

Discrimination Against  
Partially Overlapping Interference--  
Its Effect on Throughput in  
Frequency-Hopped Multiple Access Channels

by

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# Discrimination Against Partially Overlapping Interference—Its Effect on Throughput in Frequency-Hopped Multiple Access Channels

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**Abstract**—In this paper we derive the probability of correct packet reception and the resulting channel throughput achievable in an asynchronous slow-frequency-hopped multiple user channel. Reed-Solomon coding is used to correct errors caused by other-user interference in an otherwise noiseless channel. We analyze and evaluate an  $M$ -ary FSK signaling scheme, which permits the discrimination against interfering signals that are present for a sufficiently small fraction of the hop duration, and results in substantial increases in channel throughput over previous models.

## I. INTRODUCTION

THE number of users that can share a wide-band channel simultaneously by means of code division multiple access (CDMA) techniques and the resulting performance depend on the modulation/coding scheme, channel characteristics, and receiver implementation. In this paper we consider frequency-hopping (FH) spread-spectrum multiple access systems in which Reed-Solomon coding is used to correct burst errors caused by other-user interference in a packet-switched environment. We extend the model of Pursley and Hajek [1]–[4], and evaluate a signaling scheme that, for the case of a noiseless channel, provides considerable improvement in channel throughput.

Specifically, we present and analyze a model (originally discussed in [5], [6]) for other-user interference in which hops can be correctly received, despite the partial overlap by other users' signals, provided that the overlap is a sufficiently small fraction of the hop duration. This ability to discriminate against interfering signals results in substantial performance improvement. We discuss how such a scheme might be implemented, and we calculate the performance improvement.

## II. THE BASIC MODEL

We consider a wide-band frequency-hopping (FH) channel that consists of  $q$  orthogonal narrow-band frequency bins. We assume the use of noncoherent  $M$ -ary FSK; each frequency bin thus consists of  $M$  orthogonal tone positions. Each user of the channel transmits one fixed length string of symbols, called a

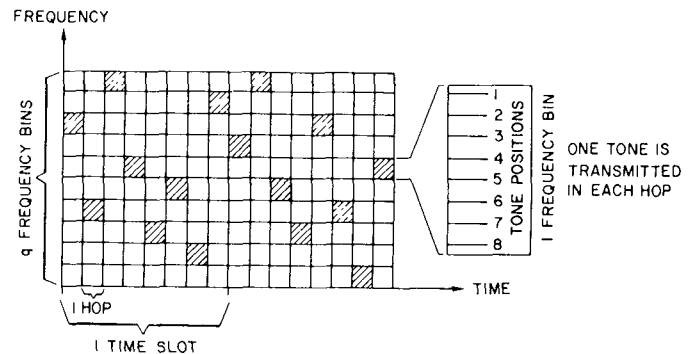


Fig. 1. Frequency-hopping channel with one  $M$ -ary FSK symbol transmitted per hop and one packet (codeword) transmitted per time slot (example shown for  $M = 8$ ).

packet, in each time slot. Each symbol, consisting of a single tone representing  $\log_2 M$  bits, is transmitted in one hop. Fig. 1 illustrates this signaling scheme. The frequency-hopping patterns are assumed to be such that each frequency bin is different from the previous one, but equally likely to be any one of the  $q - 1$  other frequencies. We assume that perfect synchronization is maintained between transmitter and receiver. Furthermore, we assume a noiseless channel model in which interference is caused only by other users.

CDMA operation is usually asynchronous at the hop level, and therefore it is possible for two or more users (even when they use orthogonal hopping patterns) to transmit simultaneously in the same frequency bin, resulting in loss of data. Such collisions are known as "hits." These hits generally result in *partial*, rather than total, overlap among tones of different users; it is assumed that the degree of overlap experienced by any pair of hops that suffer a hit is uniformly distributed over the interval from 0 to the total hop duration value. It is usually assumed that any degree of hop overlap results in the loss of the information carried in that hop. In this paper we make a departure from this assumption.

We address only the case of a fixed number of users that continuously transmit over the channel. Each user transmits one packet in each time slot. Each packet consists of a Reed-Solomon (RS) codeword of  $n = M - 1$   $M$ -ary symbols, one of which is transmitted per hop. An RS- $(n, v)$  code is capable of correcting any pattern of  $t = (n - v)/2$  symbol errors in a codeword (packet).

Considerable improvement results when the received symbols that have been affected by hits are detected and erased. In the general case in which both errors and erasures occur, a correct codeword decision can be made as long as the number of symbol erasures plus twice the number of symbol errors is not greater than  $2t$ . The detection of hits is straightforward in

Paper approved by the Editor for Computer Communication of the IEEE Communications Society. Manuscript received August 3, 1983; revised June 25, 1985. This paper was presented in part at the Fifth M.I.T./ONR Workshop on C<sup>3</sup> Systems, Monterey, CA, August 1982, the 17th Annual Conference on Information Science and Systems, Johns Hopkins University, Baltimore, MD, March 1983, the IEEE International Symposium on Information Theory, St. Jovite, P.Q., Canada, September 1983, and the 21st URSI General Assembly, Florence, Italy, September 1984.

