ABSTRACT

Title of dissertation: EQUIVARIANT VECTOR BUNDLES OVER TOPOLOGICAL TORIC MANIFOLDS AND HIRZEBRUCH–RIEMANN–ROCH THEOREM

Yong Cui, Doctor of Philosophy, 2025

Dissertation directed by: Professor Amin Gholampour
Department of Mathematics

This thesis explores the intersection of algebraic geometry and toric topology, focusing on the classification of equivariant vector bundles over topological toric manifolds and the associated Hirzebruch–Riemann–Roch theorem.

The research is presented in three primary sections. First, we classify complex equivariant vector bundles over topological toric manifolds by synthesizing techniques from differential geometry with conceptual frameworks from algebraic geometry. This work extends Klyachko's classification of toric vector bundles beyond the realm of smooth toric varieties into the topological setting. Second, we classify real equivariant vector bundles by establishing an equivariant analogue of Atiyah's equivalence between "Real" vector bundles (complex bundles with a conjugate-linear involution) and real vector bundles. Finally, we provide a geometric interpretation and proof of the combinatorial Hirzebruch–Riemann–Roch theorem, as formulated by Hattori and Masuda, specifically for topological toric manifolds. These results establish new avenues for research in toric topology—most notably regarding the equivariant stability of bundles—and demonstrate the continued utility of algebraic-geometric methods in topological contexts.

EQUIVARIANT VECTOR BUNDLES OVER TOPOLOGICAL TORIC MANIFOLDS AND HIRZEBRUCH–RIEMANN–ROCH THEOREM

by

Yong Cui

Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2025

Advisory Committee:
Professor Amin Gholampour, Chair/Advisor
Professor Patrick Brosnan
Professor Tristan Hübsch
Professor Harry Tamvakis
Professor William Gasarch, Dean's Representative

© Copyright by
Yong Cui
2025

Dedication

To my parents, for their support and for giving me the freedom to pursue my own path.

Acknowledgments

I am deeply grateful to my advisor, Professor Amin Gholampour, for his guidance, insight, and steady support throughout my doctoral studies. His mentorship has been central to my development as a mathematician, and his advice helped me navigate the technical difficulties that arose in this work.

I would like to thank Professor Patrick Brosnan, who first introduced me to toric geometry and whose early guidance opened the path that led to this dissertation. I am also grateful to Professor Tristan Hübsch, whose physical intuitions and perspective played an important role in shaping the direction of my research.

I thank the remaining members of my dissertation committee for their thoughtful comments and suggestions.

Finally, I am grateful to my colleagues, friends, and family for their patience, encouragement, and unwavering support. Their confidence in me has made this work possible.

Table of Contents

List of Tables

List of Figures

Chapter 1: Overview

1.1 Motivation

Mirror symmetry provides deep connections between algebraic geometry, symplectic geometry, and mathematical physics. One of the most tractable constructions comes from toric geometry, where Batyrev showed that reflexive polytopes naturally produce mirror pairs of Calabi–Yau hypersurfaces.

More recently, Berglund and Hübsch proposed a broader combinatorial framework that extends mirror symmetry beyond the classical convex toric setting. Their construction suggests the existence of mirror pairs associated to certain "VEX" multitopes and their transpolars, whose geometry lies outside the scope of ordinary toric varieties. They conjecture that the ambient spaces associated with VEX multitopes are certain torus manifolds.

Topological toric manifolds, a special class of torus manifolds introduced by Ishida, Fukukawa, and Masuda, form a natural generalization of smooth toric varieties and provide potential ambient spaces for Berglund–Hübsch–type mirror constructions.

This thesis studies equivariant vector bundles on topological toric manifolds and develops tools that may eventually contribute to a geometric understanding of Berglund–Hübsch transpolar mirror symmetry.

1.2 Main results

This thesis classifies all real/complex topological/smooth equivariant vector bundles over topological toric manifolds and gives a geometric meaning of the combinatorial Hirzebruch-Riemann-Roch theorem for torus manifolds proved by Hattori and Masuda in the context of topological toric manifolds.

1.2.1 Klyachko vector bundles

We define the notions of affine charts, distinguished points and orbits corresponding to cones in a topological fan, etc by mimicking their toric counterparts as shown in the following table and replace the usual total order $\geq$ on $\mathscr{R}$ by two partial orders $\geq_c, \geq_s$ that will be used to express the extendability conditions for the quotient of two character functions.

	Toric Varieties	TTM
Fans	toric fan $\overline{\Sigma}$	topological fan $\Delta = (\Sigma, \beta)$ (Def. 3.7)
(Abstract) Cones	σ	I (Def. 3.6)
Affine charts	U_σ	U_I (Def. 3.8)
character group of $\mathbb{T}$	M	$\mathscr{R}^n$ (Lemma 3.1)
cocharacter group of $\mathbb{T}$	N	$\mathscr{R}^n$ (Lemma 3.1)
orders	total order $\geq$ on $\mathbb{R}$	partial orders $\geq_c, \geq_s$ on $\mathscr{R}$ (Def. 4.3,4.9)
distinguished points	γ_σ	γ_I (Def. 4.1)
$\mathbb{T}$-orbit of γ_σ and γ_I	$\mathscr{O}(\sigma)$	$\mathscr{O}(I)$ (Def. 4.1)
closure of $\mathscr{O}(\sigma)$ and $\mathscr{O}(I)$	$V(\sigma)$	$V(I)$ (Def. 4.1)
stabilizer subgroups of γ_σ and γ_I	T_σ	$\mathbb{T}_I$ (Def. 4.4)
character groups of T_σ and $\mathbb{T}_I$	$M_\sigma = M/(\sigma^\perp \bigcap M)$	$\mathscr{R}^n/\beta_I^\perp$ (Def. 4.4, Lemma 4.9)

Table 1.1: Comparison of symbols

Then we establish an orbit-cone correspondence for topological toric manifolds:

Theorem 1.1 (4.1)**.**

1. There is a bijective correspondence

$$\Sigma \longleftrightarrow \{\mathbb{T}\text{-orbits in } X(\Delta)\}$$
$$I \mapsto \mathscr{O}(I)$$

2. For each $J \in \Sigma^{(j)}, dim_{\mathbb{C}}\mathscr{O}(J) = n - j$.

3.

$$U_I = \bigcup_{J \leq I} \mathscr{O}(J).$$

4. $J \leq I$ *iff* $\mathscr{O}(I) \subset V(J)$ *and*

$$V(J) = \bigcup_{J \leq I} \mathscr{O}(I).$$

We define a Klyachko vector bundle $\mathscr{E}$ over a topological toric manifold X to be a complex equivariant vector bundle whose restriction to each affine chart U_I is equivariantly isomorphic to the trivial bundle $U_I \times \mathbb{C}^r$. On the other hand, by modifying Klyachko's compatibility condition for the filtered vector spaces associated with a toric vector bundle, we define the category of Klyachko vector spaces as follows:

Definition 1.1 (4.6)**.** For a topological toric manifold X, the category TKVS_X is the category whose objects are complex vector spaces E with a poset P of complex subspaces $E^i(\mu)$ of E indexed by $\mu \in \mathscr{R}, i \in \Sigma^{(1)}$ which satisfy the following compatibility condition: For any $I \in \Sigma$, there exists a $\mathscr{R}^n/\beta_I^{\perp}$-grading

$$E = \bigoplus_{[\chi] \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$$

for which

$$E^i(\mu) = \sum_{\langle \chi, \beta_i \rangle \geq_c \mu} E_{[\chi]}, \ \forall i \in I.$$

A morphism $f : (E, \{E^i(\mu)\}) \to (F, \{F^i(\mu)\})$ in TKVS_X is a linear transformation $f : E \to F$ such that $f(E^i(\mu)) \subset F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$. f is an isomorphism if $E \to F$ is an isomorphism of complex vector spaces. f is a monomorphism if $E \to F$ is a monomorphism of complex vector spaces. f is an epimorphism if $E \to F$ is an epimorphism of complex vector spaces and $f(E^i(\mu)) = F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$.

Finally we show that Klyachko vector bundles are completely described by their linear algebra data.

Theorem 1.2 (4.7)**.** *The category* TKVB_X *of topological Klyachko vector bundles over the topological toric manifold X is equivalent to the category* TKVS_X *of topological Klyachko vector spaces.*

Theorem 1.3. *4.15 The category* SKVB_X *of smooth Klyachko vector bundles over the topological toric manifold X is equivalent to the category* SKVS_X.

1.2.2 Classification of complex equivariant vector bundles

We prove that all topological/smooth complex equivariant vector bundles are topological/smooth Klyachko vector bundles.

Theorem 1.4 (5.2)**.** *Let X be the* $2n$*-dimensional topological toric manifold associated with a topological fan* Δ *and* $\pi : \mathscr{E} \to X$ *a topological* $\mathbb{T}$*-equivariant complex vector bundle of rank k, then* $\mathscr{E}$ *is a topological Klyachko vector bundle.*

Theorem 1.5. *5.3 Let X be the* $2n$*-dimensional topological toric manifold associated with a topological fan* Δ *and* $\pi : \mathscr{E} \to X$ *a smooth* $\mathbb{T}$*-equivariant complex vector bundle, then* $\mathscr{E}$ *is a smooth Klyachko vector bundle.* □

1.2.3 Classification of real equivariant vector bundles

We generalize the equivalence between real vector bundles and complex vector bundles with conjugates in [1] to equivalence between real equivariant vector bundles and complex equivariant vector bundles with conjugates. By adding one more restriction on the conjugate maps τ on the filtered vector spaces in SKVS, we define the category FVSC of filtered vector spaces with conjugates as follows:

Definition 1.2. 6.2 For a topological toric manifold X, let FVSC_X be the category whose objects are pairs (E,τ) where E is a complex vector space and $\tau : E \to E$ is an anti-linear involution with a poset P of complex subspaces $E^i(\mu)$ of E indexed by $\mu \in \mathscr{R}, i \in \Sigma^{(1)}$ which satisfy the following compatibility conditions:
1. For any $I \in \Sigma$, there exists a $\mathscr{R}^n/\beta_I^{\perp}$-grading

$$E = \bigoplus_{[\chi]\in\mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$$

for which

$$E^i(\mu) = \sum_{\langle\chi,\beta_i\rangle\geq_s\mu} E_{[\chi]}, \ \ \forall i \in I.$$

2. $\tau(E_\chi) = E_{\chi^*}$ for all $[\chi] \in \mathscr{R}^n/\beta_I^{\perp}$ where $\chi^*(t) = \overline{\chi(t)}$ for all $t \in \mathbb{T}$.

Given $(E,\tau),(F,\sigma) \in \mathrm{FVSC}_X$, a morphism $f : (E,\tau) \to (F,\sigma)$ is a linear transformation $f : E \to F$ between the complex vector spaces E and F such that

$$f \circ \tau = \sigma \circ f.$$

Then we obtain a classification of all smooth real equivariant vector bundles.

