in \Lambda^1(M)$$

$$B \in \Lambda^2(M) \quad , \quad \sigma \in \Lambda^3(M)$$

where $\Lambda^k(M)$ is the set of k -forms on M . We will assume that ρ and σ are constant in time, whereas E and B are not. We claim that the following set of equations generalizes electromagnetism (i.e., Maxwell's equations) to Riemannian manifolds:

$$E_t = \delta B \quad (2.38)$$

$$B_t = -\mathbf{d}E \quad (2.39)$$

$$\delta E = -\rho \quad (2.40)$$

$$\mathbf{d}B = \sigma \quad (2.41)$$

where $\mathbf{d}$ and δ denote the exterior derivative and the co-derivative respectively.

We can write the dynamical part of the equations above as

$$\begin{bmatrix} E_t \\ B_t \end{bmatrix} = \begin{bmatrix} 0 & \delta \\ -\mathbf{d} & 0 \end{bmatrix} \begin{bmatrix} E \\ B \end{bmatrix} = W dH.$$

From the adjointness of δ and $\mathbf{d}$, we conclude that W defines a Poisson structure on $\Lambda^1(M) \times \Lambda^2(M)$. Here, the Hamiltonian H is given by

$$H(E, B) = \frac{1}{2} \int E \cdot E dx + \frac{1}{4} \int \text{Tr}(B^T B) dx.$$

In order to show that these equations generalize Maxwell's equations, we differentiate (2.38) w.r.t. time and use (2.39) to eliminate B and we obtain:

$$E_{tt} + \delta \mathbf{d} E = 0.$$

Then, we use the Laplace-deRham operator $\Delta E = \mathbf{d}\delta E + \delta \mathbf{d} E$ and (2.40) and obtain

$$E_{tt} + \Delta E = -\mathbf{d}\rho. \quad (2.42)$$

Note that this is a wave equation for the covector field E . In order to represent the dynamics in terms of a vector field, we transform the equations by using the $\sharp$ operation and get:

$$(E_{tt} + \Delta E)^\sharp = -(\mathbf{d}\rho)^\sharp.$$

Then, by using $\nabla \rho = (\mathbf{d}\rho)^\sharp$ and $(\Delta \alpha)^\sharp = -\Delta(\alpha^\sharp)$ for a covector field α we get

$$E_{tt}^\sharp - \Delta E^\sharp = -\nabla \rho$$

which is nothing but the wave equation for the vector field $E^\sharp$. Therefore, (2.38),(2.39),(2.40),(2.41) can be interpreted as a generalization of Maxwell's equations to Riemannian manifolds. Here, E is the electric field on M which is

represented as a covector field, and B is the magnetic field which we take as a 2-form on M . ρ and σ are the electrical and magnetic charge fields on the manifold respectively. Note that we take the magnetic charge σ as a 3-form whereas the electric charge ρ is only as a 0-form. We also note that the charges are preserved along the solutions of the dynamics:

$$\rho_t = -\delta E_t = -\delta^2 B = 0$$

$$\sigma_t = \mathbf{d}B_t = -\mathbf{d}^2 E = 0$$

where we used $\mathbf{d}^2 = 0$ and $\delta^2 = 0$. Therefore, our assumption that ρ and σ are time-invariant is in harmony with this formulation. If ρ and σ are given as time variant charge fields, then we modify the equations as:

$$E_t = \delta B + j$$

$$B_t = -\mathbf{d}E + k$$

where $j \in \Lambda^1(M)$ and $k \in \Lambda^2(M)$ are the electrical and the magnetic currents satisfying $\delta j = -\rho_t$ and $\mathbf{d}k = \sigma_t$ respectively.

Example: Equations of Linear Elasticity

Consider an elastic material filling a volume. Let u denotes the small displacements from the relaxed configuration of the elastic continuum. Then, the dynamics governing the evolution of the small displacement field u is given [73] by the *Navier's Equation*:

$$\rho u_{tt} - (\lambda + \mu)\nabla(\nabla \cdot u) - \mu\Delta u = 0. \quad (2.43)$$

Here, ρ is the density of the medium and λ, μ are the Lamé constants. By recalling that *grad* and *-div* are adjoint and that Δ is a symmetric operator on

vector fields, we can obtain Navier's equation as the Euler-Lagrange equation associated with the Lagrangian:

$$L(u, u_t) = \frac{1}{2} \int u_t^T u_t dx + \frac{1}{2} \int ((\lambda + \mu) u^T \nabla (\nabla \cdot u) dx + \frac{1}{2} \int \mu u^T \Delta u dx.$$

Example: Infinite Transmission Line

The equations governing the electrical dynamics of an infinite (long) transmission line [27] are given by

$$LI_t + u_x + RI = 0 \quad (2.44)$$

$$Cu_t + I_x + Gu = 0 \quad (2.45)$$

where I and u denotes the current and the voltage distribution along the transmission line. The constant inductance and capacitance distribution along the line are denoted by L and C respectively while R and G denote the leakage parameters. Infinite transmission line equations can be interpreted as a Hamiltonian system provided the leakage parameters are taken as zero. Letting $z = (u, I)$, the equations of an infinite transmission line (with $R = G = 0$) can be written as $z_t = W dH$ where

$$W(z) = \begin{pmatrix} 0 & -\frac{1}{LC} \partial_x \\ -\frac{1}{LC} \partial_x & 0 \end{pmatrix}$$

and

$$H(u, I) = \frac{1}{2} \int_{-\infty}^{\infty} Cu^2 + LI^2 dx$$

The structure W does not depend upon (u, I) , and the skew-symmetry of W follows from the fact that the differential operator ∂_x is skew-symmetric. We can calculate $d_u H = Cu$, $d_I H = LI$ and reconstruct the equations easily.

Example: Korteweg-de Vries Equation

Small amplitude surface waves in a long and shallow (water) canal [1] are mod-

elled by the *Korteweg-de Vries (KdV) equation*:

$$u_t - 6uu_x + u_{xxx} = 0. \quad (2.46)$$

Here, u is the field representing the deviation of the water surface from the parallel water level. The KdV equation can be written as a Hamiltonian system $u_t = WdH$ where $W = \partial_x$ and the Hamiltonian H is given by:

$$H(u) = \int u^3 + \frac{1}{2}u_x^2 dx.$$

Skew-symmetry of the linear operator ∂_x gives the equation its Hamiltonian character. Finally, we reconstruct the KdV equations as a Hamiltonian system:

$$u_t = WdH = \partial_x(dH) = \partial_x(3u^2 - u_{xx}) = 6uu_x - u_{xxx}.$$

Example: Beam Equation

The dynamics of a thin, uniform beam clamped at both of its ends is given by the *Euler-Bernoulli equation*:

$$\rho u_{tt} + EIu_{xxxx} = 0 \quad (2.47)$$

where $u = u(x, t)$ denotes the small displacement of the position of the beam and $x \in [0, 1]$. We take the mass density ρ and the structural flexity EI as constants along the beam. The linear operator ∂_{xxxx} associated with the (clamped ends) boundary conditions $u(0, t) = u(1, t) = u_x(0, t) = u_x(1, t) = 0$ can be shown to be symmetric. Therefore, the Euler-Bernoulli equation can be derived from the Lagrangian

$$L(u, u_t) = \frac{1}{2} \int_0^1 u_t^2 dx - \frac{1}{2} \int_0^1 uu_{xxxx} dx.$$

Example: Plate Equation

Let $w = w(x, y)$ denote the small displacement of a plate clamped at its bound-

ary. Then the equation

$$\rho h w_{tt} - \frac{\rho h^3}{12} \Delta w_{tt} + D \Delta^2 w = 0 \quad (2.48)$$

where $\Delta^2 = \Delta \cdot \Delta$ and $\Delta = \partial_x^2 + \partial_y^2$ is known as *Kirchhoff's plate equation*. This is the equation of a conservative system, and can be obtained from the Lagrangian

$$L(w, w_t) = \frac{1}{2} \int (\rho h - \frac{\rho h^3}{12}) w_t^2 dx - \frac{1}{2} \int D \omega \Delta^2 w dx$$

as the Euler-Lagrange equation.

2.9 Basic Equations of Incompressible Fluid Mechanics

In 3-d Cartesian space, the dynamics of an incompressible inviscid fluid is given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} \quad (2.49)$$

where $\mathbf{v}$ is the velocity field and ρ is the density of the fluid [21]. The incompressibility condition is incorporated as $\nabla \cdot \mathbf{v} = 0$. The scalar field p is called the *pressure field*. This equation is known as *Euler's equation for incompressible fluid flow*. The evolution of the fluid density is governed by

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \quad (2.50)$$

which is nothing but a consequence of conservation of mass. A fluid is called *homogeneous* if ρ is constant throughout the fluid. Note that if an incompressible fluid is homogenous at time t_0 , then it remains homogenous thereafter. A

homogenous, incompressible and inviscid fluid is called a *perfect fluid*. The dynamics of a perfect fluid in a Riemannian manifold is given [1] by

$$\mathbf{v}_t + \nabla_{\mathbf{v}} \mathbf{v} = -\nabla p \quad (2.51)$$

where $\mathbf{v}$ is the velocity field of the fluid on the Riemannian manifold and $\nabla_{\mathbf{v}} \mathbf{v}$ is the covariant derivative of $\mathbf{v}$ along $\mathbf{v}$.

The dynamics of an ideal fluid can be represented in different ways besides Euler's equation. The dynamical equation

$$\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times (\nabla \times \mathbf{v}) = -\frac{\nabla s}{\rho} \quad (2.52)$$

where $s = p + \frac{1}{2} \mathbf{v}^T \mathbf{v}$ is called *Bernoulli's equation for incompressible flow* [8]. It can be shown that Bernoulli's equation is equivalent to Euler's equation by using the vector identity

$$\frac{1}{2} \nabla (\mathbf{v}^T \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v}).$$

Another representation for incompressible fluid dynamics can be obtained by applying the curl operation to both sides of Bernoulli's equation. By doing so, we get:

$$\frac{\partial}{\partial t} (\nabla \times \mathbf{v}) = \nabla \times (\mathbf{v} \times (\nabla \times \mathbf{v})) - \frac{1}{\rho} \nabla \times \nabla s.$$

By defining the *vorticity field* $\mathbf{w}$ as $\mathbf{w} = \nabla \times \mathbf{v}$ and by using the fact $\text{curl} \cdot \text{grad} = 0$, we obtain:

$$\frac{\partial \mathbf{w}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{w}). \quad (2.53)$$

This equation is called the *vorticity equation*. By using the vector identity

$$\nabla \times (\mathbf{f} \times \mathbf{g}) = \mathbf{f}(\nabla \cdot \mathbf{g}) - \mathbf{g}(\nabla \cdot \mathbf{f}) - [\mathbf{f}, \mathbf{g}]_J$$

and recalling that $\nabla \cdot \mathbf{w} = \nabla \cdot \nabla \times \mathbf{v} = 0$ and $\nabla \cdot \mathbf{v} = 0$, we obtain:

$$\frac{\partial \mathbf{w}}{\partial t} = -[\mathbf{v}, \mathbf{w}]_J = -\mathcal{L}_{\mathbf{v}} \mathbf{w}. \quad (2.54)$$

In this new form, the vorticity equation is not only valid on 3-d Cartesian space but also on a Riemannian manifold where the vorticity is defined as $\mathbf{w} = \mathbf{d}\mathbf{v}$.

Neither Euler's equation nor its variants incorporate the dissipative effects which are almost always present in real life fluids. The dynamics of an incompressible viscous fluid is given by

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{\nabla p}{\rho} + \frac{\mu}{\rho} \Delta \mathbf{v} \quad (2.55)$$

which is called the *Navier-Stokes equation* [21]. Here, μ is a positive parameter and is known as the viscosity coefficient of the fluid.

The main difference between the Navier-Stokes equation and the variants of Euler's equation is that, the Navier-Stokes equation describes a dissipative dynamical system whereas the others can be interpreted as conservative (Hamiltonian) systems. We return to this point later in chapter 3, where among other things we geometrize Bernoulli's equation to Riemannian manifolds.

Chapter 3

Rigid Bodies Containing Fluids

3.1 Introduction

In this chapter, we develop a model for rigid body-fluid interaction in spacecraft and analyze the resulting model from the mechanical and geometric points of view. The structure of this chapter is as follows. In section 3.2, we develop a general model for a rigid body containing perfect fluid starting from a Lagrangian formulation followed by an implicit reduction process. Here, we take the configuration manifold of the system as the cartesian product of the rotation group $SO(3)$ and the group of volume preserving diffeomorphisms and obtain the model as the Euler-Lagrange equations. We derive the basic equations of rigid body mechanics and fluid dynamics from the rigid body-fluid model we obtained. Three equivalent representations of the model are given, which are the main objects of study of this dissertation. Constants of motion for the dynamics of rigid bodies containing fluids are identified in section 3.3. A complete classification of the equilibria of the system is given in section 3.4. In 3.5, we explore the dynamic structure of the model from a geometric mechanics point

of view. In particular, we show the model can be interpreted as a Hamiltonian system on an infinite dimensional Poisson manifold. We generalize Bernoulli's equation to a Riemannian manifold setting. Lie-Poisson and Euler-Poincare interpretations of the model are also studied in detail in this section. In the last section, we modify our model in order to incorporate the viscosity of the fluid, and investigate the dynamics of the resulting dissipative system.

3.2 Lagrangian Formulation

The Lagrangian formulation [1], [34] of mechanical systems starts with the identification of a configuration manifold $\mathcal{M}$ of the mechanical phenomenon we want to model. The second step is the determination of the Lagrangian L as a smooth function on the tangent bundle of the configuration manifold $T\mathcal{M}$, and the Lagrangian is chosen as the difference between the kinetic and the potential energies involved. The last step is to determine the dynamical equations of the system and this is done by means of the celebrated Euler-Lagrange equations:

$$\frac{d}{dt}(D_{\dot{z}}L(z(t), \dot{z}(t))) - D_zL(z(t), \dot{z}(t)) = 0$$

where $(z, \dot{z})$ are local coordinates of $T\mathcal{M}$ and D stands for the (Gateaux) derivative with respect to the subscript. The first two steps of the Lagrangian Formalism; the determination of the configuration manifold and the Lagrangian requires phenomenological knowledge of the physical system, and this constitutes the core of the physical research about nature. The last step (determination of the Euler-Lagrange equations) is nothing but a routine exercise in mathematics which only requires a sound knowledge of differentiation.

Here, we will develop a model for a rigid body containing perfect (incom-

pressible, inviscid and homogeneous) fluid via Lagrangian formalism. Before proceeding further, it is necessary to explain what we mean. For us, a rigid body containing perfect fluid is a rigid body with a cavity $\mathcal{F}$ with smooth boundary $\partial\mathcal{F}$. We assume that the cavity is fully filled by an incompressible, inviscid fluid of homogeneous mass density. This density will be denoted by ρ_F . Under these conditions, if no external force field is assumed, the center of mass of the total body-fluid mass distribution will be stationary with respect to inertial space. We pick an *inertial reference frame* sharing its origin with the center of mass. Another reference frame sharing the same origin but fixed at the body will be used, this second frame will be called the *body reference frame*. Both frames are assumed to be orthonormal and right handed. The material particles (body or fluid) will be denoted by X . $U(X, t)$ and $\eta(X, t)$ will denote the *position of particle X* at time t w.r.t. the inertial reference frame and w.r.t. the body frame respectively. With this notation, a complete indexing of the particles can be given by $\eta(X, 0) = X$. The *velocity of particle X* w.r.t. body frame will be denoted by $\dot{\eta}(X, t)$, and $\ddot{\eta}(X, t)$ will be the *acceleration of particle X* . $\mathbf{v}(x, t)$ will denote the velocity of the particle occupying the position $x = \eta(X, t)$ at time t w.r.t body frame. Since the inertial and the body frames share the same origin, the relationship between U and η is given by a rotation matrix Y formed by the direction cosines between the inertial reference frame and the body reference frame; $U(X, t) = Y(t)\eta(X, t)$. From this equality it is obvious that the cartesian product of the set of rotation matrices Y and the set of all possible configurations η for fluid particles can serve as a configuration manifold. Indeed, η can not be totally arbitrary because of the incompressibility condition on the fluid. The permissible configurations for the fluid particles will be the ones created by

incompressible velocity fields parallel to the boundary of the cavity which we denote by $\mathcal{X}_d$. We will denote the set of configurations of the incompressible fluid particles by Ψ , and will interpret it as the volume preserving diffeomorphisms of the cavity [29]. In other words, we will assume that for any t , $\eta(\cdot, t)$ is a volume preserving diffeomorphism on $\mathcal{F}$. Therefore, we will take

$$\mathcal{M} = \{(Y(t), \eta(\cdot, t)) \mid Y(t) \in SO(3), \eta(\cdot, t) \in \Psi\} = SO(3) \times \Psi$$

as the configuration manifold of a rigid body with its cavity fully filled of a perfect fluid.

3.2.1 Kinematics of the Configuration Manifold

Configuration manifold $\mathcal{M} = SO(3) \times \Psi$ is a cartesian product, so we will look at the kinematics on the rotation group and the diffeomorphism group separately.

The *rotation group* (or the *special orthogonal group*) of $\mathbb{R}^3$ is defined as

$$SO(3) = \{Y \in \mathbb{R}^{3 \times 3} \mid Y^T Y = I, \det(Y) = 1\}. \quad (3.1)$$

This is a matrix Lie group [25] for which the left and the right multiplication is given by matrix multiplication from left and right respectively. Associated Lie algebra $so(3)$ is the linear space of 3×3 skew-symmetric matrices

$$so(3) = \{\Omega \in \mathbb{R}^{3 \times 3} \mid \Omega^T = -\Omega\} \quad (3.2)$$

equipped with the Lie bracket given by matrix commutation $[A, B] = AB - BA$. We identify $so(3)$ with $\mathbb{R}^3$ by using the “hat” operation $\hat{\cdot}$. Let $\Omega \in so(3)$ and

$\omega \in \mathfrak{R}^3$, $\omega = [\omega_1, \omega_2, \omega_3]^T$ then

$$\hat{\omega} = \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} \quad (3.3)$$

defines a Lie algebra isomorphism between $so(3)$ and $\mathfrak{R}^3$ equipped with the vector cross product as Lie bracket. Let $a, b \in \mathfrak{R}^3$ then

$$\widehat{a \times b} = [\hat{a}, \hat{b}]. \quad (3.4)$$

Let $Y(t)$ be a C^1 curve lying in $SO(3)$. Then, $\dot{Y} \in T_Y SO(3)$ can always be written as $\dot{Y} = Y\Omega$ for a unique $\Omega \in so(3)$. If we interpret Y as the orientation of a rigid body, then Ω gives us the angular velocity in the body frame. Similarly, if we choose to write $\dot{Y} = WY$ for some $W \in so(3)$, then W characterizes the angular velocity in the space coordinates. We will denote the elements of the cotangent space $T_Y^* SO(3)$ by P_Y and we write $P_Y = YQ$ for $Q \in so^*(3)$ which we interpret as the momentum space of a rigid body. However, we should remind that Q unlike Ω does not have an intrinsic physical meaning. Physical interpretation of Q is only possible if we associate a Lagrangian with the motion of the rigid body. Finally, before passing to the kinematics of the volume preserving diffeomorphisms, we summarize the notation we will use in this study for the rotation group:

$$(Y, \dot{Y}) \in TSO(3), \dot{Y} \in T_Y SO(3) \quad (3.5)$$

$$(Y, P_Y) \in T^*SO(3), P_Y \in T_Y^* SO(3) \quad (3.6)$$

$$\dot{Y} = Y\Omega, \Omega \in so(3) \quad (3.7)$$

$$P_Y = YQ, Q \in so^*(3). \quad (3.8)$$

Let $\mathcal{F} \in \mathbb{R}^3$ be a domain with smooth boundary $\partial\mathcal{F}$. We will denote the set of all smooth and smoothly invertible maps $\eta : \mathcal{F} \rightarrow \mathcal{F}$ by $\tilde{\Psi}$, i.e. $\tilde{\Psi}$ is the set of diffeomorphisms of $\mathcal{F}$. This set is known to be Hilbert manifold provided η and η^{-1} have square integrable derivatives of sufficiently high order [29]. The subset of $\tilde{\Psi}$ containing the volume preserving diffeomorphisms leaving the boundary $\partial\mathcal{F}$ fixed will be denoted by Ψ . Ψ is a closed submanifold of $\tilde{\Psi}$. Both of these sets are Lie groups where the group multiplication is given by the composition [56]. The right and left multiplications on Ψ and $\tilde{\Psi}$ are given by

$$R_\phi\eta = \eta \circ \phi \tag{3.9}$$

$$L_\phi\eta = \phi \circ \eta. \tag{3.10}$$

We refer to [29], [56] for the technicalities of the diffeomorphisms groups $\tilde{\Psi}$ and Ψ as Lie groups.

The Lie algebra associated with the diffeomorphism group Ψ is the space of vector fields $\mathcal{X}$ defined on $\mathcal{F}$ [29]. For volume preserving diffeomorphisms, the associated Lie algebra is the space of divergence-free vector fields tangential to the boundary $\partial\mathcal{F}$. We denote this space by $\mathcal{X}_d$. The Lie bracket $[\cdot, \cdot]_L$ on $\mathcal{X}$ is given by the negative of the Jacobi-Lie bracket $[\cdot, \cdot]_J$ of vector fields [56]. Let $\mathbf{f}$ and $\mathbf{g} \in \mathcal{X}$, then

$$[\mathbf{f}, \mathbf{g}]_L = -[\mathbf{f}, \mathbf{g}]_J = (\mathbf{g} \cdot \nabla)\mathbf{f} - (\mathbf{f} \cdot \nabla)\mathbf{g}. \tag{3.11}$$

On divergence-free vector fields this bracket reduces to

$$[\mathbf{f}, \mathbf{g}]_L = \nabla \times (\mathbf{f} \times \mathbf{g}) \tag{3.12}$$

Now, we interpret the volume preserving diffeomorphism group as the configuration space of an incompressible continuum filling $\mathcal{F}$. Let $x = \eta(X, t)$ denote

the position of the continuum particle X . By differentiating η w.r.t. time we get the *material velocity* V_η which we naturally interpret as an object in the tangent space $T_\eta\Psi$

$$\dot{\eta}(X, t) = V_\eta(X, t). \quad (3.13)$$

We define the *spatial velocity* $\mathbf{v}$ by changing the coordinates of V_η from the material (X) to the spatial ($x = \eta(X)$) coordinates:

$$\mathbf{v}(x, t) = V_\eta(\eta^{-1}(x), t) = V_\eta \circ \eta^{-1}. \quad (3.14)$$

From these conventions, it is clear that the tangent space $T_\eta\Psi$ at point η will consist of the elements of the form $\mathbf{v} \circ \eta$ where $\mathbf{v} \in \mathcal{X}_d$. Similarly, we represent the elements M_η of the cotangent space $T_\eta^*\Psi$ by $\mathbf{m} \circ \eta$ where $\mathbf{m} \in \mathcal{X}_d^*$. Here, $\mathcal{X}_d^*$ denotes the linear space divergence-free covectors rather than the linear dual of $\mathcal{X}_d$. In this formalism, the correct interpretation for $\mathbf{v}$ and $\mathbf{m}$ will be the *velocity field* and the *momentum field* of the continuum respectively. As we pointed out before for the rigid body, to fix the momentum field we must assign a Lagrangian to the motion of the continuum. On the other hand, the velocity field of the continuum $\mathbf{v}$ can be understood without any reference to a particular energy function. Finally, we summarize the kinematics of the group Ψ of volume preserving diffeomorphisms by means of the following notations:

$$(\eta, V_\eta) \in T\Psi, \quad V_\eta = \dot{\eta} \in T_\eta\Psi \quad (3.15)$$

$$(\eta, M_\eta) \in T^*\Psi, \quad M_\eta \in T_\eta^*\Psi \quad (3.16)$$

$$V_\eta = \mathbf{v} \circ \eta, \quad \mathbf{v} \in \mathcal{X}_d \quad (3.17)$$

$$M_\eta = \mathbf{m} \circ \eta, \quad \mathbf{m} \in \mathcal{X}_d^*. \quad (3.18)$$

3.2.2 Euler-Lagrange Equations

Now, we are ready to determine the energy content of the system. The kinetic energy of the system is the sum of the kinetic energies of the individual material particles. There is no potential energy involved because neither the position of the rigid body in space nor the position of the fluid particles w.r.t. body, change the energy content of the system. Therefore, the Lagrangian L associated with a rigid body containing perfect fluid is given by:

$$L(Y, \dot{Y}, \eta, \dot{\eta}) = \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(\eta(X)) \|\dot{U}(X, t)\|^2 dX \quad (3.19)$$

$$= \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(\eta(X)) \|\dot{Y}\eta + Y\dot{\eta}\|^2 dX \quad (3.20)$$

$$= \frac{1}{2} \int_{\mathcal{B}} \rho_B \eta^T \dot{Y}^T \dot{Y} \eta dX \quad (3.21)$$

$$+ \frac{1}{2} \int_{\mathcal{F}} \rho_F (\eta^T \dot{Y}^T \dot{Y} \eta + \dot{\eta}^T Y^T Y \dot{\eta} + 2\eta^T \dot{Y}^T Y \dot{\eta}) dX \quad (3.22)$$

where $\dot{Y} \in T_Y SO(3)$ and $\dot{\eta} \in T_\eta \Psi$. Then, the Euler-Lagrange equations will be given by

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{Y}} - \frac{\delta L}{\delta Y} = 0 \quad (3.23)$$

$$\frac{d}{dt} \frac{\delta L}{\delta \dot{\eta}} - \frac{\delta L}{\delta \eta} = 0. \quad (3.24)$$

As we mentioned above, $SO(3)$ and Ψ are Lie Groups, they are not linear spaces. This creates an additional problem in the determination of the functional derivatives of the Lagrangian. To differentiate the Lagrangian with respect to $Y \in SO(3)$ we need a parametrization of the rotation group $SO(3)$. Similarly we need to know a parametrization for the volume preserving diffeomorphisms. We could use, any one of the well-known parametrizations for $SO(3)$, say Euler angles, but finding parametrizations for the volume preserving diffeomorphisms

group is a difficult task. To overcome this difficulty as well as to avoid the problems of dealing with a particular parametrization which might turn the calculations of the derivatives into a mess of symbols, here we take another way. We will enlarge the domain of definition of the Lagrangian into a larger manifold smoothly in which we could deal with differentiation easily; then we will transport the derivatives back to their original domains by projections. L is defined as a smooth function on $T\mathcal{M}$, we enlarge its domain to $T\tilde{\mathcal{M}}$ where $\mathcal{M} \subset \tilde{\mathcal{M}}$ and $T_z\mathcal{M} \subset T_z\tilde{\mathcal{M}}$ as a subspace. Then,

$$\frac{d}{dt}(D_{\dot{z}}L(z, \dot{z})) - D_zL(z, \dot{z}) = \mathcal{P}(\frac{d}{dt}(D_{\dot{z}}\tilde{L}(z, \dot{z})) - D_z\tilde{L}(z, \dot{z}))$$

where $\tilde{L} : T\tilde{\mathcal{M}} \rightarrow \mathbb{R}$, $\tilde{L}(z, \dot{z}) = L(z, \dot{z})$, $\forall (z, \dot{z}) \in T\mathcal{M}$ and $\mathcal{P}$ is the projection from $T_z^*\tilde{\mathcal{M}}$ into $T_z^*\mathcal{M}$ [11].

Here, we have $\mathcal{M} = SO(3) \times \Psi$ and we choose to enlarge the configuration manifold to $\tilde{\mathcal{M}} = GL(3) \times \tilde{\Psi}$ where $GL(3)$ is the group of invertible 3×3 matrices and $\tilde{\Psi}$ is the group of diffeomorphisms of the cavity. The enlarged configuration manifold is in the form of the cartesian product of two Lie groups as the original one. Associated Lie algebras are $gl(3)$ (set of all 3×3 matrices), and $\mathcal{X}$ (set of velocity fields defined in the cavity $\mathcal{F}$) respectively.

It is useful to note that there exists convenient decompositions of $gl(3)$ and $\mathcal{X}$ which we make use of in the following. The Lie algebra $gl(3)$ can be decomposed as

$$gl(3) = so(3) \oplus so(3)^\perp \quad (3.25)$$

where the orthogonality should be understood with respect to the trace inner product on matrices. $so(3)$ is the space of 3×3 skew-symmetric matrices, therefore $so(3)^\perp$ will be nothing but the space of 3×3 symmetric matrices. Hence,

any $X \in gl(3)$ can be written as

$$X = (X)_s + (X)_a \quad (3.26)$$

where

$$(X)_s = \frac{1}{2}(X + X^T) \in so(3)^\perp \quad (3.27)$$

$$(X)_a = \frac{1}{2}(X - X^T) \in so(3) \quad (3.28)$$

define the orthogonal projections from $gl(3)$ to the subspaces $so(3)^\perp$ and $so(3)$ respectively. This decomposition can be extended to the cotangent space $T_Y^*GL(3)$ to get the projection operator

$$\Pi : T_Y^*GL(3) \rightarrow T_Y^*SO(3) \quad (3.29)$$

$$\Pi(P_Y) = Y(Y^{-1}P_Y)_a \quad (3.30)$$

where we followed Bailleul and Levi [11].

In incompressible fluid mechanics, the following decomposition of vector fields has proven to be very useful .

Theorem 3.1 Helmholtz Decomposition *The space of vector fields $\mathcal{X}$ on $\mathcal{F} \subset \mathbb{R}^3$ with boundary $\partial\mathcal{F}$ can be orthogonally (with respect to L^2 inner product) decomposed as:*

$$\mathcal{X} = \mathcal{X}_d \oplus \mathcal{X}_g$$

where

$$\mathcal{X}_d = \{\mathbf{v} \in \mathcal{X} | \nabla \cdot \mathbf{v} = 0, \mathbf{v} \parallel \partial\mathcal{D}\}$$

$$\mathcal{X}_g = \{\mathbf{v} \in \mathcal{X} | \mathbf{v} = \nabla\phi, \phi \in C^1(\mathcal{D})\}$$

are the divergence-free vector fields parallel to the boundary and the gradient vector fields respectively.

Remark: For a proof of this theorem see [21]. We also remind that the Helmholtz decomposition is also valid for vector fields defined on Riemannian manifolds. An analog of the Helmholtz decomposition of vector fields is the Hodge decomposition for covector fields. Indeed, Hodge's theorem [2] gives a decomposition of differential forms of any order, hence it is more general than the Helmholtz decomposition.

We will denote the projection of $\mathcal{X}$ into $\mathcal{X}_d$ and $\mathcal{X}_g$ by $\mathcal{P}_d$ and $\mathcal{P}_g$ respectively. Therefore, for $\mathbf{v} \in \mathcal{X}$, we have

$$\mathbf{v} = \mathcal{P}_d(\mathbf{v}) + \mathcal{P}_g(\mathbf{v}). \quad (3.31)$$

Since the elements of the tangent space of the diffeomorphism group $\tilde{\Psi}$ are of the form $\mathbf{v} \circ \eta$ where $\mathbf{v} \in \mathcal{X}$, we can decompose [12] the tangent space $T_\eta \tilde{\Psi}$ of the diffeomorphism group as

$$T_\eta \tilde{\Psi} = \mathcal{X}_d \circ \eta \oplus \mathcal{X}_g \circ \eta. \quad (3.32)$$

Furthermore, if we identify the momentum fields $\mathbf{m}$ with the velocity fields $\mathbf{v}$ by using an equality $\mathbf{m}(x) = \mathcal{S}(\mathbf{v}(x))$ for some invertible map $\mathcal{S}$, then the cotangent space of the diffeomorphism group at η can be decomposed as

$$T_\eta^* \tilde{\Psi} = \mathcal{S}(\mathcal{X}_d) \circ \eta \oplus \mathcal{S}(\mathcal{X}_g) \circ \eta. \quad (3.33)$$

These decompositions of $gl(3)$ and $\tilde{\Psi}$ will be very useful in the determination of the functional derivatives of the Lagrangian L . First, however, we introduce some more notation in order to express the results in a compact manner. We define matrices A, B, C as:

$$A(t) = \int_{\mathcal{F}} \rho_F \eta(X, t) \eta^T(X, t) dX \quad (3.34)$$

$$B(t) = \int_{\mathcal{F}} \rho_F \dot{\eta}(X, t) \eta^T(X, t) dX \quad (3.35)$$

$$C(t) = \int_{\mathcal{B}} \rho_B \eta(X, t) \eta^T(X, t) dX. \quad (3.36)$$

From the incompressibility and homogeneity of the fluid flow, it follows that $A(t)$ does not depend on time. $B(t)$ is skew-symmetric due to the constancy of A . The time invariance of $C(t)$ can be seen by observing that the defining integral is taken only over the material particles of the rigid body.

The *moment of inertia* matrices associated with the fluid mass and the body mass are defined as

$$I_F = \int_{\mathcal{F}} \rho_F(x) (\mathbf{1} \text{tr}(xx^T) - xx^T) dx \quad (3.37)$$

$$I_B = \int_{\mathcal{B}} \rho_B(x) (\mathbf{1} \text{tr}(xx^T) - xx^T) dx \quad (3.38)$$

Recalling that $x = \eta(X)$, and $dx = dX$ (which is just an implication of incompressibility) these inertia matrices can be written as:

$$I_F = \mathbf{1} \text{Tr}(A) - A \quad (3.39)$$

$$I_B = \mathbf{1} \text{Tr}(C) - C \quad (3.40)$$

Therefore, A and C are the *coefficient of inertia* matrices of the fluid mass and the rigid body mass respectively. The moment of inertia matrix of the total system will be denoted by I_T , and it is given as $I_T = I_F + I_B$.

The functional derivatives of the Lagrangian L are obtained by lengthy but straightforward calculations. We give them by using the matrices we defined above as:

$$\frac{\delta L}{\delta \dot{Y}} = \dot{Y}(A + C) + YB \quad (3.41)$$

$$\frac{\delta L}{\delta \dot{Y}} = \dot{Y}B^T \quad (3.42)$$

$$\frac{\delta L}{\delta \eta} = \rho_F(\dot{Y}^T \dot{Y} \eta + \dot{Y}^T Y \dot{\eta}) \quad (3.43)$$

$$\frac{\delta L}{\delta \dot{\eta}} = \rho_F(\dot{\eta} + Y^T \dot{Y} \eta) \quad (3.44)$$

where the derivatives are taken w.r.t. $(Y, \dot{Y}) \in TGL(3)$ and $(\eta, \dot{\eta}) \in T\tilde{\Psi}$. We also point out that the variations are taken w.r.t. the trace inner product for $GL(3)$ variables and L^2 inner product for Ψ . Now, by using the projection operator Π (3.30), we pull back the Euler-Lagrange equations (3.23) to $TSO(3)$:

$$\begin{aligned} \Pi\left(\frac{d}{dt}\frac{\delta L}{\delta \dot{Y}} - \frac{\delta L}{\delta Y}\right) &= \Pi\left(\frac{d}{dt}(\dot{Y}(A+C) + YB) - \dot{Y}B^T\right) \\ &= \Pi(\ddot{Y}(A+C) + 2\dot{Y}B + Y\dot{B}) \\ &= \Pi(Y(\Omega^2 + \dot{\Omega})(A+C) + 2Y\Omega B + Y\dot{B}) \\ &= Y((\Omega^2 + \dot{\Omega})(A+C) + 2\Omega B + \dot{B})_a \end{aligned}$$

where we used $\dot{Y} = Y\Omega$, $\ddot{Y} = Y(\Omega^2 + \dot{\Omega})$ and constancy of matrices A, C and the skew-symmetry of B . By assuming there is no external moment acting on the system, we have:

$$((\Omega^2 + \dot{\Omega})(A+C) + 2\Omega B + \dot{B})_a = 0. \quad (3.45)$$

This differential equation captures the dynamics of a rigid body under the effect of a perfect fluid fully filling its cavity. Note that, the effect of the fluid on the rigid body is via only the A and B matrices. As we pointed out above, A is the coefficient of inertia matrix for the fluid mass. In order to clarify the physical meaning of B , as well as to represent the dynamical equations of the rigid body containing fluid in vector form instead of matrices, we will use some equalities. For symmetric $A, C \in \mathfrak{R}^{3 \times 3}$, skew-symmetric $B \in \mathfrak{R}^{3 \times 3}$ and $\omega \in \mathfrak{R}^3$ define

$$d_1 = \frac{1}{2}(A+C)\omega \times \omega \quad (3.46)$$

$$d_2 = \frac{1}{2}(\mathbf{1}tr(A + C) - (A + C))\dot{\omega} \quad (3.47)$$

$$d_3 = \frac{1}{2}(\mathbf{1}tr(B) - B)\omega. \quad (3.48)$$

Then, by means of elementary calculations, it can be shown that

$$\hat{d}_1 = (\Omega^2(A + C))_a \quad (3.49)$$

$$\hat{d}_2 = (\dot{\Omega}(A + C))_a \quad (3.50)$$

$$\hat{d}_3 = (\Omega B)_a. \quad (3.51)$$

where $\hat{\omega} = \Omega$. By using these equalities, (3.45) can be shown to be equivalent to:

$$I_T\dot{\omega} = I_T\omega \times \omega + b \times \omega - \dot{b} \quad (3.52)$$

where $b \in \mathfrak{R}^3$ is defined by $\hat{b} = 2(B)_a$. Calculating b by changing the integration variables from the material to the spatial variables we get:

$$b = \int_{\mathcal{F}} \rho_F x \times \mathbf{v} dx. \quad (3.53)$$

This is nothing but the total angular momentum of the fluid flow in the cavity w.r.t. rigid body. It is clear that the net effect of the fluid motion on the body is through the angular momentum of the fluid. If we interpret b in (3.52) as the angular momentum of momentum wheels in a rigid body, then (3.52) gives us the dynamical equation of a gyrostat [45]. Based on this equivalence, a rigid body containing fluids which fully fill the cavities of the rigid body is called *hydrostat* [65]. Furthermore, if we take $\omega = \dot{\omega} = 0$ then (3.52) reduces to Euler's equation for a rigid body. In the case of rigid gyrostats, the internal rotors have only a finite degree of freedom, however hydrostats involve infinitely many degrees of freedom associated with the fluid motion. The above dynamical equation for the rigid body motion only characterizes part of the dynamics of a rigid body

containing fluid; the effect of the fluid on the rigid body. On the other hand, the fluid motion in the cavity is affected by the rigid body which acts as the container of the fluid. This side of the interaction is determined by the Euler-Lagrange equations obtained by taking derivatives w.r.t. diffeomorphism variables:

$$\begin{aligned}\frac{d}{dt} \frac{\delta L}{\delta \dot{\eta}} - \frac{\delta L}{\delta \eta} &= \frac{d}{dt} (\rho_F (\dot{\eta} + Y^T \dot{Y} \eta)) - \rho_F (\dot{Y}^T \dot{Y} \eta + \dot{Y}^T Y \ddot{\eta}) \\ &= \rho_F (\ddot{\eta} + (Y^T \dot{Y} - \dot{Y}^T Y) \dot{\eta} + Y^T \ddot{Y} \eta)\end{aligned}$$

where we used (3.43) and (3.44). By assuming there is no external force field acting on the fluid variables, we get:

$$\rho_F (\ddot{\eta}(X, t) + (Y^T \dot{Y} - \dot{Y}^T Y) \dot{\eta}(X, t) + Y^T \ddot{Y} \eta(X, t)) = 0. \quad (3.54)$$

Note that this dynamical equation is a family of ordinary differential equations indexed by the fluid particles (X). We also note that this equation is closely related to the equation of motion for material particles in rotating reference frames [34]. In this form (material representation) the above equation is of little use, since we are not particularly interested in the fate of individual particles, but with the velocity field created by their collective motion. To determine the equations governing the dynamics in terms of the velocity field $\mathbf{v}$, we make the following substitutions:

$$\eta(X, t) \rightarrow x(t) \quad (3.55)$$

$$\dot{\eta}(X, t) \rightarrow \mathbf{v}(x, t) \quad (3.56)$$

$$\ddot{\eta}(X, t) \rightarrow \frac{d\mathbf{v}}{dt}(x, t) \quad (3.57)$$

which account for switching from the material to the spatial representation of the fluid motion. After substituting into (3.54), by using the equation $\dot{Y} = Y\Omega$

and $\ddot{Y} = Y(\Omega^2 + \dot{\Omega})$, we get

$$\rho_F \left(\frac{d\mathbf{v}}{dt} + 2\Omega\mathbf{v} + (\Omega^2 + \dot{\Omega})x \right) = 0. \quad (3.58)$$

By using

$$\frac{d\mathbf{v}}{dt} = \frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \quad (3.59)$$

and rearranging the terms we get:

$$\rho_F \frac{\partial\mathbf{v}}{\partial t} = \rho_F (-(\mathbf{v} \cdot \nabla)\mathbf{v} - 2\Omega\mathbf{v} - (\Omega^2 + \dot{\Omega})x). \quad (3.60)$$

Recall that, until this point we took η as a member of the diffeomorphism group $\tilde{\Psi}$ hence the velocity field $\mathbf{v}$ need not lie in the space of incompressible velocity fields $\mathcal{X}_d \subset \mathcal{X}$. Now, we project (3.60) into $\mathcal{X}_d$ by using the Helmholtz decomposition of vector fields:

$$\mathcal{P}_d(\rho_F \frac{\partial\mathbf{v}}{\partial t}) = \rho_F \frac{\partial\mathbf{v}}{\partial t} - \mathcal{P}_g(\rho_F \frac{\partial\mathbf{v}}{\partial t}). \quad (3.61)$$

We define the *pressure gradient* ∇p as the gradient part of $\rho_F \frac{\partial\mathbf{v}}{\partial t}$:

$$\nabla p = \mathcal{P}_g(\rho_F \frac{\partial\mathbf{v}}{\partial t}) = \mathcal{P}_g(\rho_F (-(\mathbf{v} \cdot \nabla)\mathbf{v} - 2\Omega\mathbf{v} - (\Omega^2 + \dot{\Omega})x)). \quad (3.62)$$

We also define another gradient field ∇s , which we call as the *gauge gradient*:

$$\nabla s = \nabla p + \nabla \left(\frac{1}{2} \rho_F \mathbf{v}^T \mathbf{v} \right) + \nabla \left(\frac{1}{2} \rho_F x^T \Omega^2 x \right). \quad (3.63)$$

One can show that the gauge gradient is given by

$$\nabla s = \mathcal{P}_g(\rho_F (\mathbf{v} \times (\nabla \times \mathbf{v}) - 2\Omega\mathbf{v})).$$

By using the pressure gradient, the dynamical equation for an ideal fluid in a rotating rigid body is given as:

$$\rho_F \left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} + 2\Omega\mathbf{v} + (\Omega^2 + \dot{\Omega})x \right) = -\nabla p. \quad (3.64)$$

By using the gauge gradient ∇s , we can write down the complete set of equations for a rigid body containing perfect fluid as:

$$I_T \dot{\omega} = I_T \omega \times \omega + b \times \omega - \dot{b} \quad (3.65)$$

$$\rho_F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\omega \times \mathbf{v} + \dot{\omega} \times x \right) = \nabla \left(\frac{1}{2} \rho_F \mathbf{v}^T \mathbf{v} \right) - \nabla s \quad (3.66)$$

$$b = \int_{\mathcal{F}} \rho_F x \times \mathbf{v} dx, \quad \omega \in so(3), \quad \mathbf{v} \in \mathcal{X}_d. \quad (3.67)$$

Note that, these equations are a coupled set of ordinary and partial differential equations. The interaction between the rigid body and the fluid is only through b (the angular momentum of the fluid mass w.r.t body) and ω (angular velocity of the rigid body). The dynamical equation for the fluid motion given by (3.64) is the correct formulation for the dynamics of an incompressible perfect fluid w.r.t. a reference frame rotating with angular velocity ω [36]. Furthermore, if we take $\omega = 0$ in (3.64) or in (3.66) then we get *Euler's equation for incompressible fluids* as expected:

$$\rho_F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p. \quad (3.68)$$

As we have shown in chapter 2, Euler's equation is equivalent to Bernoulli's equation

$$\rho_F \frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \rho_F \mathbf{v}) - \nabla s. \quad (3.69)$$

As we show later in this chapter the Hamiltonian structure of the ideal fluid flow is captured in this representation in a transparent way.