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IEEE Log Number 8406882.

noiseless channels with  $M$ -ary FSK signaling. The presence of energy in more than one tone position at the same time indicates the occurrence of a hit, and thus, all hits can be detected and erased.

We proceed to calculate the throughput performance of such an FH multiple user system. Consider a particular fixed user who transmits a packet in a given slot. Let there be  $k$  additional active users during that slot. We define

$$\Pr [E|k] = \Pr [\text{the packet of the fixed user is not correctly received} \mid \text{given that there are } k \text{ other users on the channel}].$$

Then the conditional throughput, given a total of  $(k + 1)$  users, can be evaluated as

$$S_{k+1} = (k + 1)(1 - \Pr [E|k]) \quad (1)$$

or, if we want to normalize with respect to the frequency bin bandwidth,

$$s_{k+1} = (1/q)S_{k+1} \quad (2)$$

which is the expected number of successful packets per time slot and frequency bin. Since each of the  $k + 1$  users transmits a packet in every time slot, the only randomness that arises in this model results from the pseudorandom nature and asynchronous timing of the FH patterns.<sup>1</sup> Also, we assume that the tone positions in each bin are orthogonally spaced; i.e., their spectral distance is the reciprocal of the symbol (hop) duration. If the tones are closer together, it is more difficult to discriminate between adjacent tone positions. Therefore, the minimum duration of a tone (i.e., the hop duration or hop dwell time) that is consistent with the orthogonal tone spacing requirement is uniquely determined by the tone spacing. If the dwell time is indeed equal to the minimum possible value that ensures tone orthogonality, we shall say that the dwell time is *matched* to the tone spacing.

Fundamental to the evaluation of  $\Pr [E|k]$  is the evaluation of the symbol error probability. We define

$$p_k = \Pr [\text{a given symbol is not received correctly} \mid \text{given that there are } k \text{ other users on the channel}].$$

This quantity depends on the model that is assumed for other-user interference as well as on  $q$ . Let us first assume, pessimistically, that all frequency hits result in symbol errors. The symbol error probability, given that  $k$  other users are simultaneously transmitting over the channel and that timing is asynchronous, was shown in [3] to be

$$p_k = 1 - (1 - 2/q)^k. \quad (3)$$

The quantity  $1/q$  represents the probability that a user chooses a specific bin for transmission during a given hop. Since there is no hop synchronization between users, partial overlap may occur either from the left or from the right, as illustrated in Fig. 2. Thus, the probability that a specific one of the other  $k$  users will interfere with the symbol of the user of interest is  $2/q$ .

The symbol error process for any user is a sequence of independent Bernoulli trials, because of the pseudorandom nature of the hopping patterns. Since the Reed-Solomon codes under consideration can tolerate  $t$  symbol errors in any  $n$ -symbol codeword, the packet error probability, given  $k$  other users, is

<sup>1</sup> If the users are bursty, we can average the conditional throughput described above with respect to the statistics of the packet generation and retransmission processes to obtain the unconditional throughput. Such a calculation is generally complicated and goes beyond the scope of this paper. (See, e.g., Hajek [2].)

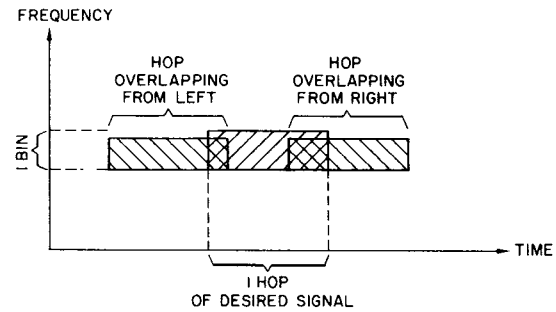


Fig. 2. A frequency hit in which two other users transmit in same frequency bin as desired signal, one overlapping from the left and one from the right.

$$\Pr [E|k] = \sum_{i=t+1}^n \binom{n}{i} p_k^i (1 - p_k)^{n-i}. \quad (4)$$

Since virtually all packet errors are detectable, we can interpret the packet error probability as a *packet erasure* probability.

If all hits are detected and the corresponding symbols erased, we simply change the lower limit of the summation in (4) to  $2t + 1$ . The quantity  $p_k$ , which now represents *symbol erasure* probability, is still given by (3).

We now turn our attention to the modified model that permits within-hit discrimination against other user interference.