Corollary 1.6. *6.4 For a topological toric manifold X, the category* REVB_X *of smooth real equivariant vector bundles over X is equivalent to* FVSC_X.

1.2.4 Hirzebruch-Riemann-Roch theorem for topological toric manifolds

We generalize the equivalence between torus invariant Cartier divisors and equivariant line bundles over smooth toric varieties to the equivalence between t-divisors and t-line bundles over topological toric manifolds and give a geometric interpretation of Hattori and Masuda's combinatorial Hirzebruch-Riemann-Roch theorem in the case of topological toric manifolds. The proof of our Hirzebruch-Riemann-Roch theorem mixes the geometric proof of toric Hirzebruch-Riemann-Roch theorem in [6] and some combinatorial identities proved in [8].

Theorem 1.7. *7.19 For a t-divisor D on a TTM $X = X(\Delta)$,*

$$\overline{\int_X} \mathrm{ch}(D)\,\mathrm{Td}(X) = \#\mathscr{P}_D.$$

Below is a table that compares three different versions of Hirzebruch-Riemann-Roch theorem.

	smooth complete TV $X_{\overline{\Sigma}}$	**TTM** $X(\Delta)$	**Torus manifolds**
Divisors	Cartier divisors $D=\sum_{\rho} a_\rho D_\rho$, $a_\rho \in \mathbb{Z}$	t-divisors $D=\sum_{i=1}^m a_i D_i$, $a_i \in \mathscr{R}$	not defined
Objects	algebraic equivariant line bundles $L=L_{\mathscr{O}_X(D)}$	smooth equivariant line bundles L_D	simple lattice multipolytopes $\mathscr{P}=(\Delta,\mathscr{F})$
tangent bundle TX	complex vector bundle	real vector bundle	real vector bundle
total Chern class	$c(TX)=\prod_{\rho\in\Sigma(1)}(1+D_\rho)$	not defined	not defined
Integrals	$\int_X$ as defined in [6]	$\overline{\int}_X$ (formula 7.1)	$\int_\Delta$ (formula 3.5)
Todd class	$\prod_{\rho\in\Sigma(1)}\frac{[D_\rho]}{1-e^{-[D_\rho]}}\in H^{\cdot}(X,\mathbb{Q})$	$\prod_{i=1}^m\frac{[\mu_i]}{1-e^{-[\mu_i]}}\in H^{\cdot}(X,\mathbb{Q})$	$\mathscr{T}(\Delta)=\sum_{g\in G_\Delta}\prod_{i=1}^d\frac{\overline{x_i}}{1-\rho_i(g)e^{-\overline{x_i}}}$
HRR	$\int_X e^{c_1(L)}\mathrm{Td}(X)=\chi(L)$	$\overline{\int}_X e^{c_1(D)}\mathrm{Td}(X)=\#\mathscr{P}_D$	$\int_\Delta e^{c_1(\mathscr{P})}\mathscr{T}(\Delta)=\#\mathscr{P}$

Table 1.2: HRRs

1.2.5 Structure of the thesis

In chapter 2 we review the basics of toric geometry, Klyachko's classification of toric vector bundles and toric Hirzebruch-Riemann-Roch theorem.

In chapter 3 we review the bijective correspondence between topological toric manifolds and topological fans, compare smooth complete toric varieties, topological toric manifolds and general torus manifolds and review Hattori and Masuda's combinatorial Hirzebruch-Riemann-Roch theorem for torus manifolds.

In chapters 4,5 we give a complete classification of complex equivariant vector bundles over topological toric manifolds.

In chapter 6 we classify all real equivariant vector bundles based on the results in chapters 4,5.

In chapter 7 we establish the correspondence between t-divisors and t-line bundles and give a geometric proof of Hirzebruch-Riemann-Roch theorem for topological toric manifolds.

Chapter 2: Toric Geometry

2.1 Toric varieties

In this chapter, we review the basics of toric geometry that will be generalized to topological toric manifolds in the following chapters.

Let $\mathbb{T}$ be the affine variety $(\mathbb{C}^*)^n$, M a lattice of rank n, i.e. a free abelian group of rank n which is isomorphic to the character group of $\mathbb{T}$ and $N := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ which is isomorphic to the cocharacter groups of $\mathbb{T}$.

Definition 2.1. A convex polyhedral cone in $N_{\mathbb{R}}$ is a set of the form

$$\sigma = \mathrm{Cone}(S) = \{\sum_{u \in S} \lambda_u u | \lambda_u \geq 0\} \subset N_{\mathbb{R}},$$

where $S \subset N_{\mathbb{R}}$ is finite. σ is strongly convex if $\sigma \bigcap (-\sigma) = \{0\}$. σ is rational if $\sigma = \mathrm{Cone}(S)$ for some finite $S \subset N$.

We will call a strongly convex rational polyhedral cone simply a cone in this thesis.

Definition 2.2. Given a cone $\sigma \subset N_{\mathbb{R}}$, its dual cone is defined by

$$\sigma^{\vee} = \{m \in M_{\mathbb{R}} | \langle m, u \rangle \geq 0 \; \forall u \in \sigma\}.$$

A subset τ of σ is a face of σ, denoted by $\tau \preceq \sigma$ if $\tau = \sigma \bigcap H_m$ where $m \in \sigma^{\vee}$ and $H_m = \{u \in N_{\mathbb{R}} | \langle m, u \rangle = 0\}$.

Definition 2.3. A fan $\overline{\Sigma}$ in $N_{\mathbb{R}}$ is a finite collection of cones $\sigma \subset N_{\mathbb{R}}$ such that:

i. Every $\sigma \in \overline{\Sigma}$ is a strongly convex rational polyhedral cone.

ii. For all $\sigma \in \overline{\Sigma}$, each face of σ is also in $\overline{\Sigma}$.

iii. For all $\sigma_1, \sigma_2 \in \overline{\Sigma}$, the intersection $\sigma_1 \bigcap \sigma_2$ is a face of each.

Given a fan $\overline{\Sigma}$, we obtain a toric variety $X_{\overline{\Sigma}}$ as follows: for each $\sigma \in \overline{\Sigma}$, the affine variety corresponding to σ is defined to be

$$U_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

where $S_\sigma := \sigma^\vee \bigcap M$. For any $\sigma_1, \sigma_2 \in \overline{\Sigma}$, the intersection $\tau = \sigma_1 \bigcap \sigma_2 \in \overline{\Sigma}$ and there exists $m \in \sigma_1^\vee \bigcap (-\sigma_2)^\vee \bigcap M$ such that

$$\sigma_1 \bigcap H_m = \tau = \sigma_2 \bigcap M$$

and hence

$$U_{\sigma_1} \supset (U_{\sigma_1})_{\chi^m} = U_\tau = (U_{\sigma_2})_{\chi^{-m}} \subset U_{\sigma_2}.$$

After glueing U_{σ_1} and U_{σ_2} along U_τ for all $\sigma_1, \sigma_2 \in \overline{\Sigma}$, the toric variety we obtain is $X_{\overline{\Sigma}}$.

2.1.1 Orbit-Cone correspondence

For a cone σ, points of U_σ are in bijective correspondence with semigroup homomorphisms $\gamma : S_\sigma \to \mathbb{C}$ and the distinguished point γ_σ of σ is the point in U_σ corresponding to the semigroup homomorphism

$$m \in S_\sigma \mapsto \begin{cases} 1 & m \in S_\sigma \bigcap \sigma^\perp \\ 0 & \text{otherwise.} \end{cases}$$

We denote the torus orbit of γ_σ by

$$\mathscr{O}(\sigma) = \mathbb{T} \cdot \gamma_\sigma \subset X_{\overline{\Sigma}}.$$

Theorem 2.1 ([6], Theorem 3.2.6.). *Let $X_{\overline{\Sigma}}$ be the toric variety of the fan $\overline{\Sigma}$ in $N_{\mathbb{R}}$. Then:*

i. There is a bijective correspondence

$$\overline{\Sigma} \longleftrightarrow \{\mathbb{T}\text{-orbits in } X_{\overline{\Sigma}}\}$$
$$I \mapsto \mathscr{O}(I).$$

ii. For each cone $\sigma \in \overline{\Sigma}, \dim \mathscr{O}(\sigma) = n - \dim \sigma$.

iii. The affine open subset U_σ is the union of orbits

$$U_\sigma = \bigcup_{\tau \preceq \sigma} \mathscr{O}(\tau).$$

iv. $\tau \preceq \sigma$ if and only if $\mathscr{O}(\sigma) \subset \overline{\mathscr{O}(\tau)}$ and

$$\overline{\mathscr{O}(\tau)} = \bigcup_{\tau \preceq \sigma} \mathscr{O}(\sigma)$$

where $\overline{\mathscr{O}(\tau)}$ denotes the closure of $\mathscr{O}(\tau)$ in the Zariski topology.

2.1.2 Divisors and equivariant line bundles

We recall the construction of an equivariant line bundles from a torus invariant Cartier divisor D on toric varieties in toric geometry in section 2.1 of [15].

For a Cartier divisor $D=\sum_{\rho\in\overline{\Sigma}(1)} a_\rho D_\rho$ on a toric variety $X_{\overline{\Sigma}}$ such that $D|_{U_\sigma}=\operatorname{div}(-l_\sigma)$ for some $-l_\sigma \in M$, the line bundle L_D corresponding to the sheaf $\mathscr{O}_{X_{\overline{\Sigma}}}(D)$ is equivariantly trivial over U_σ where the dense torus $\mathbb{T}\subset X_{\overline{\Sigma}}$ acts via

$$t(x,v) := (tx, \chi^{-l_\sigma}(t)v) \text{ for all } t\in\mathbb{T}, (x,v)\in U_\sigma\times\mathbb{C}.$$

For $\sigma,\tau\in\Delta$, the transition function

$$g_{\sigma\tau} : U_\tau\times\mathbb{C}\supset (U_\tau\bigcap U_\sigma)\times\mathbb{C}\to(U_\tau\bigcap U_\sigma)\times\mathbb{C}\subset U_\sigma\times\mathbb{C}$$

sends (x,c) to $(x,\frac{\chi^{-l_\sigma}}{\chi^{-l_\tau}}(x)c)=(x,\frac{\chi^{l_\tau}}{\chi^{l_\sigma}}(x)c)$.