The dynamics of rigid bodies containing perfect fluids, given by (3.65), (3.66) is not in the form of an evolution equation due to the existence of the terms $\dot{\omega}$ and $\dot{b}$. In order to express the dynamical equations in this form, we define a linear operator $\mathcal{K}$ from the space of covector fields $\mathcal{X}^*$ defined on the cavity $\mathcal{F}$

of the rigid body to the angular momentum space $so^*(3) \cong \mathfrak{R}^3$:

$$\mathcal{K} : \mathcal{X}^* \rightarrow so^*(3), \mathcal{K}(\mathbf{m}) = \int_{\mathcal{F}} x \times \mathbf{m} dx. \quad (3.70)$$

We calculate the adjoint operator $\mathcal{K}^* : so(3) \rightarrow \mathcal{X}$:

$$\begin{aligned} \langle \mathcal{K}(\mathbf{m}), \omega \rangle &= \langle \int_{\mathcal{F}} x \times \mathbf{m} dx, \omega \rangle = \omega^T \int_{\mathcal{F}} x \times \mathbf{m} dx \\ &= \int_{\mathcal{F}} \omega^T (x \times \mathbf{m}) dx = \int_{\mathcal{F}} \omega^T \hat{x} \mathbf{m} dx \\ &= - \int_{\mathcal{F}} \mathbf{m}^T \hat{x} \omega dx = - \int_{\mathcal{F}} \mathbf{m}^T (x \times \omega) dx \\ &= \int_{\mathcal{F}} \mathbf{m}^T (\omega \times x) dx = \langle \mathbf{m}, \mathcal{K}^*(\omega) \rangle. \end{aligned}$$

Therefore, we have

$$\mathcal{K}^*(\omega) = \omega \times x, \quad x \in \mathcal{F}. \quad (3.71)$$

By using the operators $\mathcal{K}$, $\mathcal{K}^*$ and the vector identity

$$\frac{1}{2} \nabla(\mathbf{v}^T \mathbf{v}) = (\mathbf{v} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{v}) \quad (3.72)$$

(3.65), (3.66) can be rewritten as

$$\begin{bmatrix} I_T & \rho_F \mathcal{K} \\ \rho_F \mathcal{K}^* & \rho_F \mathcal{I} \end{bmatrix} \begin{bmatrix} \dot{\omega} \\ \mathbf{v}_t \end{bmatrix} = \begin{bmatrix} (I_T \omega + \mathcal{K}(\rho_F \mathbf{v})) \times \omega \\ \mathbf{v} \times (\nabla \times (\rho_F \mathcal{K}^*(\omega) + \rho_F \mathbf{v})) - \nabla s \end{bmatrix}. \quad (3.73)$$

The form of this equation suggests to us to define some new variables. For $\mathbf{v} \in \mathcal{X}$ and $\omega \in so(3)$, we define $\mathbf{m} \in \mathcal{X}^*$ and $q \in so^*(3)$ as

$$q = I_T \omega + \mathcal{K}(\rho_F \mathbf{v}) \quad (3.74)$$

$$\mathbf{m} = \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega) \quad (3.75)$$

We will denote this mapping by $\mathcal{T} : so(3) \times \mathcal{X} \rightarrow so(3)^* \times \mathcal{X}^*$. Indeed, $\mathcal{T}$ is closely related to the Legendre transformation associated with the Lagrangian (3.19) and we will return to this point later in this chapter. Note that, if we

interpret $\mathbf{v}$ as the velocity field of the fluid and ω as the angular velocity of the rigid body then q and $\mathbf{m}$ can be interpreted as the total angular momentum of the rigid body-fluid system and the momentum field of the fluid w.r.t the inertial space respectively.

The operator $\mathcal{T}$ as well as its inverse $\mathcal{T}^{-1}$ will be very useful in the determination of various representations of the model of a rigid body containing a perfect fluid. In order to determine the operator $\mathcal{T}^{-1}$, we will use the following lemmas.

Lemma 3.1 *Let $a, r \in \mathbb{R}^3$. Then, $r \times (a \times r) = (\mathbf{1}tr(rr^T) - rr^T)a$.*

Proof: By direct calculation.

Lemma 3.2 $\rho_F \mathcal{K} \mathcal{K}^* = I_F$.

Proof: Let $a \in \mathbb{R}^3$, then we have

$$\begin{aligned}
 \rho_F \mathcal{K} \mathcal{K}^* a &= \rho_F \mathcal{K}(a \times x), \quad x \in \mathcal{F} \\
 &= \rho_F \int_{\mathcal{F}} x \times (a \times x) dx \\
 &= \int_{\mathcal{F}} \rho_F (\mathbf{1}tr(xx^T) - xx^T) a dx \\
 &= \int_{\mathcal{F}} \rho_F (\mathbf{1}tr(xx^T) - xx^T) dx a \\
 &= I_F a
 \end{aligned}$$

where we used the previous lemma in the passage from the second line to the third and the definition of I_F . ■

Corollary 3.1 $\mathcal{K}$ is bounded in the L^2 sense.

Proof: Boundedness of operator $\mathcal{K}$ follows from the finite dimensionality of I_F :

$$I_F = \rho_F \mathcal{K} \mathcal{K}^* \quad (3.76)$$

$$\|I_F\| = \|\rho_F \mathcal{K} \mathcal{K}^*\|, \quad \rho_F > 0 \quad (3.77)$$

$$\|I_F\| = \rho_F \|\mathcal{K}\| \|\mathcal{K}^*\| = \rho_F \|\mathcal{K}\|^2 \quad (3.78)$$

Therefore, $\|\mathcal{K}\| = (\frac{\|I_F\|}{\rho_F})^{\frac{1}{2}}$ is finite, hence $\mathcal{K}$ is bounded. $\blacksquare$

Remark: From the dependence of the operator $\mathcal{T}$ (3.74), (3.75) on $\mathcal{K}$ it is easy to see that $\mathcal{T}$ is a bounded operator too.

Proposition 3.1 *Let $\mathcal{T} : so(3) \times \mathcal{X} \rightarrow so^*(3) \times \mathcal{X}^*$ be given by (3.74), (3.75).*

Let $q \in so^(3)$, $\mathbf{m} \in \mathcal{X}^*$. Then, $\mathcal{T}^{-1} : so^*(3) \times \mathcal{X}^* \rightarrow so(3) \times \mathcal{X}$ is given as:*

$$\mathcal{T}^{-1}(q, \mathbf{m}) = (I_B^{-1}q - I_B^{-1}\mathcal{K}(\mathbf{m}), \frac{\mathbf{m}}{\rho_F} - \mathcal{K}^*I_B^{-1}q + \mathcal{K}^*I_B^{-1}\mathcal{K}\mathbf{m}).$$

Proof: We verify this by checking $\mathcal{T}^{-1}\mathcal{T} = \mathcal{I}$. Let $\omega \in so(3)$, $\mathbf{v} \in \mathcal{X}$.

$$\begin{aligned} \mathcal{T}^{-1}(\mathcal{T}(\omega, \mathbf{v})) &= \mathcal{T}^{-1}(I_T\omega + \mathcal{K}(\rho_F\mathbf{v}), \rho_F\mathbf{v} + \rho_F\mathcal{K}^*(\omega)) \\ &= (\bar{\omega}, \bar{\mathbf{v}}) \end{aligned}$$

where

$$\begin{aligned} \bar{\omega} &= I_B^{-1}(I_T\omega + \mathcal{K}(\rho_F\mathbf{v})) - I_B^{-1}\mathcal{K}(\rho_F\mathbf{v} + \rho_F\mathcal{K}^*(\omega)) \\ \bar{\mathbf{v}} &= \frac{1}{\rho_F}(\rho_F\mathbf{v} + \rho_F\mathcal{K}^*(\omega)) - \mathcal{K}^*I_B^{-1}(I_T\omega + \mathcal{K}(\rho_F\mathbf{v})) + \mathcal{K}^*I_B^{-1}\mathcal{K}(\rho_F\mathbf{v} + \rho_F\mathcal{K}^*(\omega)). \end{aligned}$$

By using, $I_T = I_B + I_F$ and $I_F = \rho_F \mathcal{K} \mathcal{K}^*$ we get:

$$\begin{aligned} \bar{\omega} &= I_B^{-1}((I_B + \rho_F \mathcal{K} \mathcal{K}^*)\omega + \mathcal{K}(\rho_F\mathbf{v})) - I_B^{-1}\mathcal{K}(\rho_F\mathbf{v}) - \rho_F I_B^{-1}\mathcal{K} \mathcal{K}^*(\omega) \\ &= \omega + \rho_F I_B^{-1}\mathcal{K} \mathcal{K}^*(\omega) + I_B^{-1}\mathcal{K}(\rho_F\mathbf{v}) - I_B^{-1}\mathcal{K}(\rho_F\mathbf{v}) - \rho_F I_B^{-1}\mathcal{K} \mathcal{K}^*(\omega) \\ &= \omega \end{aligned}$$

and

$$\begin{aligned}
\bar{\mathbf{v}} &= \mathbf{v} + \mathcal{K}^*(\omega) - \mathcal{K}^* I_B^{-1} ((I_B + \rho_F \mathcal{K} \mathcal{K}^*) \omega + \mathcal{K}(\rho_F \mathbf{v})) \\
&\quad + \mathcal{K}^* I_B^{-1} \mathcal{K}(\rho_F \mathbf{v}) + \rho_F \mathcal{K}^* I_B^{-1} \mathcal{K} \mathcal{K}^*(\omega) \\
&= \mathbf{v} + \mathcal{K}^*(\omega) - \mathcal{K}^*(\omega) - \rho_F \mathcal{K}^* I_B^{-1} \mathcal{K} \mathcal{K}^*(\omega) \\
&\quad - \mathcal{K}^* I_B^{-1} \mathcal{K}(\rho_F \mathbf{v}) + \mathcal{K}^* I_B^{-1} \mathcal{K}(\rho_F \mathbf{v}) + \rho_F \mathcal{K}^* I_B^{-1} \mathcal{K} \mathcal{K}^*(\omega) \\
&= \mathbf{v}.
\end{aligned}$$

■

Remark: $\mathcal{T}^{-1}$ is a bounded operator.

By using the transformation $\mathcal{T}$, (3.73) can be expressed as

$$\mathcal{T}(\dot{\omega}, \mathbf{v}_t) = ((I_T \omega + \mathcal{K}(\rho_F \mathbf{v})) \times \omega, \mathbf{v} \times (\nabla \times (\rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega))) - \nabla s).$$

Therefore, we have

$$(\dot{\omega}, \mathbf{v}_t) = \mathcal{T}^{-1}((I_T \omega + \mathcal{K}(\rho_F \mathbf{v})) \times \omega, \mathbf{v} \times (\nabla \times (\rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega))) - \nabla s)$$

and this is equivalent to the following dynamical equations:

$$\dot{\omega} = I_B^{-1}((I_T \omega + \mathcal{K}(\rho_F \mathbf{v})) \times \omega) \tag{3.79}$$

$$- I_B^{-1} \mathcal{K}(\mathbf{v} \times (\nabla \times (\rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega))) - \nabla s)$$

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times (\mathbf{v} + \mathcal{K}^*(\omega))) - \rho^{-1} \nabla s \tag{3.80}$$

$$+ \mathcal{K}^* I_B^{-1}(-(I_T \omega + \mathcal{K}(\rho_F \mathbf{v})) \times \omega)$$

$$+ \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{v} \times (\nabla \times (\rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega))) - \nabla s).$$

This is what we will call the *velocity space representation* of the dynamics of rigid bodies containing perfect fluids. Here $(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d$ and the gradient field ∇s is the gauge which constrains the flow of the system to that space.

The operators $\mathcal{T}$ and $\mathcal{T}^{-1}$ are bounded linear operators, hence they define a diffeomorphism between $so(3) \times \mathcal{X}_d$ and $\mathcal{T}(so(3) \times \mathcal{X}_d) = \mathcal{N}$. Therefore, the dynamics represented in terms of the momentum variables $(q, \mathbf{m}) = \mathcal{T}(\omega, \mathbf{v})$ is equivalent to the dynamics in terms of the velocity space variables. The dynamical equations given by (3.79) and (3.80) are equivalent to:

$$\dot{q} = q \times I_B^{-1} q - q \times I_B^{-1} \mathcal{K}(\mathbf{m}) \quad (3.81)$$

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} = & \frac{\mathbf{m}}{\rho_F} \times (\nabla \times \mathbf{m}) - \mathcal{K}^* I_B^{-1} q \times (\nabla \times \mathbf{m}) \\ & + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) \times (\nabla \times \mathbf{m}) - \nabla s \end{aligned} \quad (3.82)$$

where q and $\mathbf{m}$ are the total angular momentum of the rigid body-fluid system and the momentum field of the fluid w.r.t. inertial space respectively. We call these equations the *momentum space representation* of equations of rigid bodies containing perfect fluids. The domain of this dynamical system is the space $\mathcal{N}$ as defined above. Note that, by construction of the space $\mathcal{N}$ the gauge gradient ∇s not only restricts $(\omega, \mathbf{v})$ into the velocity space $so(3) \times \mathcal{X}_d$ but also $(q, \mathbf{m})$ into the momentum space $\mathcal{N}$. We will interpret the linear space $\mathcal{N}$ as in duality with the velocity space $so(3) \times \mathcal{X}_d$.

The dynamical equations for a rigid body containing perfect fluid do not have a simple form neither in the velocity nor in the momentum space representation. On the other hand, the following *hybrid representation* not only saves space but also reveals the geometric structure of the equations which we study later in this

chapter:

$$\dot{q} = q \times \omega \quad (3.83)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s. \quad (3.84)$$

In this section, we developed a dynamical model for a rigid body containing perfect fluid. The dynamical model is represented as three different but equivalent sets of dynamical equations: in velocity space (3.79), (3.80), in momentum space (3.81), (3.82) and in a hybrid form (3.83), (3.84).

3.3 Constants of Motion

The Hamiltonian nature of the various representations of the rigid body-fluid system will be treated later in this chapter. Here, in this section we look at some constants of motion of the system without studying the underlying geometric structure. An important aspect of rigid bodies containing ideal fluids problem is the conservative nature of the dynamics. A manifestation of this is the existence of some conserved quantities.

Energy The following scalar quantity

$$H(q, \mathbf{m}) = \frac{1}{2} \mathcal{T}^{-1}((q, \mathbf{m}), (q, \mathbf{m})) = \frac{1}{2} \langle (q, \mathbf{m}), \mathcal{T}^{-1}(q, \mathbf{m}) \rangle$$

which we refer as the *energy of the rigid body-fluid system* is a conserved quantity of the rigid bodies containing perfect fluids. Here, we use $\mathcal{T}^{-1}$ both as a linear operator and as a bilinear form. We define $\bar{H}$ as the energy defined in terms of the velocity variables:

$$\bar{H}(\omega, \mathbf{v}) = H(\mathcal{T}(\omega, \mathbf{v})) = \frac{1}{2} \langle \mathcal{T}(\omega, \mathbf{v}), (\omega, \mathbf{v}) \rangle .$$

By using the operators $\mathcal{T}$ and $\mathcal{T}^{-1}$ we defined in the previous section, H and $\bar{H}$ can be calculated as

$$H(q, \mathbf{m}) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K}(\mathbf{m}) + \frac{1}{2} \int \frac{\mathbf{m}^T \mathbf{m}}{\rho_F} + \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) dx \quad (3.85)$$

$$\bar{H}(\omega, \mathbf{v}) = \frac{1}{2} \omega^T I_T \omega + \frac{1}{2} \int \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}). \quad (3.86)$$

Note that, as it is clear from the expression for $\bar{H}$, the energy of the rigid body-fluid system is composed of three parts. The first term is the energy of the rigid motion of the system, the second term is associated with the kinetic energy of the fluid flow w.r.t. the rotating rigid body and the last term is the interaction energy between the rigid body and the fluid. This last coupling term is essential to the understanding of the interaction between the rigid body and the fluid flow. Before, showing that the energy is a conserved quantity, we show that the energy $\bar{H}$ (hence H too) is always non-negative.

Proposition 3.2 *The energy $\bar{H}$ given by (3.86) always takes non-negative values.*

Proof: We will use the formulas $I_T = I_B + I_F$ and $I_F = \rho_F \mathcal{K} \mathcal{K}^*$ to do a completions of squares:

$$\begin{aligned} \bar{H}(\omega, \mathbf{v}) &= \frac{1}{2} \omega^T I_T \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}) \\ &= \frac{1}{2} \omega^T (I_B + \rho_F \mathcal{K} \mathcal{K}^*) \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}) \\ &= \frac{1}{2} \omega^T I_B \omega + \frac{1}{2} \rho_F \omega^T \mathcal{K} \mathcal{K}^* \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}) \\ &= \frac{1}{2} \omega^T I_B \omega + \frac{1}{2} \langle \rho_F \mathcal{K}^*(\omega), \mathcal{K}^*(\omega) \rangle \\ &\quad + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \langle \omega, \mathcal{K}(\rho_F \mathbf{v}) \rangle \\ &= \frac{1}{2} \omega^T I_B \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F (((\mathcal{K}^*(\omega))^T (\mathcal{K}^*(\omega)))) dx \end{aligned}$$

$$\begin{aligned}
& + \int_{\mathcal{F}} \rho_F (\mathbf{v}^T \mathbf{v} + 2(\mathcal{K}^*(\omega))^T \mathbf{v}) dx \\
& = \frac{1}{2} \omega^T I_B \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F (\mathbf{v} + \mathcal{K}^*(\omega))^T (\mathbf{v} + \mathcal{K}^*(\omega)) dx.
\end{aligned}
\tag{3.88}$$

By identifying $\mathbf{m} = \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega)$, we get

$$\bar{H} = \frac{1}{2} \omega^T I_B \omega + \frac{1}{2} \int_{\mathcal{F}} \frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} dx \geq 0$$

where we used $I_B > 0$ and $\rho_F > 0$. ■

It is a lot easier to prove the conservation of the energy by using H (3.85) and the hybrid representation of the model (3.83), (3.84). We calculate the first variation of the energy H :

$$\begin{aligned}
\frac{\delta H}{\delta q} &= I_B^{-1} q - I_B^{-1} \mathcal{K}(\mathbf{m}) = \omega \\
\frac{\delta H}{\delta \mathbf{m}} &= \frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1} q + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) = \mathbf{v}.
\end{aligned}$$

Before calculating $\dot{H}$, we state a very simple and useful equality which we will use over and over in this work.

An identity: Let $a, b \in \mathfrak{R}^3$. Then, $a^T(b \times a) = 0$.

Now, we can calculate $\dot{H}$:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{\delta H^T}{\delta q} \dot{q} + \int_{\mathcal{F}} \frac{\delta H^T}{\delta \mathbf{m}} \mathbf{m}_t dx \\
&= \omega^T \dot{q} + \int_{\mathcal{F}} \mathbf{v}^T \mathbf{m}_t dx \\
&= \omega^T (q \times \omega) + \int_{\mathcal{F}} \mathbf{v}^T (\mathbf{v} \times (\nabla \times \mathbf{m})) dx - \int_{\mathcal{F}} \mathbf{v}^T \nabla s dx
\end{aligned}$$

where we use (3.83), (3.84) for $\dot{q}$ and $\mathbf{m}_t$. The first two terms on the right hand side of the above equality vanish due to the identity given above. The last term

vanishes because of the orthogonality of the incompressible vector fields to the gradient vector fields. Therefore, $\dot{H} = 0$; the energy H is a conserved quantity. This is only natural, since we derived the rigid body-fluid model from an Euler-Lagrange equation which does not involve any non-conservative effects on the system.

Total Angular Momentum Apart from the energy, the most important conserved quantity is the magnitude of the total angular momentum of the rigid body-fluid system. We have defined the total angular momentum $I_T\omega + \mathcal{K}(\rho_F\mathbf{v})$ as the dual variable q . The magnitude of the total angular momentum is conserved:

$$\frac{d}{dt}\|q\|^2 = 2q^T\dot{q} = 2q^T(q \times \omega) = 0.$$

Note that, here we only used (3.83). Closely related to the conservation of the magnitude of the total angular momentum is the conservation of the total momentum of the system w.r.t. the inertial space. As will be clear from the following derivation, the dynamical equation $\dot{q} = q \times \omega$ is equivalent to the conservation of total angular momentum in space, i.e. $Yq = \text{constant}$:

$$\begin{aligned} \frac{d}{dt}(Yq) &= Y\dot{q} + \dot{Y}q \\ &= Y\dot{q} + Y\Omega q \\ &= Y(\dot{q} + \Omega q) \\ &= Y(\dot{q} + \omega \times q) \\ &= Y(q \times \omega + \omega \times q) \\ &= 0. \end{aligned}$$

In other words, if we had taken the principle of conservation of total angular momentum in the space as a given, then we could have written the dynamical

equation $\dot{q} = q \times \omega$ in one shot. Instead, we started from the opposite end, and derived this principle as a by product of the Euler-Lagrange equations associated with a rigid body containing perfect fluid.

Helicity: In fluid mechanics, the scalar field $\mathbf{v}^T(\nabla \times \mathbf{v})$ associated with the velocity field $\mathbf{v}$ of a fluid is known as the *helicity density* [64]. The corresponding integral

$$\int_{\mathcal{D}} \mathbf{v}^T(\nabla \times \mathbf{v}) dx$$

is called *helicity*. Helicity is a conserved quantity of perfect fluid flow i.e. Euler's equation, if $\partial\mathcal{D} = 0$. Here, we define the integral

$$\int \mathbf{m}^T(\nabla \times \mathbf{m}) dx$$

associated with the momentum field $\mathbf{m}$ of a perfect fluid as *generalized helicity*. If we take $\mathbf{m} = \mathbf{v}$ generalized helicity reduces to the traditional helicity integral, yet generalized helicity can also be defined for fluids in rotating frames; $\mathbf{m} = \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega)$. Neither helicity nor the generalized helicity is a constant of motion for perfect fluids in the cavities of freely rotating rigid bodies. However, it is important to note that the conservation of helicity for Euler's equation for a perfect fluid is closely related to whether curl operator is symmetric or not. If $\partial\mathcal{D} = 0$, then curl operator is symmetric and as a result helicity is preserved. By the same token, generalized helicity would be a constant of motion for rotating fluids if $\partial\mathcal{D} = 0$. This implies that generalized helicity is the natural generalization of the helicity to a more general set of fluid flows including rotating fluids.

3.4 Equilibria of the System

A way of understanding the dynamics of a rigid body containing fluid is to study their equilibria. In order to characterize the equilibria of rigid bodies containing perfect fluids, we define the following subsets of the velocity space $so(3) \times \mathcal{X}_d$

$$\begin{aligned}\Sigma_1 &= \{(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d \mid \omega = 0, \mathbf{v} = 0\} \\ \Sigma_2 &= \{(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d \mid \omega = \tilde{\omega}, \mathbf{v} = 0, I_T \tilde{\omega} = \lambda \tilde{\omega}, \tilde{\omega} \neq 0\} \\ \Sigma_3 &= \{(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d \mid \omega = 0, \mathbf{v} = \tilde{\mathbf{v}}, \tilde{\mathbf{v}} \times (\nabla \times \tilde{\mathbf{v}}) = \rho_F^{-1} \nabla s, \tilde{\mathbf{v}} \neq 0\} \\ \Sigma_4 &= \{(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d \mid \omega \neq 0, \mathbf{v} \neq 0, \Psi_t(\omega, \mathbf{v}) = (\omega, \mathbf{v})\}\end{aligned}$$

where $\Psi_t(\omega, \mathbf{v})$ denotes the solution of the dynamical system starting from the initial condition $(\omega, \mathbf{v})$. By using the Legendre transformation $\mathcal{T}$ we map these subsets to the momentum space $\mathcal{N}$ and define $\bar{\Sigma}_i = \mathcal{T}(\Sigma_i)$, $i = 1, 2, 3, 4$. The sets Σ_i can be computed as:

$$\begin{aligned}\bar{\Sigma}_1 &= \{(q, \mathbf{m}) \in \mathcal{N} \mid q = 0, \mathbf{m} = 0\} \\ \bar{\Sigma}_2 &= \{(q, \mathbf{m}) \in \mathcal{N} \mid q = I_T \tilde{\omega}, \mathbf{m} = \mathcal{K}^*(\rho \tilde{\omega}), I_T \tilde{\omega} = \lambda \tilde{\omega}, \tilde{\omega} \neq 0\} \\ \bar{\Sigma}_3 &= \{(q, \mathbf{m}) \in \mathcal{N} \mid q = \mathcal{K}(\rho_F \tilde{\mathbf{v}}), \mathbf{m} = \rho_F \tilde{\mathbf{v}}, \tilde{\mathbf{v}} \times (\nabla \times \tilde{\mathbf{v}}) = \rho_F^{-1} \nabla s, \tilde{\mathbf{v}} \neq 0\} \\ \bar{\Sigma}_4 &= \{(q, \mathbf{m}) \in \mathcal{N} \mid (q, \mathbf{m}) = \mathcal{T}((\omega, \mathbf{v})), (\omega, \mathbf{v}) \in \Sigma_4\}\end{aligned}$$

Note that, by construction the sets Σ_i are mutually disjoint, so are the sets $\bar{\Sigma}_i$ since $\mathcal{T}$ is a diffeomorphism.

Proposition 3.3 *Let $\Sigma = \cup \Sigma_i$, and $\bar{\Sigma} = \cup \bar{\Sigma}_i$. Then, $(\omega, \mathbf{v})$ is an equilibrium of the dynamics of rigid bodies containing perfect fluid (3.79), (3.80) iff $(\omega, \mathbf{v}) \in \Sigma$. Similarly, $(q, \mathbf{m})$ is an equilibrium of (3.81), (3.82) iff $(q, \mathbf{m}) \in \bar{\Sigma}$.*

Proof: For computational convenience, we will use the hybrid representation

$$\dot{q} = q \times \omega$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s.$$

which is equivalent both to (3.79), (3.80) and (3.81), (3.82). Let f and g denote the right hand side of $\dot{q}$ and $\mathbf{m}_t$ respectively. Suppose $(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d$ is an equilibrium point, then there are four alternatives:

1. If $(\omega, \mathbf{v}) = (0, 0)$ then $f = 0, g = -\nabla s$. On the other hand, the gauge gradient ∇s is given by

$$\nabla s = \mathcal{P}_g(\rho_F(\mathbf{v} \times (\nabla \times \mathbf{v}) - 2\omega \times \mathbf{v})).$$

Therefore, at the null solution, $\omega = 0, \mathbf{v} = 0$, ∇s vanishes. Hence, Σ_1 is an equilibrium point.

2. Assume that there exists an equilibrium such that $\omega \neq 0, \mathbf{v} = 0$. Then, at this particular equilibrium we have $q = I_T \omega + \mathcal{K}(\rho \mathbf{v}) = I_T \omega$, hence $f = I_T \omega \times \omega = 0$. Therefore, ω should be an eigenvector of I_T . Such equilibria constitute the set Σ_2 . Furthermore, $\mathbf{v} = 0$ implies $\nabla s = 0$ hence any point in Σ_2 is an equilibrium point.

3. Let $\omega = 0, \mathbf{v} \neq 0$ be an equilibrium. Then, $f = 0$ and $g = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s = 0$. But $\omega = 0$ implies $\mathbf{m} = \rho_F \mathbf{v}$, hence $g = 0$ iff $\mathbf{v} \times (\nabla \times \mathbf{v}) = \rho_F^{-1} \nabla s$. Therefore, such equilibria lie in Σ_3 . Furthermore, any point in Σ_3 satisfies $f = 0, g = 0$, i.e. any point in Σ_3 is an equilibrium point.

4. If $\omega \neq 0, \mathbf{v} \neq 0$ is an equilibrium point, then by definition it lies in Σ_4 .

■

The equilibrium Σ_1 is the null solution of the system. It corresponds to the case where the fluid particles are stationary w.r.t. the rigid body and the rigid body

is stationary w.r.t inertial space. Σ_2 is associated with the rigid rotations of the system: the fluid mass is stationary w.r.t the body, and the whole system rotates around one of the principal axes of the total moment of inertia matrix (I_T). The equilibrium set Σ_3 corresponds to the solutions for which the rigid body is steady in space and the fluid in it is in equilibrium, i.e. velocity field of the fluid is stationary in time.

Σ_1 and Σ_2 are non-empty sets for any rigid body without any restriction on the geometry of the cavity. On the other hand, the existence and nature of Σ_3 and Σ_4 might depend upon the shape of the cavity and the relative orientation of the cavity in the rigid body. For example, for the cavities having the shape of a surface of revolution, it is possible to find equilibria in Σ_4 provided the rotation axis of the rigid body coincides with the symmetry axis of the cavity. However, for cavities of arbitrary shape, it is difficult to determine whether Σ_4 is empty or not. Before stating some interesting implications of the structure of equilibria of rigid bodies containing ideal fluids, we introduce the notion of Beltrami fields.

Definition 3.1 *Let $\mathbf{v}$ be a vector field defined on a domain $\mathcal{D} \in \mathbb{R}^3$. Then, $\mathbf{v}$ is called Beltrami field if its curl is parallel to the vector field itself:*

$$\mathbf{v} \times (\nabla \times \mathbf{v}) = 0.$$

Remark: Note that, any eigenfunction of the curl operator is a Beltrami field, but there might exist Beltrami fields which are not eigenfunctions of the curl operator. Beltrami fields are generally associated with equilibrium solutions in fluid mechanics, electromagnetism and plasmas. In the context of plasma physics, the eigenfunctions of the curl operator are known as *Chandrasekhar-Kendall eigenfunctions* and they form a complete orthonormal set for the space

of incompressible vector fields [88].

We assume $\mathbf{v}_b \neq 0$ is a Beltrami field in $\mathcal{X}_d$ and investigate $(\omega, \mathbf{v}) = (0, \mathbf{v}_b)$ as a solution of the dynamics of a rigid body containing perfect fluid. At this particular solution, (3.83), (3.84) reduce to

$$\dot{q} = 0 \quad (3.89)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v}_b \times (\nabla \times \rho_F \mathbf{v}_b) - \mathcal{P}_g(\mathbf{v}_b \times (\nabla \times \rho_F \mathbf{v}_b)). \quad (3.90)$$

Since $\mathbf{v}_b$ is a Beltrami field it is parallel to its curl. Therefore, the right hand side of $\mathbf{m}_t$ vanishes as well. In other words, $(\omega, \mathbf{v}) = (0, \mathbf{v}_b)$ is an equilibrium in Σ_3 . Note that such a solution corresponds to a rigid body stationary in space with a Beltrami flow inside the cavity. Also recall that the dynamics of ideal fluids in non-rotating containers is simply given by nothing but Euler's equation. Therefore, we have just reproduced a not well-known fact: *Beltrami flows are equilibria of perfect fluid flow.*

The fact that Σ_3 is an equilibrium set of the dynamics of a rigid body containing a perfect fluid yields an unexpected result too. Recall that $I_T = I_B + I_F$ and $q = I_T \omega + \mathcal{K}(\rho_F \mathbf{v})$. Then, the equation $\dot{q} = q \times \omega$ is equivalent to

$$I_B \dot{\omega} = I_B \omega \times \omega + I_F \omega \times \omega - I_F \dot{\omega} - \mathcal{K}(\rho_F \mathbf{v}_t) + \mathcal{K}(\rho_F \mathbf{v}) \times \omega. \quad (3.91)$$

This is nothing but the equation of a rigid body with the inertia matrix I_B under the effect of the external torque

$$T_e = I_F \omega \times \omega - I_F \dot{\omega} - \mathcal{K}(\rho_F \mathbf{v}_t) + \mathcal{K}(\rho_F \mathbf{v}) \times \omega. \quad (3.92)$$

Now, let $(\omega, \mathbf{v})$ be an equilibrium point in Σ_3 i.e. $\omega = \dot{\omega} = 0$ and $\mathbf{v}_t = 0$. Then, it is easy to see that at any such equilibrium point the external torque T_e vanishes. Also note that, since $\omega = 0$ we have a perfect fluid in a fixed container in space.

Therefore, the stationary flow $\mathbf{v}$ should be a steady flow of the Euler equation for perfect fluids. Hence, we have proven the following.

Proposition 3.4 *Consider a perfect fluid filling a domain $\mathcal{D} \subset \mathbb{R}^3$; (3.68). Let $\mathbf{v}$ be a stationary flow of the perfect fluid. Then, the torque exerted by $\mathbf{v}$ on the boundary $\partial\mathcal{D}$ is zero.*

Note that, if we lock a rigid body containing fluid then the torque exerted on the rigid body drops to $\mathcal{K}(\rho_F \mathbf{v}_t)$ which is nothing but the rate of change of the fluid momentum as expected. Therefore, if the torque exerted by a perfect fluid on its locked container is zero, then this implies that $\mathcal{K}(\rho_F \mathbf{v})$ is constant. One way of creating a situation in which the torque exerted by the fluid on the boundary $\partial\mathcal{D}$ is zero is to choose a boundaryless container. As a subset of $\mathbb{R}^3$, the only boundaryless set is $\mathbb{R}^3$ itself. So, our analysis implies that the angular momentum of an ideal fluid in $\mathbb{R}^3$ is a conserved quantity as also shown in [67].

3.5 Geometric Interpretations of the Model

In this section, we investigate the geometric structure of the various representations of rigid bodies containing perfect fluid.

3.5.1 Hamiltonian Structure

In the previous section, we have identified some constants of motion for the rigid body-fluid system which is a manifestation of the conservative nature of the dynamics. Here, we explicitly give this conservative structure as a Hamiltonian system on a Poisson manifold. We show that the dynamics of a rigid body

containing a perfect fluid can be written as a Hamiltonian system

$$\dot{z} = W_{RF}(z)dH(z)$$

where $z \in \mathcal{N} = \mathcal{T}(so(3) \times \mathcal{X}_d)$, $W_{RF}(z) : so(3) \times \mathcal{X}_d \rightarrow \mathcal{N}$ is a Poisson structure and H is a scalar function on $\mathcal{N}$.