### III. THE MODIFIED MODEL IN WHICH PARTIAL OVERLAPS CAN BE TOLERATED

We previously assumed that any tone occupancy within the same bin as the desired signal by other users, even if only partially overlapping in time with the symbol of the user of interest, caused interference. We now assume that a symbol erasure will be necessitated *if and only if, in any one (or more) of the  $M$  tone positions of the frequency bin, the amount of time overlap between the symbol (tone) of interest and those of other users exceeds a fraction  $\rho$  of the hop duration time*. Otherwise, the symbol is received correctly. This assumption is discussed later on. All other assumptions remain unchanged.

In this model we must examine each of the  $M$  tone positions of the frequency bin to determine whether any of them experience interference for more than a fraction  $\rho$  of the hop duration. Note that this interference may arise from one or more other users' signals in the same tone position whose combined overlap at the same or at opposite ends of the hop lasts for a fraction of a hop greater than  $\rho$ . The model exploits the fact that the  $M$ -ary FSK waveform is constant throughout the hop duration, unlike, e.g., the case of the serial transmission of  $\log_2 M$  bits in each hop.

To make the definition of overlap clear, we have illustrated in Fig. 3 the case of a number of overlapping signals, all of which are in the same tone position (possibly, but not necessarily, the same as that of the desired signal). The total overlap in the tone position we are considering is defined to be

$$\tau = \min(1, \tau_L + \tau_R) \quad (5)$$

where

$\tau_L$  = maximum overlap of signals from left in this tone position

$\tau_R$  = maximum overlap of signals from right in this tone position

and  $\tau_L$  and  $\tau_R$  range from 0 (if no overlap) to 1 (if total overlap). Note that we do not combine overlaps in different tone positions, but rather treat each tone position separately.

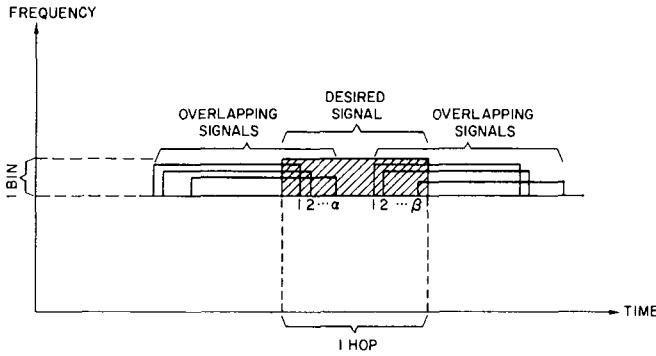


Fig. 3. The case of  $\alpha$  signals overlapping from left and  $\beta$  signals overlapping from right in a single tone position.

Overlapping signals in the same tone position as the desired signal are treated the same as those in other tone positions.

A symbol erasure is thus necessary if and only if  $\tau > \rho$  in one or more of the  $M$  tone positions. An equivalent way of viewing the overlap model is that interference in a tone position will necessitate an erasure *if and only if the clear interval of the hop (i.e., the part containing no other users) is less than  $1 - \rho$* .

The evaluation of system performance proceeds in the same manner as presented earlier for the other interference models. To evaluate the probability of incorrect packet reception, we again use (4) with a lower summation limit of  $i = 2t + 1$ . However,  $p_k$ , the symbol erasure probability given that a total of  $k + 1$  users transmit simultaneously, requires a new evaluation.

We again consider the case of  $k$  other users simultaneously using the channel along with the desired signal. We consider an arbitrary symbol of the desired signal, which corresponds to a specific frequency bin and a particular tone position therein. Suppose  $m$  out of the  $k$  other users have chosen the same frequency bin as the desired signal. Of course,  $m$  is a random variable. We can express  $p_k$  conditioned on  $m$  as follows:

$$p_k = \sum_{m=1}^k P(e|m)Q(m|k) \quad (6)$$

where

$$P(e|m) = \Pr(\text{symbol erasure} | m \text{ other users in same frequency bin as desired signal}),$$

$$Q(m|k) = \Pr(m \text{ other users in same frequency bin as desired signal} | k \text{ other users in the channel}).$$

Since the users are assumed to choose bins independently and with uniform distribution, we have

$$Q(m|k) = \binom{k}{m} \left(\frac{2}{q}\right)^m \left(1 - \frac{2}{q}\right)^{k-m}. \quad (7)$$

Thus, it remains to evaluate the quantity  $P(e|m)$ , which is calculated in the Appendix.

At this point let us discuss further the significance of the maximum tolerable overlap parameter and factors related to the implementation of systems that can tolerate some degree of hop overlap. To put things in perspective, we consider a "baseline system" in which the hop duration is "matched" to (i.e., is the reciprocal of) the tone spacing, as discussed earlier, and in which all hits necessitate erasures, regardless of overlap duration (i.e.,  $\rho = 0$ ).