2.2 Classification of toric vector bundles

Theorem 2.2 ([12],Prop 2.1.1.)**.** *Let σ be a smooth rational cone and $X=X(\sigma):=\operatorname{Spec} k[\sigma^\vee]$ the affine toric variety.*

i. All toric vector bundles on X take the form $E\times X$ where E is a linear representation space of the torus $\mathbb{T}$.

ii. Two toric vector bundles $\mathscr{E}=E\times X$ and $\mathscr{F}=F\times X$ are isomorphic if and only if the restrictions of the representations are isomorphic: $E|_{\mathbb{T}_\sigma}\simeq F|_{\mathbb{T}_\sigma}$.

iii. Define decreasing $\mathbb{Z}$-filtrations on the space E

$$E^{\alpha}(i) = \bigoplus_{\langle\chi,\alpha\rangle \geq i} E_{\chi}, \alpha \in |\sigma|,$$

where $E_{\chi} \subset E$ is the isotypical component of the character $\chi \in M$. Then the space of toral homomorphisms $\mathrm{Hom}_{\mathbb{T}}(\mathscr{E},\mathscr{F}), \mathscr{E} = E \times X, \mathscr{F} = F \times X$, is canonically isomorphic to the space of the linear operators $\varphi : E \to F$, compatible with the filtrations: $\varphi(E^{\alpha}(i)) \subset F^{\alpha}(i)$, $\forall \alpha \in |\sigma|, i \in \mathbb{Z}$.

Theorem 2.3 ([12], Theorem 2.2.1)**.** *Let $\overline{\Sigma}$ be a smooth fan and $X = X_{\overline{\Sigma}}$ the toric variety corresponding to $\overline{\Sigma}$. The category of toric vector bundles over X is equivalent to the category of vector spaces E with a family of decreasing $\mathbb{Z}$-filtrations $E^{\alpha}(i), \alpha \in |\overline{\Sigma}|$, which satisfy the following compatibility condition:*
For any $\sigma \in \overline{\Sigma}$, there exists a M_{σ}-grading $E = \bigoplus_{\chi \in M_{\sigma}} E^{[\sigma]}(\chi)$ for which

$$E^{\alpha}(i) = \sum_{\langle\chi,\alpha\rangle \geq i]} E^{[\sigma]}(\chi),\ \forall \alpha \in |\sigma|.$$

2.3 Hirzebruch-Riemann-Roch theorem for smooth complete toric varieties

Definition 2.4. The Chern roots of the tangent bundle of a smooth complete toric variety X are elements $\xi_i \in H^2(X,\mathbb{Q})$ such that

$$c(T_X) = \prod_{i=1}^{n}(1+\xi_i).$$

The Todd class of X is

$$\mathrm{Td}(X) = \prod_{i=1}^{n} \frac{\xi_i}{1-e^{-\xi_i}}.$$

Theorem 2.4. *Let $X = X_{\overline{\Sigma}}$ be a smooth complete n-dimensional toric variety with tangent bundle T_X. Then*

$$c(T_X) = \prod_{\rho \in \overline{\Sigma}(1)} (1 + [D_\rho])$$

and

$$\mathrm{Td}(X) = \prod_{i=1}^{n} \frac{[D_\rho]}{1 - e^{-[D_\rho]}} \in H^{\cdot}(X, \mathbb{Q})$$

where D_ρ is the closure of the orbit $\mathscr{O}(\rho)$ corresponding to the ray $\rho \in \overline{\Sigma}(1)$.

Definition 2.5. For a complete toric variety $X = X_{\overline{\Sigma}}$, the Euler characteristic of an invertible sheaf $\mathscr{L}$ on X is

$$\chi(\mathscr{L}) = \sum_{i=0}^{n} (-1)^i \dim H^i(X, \mathscr{L}).$$

Theorem 2.5 ([6], Theorem 13.3.8.)**.** *For an invertible sheaf $\mathscr{L}$ on a smooth complete toric variety $X = X_{\overline{\Sigma}}$, we have*

$$\chi(\mathscr{L}) = \int_X \mathrm{ch}(\mathscr{L})\,\mathrm{Td}(X)$$

and the Chern character of $\mathscr{L}$ is

$$\mathrm{ch}(\mathscr{L}) = e^{c_1(\mathscr{L})} \in H^{\cdot}(X, \mathbb{Q}).$$

Chapter 3: Toric Topology

Topological toric manifolds are generalizations of smooth complete toric varieties and torus manifolds are further generalizations of topological toric manifolds. In this chapter, we review

1. The bijective correspondence between the isomorphism classes of topological toric manifolds and the isomorphism classes of topological fans in [10].

2. Comparison among toric varieties, topological toric manifolds and torus manifolds.

3. Hattori and Masuda's combinatorial Hirzebruch-Riemann-Roch theorem for torus manifolds in [8].

Definition 3.1. A closed smooth manifold X of dimension $2n$ with an effective smooth action of $\mathbb{T}$ having an open dense orbit is a topological toric manifold if it is covered by finitely many invariant open subsets each equivariantly diffeomorphic to a direct sum of complex one-dimensional smooth representation spaces of $\mathbb{T}$.

Definition 3.2. Let M be a closed, connected, oriented, smooth manifold of dimension $2n$ with an effective smooth action of an n-dimensional torus group $G = (S^1)^n$ with non-empty fixed point set M^G. A closed, connected codimension two submanifold of M is called characteristic if it is a connected component of the set fixed pointwise by a certain circle subgroup of G and contains at least one G-fixed point. M is a torus manifold if a preferred orientation is given to each characteristic submanifold M_i.

3.1 Topological toric manifolds

3.1.1 Construction of topological toric manifolds from topological fans

Definition 3.3. Let S be the ring of 2×2 matrices of the form $\begin{bmatrix} b & 0 \\ c & v \end{bmatrix}$ where $b,c\in\mathbb{R}, v\in\mathbb{Z}$. Denote the set $\mathbb{C}\times\mathbb{Z}$ by $\mathscr{R}$. We define a ring structure on $\mathscr{R}$ via the bijection $\mathscr{R}\to S$ sending $\mu=(b+\sqrt{-1}c,v)$ to $\begin{bmatrix} b & 0 \\ c & v \end{bmatrix}$, i.e.

$$\mu_1+\mu_2 := ((b_1+b_2)+\sqrt{-1}(c_1+c_2), v_1+v_2)$$
$$\mu_1\mu_2 := (b_1b_2+\sqrt{-1}(b_1c_2+c_1v_2), v_1v_2).$$

for $\mu_i=(b_i+\sqrt{-1}c_i, v_i), i=1,2$.

For $g\in\mathbb{C}^*, \mu=(b+\sqrt{-1}c,v)\in\mathscr{R}$, define

$$g^\mu := |g|^{b+\sqrt{-1}c}(\frac{g}{|g|})^v,$$

then $(g^{\mu_1})^{\mu_2}=g^{\mu_2\mu_1}$.

The identity element $\mathbf{1}=(1,1)$ of $\mathscr{R}$ corresponds to $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. The ring $\mathscr{R}^n$ plays the roles of the character group and the cocharacter group for topological toric manifolds. For $\mu\in\mathscr{R}^n=\mathbb{C}^n\times\mathbb{Z}^n$, we also denote it by $\mu=(b(\mu)+\sqrt{-1}c(\mu), v(\mu))$ where $b(\mu),c(\mu)\in\mathbb{R}^n, v(\mu)\in\mathbb{Z}^n$.

Definition 3.4. Let G be an abelian topological group. A continuous (resp. smooth) character of G is a continuous (resp. smooth) homomorphism $G\to\mathbb{C}^*$. A continuous (resp. smooth) cocharacter of G is a continuous (resp. smooth) homomorphism $\mathbb{C}^*\to G$.

For $\alpha = (\alpha^1, \ldots, \alpha^n) \in \mathscr{R}^n$ and $\beta = (\beta^1, \ldots, \beta^n) \in \mathscr{R}^n$, define smooth characters $\chi^\alpha \in \mathrm{Hom}((\mathbb{C}^*)^n, \mathbb{C}^*)$ and smooth cocharacters $\lambda_\beta \in \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ via

$$\chi^\alpha(g_1, \ldots, g_n) := \prod_{k=1}^n g_k^{\alpha^k}, \quad \lambda_\beta(g) := (g^{\beta^1}, \ldots, g^{\beta^n}).$$

Lemma 3.1. *All continuous characters (resp. cocharacters) of $\mathbb{T}$ are of the form χ^α (resp. λ_β). In particular, every continuous character of $\mathbb{T}$ is smooth.*

Proof. Given a continuous character $\chi : \mathbb{C}^* \to \mathbb{C}^*$, because $\mathbb{C}^* \simeq \mathbb{R}_{>0} \times S^1$ as an abelian topological group, the restriction $\chi_{\mathbb{R}_{>0}}$ of χ to $\mathbb{R}_{>0}$ and the restriction χ_{S^1} of χ to S^1 are also continuous homomorphisms. $\mathbb{R}_{>0}$ is isomorphic to $(\mathbb{R}, +)$ as an abelian topological group via log, therefore $\chi_{\mathbb{R}_{>0}}(r) = r^{b+ci}$ for some $b, c \in \mathbb{R}$. Let $\pi : \mathbb{R} \to S^1$ be the covering map $\theta \mapsto e^{2\pi i\theta}$. Let $\phi : \mathbb{R} \to \mathbb{R}$ be the unique lifting of $\chi_{S^1} \circ \pi$ with $\phi(0) = 0$ so that the following diagram is commutative:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "\phi"] \arrow[d, "\pi"'] & \mathbb{R} \arrow[d, "\pi"] \\ S^1 \arrow[r, "\chi_{S^1}"] & S^1 \end{tikzcd}$$

Figure 3.1: Universal Covering

We prove that ϕ is a homomorphism. Let $\theta_1, \theta_2 \in \mathbb{R}$. We have

$$\chi_{S^1}(\pi(\theta_1 + \theta_2)) = \chi_{S^1}(\pi(\theta_1))\chi_{S^1}(\pi(\theta_2)).$$

By commutativity of the diagram, this is equivalent to

$$\pi(\phi(\theta_1+\theta_2)) = \pi(\phi(\theta_1)+\phi(\theta_2)).$$

So $f(\theta_1,\theta_2) := \phi(\theta_1+\theta_2)-\phi(\theta_1)-\phi(\theta_2) \in \mathbb{Z}$. Since f is continuous, f must be constant. This constant must be 0 because $\phi(0)=0$. Hence ϕ is a homomorphism. Let $v=\phi(1)$. It's easy to see that $\phi(n)=nv$ for $n \in \mathbb{Q}$. By continuity of ϕ, we know $\phi(r)=rv$ for all $r \in \mathbb{R}$. By continuity of the above diagram, $v \in \mathbb{Z}$. Hence $\chi(z)=z^{\mu}$ where $\mu=(b+ci,v) \in \mathscr{R}$. In general, given a continuous character $\chi:(\mathbb{C}^*)^n \to \mathbb{C}^*$, its restriction χ_i to the i-th coordinate subspace of $(\mathbb{C}^*)^n$ is χ^{α^i} for some $\alpha^i \in \mathscr{R}$. Hence $\chi=\chi^{\alpha}$ where $\alpha=(\alpha^1,\cdots,\alpha^n)$. Notice that a continuous cocharacter of $\mathbb{C}^*$ is the same as a continuous character of $\mathbb{C}^*$, so we can prove the claim about continuous cocharacters similarly. □

Definition 3.5. Define a bracket operation by

$$\langle \alpha,\beta\rangle := \sum_{k=1}^{n} \alpha^k \beta^k \in \mathscr{R}.$$

Notice that in general $\langle \alpha,\beta\rangle \neq \langle \beta,\alpha\rangle$ because the matrix product is not commutative.