Proposition 3.5 *The dynamics of a rigid body containing perfect fluid (3.81), (3.82) defined on $\mathcal{N}$ has a Hamiltonian structure. This structure is characterized by the non-canonical Poisson bracket $\{\cdot, \cdot\}_{RF}$, Poisson structure W_{RF} , and the energy H which are given as:*

$$\{F, G\}_{RF}(q, \mathbf{m}) = \frac{\delta F^T}{\delta q} \hat{q} \frac{\delta G}{\delta q} + \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta G}{\delta \mathbf{m}} dx$$

$$W_{RF}(q, \mathbf{m}) = \begin{pmatrix} W_q(q) & 0 \\ 0 & W_{\mathbf{m}}(\mathbf{m}) \end{pmatrix}$$

where

$$\begin{aligned} W_q(q)a &= q \times a = \hat{q}a \\ W_{\mathbf{m}}(\mathbf{m})\mathbf{b} &= \mathbf{b} \times (\nabla \times \mathbf{m}) - \nabla s \\ a &\in \mathfrak{R}^3, \quad \mathbf{b} \in \mathcal{X}_d \end{aligned}$$

and

$$H(q, \mathbf{m}) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K} \mathbf{m} + \frac{1}{2} \int_{\mathcal{F}} \frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} dx + \frac{1}{2} \int_{\mathcal{F}} \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K} \mathbf{m} dx.$$

Proof: First, we show that the dynamics can be written in terms of the structure W_{RF} . Then, we prove the correspondence between the bracket $\{\cdot, \cdot\}_{RF}$ and the structure W_{RF} . Finally is shown that the non-canonical bracket $\{\cdot, \cdot\}_{RF}$ on

$\mathcal{N}$ can be obtained from the canonical bracket on $T^*\mathcal{M}$ via (Poisson) reduction, where $\mathcal{M} = SO(3) \times \Psi$.

We first determine the first variation of the energy H :

$$\frac{\delta H}{\delta q} = I_B^{-1}q - I_B^{-1}\mathcal{K}\mathbf{m} \quad (3.93)$$

$$\frac{\delta H}{\delta \mathbf{m}} = -\mathcal{K}^*I_B^{-1}q + \frac{\mathbf{m}}{\rho_F} + \mathcal{K}^*I_B^{-1}\mathcal{K}\mathbf{m}. \quad (3.94)$$

Then, the dynamical equations associated with the Poisson formulation ($\dot{z} = W_{RF}dH$) should have the form:

$$\dot{q} = q \times \frac{\delta H}{\delta q}$$

$$\frac{\partial \mathbf{m}}{\partial t} = \frac{\delta H}{\delta \mathbf{m}} \times (\nabla \times \mathbf{m}) - \nabla s.$$

By substituting the first variation of the energy into these equations, we get:

$$\begin{aligned} \dot{q} &= q \times I_B^{-1}q - q \times I_B^{-1}\mathcal{K}\mathbf{m} \\ \frac{\partial \mathbf{m}}{\partial t} &= -\mathcal{K}^*I_B^{-1}q \times (\nabla \times \mathbf{m}) + \frac{\mathbf{m}}{\rho_F} \times (\nabla \times \mathbf{m}) \\ &\quad + \mathcal{K}^*I_B^{-1}\mathcal{K}\mathbf{m} \times (\nabla \times \mathbf{m}) - \nabla s \end{aligned}$$

which agrees with (3.81), (3.82).

Secondly, we establish the equivalence of the structure W_{RF} with the bracket $\{\cdot, \cdot\}_{RF}$:

$$\begin{aligned} \langle dF, W_{RF}(z)dG \rangle &= \langle \frac{\delta F}{\delta q}, W_q(q) \frac{\delta G}{\delta q} \rangle + \langle \frac{\delta F}{\delta \mathbf{m}}, W_{\mathbf{m}}(\mathbf{m}) \frac{\delta G}{\delta \mathbf{m}} \rangle \\ &= \frac{\delta F^T}{\delta q} \hat{q} \frac{\delta G}{\delta q} + \int_{\mathcal{F}} \left(\frac{\delta F^T}{\delta \mathbf{m}} \left(\frac{\delta G}{\delta \mathbf{m}} \times (\nabla \times \mathbf{m}) \right) - \frac{\delta F^T}{\delta \mathbf{m}} \nabla s \right) dx \end{aligned}$$

Now, let $\nabla \times \mathbf{m} = \mathbf{k}$, and $\hat{\mathbf{k}} = K$ then a quick calculation yields:

$$K = \frac{\partial \mathbf{m}}{\partial x} - \frac{\partial \mathbf{m}^T}{\partial x}.$$

Then by using $\mathbf{a}^T(\mathbf{b} \times (\nabla \times \mathbf{m})) = -\mathbf{a}^T(\widehat{\nabla \times \mathbf{m}})\mathbf{b}$, we obtain:

$$\begin{aligned} \langle dF, W_{RF}(z)dG \rangle_{RF} &= \frac{\delta F^T}{\delta q} \hat{q} \frac{\delta G}{\delta q} + \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta G}{\delta \mathbf{m}} dx \\ &\quad - \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \nabla s dx. \end{aligned}$$

The last term on the right hand side above vanishes due to the orthogonality of $\mathcal{X}_d$ and $\mathcal{X}_g$, and we get:

$$\langle dF, W_{RF}dG \rangle_{RF} = \{F, G\}_{RF}$$

which establishes the equivalence of the structure and bracket notations.

Finally, we will show that the bracket $\{\cdot, \cdot\}_{RF}$ can be obtained from the canonical Poisson bracket on $T^*\mathcal{M}$ by reduction, where $\mathcal{M} = SO(3) \times \Psi$.

Let $((Y, P_Y)(\eta, M_\eta)) \in T^*\mathcal{M}$. Consider the canonical bracket on $T^*\mathcal{M}$ which is given as:

$$\{F, G\}((Y, P_Y)(\eta, M_\eta)) = \frac{1}{2} Tr \left(\frac{\delta F^T}{\delta Y} \frac{\delta G}{\delta P_Y} - \frac{\delta F^T}{\delta P_Y} \frac{\delta G}{\delta Y} \right) \quad (3.95)$$

$$+ \int_{\mathcal{F}} \frac{\delta F^T}{\delta \eta} \frac{\delta G}{\delta M_\eta} - \frac{\delta F^T}{\delta M_\eta} \frac{\delta G}{\delta \eta} dX \quad (3.96)$$

where $(P_Y, M_\eta) \in T^*_{(Y, \eta)}\mathcal{M}$ and

$$P_Y = YQ, \quad Q = -Q^T, \quad Q \in so^*(3)$$

$$M_\eta(X) = \mathbf{m} \circ \eta(X) = \mathbf{m} \in \mathcal{X}_d^*.$$

Furthermore, we define q by $\hat{q} = Q$, and take $(q, \mathbf{m}) \in \mathcal{N} = T^*_{(I, id)}\mathcal{M}$.

The left action of $SO(3)$ on itself is given by left multiplication. The right action of the volume-preserving diffeomorphisms group Ψ on itself is given by a change of arguments [55]. Let $R \in SO(3)$ and $\psi \in \Psi$, then

$$(R, (Y, P_Y)) = (RY, RP_Y)$$

gives the cotangent lift of the left action of $SO(3)$ on itself by R . The cotangent lift of the right action of Ψ on itself by ψ is given by

$$(\psi, (\eta, M_\eta)) = (\eta \circ \psi, M_\eta \circ \psi)$$

$\Phi : T_{(Y, \eta)}^* \mathcal{M} \rightarrow T_{(I, id)}^* \mathcal{M} = \mathcal{N}$ as

$$\begin{aligned} \Phi((Y, P_Y), (\eta, M_\eta)) &= ((Y^{-1}Y, Y^{-1}P_Y)(\eta \circ \eta^{-1}, M_\eta \circ \eta^{-1})) \\ &= ((I, Q), (Id, \mathbf{m})) \\ &= ((I, \hat{q}), (Id, \mathbf{m})) \end{aligned}$$

Note that, this map is characterized by the cotangent lifts of the group actions given above, hence it is canonical. We pull back the canonical Hamiltonian structure $\{\cdot, \cdot\}$ from $T^* \mathcal{M}$ to $\mathcal{N}$ by using Φ . By applying the chain rule we have:

$$\begin{aligned} \{F \circ \Phi, G \circ \Phi\} &= \frac{1}{2} Tr((\frac{\delta Q}{\delta Y} \frac{\delta F}{\delta Q})^T (\frac{\delta Q}{\delta P_Y} \frac{\delta G}{\delta Q}) - (\frac{\delta Q}{\delta P_Y} \frac{\delta F}{\delta Q})^T (\frac{\delta Q}{\delta Y} \frac{\delta G}{\delta Q})) \\ &+ \int_{\mathcal{F}} (\frac{\delta \mathbf{m}}{\delta \eta} \frac{\delta F}{\delta \mathbf{m}})^T (\frac{\delta \mathbf{m}}{\delta M_\eta} \frac{\delta G}{\delta \mathbf{m}}) - (\frac{\delta \mathbf{m}}{\delta M_\eta} \frac{\delta F}{\delta \mathbf{m}})^T (\frac{\delta \mathbf{m}}{\delta \eta} \frac{\delta G}{\delta \mathbf{m}}) dX. \end{aligned}$$

We have the equalities:

$$\begin{aligned} \frac{\delta Q}{\delta Y} &= P_Y \quad , \quad \frac{\delta Q}{\delta P_Y} = Y \\ \frac{\delta \mathbf{m}}{\delta M_\eta} &= \mathcal{I} \quad , \quad \frac{\delta \mathbf{m}}{\delta \eta} = \frac{\partial \mathbf{m}}{\partial x}. \end{aligned}$$

Substitution of these terms into the bracket and some rearrangement yields

$$\{F \circ \Phi, G \circ \Phi\} = \frac{1}{2} Tr(\frac{\delta F^T}{\delta Q} (Q^T Y^T Y - Y^T Y Q) \frac{\delta G}{\delta Q}) \quad (3.97)$$

$$+ \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \frac{\partial \mathbf{m}^T}{\partial x} \mathcal{I} \frac{\delta G}{\delta \mathbf{m}} - \frac{\delta F^T}{\delta \mathbf{m}} \mathcal{I} \frac{\partial \mathbf{m}}{\partial x} \frac{\delta G}{\delta \mathbf{m}} dX. \quad (3.98)$$

By using $Y^T Y = I$ and by changing the variable of integration from the material (X) to the spatial (x) variables while keeping in mind the incompressibility

condition, which implies $dX = dx$, we get:

$$\{F \circ \Phi, G \circ \Phi\} = \frac{1}{2} Tr \left(\frac{\delta F^T}{\delta Q} (Q^T - Q) \frac{\delta G}{\delta Q} \right) + \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta G}{\delta \mathbf{m}} dx.$$

Furthermore, if we represent Q as a vector q by using the equation $\hat{q} = Q$, then after some more calculations we obtain

$$\begin{aligned} \{F \circ \Phi, G \circ \Phi\} &= \frac{\delta F^T}{\delta q} \hat{q} \frac{\delta G}{\delta q} + \int_{\mathcal{F}} \frac{\delta F^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta G}{\delta \mathbf{m}} dx \\ &= \{F, G\}_{RF}(q, \mathbf{m}) = \langle dF, W_{RF} dG \rangle_{RF}. \end{aligned}$$

■

The Hamiltonian structures of rigid body motion and perfect fluid flow are well known [53], [56]. Here, we have shown that the interactive dynamics of rigid body and perfect fluid in the case of a rigid body containing a perfect fluid also has a Hamiltonian structure. We note that although the rigid body-fluid bracket is in decoupled form, the dynamics are coupled since the Hamiltonian of the system has a coupling term.

3.5.2 Generalized Bernoulli's Equation

Here, we consider only the fluid part of the dynamics of a rigid body containing perfect fluid given in the hybrid form:

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s.$$

In order to generalize this equation which is given in Bernoulli's form, we first make some rearrangement of the terms in the above equation:

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} &= \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s \\ &= -(\nabla \times \mathbf{m}) \times \mathbf{v} - \nabla s \\ &= -(\widehat{\nabla \times \mathbf{m}}) \mathbf{v} - \nabla s. \end{aligned}$$

By recalling $\text{curl}(\mathbf{m}) = -(\widehat{\nabla \times \mathbf{m}}) = \mathbf{d}\mathbf{m}$, we have

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial t} &= (\text{curl}(\mathbf{m}))\mathbf{v} - \nabla s \\ &= (\mathbf{d}\mathbf{m})\mathbf{v} - \nabla s \\ &= -\mathbf{i}_{\mathbf{v}}(\mathbf{d}\mathbf{m}) - ds \end{aligned}$$

where we used the definition of contraction operator $\mathbf{i}_{\mathbf{v}}$ and the fact $\nabla s = ds$ in a cartesian space. Our claim is that the resulting equation

$$\mathbf{m}_t = -\mathbf{i}_{\mathbf{v}}(\mathbf{d}\mathbf{m}) - ds \quad (3.99)$$

which we call the *generalized Bernoulli's equation* for the perfect fluid flow, not only characterizes a perfect fluid in $\mathbb{R}^3$ but also in a Riemannian manifold. To validate this claim, we will show that the Euler's equation for a perfect fluid in a Riemannian manifold [1]

$$\mathbf{v}_t = -\nabla_{\mathbf{v}}\mathbf{v} - \nabla p \quad (3.100)$$

can be obtained from the generalized Bernoulli's equation.

Proposition 3.6 *Let the momentum field $\mathbf{m} \in \mathcal{X}_d^*$ of a perfect fluid in a Riemannian manifold be related to the velocity field $\mathbf{v} \in \mathcal{X}_d$ by $\mathbf{m} = \mathbf{v}^\flat$. Then, the generalized Bernoulli's equation (3.99) reduces to Euler's equation (3.100).*

Proof: By substituting $\mathbf{m} = \mathbf{v}^\flat$ into (3.99), we get:

$$\mathbf{v}_t^\flat = -\mathbf{i}_{\mathbf{v}}(\mathbf{d}\mathbf{v}^\flat) - ds.$$

We use the Cartan's formula $\mathcal{L}_{\mathbf{v}}\alpha = \mathbf{d}\mathbf{i}_{\mathbf{v}}\alpha + \mathbf{i}_{\mathbf{v}}\mathbf{d}\alpha$ to obtain :

$$\mathbf{v}_t^\flat = \mathbf{d}(\mathbf{i}_{\mathbf{v}}\mathbf{v}^\flat) - \mathcal{L}_{\mathbf{v}}\mathbf{v}^\flat - ds.$$

Then, by using the formula $\mathcal{L}_{\mathbf{v}}(\mathbf{v}^b) = (\nabla_{\mathbf{v}}\mathbf{v})^b + \frac{1}{2}d(\|\mathbf{v}\|^2)$ [1] we get:

$$\begin{aligned}\mathbf{v}_t^b &= -(\nabla_{\mathbf{v}}\mathbf{v})^b - \frac{1}{2}d(\|\mathbf{v}\|^2) + d\mathbf{i}_{\mathbf{v}}\mathbf{v}^b - ds \\ &= -(\nabla_{\mathbf{v}}\mathbf{v})^b - d(\frac{1}{2}\|\mathbf{v}\|^2 - \mathbf{i}_{\mathbf{v}}\mathbf{v}^b + s).\end{aligned}$$

Now, we define $p = s - \frac{1}{2}\|\mathbf{v}\|^2$. Then, by using $\mathbf{i}_{\mathbf{v}}\mathbf{v}^b = \|\mathbf{v}\|^2$ we obtain:

$$\begin{aligned}\mathbf{v}_t^b &= -(\nabla_{\mathbf{v}}\mathbf{v})^b - dp \\ &= -(\nabla_{\mathbf{v}}\mathbf{v})^b - (\nabla p)^b.\end{aligned}$$

We transform these covectors to vectors by using the $\sharp$ operation and obtain the generalized Euler's equation:

$$\mathbf{v}_t = -\nabla_{\mathbf{v}}\mathbf{v} - \nabla p.$$

■

Remark: Euler's equation (3.100) which defines the dynamics of an incompressible fluid in a Riemannian manifold is well known [1], [56]. On the other hand, the generalization of Bernoulli's equation to Riemannian manifolds seems to be novel. We would like to emphasize that Euler's equation only describes the dynamics of a perfect fluid in a Riemannian manifold, however the generalized Bernoulli's equation is of a more general nature and capable of describing fluid motion in rotating frames too. We also note that Bernoulli's equation does not depend upon the particular Riemannian structure on the manifold, since the exterior derivative and contraction operation are defined without any reference to a Riemannian structure.

The generalized Bernoulli's equation also yields a *generalized vorticity equation*:

$$\mathbf{w}_t = -\mathcal{L}_{\mathbf{v}}\mathbf{w}.$$

Proposition 3.7 *Let the vorticity field $\mathbf{w} \in \Lambda^2$ be defined as $\mathbf{w} = d\mathbf{m}$ where $\mathbf{m}$ is the momentum field of a perfect fluid. Then, the generalized vorticity equation follows from the generalized Bernoulli's equation.*

Proof: First we take exterior derivative of both sides of the generalized Bernoulli's equation (3.99) and obtain:

$$d\mathbf{m}_t = -d(\mathbf{i}_v(d\mathbf{m})) - d^2s.$$

By using Cartan's formula $\mathcal{L}_v\alpha = d\mathbf{i}_v\alpha + \mathbf{i}_vd\alpha$ and $d^2 = 0$, we get

$$\begin{aligned} d\mathbf{m}_t &= -\mathcal{L}_v(d\mathbf{m}) + \mathbf{i}_vd^2\mathbf{m} \\ &= -\mathcal{L}_v(d\mathbf{m}) \end{aligned}$$

where we used $d^2 = 0$ once again. Then, by using $\mathbf{w} = d\mathbf{m}$ we get the vorticity equation:

$$\mathbf{w}_t = -\mathcal{L}_v\mathbf{w}.$$

■

Remark: The vorticity equation is valid in rotating frames too, since the vorticity $\mathbf{w}$ is defined via the momentum field $\mathbf{m}$ not by the velocity field $\mathbf{v}$.

Finally, we point out that the Hamiltonian structure of the generalized Bernoulli's equation is associated with the Poisson bracket

$$\{F, G\}(\mathbf{m}) = \int (d\mathbf{m})(dF, dG)dx.$$

Note that this is nothing but the generalization of the fluid bracket part of the rigid body-fluid bracket.

3.5.3 Lie-Poisson Structure

The dynamics of a rigid body containing fluid can be interpreted as a Lie-Poisson equation on $\mathcal{N}$ which can be interpreted as in duality with $so(3) \times \mathcal{X}_d$. Lie-Poisson structures are related to linear Poisson brackets on duals of Lie algebras and their history goes back back to the late nineteenth century and Sophus Lie [56]. However, the close relationships between the Lie-Poisson structures, symmetries of dynamical systems and the geometric reduction methods has been revealed in the last two decades. Here, first we present some generalities about the Lie-Poisson concepts which we extracted from [56]. Then, we interpret the dynamics of a rigid body containing perfect fluid as a Lie-Poisson system.

Let G be a Lie group and $\mathcal{G}$ be its Lie algebra. For $a, b \in \mathcal{G}$ let $[a, b]$ denotes the Lie bracket of a and b . Let $\mathcal{G}^*$ denote the dual space of $\mathcal{G}$.

Definition 3.2 *A dynamical equation defined on $\mathcal{G}^*$ is called a Lie-Poisson equation if it can be written as*

$$\frac{d\mu}{dt} = ad_{\frac{\delta h}{\delta \mu}}^* \mu$$

for some $h : \mathcal{G}^* \rightarrow \mathbb{R}$, where $\mu \in \mathcal{G}^*$ and ad^* is the coadjoint map on $\mathcal{G}^*$.

Lie-Poisson equations are closely related to Lie-Poisson brackets on $\mathcal{G}^*$.

Definition 3.3 *Let $f, g \in C^\infty(\mathcal{G}^*)$. Then, the bracket*

$$\{\cdot, \cdot\}_\pm : C^\infty(\mathcal{G}^*) \times C^\infty(\mathcal{G}^*) \rightarrow C^\infty(\mathcal{G}^*)$$

given by

$$\{f, g\}_\pm(\mu) = \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu} \right] \right\rangle$$

is called $+$ ($-$) Lie-Poisson bracket on $\mathcal{G}$. $\mathcal{G}^*$ equipped with $+$ and $-$ bracket are denoted by $\mathcal{G}_+$ and $\mathcal{G}_-$ respectively.

The important fact is that Lie-Poisson brackets satisfies the axioms of Poisson brackets. Being such, for given differentiable hamiltonian functions on $\mathcal{G}^*$, they define Hamiltonian systems on $\mathcal{G}^*$. Hamiltonian systems induced by Lie-Poisson brackets are given by Lie-Poisson equations.

Proposition 3.8 *The equations of motion for the Hamiltonian system associated with the Hamiltonian $h : \mathcal{G}^* \rightarrow \mathbb{R}$ with respect to the $\pm$ Lie-Poisson brackets on $\mathcal{G}^*$ are given by the Lie-Poisson equations*

$$\frac{d\mu}{dt} = \mp ad_{\frac{\delta h}{\delta \mu}}^* \mu.$$

Proof: Let $f, h \in C^\infty(\mathcal{G}^*)$. Then, the evolution of a function $f : \mathcal{G}^* \rightarrow \mathbb{R}$ along a Lie-Poisson equation is given by

$$\frac{df}{dt} = \{f, h\}_\pm = \left\langle \frac{\delta f}{\delta \mu}, \frac{\delta \mu}{\delta t} \right\rangle.$$

We also have,

$$\begin{aligned} \{f, h\}_\pm &= \pm \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta h}{\delta \mu} \right] \right\rangle \\ &= \pm \left\langle \mu, -ad_{\frac{\delta h}{\delta \mu}} \frac{\delta f}{\delta \mu} \right\rangle \\ &= \mp \left\langle ad_{\frac{\delta h}{\delta \mu}}^* \mu, \frac{\delta f}{\delta \mu} \right\rangle \\ &= \left\langle \frac{\delta f}{\delta \mu}, \mp ad_{\frac{\delta h}{\delta \mu}} \mu \right\rangle. \end{aligned}$$

The result follows from the non-degeneracy of the pairing and the arbitrariness of f . ■

Another important aspect of the Lie-Poisson equations is that they are related to left or right invariant Hamiltonian systems on T^*G . Let λ and ρ be the cotangent lifts of the left and right translations of the group respectively:

$$\lambda : T^*G \rightarrow G \times \mathcal{G}^* \cong T^*G$$

$$\rho : T^*G \rightarrow G \times \mathcal{G}^* \cong T^*G$$

$$\lambda((g, \alpha_g)) = (g, T_e^* L_g \alpha_g)$$

$$\rho((g, \alpha_g)) = (g, T_e^* R_g \alpha_g)$$

where $\alpha_g \in T_g^*G$ and $g \in G$. Then, the following proposition characterizes Lie-Poisson reduction which is nothing but a special case of Poisson reduction applied to Hamiltonian systems having full group symmetry.

Proposition 3.9 Lie-Poisson Reduction *Let $H : T^*G \cong G \times \mathcal{G}^* \rightarrow \mathfrak{R}$ be a left (respectively right) invariant energy function on T^*G , i.e. ;*

$$H \circ \lambda = H \quad (H \circ \rho = H).$$

*Then, the Hamiltonian equation on T^*G associated with H reduces to the Lie-Poisson equation*

$$\frac{d\mu}{dt} = \pm ad_{\frac{\delta h}{\delta \mu}}^* \mu$$

on $\mathcal{G}^$ with the reduced energy $h : \mathcal{G}^* \rightarrow \mathfrak{R}$ given by the restriction of H at the identity element of the group G ;*

$$h(\mu) = H(id, \mu).$$

In other words, a left invariant hamiltonian on T^*G induces Lie-Poisson dynamics on $\mathcal{G}_-^*$ while a right invariant one induces Lie-Poisson dynamics on $\mathcal{G}_+^*$.

Proposition 3.10 *The rigid body-fluid bracket given by*

$$\{f, g\}_{RF}(q, \mathbf{m}) = \frac{\delta f}{\delta q} \hat{q} \frac{\delta g}{\delta q} + \int_{\mathcal{F}} \frac{\delta f^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta g}{\delta \mathbf{m}} dx$$

is the sum of two Lie-Poisson brackets:

$$\begin{aligned} \{f, g\}_{RF}(q, \mathbf{m}) &= \{f, g\}_R(q) + \{f, g\}_F(\mathbf{m}) \\ &= - \langle q, [\frac{\delta f}{\delta q}, \frac{\delta g}{\delta q}] \rangle + \langle \mathbf{m}, [\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}}]_L \rangle \end{aligned}$$

where $[\cdot, \cdot]$ is the Lie bracket on $so(3) \cong \mathfrak{R}^3$, and $[\cdot, \cdot]_L$ is the Lie bracket on $\mathcal{X}_d$.

Proof: We first show $\{f, g\}_R$ is the minus Lie-Poisson bracket on $\mathfrak{R}^3$ interpreted as the dual of the Lie algebra $so(3)$. By using elementary matrix and vector manipulations, we get

$$\begin{aligned}\{f, g\}_R(q) &= \frac{\delta f^T}{\delta q} \hat{q} \frac{\delta g}{\delta q} = q^T \left(\frac{\delta g}{\delta q} \times \frac{\delta f}{\delta q} \right) \\ &= -q^T \left(\frac{\delta f}{\delta q} \times \frac{\delta g}{\delta q} \right) \\ &= - \langle q, [\frac{\delta f}{\delta q}, \frac{\delta g}{\delta q}] \rangle.\end{aligned}$$

In order to obtain the fluid bracket $\{f, g\}_F$ in the Lie-Poisson form, we proceed as

$$\begin{aligned}\{f, g\}_F(\mathbf{m}) &= \int \frac{\delta f^T}{\delta \mathbf{m}} \left(\frac{\partial \mathbf{m}^T}{\partial x} - \frac{\partial \mathbf{m}}{\partial x} \right) \frac{\delta g}{\delta \mathbf{m}} dx \\ &= - \int_{\mathcal{F}} \frac{\delta f^T}{\delta \mathbf{m}} (\nabla \times \mathbf{m}) \frac{\delta g}{\delta \mathbf{m}} dx \\ &= \int_{\mathcal{F}} (\nabla \times \mathbf{m})^T \left(\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta g}{\delta \mathbf{m}} \right) dx.\end{aligned}$$

Now, we add the following zero term

$$\int_{\partial \mathcal{F}} \mathbf{m} \cdot (\mathbf{n} \times \left(\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta g}{\delta \mathbf{m}} \right)) ds$$

to $\{f, g\}_F$. Note that, this integral is identically zero due to the parallelness of $\frac{\delta f}{\delta \mathbf{m}}$ and $\frac{\delta g}{\delta \mathbf{m}}$ to the boundary $\partial \mathcal{F}$. Then, by using the equality

$$\int_{\mathcal{F}} \mathbf{a} \cdot (\nabla \times \mathbf{b}) dx = \int_{\mathcal{F}} \mathbf{b} \cdot (\nabla \times \mathbf{a}) dx + \int_{\partial \mathcal{F}} \mathbf{a} \cdot (\mathbf{n} \times \mathbf{b}) ds$$

we obtain

$$\{f, g\}_F(\mathbf{m}) = \int_{\mathcal{F}} \mathbf{m} \cdot (\nabla \times \left(\frac{\delta f}{\delta \mathbf{m}} \times \frac{\delta g}{\delta \mathbf{m}} \right)) dx.$$

This time, by using the vector identity

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = -[\mathbf{a}, \mathbf{b}]_J - (\nabla \cdot \mathbf{a})\mathbf{b} + (\nabla \cdot \mathbf{b})\mathbf{a}$$

and divergence-freeness of $\frac{\delta f}{\delta \mathbf{m}}$ and $\frac{\delta g}{\delta \mathbf{m}}$ we get

$$\begin{aligned}\{f, g\}_F(\mathbf{m}) &= - \int_{\mathcal{F}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}} \right]_J dx \\ &= \int_{\mathcal{F}} \mathbf{m} \cdot \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}} \right]_L dx \\ &= \langle \mathbf{m}, \left[\frac{\delta f}{\delta \mathbf{m}}, \frac{\delta g}{\delta \mathbf{m}} \right]_L \rangle\end{aligned}$$

by using the fact that the Lie bracket on $\mathcal{X}_d$ is given by the negative of the Jacobi-Lie bracket. ■

The dynamics of rigid bodies containing fluids can be interpreted as Lie-Poisson equations on the momentum space $\mathcal{N} = \mathcal{T}(so(3) \times \mathcal{X}_d)$. To show this, first we determine the co-adjoint maps of $so(3)$ and $\mathcal{X}_d$.

Lemma 3.3 *Let $\mathcal{G} = so(3)$ be equipped with the Lie bracket $[f, g] = f \times g$. Then, the coadjoint map on $so^*(3)$ is given by $ad_f^* h = h \times f$.*

Proof: Let $f, g \in so(3) \cong \mathbb{R}^3$ and $h \in so^*(3) \cong \mathbb{R}^3$. Then,

$$\begin{aligned}\langle h, ad_f g \rangle &= \langle h, [f, g] \rangle \\ &= \langle h, f \times g \rangle \\ &= \langle h, \hat{f} g \rangle \\ &= \langle \hat{f}^T h, g \rangle \\ &= \langle h \times f, g \rangle \\ &= \langle ad_f^* h, g \rangle\end{aligned}$$

Therefore, $ad_f^* h = h \times f$. ■

Lemma 3.4 *Let $\mathcal{G} = \mathcal{X}_d$ be equipped with the Lie bracket $[\mathbf{f}, \mathbf{g}]_L = \nabla \times (\mathbf{f} \times \mathbf{g})$. Then, the coadjoint map is given by*

$$ad_{\mathbf{f}}^* \mathbf{h} = (\nabla \times \mathbf{h}) \times \mathbf{f} + \nabla s$$

where ∇s is the negative of the projection of $(\nabla \times \mathbf{h}) \times \mathbf{f}$ to gradient vector fields.

Proof: Let $\mathbf{f}, \mathbf{g} \in \mathcal{X}_d$ and $\mathbf{h} \in \mathcal{X}_d^*$. Then,

$$\begin{aligned} \langle \mathbf{h}, ad_{\mathbf{f}} \mathbf{g} \rangle &= \langle \mathbf{h}, [\mathbf{f}, \mathbf{g}]_L \rangle \\ &= \langle \mathbf{h}, \nabla \times (\mathbf{f} \times \mathbf{g}) \rangle \\ &= \langle \mathbf{h}, \nabla \times (\mathbf{f} \times \mathbf{g}) \rangle . \end{aligned}$$

Then, by using the equality

$$\int_{\mathcal{F}} \mathbf{a} \cdot (\nabla \times \mathbf{b}) dx = \int_{\mathcal{F}} (\nabla \times \mathbf{a}) \cdot \mathbf{b} dx + \int_{\partial \mathcal{F}} \mathbf{a} \cdot (\mathbf{n} \times \mathbf{b}) ds$$

we get

$$\langle \mathbf{h}, ad_{\mathbf{f}} \mathbf{g} \rangle = \int_{\mathcal{F}} (\nabla \times \mathbf{h}) \cdot (\mathbf{f} \times \mathbf{g}) dx + \int_{\partial \mathcal{F}} \mathbf{h} \cdot (\mathbf{n} \times (\mathbf{f} \times \mathbf{g})) dx.$$

The last term on the right drops since $\mathbf{f}$ and $\mathbf{g}$ are parallel to the boundary $\partial \mathcal{F}$.

Then, we obtain

$$\begin{aligned} \langle \mathbf{h}, ad_{\mathbf{f}} \mathbf{g} \rangle &= \int (\nabla \times \mathbf{h})^T (\mathbf{f} \times \mathbf{g}) dx \\ &= \int (\nabla \times \mathbf{h})^T \hat{\mathbf{f}} \mathbf{g} dx \\ &= \langle \hat{\mathbf{f}}^T (\nabla \times \mathbf{h}), \mathbf{g} \rangle . \end{aligned}$$

Since $\mathbf{g} \in \mathcal{X}_d$, we can add a gradient field ∇s without breaking the equality

$$\langle \mathbf{h}, ad_{\mathbf{f}} \mathbf{g} \rangle = \langle \hat{\mathbf{f}}^T (\nabla \times \mathbf{h}) + \nabla s, \mathbf{g} \rangle = \langle ad_{\mathbf{f}}^* \mathbf{h}, \mathbf{g} \rangle .$$

Therefore,

$$\langle \hat{\mathbf{f}}^T (\nabla \times \mathbf{h}) + \nabla s - ad_{\mathbf{f}}^* \mathbf{h}, \mathbf{g} \rangle = 0$$

and

$$\begin{aligned} ad_{\mathbf{f}}^* \mathbf{h} &= \hat{\mathbf{f}}^T (\nabla \times \mathbf{h}) + \nabla s \\ &= -\mathbf{f} \times (\nabla \times \mathbf{h}) + \nabla s \\ &= (\nabla \times \mathbf{h}) \times \mathbf{f} + \nabla s. \end{aligned}$$

■

Remark: The coadjoint map we obtained agrees with the one given in [70].

Now, by using the co-adjoint maps, we can represent the dynamics of rigid bodies containing ideal fluids as:

$$\dot{q} = q \times \omega = ad_{\omega}^* q$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s = -ad_{\mathbf{v}}^* \mathbf{m}$$

Recalling that, $\omega = \frac{\delta H}{\delta q}$, $\mathbf{v} = \frac{\delta H}{\delta \mathbf{m}}$ we get;

$$(q, \mathbf{m})_t = (ad_{\frac{\delta H}{\delta q}}^* q, -ad_{\frac{\delta H}{\delta \mathbf{m}}}^* \mathbf{m}).$$

Note that, this equation is not a Lie-Poisson equation. However, if we change the Lie algebra bracket on vector fields from $[\cdot, \cdot]_L = -[\cdot, \cdot]_J$ to $[\cdot, \cdot]_L = [\cdot, \cdot]_J$, then we also have to change sign of the coadjoint map. With such a modification the momentum space $\mathcal{N}$ can be interpreted as dual to the Lie algebra $\mathcal{G} = so(3) \times \mathcal{X}_d$ with the coadjoint map on $\mathcal{G}^* = \mathcal{N}$ given as:

$$ad_{(\omega, \mathbf{v})}^*(q, \mathbf{m}) = (q \times \omega, \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s) \in \mathcal{N}.$$

With this modification, rigid body-fluid dynamics acquire the structure of a Lie-Poisson equation.

Proposition 3.11 *The dynamics of a rigid body containing a perfect fluid (3.81), (3.82) written in terms of the momentum variables $(q, \mathbf{m})$ are Lie-Poisson equations on $\mathcal{G}^* = \mathcal{N}$ associated with the reduced energy function h is given by*

$$h(q, \mathbf{m}) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K} \mathbf{m} + \frac{1}{2} \int_{\mathcal{F}} \left(\frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} \right) dx + \frac{1}{2} \int_{\mathcal{F}} \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K} \mathbf{m} dx.$$

Proof: The co-adjoint action on $\mathcal{N}$ is given by

$$ad_{(\omega, \mathbf{v})}^*(q, \mathbf{m}) = (q \times \omega, \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s).$$

Therefore, the Lie-Poisson equation on $\mathcal{N}$ associated with the energy h is determined by

$$\begin{aligned} (q, \mathbf{m})_t &= ad_{(\frac{\delta h}{\delta q}, \frac{\delta h}{\delta \mathbf{m}})}^*(q, \mathbf{m}) \\ &= (q \times \frac{\delta h}{\delta q}, \frac{\delta h}{\delta \mathbf{m}} \times (\nabla \times \mathbf{m}) - \nabla s) \end{aligned}$$

We calculate the derivatives of h w.r.t. q and $\mathbf{m}$

$$\begin{aligned} \frac{\delta h}{\delta q} &= I_B^{-1} q - I_B^{-1} \mathcal{K}(\mathbf{m}) = \omega \\ \frac{\delta h}{\delta \mathbf{m}} &= \frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1} q + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) = \mathbf{v}. \end{aligned}$$

Therefore, we obtain

$$(q, \mathbf{m})_t = (q \times \omega, \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s)$$

which is the hybrid representation of the dynamics of rigid bodies containing fluid and equivalent to (3.81), (3.82). ■

3.5.4 Euler-Poincare Structure

Euler-Poincare equations can be interpreted as dual to Lie-Poisson equations. As opposed to Lie-Poisson equations, Euler-Poincare equations are defined on a Lie algebra $\mathcal{G}$ rather than on the dual $\mathcal{G}^*$. Some references which we draw from are [57], [58], [16].

Definition 3.4 *A dynamical equation is called an Euler-Poincare equation if it can be written as*

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = ad_{\xi}^* \frac{\delta l}{\delta \xi}$$

where $\xi \in \mathcal{G}$, $l : \mathcal{G} \rightarrow \mathbb{R}$ and ad^* is the coadjoint map on $\mathcal{G}^*$.

Euler-Poincare and Lie-Poisson equations characterize the very same dynamics. Indeed, if we define $\mu \in \mathcal{G}^*$ by $\mu = \frac{\delta l}{\delta \xi}$, then an Euler-Poincare equation becomes a Lie-Poisson equation. Furthermore, the following hybrid representation gives both Lie-Poisson and Euler-Poincare equations in a single equation:

$$\frac{d\mu}{dt} = ad_{\xi}^* \mu.$$

This implies that, $\xi \in \mathcal{G}$ should be defined via equation $\xi = \frac{\delta h}{\delta \mu}$ where h is given by

$$h(\mu) = \langle \mu, \xi \rangle - l(\xi).$$

Note that this is nothing but a Lagrangian mechanics formulation restricted to the algebra level. Euler-Poincare equations are related to particular Lagrangian systems defined on TG . Let $\tilde{\lambda}$ and $\tilde{\rho}$ be the tangent lifts of the left and right translations of the group G ;

$$\tilde{\lambda} : TG \rightarrow G \times \mathcal{G} \cong TG$$

$$\tilde{\rho} : TG \rightarrow G \times \mathcal{G} \cong TG$$

$$\tilde{\lambda}((g, v_g)) = (g, T_e L_g v_g)$$

$$\tilde{\rho}((g, v_g)) = (g, T_e R_g v_g)$$

where $v_g \in T_g G$. Then, we give the following proposition from [57].