Then we can view the system for which  $\rho > 0$  as one that employs a more sophisticated receiver. For example, a

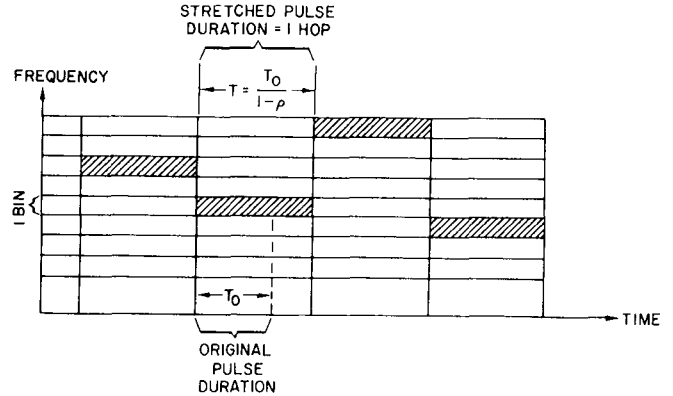


Fig. 4. "Stretched pulse" frequency-hopping signal.

possible implementation would consist of banks of matched filters, one for each tone position. The outputs of these filters would be examined throughout the hop duration. If the time derivative of the output were zero, the decision would be made that there is no signal at that tone frequency at that instant of time, independent of the total energy accumulated thus far. A signal would be declared present as long as the time derivative was sufficiently large.

The fact that partial overlaps can be tolerated suggests that we may be dwelling longer than we really have to in each frequency bin. In so doing we may be wasting energy (since only a fraction  $1 - \rho$  of the hop appears to be really needed) and transmitting data slower than we may be able to (by the same factor of  $1 - \rho$ ). We note, however, that a shortening of the dwell time results in an increase in the orthogonal tone spacing and, therefore, a decrease in the number of disjoint frequency bins (again by the factor of  $1 - \rho$ ) that are available within the same given fixed total bandwidth if we are to maintain a matched system.

In fact, another way of achieving tolerance to partial overlaps would be by means of slowing down the hopping rate and, thus, dwelling longer at each hop. In Fig. 4 we illustrate such a system in which the hop duration, initially  $T_0$ , is lengthened to  $T = T_0 / (1 - \rho)$  to permit a tolerable hop overlap of  $\rho$ . The data rate is thus reduced by a factor of  $(1 - \rho)$ , and therefore, the throughput measure must reflect this same factor of decrease. We assume that in this case the number of frequency bins remains constant at  $q$ . The performance of such a stretched pulse system is discussed in the next section. Other interpretations and ways of achieving the ability to withstand partial overlap interference are discussed in detail in [6].

#### IV. PERFORMANCE EVALUATION

The performance measures we have considered are the probability of incorrect packet reception (evaluated using (4) with the appropriate limits used in the summation), and the resulting channel throughput [obtained through the use of (1)]. Using the results of the Appendix, we can calculate these quantities. Fig. 5 illustrates the packet erasure probability as a function of  $\rho$  for  $q = 100$  frequency bins,  $k + 1 = 50$  simultaneous users, and several alphabet sizes ( $M$ ) where we are using RS- $(M - 1, (M - 2)/2)$  codes, which have rate approximately equal to  $1/2$ . Note that for  $\rho$  less than about 0.4, the shorter codes result in lower packet erasure probability. In most practical applications, however, where there are symbol errors resulting from channel noise, the use of longer codes is preferable because of their greatly improved ability to detect uncorrectable codeword errors. We note, however, that for the  $M$ -ary FSK signaling scheme considered here, higher alphabet sizes are less bandwidth efficient than lower ones; for

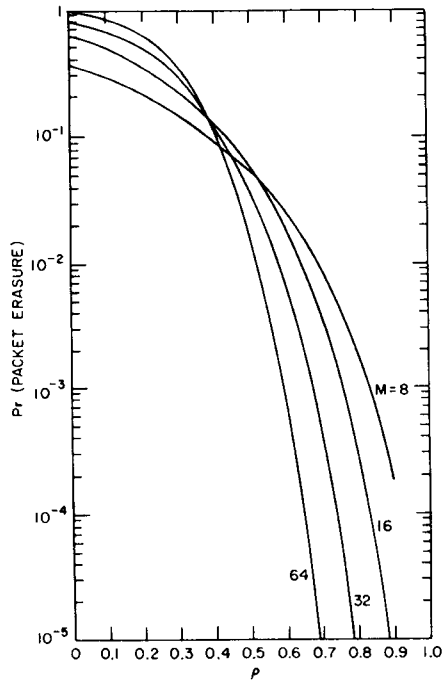


Fig. 5. Packet erasure probability for a noiseless asynchronous multiple user FH channel in which partial hits whose combined total overlap in any tone position is less than  $\rho$  can be ignored;  $M$ -ary FSK signaling; RS- $(M-1, (M-2)/2)$  coding; 100 frequency bins; 50 users.