For a family $\{\alpha_i\}_{i=1}^n$ of n elements in $\mathscr{R}^n$, define its dual to be a family $\{\beta_i\}_{i=1}^n$ such that $\langle \alpha_i,\beta_j\rangle = \delta_{ij}\mathbf{1}$ for all i,j, which is equivalent to

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \dots & \alpha_1^n \\ \alpha_2^1 & \alpha_2^2 & \dots & \alpha_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^1 & \alpha_n^2 & \dots & \alpha_n^n \end{bmatrix} \begin{bmatrix} \beta_1^1 & \beta_2^1 & \dots & \beta_n^1 \\ \beta_1^2 & \beta_2^2 & \dots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \dots & \beta_n^n \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{bmatrix} \tag{3.1}$$

where $\alpha_j = (\alpha_j^1,\dots,\alpha_j^n), \beta_j = (\beta_j^1,\dots,\beta_j^n)^T$.

For $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$ with $\alpha_i,\beta_i \in \mathscr{R}^n$, we define group endomorphisms $\bigoplus_{i=1}^n \chi^{\alpha_i}$

and $\prod_{i=1}^n \lambda_{\beta_i}$ of $(\mathbb{C}^*)^n$ by

$$\begin{aligned}
&(\bigoplus_{i=1}^n \chi^{\alpha_i})(g_1,\dots,g_n) := (\chi^{\alpha_1}(g_1,\dots,g_n),\dots,\chi^{\alpha_n}(g_1,\dots,g_n)) \\
&(\prod_{i=1}^n \lambda_{\beta_i})(g_1,\dots,g_n) := \prod_{i=1}^n \lambda_{\beta_i}(g_i).
\end{aligned} \tag{3.2}$$

Lemma 3.2 (cf. Lemma 2.4 of [10])**.** *If $\{b_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ are bases of $\mathbb{R}^n$ and $\mathbb{Z}^n$ respectively, then $\{\beta_i\}_{i=1}^n$ has a dual set $\{\alpha_i\}_{i=1}^n$, i.e. $\langle \alpha_i, \beta_j \rangle = \delta_{ij}\mathbf{1}$ for any i, j.*

Lemma 3.3 (cf. Lemma 2.3 of [10])**.** *If $\{\alpha_i\}_{i=1}^n$ is dual to $\{\beta_i\}_{i=1}^n$, then the composition*

$$(\prod_{i=1}^n \lambda_{\beta_i})(\bigoplus_{i=1}^n \chi^{\alpha_i}) \colon (\mathbb{C}^*)^n \to (\mathbb{C}^*)^n$$

is the identity, in particular, both $\bigoplus_{i=1}^n \chi^{\alpha_i}$ and $\prod_{i=1}^n \lambda_{\beta_i}$ are automorphisms of $(\mathbb{C}^*)^n$.

Definition 3.6. An abstract simplicial complex Σ is a collection of non-empty finite subsets of a set S such that for every set $I \in \Sigma$ and $J \subset I$, we have $J \in \Sigma$. Let Σ be an abstract simplicial complex. If Σ contains only finitely many elements, then it is called a finite abstract simplicial complex. The dimension of $I \in \Sigma$ is $|I|-1$. If $|I| \leq n$ for all $I \in \Sigma$ and there exists $I_0 \in \Sigma$ such that $|I_0| = n$, then Σ is called an abstract simplicial complex of dimension $n-1$. An augmented abstract finite simplicial complex Σ of dimension $n-1$ is a set $\Sigma' \bigcup \{\emptyset\}$ where Σ' is an abstract finite simplicial complex of dimension $n-1$. For $i \in \mathbb{N}$, denote by $\Sigma^{(i)}$ the set of elements $I \in \Sigma$ such that $|I| = i$. In particular, we denote the vertex set by $\Sigma^{(1)}$. We call the elements in Σ abstract cones.

Definition 3.7. Let Σ be an augmented abstract finite simplicial complex of dimension $n-1$ and let

$$\beta \colon \Sigma^{(1)} \to \mathscr{R}^n = \mathbb{C}^n \times \mathbb{Z}^n$$

where $\Sigma^{(1)}$ denotes the vertex set of Σ. We abbreviate an element $\{i\} \in \Sigma^{(1)}$ as i and $\beta(\{i\})$

as β_i and express $\beta_i = (b_i + \sqrt{-1}c_i, v_i) \in \mathbb{C}^n \times \mathbb{Z}^n$. Denote the cone spanned by $b_i, i \in I$ by $\angle b_I$. Then the pair (Σ, β) is called a *(simplicial) topological fan* of dimension n if the following are satisfied.

1. b_i's for $i \in I$ are linearly independent whenever $I \in \Sigma$, and $\angle b_I \cap \angle b_J = \angle b_{I \cap J}$ for any $I, J \in \Sigma$. (In short, the collection of cones $\angle b_I$ for all $I \in \Sigma$ is an ordinary simplicial fan although b_i's are not necessarily in $\mathbb{Z}^n$.)

2. Each v_i is primitive and v_i's for $i \in I$ are linearly independent (over $\mathbb{R}$) whenever $I \in \Sigma$.

A topological fan Δ of dimension n is *complete* if $\bigcup_{I \in \Sigma} \angle b_I = \mathbb{R}^n$ and *non-singular* if the v_i's for $i \in I$ form a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ whenever $I \in \Sigma$.

Construction. Let $\Delta = (\Sigma, \beta)$ be a non-singular topological fan of dimension n. We take the vertex set $\Sigma^{(1)}$ as $[m] = \{1, 2, \ldots, m\}$. Because Σ is a fan, for all $I \in \Sigma$, we have $I \subset [m]$. For $I \subset [m]$, we set $I^c := \Sigma^{(1)} \backslash I$ and

$$U(I) := \{(z_1, \ldots, z_m) \in \mathbb{C}^m \mid z_i \neq 0 \text{ for } \forall i \notin I\} = \mathbb{C}^I \times (\mathbb{C}^*)^{I^c}.$$

Note that $U(I) \cap U(J) = U(I \cap J)$ for any $I, J \in [m]$ and $U(I) \subset U(J)$ if and only if $I \subset J$. We define

$$U(\Sigma) := \bigcup_{I \in \Sigma} U(I).$$

Let

$$\lambda : (\mathbb{C}^*)^m \to (\mathbb{C}^*)^n$$

be the homomorphism defined by

$$\lambda(h_1, \ldots, h_m) := \prod_{k=1}^{m} \lambda_{\beta_k}(h_k).$$

Lemma 3.4. *λ is surjective.*

Proof. Given $I \in \Sigma^{(n)}$, let $\{\alpha_i^I\}_{i\in I}$ be the dual to $\{\beta_i\}_{i\in I}$. We prove that $\bigoplus_{i\in I}\chi^{\alpha_i^I}\circ\lambda$ is surjective. By Lemma 3.3 $\bigoplus_{i\in I}\chi^{\alpha_i^I}$ is a bijection, hence λ is surjective.

Surjectivity of $\bigoplus_{i\in I}\chi^{\alpha_i^I}\circ\lambda$: Given $(z_1,\ldots,z_n)\in(\mathbb{C}^*)^n=(\mathbb{C}^*)^I$, let $h_k=z_k$ if $k\in I$ and $h_k=1$ otherwise, then

$$\chi^{\alpha_i^I}\circ\lambda(h_1,\ldots,h_m)=\prod_{k=1}^{m}h_k^{\langle\alpha_i^I,\beta_k\rangle}=z_i.$$

Hence $\bigoplus_{i\in I}\chi^{\alpha_i^I}\circ\lambda(h_1,\ldots,h_m)=(z_1,\ldots,z_n)$. □

Lemma 3.5 (cf. Lemma 4.1 of [10])**.**

$$\operatorname{Ker}\lambda=\{(h_1,\ldots,h_m)\in(\mathbb{C}^*)^m \mid \prod_{k=1}^{m}h_k^{\langle\alpha,\beta_k\rangle}=1 \quad \textit{for any } \alpha\in\mathscr{R}^n\}. \tag{3.3}$$

Using $\{\alpha_i^I\}_{i\in I}$ for $I\in\Sigma^{(n)}$, which is dual to $\{\beta_i\}_{i\in I}$, we have

$$\operatorname{Ker}\lambda=\{(h_1,\ldots,h_m)\in(\mathbb{C}^*)^m \mid h_i\prod_{k\notin I}h_k^{\langle\alpha_i^I,\beta_k\rangle}=1 \quad \textit{for any } i\in I\}. \tag{3.4}$$

Definition 3.8. We call

$$X(\Delta):=U(\Sigma)/\operatorname{Ker}\lambda$$

equipped with the quotient topology together with the natural action of $\mathbb{T}=(\mathbb{C}^*)^n\simeq(\mathbb{C}^*)^m/\operatorname{Ker}\lambda$ the topological toric manifold associated with Δ.

Denote $U(I)/\operatorname{Ker}\lambda$ by U_I and the representation space of $\bigoplus_{i\in I}\chi^{\alpha_i^I}$ by $\bigoplus_{i\in I}V(\chi^{\alpha_i^I})\simeq\mathbb{C}^I$. The following theorem confirms that $X(\Delta)$ is a smooth manifold.

Theorem 3.6 ([10], Lemma 5.1, Lemma 5.2)**.** *$X(\Delta)$ is a smooth manifold with a smooth*

$\mathbb{T}$*-action whose equivariant local charts are the maps*

$$\varphi_I : U_I \to \bigoplus_{i\in I} V(\chi^{\alpha_i^I})$$

defined via

$$\varphi_I(z_1,\ldots,z_m) := (\prod_{k=1}^{m} z_k^{\langle\alpha_i^I,\beta_k\rangle})_{i\in I}$$

for all $I \in \Sigma^{(n)}$ *and whose* j*-th component of transition function* $\varphi_J\varphi_I^{-1} : \varphi_I(U_I \bigcap U_J) \to \varphi_J(U_J)$ *is given by*

$$\prod_{i\in I} w_i^{\langle\alpha_j^J,\beta_i\rangle}.$$

for all $I,J \in \Sigma^{(n)}$.

Remark 1: Because of theorem 3.6, sometimes we tacitly identify U_I with $\bigoplus_{i\in I} V(\chi^{\alpha_i^I})$, e.g. in lemma 4.5.

3.1.2 Construction of topological fans from topological toric manifolds

Definition 3.9. A closed connected smooth submanifold of a topological toric manifold X of real codimension two is characteristic if it is fixed pointwise by some $\mathbb{C}^*$-subgroup of $\mathbb{T}$. There are finitely many characteristic submanifolds denoted by $X_1,\cdots,X_m$.

Definition 3.10. An omniorientation on a topological toric manifold X is a choice of an orientation on each characteristic submanifolds X_i together with the canonical orientation on X induced from the orientation of the open dense orbit which is identified with $(\mathbb{C}^*)^n$.