Proposition 3.12 Lagrangian Lie-Poisson Reduction *Let $L : TG \cong G \times \mathcal{G} \rightarrow \mathfrak{K}$ be a left(respectively right) invariant Lagrangian on G , i.e.;*

$$L \circ \tilde{\lambda} = L \quad (L \circ \tilde{\rho} = L).$$

Then, the Euler-Lagrange equations on TG associated with the Lagrangian L reduces to Euler-Poincare equations

$$\frac{d}{dt} \frac{\delta l}{\delta \xi} = \pm ad_{\xi}^* \frac{\delta l}{\delta \xi}$$

on $\mathcal{G}$ where the reduced Lagrangian $l : \mathcal{G} \rightarrow \mathfrak{K}$ is defined by restricting the Lagrangian L at the unity element of G ;

$$l(\xi) = L(id, \xi).$$

This proposition which characterizes the Lagrange reduction on Lie groups with full symmetry is dual to the Lie-Poisson reduction [56]. For other aspects of the Lagrangian reduction we refer to [57], [58], [87].

Proposition 3.13 *The dynamics of rigid bodies containing perfect fluids (3.79), (3.80) are Euler-Poincare equations on $\mathcal{G} = so(3) \times \mathcal{X}_d$ associated with the reduced lagrangian*

$$l(\omega, \mathbf{v}) = \frac{1}{2} \omega^T I_T \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}).$$

Correctness of this proposition comes from that l as given in the proposition is nothing but h given in Proposition 3.11 written in terms of $(\omega, \mathbf{v})$, and the equivalence of (3.79), (3.80) and (3.81), (3.82) under the change of coordinates given by $\mathcal{T}$. Here, we show that, l as it is given in the above proposition can be obtained from the Lagrangian L (3.19) of the rigid body-fluid system by restricting it to the tangent space of $\mathcal{M} = SO(3) \times \Psi$ at the unity element

$(Y, \eta) = (I, id)$. First, we state two lemmas which we will use as notational tools.

Lemma 3.5

$$\int_{\mathcal{B}+\mathcal{F}} \rho(x) \|\Omega x\|^2 dx = \omega^T I_T \omega.$$

Proof:

$$\begin{aligned} \int_{\mathcal{B}+\mathcal{F}} \rho(x) \|\Omega x\|^2 dx &= \int_{\mathcal{B}+\mathcal{F}} \rho(x) (\Omega x)^T (\Omega x) dx \\ &= \int_{\mathcal{B}+\mathcal{F}} \rho(x) (\mathcal{K}^*(\omega))^T (\mathcal{K}^*(\omega)) dx \\ &= \langle \rho \mathcal{K}^*(\omega), \mathcal{K}^*(\omega) \rangle \\ &= \langle \omega, \mathcal{K}(\rho \mathcal{K}^*(\omega)) \rangle \\ &= \langle \omega, I_T \omega \rangle = \omega^T I_T \omega. \end{aligned}$$

■

Lemma 3.6

$$\int_{\mathcal{F}} \rho_F \mathbf{v}^T \Omega x dx = \omega^T \mathcal{K}(\rho_F \mathbf{v}).$$

Proof:

$$\begin{aligned} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \Omega x dx &= \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathcal{K}^*(\omega) dx \\ &= \langle \rho_F \mathbf{v}, \mathcal{K}^*(\omega) \rangle \\ &= \langle \mathcal{K}(\rho_F \mathbf{v}), \omega \rangle \\ &= \omega^T \mathcal{K}(\rho_F \mathbf{v}). \end{aligned}$$

■

Now, we recall that the Lagrangian $L : T(SO(3) \times \Psi) \rightarrow \mathfrak{R}$ of the rigid body-fluid system which is given as

$$L(Y, \dot{Y}, \eta, \dot{\eta}) = \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(\eta(X)) \|\dot{Y}\eta + Y\dot{\eta}\|^2 dX.$$

Remembering that $\dot{Y} = Y\Omega$ and $\dot{\eta} = \mathbf{v} \circ \eta$ we have

$$L(Y, \dot{Y}, \eta, \dot{\eta}) = \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(\eta(X)) \|Y\Omega\eta + Y(\mathbf{v} \circ \eta)\|^2 dX.$$

By setting $Y = I$ and $\eta = id$ we obtain

$$L(I, \Omega, id, \mathbf{v}) = \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(X) \|\Omega X + \mathbf{v}(X)\|^2 dX.$$

Now, we change the variable of integration from X to x which accounts for passing from the material to the spatial domain, and we get

$$\begin{aligned} L(I, \Omega, id, \mathbf{v}) &= \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(x) \|\Omega x + \mathbf{v}(x)\|^2 dx \\ &= \frac{1}{2} \int_{\mathcal{B}+\mathcal{F}} \rho(x) \|\Omega x\|^2 + \int_{\mathcal{F}} \rho_F \mathbf{v}^T \Omega x dx \\ &\quad + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx. \end{aligned}$$

Then, by using the lemmas we stated above, we obtain

$$L(I, \Omega, id, \mathbf{v}) = \frac{1}{2} \omega^T I_T \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}) = l(\omega, \mathbf{v})$$

Note that, all this analysis clearly points out that the reduction process which we have carried out in section 2 of this chapter was indeed a Lagrangian reduction albeit conducted in a “bare hands” fashion. We also note that, although the Hamiltonian (Poisson) and the Lagrangian reduction are dual processes, for the rigid body-fluid problem the Lagrangian approach was more natural since we did not have the interpretation of the momentum variables beforehand.

3.6 Rigid Bodies Containing Viscous Fluids

The model we studied up to this point assumes a perfect (incompressible, inviscid and homogeneous) fluid filling the cavities of a rigid body. In the real world, there is an important dissipative effect on the flow of fluids: viscosity. In this section, we modify the equations of a rigid body containing perfect fluid to incorporate the effect of viscosity into the dynamics. Indeed, this will be an easy task, since we developed the model in a conservative framework by neglecting any dissipative effects. Therefore, any non-conservative effect on the dynamics, can be incorporated into the equations by adding these non-conservative forces into the equations. The viscous friction is due to the friction between fluid particles traveling with different velocities [21]. The forces created by this effect are modelled by the $\Delta \mathbf{v}$ where Δ is the Laplacian operator acting on vector fields. Indeed, this is not an arbitrary choice. It is the natural result of assuming a linear but otherwise arbitrary stress-strain relationship for the fluid material. It is a basic result of continuum mechanics [59] that, the linear stress-strain relationships involve only three parameters: p, μ, ν where μ and ν are called as the first and the second viscosity coefficients and they are generally assumed as fixed constants to model a homogeneous material. On the other hand, p is treated as “gauge” to fix the conditions we like to impose on the flow such as incompressibility and/or boundary conditions. So, p is assumed to be a scalar field. Indeed, this scalar field is the pressure field appearing in the various equations of fluid mechanics. In our model, the gradient field ∇s plays a similar role, and given ∇s and the other dynamical variables we could calculate the pressure gradient ∇p and vice versa. In the case of incompressible viscous flows,

the second viscosity coefficient does not appear in the viscous force. Let

$$\rho \frac{d\mathbf{v}}{dt} = \mathcal{F}(\mathbf{v}, t)$$

denotes the equations describing the dynamics of an incompressible, homogeneous fluid with a linear stress-strain relationship such that $\mu = 0$. Other than the linear fluid with $\mu = 0$, we do not pose any restrictions. The fluid may be under the effect of some external forces (conservative or non-conservative). As long as they do not effect the linear constitutive relation of the fluid material we summarize all such effects by parametrizing the right hand side of the above dynamical equation by the time variable t . Then, if we take $\mu \neq 0$, we have to modify the equations to:

$$\rho_F \frac{d\mathbf{v}}{dt} = \mathcal{F}(\mathbf{v}, t) + \mu \Delta \mathbf{v}.$$

All these follow from the Newtonian interpretation of $\frac{d\mathbf{v}}{dt}$ as the acceleration field of the fluid. As a result of these arguments, we will use the following equations to describe the dynamics of rigid bodies containing incompressible, viscous, homogeneous fluids:

$$\dot{\mathbf{q}} = \mathbf{q} \times \boldsymbol{\omega} \quad (3.101)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s + \mu \Delta \mathbf{v} \quad (3.102)$$

where μ is the second viscosity coefficient of the fluid. Furthermore, we impose a boundary condition on $\mathbf{v}$ such that it should adhere to the boundary of the cavity. Therefore, the field s should be chosen such that $\mathbf{v}$ vanishes at the boundary. It can be shown that, the equation for the momentum field reduces to the Navier-Stokes equation for incompressible fluids

$$\rho_F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right) = -\nabla p + \mu \Delta \mathbf{v} \quad (3.103)$$

if we take $\omega = \dot{\omega} = 0$. Conversely, (3.102) is nothing but Navier Stokes equation in a rotating reference frame albeit expressed in terms of the momentum field $\mathbf{m}$ instead of the velocity field $\mathbf{v}$.

The viscosity changes the nature of the dynamical equations in a radical way. As expected, the dynamics of rigid bodies containing viscous fluid is not conservative. The first and foremost manifestation of this is the decreasing of the energy. In order to show this we will need the following lemma.

Lemma 3.7 *Let $\mathbf{v}$ be a smooth vector field on $\mathcal{D} \subset \mathbb{R}^3$ which vanishes on the boundary $\partial\mathcal{D}$. Then,*

$$\int_{\mathcal{D}} \mathbf{v} \cdot \Delta \mathbf{v} dx \leq -\lambda \int_{\mathcal{D}} \mathbf{v} \cdot \mathbf{v} dx$$

for some $\lambda > 0$.

Proof: First, by making use of the vector identity [21]

$$\nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) = \nabla \mathbf{v} \cdot \nabla \mathbf{v} + \mathbf{v} \cdot \Delta \mathbf{v}$$

we get

$$\int_{\mathcal{D}} \mathbf{v} \cdot \Delta \mathbf{v} dx = - \int_{\mathcal{D}} \|\nabla \mathbf{v}\|^2 dx + \int_{\mathcal{D}} \nabla \cdot (\mathbf{v} \cdot \nabla \mathbf{v}) dx.$$

From the divergence theorem and the fact that $\mathbf{v}$ vanishes on the boundary $\partial\mathcal{D}$, it follows that the second term on the right vanishes. Then, by using the Poincare inequality

$$\int_{\mathcal{D}} \|\nabla \mathbf{v}\|^2 dx \geq \lambda \int_{\mathcal{D}} \|\mathbf{v}\|^2 dx, \quad \lambda \geq 0$$

we get

$$\int_{\mathcal{D}} \mathbf{v} \cdot \Delta \mathbf{v} dx = - \int_{\mathcal{D}} \|\nabla \mathbf{v}\|^2 dx \leq -\lambda \int_{\mathcal{D}} \|\mathbf{v}\|^2 dx.$$

■

Remark: The smallest λ satisfying the inequality in the above lemma is the negative of largest eigenvalue of the Laplacian operator. Recalling that Δ is a negative operator, λ is always positive.

Now, we calculate the energy H (3.85) along the solutions of the viscous model (3.101), (3.102):

$$\begin{aligned}
\dot{H} &= \frac{\delta H^T}{\delta q} \dot{q} + \int_{\mathcal{F}} \frac{\delta H^T}{\delta \mathbf{m}} \mathbf{m}_t \\
&= \omega^T \dot{q} + \int_{\mathcal{F}} \mathbf{v}^T \mathbf{m}_t dx \\
&= \omega^T (q \times \omega) + \int_{\mathcal{F}} \mathbf{v}^T (\mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s + \mu \Delta \mathbf{v}) dx \\
&= \mu \int_{\mathcal{F}} \mathbf{v}^T \Delta \mathbf{v} dx \leq -\lambda \mu \int_{\mathcal{F}} \mathbf{v}^T \mathbf{v} dx
\end{aligned}$$

which is obtained by using the lemma, the orthogonality of the gradient and the incompressible vector fields and the simple fact $a^T(b \times a) = 0$ for any $a, b \in \mathfrak{R}^3$.

Decreasing of the energy comes from the positiveness of λ and μ :

$$\dot{H} \leq -\lambda \mu \|\mathbf{v}\|^2 \leq 0. \quad (3.104)$$

Remark: The largest eigenvalue of the Laplacian operator is inversely proportional to the size of the container $\mathcal{F}$ [44]. Hence, the energy of a viscous fluid dissipates faster in small containers.

By using the fact that the energy H is non-negative we conclude that the viscosity dissipates all the energy of the fluid w.r.t the rigid body;

$$\lim_{t \rightarrow \infty} \int \mathbf{v}^T \mathbf{v} dx = 0. \quad (3.105)$$

Of course, this does not necessarily imply that all the energy of the rigid body-fluid system goes to zero. Recall the total energy of the system:

$$H(q, \mathbf{m}) = \bar{H}(\omega, \mathbf{v}) = \frac{1}{2} \omega^T I_T \omega + \frac{1}{2} \int_{\mathcal{F}} \rho_F \mathbf{v}^T \mathbf{v} dx + \omega^T \mathcal{K}(\rho_F \mathbf{v}). \quad (3.106)$$

Then, from the fact that $\mathcal{K}$ is a bounded operator, it yields that

$$\lim_{t \rightarrow \infty} H(q(t), \mathbf{m}(t)) = \lim_{t \rightarrow \infty} \frac{1}{2} \omega^T I_T \omega. \quad (3.107)$$

Note that, this shows that the motion of the rigid body-fluid system approaches to the motion of the whole system as a rigid body. Indeed, we can show that the rotation of the whole ensemble as a rigid body should be along one of the principal axes of the total mass distribution I_T . We substitute $\mathbf{v} = 0$ into (3.101) and by using $q = I_T \omega + \mathcal{K}(\rho_F \omega)$ we get

$$I_T \dot{\omega} = I_T \omega \times \omega. \quad (3.108)$$

We substitute $\mathbf{v} = 0$ into (3.102), then by using $\mathbf{m} = \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega)$ and the fact that $\nabla s = 0$ provided $\mathbf{v} = 0$ we obtain

$$\dot{\omega} \times x = 0 \quad \forall x \in \mathcal{F}. \quad (3.109)$$

The second equation implies $\dot{\omega} = 0$. Then by using the first equation we conclude that the angular velocity of the rigid body containing incompressible viscous fluid approaches to one of the eigenvectors of the total moment of inertia tensor I_T . This result is known as the *Zhukovskiy's theorem* [65].

Remark: In the infinite dimensional systems literature, finite dimensional manifolds asymptotically attracting all solutions of the systems are known as *inertial manifolds*. In the case of rigid bodies containing viscous fluids, the set $(\omega, \mathbf{v}) = (\omega_e, 0) \subset so(3) \times \mathcal{X}_d$ where ω_e is an eigenvector of I_T is a finite dimensional subset of the phase space. A similar phenomenon is reported to be present for some models of rigid bodies with dissipative elastic attachments [86]. The underlying feature common to both problems is the dissipation.

A remarkable fact is that the magnitude of total angular momentum $\|q\|$ is preserved despite the presence of viscosity:

$$\frac{d}{dt}\|q\|^2 = 2q^T \dot{q} = 2q^T (q \times \omega) = 0 \quad (3.110)$$

where we only used (3.101). Note that, we are able to show this very easily since in terms of the momentum variables the form of the rigid body equation does not change in the case of viscous fluids. Therefore, we have:

$$\begin{aligned} \|q(0)\| &= \|I_T \omega(0) + \mathcal{K}(\rho \mathbf{v}(0))\| \\ &= \lim_{t \rightarrow \infty} \|I_T \omega(t) + \mathcal{K}(\rho \mathbf{v}(t))\| \\ &= \lim_{t \rightarrow \infty} \|I_T \omega(t)\|. \end{aligned}$$

From the definiteness of the inertia matrix I_T , we get the following: ω goes to zero iff the initial total angular momentum is zero. Also, the energy H (3.86) goes to zero iff the initial total angular momentum of the system is zero, since the total energy of the system asymptotically approach to energy of the rigid rotation of the system.

3.6.1 Equilibria of the Dissipative Model

Let Σ_v be the set of all equilibrium points of the dynamics of a rigid body containing a viscous fluid ($\mu \neq 0$) expressed in terms of the velocity space variables. Then,

$$\Sigma_v = \Sigma_1 \cup \Sigma_2$$

where Σ_1 and Σ_2 are as defined before in section 4. To validate this claim, we present the following arguments.

As we have shown before, for rigid bodies containing viscous fluids $\|\mathbf{v}\|$ goes to zero asymptotically in time. It is clear that if $(\omega_e, \mathbf{v}_e)$ is an equilibrium of the rigid body-viscous fluid system, then $\|\mathbf{v}_e\| = 0$. Now, recall that the dynamical equation $\dot{q} = q \times \omega$ is equivalent to

$$I_T \dot{\omega} = I_T \omega \times \omega + \mathcal{K}(\rho_F \mathbf{v}) \times \omega - \mathcal{K}(\rho_F \mathbf{v}_t). \quad (3.111)$$

It is easy to see that, by using the boundedness of $\mathcal{K}$, at an equilibrium $(\omega_e, \mathbf{v}_e)$ this equation drops to the equality

$$0 = I_T \omega_e \times \omega_e. \quad (3.112)$$

This equality can be satisfied in two different ways. $\omega_e = 0$ satisfies this equality, and this solution is associated with the null solution (Σ_1) . Or, ω_e can be an eigenvector of the total inertia matrix I_T , and such steady solutions characterize the rigid rotations equilibria Σ_2 . Furthermore, by using the fact that the total energy goes to zero iff the initial angular momentum of the total system is zero, and that the energy $H = 0$ iff $(\omega, \mathbf{v}) = (0, 0) = \Sigma_1$, we conclude:

$$\begin{aligned} \|q(0)\| \neq 0 &\Rightarrow (\omega(t), \mathbf{v}(t)) \rightarrow \Sigma_2 \\ \|q(0)\| = 0 &\Rightarrow (\omega(t), \mathbf{v}(t)) \rightarrow \Sigma_1 \end{aligned}$$

It is easy to see that the null solution (Σ_1) is a stable equilibrium. This can be shown by choosing the energy H (which is quadratic and positive) as a Lyapunov function and by showing that it is non-increasing (which we have shown already.) On the other hand, the investigation of the stability status of the rigid rotations (Σ_2) equilibria is not trivial at all. We study the stability of rigid rotations in the next chapter both for the ideal and viscous fluid cases.

Chapter 4

Stability and Control

In this chapter, we present some stability and control notions of relevance to generic mechanical systems and study some stability and control problems related to the dynamics of a rigid body containing incompressible fluid. The structure of this chapter is as follows. The first three sections are expository in nature presenting basic definitions and theorems. In section 4.1, we present several different notions of stability and give a stability theorem in a Banach space setting, which is essentially a reformulation of V. Arnold's convexity conditions which guarantee nonlinear stability as opposed to formal stability in mechanical systems. Section 4.2 presents some stability methods which specifically exploit the conservative nature of mechanical systems. In section 4.3, we formally develop two control methods for Lagrangian systems by using energy and dissipation concepts. The control methods given in this section are not novel approaches, yet our presentation is not standard and emphasizes the generic aspects of the frequently used control approaches in mechanical systems. The last four sections of this chapter contain original results of our research, and constitute some of the main contributions of this dissertation. In section 4.4, we study hamiltonian

systems with purely quadratic energy and casimir terms. By using the ratio of energy and casimir as a Lyapunov function, we are able to obtain some new stability results about n-dimensional rigid body equation, Beltrami flows and rigid bodies containing fluid. Section 4.5 addresses the stability of rigid rotations of a rigid body containing incompressible fluid. Using the energy-casimir methodology we obtain sufficient conditions for stability of rigid rotations of a rigid body-fluid system. We also study the effect of viscosity on stability. Sections 4.6 and 4.7 deal with velocity control and attitude control of a rigid body-fluid system respectively. Both of these problems are posed and studied for the first time in this dissertation. The velocity control problem, i.e. stabilization of rigid rotations of a rigid body-fluid system is solved using the stability results obtained in section 4.5. Being such, it can be interpreted as an application of the stabilization by energy-casimir methodology. In the same vein, our solution to the attitude control problem depends heavily upon the mechanical nature of the system. Here, we developed a control law by shaping the energy of the system.

4.1 Stability Notions

In the context of mechanical systems, different but related notions of stability are used. We present the following stability notions along the lines of [37].

Spectral Stability Consider a dynamical system $\dot{u} = \mathcal{F}(u)$. Let u_e be an equilibrium point i.e., $\mathcal{F}(u_e) = 0$. Then, u_e is said to be *spectrally stable* if the spectrum of the linear operator $D\mathcal{F}(u_e)$ has no strictly positive real parts. If an equilibrium point of a Hamiltonian system is spectrally stable, then the associated spectrum should lie on the imaginary axis (neutral stability) since

the spectrum of a Hamiltonian system is necessarily symmetric with respect to the imaginary axis. Spectral stability does not imply much about the dynamics of a system (not even boundedness of the perturbed solutions) other than the obvious fact that absence of spectral stability implies instability.

Linear Stability Consider the linearization $(\delta \dot{u}) = (D\mathcal{F})(u_e)\delta u$ around u_e . If for any given $\epsilon > 0$, we can find $\gamma > 0$ such that $\|\delta u(0)\| < \gamma$ implies $\|\delta u(t)\| < \epsilon$ for $t > 0$, then u_e is called *linearly stable*. Linear stability implies spectral stability, but the converse is not true. Any linear system which has no eigenvalue on the open right half plane, but has repeating purely imaginary roots in its minimal polynomial could serve as a counter-example. If the spectrum is purely imaginary, then non-repeating eigenvalues implies linear stability.

Formal Stability An equilibrium u_e is called *formally stable* if there exists a function which is constant along the solutions of $\dot{u} = \mathcal{F}(u)$ and whose first variation vanishes at u_e while the second variation is definite. Although this is a poor characterization of the Lyapunov stability criterion, the notion of formal stability is widely used in the stability of infinite dimensional mechanical systems, since it is necessary for nonlinear stability and implies linearized stability.

Nonlinear Stability This is the rigorous notion of Lyapunov stability. An equilibrium u_e is said to be *nonlinearly stable* (or just *stable*) if for any given $\epsilon > 0$, there exist $\gamma > 0$ such that $\|u(0) - u_e\| < \gamma$ implies $\|u(t) - u_e\| < \epsilon$ for $t > 0$.

Due to the conservative nature of Hamiltonian systems neither spectral stability nor linear stability concepts can be used to determine exact stability results for the conservative dynamics. However, they can be useful to find conditions of instability.

In finite dimensional dynamical systems, formal stability implies nonlinear stability. But this is no longer true for infinite dimensional systems. In order to resolve the difference between formal stability and nonlinear stability as well as to clarify the nuances between finite and infinite dimensional cases, we state the following theorem which is essentially a reformulation of the convexity conditions of Arnold for nonlinear stability [37].

Theorem 4.1 *Consider the dynamical system described by the vector field $\dot{z} = \mathcal{F}(z)$, $z \in M$ where M is a Banach space with norm $\|\cdot\|$ and $\mathcal{F} : M \rightarrow M$. Let z_e be an equilibrium point, $\mathcal{F}(z_e) = 0$, and $\mathcal{O}$ be an open set around z_e . Then z_e is (nonlinearly) stable with respect to norm $\|\cdot\|$, provided there exists a twice Frechet differentiable function $L : \mathcal{O} \rightarrow \mathbb{R}$ satisfying:*

1. $DL(z_e)\delta z = 0$
2. $c_1\|\delta z\|^2 \geq D^2L(z_e)(\delta z, \delta z) \geq c_2\|\delta z\|^2$
3. $\frac{dL}{dt} \leq 0$

for some c_1, c_2 s.t. $\infty > c_1 \geq c_2 > 0$ and for any $\delta z \in M$.

Proof: Since L is twice Frechet differentiable, we can expand L as:

$$L(z) = L(z_e) + DL(z_e)\delta z + \frac{1}{2}D^2L(z_e)(\delta z, \delta z) + o(\|\delta z\|^2). \quad (4.1)$$

By using the first condition ($DL(z_e)\delta z = 0$), we get:

$$\frac{1}{2}D^2L(z_e)(\delta z, \delta z) = L(z) - L(z_e) - o(\|\delta z\|^2). \quad (4.2)$$

By using the second condition, we can write:

$$2c_1\|\delta z\|^2 \geq L(z) - L(z_e) - o(\|\delta z\|^2) \geq 2c_2\|\delta z\|^2. \quad (4.3)$$

Now, we choose an ϵ such that $2c_2 > \epsilon > 0$ and

$$\epsilon \|\delta z\|^2 \geq o(\|\delta z\|^2) \geq -\epsilon \|\delta z\|^2 \quad (4.4)$$

for sufficiently small $\|\delta z\|$. Then, we get the inequality:

$$(2c_1 + \epsilon) \|\delta z\|^2 \geq L(z) - L(z_e) \geq (2c_2 - \epsilon) \|\delta z\|^2. \quad (4.5)$$

By identifying $\delta z(t) = z(t) - z_e$, and calculating the above inequality at $t = 0$ we get:

$$(2c_1 + \epsilon) \|\delta z(0)\|^2 \geq L(z(0)) - L(z_e). \quad (4.6)$$

From the third condition on L we have:

$$L(z(0)) - L(z_e) \geq L(z(t)) - L(z_e) \quad (4.7)$$

and finally using (4.5) again we obtain

$$L(z(t)) - L(z_e) \geq (2c_2 - \epsilon) \|\delta z(t)\|^2. \quad (4.8)$$

Therefore, by combining (4.6), (4.7) and (4.8) we obtain

$$(2c_2 - \epsilon) \|\delta z(t)\|^2 \leq (2c_1 + \epsilon) \|\delta z(0)\|^2 \quad (4.9)$$

and

$$\|\delta z(t)\|^2 \leq \left(\frac{2c_1 + \epsilon}{2c_2 - \epsilon} \right) \|\delta z(0)\|^2. \quad (4.10)$$

This establishes the stability of z_e with respect to norm $\|\cdot\|$. Furthermore, if the inequality used in the choice of ϵ is satisfied for $\|\delta z\|^2 < \epsilon_1$, then the domain of validity for the bound on the perturbations is assured for

$$\|\delta z(0)\|^2 \leq \epsilon_1 \left(\frac{2c_1 + \epsilon}{2c_2 - \epsilon} \right)^{-1}.$$

■

Remark: This theorem is just a restatement of the well-known characterization of Lyapunov stability: the existence of a non-increasing function which has a strict relative minimum at an equilibrium point is sufficient for stability. It is important to note that DL and D^2L are not just the first and second variations respectively, they should be the Frechet derivatives. Otherwise, the axioms of the theorem do not guarantee that L assumes a strict relative minimum at z_e , hence no conclusion about the stability can be drawn. We illustrate this point by an example which we adopted from [2]. Consider a dynamical system

$$u_t = \mathcal{F}(u) \quad (4.11)$$

where $u \in L^2([0, 1])$ and $u_e = 0$ is an equilibrium point. We further assume that the functional

$$\phi(u) = \frac{1}{2} \int_0^1 u^2(x) - u^4(x) dx \quad (4.12)$$

is non-increasing along the solutions of the system. The first and the second variations of ϕ calculated at $u_e = 0$ are given by

$$D\phi(0) = 0 \quad (4.13)$$

$$D^2\phi(0)(v, v) = \int_0^1 v^2(x) dx. \quad (4.14)$$

It is clear that the first variation vanishes and the second variation is positive definite. One may be tempted to think that these imply the stability of $u_e = 0$. Indeed, this need not be the case since ϕ can take negative values even if u is arbitrarily close to $u_e = 0$. Let $\epsilon \in (0, 1)$, we define u_ϵ as:

$$u_\epsilon(x) = \begin{cases} 2 & x \in [0, \epsilon] \\ 0 & x \in (\epsilon, 1]. \end{cases}$$

We calculate ϕ at $u = u_\epsilon$

$$\begin{aligned}\phi(u_\epsilon) &= \frac{1}{2} \int_0^\epsilon 4dx - \frac{1}{2} \int_0^\epsilon 16dx \\ &= \frac{4\epsilon}{2} - \frac{16\epsilon}{2} \\ &= -6\epsilon < 0.\end{aligned}$$

Obviously, u_ϵ can take negative values for arbitrarily small $\|u_\epsilon\|$ hence $u_e = 0$ is not a minimum of ϕ and we cannot conclude anything about the stability from this analysis. A simple calculation might reveal that $D^2\phi$ as given above is only the second variation of ϕ , not the second Frechet derivative. Therefore, the notion of formal stability is meaningful only if the variations are taken in the Frechet sense. In finite dimensional spaces, any multilinear function is continuous, therefore n-th variation is also the n-th Frechet derivative provided it is a multilinear function.

Remark: It is useful to recall that the notion of stability depends upon a given norm. On a finite dimensional space, any norm is equivalent to any other norm defined on the same space. This is no longer true in the realm of infinite dimensional spaces. Therefore, it is possible that an equilibrium of an infinite dimensional system might be stable w.r.t. one norm yet unstable w.r.t another norm. We also note that if an equilibrium is stable w.r.t. a norm, then it is stable w.r.t. any equivalent norm. In infinite dimensional systems, it is essential to mention in which norm a stability result holds.

Remark: The first two hypotheses of theorem 4.1 only assure that L assumes a strict relative minimum at the equilibrium point u_e . This does not imply any restrictions on the dynamics. The stability follows from the non-increasing nature ($\dot{L} \leq 0$) of the Lyapunov function L along the solutions of the system. It is

clear that we presume the existence of $\dot{L}$, i.e. existence of C^1 solutions. As long as such solutions exist the theorem gives sufficient conditions for their stability.

Remark: This theorem is valid not only for dynamics defined on Banach spaces but also for dynamics defined on Banach manifolds, since essentially Lyapunov stability is a local property of a vector field. The content of the theorem is essentially identical to Arnold's convexity conditions for proving nonlinear stability [37].

4.2 Stability Methods for Mechanical Systems

The stability theorem 4.1 can be used to determine stability results for a wide class of dynamical system. However in the case of mechanical systems, the conservative nature of the dynamics help us to find Lyapunov function candidates more easily. Here, we give two well-known and useful stability tests for mechanical systems. We consider mechanical systems written in the form

$$\dot{z} = W(z)dH(z) \tag{4.15}$$

where W is a Poisson structure and H is the Hamiltonian. We remind that this is the most general formulation of the conservative mechanical systems. After appropriate transformations, Lagrangian and Symplectic systems can be cast in this form.

First, we consider the stability of an equilibrium point z_e which corresponds to a critical point of the Hamiltonian H . For such equilibria, the following well-known criterion provides a simple yet effective test for stability.

Lagrange-Dirichlet Criterion Let z_e be a critical point of the Hamiltonian H i.e. $dH(z_e) = 0$. Then z_e is a stable equilibrium point of (4.15), provided the

second derivative D^2H of the Hamiltonian calculated at z_e is either a positive or negative definite operator.

Since the first variation DH vanishes at the equilibrium z_e and the second variation $D^2H(z_e)$ is positive (or negative) definite, we can use H (or $-H$) as a Lyapunov function. The stability follows from the conservation of the Hamiltonian H along the solutions of (4.15).

Some mechanical systems can be written in the form (4.15) where $z = (x, p_x)$ lies in the tangent bundle T^*M of a Riemannian manifold M and W is the canonical Poisson structure on the cotangent bundle with the Hamiltonian

$$H(x, p_x) = \frac{1}{2} \ll p_x, p_x \gg + V(x). \quad (4.16)$$

For such systems the only possible equilibria are of the form $(x, p_x) = (x_e, 0)$ where x_e is a critical point of the potential energy V . Such an equilibrium point is stable if x_e is a strict minimum of V .

The application of the Lagrange-Dirichlet criterion is limited to the equilibria on which the Hamiltonian H assumes an extremal value. However, there are interesting and important dynamical problems in which an equilibrium z_e is not associated with a critical point of H . Such a situation is only possible if the Poisson structure W has a non-empty kernel. Assume $W(z_e)dH(z_e) = 0$, $dH(z_e) \neq 0$. For such equilibria the Hamiltonian itself cannot be used as a Lyapunov function but a combination of the Hamiltonian and Casimirs can be used to assess stability. Let C be a Casimir, i.e., $W(z_e)dC(z_e) = 0$ and let ϕ be a smooth function. Then, the following method can be used to test for stability of z_e .

Energy-Casimir Method Choose a smooth function ϕ such that the first derivative of $H + \phi(C)$ vanishes at the equilibrium point z_e . Then, z_e is stable

provided the second derivative $D^2(H+\phi(C))$ is either positive or negative definite operator.

The energy-casimir method was introduced by Arnold [7] for the stability of stationary flows of ideal fluids. Later applications of this method include [37] (various fluid and plasma problems), [43], [71] (rigid bodies with flexible appendages), [14] (rigid bodies with internal rotors).

Here, we would like to emphasize that the first and second derivatives in both the Lagrange-Dirichlet and energy-casimir methods should be taken in the Frechet sense. If they are not Frechet derivatives but weaker kinds of derivatives, then only formal stability can be assessed by these methods.

There exist mechanical systems where there is no Casimir functions, but only some constants of motion associated with the symmetries of the mechanical system. In such cases, the *energy-momentum method* is relevant and it can be applied to test the stability of the relative equilibria. See [74], [75] for a detailed treatment of this method and some of its applications.

4.3 Control of Mechanical Systems

In this section, we formally develop two control methods for asymptotic stabilization of mechanical systems represented in Lagrangian form. We do that to emphasize the role of the energy dissipation in stabilization of mechanical systems. Our derivations are formal to simplify the exposition, but with suitable hypotheses they can be cast in rigorous form. The methods we introduce are not novel; in one form or another they can be found in the literature.

4.3.1 Static Dissipative Controllers

We consider a Hilbert space Z which we will take as the configuration space of a Lagrangian system. For $z \in Z$, the objects in the tangent space at z , which we identify with Z itself, will be denoted by z_t and be interpreted as “velocity” of a mechanical system. The most general quadratic Lagrangian on the tangent bundle $TZ \cong Z \times Z$ can be written as:

$$L(z, z_t) = \frac{1}{2} \langle z_t, \mathcal{A}z_t \rangle + \langle z_t, \tilde{\mathcal{B}}z \rangle - \frac{1}{2} \langle z, \mathcal{C}z \rangle \quad (4.17)$$

where $\mathcal{A}$, $\tilde{\mathcal{B}}$, $\mathcal{C}$ are linear operators on Z . Because of the way they enter into the Lagrangian, we could take $\mathcal{A}$ and $\mathcal{C}$ as symmetric operators without losing generality. The associated Euler-Lagrange equation is given by

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + \mathcal{C}z = 0 \quad (4.18)$$

where $\mathcal{B} = \tilde{\mathcal{B}} - \tilde{\mathcal{B}}^*$. Note that $\mathcal{B}$ will be an anti-symmetric operator. We consider the functional

$$H(z, z_t) = \frac{1}{2} \langle z_t, \mathcal{A}z_t \rangle + \frac{1}{2} \langle z, \mathcal{C}z \rangle. \quad (4.19)$$

We note that, H as given above is not the Hamiltonian of the system which is related to the Lagrangian L via the Legendre transformation. Indeed, H is the Hamiltonian of the system w.r.t. a non-canonical Poisson structure which involves the gyroscopic term $\mathcal{B}$. We calculate H along the solutions of the Lagrangian system (4.18):

$$\begin{aligned} \dot{H} &= \frac{1}{2} \langle z_{tt}, \mathcal{A}z_t \rangle + \frac{1}{2} \langle z_t, \mathcal{A}z_{tt} \rangle + \frac{1}{2} \langle z_t, \mathcal{C}z \rangle + \frac{1}{2} \langle z, \mathcal{C}z_t \rangle \\ &= \frac{1}{2} \langle \mathcal{A}z_{tt}, z_t \rangle + \frac{1}{2} \langle z_t, \mathcal{A}z_{tt} \rangle + \frac{1}{2} \langle z_t, \mathcal{C}z \rangle + \frac{1}{2} \langle \mathcal{C}z, z_t \rangle \\ &= \langle z_t, \mathcal{A}z_{tt} \rangle + \langle z_t, \mathcal{C}z \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle z_t, \mathcal{A}z_{tt} + \mathcal{C}z \rangle \\
&= \langle z_t, -\mathcal{B}z_t \rangle = 0
\end{aligned}$$

where we used the symmetry of $\mathcal{A}$ and $\mathcal{C}$ in passing from the first line to the second, and the anti-symmetry of $\mathcal{B}$ in the last line. Note that H is a conserved quantity of the Lagrangian system (4.18). Furthermore, if $\mathcal{A}$ and $\mathcal{C}$ are positive operators, then the functional H qualifies as a Lyapunov function candidate and the constancy of H implies the stability of the null solution ($z = 0, z_t = 0$) of (4.18). In most physical models such as wave equations, beam equations, plate equations etc., the operator $\mathcal{A}$ is positive. Hence, in most Lagrangian models stability depends upon the positivity of the operator $\mathcal{C}$ which is related to the potential forces acting on the system. The Lagrange-Dirichlet principle captures this as: an equilibrium of a (Lagrangian) mechanical system is stable provided the potential field assumes a minimum at the equilibrium. Indeed, this principle can be extended to mechanical systems which are not necessarily in Lagrangian form by suitably interpreting the potential term on a reduced space [15].

The stability analysis given above immediately suggests a stabilization method for Lagrangian systems. We will consider the forced Lagrangian system

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + \mathcal{C}z = \mathcal{D}^*u \quad (4.20)$$

$$y = \mathcal{D}z \quad (4.21)$$

where $u \in U$ is the control and $y \in Y$ is the observation. We assume $\mathcal{A}$ is symmetric and positive, $\mathcal{B}$ is anti-symmetric and $\mathcal{C}$ is symmetric. Here, we also assume that the control space U and the observation space Y can be modeled as Hilbert spaces. The observation operator $\mathcal{D}$ is a linear map from Z to Y , similarly the control input operator $\mathcal{D}^*$ is a linear map from U to Z . This

implies that U and Y are dual spaces. In applications, this input-output model is associated with co-located sensors and actuators. We note that under this specific control-observation structure the “transfer function” operator from u to y is given by

$$G(s) = \mathcal{D}(\mathcal{A}s^2 + \mathcal{B}s + \mathcal{C})^{-1}\mathcal{D}^*. \quad (4.22)$$

It is easy to check that $G(s)$ satisfies the equality

$$G(s) = G^*(-s). \quad (4.23)$$

We call such transfer functions Hamiltonian.