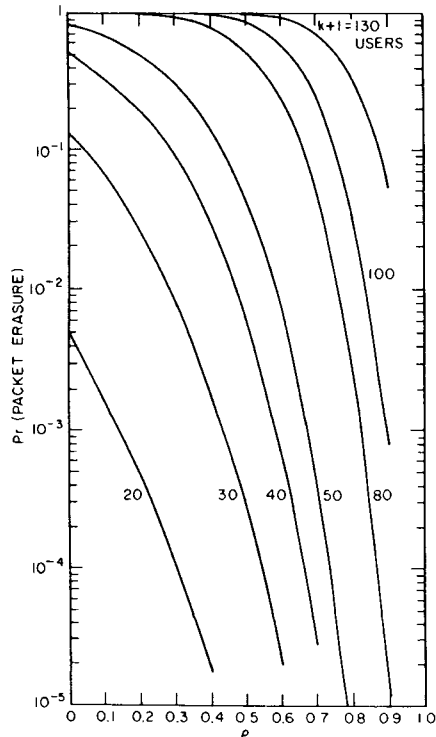


Fig. 6. Packet erasure probability for a noiseless asynchronous multiple user FH channel in which partial hits whose combined total overlap in any tone position is less than  $\rho$  can be ignored; 32-ary FSK signaling; RS-(31, 15) coding; 100 frequency bins.

a fixed hopping rate the bandwidth is proportional to  $M$ , whereas the data rate is proportional to  $\log_2 M$ .

Figs. 6 and 7 illustrate the packet erasure probability and throughput per frequency bin as a function of  $\rho$  for  $q = 100$  frequency bins,  $M = 32$ , and RS-(31, 15) coding for several

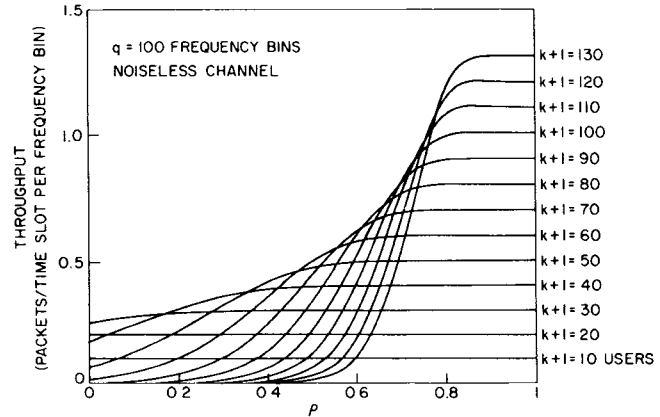


Fig. 7. Throughput per frequency bin of a noiseless asynchronous multiple user FH channel in which partial hits whose combined total overlap in any tone position is less than  $\rho$  can be ignored; 32-ary FSK signaling; RS-(31, 15) coding; 100 frequency bins.

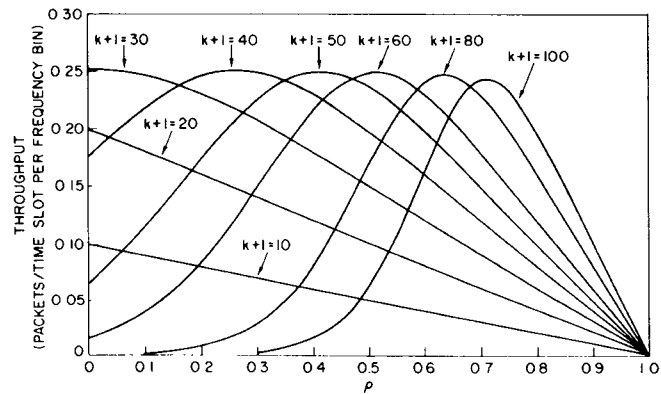


Fig. 8. Normalized throughput per frequency bin of a noiseless asynchronous multiple user FH stretched pulse system with fixed bin bandwidth; partial hits whose combined total overlap in any tone position is less than  $\rho$  can be ignored; 32-ary FSK signaling; RS-(31, 15) coding; 100 frequency bins.

values of  $k+1$ . The case of  $\rho = 0$  corresponds to the "baseline" system. As  $\rho$  approaches 1, the total channel throughput (summed over all frequency bins) approaches the number of users; the throughput per frequency bin can actually be greater than one packet per time slot! It is difficult to estimate what values of  $\rho$  might be achievable in a practical system. Realistic values would depend on hopping rates and hardware implementation, as well as on the channel model.

In Section III we considered an interpretation of the idea of tolerable partial overlap in terms of stretching the hop duration while keeping the frequency bin bandwidth fixed. The throughput performance of this system is obtained by multiplying the curves of Fig. 7 by the factor  $(1 - \rho)$  to reflect the fact that the ability to tolerate partial overlap results from a pulse stretching that lowers the data rate. The resulting throughput is illustrated in Fig. 8. Note that there is an optimum value of  $\rho$  that varies with the number of users. For small values of  $k+1$  the optimum value of  $\rho$  is 0, indicating that we should not "slow down" our system.