Let X be an omnioriented topological toric manifold. Define

$$\Sigma(X) := \{I \subset [m] | X_I := \bigcap_{i\in I} X_i \neq \emptyset\} \bigcup \{\emptyset\}.$$

Lemma 3.7 ([10], Lemma 3.3)**.** *For each $i \in [m] = \Sigma^{(1)}(X)$, there is a unique $\beta_i(X) \in \mathscr{R}^n$ such that $\lambda_{\beta_i(X)}(\mathbb{C}^*)$ fixes X_i pointwise and $\lambda_{\beta_i(X)}(g)_*\xi = g\xi$ for any $g \in \mathbb{C}^*, \xi \in \nu_i$ where $\lambda_{\beta_i(X)}(g)_*$ denotes the differential of $\lambda_{\beta_i(X)}(g)$ and the right hand side denotes scalar multiplication by g.*

Write $\beta_i(X)$ as

$$\beta_i(X) = (b_i(X) + \sqrt{-1}c_i(X), v_i(X)) \in \mathscr{R}^n = \mathbb{C}^n \times \mathbb{Z}^n$$

and define

$$\beta : \Sigma^{(1)} \to \mathscr{R}^n = \mathbb{C}^n \times \mathbb{Z}^n$$

by $\beta(\{i\}) = \beta_i$.

Theorem 3.8 ([10], Lemma 3.8)**.** $\Delta(X) = (\Sigma(X), \beta(X))$ *is a complete non-singular topological fan of dimension n.*

3.1.3 Classification of topological toric manifolds

Definition 3.11. Let $\Delta = (\Sigma, \beta), \Delta' = (\Sigma', \beta')$ be topological fans. Δ, Δ' are equivalent, denoted $[\Delta] = [\Delta']$, if there is an isomorphism $\sigma : \Sigma \to \Sigma'$ such that $\beta'_{\sigma(i)} = \beta_i$ for any $i \in \Sigma^{(1)}$.

Theorem 3.9 ([10], Theorem 8.1)**.** *The category of isomorphism classes of omnioriented topological toric manifolds of dimension* $2n$ *is equivalent to the category of isomorphism classes of complete, non-singular topological fans of dimension n.*

3.2 Smooth complete toric varieties vs. topological toric manifolds vs. torus manifolds

If Δ is a non-singular complete topological fan such that $\beta_i = (v_i, v_i) \in \mathbb{Z}^n \times \mathbb{Z}^n$ for each $i \in \Sigma^{(1)}$, then $X(\Delta)$ is just the toric variety associated with the fan Σ' whose cones are all $\sigma_I = \angle(\{v_i | i \in I\})$. Hence every smooth complete toric variety is an omnioriented topological toric manifold.

For an omnioriented topological toric manifold X, if we restrict the $\mathbb{T}$-action to $G = (S^1)^n$-action, then X is a torus manifold.

Example 3.1 ([10] Example in section 5)**.** $\mathbb{CP}^2 \# \mathbb{CP}^2$ is a topological toric manifold but not a toric variety because it does not admit any almost complex structure.

Theorem 3.10 ([10, Prop 8.3])**.** *Let X be an omnioriented TTM and let $\Delta(X) = (\Sigma(X), \beta(X))$ be the complete non-singular topological fan associated with X. We may assume that the vertex set $\Sigma^{(1)}(X)$ of $\Sigma(X)$ is $[m]$ and express $\beta(X)(i) = (b_i(X) + \sqrt{-1}c_i(X), v_i(X)) \in \mathbb{C}^n \times \mathbb{Z}^n$ for $i \in [m] = \Sigma^{(1)}(X)$. Then*

$$H^*(X;\mathbb{Z}) = \mathbb{Z}[\mu_1, \ldots, \mu_m]/\mathscr{I}$$

where $\mu_i \in H^2(X;\mathbb{Z})$ and $\mathscr{I}$ is the ideal generated by the following two types of elements:

1. $\prod_{i \in I} \mu_i$ *for $I \notin \Sigma(X)$,*

2. $\sum_{i=1}^{m} \langle u, v_i(X) \rangle \mu_i$ *for any $u \in \mathbb{Z}^n$.*

Here μ_i is actually the Poincaré dual of the t-divisor D_i in X.

Example 3.2 ([13], Example 3.2)**.** For $n \geq 2$, let S^{2n} be the $2n$-sphere identified with the following subset in $\mathbb{C} \times \mathbb{R}$:

$$\{(z_1, \cdots, z_n, y) \in \mathbb{C} \times \mathbb{R} | |z_1|^2 + \cdots + |z_n|^2 + y^2 = 1\}$$

with the G-action $(t_1, \cdots, t_n) \cdot (z_1, \cdots, z_n, y) = (t_1 z_1, \cdots, t_n z_n, y)$ is a torus manifold but not a topological toric manifold because its cohomology ring is not generated by degree 2 elements.

3.3 Combinatorial Hirzebruch-Riemann-Roch theorem for torus manifolds

3.3.1 Multipolytopes

We recall some general facts about multi-polytopes in sections 2,4,5,7 of [8]. We use Δ to denote a multi-fan in this subsection and will explain the abuse of notations of multi-fans and topological fans.

Fix the lattice $N := H_2(BG, \mathbb{Z}) \cong \mathbb{Z}^n$ and denote $N \otimes \mathbb{R}$ by $N_{\mathbb{R}}$. A strongly convex rational polyhedral cone in $N_{\mathbb{R}}$ is simply called a cone in N. Denote by $\mathrm{Cone}(N)$ the space of all cones in N with a partial order $\prec$ defined by: $\tau \prec \sigma$ if and only if τ is a proper face of σ.

Definition 3.12. A multi-fan in N is a triple $\Delta = (\Sigma, C, w^{\pm})$ where

i. $(\Sigma, <)$ is a finite poset with a unique minimum element $*$.

ii. $C : \Sigma \to \mathrm{Cone}(N)$ is a map satisfying the following properties:

(1) $C(*) = \{0\}$

(2) If $I < J$ for $I, J \in \Sigma$, then $C(I) \prec C(J)$.

(3) For any $J \in \Sigma$, the map C restricted on $\{I \in \Sigma | I \leq J\}$ is an isomorphism of ordered sets onto $\{\kappa \in \mathrm{Cone}(N) | \kappa \preceq C(J)\}$.

iii. $w^{\pm}:\Sigma^{(\mathrm{n})}:=\{I\in\Sigma|\dim C(I)=n\}\to\mathbb{Z}_{\geq 0}$ such that $w^{+}(I)>0$ or $w^{-}(I)>0$ for all $I\in\Sigma^{(\mathrm{n})}$.

A multi-fan Δ' is simplicial (resp. non-singular) if every cone in $C(\Sigma)$ is simplicial (resp. non-singular).

Let $\Delta=(\Sigma,C,w^{\pm})$ be a multi-fan and $w:=w^{+}-w^{-}$. A vector $v\in N_{\mathbb{R}}$ is called generic if v does not lie on any linear subspace spanned by a cone in $C(\Sigma)$ of dimension less than n. For a generic v, define

$$d_v=\sum_{v\in C(I)}w(I)$$

where the sum is understood to be zero if there is no such I.

Definition 3.13. We say Δ is pre-complete if $\Sigma^{(\mathrm{n})}\neq\emptyset$ and d_v is independent of the choice of generic vectors v. This integer is called the degree of Δ and is denoted by $\deg(\Delta)$.

For each $K\in\Sigma$, we set

$$\Sigma_K:=\{J\in\Sigma|K\leq J\}.$$

It inherits the partial order from Σ and K is the unique minimum element in Σ_K. A map

$$C_K:\Sigma_K\to\mathrm{Cone}(N^{C(K)})$$

sending $J\in\Sigma_K$ to the cone $C(J)$ projected on $(N^{C(K)})_{\mathbb{R}}$ satisfies the three properties above required for C where $N^{C(K)}:=N/\langle C(K)\bigcap N\rangle$ is the quotient lattice of N by the sublattice generated by $C(K)\bigcap N$. Define

$$w_K^{\pm}:\Sigma_K^{n-|K|}:=\{J\in\Sigma^{(\mathrm{n})}|K\leq J,\dim C_K(J)=n-|K|\to\mathbb{Z}_{\geq 0}$$

to be the restrictions of $w^{\pm}$ to $\Sigma_K^{n-|K|}$. The triple $\Delta_K:=(\Sigma,C_K,w_K^{\pm})$ is a multi-fan in $N^{C(K)}$ and the projected multi-fan with respect to $K\in\Sigma$.

Definition 3.14. Δ is complete if Δ_K is pre-complete for any $K \in \Sigma$.

Definition 3.15. Let $\Delta = (\Sigma, C, w^{\pm})$ be a complete multi-fan and let $\mathscr{F} : \Sigma^{(1)} \to \mathrm{HP}(N_{\mathbb{R}}^*)$ be a map such that the affine hyperplane $\mathscr{F}(I)$ is perpendicular to the half line $C(I)$ for each $I \in \Sigma^{(1)}$. We call a pair $(\Delta, \mathscr{F})$ a multi-polytope and denote it by $\mathscr{P}$. The dimension of a multi-polytope $\mathscr{P}$ is defined to be the dimension of the multi-fan Δ. A multi-polytope $\mathscr{P}$ is simple if Δ is simplicial.

For a simple multi-polytope $\mathscr{P}$, we assume that $\Sigma^{(1)} = \{1, \cdots, m\}$ and Σ consists of subsets of $[m]$. Denote by v_i a nonzero vector in the 1-dimensional cone $C(\{i\})$ and by F_i the cone $\mathscr{F}(\{i\})$ and set

$$F_I := \bigcap_{i \in I} F_i \text{ for } I \in \Sigma.$$

If $|I| = n$, then F_I is a point, denoted by u_I. Suppose $I \in \Sigma^{(\mathrm{n})}$, then $\{v_i | i \in I\}$ forms a basis of $N_{\mathbb{R}}$ and denote its dual basis of $N_{\mathbb{R}}^*$ by $\{u_i^I | i \in I\}$, i.e. $\langle u_i^I, v_j \rangle = \delta_{ij}$. Take a generic vector $v \in N_{\mathbb{R}}$, then $\langle u_i^I, v \rangle \neq 0$ for all $I \in \Sigma^{(\mathrm{n})}, i \in I$. Set

$$(-1)^I := (-1)^{\#\{i \in I | \langle u_i^I, v \rangle > 0\}}, (u_i^I)^+ := \begin{cases} u_i^I & \text{if } \langle u_i^I, v \rangle > 0 \\ -u_i^I & \text{if } \langle u_i^I, v \rangle < 0. \end{cases}$$

Denote by $C^*(I)^+$ the cone in $N_{\mathbb{R}}^*$ spanned by $(u_i^I)^+$'s $(i \in I)$ with apex at u_I and by ϕ_I its characteristic function.