We will consider linear controls of the form

$$u = -\mathcal{N}y - \mathcal{M}y_t \quad (4.24)$$

where $\mathcal{N}, \mathcal{M} : Y \rightarrow U$ are linear operators. Substitution of (4.24) into (4.20) yields the closed loop equation:

$$\mathcal{A}z_{tt} + (\mathcal{B} + \mathcal{D}^*\mathcal{M}\mathcal{D})z_t + (\mathcal{C} + \mathcal{D}^*\mathcal{N}\mathcal{D})z = 0. \quad (4.25)$$

We take

$$H(z, z_t) = \frac{1}{2} \langle z_t, \mathcal{A}z_t \rangle + \frac{1}{2} \langle z, (\mathcal{C} + \mathcal{D}^*\mathcal{N}\mathcal{D})z \rangle \quad (4.26)$$

as a Lyapunov function candidate. Note that, this requires

$$\langle z, (\mathcal{C} + \mathcal{D}^*\mathcal{N}\mathcal{D})z \rangle > 0 \quad \forall z \in Z. \quad (4.27)$$

The time rate of change of H is given by:

$$\dot{H} = - \langle z_t, \mathcal{D}^*\mathcal{M}\mathcal{D}z_t \rangle = - \langle \mathcal{D}z_t, \mathcal{M}\mathcal{D}z_t \rangle. \quad (4.28)$$

We observe that if $\mathcal{M}$ is a positive operator then H is non-increasing and

$$z_t \rightarrow \ker(\mathcal{D}) \quad (4.29)$$

as time tends to infinity. Therefore, (z, z_t) asymptotically approaches an invariant set in the subspace $z_t \in \ker(\mathcal{D})$. If $\ker \mathcal{D} = \{0\}$, then $z_t \rightarrow 0$ and by inserting $z_t = 0$ into (4.25), we get

$$(\mathcal{C} + \mathcal{D}^* \mathcal{N} \mathcal{D})z = 0. \quad (4.30)$$

Recalling condition (4.27), we obtain $z = 0$. Therefore, by choosing $\mathcal{M}$ as a positive operator, and by choosing $\mathcal{N}$ such that it satisfies the condition (4.27), the asymptotic stability of the null solution is assured provided $\ker(\mathcal{D}) = \{0\}$. For a finite dimensional model, this condition on the kernel means we have to have as many controls as the degrees of freedom of the mechanical system. Therefore, $\ker \mathcal{D} = \{0\}$ is a very restrictive condition. We find a less restrictive sufficient condition for the asymptotic stabilization of the null solution which does not require $\ker \mathcal{D} = \{0\}$. As we have shown above, positivity of $\mathcal{M}$ and the inequality (4.27) are sufficient for $z_t \rightarrow \ker(\mathcal{D})$. In this subspace, the closed loop system (4.25) reduces to

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + (\mathcal{C} + \mathcal{D}^* \mathcal{N} \mathcal{D})z = 0 \quad (4.31)$$

$$y = \mathcal{D}z. \quad (4.32)$$

Therefore, the zero state observability of this system is sufficient for the asymptotic stability of the null solution ($z = 0, z_t = 0$) provided (4.27) holds for some $\mathcal{N}$. We note that a necessary condition for (4.27) to hold is that $\mathcal{C}$ restricted to the subspace $z_t \in \ker \mathcal{D}$ be positive. Also note that, if $\mathcal{C}$ is a positive operator then (4.27) is satisfied for $\mathcal{N} = 0$, and for any positive operator $\mathcal{M}$ the control $u = -\mathcal{M}y$ achieves the asymptotical stabilization provided the original Lagrangian system (4.20) is zero state observable. If $\mathcal{C}$ is not a positive operator, asymptotic stabilization depends upon whether we could shape the potential of

the Lagrangian system while keeping the system observable from the output $y = \mathcal{D}z$.

Remark: A similar method for the stabilization of mechanical systems is proposed by Van der Schaft [26] where the problem is studied in a Hamiltonian setting as opposed to a Lagrangian one. Furthermore, [26] considers finite dimensional but nonlinear mechanical systems, yet the essential strategy is the same: if the potential energy assumes a minimum at the equilibrium then adding dissipation to the system is enough for stabilization provided the system is observable. If the equilibrium is not associated with a minimum of the potential energy, then we first try to shape the potential by using the controls, then add dissipation.

Example: Control of an Euler-Bernoulli Beam We consider an Euler-Bernoulli beam clamped at its ends $x = 0$ and $x = 1$

$$\begin{aligned}\rho \frac{\partial^2 \omega}{\partial t^2} + EI \frac{\partial^4 \omega}{\partial x^4} &= \frac{\partial^2 u}{\partial x^2} \\ y &= \frac{\partial^2 \omega}{\partial x^2}\end{aligned}$$

where ρ and EI are some constants of the beam. Here, $x \in [0, 1]$ and we will assume the deflection of the beam ω , the control input u , and the observation signal y lie in $L^2[0, 1]$. Note that we only observe the “curvature” along the beam, and dually we are only allowed to manipulate the curvature. This is a natural model for sensing and actuation through “smart materials.” We will assume the natural boundary conditions associated with the clamped ends:

$$\omega(0, t) = \omega(1, t) = 0$$

$$\omega_x(0, t) = \omega_x(1, t) = 0.$$

First, we observe that in order to study this system in our framework the natural candidates are

$$\mathcal{A} = \rho \mathcal{I} , \mathcal{B} = 0 , \mathcal{C} = EI \partial_{xxxx} , \mathcal{D} = \partial_{xx} .$$

Since the output is given as $y = \omega_{xx} = \mathcal{D}\omega$, we should show that $\mathcal{D}$ is a symmetric operator in order to show that the system fits into our framework.

Lemma 4.1 *$\mathcal{D} = \partial_{xx}$ is symmetric and negative.*

Proof: By using integration by parts, we have

$$\begin{aligned} \langle a, \mathcal{D}b \rangle &= \int_0^1 ab_{xx} dx \\ &= ab_x \Big|_0^1 - \int_0^1 a_x b_x dx . \end{aligned}$$

Recalling the boundary conditions $b_x(0, t) = b_x(1, t) = 0$, we get

$$\begin{aligned} \langle a, \mathcal{D}b \rangle &= - \int_0^1 a_x b_x dx = - \int_0^1 b_x a_x dx \\ &= \int_0^1 b a_{xx} = \langle b, \mathcal{D}a \rangle \\ &= \langle \mathcal{D}a, b \rangle . \end{aligned}$$

This establishes the symmetry of $\mathcal{D}$. Furthermore, we have

$$\langle a, \mathcal{D}a \rangle = - \int_0^1 a_x a_x dx = - \int_0^1 a_x^2 dx \leq 0$$

which shows the non-positivity of $\mathcal{D}$. From the boundary conditions, we get $a_x = 0$ iff $a = 0$. Therefore, $\langle a, \mathcal{D}a \rangle = 0$ iff $a = 0$. Hence, $\mathcal{D}$ is negative. ■

Lemma 4.2 *$\mathcal{C} = EI \partial_{xxxx}$ is symmetric and positive.*

Proof: This follows from $EI > 0$, symmetry of $\mathcal{D}$, $\ker(\mathcal{D}) = 0$ and $\mathcal{C} = EI \mathcal{D}^2$.

Note that since $\mathcal{C} = EI\partial_{xxxx} > 0$, (4.27) is trivially satisfied for $\mathcal{N} = 0$. So it is sufficient to choose $\mathcal{N} = 0$ and $\mathcal{M} > 0$ for stabilization of the null solution. For simplicity, we choose $\mathcal{M} = \alpha\mathcal{I}$ for some $\alpha > 0$. Our framework suggests that the control

$$u = -\mathcal{M}y_t - \mathcal{N}y = -\alpha y_t = -\alpha\omega_{xxt} \quad (4.33)$$

stabilizes the vibrations of the clamped beam. The closed loop equation under this control input is given by

$$\rho\frac{\partial^2\omega}{\partial t^2} + \alpha\frac{\partial^5\omega}{\partial t\partial x^4} + EI\frac{\partial^4\omega}{\partial x^4} = 0. \quad (4.34)$$

We observe that the closed loop equation is the equation of motion for a beam with viscous (Kelvin-Voigt) damping. It is known that this equation generates a semigroup in a suitably defined Hilbert space [19].

4.3.2 Dynamic Dissipative Controllers

The controllers we proposed for Lagrangian systems in the previous section are static dissipative controllers. Being such they can be thought of as having an infinite bandwidth. Here, we will develop dynamic dissipative controllers which do not have this problem. We consider forced Lagrangian systems of the form:

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + \mathcal{C}z = -\mathcal{D}^*u \quad (4.35)$$

$$y = \mathcal{D}z_t \quad (4.36)$$

where $\mathcal{A} = \mathcal{A}^* > 0$, $\mathcal{B} = -\mathcal{B}^*$ and $\mathcal{C} = \mathcal{C}^* > 0$ are linear operators on a Hilbert space Z . We consider a linear dynamic output controller

$$x_t = \mathcal{P}x + \mathcal{R}y \quad (4.37)$$

$$u = \mathcal{S}x + \mathcal{T}y \quad (4.38)$$

where $x \in X$. Here, X is the state space of the controller which we assume to have a Hilbert space structure. The controller is determined by the linear operators:

$$\mathcal{P} : X \rightarrow X \quad , \quad \mathcal{R} : Y \rightarrow X \quad (4.39)$$

$$\mathcal{S} : X \rightarrow U \quad , \quad \mathcal{T} : Y \rightarrow U \quad (4.40)$$

Under the proposed controller (4.37), the closed loop equation becomes

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + \mathcal{C}z = -\mathcal{D}^* \mathcal{S}x - \mathcal{D}^* \mathcal{T} \mathcal{D} z_t \quad (4.41)$$

$$x_t = \mathcal{P}x + \mathcal{R} \mathcal{D} z_t. \quad (4.42)$$

We investigate the stability of the null solution ($z = 0, z_t = 0, x = 0$) of the closed loop system via the Lyapunov function candidate

$$V(z, z_t, x) = \frac{1}{2} \langle z_t, \mathcal{A}z_t \rangle + \frac{1}{2} \langle z, \mathcal{C}z \rangle + \frac{1}{2} \langle x, \mathcal{M}x \rangle \quad (4.43)$$

where we choose $\mathcal{M} : X \rightarrow X$ as a symmetric positive operator. We calculate $\dot{V}$ along the solutions of (4.41), (4.42):

$$\begin{aligned} \dot{V} &= \langle \mathcal{A}z_{tt}, z_t \rangle + \langle \mathcal{C}z, z_t \rangle + \langle x, \mathcal{M}x_t \rangle \\ &= \langle \mathcal{A}z_{tt} + \mathcal{C}z, z_t \rangle + \langle x, \mathcal{M}(\mathcal{P}x + \mathcal{R} \mathcal{D} z_t) \rangle \\ &= \langle -\mathcal{B}z_t - \mathcal{D}^* \mathcal{S}x - \mathcal{D}^* \mathcal{T} \mathcal{D} z_t, z_t \rangle \\ &\quad + \frac{1}{2} \langle x, (\mathcal{P}^* \mathcal{M} + \mathcal{M} \mathcal{P})x \rangle + \langle x, \mathcal{M} \mathcal{R} \mathcal{D} z_t \rangle \\ &= -\langle \mathcal{T} \mathcal{D} z_t, \mathcal{D} z_t \rangle + \frac{1}{2} \langle x, (\mathcal{P}^* \mathcal{M} + \mathcal{M} \mathcal{P})x \rangle \\ &\quad + \langle (\mathcal{R}^* \mathcal{M} - \mathcal{S})x, \mathcal{D} z_t \rangle \end{aligned}$$

where we used the symmetry of operators $\mathcal{A}, \mathcal{C}, \mathcal{M}$ and the anti-symmetry of $\mathcal{B}$.

We assume that the controller parameters $\mathcal{P}, \mathcal{R}, \mathcal{S}, \mathcal{T}$ satisfy the equalities

$$\langle x, (\mathcal{P}^* \mathcal{M} + \mathcal{M} \mathcal{P})x \rangle = -\langle \mathcal{L}x, \mathcal{L}x \rangle - \langle x, \mathcal{Q}x \rangle \quad (4.44)$$

$$\langle y, \mathcal{S}x \rangle = \langle \mathcal{R}y, \mathcal{M}x \rangle + \langle \mathcal{W}y, \mathcal{L}x \rangle \quad (4.45)$$

$$\langle \mathcal{W}y, \mathcal{W}y \rangle = \langle y, \mathcal{T}y \rangle + \langle \mathcal{T}y, y \rangle \quad (4.46)$$

for $\forall x \in X, \forall y \in Y$, and for some linear operators $\mathcal{L}, \mathcal{Q} : X \rightarrow X$, $\mathcal{W} : Y \rightarrow X$ where $\mathcal{Q}$ is symmetric positive. If the controller parameters satisfy these equalities, then one can show that

$$\begin{aligned} \dot{V} &= -\frac{1}{2} \langle \mathcal{L}x, \mathcal{L}x \rangle - \frac{1}{2} \langle \mathcal{L}x, \mathcal{W}\mathcal{D}z_t \rangle - \frac{1}{2} \langle \mathcal{W}\mathcal{D}z_t, \mathcal{L}x \rangle \\ &\quad - \frac{1}{2} \langle \mathcal{W}\mathcal{D}z_t, \mathcal{W}\mathcal{D}z_t \rangle - \frac{1}{2} \langle x, \mathcal{Q}x \rangle \\ &= -\frac{1}{2} \|\mathcal{L}x\|^2 - \langle \mathcal{L}x, \mathcal{W}\mathcal{D}z_t \rangle - \frac{1}{2} \|\mathcal{W}\mathcal{D}z_t\|^2 - \frac{1}{2} \langle x, \mathcal{Q}x \rangle \\ &= -\frac{1}{2} \|\mathcal{L}x + \mathcal{W}\mathcal{D}z_t\|^2 - \frac{1}{2} \langle x, \mathcal{Q}x \rangle \leq 0 \end{aligned}$$

Note that since $V > 0$ and $\mathcal{Q}$ is positive, the result above implies that the solutions of (4.41), (4.42) approach to an invariant set in the subspace

$$x = 0, \mathcal{W}\mathcal{D}z_t = 0. \quad (4.47)$$

By using (4.46), it can be shown that the closed loop dynamics (4.41), (4.42) reduces to

$$\mathcal{A}z_{tt} + \mathcal{B}z_t + \mathcal{C}z = 0 \quad (4.48)$$

$$\mathcal{R}\mathcal{D}z_t = 0 \quad (4.49)$$

on the subspace specified above. Note that (4.49) implies $(z = 0, z_t = 0)$ iff (4.48) (which is nothing but the uncontrolled Lagrangian system) is zero state observable from the modified read-out map $y = \mathcal{R}\mathcal{D}z_t$. Therefore, we give the following design procedure for the stabilization of (4.35), (4.36).

Assumption: The Lagrangian system (4.35), (4.36) is zero-state observable.

Design Parameters: Choose a Hilbert space X and a linear operator $\mathcal{R} : Y \rightarrow$

X such that the Lagrangian system is zero-state observable from the read-out map $y = \mathcal{R}\mathcal{D}z_t$. Choose linear operators

$$\mathcal{L} : X \rightarrow X, \mathcal{W} : Y \rightarrow X, \mathcal{Q} : X \rightarrow X$$

s.t. $\mathcal{Q}$ is symmetric and positive. Determine linear operators

$$\mathcal{P} : X \rightarrow X, \mathcal{S} : X \rightarrow Y, \mathcal{T} : Y \rightarrow U$$

s.t. (4.44), (4.45), (4.46) are satisfied for some symmetric positive operator $\mathcal{M}$ on X .

Controller: The operators $(\mathcal{P}, \mathcal{R}, \mathcal{S}, \mathcal{T})$ define a controller (4.37) which asymptotically stabilizes the null solution of the closed loop Lagrangian system (4.41), (4.42).

Now a few words about the structure of the controller are in order. The equations (4.44), (4.45), (4.46), which we assumed to be satisfied by the controller parameters are nothing but a straight forward generalization of the state space conditions for strict positive realness of finite dimensional linear transfer functions. If the output space Y and the controller state space X are finite dimensional, then these conditions characterize finite dimensional strictly positive real transfer functions [6], [60]. It is remarkable that since these conditions on the controller do not involve any parameter of the Lagrangian system, asymptotic stabilization can be achieved by any strictly passive controller provided the observability condition holds. Even if the observability condition does not hold, there is no Lagrangian system with $\mathcal{A} > 0$, $\mathcal{C} > 0$ which could be destabilized by the dissipative controllers defined above. This is a remarkable degree of robustness with respect to the uncertainty of both the Lagrangian model and the controller. We also note that, if the original Lagrangian system (4.35), (4.36)

is observable from the read-out $\mathcal{D}z_t$, then we could always choose $\mathcal{R}$ such that it is also observable from $y = \mathcal{R}\mathcal{D}z_t$ by taking $X = Y$ and $\mathcal{R}$ as the identity map. In particular, if the Lagrangian system is observable from a finite number of outputs, then it can be asymptotically stabilized by any finite dimensional strictly passive controller.

Remark: The observability condition, which is necessary to apply this framework to a specific problem is an essential one. For any output control scheme applied to state-space models, similar assumptions are inevitable. Such assumptions for the stabilization of mechanical system also appear in [26], [76].

Remark: The stabilization of a wave equation by using a passive controller has been studied in [66]. We are inspired by this work and we generalized this idea to generic Lagrangian models. A similar framework has been proposed also in [39] in the context of finite dimensional models for large flexible space structure control.

We close this section by applying the dissipative controller concept to a wave equation.

Example: Control of a Wave Equation We consider a wave equation under the control of the scalar input $u(t) \in \mathfrak{R}$

$$\frac{\partial \omega}{\partial t^2} - \frac{\partial^2 \omega}{\partial x^2} = -a(x)u \quad (4.50)$$

where $x \in [0, 1]$, $a \in \mathbf{L}^2[0, 1]$ and $\omega = \omega(x, t)$. We investigate the wave equation with boundary conditions

$$\omega(0, t) = 0, \quad \omega(1, t) = 0.$$

First, we observe that this wave equation conforms to our Lagrangian control

formalism with

$$\mathcal{A} = \mathcal{I} , \mathcal{B} = 0 , \mathcal{C} = -\partial_{xx} , \mathcal{D}^* = a(x).$$

Obviously, $\mathcal{A} = \mathcal{A}^*$ and $\mathcal{B} = -\mathcal{B}^*$. Furthermore, $\mathcal{C}$ is symmetric and positive. In order to design a passive controller, first we have to find a read-out map which is dual to the input map $\mathcal{D}^* = a$.

Lemma 4.3 *Let $a, \omega \in \mathbf{L}^2[0, 1]$ and $\mathcal{D} : \mathbf{L}^2[0, 1] \rightarrow \mathfrak{R}$ be defined by*

$$\mathcal{D}\omega = \int_0^1 a(x)\omega(x)dx.$$

Then, the adjoint operator $\mathcal{D}^ : \mathfrak{R} \rightarrow \mathbf{L}^2[0, 1]$ is given by*

$$\mathcal{D}^*u = a(x)u.$$

Proof: Let $\omega \in \mathbf{L}^2[0, 1]$ and $u \in \mathfrak{R}$, then

$$\begin{aligned} \langle \mathcal{D}\omega, u \rangle &= \int_0^1 a(x)\omega(x)dxu \\ &= \int_0^1 \omega(x)a(x)udx \\ &= \langle \omega, \mathcal{D}^*u \rangle . \end{aligned}$$

■

Therefore, in our framework the controller input will be the scalar read-out signal given by

$$y(t) = \int_0^1 a(x)\omega_t(x, t)dx.$$

One can show that [24], the wave equation (4.50) is zero-state observable from the output y provided

$$\int_0^1 a(x)f_n(x)dx \neq 0 \tag{4.51}$$

for $n = 1, 2, 3, \dots$ where f_n is a complete orthogonal basis for $\mathbf{L}^2[0, 1]$. Therefore, in order to continue with the design procedure, we will assume that the function

a satisfies this condition. Indeed, it is easy to find such functions which account for the distributed effect of the actuator. For example, let $\{a_n\} \in \ell^2$, and $a_n \neq 0$, $n = 1, 2, 3, \dots$, then the function $a(x) = \sum_n a_n f_n(x)$ satisfies (4.51). Now, assuming a is chosen such that the observability condition holds, we choose a finite dimensional state space for the controller, i.e., $X = \mathbb{R}^k$ for a fixed k . Then, for any non-zero vector $r \in \mathbb{R}^k$, the Lagrangian system is observable from $\tilde{y} = ry$ since it is observable from the output y . Since the input space $U = \mathbb{R}$, output space $Y = \mathbb{R}$ and the controller state space $X = \mathbb{R}^k$ are finite dimensional, the conditions given by (4.44), (4.45), (4.46) are satisfied by any k dimensional strictly positive transfer function. And, any linear SISO controller $u = Gy$ where G is a strictly positive transfer function stabilizes the null solution of the wave equation.

Remark: There exist scalar fields on $[0, 1]$ which vanish outside of an arbitrarily small interval, yet satisfy the observability condition (4.51). This implies that the passive controller strategies might work even when only point measurements and point actuations are allowed. Such an example, which incorporates a passive controller in a boundary control problem has been studied in [66].

Remark: Both the static and the dynamic dissipative controllers we introduced in this section are especially tailored for the control of mechanical systems. As opposed to the application of generic control methods (such as infinite dimensional versions of LQ and H^∞ control theories) to mechanical control problems, these passive controllers show a greater degree of robustness. As it is implied from our analysis, the passive controllers are almost universal for the stabilization of Lagrangian systems. This is not surprising since a passive controller is nothing but a dynamic mechanism for imitating dissipation.

4.4 A Modified Energy-Casimir Method

We consider Hamiltonian systems $\dot{z} = W(z)dH$ where W is a Poisson structure and z lies in a Hilbert space Z . We further assume that the system has purely quadratic Hamiltonian H and a purely quadratic casimir C . Therefore, we have

$$\begin{aligned} H(z) &= \frac{1}{2} \langle z, \mathcal{H}z \rangle \\ C(z) &= \frac{1}{2} \langle z, \mathcal{C}z \rangle \\ 0 &= W(z)dC. \end{aligned}$$

for some symmetric linear operators $\mathcal{H}$ and $\mathcal{C}$. Although this is a limited framework, some interesting mechanical systems such as rigid body dynamics, perfect fluid flow and rigid bodies containing perfect fluids can be cast in this form. It is easy to see that H and C are constant along the solutions of the Hamiltonian system:

$$\dot{H} = \langle dH, \dot{z} \rangle = \langle dH, WdH \rangle = 0$$

and

$$\begin{aligned} \dot{C} &= \langle dC, \dot{z} \rangle = \langle dC, WdH \rangle \\ &= \langle W^*dC, dH \rangle = - \langle WdC, dH \rangle \\ &= 0. \end{aligned}$$

We note that the conservation of H and C only depends upon the skew-symmetry of the Poisson tensor W . We also note that z_e is an equilibrium point of the Hamiltonian system iff $dH(z_e)$ lies the kernel of $W(z_e)$. If C is a casimir function, then $dC(z_e)$ lies in the same kernel too. Therefore, some (but not all) equilibrium points are characterized by the parallelness of dH and dC :

$$\alpha dH(z_e) = dC(z_e), \quad \alpha \in \mathfrak{R}. \quad (4.52)$$

The energy and casimir terms are quadratic, hence the generalized eigenvalue-eigenvector equation

$$(\alpha\mathcal{H} - \mathcal{C})z_e = 0 \quad (4.53)$$

characterizes a particular class of equilibria which we want to study for stability.

We investigate the plausibility of functions of the form

$$L(z) = \frac{H(z)}{C(z)}. \quad (4.54)$$

as Lyapunov function candidates in order to determine the stability of equilibria characterized by (4.53). We will assume $H(z_e) > 0$ and $C(z_e) \neq 0$. Note that, this implies we are interested in some non-zero equilibria. In order to use L as a Lyapunov function, its first variation DL should vanish at the equilibrium z_e and the second variation D^2L should be definite. If D^2L is positive, then it could be used as a Lyapunov function since $L = H/C$ is constant of motion. If D^2L is negative, then we could use $\tilde{L} = -H/C$ as a Lyapunov function to show stability. The first and second variations of L are given by

$$DL = \frac{DH \cdot C - DC \cdot H}{C^2} \quad (4.55)$$

and

$$D^2L = \frac{(D^2H \cdot C - D^2C \cdot H + DH \cdot DC - DC \cdot DH)C^2}{C^4} - \frac{(DH \cdot C - DC \cdot H)2DC}{C^4}. \quad (4.56)$$

We are particularly interested in with the equilibria at which $dH \parallel dC$. At any such equilibrium, DL vanishes automatically and the second variation reduces to

$$D^2L = \frac{D^2H \cdot C - D^2C \cdot H}{C^2}.$$

If the energy and casimir are given by $H(z) = \frac{1}{2} \langle z, \mathcal{H}z \rangle$ and $C(z) = \frac{1}{2} \langle z, \mathcal{C}z \rangle$ respectively, then (4.55) and (4.56) are equivalent to

$$\alpha C(z_e) DL(z_e) = (\alpha \mathcal{H} - \mathcal{C})z_e \quad (4.57)$$

$$\alpha C(z_e) D^2 L(z_e) = (\alpha \mathcal{H} - \mathcal{C}) \quad (4.58)$$

where $\alpha = C(z_e)/H(z_e)$. Note that, since $C(z_e) \neq 0$ and $H(z_e) > 0$, $DL(z_e)$ vanishes iff $(\alpha \mathcal{H} - \mathcal{C})z_e = 0$. Similarly, the definiteness status of $D^2 L(z_e)$ is identical to that of the operator $(\alpha \mathcal{H} - \mathcal{C})$. We also observe that $D^2 L(z_e)$ cannot be definite if $DL(z_e) = 0$ since this contradicts with $z_e \in \ker(\alpha \mathcal{H} - \mathcal{C})$. On the other hand, $(\alpha \mathcal{H} - \mathcal{C})$ can be definite on the subspace perpendicular to z_e which we denote by $z_e^\perp$. Let $(\alpha \mathcal{H} - \mathcal{C})_{z_e^\perp}$ denote the operator $(\alpha \mathcal{H} - \mathcal{C})$ restricted to the subspace $z_e^\perp$. Suppose, $(\alpha \mathcal{H} - \mathcal{C})_{z_e^\perp}$ is a positive operator. Then, $L = H/C$ becomes a valid Lyapunov function candidate on the invariant set

$$\{z \in Z \mid C(z) = \frac{1}{2} \langle z, \mathcal{C}z \rangle\}.$$

Therefore,

$$(\alpha \mathcal{H} - \mathcal{C})z_e = 0 \quad (4.59)$$

$$(\alpha \mathcal{H} - \mathcal{C})_{z_e^\perp} > 0 \quad (4.60)$$

imply the stability of equilibrium z_e with respect to perturbations in the casimir leaf passing through z_e . To determine the stability w.r.t arbitrary perturbations, first we observe that the equilibria characterized by equation $(\alpha \mathcal{H} - \mathcal{C})z_e = 0$ is a continuum in the form of a linear space. Particularly, if z_e is an equilibrium so is $(1 \pm \epsilon)z_e$ for small ϵ . Similarly, if (4.59), (4.60) are satisfied for z_e then they are satisfied for the equilibrium $(1 + \epsilon)z_e$ too. Therefore, (4.59), (4.60) are sufficient for the stability of z_e not only w.r.t. perturbations in the casimir leaf

but w.r.t. any perturbation. If $(\alpha\mathcal{H} - \mathcal{C})_{z_e^\perp}$ is a negative operator, then the same arguments are valid if we choose $\tilde{L} = -H/C$ as the Lyapunov function. This test is not conclusive if $(\alpha\mathcal{H} - \mathcal{C})_{z_e^\perp}$ is not a definite operator. However, there is a class of dissipative systems which can be obtained from the Hamiltonian systems by adding a very particular dissipative field, for which this stability test is conclusive even if the second variation $(\alpha\mathcal{H} - \mathcal{C})_{z_e^\perp}$ is not definite. Consider the perturbed Hamiltonian system

$$\dot{z} = W(z)dH - \gamma W^*(z)W(z)dH$$

where $\gamma > 0$ is a real number. It is not very difficult to check that along the solutions of this perturbed system the energy H decreases but casimirs are preserved. Furthermore, under this kind of perturbations the equilibria remain intact. For a detailed study of this class of dissipative systems where z lies in a Lie algebra and W is a Lie-Poisson structure, see [16]. Also see [77], [78] for casimir preserving dissipative perturbation for the perfect fluid flow. If $(\alpha\mathcal{H} - \mathcal{C})_{z_e^\perp}$ is positive then, the stability of z_e is not lost under casimir preserving dissipations since we get $\dot{L} \leq 0$. On the other hand if $(\alpha\mathcal{H} - \mathcal{C})_{z_e^\perp}$ is negative, then a stable equilibrium turns into an unstable one since $\dot{C} = 0$, $\dot{H} \leq 0$ imply that $\dot{\tilde{L}} = -\dot{L} \geq 0$. And, instability follows from Cetaev's theorem.

Remark: This stability test given by (4.59), (4.60) is similar to the energy-casimir stability method, with the exception that here we calculate the definiteness of the second variation not on the full space but on an invariant casimir leaf. Due to the specific choice of the Lyapunov function, the definiteness could be checked via an eigenvalue-eigenvector problem for systems having quadratic energy and casimir terms. Of course, Lyapunov function candidates of the form H/C can be tried for any Hamiltonian system; but then there will be no guaran-

are nothing but steady spins around the principal axes of the rigid body. Indeed, by identifying $q = I\omega$ and $\alpha = \lambda_e$. Therefore, the equilibria at which $dH \parallel dC$

$$(4.63) \quad I\omega_e = \lambda_e \omega_e$$

which is equivalent to the eigenvalue equation

$$(4.62) \quad (\alpha \mathcal{H} - C)q_e = (\alpha I^{-1} - \mathbf{1})q_e = 0$$

to determine the non-zero equilibria of rigid body motion, we solve $\lambda_1 > \lambda_2 > \lambda_3 > 0$ be the eigenvalues of the moment of inertia matrix I . In order operators defining the energy and casimir are given by $\mathcal{H} = I^{-1}$ and $C = \mathbf{1}$. Let It is also easy to check that $C = \frac{1}{2}q^T q$ is a casimir of this system. Therefore, the and the Hamiltonian H are given by $W(q) = q \times$ and $H = \frac{1}{2}q^T I^{-1} q$ respectively. The dynamics can be written as $\dot{q} = W(q)dH$ where the Poisson structure W respectively and I is the symmetric positive definite moment of inertia matrix. where ω and q are the angular velocity and angular momentum of the rigid body

$$(4.61) \quad \dot{q} = q \times \omega, \quad q = I\omega$$

Example: Rigid Body Free motion of a rigid body is described by

Finally, we apply this modified energy-casimir stability test to some examples. heavy top see [49], [50].

effort to use the energy plus two different casimir functions for the stability of a the conditions assuring $DL = 0$ and $D^2L > 0$ can be extremely difficult. For an formed by using all the casimirs could yield less conservative results, but finding this test can be tried for different casimirs. In principle, a Lyapunov function independent casimir functions associated with a Poisson structure. In such cases, tee that DL will vanish at the equilibria. We remind that there may be several

these are the only non-zero equilibria for the free motion of a rigid body. To determine the stability status of the rotations we should check the definiteness of

$$(\alpha \mathcal{H} - \mathcal{C})^{q_{\perp}^2} = (\lambda_e I^{-1} - \mathbf{1})^{q_{\perp}^2}.$$

One can easily check that the operator given above is positive definite for $\lambda_e = \lambda_1$ and negative definite for $\lambda_e = \lambda_3$. It is indefinite for $\lambda_e = \lambda_2$. Therefore, rotation of a rigid body about the axes associated with the smallest and the largest eigenvalues of the inertia matrix are stable. Among these only the rotation along the “short axis” (i.e. the eigenvector associated with the largest eigenvalue of I) remains stable under the momentum preserving dissipations. Other equilibria turns unstable under momentum preserving dissipations.

Example: n-Dimensional Rigid Body Let $M \in so^*(n)$, and $[\cdot, \cdot]$ be the matrix commutator. Then, the differential equation

$$\dot{M} = [M, \Phi^{-1}M] \tag{4.64}$$

is known as the n-dimensional rigid body equations [72] where $\Phi^{-1} : so^*(n) \rightarrow so(n)$ is a symmetric positive definite linear operator. The skew-symmetric $n \times n$ matrix M represents the angular momentum of the n-dimensional rigid body. Furthermore, if we identify $\Omega = \Phi^{-1}M \in so(3)$ as the angular velocity of the n-dimensional rigid body, then the correct interpretation for Φ will be the moment of inertia operator of the n-dimensional body. We consider the eigenvalue equation

$$\Phi \Omega = \lambda \Omega. \tag{4.65}$$

This equation has $n(n-1)/2$ independent solutions, since the dimension of $so(n)$ is $n(n-1)/2$ and Φ is a symmetric operator. It is easy to see that if Ω_e is a

$$\begin{aligned} C(v) &= \frac{1}{2} \int \mathbf{v}^T \Delta (\mathbf{v} \times \mathbf{v}) dx. \\ H(v) &= \frac{1}{2} \int \mathbf{v}^T \mathbf{v} dx \end{aligned}$$

nian system with the energy H and casimir C given by defines the dynamics of a perfect fluid in $\mathfrak{R}^3$. Bernoulli's equation is a Hamilito-

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{v}) - \nabla s, \quad \nabla \cdot \mathbf{v} = 0$$

Example: Perfect Fluid Flow Bernoulli's equation [8]

the casimir $tr(M^t M)$.

with the largest eigenvalue of Φ) is robust with respect to dissipations preserving the stability of rotations along the "shortest axis" (the eigendirection associated "shortest" and the "longest" axis of an n -dimensional rigid body is stable. And similar to the 3-d rigid body case we could say that only the rotations along the $i = 1$. At any other equilibria, this operator will be non-definite. Therefore, This operator is negative definite for $i = n(n-1)/2$ and positive definite for

$$(\alpha \mathcal{H} - C)_{z_i} = (\lambda_i \Phi^{-1} - \mathbf{1})_{M_i^+}.$$

these rotations are determined by the definiteness of tations along the eigen-axes of the n -dimensional rigid body. The stability of is equivalent to (4.65). We note that these equilibria are associated with the ro- operator Φ . Then, the equation which determines the equilibria $(\alpha \mathcal{H} - C)M_e = 0$ Let, $\lambda_1 > \lambda_2 > \dots > \lambda_{n(n-1)/2} > 0$ be the eigenvalues of the moment of inertia tity operation on $n \times n$ matrices) and the inner product is given by $tr((\cdot)^T(\cdot))$. that $\frac{1}{2}tr(M^T M)$ is a casimir. Therefore, in our notation, $\mathcal{H} = \Phi^{-1}$, $C = \mathbf{1}$ (iden- $W(M) = [M, \cdot]$, and energy $H(M) = \frac{1}{2}tr(M^T \Phi^{-1} M)$. It is also easy to check dynamics of an n -dimensional rigid body is Hamiltonian w.r.t. Poisson structure solution to (4.65) then $M_e = \Phi \Omega_e$ is an equilibrium of (4.64). Furthermore, the

The casimir function C is known as helicity integral [64]. Therefore, we have $\mathcal{H} = \mathcal{I}$ and $\mathcal{C} = \text{curl}$. Then, the equation $(\alpha\mathcal{H} - \mathcal{C})\mathbf{v}_e = 0$ characterizes nothing but the Beltrami flows:

$$\nabla \times \mathbf{v}_e = \alpha \mathbf{v}_e$$

which are equilibria of the perfect fluid flow as we showed in chapter 3, section 4. Let $\alpha = \lambda_i$ be an eigenvalue of the curl operator associated with the beltrami flow $\mathbf{v}_i$. Then, the definiteness of the operator

$$(\alpha\mathcal{H} - \mathcal{C})_{\mathbf{v}_i^\perp} = (\lambda_i\mathcal{I} - \text{curl})_{\mathbf{v}_i^\perp}$$

determines the stability of Beltrami flow $\mathbf{v}_i$. At this point, we recall that the Laplacian operator on vector fields are given by

$$\Delta = \text{grad} \cdot \text{div} - \text{curl} \cdot \text{curl}.$$

Note that, the Laplacian acting on the divergence-free vector fields reduces to $\Delta = -\text{curl} \cdot \text{curl}$. The eigenvalues of the Laplacian operator are not bounded, so are the eigenvalues of curl operator. Therefore, $(\lambda_i\mathcal{I} - \text{curl})_{\mathbf{v}_i^\perp}$ cannot be definite for any $\lambda_i < \infty$. This does not imply anything about the stability of Beltrami flows as the equilibria of Bernoulli's equation. However, if we perturb the Bernoulli's equation by adding a helicity preserving dissipation [77], [78] then any Beltrami flow become an unstable equilibrium of the perturbed system. Since this result holds even for infinitesimally small helicity preserving dissipations, Beltrami flows cannot be stable equilibria of Bernoulli's equation either. We remind that, our stability test requires $C(\mathbf{v}_e) \neq 0$. Therefore, the instability of gradient flows (which are Beltrami flows with zero helicity) are not claimed.