As the number of users increases, the optimum value of  $\rho$  increases and the maximum achievable throughput decreases slightly. However, as  $k+1$  increases, the performance becomes increasingly sensitive to the value of  $\rho$ ; thus,  $\rho$  can be chosen to produce a high value of throughput only if there is a good estimate of the number of active users.

## V. CONCLUSIONS

In this paper we have considered frequency-hopping (FH) spread-spectrum multiple access systems in which Reed-Solomon coding is used to correct burst errors caused by other-user interference in a packet-switched environment. Under the model considered, each packet is encoded as a Reed-Solomon codeword, one symbol of which is transmitted in each hop. We have considered a noiseless channel model in which the only interference is that caused by other users.

We have assumed that frequency hits can be detected. Furthermore, hops (symbols) can be correctly received, despite partial overlaps by other users' signals, provided that the overlap is a sufficiently small fraction of the hop duration. Such an interference model is valid, provided that the signal remains constant throughout each hop duration and that a sufficiently sophisticated receiver is used.

For  $M$ -ary FSK signaling, we have derived exact expressions for the probability of successful symbol reception and the resulting probability of correct packet reception and channel throughput as a function of the number of channel users, number of frequency bins, alphabet size, and tolerable symbol overlap  $\rho$ . In [6] we have also addressed the case in which several binary FSK tones are transmitted in parallel, as well as various interpretations of the capability of tolerating partial overlap, along with the implications on spectral efficiency. The ability to discriminate against interfering signals that are present for a sufficiently small fraction of the hop duration results in dramatic increases in throughput as  $\rho$  is increased from 0 to 1.

## APPENDIX

## EVALUATION OF THE CONDITIONAL SYMBOL ERASURE PROBABILITY

In this Appendix we evaluate

$$P(e|m) = \Pr [\text{symbol erasure} | \text{given } m \text{ other users in same frequency bin as desired signal}]$$

which is the same as the conditional probability that the overlap in one or more tone positions is greater than  $\rho$ , given  $m$ . Note that when two or more overlapping signals share the same tone position, we must consider their combined overlap  $\tau$ , which was defined in Section III. We must therefore characterize the distribution of the  $m$  other users' signals among the tone positions, and then, for each such distribution of other-users' signals, evaluate the probability that the overlap is indeed greater than  $\rho$  in one or more tone positions. To do so we define a "state" variable  $\mathbf{n}_m$  to represent the tone position occupancy distribution within a frequency bin in which  $m$  other signals are present in addition to the desired one. We let

$$\mathbf{n}_m = (n_m(1), n_m(2), \dots, n_m(m)) \quad (\text{A.1})$$

where

$n_m(j)$  = number of distinct tone positions in the bin, each containing exactly  $j$  other signals, or equivalently, number of "bunches" of exactly  $j$  users occupying distinct tone positions in the bin.

For example, consider the case of  $m = 5$ . The state  $\mathbf{n}_5 = (3, 1, 0, 0, 0)$  corresponds to a realization in which one tone position is occupied by two other users and three tone positions are singly occupied by other users.

In general, the states  $\mathbf{n}_m$  that are realizable must satisfy two physical constraints. First, since there are a total of  $m$  other signals in the same frequency bin, we must have

$$\sum_{j=1}^m j n_m(j) = m. \quad (\text{A.2})$$

Also, the total number of occupied tone positions, which we denote by  $\hat{n}$ , cannot be greater than either the  $M$ -ary alphabet size or the number of other users in the frequency bin, i.e.,

$$\hat{n} = \sum_{j=1}^m n_m(j) \leq \min(m, M). \quad (\text{A.3})$$

The conditional symbol erasure probability, given  $m$ , can then be expressed as

$$P(e|m) = 1 - \sum_{\mathbf{n}_m} \Pr(\text{symbol success} | \mathbf{n}_m) R(\mathbf{n}_m) \quad (\text{A.4})$$

where  $R(\mathbf{n}_m)$  is the probability of state  $\mathbf{n}_m$  occurring. The evaluation of  $P(e|m)$  therefore requires 1) the enumeration of all possible states  $\mathbf{n}_m$ , to perform the indicated summation; 2) the evaluation of the conditional probability distribution  $R(\mathbf{n}_m)$  for each state  $\mathbf{n}_m$ ; and 3) the evaluation of  $\Pr(\text{symbol success} | \mathbf{n}_m)$  for each state  $\mathbf{n}_m$ .

## A. Enumeration of the States

Given that there are  $M$  tone positions, we must enumerate the states  $\mathbf{n}_m$  that can occur for any particular value of  $m$ , i.e., all states for which (A.2) and (A.3) are satisfied. This problem is equivalent to the construction of Young's lattice [7]. We proceed iteratively as follows.