Definition 3.16. The Duistermaat-Heckman function $DH_{\mathscr{P}}$ on $N_{\mathbb{R}}^* \backslash \bigcup_{i=1}^m F_i$ associated with $\mathscr{P}$ is defined to be

$$\mathrm{DH}_{\mathscr{P}} := \sum_{I \in \Sigma^{(\mathrm{n})}} (-1)^I w(I) \phi_I.$$

Lemma 3.11 ([8, Lemma 5.4])**.** *The support of* $\mathrm{DH}_{\mathscr{P}}$ *is bounded and is independent of the choice of the generic vector* $v \in N_{\mathbb{R}}$.

For a simple lattice multi-polytope $\mathscr{P} = (\Delta, \mathscr{F})$, we further require v_i to be a primitive integral vector in the half line $C(\{i\})$. Then we slightly move F_i in the direction of v_i for each i, so that we obtain a map $\mathscr{F}_+ : \Sigma^{(1)} \to \mathrm{HP}(N^*_{\mathbb{R}})$ and denote the multi-polytope $(\Delta, \mathscr{F}_+)$ by $\mathscr{P}_+$. Since $\mathscr{F}_+(\{i\})$'s miss the lattice N^*, $\mathrm{DH}_{\mathscr{P}_+}$ is defined on N^*.

Definition 3.17.

$$\#\mathscr{P} := \sum_{u \in N^*} \mathrm{DH}_{\mathscr{P}_+}(u).$$

3.3.2 Combinatorial equivariant cohomology

This subsection is a review of §8 of [8].

Let $\mathscr{P} = (\Delta, \mathscr{F})$ be a simple lattice multi-polytope associated with integers $c_i's$ by

$$\mathscr{F}(\{i\}) = \{u \in H^2(BG; \mathbb{R}) | \langle u, v_i \rangle = c_i\}.$$

Define the equivariant cohomology of Δ as

$$H^*_G(\Delta) := \mathbb{Z}[x_1, \cdots, x_m]/(x_I | I \notin \Sigma)$$

where $x_I = \prod_{i \in I} x_i$ and the degree of x_i is two. Define a group homomorphism

$$\begin{aligned} \pi^* : H^2(BG, \mathbb{Z}) &\to H^2_G(\Delta) \\ u &\mapsto \sum_{i=1}^{m} \langle u, v_i \rangle x_i \end{aligned}$$

where $\langle , \rangle$ is the natural pairing between cohomology and homology and extend it to an algebra homomorphism

$$\pi^* : H^*(BG, \mathbb{Z}) \to H^*_G(\Delta).$$

π^* defines a $H^*(BG,\mathbb{Z})$-module structure on $H^*_G(\Delta)$.

Lemma 3.12 ([8, Lemma 8.1])**.** *Any element in $H^*_G(\Delta)\otimes\mathbb{Q}$ can be written in the form $\sum_{J\in\Sigma}\pi^*(a_J)x_J$ with $a_J\in H^*(BG,\mathbb{Q})$.*

For $I\in\Sigma^{(\mathrm{n})}$ let $\{u_i^I|i\in I\}=\{-q(\alpha_i^I)|i\in I\}$ be the dual basis of $\{v_i|i\in I\}$ as before. Define a ring homomorphism $\iota_I^*:H^*_G(\Delta)\otimes\mathbb{Q}\to H^*(BG,\mathbb{Q})$ by

$$\iota_I^*(x_i)=\begin{cases}u_i^I & \text{if } i\in I,\\ 0 & \text{otherwise.}\end{cases}$$

Lemma 3.13 ([8, Lemma 8.2])**.** *ι_I^* is an $\hat{H}^{\cdot}_G(\{\mathrm{pt}\},\mathbb{Q})$-module map.*

For $I\in\Sigma^{(\mathrm{n})}$, define G_I to be the image of

$$\{(a_1,\cdots,a_m)\in\mathbb{R}^m|\sum_{i=1}^m a_iv_i\in N, a_j=0\text{ for } j\notin I\}$$

under the projection $\mathbb{R}^m\to\mathbb{R}^m/\mathbb{Z}^m$.

Definition 3.18. The combinatorial generalized equivariant Gysin map (called the index map in [8]) is defined by

$$\begin{aligned}\pi_!:H^*_G(\Delta)\otimes\mathbb{Q}&\to H^*(BG,\mathbb{Q})\\ x&\mapsto\sum_{I\in\Sigma^{(\mathrm{n})}}\frac{w(I)\iota_I^*(x)}{|G_I|\prod_{i\in I}u_i^I}.\end{aligned}$$

The cohomology ring of the multi-fan Δ of $\mathscr{P}$ is defined by

$$H^*(\Delta):=\mathbb{Z}[x_1,\cdots,x_m]/\mathfrak{A}$$

where $\mathfrak{A}$ is the ideal generated by all

(1) x_I for $I \notin \Sigma$,

(2) $\sum_{i=1}^{m} \langle u, v_i \rangle x_i$ for $u \in N$.

$\pi_!$ induces a homomorphism

$$\int_\Delta : H^*(\Delta) \otimes \mathbb{Q} \to H^*(BG, \mathbb{Q})/H^2(BG, \mathbb{Q}) = \mathbb{Q}. \tag{3.5}$$

For $i \in [m]$, let

$$\begin{aligned} \rho_i : \mathbb{R}^m/\mathbb{Z}^m &\to \mathbb{C}^* \\ (a_1, \cdots, a_m) &\mapsto \exp(2\pi\sqrt{-1}a_i). \end{aligned}$$

Denote by G_Δ the union of all G_I where I runs over all $I \in \Sigma^{(\mathrm{n})}$.

Definition 3.19. We define the equivariant Todd class $\mathscr{T}^G(\Delta)$ of Δ to be

$$\mathscr{T}^G(\Delta) := \sum_{G_\Delta} \prod_{i=1}^{m} \frac{x_i}{1 - \rho_i(g) e^{-x_i}},$$

and the Todd class $\mathscr{T}(\Delta)$ of Δ by

$$\mathscr{T}(\Delta) := \sum_{G_\Delta} \prod_{i=1}^{m} \frac{\overline{x_i}}{1 - \rho_i(g) e^{-\overline{x_i}}},$$

where $\overline{x_i}$ is the image of $x_i \in H_G^*(\Delta)$ in $H^*(\Delta)$.

3.3.3 Combinatorial Hirzebruch-Riemann-Roch theorem

Theorem 3.14 ([8], Theorem 8.5)**.** *If $\mathscr{P}$ is a simple lattice multi-polytope, then $\int_\Delta e^{c_1(\mathscr{P})} \mathscr{T}(\Delta) = \#(\mathscr{P})$.*

Chapter 4: Klyachko vector bundles

4.1 Orbit-Cone correspondence

Definition 4.1. Let $\Delta = (\Sigma, \beta)$ be a non-singular topological fan of dimension n. For each $I \in \Sigma$, we call

$$\gamma_I = [0^I \times 1^{\Sigma^{(1)} \setminus I}] \in U(I)/\mathrm{Ker}\,\lambda$$

the distinguished point corresponding to I and we denote its $\mathbb{T}$-orbit by

$$\mathscr{O}(I) := \mathbb{T} \cdot \gamma_I = (0^I \times (\mathbb{C}^*)^{\Sigma^{(1)} \setminus I})/\mathrm{Ker}\,\lambda$$

and the closure of $\mathscr{O}(I)$ in $X(\Delta)$ by

$$V(I) := \overline{\mathscr{O}(I)} = (0^I \times \mathbb{C}^{\Sigma^{(1)} \setminus I})/\mathrm{Ker}\,\lambda \subset X(\Delta).$$

The following theorem is a direct generalization of Theorem 3.2.6 of [6].

Theorem 4.1.

1. *There is a bijective correspondence*

$$\Sigma \longleftrightarrow \{\mathbb{T}\text{-}\mathit{orbits\ in}\ X(\Delta)\}$$
$$I \mapsto \mathscr{O}(I)$$

2. *For each $J \in \Sigma^{(j)}, dim_{\mathbb{C}} \mathscr{O}(J) = n - j$.*

3. $$U_I = \bigcup_{J \leq I} \mathscr{O}(J).$$

4. *$J \leq I$ iff $\mathscr{O}(I) \subset V(J)$ and*
$$V(J) = \bigcup_{J \leq I} \mathscr{O}(I).$$

Proof.

1. Let $\mathscr{O}$ be a $\mathbb{T}$-orbit in $X(\Delta)$ and $\gamma = [\gamma^1, \ldots, \gamma^m] \in \mathscr{O}$. Let $I = \{i \in \Sigma(1) | \gamma^i = 0\}$, then $\mathscr{O} \bigcap \mathscr{O}(I) \neq \emptyset$. Hence $\mathscr{O} = \mathscr{O}(I)$.

2. Given $J \in \Sigma^{(j)}$, let $I \in \Sigma^{(n)}$ such that $J \leq I$. Then $\varphi_I(\mathscr{O}(J)) = 0^J \times (\mathbb{C}^*)^{I \backslash J}$, hence $dim_{\mathbb{C}} \mathscr{O}(J) = n - j$.

3. follows from (a).

4. If $J \leq I$, then by definition $\mathscr{O}(I) \subset V(J)$. Conversely if $\mathscr{O}(I) \subset V(J)$, then $\Sigma(1) \backslash I \subset \Sigma(1) \backslash J$, i.e. $J \leq I$.

□

4.2 Topological Klyachko vector bundles

Definition 4.2. A topological Klyachko vector bundle of rank r over an n-dimensional topological toric manifold $X(\Delta)$ associated with a topological toric fan $\Delta = (\Sigma, \beta)$ is a $\mathbb{T}$-equivariant topological complex vector bundle $\mathscr{E} \to X(\Delta)$ whose restriction $\mathscr{E}|_{U_I}$ is equivariantly isomorphic to $U_I \times \mathbb{C}^r$ for all $I \in \Sigma$ and whose action map $\mathbb{T} \times \mathscr{E} \to \mathscr{E}$ is continuous.