Example: Rigid Bodies Containing Perfect Fluid The model (3.81), (3.82)

we developed for a rigid body containing perfect fluid in Chapter 3 falls into the category of Hamiltonian systems with purely quadratic energy and casimir. Let $z = (q, \mathfrak{m}) \in \mathcal{N}$ where $\mathcal{N}$ is a linear space as defined in chapter 3, section 2, then the energy $H = \frac{1}{2} \langle z, \mathcal{H}z \rangle$ given by (3.85) and casimir $C = \frac{1}{2} \langle z, \mathcal{C}z \rangle$ of rigid bodies containing fluids can be written by using the linear operators $\mathcal{H}$

and $\mathcal{C}$ which are given as:

$$\mathcal{H} = \begin{bmatrix} I_B^{-1} & -\mathcal{K}^* I_B^{-1} \frac{1}{\rho^F} + \mathcal{K}^* I_B^{-1} \mathcal{K} \\ -I_B^{-1} \mathcal{K} & \end{bmatrix}$$

$$\mathcal{C} = \begin{bmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Note that $\mathcal{H}$ is T^{-1} as defined in chapter 3. And $C = \|q\|_2^2$. We calculate $(\alpha \mathcal{H} - C)z_e = 0$ in order to determine the equilibria at which $\alpha dH = dC$:

$$\alpha(I_B^{-1}q_e - I_B^{-1}\mathcal{K}(\mathfrak{m}_e)) = q_e$$

$$\alpha(-\mathcal{K}^* I_B^{-1}q_e + \frac{\mathfrak{m}}{\rho} + \mathcal{K}^* I_B^{-1}\mathcal{K}(\mathfrak{m})) = 0.$$

These equations are equivalent to

$$\alpha \omega_e = q_e$$

$$\alpha \mathbf{v}_e = 0.$$

Assuming $\alpha \neq 0$, we get $\mathbf{v}_e = 0$ and $q_e = I_{T\omega_e} + \mathcal{K}(\rho^F \mathbf{v}_e) = I_{T\omega_e}$. Therefore, we get

$$\alpha \omega_e = I_{T\omega_e}.$$

Hence, ω_e is an eigenvector of I_T and $\alpha = \lambda_e$ is the corresponding eigenvalue. Note that our assumption $\alpha \neq 0$ is justified since I_T is positive definite. Therefore, it is clear that the equilibria at which the gradients of the energy and the casimir are parallel are the rigid rotations equilibria which we defined in chapter 3, section 4. In order to apply the stability test we proposed to the stability of rigid rotations, we check the definiteness of $(\alpha\mathcal{H}-\mathcal{C})$ on the subspace orthogonal to the rigid rotation equilibrium. Note that, since $\alpha = \lambda_e$ is an eigenvalue of a positive definite matrix I_T , the definiteness of $(\lambda_e\mathcal{H}-\mathcal{C})$ is equivalent to the definiteness of

$$(\mathcal{H}-\lambda_e^{-1}\mathcal{C}) = \begin{bmatrix} I_B^{-1} - \frac{\lambda_e}{1} & -\mathcal{K}^* I_B^{-1} \frac{\partial}{\partial \mathbf{p}} \\ -\mathcal{K}^* I_B^{-1} \frac{\partial}{\partial \mathbf{p}} & +\mathcal{K}^* I_B^{-1} \mathcal{K} \end{bmatrix}.$$

We calculate the associated quadratic form:

$$(\mathcal{H}-\lambda_e^{-1}\mathcal{C})[(q, \mathbf{m}), (q, \mathbf{m})] = q^T (I_B^{-1} - \frac{\lambda_e}{1}) q - 2q^T I_B^{-1} \mathcal{K}(\mathbf{m}) + \int \frac{1}{\rho^F} \mathbf{m}^T \mathbf{m} + \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) d\mathbf{x}$$

Let $\tilde{q} \in \mathfrak{R}^3$ denote the vectors in the subspace perpendicular to q_e . Then, after

some work, we can show that

$$(\mathcal{H}-\lambda_e\mathcal{C})[(q, \mathbf{m}), (q, \mathbf{m})] \geq \lambda_{min}(I_B^{-1} - \frac{\lambda_e}{1}) \|\tilde{q}\|_2^2$$

$$-2\|(I_B^{-1} \mathcal{K})^{q_e^\perp} \|\|\mathbf{m}\| + \frac{1}{\rho^F} \|\mathbf{m}\|_2^2$$

One can also check that the right hand side of the above inequality is strictly

greater than zero provided

$$(4.66) \quad \lambda_{min}(I_B^{-1} - \frac{\lambda_e}{1})^{q_e^\perp} > \rho^F \|(I_B^{-1} \mathcal{K})^{q_e^\perp}\|_2.$$

Therefore, this is a sufficient condition for the stability of rigid rotation $q_e = I_T \omega_e = \lambda_e \omega_e$, $\mathbf{m}_e = \rho^F \mathcal{K}^*(\omega_e)$ of a rigid body containing perfect fluid. Furthermore, this same condition is also sufficient for the stability of rigid rotations

$$e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_3 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

of traceless 2×2 real matrices. Let e_1, e_2, e_3 be a basis for $sl(2)$:

Example: A Dynamical Equation on $sl(2)$ Consider the Lie algebra $sl(2)$

its stability under such dissipations provided $z_{min}^T C z_{min} < 0$. even under casimir preserving dissipative perturbations. Similarly, z_{min} retains and $z_{max} = P y_{max}$ are stable equilibria. If $z_{max}^T C z_{max} > 0$ then, z_{max} is stable tively. Let y_{min} and y_{max} be the associated eigenvectors. Then, $z_{min} = P y_{min}$ Let λ_{min} and λ_{max} be the minimum and maximum eigenvalues of $P^T C P$ respec- under the transformation $P y = z$. We assume $P^T C P$ has distinct eigenvalues.

$$(aI - P^T C P)y = 0$$

is equivalent to

$$(aH - C)z = 0$$

matrix P such that $P^T H P = I$. Therefore, the equality

$C = \frac{1}{2} z^T C z$ where $H > 0$ and C are symmetric matrices. Then, there exists a

Finally, we consider a Hamiltonian system on $\mathfrak{R}^n$ with $H = \frac{1}{2} z^T H z$ and

The next section will also provide the calculations we skipped in this section.

an energy-casimir point of view and derive a very similar condition of stability. fluids. In the next section, we approach to the stability of rigid rotations from perturbation to the conservative dynamics of the rigid bodies containing perfect In other words, the dynamic effect of the viscosity is a momentum preserving section 6, the viscosity has no effect on the conservation of the casimir $\|q\|^2$. of a rigid body containing viscous fluid, since as we have shown in chapter 3,

Then, the commutation relations

$$[e_i, e_i] = 0, [e_1, e_2] = e_3$$

$$[e_1, e_3] = -2e_1, [e_2, e_3] = 2e_2$$

determine the structure constants of the algebra which are defined via $[e_i, e_j] = \sum c_{ij}^k e_k$. Let $x_i \in \mathfrak{g}$, then $x = x_1 e_1 + x_2 e_2 + x_3 e_3 \in sl(2) \approx \mathfrak{g}^3$ and the corresponding Lie-Poisson bracket $W_{ij}(x) = c_{ij}^k x_k$ on the Lie algebra $sl(2) \cong \mathfrak{g}^3$ is

given by

$$W(x) = \begin{bmatrix} 0 & x_3 & -2x_1 \\ -x_3 & 0 & 2x_2 \\ 2x_1 & -2x_2 & 0 \end{bmatrix}.$$

It is easy to check that $C(x) = x_1 x_2 + \frac{1}{2} x_3^2$ is a casimir of this Poisson structure. We will investigate stability of the equilibria of the differential equation

$$\dot{x}_1 = x_3 x_2 - 2x_1 x_3$$

$$\dot{x}_2 = 2x_2 x_3 - x_1 x_3$$

$$\dot{x}_3 = 2x_1^2 - 2x_2^2$$

One can check that this is a Hamiltonian system w.r.t the Lie-Poisson bracket given above and the Hamiltonian function $H = \frac{1}{2} \|x\|^2$. It is clear that this system accepts quadratic energy and casimir determined by the matrices

$$\mathcal{H} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$$

The equilibria at which the gradients of the energy and the casimir are parallel to each other are determined by the eigenvalue equation:

$$(\lambda \mathcal{H} - \mathcal{C})x = 0.$$

$$\dot{q} = q \times I_B^{-1} q - q \times I_B^{-1} \mathcal{K}(\mathbf{m}) \quad (4.67)$$

dynamical equations

3. This model, expressed in terms of momentum variables, is given by the We developed a rigorous model for a rigid body containing perfect fluid in chapter

4.5 Stability of Rigid Bodies Containing Fluid

the equilibria of some systems can be determined by using a simple method. flow around any equilibrium in $\mathcal{E}_2$. It is remarkable that stability status of all even in the absence of dissipation; and this can be shown by a linearizing the of $\mathcal{E}_2$ under casimir preserving dissipations. Indeed, any point in $\mathcal{E}_2$ is unstable $C(x) > 0$ if $x \in \mathcal{E}_1$ and $C(x) < 0$ if $x \in \mathcal{E}_3$. Our test also predicts the instability in $\mathcal{E}_1$ nor in $\mathcal{E}_3$ loses its stability under casimir preserving dissipations because largest eigenvalue of $P^T C P = C$. Furthermore, the equilibrium points neither point in $\mathcal{E}_1$ or $\mathcal{E}_3$ are stable, since they are associated with the smallest and the where $\mathcal{E}_i$ is associated the λ_i . From our previous arguments, any equilibrium

$$\begin{aligned} \mathcal{E}_3 &= \{(x_1, x_2, x_3) = (a, -a, 0), a \in \mathbb{R}\} \\ \mathcal{E}_2 &= \{(x_1, x_2, x_3) = (0, 0, a), a \in \mathbb{R}\} \\ \mathcal{E}_1 &= \{(x_1, x_2, x_3) = (a, a, 0), a \in \mathbb{R}\} \end{aligned}$$

Therefore, there are three sets of equilibria :

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

corresponding eigenvectors are:

The eigenvalues of C can be calculated as $\lambda_1 = 1, \lambda_2 = 0.5, \lambda_3 = -1$. And, the

$$(\omega_e, \mathbf{v}_e) = (\omega_e, 0).$$

where $I_T \omega_e = \lambda_e \omega_e$. In the velocity space, these equilibria are associated with

$$(q_e, \mathbf{m}_e) = (I_T \omega_e, \rho_F \mathcal{K}^*(\omega_e))$$

correspond to

viscous fluid. In the model given by (4.67), (4.68), the rigid rotations equilibria only for a rigid body containing ideal fluid, but also for a rigid body containing have shown in chapter 3, sections 4 and 6, the rigid rotations are equilibria not while the fluid in the cavity is stationary with respect to the rigid body. As we fluid system along the principal axes of the total moment of inertia matrix I_T , of (4.67), (4.68). Rigid rotations correspond to the rotations of the rigid body- In this section, we study nonlinear stability of the rigid rotations equilibria

$$+ \frac{1}{2} \int \frac{\mathbf{m}^T \mathbf{m}}{\rho_F} + \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) dx.$$

$$= \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K}(\mathbf{m}) \quad (4.71)$$

$$H(q, \mathbf{m}) = \frac{1}{2} < (q, \mathbf{m}), T^{-1}(q, \mathbf{m}) > \quad (4.70)$$

containing ideal incompressible fluid is a Hamiltonian system with the energy:

We have also shown (in chapter 3, section 5) that, the dynamics of a rigid body

$$T(\omega, \mathbf{v}) = (q, \mathbf{m}) = (I_T \omega + \mathcal{K}(\rho_F \mathbf{v}), \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega)). \quad (4.69)$$

space $\mathcal{N}$. This operator was given in chapter 3, section 2 as:

a diffeomorphism between the velocity space ($so(3) \times \mathcal{K}_d$) and the momentum

where $(q, \mathbf{m}) \in \mathcal{N} = T(so(3) \times \mathcal{K}_d)$. Here, T is a linear operator which defines

$$\frac{\partial \mathbf{m}}{\partial t} = \left(\frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1} q + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) \right) \times (\nabla \times \mathbf{m}) - \nabla s \quad (4.68)$$

In order to determine suitable sufficient conditions for stability of the rigid rotations, we will proceed as follows. We consider functions of the following form as Lyapunov function candidates:

$$L(q, \mathbf{m}) = H(q, \mathbf{m}) + \phi\left(\frac{1}{2}\|q\|^2\right).$$

Here, H is the energy of the system as given by (4.71), $\|q\|$ is the magnitude of the total angular momentum of the rigid body-fluid system, and ϕ is a smooth function. The function L , which we will use as a Lyapunov function in the sequel, will be preserved along solutions of (4.67), (4.68) since both the energy and the angular momentum are constants of motion. Recall that, $\|q\|^2$ is a casimir of the Hamiltonian system, therefore our approach could be interpreted as an energy-casimir approach to stability. Note that, we only assumed ϕ is a smooth function. In the following, we put further restrictions on ϕ such that the first variation DL of the function L vanishes at the equilibria corresponding to rigid rotations, while the second variation D^2L is a positive operator. Indeed, we will seek conditions which make D^2L not only a positive but also a bounded operator with bounded inverse. The conditions we will impose on L will make L a Lyapunov function candidate, and the constancy of L along the solutions will prove the stability of rigid rotations under the imposed conditions. Note that, the only free parameter in L is the smooth function ϕ . Conditions imposed on L will be formulated as restrictions on the function ϕ as well as some conditions which should be satisfied by the system parameters. The conditions on the system parameters will determine sufficient conditions for the stability of a rigid rotation equilibrium.

First Variation The first variation condition $DL(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m})] = 0$ is

satisfied provided:

$$\frac{\delta L}{\delta q} = 0 \quad , \quad \frac{\delta L}{\delta \mathbf{m}} = 0.$$

These variations, taken with respect to the norm characterized by $\|(q, \mathbf{m})\|^2 = q^T q + \int \mathbf{m}^T \mathbf{m} dx$ can be given as

$$\frac{\delta L}{\delta q} = \frac{\delta H}{\delta q} + \frac{\delta \phi}{\delta q} = I_B^{-1} q - I_B^{-1} \mathcal{K}(\mathbf{m}) + \phi'(\frac{1}{2}\|q\|^2)q \quad (4.72)$$

$$\frac{\delta L}{\delta \mathbf{m}} = \frac{\delta H}{\delta \mathbf{m}} + \frac{\delta \phi}{\delta \mathbf{m}} = \frac{\mathbf{m}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) - \mathcal{K}^* I_B^{-1} q. \quad (4.73)$$

Calculating these variations at an arbitrary equilibrium $(q_e, \mathbf{m}_e)$, and equating to zero, we get:

$$I_B^{-1} q_e - I_B^{-1} \mathcal{K}(\mathbf{m}_e) + \phi'(\frac{1}{2}\|q_e\|^2)q_e = 0 \quad (4.74)$$

$$\frac{\mathbf{m}_e}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}_e) - \mathcal{K}^* I_B^{-1} q_e = 0. \quad (4.75)$$

The left hand side of (4.75) is equivalent to the equilibrium value of the fluid velocity field $\mathbf{v}_e$ by definition. Therefore, (4.75) is satisfied trivially for any rigid rotation equilibrium $(\omega_e, \mathbf{v}_e) = (\omega_e, 0)$, $I_T \omega_e = \lambda_e \omega_e$. By recalling $I_B^{-1} q_e - I_B^{-1} \mathcal{K}(\mathbf{m}_e) = \omega_e$, we observe that (4.74) is equivalent to

$$\omega_e + \phi'(\frac{1}{2}\|q_e\|^2)q_e = 0.$$

At a rigid rotation equilibrium, we have $q_e = I_T \omega_e = \lambda_e \omega_e$. Therefore, ϕ should be chosen such that

$$\phi'(\frac{1}{2}\|q_e\|^2) = -\frac{1}{\lambda_e} \quad (4.76)$$

in order to force $DL(q_e, \mathbf{m}_e) = 0$.

Second Variation The second variation $D^2 L$ is determined by the functional derivatives:

$$\frac{\delta^2 L}{\delta q^2} = I_B^{-1} + \phi'(\frac{1}{2}\|q\|^2)\mathbf{1} + \phi''(\frac{1}{2}\|q\|^2)qq^T \quad (4.77)$$

$$\frac{\delta^2 L}{\delta q \delta \mathbf{m}} = -\mathcal{K}^* I_B^{-1} \quad (4.78)$$

$$\frac{\delta^2 L}{\delta \mathbf{m} \delta q} = -I_B^{-1} \mathcal{K} \quad (4.79)$$

$$\frac{\delta^2 L}{\delta \mathbf{m}^2} = \frac{\mathcal{I}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}. \quad (4.80)$$

The second variation $D^2 L$ calculated at an equilibrium point $(q_e, \mathbf{m}_e)$ is given by

$$\begin{aligned} D^2 L(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m})(\delta q, \delta \mathbf{m})] &= \delta q^T (I_B^{-1} + \phi'(\frac{1}{2}\|q_e\|^2)\mathbf{1} + \phi''(\frac{1}{2}\|q_e\|^2)q_e q_e^T) \delta q \\ &\quad + \int_{\mathcal{F}} (\frac{1}{\rho_F} \delta \mathbf{m}^T \delta \mathbf{m} + \delta \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\delta \mathbf{m})) dx \\ &\quad - \int_{\mathcal{F}} \delta \mathbf{m}^T \mathcal{K}^* I_B^{-1} \delta q dx - \delta q^T I_B^{-1} \mathcal{K}(\delta \mathbf{m}). \end{aligned}$$

Now, we will show that there exist $c_1, c_2 \in \mathfrak{R}$, $\infty > c_1 \geq c_2 > 0$ satisfying the inequality

$$c_1 \|(\delta q, \delta \mathbf{m})\|^2 \geq D^2 L(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m}), (\delta q, \delta \mathbf{m})] \geq c_2 \|(\delta q, \delta \mathbf{m})\|^2$$

provided we put further restrictions on ϕ . First, we recall that $L = H + \phi$, and $H = \frac{1}{2} \langle (q, \mathbf{m}), \mathcal{T}^{-1}(q, \mathbf{m}) \rangle$. Therefore, $D^2 L = D^2 H + D^2 \phi$ and by using (4.76), we can write

$$\begin{aligned} D^2 L(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m}), (\delta q, \delta \mathbf{m})] &= \langle (\delta q, \delta \mathbf{m}), \mathcal{T}^{-1}(\delta q, \delta \mathbf{m}) \rangle \\ &\quad + \delta q^T (\phi''(\frac{1}{2}\|q_e\|^2)q_e q_e^T - \frac{1}{\lambda_e}) \delta q. \end{aligned}$$

Note, that $D^2 L$ differs from $\mathcal{T}^{-1}$ only by a quadratic form defined on a finite dimensional space. Therefore, provided ϕ'' is finite and $\lambda_e \neq 0$, boundedness of the operator $\mathcal{T}^{-1}$ implies the boundedness of $D^2 L$. On the other hand, as we have shown in chapter 3, $\mathcal{T}^{-1}$ is a bounded operator. Therefore, there exist $c_1 < \infty$ such that

$$c_1 \|(\delta q, \delta \mathbf{m})\|^2 \geq D^2 L(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m}), (\delta q, \delta \mathbf{m})].$$

Now, we try to find sufficient conditions to satisfy the inequality:

$$D^2L(q_e, \mathbf{m}e)[(\delta q, \delta \mathbf{m}), (\delta q, \delta \mathbf{m})] \geq c_2 \|(\delta q, \delta \mathbf{m})\|^2 \quad (4.81)$$

for some $c_2 > 0$. To this end, we define two new parameters α_ϕ and γ :

$$\alpha_\phi = \lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e} + \phi'' q_e q_e^T) \quad (4.82)$$

$$\gamma = \|I_B^{-1} \mathcal{K}\|. \quad (4.83)$$

Then, we have the following inequalities:

$$\delta q^T (I_B^{-1} - \frac{\mathbf{1}}{\lambda_e} + \phi'' q_e q_e^T) \delta q \geq \lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e} + \phi'' q_e q_e^T) \|\delta q_e\|^2 \quad (4.84)$$

$$= \alpha_\phi \|\delta q\|^2 \quad (4.85)$$

and

$$\delta q^T I_B^{-1} \mathcal{K}(\delta \mathbf{m}) \leq \|I_B^{-1} \mathcal{K}\| \|\delta q\| \|\delta \mathbf{m}\| = \gamma \|\delta q\| \|\delta \mathbf{m}\|. \quad (4.86)$$

Now, we do a simple calculation

$$\langle \mathbf{m}, \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) \rangle = \langle \mathcal{K}(\mathbf{m}), I_B^{-1} \mathcal{K}(\mathbf{m}) \rangle \quad (4.87)$$

$$= (\mathcal{K}(\mathbf{m}))^T I_B^{-1} (\mathcal{K}(\mathbf{m})) \quad (4.88)$$

$$\geq \lambda_{\min}(I_B^{-1}) \|\mathcal{K}(\mathbf{m})\|^2. \quad (4.89)$$

Then, from the positive definiteness of the moment of inertia matrix I_B , we get

$\mathcal{K}^* I_B^{-1} \mathcal{K} \geq 0$. From this result, we obtain another inequality:

$$\int_{\mathcal{F}} \left(\frac{\delta \mathbf{m}^T \delta \mathbf{m}}{\rho_F} + \delta \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K} \delta \mathbf{m} \right) dx \geq \frac{1}{\rho_F} \int_{\mathcal{F}} \delta \mathbf{m}^T \delta \mathbf{m} dx = \frac{1}{\rho_F} \|\delta \mathbf{m}\|^2 \quad (4.90)$$

Furthermore, by using $(I_B^{-1} \mathcal{K})^* = \mathcal{K}^* I_B^{-1}$, we get

$$\int_{\mathcal{F}} \delta \mathbf{m}^T \mathcal{K}^* I_B^{-1} \delta q dx + \delta q^T I_B^{-1} \mathcal{K}(\delta \mathbf{m}) = 2 \delta q^T I_B^{-1} \mathcal{K}(\delta \mathbf{m}) \quad (4.91)$$

$$\leq 2\gamma \|\delta q\| \|\delta \mathbf{m}\|. \quad (4.92)$$

Therefore, by using the inequalities given by (4.84), (4.86), (4.90), (4.92) we obtain:

$$D^2L(q_e, \mathbf{m}_e)[(\delta q, \delta \mathbf{m}), (\delta q, \delta \mathbf{m})] \geq \alpha_\phi \|\delta q\|^2 - 2\gamma \|\delta q\| \|\delta \mathbf{m}\| + \frac{1}{\rho_F} \|\delta \mathbf{m}\|^2.$$

Note that, the right hand side of the above inequality is quadratic in $(\|\delta q\|, \|\delta \mathbf{m}\|)$, therefore we can write the right hand side as

$$\begin{bmatrix} \|\delta q\| & \|\delta \mathbf{m}\| \end{bmatrix} \begin{bmatrix} \alpha_\phi & -\gamma \\ -\gamma & \frac{1}{\rho_F} \end{bmatrix} \begin{bmatrix} \|\delta q\| \\ \|\delta \mathbf{m}\| \end{bmatrix}.$$

Let $\beta \in \Re$ be the minimum eigenvalue of this quadratic form. Then, the inequality

$$\begin{aligned} D^2L(q_e, m_e)[(\delta q, \delta m), (\delta q, \delta m)] &\geq \beta(\|\delta q\|^2 + \|\delta \mathbf{m}\|^2) \\ &= \beta\|(\delta q, \delta \mathbf{m})\|^2 \end{aligned}$$

is satisfied for $\beta > 0$, provided the conditions

$$\alpha_\phi > 0 \tag{4.93}$$

$$\alpha_\phi > \rho_F \gamma^2 \tag{4.94}$$

hold. Note that, since ρ_F is positive it suffices to check only (4.94). Also note that α_ϕ depends upon both the function ϕ and the system parameters. Let ϕ'' be chosen as positive, then $\phi'' q_e q_e^T$ be a positive semi-definite matrix. Therefore, we get

$$\alpha_\phi = \lambda_{\min}(I_B^{-1} - \frac{1}{\lambda_e} + \phi'' q_e q_e^T) \geq \lambda_{\min}(I_B^{-1} - \frac{1}{\lambda_e}). \tag{4.95}$$

Consequently, the inequality $\alpha_\phi > \rho_F \gamma^2$, which assures that the spectrum of D^2L is bounded below from zero, is satisfied provided

$$\alpha_\phi > \lambda_{\min}(I_B^{-1} - \frac{1}{\lambda_e}) > \rho_F \gamma^2 = \rho_F \|I_B^{-1} \mathcal{K}\|^2 \tag{4.96}$$

Therefore, a rigid rotation equilibrium is stable if inequality (4.96) holds. We present this result as a theorem.

Theorem 4.2 *Let $(q_e, \mathbf{m}_e)$ be a rigid rotation equilibrium of a rigid body containing ideal incompressible fluid (4.67), (4.68). Then, this equilibrium is stable w.r.t. the norm $\|(q, \mathbf{m})\|^2 = q^T q + \int \mathbf{m}^T \mathbf{m} dx$, provided*

$$\lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e}) > \rho_F \|I_B^{-1} \mathcal{K}\|^2.$$

Remark: Note that, the inequality condition of this theorem is very similiar in form to the stability condition

$$\lambda_{\min}((I_B^{-1} - \frac{\mathbf{1}}{\lambda_e})_{q_e^\perp}) > \rho_F \|(I_B^{-1} \mathcal{K})_{q_e^\perp}\|^2 \quad (4.97)$$

we obtained in the previous section by using the ratio of the energy to momentum casimir as a Lyapunov function. Indeed, the inequality given in theorem 4.2 implies (4.97).

Remark: To the best of our knowledge, this is the first exact result in the literature for the stability of rigid rotations which considers the dynamics of rigid bodies containing fluids as infinite dimensional models. Some stability conditions for the same problem have been developed in [65]. However, those results were derived either by using a finite dimensional model for the rigid body-fluid system or by checking the stability only w.r.t. a finite number of the dynamical variables.

Remark: Stability of the rigid bodies with flexible appendages is studied by Krishnaprasad and Marsden [43], where they developed some stability conditions for the rigid rotations of a rigid body-flexible beam system. They proved that rigid rotations along the “short axis” is stable provided the rotation rate is

smaller than a specified value which depends upon the structure of the beam. Compared with this result, our stability condition only depends upon the direction of the rotations not on the rotation rate. We believe, this reflects an inherent difference between the rigid body-fluid and rigid body-flexible appendage dynamics. In this particular sense, a flexible appendage is more complex than a perfect fluid.

In order to interpret the meaning and implications of theorem 4.2 we first recall the fact $\|I_F\| = \rho_F \|\mathcal{K}\|^2$ (from chapter 3, section 2). From this we get:

$$\begin{aligned} \gamma = \|I_B^{-1} \mathcal{K}\| &\leq \|I_B^{-1}\| \|\mathcal{K}\| \\ &= \|I_B^{-1}\| (\rho_F^{-1} \|I_F\|)^{\frac{1}{2}}. \end{aligned}$$

Therefore, $\rho_F \gamma^2 = \rho_F \|I_B^{-1} \mathcal{K}\|^2 \leq \|I_B^{-1}\|^2 \|I_F\|$. And, the inequality $\alpha_\phi > \rho_F \gamma^2$ is satisfied provided

$$\lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e}) > \|I_B^{-1}\|^2 \|I_F\|. \quad (4.98)$$

Note that this nothing but another sufficient condition for stability of rigid rotations. This condition is more conservative than the inequality of the theorem, on the other hand easier to check since (4.98) is given in terms of two finite dimensional operators I_F and I_B .

Remark: If we take the fluid mass as zero $\rho_F \rightarrow 0$, then $I_T \rightarrow I_B$ and by using (4.97) we get a sufficient condition for the stability of steady rotations of a rigid body as:

$$\lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e})_{q_e^\perp} > 0.$$

We note that this condition is identical to the stability condition for the rotations of a rigid body obtained via an energy-momentum method in [90]. Also note that, the above condition is always satisfied if λ_e is the strictly maximum

eigenvalue of the inertia tensor I_B . This also implies that rigid rotations of a rigid body containing perfect fluid is stable provided the total fluid mass (or equivalently $\|I_F\|$) is sufficiently small.

Remark: We remind that, we obtained the conditions of stability for rigid rotations by using the momentum space representation of rigid body-fluid system. Theorem 4.2 gives a sufficient condition for stability w.r.t the norm $\|(q, \mathbf{m})\|^2 = q^T q + \int \mathbf{m}^T \mathbf{m} dx$. On the other hand, the dynamics can also be expressed in the velocity space (3.79), (3.80) by using the variables $(\omega, \mathbf{v})$. Indeed, provided the inequality of the theorem is satisfied, a rigid rotations equilibrium is also stable with respect to velocity space perturbations i.e. w.r.t. the norm given by $\|(\omega, \mathbf{v})\|^2 = \omega^T \omega + \int \mathbf{v}^T \mathbf{v} dx$. We note that, this is not automatic. Rather it depends upon the fact that momentum and velocity variables are related by a bounded linear operator $(\mathcal{T})$ with bounded inverse. This implies $\|(\omega, \mathbf{v})\|$ is bounded iff $\|(q, \mathbf{m})\|$ is bounded, i.e. small momentum perturbations are indeed small velocity perturbations.

Remark: Due to the difficulty of the problem of existence and uniqueness of 3-d Euler's equation, we do not address this problem in this study. Hence, the stability results in this section as well as the ones that will be given in the following sections are conditional upon the existence of solutions of the relevant dynamical equations.

4.5.1 Effect of Viscosity on Stability

The dynamics of a rigid body containing incompressible viscous fluid is given by

$$\dot{q} = q \times I_B^{-1} q - q \times I_B^{-1} \mathcal{K}(\mathbf{m}) \quad (4.99)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \left(\frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1} q + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) \right) \times (\nabla \times \mathbf{m}) - \nabla s + \mu \Delta \mathbf{v} \quad (4.100)$$

where $\mu > 0$ is the viscosity coefficient of the fluid and

$$\mathbf{v} = \frac{\mathbf{m}}{\rho_F} - I_B^{-1} \mathcal{K}(\mathbf{m}) + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}).$$

We have studied this equation in chapter 3, section 6 and shown that (apart from the null solution) rigid rotations are the only equilibrium of (4.99), (4.100). We have also shown that the magnitude of total angular momentum $\|q\|$ is conserved despite the fact that the energy of the system is dissipated in due to the viscosity. Now, we recall that the hypothesis of theorem 4.2 only assures that the function

$$L(q, \mathbf{m}) = H(q, \mathbf{m}) + \phi\left(\frac{1}{2}\|q\|^2\right)$$

is a valid Lyapunov function candidate for the rigid rotations equilibria provided ϕ is chosen appropriately. The stability comes from the constancy of L along the solutions of the rigid body containing perfect fluid. Therefore, the very same Lyapunov function is also a Lyapunov function candidate for the viscous fluid case. The only difference is that the function L is no longer a constant of motion. By using our previous analysis in chapter 3, section 6, we can easily calculate $\dot{L}$ along the solutions of (4.99), (4.100):

$$\begin{aligned} \dot{L} &= \dot{H} + \dot{\phi} = \dot{H} \\ &= \mu \int \mathbf{v}^T \Delta \mathbf{v} dx \leq -\lambda_1 \mu \int \mathbf{v}^T \mathbf{v} dx \leq 0. \end{aligned}$$

where λ_1 is the smallest eigenvalue of $-\Delta$. Therefore, $\dot{L} \leq 0$ and a rigid rotation of a rigid body containing incompressible viscous fluid is stable provided

$$\lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e}) > \rho_F \|I_B^{-1} \mathcal{K}\|^2.$$

If we restrict our attention to smooth $\mathbf{v}$ solutions, then, it is easy to see that $\mathbf{v} \rightarrow 0$ in time. As we have already shown in chapter 3, the only consistent solution of rigid bodies containing viscous fluids with $\mathbf{v} = 0$, is the rigid rotation equilibria i.e.; $(\omega_e, \mathbf{v}_e) = (\omega_e, 0)$, $I_T \omega_e = \lambda_e \omega_e$. However, this fact together with $\dot{L} \leq 0$ doesn't imply the asymptotic stability of a rigid rotation $(\omega_e, 0)$, since the rigid rotations are not isolated equilibria. In other words, after small perturbations to a rigid rotation, the system returns to rotate along the same axis but possibly with a slightly different speed. However, if the perturbation is small, then the shift in the rotation rate ω cannot be large.

4.6 Velocity Control Problem

The free dynamics of a rigid body containing ideal incompressible fluid allows the rigid rotations along the principal axes of the total moment of inertia matrix I_T as equilibria. Our previous analysis in section 4.5 has produced some sufficient conditions for the stability of these equilibria. As implied from our analysis, stability of a rigid rotation along the short axis is assured for small fluid mass. On the other hand, the rotations along the middle and long axes are not necessarily stable even for infinitesimally small fluid mass. Furthermore, a rotation around an arbitrary axis need not be an equilibrium point of the system, let alone stable. In this section, we will study feedback controls which stabilize the rigid rotations associated with arbitrary angular velocities.

Let $u \in \mathbb{R}^3$ be an external control torque acting on a rigid body. Then, the equations of motion describing a rigid body containing perfect fluid, expressed

in terms of the velocity variables, are given by

$$I_T \dot{\omega} = I_T \omega \times \omega + \mathcal{K}(\rho_F \mathbf{v}) \times \omega - \mathcal{K}(\rho_F \mathbf{v}_t) + u \quad (4.101)$$

$$\rho_F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\omega \times \mathbf{v} + \dot{\omega} \times x \right) = -\nabla p - \nabla \left(\frac{1}{2} \rho_F x^T \hat{\omega}^2 x \right) \quad (4.102)$$

where $(\omega, \mathbf{v}) \in so(3) \times \mathcal{X}_d$. We pose the following problem.

Velocity Control Problem Let $\omega_0 \in \mathbb{R}^3$ be a fixed vector. Develop a control law $u = f_{\omega_0}(\omega, \mathbf{v})$ which forces $(\omega_e, \mathbf{v}_e) = (\omega_0, 0)$ to be a stable equilibrium of a rigid body containing ideal incompressible fluid (4.101), (4.102).

We will restrict ourselves to the controls of the form $u = f_{\omega_0}(\omega, \mathbf{v}) = f_{\omega_0}(\omega)$, since the availability of the fluid velocity field information is not a realistic assumption whereas the angular velocity of a rigid body can be measured accurately. We approach this control problem in the same spirit we treated the stability problem. We try to exploit the mechanical nature of the problem in order to develop a control strategy. We will proceed in three steps to develop a feedback control law which stabilizes any rotation around a given axis. First, we start with a class of controls which do not destroy the Hamiltonian nature of the dynamics. Then, we further restrict this class of controls such that a given rigid rotation $\omega_0 \in \mathbb{R}^3, \mathbf{v} = 0$ is an equilibrium of the dynamics given by (4.101), (4.102). In the last step, we will determine the stabilizing control laws among the class which makes the rigid rotation an equilibrium, by using an energy-casimir stability analysis.

Step 1 We recall the interrelation between the momentum and velocity variables:

$$q = I_T \omega + \mathcal{K}(\rho_F \mathbf{v})$$

$$\mathbf{m} = \rho_F \mathcal{K}^*(\omega) + \rho_F \mathbf{v}.$$

The velocity variables are given in terms of the momentum variables as

$$\begin{aligned}\omega &= I_B^{-1} q - I_B^{-1} \mathcal{K}(\mathbf{m}) \\ \mathbf{v} &= \frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1} q + \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m})\end{aligned}$$

Then, (4.101), (4.102) can be represented in a hybrid form as:

$$\dot{q} = q \times \omega + u \quad (4.103)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s \quad (4.104)$$

where $(q, \mathbf{m}) \in \mathcal{N} = \mathcal{T}(so(3) \times \mathcal{X}_d)$. We know that the free dynamics, i.e. $u = 0$, of the system is associated with the conserved energy

$$\begin{aligned}H(q, \mathbf{m}) &= \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K}(\mathbf{m}) \\ &+ \frac{1}{2} \int \frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} dx + \frac{1}{2} \int \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) dx.\end{aligned} \quad (4.105)$$

Then, it is easy to show that for the forced dynamics, the time rate of change of the energy is given by

$$\frac{dH}{dt} = \omega^T u$$

under the action of control u . It is obvious that, the energy H remains constant under the control inputs of the form $u(\omega) = G(\omega)\omega$ where $G(\omega) = -G^T(\omega)$. The conservation of energy does not necessarily imply that the the resulting closed loop system (4.103), (4.104) is Hamiltonian, but we will show that this is correct provided $G(\omega) = G$ is a constant skew-symmetric matrix. We define $g \in \mathfrak{R}^3$ via $\hat{g} = G$. Then, $u = G\omega = g \times \omega$. Futhermore, we can always write $u = g \times \omega - \dot{g}$ since $\dot{g} = 0$. In this notation, the controlled dynamics of a rigid body containing

ideal incompressible fluid is given by:

$$\dot{q} = q \times \omega + g \times \omega - \dot{g} \quad (4.106)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s \quad (4.107)$$

We define new variables $\tilde{q}$ and $\tilde{\mathbf{m}}$ as:

$$\tilde{q} = q + g \quad (4.108)$$

$$\tilde{\mathbf{m}} = \mathbf{m}. \quad (4.109)$$

In terms of these modified momentum variables (4.106), (4.107) can be expressed as:

$$\dot{\tilde{q}} = \tilde{q} \times \omega \quad (4.110)$$

$$\frac{\partial \tilde{\mathbf{m}}}{\partial t} = \mathbf{v} \times (\nabla \times \tilde{\mathbf{m}}) - \nabla s \quad (4.111)$$

where the relations between the velocity and modified momentum variables are given by

$$\omega = I_B^{-1} \tilde{q} - I_B^{-1} g - I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}}) \quad (4.112)$$

$$\mathbf{v} = \frac{\tilde{\mathbf{m}}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}}) - \mathcal{K}^* I_B^{-1} \tilde{q} + \mathcal{K}^* I_B^{-1} g. \quad (4.113)$$

The closed loop dynamics can be written solely in terms of the modified momentum variables as:

$$\dot{\tilde{q}} = \tilde{q} \times (I_B^{-1} \tilde{q} - I_B^{-1} g - I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}})) \quad (4.114)$$

$$\frac{\partial \tilde{\mathbf{m}}}{\partial t} = \left(\frac{\tilde{\mathbf{m}}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}}) - \mathcal{K}^* I_B^{-1} \tilde{q} + \mathcal{K}^* I_B^{-1} g \right) \times (\nabla \times \tilde{\mathbf{m}}) - \nabla s \quad (4.115)$$

These dynamical equations can be interpreted as a Hamiltonian system. The equations (4.114), (4.115) can be written as

$$(\dot{\tilde{q}}, \dot{\tilde{\mathbf{m}}}_t) = W_g(\tilde{q}, \tilde{\mathbf{m}}) dH_g$$

where W_g is a Poisson structure, and H_g is the energy of the controlled system which incorporates the effect of the control $u = g \times \omega$. The Poisson structure

$$W_g(\tilde{q}, \tilde{m}) = \begin{pmatrix} W_{\tilde{q}}(\tilde{q}) & 0 \\ 0 & W_{\tilde{m}}(\tilde{m}) \end{pmatrix}$$

is determined by the skew-symmetric linear operators

$$W_{\tilde{q}}(\tilde{q})a = \tilde{q} \times a$$

$$W_{\tilde{m}}(\tilde{m})\mathbf{b} = \mathbf{b} \times (\nabla \times \tilde{m}) - \nabla s$$

where $a \in so(3)$ and $\mathbf{b} \in \mathcal{X}_d$. The energy H_g is given by

$$\begin{aligned} H_g(\tilde{q}, \tilde{m}) &= \frac{1}{2} \tilde{q}^T I_B^{-1} \tilde{q} - \tilde{q}^T I_B^{-1} \mathcal{K}(\tilde{m}) + \frac{1}{2} \int \frac{1}{\rho_F} \tilde{m}^T \tilde{m} dx \\ &+ \frac{1}{2} \int \tilde{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\tilde{m}) dx - \tilde{q}^T I_B^{-1} g + \int \tilde{m}^T \mathcal{K}^* I_B^{-1} g dx \end{aligned}$$

We observe that, H_g differs from the energy of the free dynamics (4.105) only by a linear term in form. We also note that, the structure W_g is identical in form with the Poisson structure W of the rigid body-fluid bracket we defined in chapter 3, section 5. Particularly, we have $W_g(\tilde{q}, \tilde{m}) = W(\tilde{q}, \tilde{m})$. Therefore, W_g is a Poisson structure too. We calculate the functional derivatives of H_g :

$$\frac{\delta H_g}{\delta \tilde{q}} = I_B^{-1} \tilde{q} - I_B^{-1} g - I_B^{-1} \mathcal{K}(\tilde{m}) = \omega$$

$$\frac{\delta H_g}{\delta \tilde{m}} = \frac{\tilde{m}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\tilde{m}) - \mathcal{K}^* I_B^{-1} \tilde{q} + \mathcal{K}^* I_B^{-1} g = \mathbf{v}.$$

Therefore, it is easy to see that

$$\begin{pmatrix} \dot{\tilde{q}} \\ \tilde{m}_t \end{pmatrix} = \begin{pmatrix} W_{\tilde{q}}(\tilde{q}) & 0 \\ 0 & W_{\tilde{m}}(\tilde{m}) \end{pmatrix} \begin{pmatrix} \frac{\delta H_g}{\delta \tilde{q}} \\ \frac{\delta H_g}{\delta \tilde{m}} \end{pmatrix}$$

yields

$$\begin{aligned}\dot{\tilde{q}} &= \tilde{q} \times \omega \\ \frac{\partial \tilde{\mathbf{m}}}{\partial t} &= \mathbf{v} \times (\nabla \times \tilde{\mathbf{m}}) - \nabla s.\end{aligned}$$

which is equivalent to (4.114), (4.115). Therefore, our claim that the closed loop dynamics is hamiltonian is justified. As a direct result of the hamiltonian structure, H_g is constant along the solutions of (4.114), (4.115). We also note that $\|\tilde{q}\|$ is another constant of motion for the dynamics of the closed loop system.