Assume we know all states for a given value of  $m$ . [For example, we may start with the trivial case of  $m = 1$ , for which the only state is  $\mathbf{n}_1 = (1)$ .] For each state  $\mathbf{n}_m$  that is consistent with the presence of  $m$  other users in the frequency bin of interest, we determine the states  $\mathbf{n}_{m+1}$  that can be generated as one additional user is added to the system. To do so, we first consider the case in which the new  $(m + 1)$ st user has chosen a tone position not previously chosen by any of the first  $m$  users. The number of singly occupied tone positions thus increases by 1 while the number of tone positions containing  $i$  other users ( $i = 2, 3, \dots, m$ ) remains unchanged. The new states are thus generated by the following procedure.

For each state  $\mathbf{n}_m$ , set

$$\begin{aligned} n_{m+1}(1) &= n_m(1) + 1, \\ n_{m+1}(i) &= n_m(i), \quad i = 2, 3, \dots, m. \end{aligned} \quad (\text{A.5})$$

We now consider the case in which the new  $(m + 1)$ st user has chosen one of the tone positions already chosen by one of the first  $m$  users. If that tone position already contained  $i$  users, then it would now contain  $i + 1$  users, thus decrementing the number containing  $i$  users by 1 while incrementing the number containing  $i + 1$  by 1. The new states are thus generated as follows:

For each state  $\mathbf{n}_m$  and every  $n_m(i) \neq 0$ , set

$$\begin{aligned} n_{m+1}(i) &= n_m(i) - 1 \\ n_{m+1}(i+1) &= n_m(i+1) + 1. \end{aligned} \quad (\text{A.6})$$

Note that duplicate identical states are generated by this process, because two different  $\mathbf{n}_m$  states can evolve to the same  $\mathbf{n}_{m+1}$  successor state. Such duplicate states are easily recognized and discarded. Also, states for which the number of occupied tone positions ( $\hat{n}$ ) is greater than the alphabet size  $M$  may be generated. These are also discarded. As an example, we show in Table I the resulting states for  $m \leq 5$  (and  $M \geq m$ ).

TABLE I  
ENUMERATION OF STATES FOR  $m \leq 5$  AND  $M \geq m$

$m = 1$ :	(1)
$m = 2$ :	(2,0), (0,1)
$m = 3$ :	(3, 0, 0), (1, 1, 0), (0, 0, 1)
$m = 4$ :	(4, 0, 0, 0), (2, 1, 0, 0), (0, 2, 0, 0), (1, 0, 1, 0), (0, 0, 0, 1)
$m = 5$ :	(5, 0, 0, 0, 0), (3, 1, 0, 0, 0), (1, 2, 0, 0, 0), (2, 0, 1, 0, 0), (0, 1, 1, 0, 0), (1, 0, 0, 1, 0), (0, 0, 0, 0, 1)

### B. The State Probability Distribution: $R(n_m)$

We first observe that we are not interested in *which* of the  $M$  tone positions in a frequency bin are occupied by a specific number of signals, but only in the *number* of such tone positions. Thus, each state  $n_m$  corresponds to several different realizations, all of which are equally likely.

Our approach is to consider  $m$  users transmitting in the same frequency bin as the desired signal, each of which places a signal into one of the  $M$  tone positions; each tone position is chosen equally likely with probability  $1/M$ . We now consider the sequence in which some subset of the  $M$  tone positions is filled as we examine the  $m$  users, which are numbered from 1 to  $m$ . We want to realize the state  $n_m = (n_m(1), n_m(2), \dots, n_m(m))$ .

There are numerous ways to do so. For example, let the first  $n_m(1)$  users all choose different tone positions; these are followed by  $n_m(2)$  pairs of users, such that the two members of each pair choose the same tone position as each other, but different from all those previously chosen; these are followed by  $n_m(3)$  groups of three users, such that the three members of each triplet choose the same tone position as each other, but different from all those previously chosen, etc. This realization is denoted by  $I_m$ . The probability of any other specific realization corresponding to the same state  $n_m$  is identical to that of  $I_m$ .<sup>2</sup>

We first consider the  $n_m(1)$  singly occupied tone positions. The first user chooses a tone position at random; thus, the probability that he picks a previously unchosen tone position is simply  $M/M = 1$ . The second user also chooses a tone position at random; the probability that he picks a previously unchosen tone position is  $(M-1)/M$ . Continuing in the same manner, user  $n_m(1)$  chooses a tone position that was not previously chosen with probability  $[M - (n_m(1) - 1)]/M$ . Thus,

Pr (first  $n_m(1)$  users choose different tone positions)

$$= \frac{M}{M} \frac{(M-1)}{M} \frac{(M-2)}{M} \dots \frac{(M - (n_m(1) - 1))}{M}. \quad (\text{A.7})$$