Lemma 4.2. *A continuous character $\chi^\alpha : \mathbb{C}^* \to \mathbb{C}^*$ where $\alpha = (b + c\sqrt{-1}, v) \in \mathscr{R}$ can be continuously extended to $\mathbb{C}$ iff $b > 0$ or $b = c = v = 0$.*

Proof. In this proof, I will use i for $\sqrt{-1}$. Given $\chi^\alpha : \mathbb{C}^* \to \mathbb{C}^*$ with $\alpha = (b + ci, v)$ and $g = \rho e^{\theta i} \in \mathbb{C}^*$ where $\rho \in \mathbb{R}_{>0}$, we have $\chi^\alpha(g) = \rho^{b+ci} e^{i\theta v}$. If $b > 0$, then $|\chi^\alpha(g)| = \rho^b \to 0$ as $g \to 0$, therefore $\chi^\alpha(g) \to 0$ as $g \to 0$. If $b < 0$, then $|\chi^\alpha(g)| = \rho^b \to +\infty$ as $g \to 0$, therefore $\chi^\alpha(g)$ diverges as $g \to 0$. If $b = 0$, suppose $c \neq 0$, then let $g(t) = t, t \in (0, +\infty)$, we have $\chi^\alpha(g(t)) = e^{ic \ln t}$ diverges as $t \to 0^+$. Therefore $c = 0$ when χ^α can be continuously extended to $\mathbb{C}$ and $b = 0$. When $b = c = 0$, let $g(t) = \frac{e^{ti}}{t}, t \in (0, +\infty)$, then $\chi^\alpha(g(t))$ converges as $t \to +\infty$ if and only if $v = 0$. Therefore when $b = 0$, χ^α can be continuously extended to $\mathbb{C}$ if and only if $c = v = 0$. $\square$

Definition 4.3. We can define a partial order $\geq_c$ on $\mathscr{R}$ as follows: For $\alpha = (b + c\sqrt{-1}, v) \in \mathscr{R}$, we say $\alpha \geq_c 0 = (0 + 0\sqrt{-1}, 0)$ if $b > 0$ or $b = c = v = 0$. For $\alpha, \beta \in \mathscr{R}$, we say $\alpha \geq_c \beta$ if $\alpha - \beta \geq_c 0$.

Proposition 4.3. *$\geq_c$ is a partial order.*

Proof. Reflexivity is clear. Anti-symmetry: suppose $\alpha_i = (b_i + c_i\sqrt{-1}, v_i) \in \mathscr{R}, i = 1, 2$ and $\alpha_1 \geq_c \alpha_2, \alpha_2 \geq_c \alpha_1$, then $b_1 - b_2 \geq 0, b_2 - b_1 \geq 0$, therefore $b_1 - b_2 = c_1 - c_2 = v_1 - v_2 = 0$. Transitivity: Suppose $\alpha_i = (b_i + c_i\sqrt{-1}, v_i) \in \mathscr{R}, i = 1, 2, 3$ and $\alpha_1 \geq_c \alpha_2, \alpha_2 \geq_c \alpha_3$. There are four cases: 1) $b_1 > b_2, b_2 > b_3$, 2) $b_1 > b_2, \alpha_2 = \alpha_3$, 3) $\alpha_1 = \alpha_2, b_2 > b_3$, 4) $\alpha_1 = \alpha_2 = \alpha_3$. In the first three cases, we have $b_1 > b_3$, hence $\alpha_1 \geq_c \alpha_3$. The case 4 obviously satisfies $\alpha_1 \geq_c \alpha_3$. $\square$

Lemma 4.4. *If $I \in \Sigma^{(n)}$ and $\alpha \in \mathscr{R}^n$, then $\alpha = \Sigma_{i \in I} \lambda_i \alpha_i^I$ where $\lambda_i = \langle \alpha, \beta_i \rangle$ and $\{\alpha_i^I\}_{i \in I}$ is dual to $\{\beta_i\}_{i \in I}$.*

Proof. Let $\beta = \alpha - \Sigma_{i \in I} \lambda_i \alpha_i^I$, then $\langle \beta, \beta_i \rangle = 0$ for all i. The duality relation $\langle \alpha_i, \beta_j \rangle = \delta_{ij} \mathbf{1}$

is equivalent to

$$\begin{bmatrix} \alpha_1^1 & \alpha_1^2 & \dots & \alpha_1^n \\ \alpha_2^1 & \alpha_2^2 & \dots & \alpha_2^n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_n^1 & \alpha_n^2 & \dots & \alpha_n^n \end{bmatrix} \begin{bmatrix} \beta_1^1 & \beta_2^1 & \dots & \beta_n^1 \\ \beta_1^2 & \beta_2^2 & \dots & \beta_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_1^n & \beta_2^n & \dots & \beta_n^n \end{bmatrix} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \mathbf{1} \end{bmatrix} \tag{4.1}$$

where $\beta_j = (\beta_j^1, \dots, \beta_j^n)^T$, therefore the matrix $(\beta_j^i)_{ij}$ is invertible as a $2n \times 2n$ matrix with coefficients in $\mathbb{R}$, hence $\beta = 0$. $\square$

The following lemma is used to derive the compatibility condition that guarantees that we can extend the transition functions of the equivariant vector bundles from the open dense subset $\mathbb{T} \bigcap U_I \bigcap U_J$ to $U_I \bigcap U_J$.

Lemma 4.5. *If $I \in \Sigma$, then $\chi^\alpha : \mathbb{T} \to \mathbb{C}^*, \alpha \in \mathscr{R}^n$ can be continuously extended to U_I iff $\langle \alpha, \beta_i \rangle \geq_c 0 \ \ \forall i \in I$.*

Proof. Given $I \in \Sigma$, by lemma 4.4, we may assume that $\alpha = \Sigma_{i \in I} \lambda_i \alpha_i^I$ where $\lambda_i = \langle \alpha, \beta_i \rangle$. χ^α can be continuously extended to U_I iff $\chi^\alpha \circ \lambda(h_1, \cdots, h_m) = \prod_{k=1}^m h_k^{\langle \alpha, \beta_k \rangle}$ can be continuously extended to $\mathbb{C}^I \times (\mathbb{C}^*)^{\Sigma^{(1)} \setminus I}$ iff $\langle \alpha, \beta_i \rangle \geq_c 0 \ \ \forall i \in I$. $\square$

Definition 4.4. For each $I \in \Sigma$, let $\mathbb{T}_I = ((\mathbb{C}^*)^I \times \mathbf{1}^{\Sigma^{(1)} \setminus I}) / \mathrm{Ker}\, \lambda$ be the stabilizer of γ_I and $\beta_I^\perp := \{\alpha \in \mathscr{R}^n | \langle \alpha, \beta_i \rangle = 0 \ \ \forall i \in I\}$.

The following lemma is used in Theorem 4.7 to prove that the equivariant vector bundle we get is independent of the choice of extensions of φ_I. It generalizes Prop 2.1.1 of [12].

Lemma 4.6. *Let $X = U_I$ be an affine chart of a topological toric manifold corresponding to a cone $I \in \Sigma^{(n)}$.*

i) Define a family of subspaces of a complex $\mathbb{T}$-module E by

$$E^i(\mu) = \bigoplus_{\langle\chi,\beta_i\rangle \geq_c \mu} E_\chi, i \in I,$$

where $E_\chi \subset E$ is the isotypical component of the continuous character $\chi \in \mathscr{R}^n$. Similarly define $\{F^i(\mu)|i \in I, \mu \in \mathscr{R}\}$ for another complex $\mathbb{T}$-module F. Then the space of equivariant homomorphisms $\mathrm{Hom}_{\mathbb{T}}(X \times E, X \times F)$ is canonically isomorphic to the space of linear operators $\varphi : E \to F$ satisfying $\varphi(E^i(\mu)) \subset F^i(\mu)$ for all $i \in I$.

ii) Two topological Klyachko vector bundles $X \times E$ and $X \times F$ are isomorphic if and only if the restrictions of the continuous representations are isomorphic $E\mid_{T_I} \simeq F\mid_{T_I}$.

Proof. i): All topological Klyachko vector bundles on U_I decompose into a sum of line bundles by definition. Therefore we can assume that $dimE = dimF = 1$. A bundle morphism $f : X \times E \to X \times F$ is a family of linear maps $\varphi_x : E \to F, x \in X$. The equivariance condition implies that for $\chi_E, \chi_F \in \mathscr{R}^n, e \in E$, we have $\varphi_{tx}(te) = t\varphi_x(e)$, i.e. $\chi_E(t)\varphi_{tx}(e) = \chi_F(t)\varphi_x(e)$. Fix an arbitrary point x_0 in the open dense orbit and let $\varphi = \varphi_{x_0}$. Then $f(tx_0, e) = (tx_0, \chi_E^{-1}\chi_F(t)\varphi(e))$ for all $t \in \mathbb{T}$. Since f is continuous on X, either $\varphi = 0$ or $\chi_E^{-1}\chi_F$ can be continuously extended to X. By lemma 4.5, the latter condition is equivalent to

$$\langle\chi_F, \beta_i\rangle \geq_c \langle\chi_E, \beta_i\rangle, \forall i \in I.$$

In both cases φ satisfies $\varphi(E^i(\mu)) \subset F^i(\mu)$.

Conversely, if $\varphi : E \to F$ satisfies $\varphi(E^i(\mu)) \subset F^i(\mu)$, then

$$f(tx_0, e) = (tx_0, \chi_E^{-1}\chi_F(t)\varphi(e))$$

defines a continuous bundle morphism $f : X \times E \to X \times F$ by lemma 4.5.

ii) If the bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic, then $E\mid_{\mathbb{T}} = \mathscr{E}(\gamma_I) \simeq \mathscr{F}(\gamma_I) = F\mid_{\mathbb{T}}$. (Recall that γ_I is the unique fixed point in U_I for the top dimensional cone $I \in \Sigma^{(n)}$ and $\mathscr{E}(\gamma_I)$ denotes the fiber of $\mathscr{E}$ over γ_I.) Conversely, if $E\mid_{\mathbb{T}} = F\mid_{\mathbb{T}}$, then the posets of subspaces $\{E^i(\mu) | i \in I, \mu \in \mathscr{R}\}$ and $\{F^i(\mu) | i \in I, \mu \in \mathscr{R}\}$ are isomorphic. By i), the bundles $\mathscr{E}$ and $\mathscr{F}$ are isomorphic. $\square$

Definition 4.5. For a topological toric manifold X, the category TKVB_X is the category whose objects are topological Klyachko vector bundles over X. A morphism $f : \mathscr{E} \to \mathscr{F}$ in TKVB_X is a morphism of topological complex vector bundles over X that is $\mathbb{T}$-equivariant.

Definition 4.6. For a topological toric manifold X, the category TKVS_X is the category whose objects are complex vector spaces E with a poset P of complex subspaces $E^i(\mu)$ of E indexed by $\mu \in \mathscr{R}, i \in \Sigma^{(1)}$ which satisfy the following compatibility condition:
For any $I \in \Sigma$, there exists a $\mathscr{R}^n/\beta_I^{\perp}$-grading

$$E = \bigoplus_{[\chi] \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$$

for which

$$E^i(\mu) = \sum_{\langle \chi, \beta_i \rangle \geq_c \mu} E_{[\chi]}, \ \forall i \in I.$$

A morphism $f : (E, \{E^i(\mu)\}) \to (F, \{F^i(\mu)\})$ in TKVS_X is a linear transformation $f : E \to F$ such that $f(E^i(\mu)) \subset F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$. f is an isomorphism if $E \to F$ is an isomorphism of complex vector spaces. f is a monomorphism if $E \to F$ is a monomorphism of complex vector spaces. f is an epimorphism if $E \to F$ is an epimorphism of complex vector spaces and $f(E^i(\mu)) = F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$.

Theorem 4.7. *The category* TKVB_X *of topological Klyachko vector bundles over the topological toric manifold* $X = X(\Delta)$ *is equivalent to the category* TKVS_X.