Remark: Under the action of the proposed class of controls, i.e. $u = g \times \omega$, the dynamics of rigid bodies containing fluids preserve their conservative structure, but we had to redefine the dynamical variables as well as changed the energy of the system. We note that, the total angular momentum of the system q and the modified momentum variable $\tilde{q}$ differ only by a constant term g . Therefore, g can be interpreted as a momentum shift. Indeed, (4.106) has the form of equations of a gyrostat. Therefore, the controls of the form $u = g \times \omega$ can be implemented by placing a momentum wheel in the body and spinning it such that its angular momentum w.r.t. rigid body is g . Of course, this control can also be implemented by using external torque jets on the rigid body.

Step 2 We show that, it is possible to assign an arbitrary rigid rotation as an equilibrium point of the controlled system.

Claim: Let $\omega_0 \in \mathbb{R}^3$ be a given angular velocity vector. Let g_{ω_0} be chosen s.t. $g_{\omega_0} = (a\mathbf{1} - I_T)\omega_0, a \in \mathbb{R}$. Then, the control $u = g_{\omega_0} \times \omega$ makes $(\omega, \mathbf{v}) = (\omega_0, 0)$ an equilibrium point of the controlled system (4.101), (4.102).

Proof: Under the suggested control input, (4.101), (4.102) can be written as:

$$I_T \dot{\omega} = I_T \omega \times \omega + \mathcal{K}(\rho_F \mathbf{v}) \times \omega - \mathcal{K}(\rho_F \mathbf{v}_t) + (a\mathbf{1} - I_T)\omega_0 \times \omega$$

$$\rho_F \left(\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} + 2\omega \times \mathbf{v} + \dot{\omega} \times x \right) = -\nabla p - \nabla \left(\frac{1}{2} x^T \dot{\omega}^2 x \right).$$

First, we note that the right hand side of the second equation above vanishes provided $\mathbf{v} = 0$ and $\dot{\omega} = 0$. Then, by substitution it is easy to see that $(\omega, \mathbf{v}, \dot{\omega}, \mathbf{v}_t) = (\omega_0, 0, 0, 0)$ satisfies the equations above. Therefore, $(\omega, \mathbf{v}) = (\omega_0, 0)$ is an equilibrium point of (4.101), (4.102). $\blacksquare$

We remind that (4.101), (4.102) are dynamically equivalent to (4.114), (4.115). Therefore, $\tilde{q}_e = I_T \omega_0 + g_{\omega_0}$, $\tilde{\mathbf{m}}_e = \rho_F \mathcal{K}^*(\omega_0)$ is an equilibrium of (4.114, 4.115).

Step 3 Up to this point, we have shown that under the feedback control $u = (a\mathbf{1} - I_T)\omega_0 \times \omega$, the closed loop system can be written as a Hamiltonian system which takes $\omega = \omega_0, \mathbf{v} = 0$ as an equilibrium point. Now, we show that it is possible to choose the free parameter $a \in \mathfrak{R}$ in the control law, such that this equilibrium is stable. We will do this by an energy-casimir analysis which will be very similiar to the one we have carried out in the previous section of this chapter.

We consider the function

$$V(\tilde{q}, \tilde{\mathbf{m}}) = H_g(\tilde{q}, \tilde{\mathbf{m}}) + \phi\left(\frac{1}{2}\|\tilde{q}\|^2\right)$$

which is conserved along the solutions of the controlled equation (4.114), (4.115) or equivalently (4.101), (4.102), because H_g and $\|\tilde{q}\|$ are constants of motion. We will show that the parameter a , which enters into V via H_g can be chosen such that V is a Lyapunov function. In other words, we are looking for the values of a which makes the first variation DV vanish at the given rigid rotation while making D^2V positive definite at this equilibrium.

First Variation We calculate functional derivatives of V :

$$\frac{\delta V}{\delta \tilde{q}} = I_B^{-1} \tilde{q} - I_B^{-1} g - I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}}) + \phi'\left(\frac{1}{2}\|\tilde{q}\|^2\right) \tilde{q}$$

$$\frac{\delta V}{\delta \tilde{\mathbf{m}}} = \frac{\tilde{\mathbf{m}}}{\rho_F} + \mathcal{K}^* I_B^{-1} \mathcal{K}(\tilde{\mathbf{m}}) - \mathcal{K}^* I_B^{-1} \tilde{q} + \mathcal{K}^* I_B^{-1} g.$$

Calculating these derivatives at a rigid rotation $(\omega, \mathbf{v}) = (\omega_0, 0)$ or equivalently at $(\tilde{q}, \tilde{\mathbf{m}}) = (I_T \omega_0 + g_{\omega_0}, \mathcal{K}^*(\rho_F \omega_0))$, and equating to zero, we get

$$\omega_0 + \phi'(\frac{1}{2}\|\tilde{q}_e\|^2)\tilde{q}_e = 0$$

$$\mathbf{v}_e = 0.$$

Substitution of $\tilde{q}_e = I_T \omega_0 + g_{\omega_0} = I_T \omega_0 + (a\mathbf{1} - I_T)\omega_0$ into the first equality yields

$$\omega_0 + \phi'(\frac{1}{2}\|\tilde{q}_e\|^2)a\omega_0 = 0.$$

Therefore, the fuction ϕ should be chosen such that

$$\phi'(\frac{1}{2}\|\tilde{q}_e\|^2) = -\frac{1}{a}$$

in order to comply with the first variation condition.

Second Variation As we observed before, the energy H (4.105) of the free dynamics of a rigid body containing perfect fluid differs from the energy H_g of the closed loop system in form only by a linear term. Since both H and H_g are quadratic, their second variations will be identical. By using this fact and by drawing from the derivations we performed in the previous section, the second variation D^2V can be written down by using a formal matrix representation as:

$$D^2V(\tilde{q}_e, \tilde{\mathbf{m}}_e) = \begin{pmatrix} I_B^{-1} - \frac{1}{a} + \phi''\tilde{q}_e\tilde{q}_e^T & -I_B^{-1}\mathcal{K} \\ -\mathcal{K}^*I_B^{-1} & \frac{\mathcal{I}}{\rho_F} + \mathcal{K}^*I_B^{-1}\mathcal{K} \end{pmatrix}$$

where we used the first variation condition $\phi' = -\frac{1}{a}$. As in the previous section, the boundedness of the operator D^2V follows from the boundedness of the operator $\mathcal{T}^{-1}$ provided $a \neq 0$ and ϕ'' is finite. We define

$$\tilde{\alpha} = \lambda_{\min}(I_B^{-1} - \frac{1}{a} + \phi''\tilde{q}_e\tilde{q}_e^T)$$

$$\tilde{\gamma} = \|\mathcal{K}^*I_B^{-1}\mathcal{K}\|.$$

Then, by using the exactly same arguments as in the previous section, it can be shown that

$$D^2V(\tilde{q}_e, \tilde{\mathbf{m}}_e)[(\delta\tilde{q}, \delta\tilde{\mathbf{m}}), (\delta\tilde{q}, \delta\tilde{\mathbf{m}})] \geq \beta\|(\delta\tilde{q}, \delta\tilde{\mathbf{m}})\|^2$$

for some $\beta > 0$ provided $\tilde{\alpha} > \rho_F \tilde{\gamma}^2 > 0$. In other words, the second variation condition on D^2V is satisfied provided

$$\lambda_{\min}(I_B^{-1} - \frac{1}{a} + \phi'' \tilde{q}_e \tilde{q}_e^T) > \rho_F \|I_B^{-1} \mathcal{K}\|^2. \quad (4.116)$$

We have the inequality

$$\lambda_{\min}(I_B^{-1} - \frac{1}{a} + \phi'' \tilde{q}_e \tilde{q}_e^T) \geq \lambda_{\min}(I_B^{-1} - \frac{1}{a}) \quad (4.117)$$

provided $\phi'' > 0$. Therefore, (4.116) is satisfied if $\lambda_{\min}(I_B^{-1} - \frac{1}{a}) > \rho_F \|I_B^{-1} \mathcal{K}\|^2$. Consequently, we have proved the following theorem which provides an answer to the velocity control problem we posed at the begining of this section.

Theorem 4.3 *Let $a \in \Re$ be chosen such that the inequality*

$$\lambda_{\min}(I_B^{-1} - \frac{1}{a}) > \rho_F \|I_B^{-1} \mathcal{K}\|^2.$$

is satisfied. Then, the linear control law

$$u(\omega) = (a\mathbf{1} - I_T)\omega_0 \times \omega$$

stabilizes the rigid rotation equilibrium $(\omega, \mathbf{v}) = (\omega_0, 0)$ of a rigid body containing ideal incompressible fluid (4.101), (4.102).

In order to understand the stabilizing effect of the proposed control law, we look at how it works if ω_0 is chosen as an eigenvector of I_T . This corresponds to the problem of stabilizing the rotations about a principal axis of the rigid

body-fluid system. Let $\omega_0 = \omega_e$, where $I_T \omega_e = \lambda_e \omega_e$. Then, the proposed control is given by

$$u(\omega) = (a\mathbf{1} - I_T)\omega_0 \times \omega = (a - \lambda_e)\omega_e \times \omega.$$

We define $d = a - \lambda_e$. Note that, $|d|$ is a measure of the gain of the controller. $|d| = 0$ corresponds to the case in which there is no controller. Large $|d|$ is associated with large gains, hence with large control efforts. The inequality (4.116), which assures the stability of the rigid rotations under the action of the controller can be written as

$$\lambda_{\min}(I_B^{-1} - \frac{\mathbf{1}}{\lambda_e + d} + \phi'' \tilde{q}_e \tilde{q}_e^T) > \rho_F \|I_B^{-1} \mathcal{K}\|^2. \quad (4.118)$$

by using $a = \lambda_e + d$. Note that if we take $d = 0$, then (4.118) becomes equivalent to the inequality condition of theorem 4.2 which assures the stability of an uncontrolled rigid rotation as expected. We also note that, the net effect of the proposed controller is to shift the value of λ_e which can be interpreted as the principal moment of inertia in direction ω_e . Hence, the controller changes the moment of inertia component associated with the rotation direction in order to stabilize a rotation. For a rigid body, i.e. $\rho_F = 0$, negative and large d satisfies (4.118) hence stabilizes the rotation. This means, in order to stabilize the rotation of a rigid body along a principal axis, we can place a momentum wheel in the rigid body which is aligned in the direction of rotation and spin it in the opposite direction with high speed. This method, known as the dual spin control is a widely used stabilization method for the rotations of spacecraft and had been studied in [42]. As is evident from our analysis, the same strategy basically works also for rigid bodies containing fluids. But in this case, the amount of spin should be chosen more carefully. By using theorem 4.116, it is easy to

show that if the controller parameter a lies in the range given by the inequality

$$0 > a > -(\rho_F \|I_B^{-1} \mathcal{K}\|^2)^{-1}$$

then the specified controller stabilizes the rotation of a rigid body-fluid system around any of the principal axes of the total mass distribution.

Remark: We developed a controller to stabilize the rotations of a rigid body-fluid system by applying a feedback which preserves the hamiltonian structure of the equations. Stability and control of rigid bodies in hamiltonian settings had been studied in various works including [42], [14], [17], [3], [90]. The underlying idea in all these studies is to solve the stabilization problems in a conservative setting. The main advantage of this approach is to be able to use stability methods like energy-casimir, energy-momentum, etc. to address the stabilization problems. In particular, [42] discusses the dual-spin problem for rigid spacecraft and interpretation of the system as Lie-Poisson equations. In [14] and [17] it has been shown that the rotations around the intermediate axis can be stabilized by applying a quadratic structure preserving torque feedback along the major or the minor axis. The stabilization of uniform rotations of a rigid body around an arbitrary axis is studied in [90] by working out the problem in an energy-momentum framework. Our solution to the velocity control of rigid bodies containing fluids is inspired by these studies. As opposed to the previous studies, which investigated finite dimensional dynamics of rigid bodies, here we solved an infinite dimensional stabilization problem for the rigid body-fluid interaction.

4.6.1 Effect of Viscosity

If the fluid in the cavity is viscous, then the dynamics of the controlled system can be written as

$$\dot{\tilde{q}} = \tilde{q} \times \omega \quad (4.119)$$

$$\frac{\partial \tilde{\mathbf{m}}}{\partial t} = \mathbf{v} \times (\nabla \times \tilde{\mathbf{m}}) - \nabla s + \mu \Delta \mathbf{v} \quad (4.120)$$

where μ is the viscosity coefficient of the fluid. We recall that, $\tilde{\mathbf{m}} = \mathbf{m}$. Therefore, as we have pointed out in chapter 3, (4.120) is equivalent to the Navier-Stokes equations in an accelerating reference frame. We also note that the only difference between (4.119), (4.120) and (4.110), (4.111) is the viscous term $\mu \Delta \mathbf{v}$. We know that $(\omega, \mathbf{v}) = (\omega_0, 0)$ is an equilibrium of (4.110), (4.111), therefore it is also an equilibrium of (4.119), (4.120) since if $\mathbf{v} = 0$ then the viscous term drops and the two equations becomes identical. We also recall that the inequality condition in theorem 4.3 only assures that V is a valid Lyapunov function satisfying the first and second variation conditions at a rigid rotation equilibrium. Conclusion of the theorem depends upon the fact that V is constant along the solutions. If the controller parameter a is chosen as it is specified by theorem 4.3, then the very same controller also works in the viscous case. Since this time V is not constant but non-increasing:

$$\begin{aligned} \dot{V} &= \dot{H}_g + \dot{\phi} = \dot{H}_g \\ \dot{V} &= \frac{\delta H_g^T}{\delta \tilde{q}} \dot{\tilde{q}} + \int_{\mathcal{F}} \frac{\delta H_g^T}{\delta \tilde{\mathbf{m}}} \tilde{\mathbf{m}}_t dx \\ &= \omega^T \dot{\tilde{q}} + \int_{\mathcal{F}} \mathbf{v}^T \tilde{\mathbf{m}}_t dx \\ &= \mu \int_{\mathcal{F}} \mathbf{v}^T \Delta \mathbf{v} dx \leq -\mu \lambda_1 \int_{\mathcal{F}} \mathbf{v}^T \mathbf{v} dx \end{aligned}$$

where λ_1 is the smallest eigenvalue of $-\Delta$. The viscosity coefficient μ and λ_1 are positive, hence $\dot{L} \leq 0$. Therefore, for the proposed controller the effect of viscosity is not destabilizing rather it helps to the stability of rotations. But, this is by no means a manifestation of the stabilizing effect of dissipation. Indeed, control and stability results developed in a conservative framework need not be robust w.r.t. dissipative perturbations. For an interesting discussion of dissipation induced instabilities, see [15].

4.7 Attitude Control Problem

The attitude control problem of rigid bodies received considerable attention in control theory circles. Some of these efforts are exemplified by the papers [63], [28], [23], [84], [85], [20], [81]. These works use various methods ranging from feedback linearization to Lyapunov techniques. On the other hand, the attitude control problem for a rigid body containing fluid has not been studied before. This is partially because of the absence of good models describing the dynamics of rigid body-fluid interaction. Having developed such models in this study, we can address the attitude control problem for a rigid body containing fluid. We will study the following problem in this section.

Attitude Control Problem: Consider the dynamics of a rigid body containing incompressible fluid. Let $Y \in SO(3)$ denote the orientation of the system w.r.t. an inertial reference frame. Let $Y_0 \in SO(3)$ be a given orientation. Develop a control method which forces $Y(t) \rightarrow Y_0$ as $t \rightarrow \infty$.

As in any control problem, we have two alternatives: open loop control (motion planning) and closed loop (feedback) control. In principle, motion planning

on the rotation group $SO(3)$ can be interpreted as a trivial problem. Since, the kinematics on $SO(3)$ is given by the equation $\dot{Y} = Y\Omega$ and Y is non-singular, any desired trajectory $Y_d(t)$ can be achieved by choosing $\Omega(t) = Y_d^T(t)\dot{Y}_d(t)$. Of course, this assumes that we have full control authority over the angular velocity ω . In the case of a rigid body containing fluid, the effect of the fluid motion on the rigid body manifests itself as a disturbance on the angular velocity. Therefore, it is not plausible to consider the angular velocity Ω as a means of control for the orientation of the rigid body-fluid system.

As far as the feedback control is concerned, the only feasible control input is the external torque applied on the rigid body. Furthermore, only the angular velocity ω and the attitude information Y could be used to manipulate the control torque, since only these are the dynamical variables which can be measured accurately.

As we have shown in chapter 3, the kinematic equation $\dot{Y} = Y\Omega$ drops from the dynamics of a rigid body containing fluid due to the dynamical symmetries inherent in the system. Now, we have to incorporate this kinematic equation back into the dynamics, since we try to control the orientation of the system. In this “symbolic” form, the equation $\dot{Y} = Y\Omega$ is of little help in order to develop a control method for the attitude control problem. In order to deal with the attitude control of a rigid body, we choose a specific representation for $SO(3)$. Our choice will be the Euler parameters, although any other parametrization could be used. In order to introduce Euler parameters, we first present a well-known theorem of rigid body kinematics.

Theorem 4.4 Euler *Let $C \in SO(3)$. Then, we can write C as*

$$C = \mathbf{1} + (1 - \cos\phi)\hat{\mathbf{a}}\hat{\mathbf{a}} - \sin\phi\hat{\mathbf{a}}$$

for some $\mathbf{a} \in \mathbf{S}^2$ and $\phi \in \mathbf{S}^1$.

Remark: In classical terms, the Euler theorem codifies the fact that any orientation of a rigid body (with a point fixed) can be obtained by rotating the rigid body along an axis passing through the fixed point. In geometric terms, the Euler theorem is related to the fact that $\mathbf{S}^2 = SO(3)/SO(2) = SO(3)/\mathbf{S}^1$. For more on Euler theorem and in general kinematics on $SO(3)$ see [61], [5].

Euler Parameters The Euler parameter (ϵ, η) associated with $C \in SO(3)$ are defined as

$$\epsilon = \mathbf{a}\sin\left(\frac{\phi}{2}\right) \quad (4.121)$$

$$\eta = \cos\left(\frac{\phi}{2}\right). \quad (4.122)$$

The rate of change of Euler parameters [38] are given by:

$$\dot{\epsilon} = \frac{1}{2}(\epsilon \times \omega + \eta\omega) \quad (4.123)$$

$$\dot{\eta} = -\frac{1}{2}\epsilon^T\omega \quad (4.124)$$

where $\hat{\omega} = \Omega$ is the angular velocity in body coordinates and $\Omega = C\dot{C}^T$. Here, C is the matrix of direction cosines which determines the orientation of the inertial reference frame w.r.t. the body frame. The rotation matrix Y , which in our notation gives the orientation of the body frame w.r.t. inertial frame is determined by $Y = C^T$. We note that, the attitude kinematics expressed in terms of the Euler parameters are linear. This linearity is an advantage from a computational viewpoint and lies behind the popularity of the Euler parameters in real-time applications. The price we have to pay for this linearity

is to have to use four parameters instead of the minimal three as in the Euler angles parametrization of $SO(3)$. However, the four Euler parameters cannot be totally arbitrary. From their definitions, it is easy to check that (ϵ, η) should satisfy the condition

$$\epsilon^T \epsilon + \eta^2 = 1. \quad (4.125)$$

In other words, Euler parameters lie on the sphere $\mathbf{S}^3$. We can easily check that, $\mathbf{S}^3$ is invariant under the dynamics defined by (4.123), (4.124).

By using the Euler parameters, we give the full attitude dynamics of a rigid body containing incompressible fluid as:

$$\dot{q} = q \times \omega + u \quad (4.126)$$

$$\frac{\partial \mathbf{m}}{\partial t} = \mathbf{v} \times (\nabla \times \mathbf{m}) - \nabla s + \mu \Delta \mathbf{v} \quad (4.127)$$

$$\dot{\epsilon} = \frac{1}{2}(\epsilon \times \omega + \eta \omega) \quad (4.128)$$

$$\dot{\eta} = -\frac{1}{2}\epsilon^T \omega \quad (4.129)$$

where u is the control torque acting on the rigid body and (ϵ, η) are the Euler parameters associated with the matrix of direction cosines of the inertial reference frame w.r.t. the body frame. Again, here $\mu \geq 0$ is the viscosity coefficient of the fluid. Without loss of generality, we will choose the inertial reference frame as the desired orientation, i.e. $Y_0 = \mathbf{1}$. We note that, the Euler parameters associated with the reference orientation $Y_0 = \mathbf{1}$ are $\epsilon_0 = 0$, $\eta_0 \in \{-1, 1\}$.

We will develop a feedback control law which forces the orientation of the system to approach to the reference orientation $Y_0 = \mathbf{1}$, by performing a Lyapunov stability analysis. We start with the following function which we will subsequently use as a Lyapunov function candidate:

$$L(q, \mathbf{m}, \epsilon, \eta) = H(q, \mathbf{m}) + \tilde{L}(\epsilon, \eta) \quad (4.130)$$

where $H : \mathcal{N} \rightarrow \mathfrak{R}$ is the energy of the rigid body containing incompressible fluid

$$H(q, \mathbf{m}) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \mathcal{K}(\mathbf{m}) \quad (4.131)$$

$$; ; + \frac{1}{2} \int \frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} dx + \frac{1}{2} \int \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) dx.$$

and $\tilde{L} : \mathbf{S}^3 \rightarrow \mathfrak{R}$ is given by

$$\tilde{L}(\epsilon, \eta) = \epsilon^T A \epsilon + c \eta^2. \quad (4.132)$$

Here, A is a 3×3 symmetric positive definite matrix with eigenvalues $\sigma_1 > \sigma_2 > \sigma_3 > 0$. The parameter $c \in \mathfrak{R}$ will be chosen s.t. $\sigma_3 > c > 0$. The importance of this choice will be evident in the sequel.

First, we investigate the points at which DL vanishes. We know that,

$$D_q H = \omega, \quad D_{\mathbf{m}} H = \mathbf{v}.$$

Therefore, $\omega = 0, \mathbf{v} = 0$ is necessary for DL to vanishes. In order to determine the set on which $D\tilde{L}$ vanishes, we consider the following extremum problem:

$$ext(\epsilon^T A \epsilon + c \eta^2)$$

subject to the constraint

$$\epsilon^T \epsilon + \eta^2 = 1.$$

By using a Lagrange multiplier λ , we write an equivalent extremum problem:

$$extM(\epsilon, \eta, \lambda) = ext(\epsilon^T A \epsilon + c \eta^2 - \lambda(\epsilon^T \epsilon + \eta^2 - 1)).$$

To solve this unconstrained extremum problem, we take the first partials and equate them to zero:

$$\frac{\partial M}{\partial \epsilon} = 2A\epsilon - 2\lambda\epsilon = 0 \quad (4.133)$$

$$\frac{\partial M}{\partial \eta} = 2c\eta - 2\lambda\eta = 0 \quad (4.134)$$

$$\frac{\partial M}{\partial \lambda} = -\epsilon^T \epsilon - \eta^2 + 1 = 0. \quad (4.135)$$

Equations (4.133), (4.134), (4.135) are equivalent to the eigenvalue problem

$$\begin{pmatrix} A & 0 \\ 0 & c \end{pmatrix} \begin{pmatrix} \epsilon \\ \eta \end{pmatrix} = \lambda \begin{pmatrix} \epsilon \\ \eta \end{pmatrix}$$

where (ϵ, η) is a normalized eigenvector, i.e. $(\epsilon, \eta) \in \mathbf{S}^3$. Let ϵ_i denotes a normalized eigenvector of A associated with the eigenvalue σ_i . Then, the solutions to this eigenvalue problem are given as

$$\lambda_1 = \sigma_1 \quad , \quad (\epsilon, \eta) = (\pm\epsilon_1, 0) \quad (4.136)$$

$$\lambda_2 = \sigma_2 \quad , \quad (\epsilon, \eta) = (\pm\epsilon_2, 0) \quad (4.137)$$

$$\lambda_3 = \sigma_3 \quad , \quad (\epsilon, \eta) = (\pm\epsilon_3, 0) \quad (4.138)$$

$$\lambda_4 = c \quad , \quad (\epsilon, \eta) = (0, \pm 1). \quad (4.139)$$

These eight points are the points on the Euler parameter space at which the first variation $D\tilde{L}$ vanishes. We calculate $\tilde{L}$ at these critical points:

$$(\epsilon, \eta) = (\pm\epsilon_i, 0) \Rightarrow \tilde{L}(\epsilon, \eta) = \sigma_i$$

$$(\epsilon, \eta) = (0, \pm 1) \Rightarrow \tilde{L}(\epsilon, \eta) = c$$

where $i \in \{1, 2, 3\}$. Recall that, we have assumed $\sigma_1 > \sigma_2 > \sigma_3 > c > 0$. Therefore, the critical points $(\epsilon, \eta) = (\pm\epsilon_1, 0)$ are the global maxima and $(\epsilon, \eta) = (0, \pm 1)$ are the global minima of the function $\tilde{L}$. The other four critical points are saddle points of $\tilde{L}$. We also recall that, the energy $H(q, \mathbf{m})$ is a quadratic function characterized by a positive definite operator. Hence, $(q = 0, \mathbf{m} = 0)$ is the global minimum of H . Therefore, the function $L = H + \tilde{L}$ assumes its global minima at the points

$$(q, \mathbf{m}, \epsilon, \eta) = (0, 0, 0, \pm 1).$$

Note that, these minima are associated with the reference orientation $Y_0 = \mathbf{1}$. We also note that, the function L assumes its strict minimum at these points since both H and $\tilde{L}$ are analytic functions.

Now, we consider the evolution of the Lyapunov function candidate L along the solution of (4.126), (4.127), (4.128), (4.129) :

$$\begin{aligned}\dot{L} &= \frac{\delta H^T}{\delta q} \dot{q} + \int \frac{\delta H^T}{\delta \mathbf{m}} \mathbf{m}_t dx + 2\dot{\epsilon}^T A \epsilon + 2c\dot{\eta}\eta \\ &= \omega^T u + \mu \int \mathbf{v}^T \Delta \mathbf{v} dx + \epsilon^T A (\epsilon \times \omega + \eta \omega) - c\eta \epsilon^T \omega \\ &= \omega^T (u + (\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon) + \mu \int \mathbf{v}^T \Delta \mathbf{v} dx\end{aligned}$$

where we used the previous calculations in chapter 3 in order to avoid duplication.

Let the control u be chosen as:

$$u = -(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon - M(\omega)\omega \quad (4.140)$$

where $M(\omega) = M^T(\omega) > k\mathbf{1}$ for some positive parameter k . We recall that

$$\int \mathbf{v}^T \Delta \mathbf{v} dx \leq -\lambda_1 \int \mathbf{v}^T \mathbf{v} dx$$

where $\lambda_1 > 0$ is the smallest eigenvalue of $-\Delta$. Then, we get

$$\frac{dL}{dt} \leq -k\omega^T \omega - \mu\lambda_1 \int \mathbf{v}^T \mathbf{v} dx \leq 0. \quad (4.141)$$

Therefore, $\dot{L}$ is non-increasing and we have proved the following local stability result.

Theorem 4.5 *The equilibrium points $(q, \mathbf{m}, \epsilon, \eta) = (0, 0, 0, \pm 1)$ of the attitude dynamics of a rigid body containing incompressible fluid (4.126), (4.127), (4.128), (4.129) are stable under the control law*

$$u = (\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon - M(\omega)\omega$$

provided $M(\omega) = M^T(\omega) > k\mathbf{1}$, $A = A^T > 0$, $\lambda_{\min}(A) > c$ and $k > 0$.

This results holds for both the viscous ($\mu > 0$) and the inviscid ($\mu = 0$) case. The controller specified in the theorem achieves much more than the content of the theorem. In order to understand the asymptotic behavior of the closed loop system, first we look at the inviscid fluid case. If $\mu = 0$, then (4.141) and the positiveness of function L implies

$$\lim_{t \rightarrow \infty} \omega = 0. \quad (4.142)$$

Now, by recalling that $I_T = I_B + I_F$ and $q = I_T \omega + \mathcal{K}(\mathbf{v})$, we can rewrite (4.126) as

$$I_B \dot{\omega} = I_B \omega \times \omega + T_f + T_c$$

where T_f and T_c are the torques exerted on the rigid body by the fluid and the control u respectively. These torques are given by

$$T_c = -(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon - M(\omega)\omega \quad (4.143)$$

$$T_f = I_F \omega \times \omega - I_F \dot{\omega} - \mathcal{K}(\rho_F \mathbf{v}_t) + \mathcal{K}(\rho_F \mathbf{v}) \times \omega. \quad (4.144)$$

Equation (4.142) implies that, the rigid body comes to a stop as $t \rightarrow \infty$. Therefore, the two torques on the rigid body should cancel each other:

$$T_f + T_c = 0$$

as time approaches infinity. This implies the equation

$$I_F \dot{\omega} + \mathcal{K}(\rho_F \mathbf{v}_t) = -(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon \quad (4.145)$$

should hold at the steady state reached at $t = \infty$. By recalling, $\mathbf{m} = \rho_F \mathbf{v} + \rho_F \mathcal{K}^*(\omega)$ and $I_F = \rho_F \mathcal{K} \mathcal{K}^*$, we can write (4.145) as

$$\mathcal{K}(\mathbf{m}_t) = -(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon \quad (4.146)$$

Let $d \in \mathbb{R}^3$ denotes the right hand side of (4.146). d will be a constant vector since $\dot{\epsilon} = 0$, $\dot{\eta} = 0$ at the steady state. Integrating both side of this equality w.r.t. time we obtain:

$$\mathcal{K}(\mathbf{m}) = dt + d_0.$$

Taking the norm of both sides, and squaring them we get:

$$\|\mathcal{K}(\mathbf{m})\|^2 = t^2 d^T d + 2td^T d_0 + d_0^T d_0.$$

Therefore, if $d \neq 0$ then

$$\lim_{t \rightarrow \infty} \|\mathcal{K}(\mathbf{m})\|^2 = \infty. \quad (4.147)$$

On the other hand, $\mathcal{K}$ is a bounded operator. Hence, (4.147) is satisfied iff $\lim_{t \rightarrow \infty} \|\mathbf{m}\| = \infty$. But, this contradicts with $\dot{L} \leq 0$. Therefore, d should be zero. And, we get the equations

$$\lim_{t \rightarrow \infty} (\dot{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = 0 \quad (4.148)$$

$$\lim_{t \rightarrow \infty} \mathcal{K}(\mathbf{m}_t) = 0. \quad (4.149)$$

which describes the asymptotics of the closed loop dynamics. Of these two, (4.149) is not very informative, since it is equivalent to

$$\lim_{t \rightarrow \infty} \mathcal{K}(\rho_F \mathbf{v}) = \text{constant}.$$

Indeed, this is nothing but the conservation of angular momentum of the fluid. Therefore, (4.149) doesn't reveal much about the asymptotics of the velocity field of the fluid. On the other hand, (4.148) implies that the orientation of the system approaches to the set characterized by the equation

$$(\dot{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = 0.$$

Indeed, this set is finite and coincides with the critical points of function $\tilde{L}$ (4.136), (4.137), (4.138), (4.139).

Claim: Let A and c be as specified in theorem 4.5. Let $(\epsilon, \eta) \in \mathbf{S}^3$. Then the equation $(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = 0$ is satisfied iff $D\tilde{L} = 0$ where $\tilde{L} = \epsilon^T A \epsilon + c\eta^2$.

Proof: We divide the solutions $(\epsilon, \eta) \in \mathbf{S}^3$ to $(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = 0$ into four disjoint sets:

$$\begin{aligned} s_1 &= \{\epsilon = 0, \eta = 0\} \\ s_2 &= \{\epsilon \neq 0, \eta = 0\} \\ s_3 &= \{\epsilon = 0, \eta \neq 0\} \\ s_4 &= \{\epsilon \neq 0, \eta \neq 0\} \end{aligned}$$

The set s_1 is empty since it does not lie in $\mathbf{S}^3$. Consider the solutions in the set s_4 . If a solution lies in s_4 , then $\epsilon \neq 0$ should be in the null space of the matrix

$$(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))$$

By multiplying this matrix from left and from right by ϵ , we get

$$\epsilon^T (\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = \epsilon^T \hat{\epsilon}^T A \epsilon + \eta \epsilon^T (A - c\mathbf{1})\epsilon = 0.$$

The first term on the right hand side is identically zero, since $\epsilon^T \hat{\epsilon}^T = 0$. On the other hand, $\eta \epsilon^T (A - c\mathbf{1})\epsilon$ cannot be zero since $\lambda_{\min}(A) > c$, $\eta \neq 0$ and $\epsilon \neq 0$. Therefore, no solution satisfying the equation can be in s_4 . Now, consider a solution in s_2 . Since $\eta = 0$ the equality is satisfied iff $\hat{\epsilon}^T A \epsilon = 0$. This in turn implies that ϵ should be an eigenvector of A . To satisfy the constraint, we should have $\epsilon^T \epsilon = 1$. We know that, such solutions satisfy $D\tilde{L} = 0$. Finally, any element in s_3 trivially satisfies the equality. To comply with the constraint, we

need $|\eta| = 1$. The solutions in the set s_4 satisfy $D\tilde{L} = 0$. Finally, we observe that any solution of $D\tilde{L} = 0$, i.e. (4.136), (4.137), (4.138), (4.139), either is in s_2 or in s_3 . This concludes the proof of the claim. $\blacksquare$

Therefore, by using the claim and (4.148), we get the following theorem.

Theorem 4.6 *Under the action of the controller described in theorem 4.5, the Euler parameters associated with the orientation of a rigid body containing ideal fluid approach to the finite set characterized by the equation $D\tilde{L}(\epsilon, \eta) = 0$. The eight points in this set are given by (4.136), (4.137), (4.138), (4.139).*

Remark: We have shown that the proposed controller asymptotically drives the Euler parameters to a finite set. As we have shown before, the reference orientation $Y_0 = \mathbf{1}$ is associated with two of these points. For the ideal fluid case, the controller doesn't cause the velocity field to fade, rather it cancels the disturbance of the fluid flow on the rigid body asymptotically in time. Note that this theorem does not claim anything about the asymptotics of the velocity field $\mathbf{v}$.

4.7.1 Effect of Viscosity

We take the case of viscous fluid ($\mu > 0$) and restrict our discussion to smooth ($\mathbf{v}$) solutions. Under this assumption, (4.141) implies

$$\lim_{t \rightarrow \infty} \omega = 0 \quad (4.150)$$

$$\lim_{t \rightarrow \infty} \mathbf{v} = 0. \quad (4.151)$$

By using our previous analysis, (4.150) implies that the Euler variables approach to the set

$$(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon = 0$$

as in the ideal fluid case. On the other hand, (4.150) together with (4.151) implies

$$\lim_{t \rightarrow \infty} q = 0, \quad \lim_{t \rightarrow \infty} \mathbf{m} = 0.$$

Therefore, by using the claim we proved before, we conclude that all (smooth) solutions of (4.126), (4.127), (4.128), (4.129) approach to one of the following equilibrium points

$$(q, \mathbf{m}, \epsilon, \eta) = (0, 0, \pm\epsilon_1, 0) \quad (4.152)$$

$$(q, \mathbf{m}, \epsilon, \eta) = (0, 0, \pm\epsilon_2, 0) \quad (4.153)$$

$$(q, \mathbf{m}, \epsilon, \eta) = (0, 0, \pm\epsilon_3, 0) \quad (4.154)$$

$$(q, \mathbf{m}, \epsilon, \eta) = (0, 0, 0, \pm 1) \quad (4.155)$$

As we have already shown before, (4.155) are stable equilibria for the ideal fluid case. For viscous case, (4.155) turns into locally asymptotically stable equilibria, since $\dot{L} = 0$ iff $\omega = 0$ and $\mathbf{v} = 0$ and the equilibria given above are isolated from each other. We have also shown that $(\epsilon, \eta) = (\pm\epsilon_1, 0)$ are the maxima of $\tilde{L}$, and $(\pm\epsilon_2, 0), (\pm\epsilon_3, 0)$ are saddle points. By recalling that $(q, \mathbf{m}) = (0, 0)$ is the minimum of the energy H and $L = H + \tilde{L}$, we conclude that equilibria given by (4.152), (4.153) and (4.154) are all saddle points of function L . Therefore, the second variation D^2L will be an indefinite operator calculated at each of these six equilibria. This observation, together with $\dot{L} \leq 0$ implies that (4.152), (4.153), (4.154) are unstable. We summarize these results as a theorem.

Theorem 4.7 *Consider the smooth solutions of a rigid body containing viscous fluid (4.126), (4.127), (4.128), (4.129) under the effect of the controller specified in theorem 4.5. Then, the solutions approach to one of the equilibria given by*

(4.152), (4.153), (4.154), (4.155) as time tends infinity. Among these equilibria, only (4.155) i.e. the equilibria associated with the reference orientation are stable. The other equilibria are unstable.

The orientation associated with the equilibria (4.155) is the reference orientation $Y = \mathbf{1}$. Therefore, the control law we proposed effectively solves the attitude control problem for a rigid body containing incompressible viscous fluid. However, we note that the stability of the reference orientation is not in a global sense. The solutions starting from the unstable equilibria stay there. The obstruction of the global asymptotic stability is related to the topology of the rotation group $SO(3)$. Before, clarifying this statement we will make some observations about the nature of the controller we proposed. The control law

$$u = -(\hat{\epsilon}^T A + \eta(A - c\mathbf{1}))\epsilon - M(\omega)\omega$$

shows some similarities to the dissipative control methods we introduced in section 3. The control u can be interpreted as the sum of two parts. The first part, which only involves the orientation variables (ϵ, η) , plays the role of a function which shapes the energy of the uncontrolled dynamics, such that the reference orientation corresponds to the global minimum of the shaped energy function (or the Lyapunov function). This term can also be interpreted as a direction command which forces the rigid body to rotate in a “correct direction” such that it approaches to the reference orientation. The second part which only involves the angular velocity of the rigid body is a dissipation term and responsible for the convergence of the solutions. Indeed, we could develop such control laws by using different parametrizations of the rotation group $SO(3)$. Better yet we give the following abstract framework, which could be used with any parametrization

of $SO(3)$. Let, $F(Y)$ be an analytic function on $SO(3)$ assuming its global minimum at $Y = I$. Then, consider $L = H + F$ as a candidate Lyapunov function. Under the effect of a control u , we get:

$$\dot{L} = \omega^T u - \mu \lambda_1 \|\mathbf{v}\|^2 + \text{tr}\left(\frac{\delta F^T}{\delta Y} Y \Omega\right)$$

where $\hat{\omega} = \Omega$. Note that, the last term in the right hand side is linear in $\hat{\omega} = \Omega$. Hence, we could write it as $f(Y)^T \omega$ for some vector valued function f on $SO(3)$ and obtain

$$\dot{L} = \omega^T (u + f(Y)) - \mu \lambda_1 \|\mathbf{v}\|^2.$$

Choose the control u as:

$$u = -f(Y) + d(\omega)$$

where $d(\omega)$ is a dissipative vector field, i.e.

$$\omega^T d(\omega) \leq -k \|\omega\|^2$$

for some $k > 0$. Therefore, we obtain

$$\dot{L} \leq -k \|\omega\|^2 - \mu \lambda_1 \|\mathbf{v}\|^2 \leq 0.$$

Then, we can show that the invariant set associated with $\omega = 0$, $\mathbf{v} = 0$ is given by $q = 0$, $\mathbf{m} = 0$, $f(Y) = 0$. Since the function F is chosen such that it assumes its global minimum at the reference orientation $Y_0 = \mathbf{1}$, the reference orientation lies in the above invariant set. However, the reference orientation cannot be the only solution of $f(Y) = 0$. This is related to the fact that any smooth function on the compact set $SO(3)$ assumes both its minima and maxima on $SO(3)$. Therefore, regardless of the choice for the smooth function F , the controller of the form given above cannot achieve global convergence to the reference orientation. However,

from a practical point of view this is not a major problem, since the points which do not converge to the reference orientation will not form a dense set.