Given that the first  $n_m(1)$  users have chosen different tone positions, we now evaluate the probability that we then have  $n_m(2)$  pairs of users that choose the same tone position as each other, but different from previously chosen tone positions. Thus, user  $n_m(1) + 1$  must choose a tone position that is different from that of the first  $n_m(1)$  signals, an event that occurs with probability  $[M - n_m(1)]/M$ . User  $n_m(1) + 2$  must choose the *same* tone position as user  $n_m(1) + 1$ , an event that occurs with probability  $1/M$ . Similarly, user  $n_m(1) + 3$  must choose a tone position not previously chosen (resulting in a probability of  $[M - n_m(1) - 1]/M$ ), and user  $n_m(1) + 4$  must choose the same tone position as user  $n_m(1) + 3$  (resulting in  $1/M$ ).

We then consider  $n_m(3)$  triplets of users,  $n_m(4)$  quadruples, and so on. Continuing in this way, we finally obtain

<sup>2</sup> This can be easily seen by a renumbering of the users.

$$\text{Pr } (I_m) = \frac{M!}{M^m (M - \hat{n})!} \quad (\text{A.8})$$

where  $\hat{n}$  is the total number of occupied tone positions, as given in (A.3). Thus, we can write

$$R(n_m) = N(n_m) \text{Pr } (I_m) \quad (\text{A.9})$$

where  $N(n_m)$  is the number of equally likely realizations in the class defined by  $n_m$ .

The determination of  $N(n_m)$  is identical to that of finding the number of different partitions of a set of  $m$  objects into classes of  $n_m(j)$  groups, each group having  $j$  objects, for  $j = 1, 2, \dots, m$ . From [7] we have

$$N(n_m) = \frac{m!}{\prod_{j=1}^m (j!)^{n_m(j)} n_m(j)!}. \quad (\text{A.10})$$

Finally, combining with (A.9), we obtain

$$R(n_m) = \frac{M! m!}{M^m (M - \hat{n})! \prod_{j=1}^m (j!)^{n_m(j)} n_m(j)!}. \quad (\text{A.11})$$

### C. The Evaluation of Symbol Success Probability

This requires two steps:

a) evaluate  $\text{Pr } (\tau < \rho | j)$ , the probability that the total overlap  $\tau$  is less than  $\rho$  in a particular tone position, given that  $j$  users transmit in that tone position;

b) evaluate the resulting conditional symbol success probability, given state  $n_m$ .

To evaluate  $\text{Pr } (\tau < \rho | j)$ , let us fix a particular tone position. Let  $\alpha$  be the number of other users overlapping at the left (leading) edge and  $\beta = j - \alpha$  the number of other users overlapping at the right (trailing) edge of the symbol. Note that it is equally likely for an overlapping signal to be at the leading or trailing end of the hop. We define

$X(a)$  = time overlap with the symbol of interest of the  $a$ th interfering symbol from the left,  
 $a = 1, \dots, \alpha$

$Y(b)$  = time overlap with the symbol of interest of the  $b$ th interfering symbol from the right,  
 $b = 1, \dots, \beta$ .

The  $X(a)$ 's and  $Y(b)$ 's are independent and uniformly distributed over the interval  $(0, 1)$ . We also define

$$\tau_L = \max_a (X(a))$$

$$\tau_R = \max_b (Y(b)).$$

The combined total overlap of all users in the particular tone position we are considering is therefore

$$\tau = \min (1, \tau_L + \tau_R). \quad (\text{A.12})$$

As a result of the independence of  $\tau_L$  and  $\tau_R$ , the conditional density of  $\tau$  is simply the convolution of their densities. The evaluation of  $\text{Pr } (\tau < \rho | j)$  is routinely obtained (see [6]) to be

$$\text{Pr } (\tau < \rho | j) = (j+1) \left( \frac{\rho}{2} \right)^j. \quad (\text{A.13})$$

To evaluate the Pr (symbol success  $| n_m$ ), we define

$T_j$  = Pr (in each of the  $n_m(j)$  tone positions, each containing exactly  $j$  other users, we have total overlap with symbol of interest  $< \rho$ )



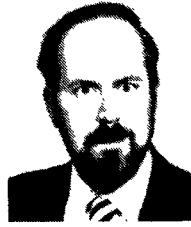
$$= [\Pr(\tau < \rho | j)]^{n_m(j)} = \left[ (j+1) \left( \frac{\rho}{2} \right)^j \right]^{n_m(j)} \quad (\text{A.14})$$

where we have made use of the independence among the total overlap variables in different tone positions, given the number of users in each tone position. Making further use of this independence, we write

$$\Pr(\text{symbol success} | n_m) = \prod_{j=1}^m T_j = \rho^m \prod_{j=1}^m \left[ (j+1) \left( \frac{1}{2} \right)^j \right]^{n_m(j)}. \quad (\text{A.15})$$

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