Remark: As seen in this new light, our approach has similar flavor to the navigation function framework of Koditschek [40], [41]. Our Lyapunov function L is not exactly a navigation function in the sense of Koditschek. However, their functionality is similar in creating a function whose gradient shows a “correct direction” to move.

Remark: Some attitude control methods for rigid bodies which also use Euler parameters are [81], [84], [85]. As is evident from our derivation, the proposed controller also covers the pure rigid body case. We note that our control law does not involve any system parameter. Therefore, it has some inherent robustness w.r.t. the model parameters. Furthermore, the proposed controller is scalable, i.e. if u chosen as specified as in this section, then $\tilde{u} = \alpha u$ for any $\alpha > 0$ also works for the attitude control problem.

Chapter 5

Conclusions and Future Directions

In this dissertation, we have studied the dynamics, stability and control of spacecraft with fluid-filled containers. We made extensive use of the tools and approaches of geometric mechanics both in the modeling of the system and in addressing of the stability and control problems. We have studied the mechanical and geometric structure of the system in great detail, and used this structural information to develop stability and control results for spacecraft containing fluid.

In chapter 3, we have developed a unified model for the dynamics of a rigid body with fluid-filled cavities. We obtained the model by starting from the Euler- Lagrange equations followed by a hidden reduction process which divided out the dynamical symmetries of the system. Since our starting point was very elementary, we were able to derive the basic equations of the rigid body dynamics and fluid mechanics as special cases of the complete model for the rigid body-fluid system. The model is presented in three different but equivalent ways; in the velocity space, in the momentum space and in a hybrid form. Each of these representations has certain merits and we used them interchangeably through-

out the dissertation to study the dynamics and stability problems. We identified the Hamiltonian structure of the equations in the momentum space as an infinite dimensional Lie-Poisson system. Here, we have shown that the momentum space can be interpreted as in duality with the Lie algebra of the cartesian product of $so(3)$ with the space of incompressible velocity fields. We also identify the Euler-Poincare structure of the model. With this interpretation, we showed that our derivation of the model was indeed a Lagrangian reduction process. This constitutes the first application of the Lagrangian reduction process to a non-trivial infinite dimensional mechanical system. Also based on the momentum space representation of the model, we have generalized Bernoulli's equation for incompressible fluid flow to a Riemannian manifold setting. This generalized Bernoulli equation transparently shows the Hamiltonian structure of the dynamics and it is more general than Euler's equation for incompressible fluid flow on Riemannian manifolds. We incorporated viscosity into the conservative model by using the Laplacian operator, and showed that the motion of a rigid body containing a viscous fluid approaches to a rigid rotation asymptotically in time.

In chapter 4, we considered the stability and control of spacecraft with fluid-filled containers. Here, we made extensive use the mechanical structure of the model to study the dynamical problems. In the derivation of the stability result in section 5, the conservation of the energy and the magnitude of angular momentum of the system played an important role. By using an energy-casimir approach we showed that for a rigid body containing fluid, rotation about the short axis is stable provided fluid mass is sufficiently small. In section 6, we used the results of the previous section to develop a control method to stabilize the rigid rotation of the rigid body-fluid system along a given axis. Here, again

we used the conservative nature of the model and developed a linear feedback controller which doesn't destroy the conservative nature of the open loop dynamics. For the attitude control problem, which we considered in section 7, we used the Euler parameters to parametrize the rotation group $SO(3)$. We developed a feedback controller by shaping the energy function of the system with torque controls. Here, we used the shaped energy function as a Lyapunov function to derive the stability properties of the closed loop system. A key feature of the attitude controller we proposed is that it does not depend upon model parameters, hence it has an intrinsic robustness w.r.t. model parameters. In all the stability and control problems we considered in this chapter, the solutions were developed by using the conservative model of the rigid body-fluid system. Later, we showed that the proposed approaches are robust when we account for the viscosity of the fluid for each case.

This dissertation considered various aspects of the mechanics, geometry, stability and control of a single complex mechanical system. Although the approaches taken here, and the results obtained might seem special to a specific mechanical system, we point out the following directions for future work.

In this dissertation, we studied the dynamics and control of rigid bodies with fluid-filled containers. This can be interpreted as a special case of a rigid body with partially filled containers. Although a rigorous formulation of the liquid slosh problem is difficult due to free fluid surface, the geometric and mechanical tools we used in this study may prove useful for the partially filled case too.

Another direction for generalization is to generalize the equations for a rigid body containing fluid to an n -dimensional setting. This would be a coupled model of the generalized Bernoulli's equation with a generalized n -dimensional

gyrostat.

We remind that generalized Bernoulli's equation is more general than Euler's equation. In particular, Bernoulli's equation is capable of describing the dynamics of an incompressible fluid in rotating reference frames, hence it can be a starting point to geometrize the theory of rotating fluids [36].

Our model for a rigid body containing fluid is also relevant to the dynamics of celestial objects with non-rigid interiors (such as planet earth). Although such mechanical problems have been subject to some classical studies, the geometric approach we took in this study might be useful to reconsider this classical problem.

A direction which might be useful from an engineering point of view is the investigation of the effect of container shape on the stability of spacecraft. The stability result which we developed in chapter 4, section 5 for the rigid rotations of a spacecraft with fluid can be used to study the optimal shapes for containers which allows the maximum the amount of fluid that can be stored in a spacecraft without causing instability.

In chapter 4, section 6, our approach to the stability of rigid bodies containing fluid was the energy-Casimir method. One might also try the energy-momentum method [53], to address stability. Such an approach is, in principle, more general and can be used to study the exotic equilibria of the system which we define in chapter 3, section 4.

The approach we used to study the velocity control problem of a rigid body-fluid system could also prove useful for rigid bodies with flexible appendages. Such a study could generalize the results of [43], [86].

In the process of this study, we are convinced that the forces associated with

a rotating frame on a conservative system can always be incorporated into the dynamical equations without breaking the conservative nature of the equations. We also think that this might be a good way of geometrizing (generalizing) the equations of some physical phenomena.

In this study, there are some formal results which could be treated in a more rigorous sense. In chapter 4 section 4, we give Bernoulli's equation as an example of a Hamiltonian system with quadratic energy and casimir. As a result of a formal analysis, we showed that Beltrami flows cannot be stable equilibria of the Euler's equation. To our knowledge, this formal result is novel and certainly it deserves further investigation. We also think that Beltrami flows (which seem to be a forgotten notion in fluid mechanics) might be of some use in understanding turbulence in fluids.

Another part which deserves further study is the role of (infinite dimensional) positive real controllers for the stabilization of Lagrangian systems. Of course, the natural tool to study this framework in a rigorous sense is the theory of semi-groups for evolution equations.

In our consideration of the attitude control problem, we realized that the global control theory for systems defined on manifolds is far from being a well-established subject. However, we believe that the global control concepts on Lie groups might be an important special case for a more general global control theory on manifolds.

The pseudo models, which we introduced in the appendix, for rigid bodies containing fluids might be used as toy models to study the qualitative dynamics of fluids in rotating frames.

Appendix A

Pseudo Models for Rigid Bodies Containing Fluid

In this appendix, we develop a family of finite dimensional models which “approximate” in a qualitative sense the dynamics of a rigid body containing fluid. The dynamical equations of a rigid body containing ideal incompressible fluid, expressed in terms of the momentum space variables can be written as:

$$\dot{q} = W_q(q)(I_B^{-1}q - I_B^{-1}\mathcal{K}(\mathbf{m})) \quad (\text{A.1})$$

$$\frac{\partial \mathbf{m}}{\partial t} = W_{\mathbf{m}}(\mathbf{m})\left(\frac{\mathbf{m}}{\rho_F} - \mathcal{K}^* I_B^{-1}q + \mathcal{K}^* I_B^{-1}\mathcal{K}(\mathbf{m})\right) \quad (\text{A.2})$$

where q and $\mathbf{m}$ are the angular momentum of the total rigid body-fluid system and the momentum field of the fluid respectively. The relations between the momentum space variables $(q, \mathbf{m})$ and the velocity space variables $(\omega, \mathbf{v})$ are given by

$$q = I_T\omega + \mathcal{K}(\rho_F\mathbf{v}) \quad (\text{A.3})$$

$$\mathbf{m} = \rho_F\mathbf{v} + \rho_F\mathcal{K}^*(\omega). \quad (\text{A.4})$$

The operators $W_q(q)$ and $W_{\mathbf{m}}(\mathbf{m})$ appearing in (A.1), (A.2) are Poisson structures given by

$$W_q(q)a = q \times a \quad (\text{A.5})$$

$$W_{\mathbf{m}}(\mathbf{m})\mathbf{b} = \mathbf{b} \times (\nabla \times \mathbf{m}) - \nabla s \quad (\text{A.6})$$

where $a \in \mathfrak{R}^3 \cong so(3)$ and $\mathbf{b} \in \mathcal{X}_d$. As we have shown in chapter 3, we can write (A.1), (A.2) as a Hamiltonian system

$$\dot{z} = W(z)dH(z) \quad (\text{A.7})$$

where $z = (q, \mathbf{m}) \in \mathcal{N} = \mathcal{T}(so(3) \times \mathcal{X}_d)$. The Poisson structure W is given by

$$W(q, \mathbf{m}) = \begin{pmatrix} W_q(q) & 0 \\ 0 & W_{\mathbf{m}}(\mathbf{m}) \end{pmatrix}$$

and the Hamiltonian H is

$$H(q, \mathbf{m}) = \frac{1}{2}q^T I_B^{-1}q - q^T I_B^{-1}\mathcal{K}(\mathbf{m}) + \frac{1}{2} \int_{\mathcal{F}} \frac{1}{\rho_F} \mathbf{m}^T \mathbf{m} dx \quad (\text{A.8})$$

$$+ \frac{1}{2} \int_{\mathcal{F}} \mathbf{m}^T \mathcal{K}^* I_B^{-1} \mathcal{K}(\mathbf{m}) dx. \quad (\text{A.9})$$

We make the following observations about the structure of these equations.

- The momentum space variable $z = (q, \mathbf{m})$ lies in the linear space $\mathcal{N}$. This linear space can be interpreted as in duality with the Lie algebra $so(3) \times \mathcal{X}_d$ (see chapter 3, section 5).
- The Poisson structure $W(z)$ is linear in z and it can be interpreted as a Lie-Poisson structure on $\mathcal{N}$ (see chapter 3, section 5).
- The Hamiltonian H is a quadratic function on $\mathcal{N}$.

These are the features of the rigid body-fluid model which we would like to preserve in the finite dimensional approximations. In particular, we are interested to approximate (A.1), (A.2) with a finite dimensional Hamiltonian system

$$\dot{\tilde{z}} = \tilde{W}(\tilde{z})d\tilde{H}(\tilde{z}) \quad (\text{A.10})$$

where the following conditions are satisfied.

- The variable $\tilde{z}$ lies in a linear space which can be interpreted as $so^*(3) \times \mathcal{G}^*$ where $\mathcal{G}^*$ is the dual of a finite dimensional Lie algebra $\mathcal{G}$.
- $\tilde{W}(\tilde{z})$ is a Lie-Poisson structure on $so^*(3) \times \mathcal{G}^*$.
- The approximate Hamiltonian $\tilde{H}$ is a quadratic function on $so^* \times \mathcal{G}^*$.

In order to develop finite dimensional models, we consider the linear space of “divergence-free” covector fields defined on the cavity of the rigid body:

$$\Phi = \{\mathbf{m} \in \Lambda^1 \mid \delta \mathbf{m} = 0\}.$$

We remind that the co-differential operator δ acting on covector fields is analogous to the divergence operator acting on vector fields. In particular, $\delta \mathbf{m} = 0$ iff $\nabla \cdot \mathbf{m}^\sharp = 0$. The linear space Φ of divergence-free covector fields is a Lie algebra (see Lichnerowicz [47]) with the Lie bracket

$$[\mathbf{a}, \mathbf{b}] = \delta(\mathbf{a} \wedge \mathbf{b}).$$

Let $\{\phi_i\}$ be an orthonormal basis for Φ . Let $\mathbf{a}, \mathbf{b} \in \Phi$ be expanded as

$$\mathbf{a} = \sum_i a_i \phi_i \quad , \quad \mathbf{b} = \sum_i b_i \phi_i$$

where $a_i, b_i \in \mathfrak{R}$. Then, we have:

$$\begin{aligned} [\mathbf{a}, \mathbf{b}] &= \delta(\mathbf{a} \wedge \mathbf{b}) \\ &= \delta\left(\sum_i a_i \phi_i \wedge \sum_j b_j \phi_j\right) \\ &= \delta\left(\sum_i \sum_j a_i b_j \phi_i \wedge \phi_j\right). \end{aligned}$$

Define $S_{ij} \in \Lambda^2$ by $S_{ij} = \phi_i \wedge \phi_j$. Then, we get:

$$\begin{aligned} [\mathbf{a}, \mathbf{b}] &= \delta\left(\sum_i \sum_j a_i b_j S_{ij}\right) \\ &= \sum_i \sum_j a_i b_j \delta(S_{ij}). \end{aligned}$$

Note that, since $\delta^2 = 0$ we can write

$$\delta(S_{ij}) = \sum_k L_{ij}^k \phi_k$$

where $L_{ij}^k \in \mathfrak{R}$. We note that $L_{ij}^k = -L_{ji}^k$ due to the definition of S_{ij} . Here, we remind that ϕ_i is a one-form (covector), S_{ij} is a two-form and each L_{ij}^k is just a real number. We have:

$$\begin{aligned} [\mathbf{a}, \mathbf{b}] &= \sum_i \sum_j \sum_k L_{ij}^k a_i b_j \phi_k \\ &= \sum_k c_k \phi_k \end{aligned}$$

where $c_k = \sum_i \sum_j L_{ij}^k a_i b_j$ $k = 1, 2, 3, \dots$. Therefore, if $\{a_i\}$ and $\{b_i\}$ are the sequences formed by the expansion coefficients of $\mathbf{a}$ and $\mathbf{b}$ respectively, then we can define a Lie bracket $[\cdot, \cdot]_*$ on these sequences by:

$$[\{a_i\}, \{b_i\}]_* = \{c_i\}$$

where the components of the sequence $\{c_i\}$ are as defined above. Let τ be the linear map giving the expansion coefficients of a divergence-free covector field in

terms of the orthonormal basis $\{\phi_i\}$. Then, by construction τ is a Lie algebra isomorphism between Φ and the sequence space $\tau(\Phi)$, i.e.;

$$[\tau(\mathbf{a}), \tau(\mathbf{b})]_* = \tau(\delta(\mathbf{a} \wedge \mathbf{b})).$$

We note that, by construction of the Lie bracket $[\cdot, \cdot]_*$, the structure constants of the Lie algebra $\tau(\Phi)$ is given by L_{ij}^k . Now, we define an infinite column vector $\bar{m}$ as the sequence formed from the expansion coefficients of $\mathbf{m}$, i.e.;

$$\bar{m} = \{m_i\} = \tau(\mathbf{m}).$$

We also define $\bar{K} \in \mathfrak{R}^{3 \times \infty}$ by

$$\bar{K}(\bar{m}) = \mathcal{K}(\tau^{-1}(\bar{m}))$$

where $\mathcal{K}$ (see chapter 3, section 2) is given as

$$\mathcal{K}(\mathbf{m}) = \int_{\mathcal{F}} x \times \mathbf{m} dx.$$

Then, the operator $\bar{K}$ is determined by

$$\bar{K}_i = \int_{\mathcal{F}} x \times \phi_i(x) dx \quad i : 1, 2, 3, \dots$$

where $\bar{K}_i$ is the i-th column of $\bar{K}$.

We define $\bar{H}$ by $\bar{H}(q, \bar{m}) = H(q, \tau^{-1}(\bar{m}))$. Then, by using (A.8), the definition of $\bar{K}$ and the orthonormality of $\{\phi_i\}$, $\bar{H}$ can be written as

$$\bar{H}(q, \bar{m}) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} \bar{K}(\bar{m}) + \frac{1}{2} \frac{1}{\rho_F} \bar{m}^T \bar{m} + \frac{1}{2} \bar{m}^T \bar{K}^T I_B^{-1} \bar{K}(\bar{m}).$$

Furthermore, we can write (A.1), (A.2) in terms of $(q, \bar{m})$ as

$$\dot{q} = \bar{W}_q(q)(I_B^{-1} q - I_B^{-1} \bar{K}(\bar{m})) \tag{A.11}$$

$$\dot{\bar{m}} = \bar{W}_{\bar{m}}(\bar{m})\left(\frac{\bar{m}}{\rho_F} - \bar{K}^T I_B^{-1} q + \bar{K}^T I_B^{-1} \bar{K}(\bar{m})\right) \quad (\text{A.12})$$

where $\bar{W}_q = W_q$ is as given by (A.5) and $\bar{W}_{\bar{m}}$ is an skew-symmetric infinite matrix with the coefficients

$$(\bar{W}_{\bar{m}})_{ij}(\bar{m}) = \sum_k L_{ij}^k \bar{m}_k \quad i, j : 1, 2, 3, \dots$$

We note that, (A.11), (A.12) are only another representation of the full equation set (A.1), (A.2) in terms of the expansion coefficients $\bar{m}$ of the covector field $\mathbf{m}$. Now, in order to obtain a finite dimensional model, we will truncate the infinite vector $\bar{m}$. We define $m \in \Re^n$ as the column vector formed from the first n entries of $\bar{m}$. We also define $K \in \Re^{3 \times n}$ by truncating $\bar{K}$:

$$K_i = \bar{K}_i \quad i = 1, 2, 3, \dots, n$$

where K_i is the i -th column of K . Finally, we truncate $\bar{W}_{\bar{m}}$ and define an $n \times n$ matrix $\tilde{W}_m$ as

$$(\tilde{W}_m)_{ij}(m) = \sum_k L_{ij}^k m_k \quad i, j : 1, 2, 3, \dots, n.$$

We note that $\tilde{W}$ is a skew-symmetric matrix since L_{ij}^k is symmetric in i, j . We also note that although L_{ij}^k are the structure constants of the infinite dimensional Lie algebra of the divergence-free covector fields, the truncated version does not necessarily give the structure constants of a finite dimensional Lie algebra. Indeed, a necessary condition for this is the existence of a finite dimensional ideal of the Lie algebra Φ . In terms of the truncated structures, we form a finite dimensional approximate model for the rigid body-fluid system:

$$\dot{q} = \tilde{W}_q(q)(I_B^{-1} q - I_B^{-1} K(m)) \quad (\text{A.13})$$

$$\dot{m} = \tilde{W}_m(m)\left(\frac{m}{\rho_F} - K^T I_B^{-1} q + K^T I_B^{-1} K(m)\right) \quad (\text{A.14})$$

where $\tilde{W}_q(q) = W_q(q)$ and $\tilde{W}_m(m)$ is the skew-symmetric (not necessarily Poisson) structure on $\mathfrak{R}^n$ as defined above. We note that, the above approximate model retains most of the structure of the original equations, yet the Hamiltonian structure is lost due the fact that $\tilde{W}_m$ is not necessarily a Poisson structure. Here, we fix this deficiency in a way which involves some arbitrary choices, and call the resulting equations a “pseudo model”. The equations (A.13), (A.14) will be called a pseudo model for a rigid body containing ideal incompressible fluid if the model parameters are chosen according to the following points.

- Let I_B be as in the full model (A.1), (A.2).
- Let $\tilde{W}_q = W_q$ be the usual Lie-Poisson structure on $so(3)$ as it is given by (A.5).
- Pick a n -dimensional Lie algebra $\mathcal{G}$ with its structure constants c_{ij}^k .
- Form $\tilde{W}_m$ as $(\tilde{W}_m)_{ij}(m) = \sum_k c_{ij}^k m_k$. Note that, $\tilde{W}_m$ will be a Lie-Poisson structure on $\mathcal{G}^*$ since c_{ij}^k are the structure constants of $\mathcal{G}$.
- Given the inertia matrix I_F of the fluid mass, choose $K \in \mathfrak{R}^{3 \times n}$ s.t. $I_F = \rho_F K K^T$. This is essential to keep the consistency between the velocity space and momentum space representations of the pseudo models.

Then, with these choices (A.13), (A.14) becomes a finite dimensional Hamiltonian system with respect to the Lie-Poisson bracket

$$\tilde{W}(q, m) = \begin{pmatrix} \tilde{W}_q(q) & 0 \\ 0 & \tilde{W}_m(m) \end{pmatrix}$$

and the Hamiltonian $\tilde{H}$

$$\tilde{H}(q, m) = \frac{1}{2} q^T I_B^{-1} q - q^T I_B^{-1} K m + \frac{1}{2} \frac{1}{\rho_F} m^T m + \frac{1}{2} m^T K^T I_B^{-1} K m.$$

Remark: We emphasize that finite dimensional pseudo models characterize the rigid body dynamics in an exact way but the dynamics of the fluid motion is captured only in a qualitative sense. Yet, the essential features of the full dynamical system are preserved. For example, by construction our pseudo models retain the Hamiltonian nature of the original dynamics as well as the Lie-Poisson structure. Furthermore, with some work it can be checked that the equilibrium structure of the full model and the pseudo models are similar, despite the arbitrary choice of the Lie algebra $\mathcal{G}$. For example, pseudo models accept rigid rotations as equilibria and the stability status of the rigid rotations can be worked out by using almost identical steps as in chapter 4. Indeed, all control methods we developed in chapter 4 can be developed by relying on the pseudo models too. Of course, this is not a merit of the pseudo models we presented here, but a manifestation of the fact that our stability and control results in chapter 4 depends heavily upon the “structure” of the equations rather than the parameters appearing in the models.

Remark: In order to form a pseudo model, we make two arbitrary choices; for $\mathcal{G}$ and for K . Choosing a specific Lie algebra $\mathcal{G}$ means choosing $\mathcal{G}$ as the phase space for the fluid flow instead $\mathcal{X}_d$. If we choose $\mathcal{G} = so(3)$ then the resulting pseudo model (A.13), (A.14) becomes a Hamiltonian system on $so(3) \times so(3) \cong so(4)$. Indeed, such a model appears in a book of Fomenko [31] as a qualitative model for a rigid body containing incompressible fluid. Our pseudo models are more structured than the one given by Fomenko. His qualitative model is just a Lie-Poisson system on $so(4)$ with a generic quadratic Hamiltonian. However, here we know the exact form of the energy of a rigid body containing fluid, and we only approximated the parameters related to the fluid part of the model.

Remark: A similiar pseudo-modelling approach [22] exists for the dynamics of elastic bodies where the configuration manifold for an elastic body is taken as a finite dimensional manifold by choice. Lewis and Simo [46] considered such a pseudo-rigid body where they take $GL(3)$ as the configuration manifold of an elastic continuum, and they studied some stability problems by using the resulting pseudo model.

Remark: We also emphasize that the effect of viscosity can be incorporated in to the pseudo models easily. We can qualitatively approximate the Laplacian operator Δ with an n dimensional negative definite matrix R which we add to the right hand side of (A.14):

$$\dot{m} = \tilde{W}_m(m) \left(\frac{m}{\rho_F} - K^T I_B^{-1} q + K^T I_B^{-1} K(m) \right) + Rv.$$

Expressing this model in terms of the velocity field approximant

$$v = \frac{m}{\rho_F} + K^T I_B^{-1} q + K^T I_B^{-1} K m \in \mathfrak{R}^n$$

we get:

$$\rho_F \dot{v} = \tilde{W}_m(\rho_F v) v + \tilde{W}_m(\rho_F K^T(\omega)) v - \rho_F K^T(\dot{\omega}) + Rv.$$

This dynamical equation approximates the Navier-Stokes equations w.r.t. a reference frame rotating with angular velocity ω . Finite dimensional quadratic equations with linear dissipative terms have been studied as finite dimensional approximations to Navier-Stokes equations in [18], [32], [33]. The finite dimensional models in these works have similar structures (quadratic conservative systems with linear dissipative terms) and they can be represented in the above form if we take $\omega = 0$, $\dot{\omega} = 0$.

Bibliography

- [1] R. Abraham and J. E. Marsden. *Foundation of Mechanics*. Addison-Wesley, 2nd edition, 1985.
- [2] R. Abraham, J. E. Marsden, and T. Ratiu. *Manifolds, Tensor Analysis and Applications*, volume 75 of *Applied Mathematical Sciences*. Springer-Verlag, 2nd edition, 1988.
- [3] D. Aeyels. On stabilization by means of the energy-casimir method. *Syst. Cont. Lett.*, 18:325–328, 1992.
- [4] B. N. Agrawal. *Design of Geosynchronous Spacecraft*. Prentice-Hall, 1986.
- [5] S. L. Altmann. *Rotations, Quaternions and Double Groups*. Oxford University Press, 1986.
- [6] B. D. O. Anderson. A system theory criterion for positive real matrices. *J. SIAM Control*, 5(2):171–182, 1967.
- [7] V. I. Arnold. Conditions for nonlinear stability of stationary plane curvilinear flows of an ideal fluid. *Soviet Math. Doklady*, 5:773–777, 1965.
- [8] V. I. Arnold. *Mathematical Methods of Classical Mechanics*, volume 60 of *Graduate Text in Mathematics*. Springer-Verlag, 2nd edition, 1989.

- [9] A. M. Arthurs. *Complementary Variational Principles*. Oxford University Press, 1980.
- [10] A. Avez. *Differential Calculus*. John Wiley and Sons, 1986.
- [11] J. Baillieul and M. Levi. Rotational elastic dynamics. *Physica D*, 27:43–62, 1987.
- [12] D. Bao and T. Ratiu. On the geometrical origins and the solutions of a degenerate Monge-Ampere equation. In *Proceedings of Symposia in Pure Mathematics*, volume 54, pages 55–68. AMS, 1993.
- [13] P. Blanchard and E. Bruning. *Variational Principles in Mathematical Physics: A Unified Approach*. Springer-Verlag, 1992.
- [14] A. Bloch and J. E. Marsden. Stabilization of rigid body dynamics by the energy-casimir method. *Syst. Cont. Lett.*, 14:341–346, 1990.
- [15] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu. Dissipation induced instabilities. *Ann. Inst. Henri Poincare*, 11(1):37–90, 1994.
- [16] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and T. S. Ratiu. The Euler-Poincare equation and double bracket dissipation. Preprint, 1994.
- [17] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and G. Sanchez de Alvarez. Stabilization of rigid body dynamics by internal and external torques. *Automatica*, 28(4):745–756, 1992.
- [18] C. Boldrighini and V. Franceschini. A five-mode truncation of the plane Navier-Stokes equation. *Comm. Math. Phys.*, 64:159–170, 1979.

- [19] J. Bontsema, R. F. Curtain, and J. M. Schumacher. Robust control of flexible structures: a case study. *Automatica*, 24(2):177–186, 1988.
- [20] C. I. Byrnes and A. Isidori. On the attitude stabilization of rigid spacecraft. *Automatica*, 27(1):87–95, 1991.
- [21] A. J. Chorin and J. E. Marsden. *An Introduction to Fluid Dynamics*, volume 4 of *Texts in Applied Mathematics*. Springer-Verlag, 2nd edition, 1990.
- [22] H. Cohen. *The Theory of Pseudo-Rigid Bodies*, volume 33 of *Springer Tracts in Natural Philosophy*. Springer-Verlag, 1988.
- [23] P. E. Crouch. Spacecraft attitude control and stabilization. *IEEE Trans. Autom. Cont.*, 29(4):321–331, 1984.
- [24] R. Curtain and H. Zwart. An introduction to infinite dimensional linear systems theory. Book in preperation, 1992.
- [25] M. L. Curtis. *Matrix Groups*. Springer-Verlag, 2nd edition, 1984.
- [26] A. J. Van der Schaft. Stabilization of hamiltonian systems. *Nonlinear Analysis, Theory, Methods and Applications*, 10(10):1021–1035, 1986.
- [27] P. DuChateau and D. Zachmann. *Partial Differential Equations*. McGraw-Hill, 1986.
- [28] T. A. W. Dwyer. Exact nonlinear control of large angle rotational maneuvers. *IEEE Trans. Autom. Cont.*, 29(9), 1984.
- [29] D. G. Ebin and J. E. Marsden. Group of diffeomorphisms and the motion of an incompressible fluid. *Ann. Math.*, 92:102–163, 1970.

- [30] H. Flanders. *Differential Forms with Applications to Physical Problems*. Dover, 1989.
- [31] A. T. Fomenko. *Integrability and Nonintegrability in Geometry and Mechanics*. Kluwer Academic Publishers, 1988.
- [32] V. Franceschini and C. Tebaldi. Sequences of infinite bifurcations and turbulence in a five-mode truncation of Navier-Stokes equation. *Journal of Statistical Physics*, 21(6):707–726, 1979.
- [33] V. Franceschini and C. Tebaldi. A seven-mode truncation of the plane incompressible Navier-Stokes equation. *Journal of Statistical Physics*, 25(3):397–417, 1981.
- [34] H. Goldstein. *Classical Mechanics*. Addison-Wesley, 2nd edition, 1980.
- [35] A. L. Greensite. *Analysis and Design of Spacecraft Vehicle Flight Control Systems*. Spartan Books, 1970.
- [36] H. P. Greenspan. *The Theory of Rotating Fluids*. Cambridge University Press, 1968.
- [37] D. D. Holm, J. E. Marsden, T. Ratiu, and A. Weinstein. Nonlinear stability fluid and plasma equilibria. *Physics Reports*, 123(1,2):1–116, 1985.
- [38] P. C. Hughes. *Spacecraft Attitude Dynamics*. John Wiley and Sons, 1986.
- [39] S. Joshi and P. Maghami. Dissipative compensators for flexible spacecraft. In *Proceedings of American Control Conference*, pages 1955–1961, 1990.

- [40] D. E. Koditschek. The control of natural motion in mechanical systems. *ASME Journal of Dynamical Systems, Measurement and Control*, 113:547–551, 1991.
- [41] D. E. Koditschek. Some applications of natural motion control. *ASME Journal of Dynamical Systems, Measurement and Control*, 113:552–557, December 1991.
- [42] P. S. Krishnaprasad. Lie-Poisson structures, dual-spin spacecraft and asymptotic stability. *Nonlinear Analysis, Theory, Methods and Applications*, 9(10):1011–1035, 1985.
- [43] P. S. Krishnaprasad and J. E. Marsden. Hamiltonian structures and stability for rigid bodies with flexible attachments. *Arch. Rational Mech. Anal.*, 98:71–93, 1987.
- [44] O. A. Ladyzhenskaya. *The Mathematical Theory of Viscous Incompressible Flow*. Gordon and Breach, 1963.
- [45] E. Leimanis. *The General Problem of the Motion of Coupled Rigid Bodies About a Fixed Point*. Springer-Verlag, 1965.
- [46] D. Lewis and J. C. Simo. Nonlinear stability of rotating psudeo-rigid bodies. *Proc. Roy. Soc. Lond., A* 427:281–319, 1990.
- [47] A. Lichnerowicz. Differential geometry and deformations. In A. Crumeyrolle and J. Grifone, editors, *Symplectic Geometry*, pages 107–121. Pitman Advanced Pub., 1983.

- [48] P. Likins. Spacecraft attitude dynamics and control - a personal perspective on early developments. *J. Guidance, Control and Dynamics*, 9(2):129–134, March-April 1986.
- [49] J. H. Maddocks. On the stability of relative equilibria. *IMA Journal of Applied Mathematics*, 46:71–99, 1991.
- [50] J. H. Maddocks and R. L. Sachs. Constrained variational principles and stability in hamiltonian systems. Preprint, 1994.
- [51] J. Marsden and A. Weinstein. Reduction of symplectic manifolds with symmetry. *Reports on Mathematical Physics*, 5(1):121–130, 1974.
- [52] J. E. Marsden. *Lectures on Geometric Methods in Mathematical Physics*, volume 37 of *CBMS-NSF*. SIAM, 1981.
- [53] J. E. Marsden. *Lectures on Mechanics*, volume 174 of *London Mathematical Society Lecture Notes*. Cambridge University Press, 1992.
- [54] J. E. Marsden and T. Ratiu. Reduction of Poisson manifolds. *Letters in Mathematical Physics*, 11:161–169, 1986.
- [55] J. E. Marsden, T. Ratiu, and A. Weinstein. Reduction and hamiltonian structures on duals of semidirect product Lie algebras. *Contemporary Mathematics*, 28, 1984.
- [56] J. E. Marsden and T. S. Ratiu. *Mechanics and Symmetry*. Springer-Verlag, 1994. To be published.
- [57] J. E. Marsden and J. Scheurle. Lagrangian reduction and the double pendulum. *ZAMP*, 44:17–43, 1993.

- [58] J. E. Marsden and J. Scheurle. The reduced Euler-Lagrange equations. Preprint, 1993.
- [59] G. E. Mase. *Continuum Mechanics*. McGraw-Hill, 1970.
- [60] N. Matsumato and S.I. Takahashi. System-theoretical representations of linear passive networks. Preprint, 1992.
- [61] J. M. McCarthy. *An Introduction to Theoretical Kinematics*. MIT Press, 1990.
- [62] J. P. McKelvey. The case of the curious curl. *Am. J. Phys.*, 58(4):306–310, April 1990.
- [63] G. Meyer. Design and global analysis of spacecraft attitude control systems. Technical Report TR R-361, NASA, 1971.
- [64] H. K. Moffatt. The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics*, 35(1):117–129, 1969.
- [65] N. N. Moiseyev and V. V. Rumyantsev. *Dynamic Stability of Bodies Containing Fluid*. Springer-Verlag, 1968.
- [66] Ö. Morgül. On the stabilization of the wave equation. In R. F. Curtain, editor, *Analysis and Optimization of Systems: State Space and Frequency Domain Approaches for Infinite Dimensional Systems*, pages 531–542. Springer-Verlag, 1993.
- [67] P. J. Olver. A nonlinear hamiltonian structure for the Euler equation. *JMA*, 89:233–250, 1982.

- [68] P. J. Olver. *Applications of Lie Groups to Differential Equations*, volume 107 of *Graduate Text in Mathematics*. Springer-Verlag, 1986.
- [69] B. O’neill. *Semi-Riemannian Geometry*. Academic Press, 1983.
- [70] V. Penna and M. Spera. On coadjoint orbits of rotational perfect fluids. *J. Math. Phys.*, 33(3):901–909, March 1992.
- [71] T. A. Posbergh, P. S. Krishnaprasad, and J. E. Marsden. Stability analysis of a rigid body with a flexible attachment using the energy-casimir method. *Contemporary Mathematics*, 68:253, 1987.
- [72] T. Ratiu. The motion of the free n -dimensional rigid body. *Indiana University Mathematics Journal*, 29(4):609–629, 1980.
- [73] L. A. Segel. *Mathematics Applied to Continuum Mechanics*. Dover, 1987.
- [74] J. C. Simo, D. Lewis, and J. E. Marsden. Stability of relative equilibria part 1: Reduced energy-momentum method. *Arch. Rational. Mech. Anal.*, 115:15–59, 1991.
- [75] J. C. Simo, T. A. Posbergh, and J. E. Marsden. Stability of relative equilibria part 2: Application to nonlinear elasticity. *Arch. Rational Mech. Anal.*, 115:61–100, 1991.
- [76] M. Slemrod. Feedback stabilization of a linear control system in Hilbert space with an a priori bounded control. *Math. Control Signals Systems*, 2:265–285, 1989.

- [77] G. K. Vallis, G. F. Carnevale, and T. G. Shepherd. A natural method for finding stable states of hamiltonian systems. In *Topological Fluid Dynamics, 1989 IUTAM Proceedings*, pages 429–439, 1989.
- [78] G. K. Vallis, G. F. Carnevale, and W. R. Young. Extremal energy properties and construction of stable solutions of the Euler equations. *Journal of Fluid Mechanics*, 207:133–152, 1989.
- [79] L. S. Wang and P. S. Krishnaprasad. Gyroscopic stabilization and control. *J. Nonlinear Sci.*, 2:367–415, 1992.
- [80] A. Weinstein. The local structure of poisson manifolds. *J. Differential Geometry*, 18:523–557, 1983.
- [81] J. T.-Y. Wen and K. Kreutz-Delgado. The attitude control problem. *IEEE AC Trans.*, 36(10):1148–1162, October 1991.
- [82] J. R. Wertz. *Spacecraft Attitude Determination and Control*. D. Reidel Publishing Company, Holland, 1985.
- [83] J. R. Wertz and W. J. Larson, editors. *Space Mission Analysis and Design*. Kluwer Academic Publishers, Holland, 1991.
- [84] B. Wie and P. M. Barba. Quaternion feedback for spacecraft large angle maneuvers. *J. Guidance, Control and Dynamics*, 8(3):360–365, 1985.
- [85] B. Wie, H. Weiss, and A. Arapostathis. Quaternion feedback regulator for spacecraft eigenaxis rotations. *J. Guidance, Control and Dynamics*, 12(3):375–380, 1989.

- [86] C. Z. Xu and J. Baillieul. Stabilizability and stabilization of a rotating body-beam system with torque control. *IEEE AC Trans.*, 38(12):1754–1765, December 1993.
- [87] R. Yang. *Nonholonomic Geometry, Mechanics and Control*. PhD thesis, University of Maryland, College Park, 1992.
- [88] Z. Yoshida. Eigenfunction expansions associated with the curl derivatives in cylindrical geometries: completeness of Chandrasekhar-Kendall eigenfunctions. *J. Math. Phys.*, 33(4):1252–1256, April 1992.
- [89] Z. Yoshida and Y. Giga. Remarks on spectra of operator rot. *Math. Z.*, 204:235–245, 1990.
- [90] R. Zhao and T. A. Posbergh. Feedback stabilization of uniform rigid body motion. *Syst. Cont. Lett.*, 22:39–45, 1994.