Corollary 4.8. *Let $\Phi : \mathrm{TKVB}_X \to \mathrm{TKVS}_X$ be an equivalence in the proof of Theorem 4.7, then a morphism $f : \mathscr{E} \to \mathscr{F}$ in TKVB_X is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if $\Phi(f)$ is a monomorphism (resp. epimorphism, resp. isomorphism).*

Definition 4.7. For each $I \in \Sigma$, we denote by $\mathrm{Hom}_{cont}(\mathbb{T}_I, \mathbb{C}^*)$ the group of continuous homomorphisms $\mathbb{T}_I \to \mathbb{C}^*$ and by $\mathrm{Hom}_{sm}(\mathbb{T}_I, \mathbb{C}^*)$ the group of smooth homomorphisms $\mathbb{T}_I \to \mathbb{C}^*$.

The following lemma explains the meaning of the index set $\mathscr{R}^n/\beta_I^{\perp}$ in the grading of E.

Lemma 4.9. *For each $I \in \Sigma, \mathrm{Hom}_{cont}(\mathbb{T}_I, \mathbb{C}^*) = \mathrm{Hom}_{sm}(\mathbb{T}_I, \mathbb{C}^*) \simeq \mathscr{R}^n/\beta_I^{\perp}$ as abelian groups.*

Proof. Notice that $\mathbb{T}_I \simeq (\mathbb{C}^*)^k$ where $k = |I|$, therefore by the proof of lemma 3.1, $\mathrm{Hom}_{cont}(\mathbb{T}_I, \mathbb{C}^*) = \mathrm{Hom}_{sm}(\mathbb{T}_I, \mathbb{C}^*)$. For the isomorphism, on the one hand, given $\alpha \in \mathscr{R}^n$, we have $\chi^\alpha : \mathbb{T} \to \mathbb{C}^*$ such that $\chi^\alpha(\lambda(h_1, \dots, h_m)) = \prod_{k=1}^m h_k^{\langle \alpha, \beta_k \rangle}$. In particular, if $\alpha \in \beta_I^{\perp}$, then $\chi^\alpha(\mathbb{T}_I) = \mathbf{1}$. Therefore by restricting to $\mathbb{T}_I$, $\alpha \in \mathscr{R}^n/\beta_I^{\perp}$ gives rise to $\chi^\alpha : \mathbb{T}_I \to \mathbb{C}^*$. On the other hand, given $\chi : \mathbb{T}_I \to \mathbb{C}^*$ where $I \in \Sigma$, choose an $I_0 \in \Sigma^{(n)}$ that contains I. Suppose $\chi(\lambda(h_1, \dots, h_m)) = \prod_{k \in I} h_k^{\lambda_k}$ where $[h_1, \dots, h_m] \in \mathbb{T}_I$ (the bracket denotes the equivalence class). Let $\alpha = \Sigma_{k \in I} \lambda_k \alpha_k^{I_0}$ where α^{I_0} is dual to β_{I_0}, then $\chi = \chi^\alpha : \mathbb{T}_I \to \mathbb{C}^*$. It is easy to check that these two assignments are inverse to each other, hence the result. □

Lemma 4.10. *There is a canonical splitting $\mathbb{T} \simeq \mathbb{T}_I \times \iota_I(\mathscr{O}(I))$ where $\iota_I : \mathscr{O}(I) \to \mathbb{T}$ is an embedding for each $I \in \Sigma^{(n)}$.*

Proof. Given $I \in \Sigma$, pick an arbitrary $I_0 \in \Sigma^{(n)}$ that contains I. We have the following exact sequence

$$\mathbf{1} \to \mathbb{T}_I \to \mathbb{T} = (\mathbb{C}^*)^n \simeq (\mathbb{C}^*)^m/\ker\lambda \to \mathscr{O}(I) \to \mathbf{1}.$$

The morphism $f_1 : \mathbb{T}_I \to \mathbb{T}$ is $f_1([h_1, \ldots, h_m]) = [w]$ where $w_i = h_i$ if $i \in I$ and $w_i = 1$ if $i \in I_0 \backslash I$. The morphism $f_2 : (\mathbb{C}^*)^{I_0} \to \mathscr{O}(I)$ is defined as follows: given $g = (g_i)_{i \in I_0} \in (\mathbb{C}^*)^{I_0}$, let $h_i = \chi^{\alpha_i^{I_0}}(g)$ for $i \in I_0$ and $h_i = 1$ otherwise, then $\lambda(h) = g$. Define $f_2(g) = [w]$ where $w_i = 0$ for $i \in I$ and $w_i = h_i$ for $i \in \Sigma^{(1)} \backslash I$. It is easy to check that the above sequence is exact. We define ι_I below: Given $w \in \mathscr{O}(I)$, let $h \in 0^I \times 1^{\Sigma^{(1)} \backslash I}$ such that $[h] = w$, then let $\iota_I(w)_i = 1$ if $i \in I$ and $\iota_I(w)_i = h_i$ otherwise. Then $f_2 \circ \iota = id$. Hence the exact sequence splits. □

Proof of Theorem 4.7. Klyachko's original proof goes through with the previous lemmas:

I. Construction of a functor $\Phi : \mathrm{TKVB}_X \to \mathrm{TKVS}_X$:

Let $\mathscr{E}$ be a smooth Klyachko vector bundle over X. Let $X = \bigcup_{I \in \Sigma} U_I$ be the open covering by the open invariant subsets U_I corresponding to $I \in \Sigma$. By definition, we have $\mathscr{E}|_{U_I} \simeq U_I \times \mathscr{E}(\gamma_I)$ for all $I \in \Sigma$. Because U_I is $\mathbb{T}_I$-invariant and $\mathbb{T}$-invariant, we have

$$\varphi_I : \mathbb{T}_I \to \mathrm{Aut}(\mathscr{E}(\gamma_I))$$

which can be extended to

$$\varphi_I : \mathbb{T} \to \mathrm{Aut}(\mathscr{E}(\gamma_I))$$

by lemma 4.10. The transition functions $f_{IJ} : U_I \bigcap U_J \to \mathrm{Hom}(\mathscr{E}(\gamma_I), \mathscr{E}(\gamma_J))$ satisfy the cocycle conditions $f_{IJ} f_{JK} f_{KI} = \mathbf{1}$ and $f_{IJ} f_{JI} = \mathbf{1}$ and the equivariance condition

$$f_{IJ}(tx) = \varphi_I(t) f_{IJ}(x) \varphi_J(t)^{-1}.$$

Let x_0 be an arbitrary point in the open dense orbit $\mathscr{O}(\emptyset)$. We identify all fibers of $\mathscr{E}$ with $E = \mathscr{E}(x_0)$ via f_{IJ}, i.e. we assume that $f_{IJ}(x_0) = \mathbf{1}$. Then $f_{IJ}(tx_0) = \varphi_I(t)\varphi_J(t)^{-1}$ and the map $\varphi_I(t)\varphi_J(t)^{-1}$ can be smoothly extended from $\mathbb{T}$ to $U_I \bigcap U_J$. We will interpret this extendability condition in terms of inclusion of filtrations below. First, as a representation space of $\mathbb{T}_I$, the space E admits a weight decomposition

$$E = \bigoplus_{[\chi] \in \mathscr{R}^n/\beta_I^\perp} E_{[\chi]}$$

where $\chi \in \mathscr{R}^n/\beta_I^\perp$ is considered as a map $\mathbb{T}_I \to \mathbb{C}^*$ by lemma 4.9 and $E_{[\chi]} := \{x \in E | t \cdot x = \chi(t)x \ \ \forall t \in \mathbb{T}_I\}$. Define a family of subspaces of E by

$$E^{i,I}(\mu) = \sum_{[\chi] \in \mathscr{R}^n/\beta_I^\perp, \langle \chi, \beta_i \rangle \geq_c \mu} E_{[\chi]}, \ \ \forall i \in I.$$

We prove that $\varphi_I(t)\varphi_J(t)^{-1}$ can be continuously extended from $\mathbb{T}$ to $U_I \bigcap U_J$ iff $E^{i,J}(\mu) \subset E^{i,I}(\mu)$ for all $i \in I \bigcap J$ and $\mu \in \mathscr{R}$. Indeed, let $e_i, i \in \Lambda_I$ be a diagonalizing basis of the representation φ_I, i.e. $\varphi_I(t)e_i = \chi_i(t)e_i, \chi_i \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$ and let $f_j, j \in \Lambda_J$ be a diagonalizing basis of the representation φ_J, i.e. $\varphi_J(t)f_j = \psi_j(t)f_j, \psi_j \in \mathrm{Hom}(\mathbb{T}, \mathbb{C}^*)$. Let $f_j = \Sigma_i a_{ij} e_i$. Then

$$\varphi_I \varphi_J^{-1} f_j = \Sigma_i a_{ij} \chi_i \psi_j^{-1} e_i.$$

By lemma 4.5, the character $\chi_i \psi_j^{-1}$ can be continuously extended to $U_I \bigcap U_J$ iff $\langle \chi_i, \beta_k \rangle \geq_c \langle \psi_j, \beta_k \rangle$ for all $k \in I \bigcap J$. In other words,

$$E^{i,J}(\mu) \subset E^{i,I}(\mu) \ \ \forall i \in I \bigcap J \text{ and } \mu \in \mathscr{R}.$$

As a consequence, $E^{i,I}(\mu) = E^{i,J}(\mu)$ for all $i \in I, J$ because both f_{IJ} and f_{JI} can be continuously extended to $U_I \bigcap U_J$. Therefore we denote $E^{i,I}(\mu)$ simply by $E^i(\mu)$. Define $\Phi(\mathscr{E}) := (E, \{E^i(\mu) | i \in \Sigma^{(1)}, \mu \in \mathscr{R}\})$.

Suppose $f : \mathscr{E} \to \mathscr{F}$ is a $\mathbb{T}$-equivariant map between two topological Klyachko vector bundles $\mathscr{E},\mathscr{F}$ over X. For each $I \in \Sigma^{(n)}$, we have the following commutative diagram

$$\begin{tikzcd} \mathcal{E}|_{U_I} \arrow[r, "f"] \arrow[d, "\simeq"] & \mathcal{F}|_{U_I} \arrow[d, "\simeq"] \\ U_I \times E \arrow[r] & U_I \times F \end{tikzcd}$$

Figure 4.1: Location Description of f

Define $\Phi(f)$ to be $f\,|_{x_0}\colon E \to F$. Suppose $\{E^i(\mu)|i \in \Sigma^{(1)}, \mu \in \mathscr{R}\}$ comes from a grading $E = \bigoplus_{\chi \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$ and $\{F^i(\mu)|i \in \Sigma^{(1)}, \mu \in \mathscr{R}\}$ comes from $F = \bigoplus_{\chi \in \mathscr{R}^n/\beta_I^{\perp}} F_{[\chi]}$, then by equivariance of f, we must have $\Phi(f)(E^i(\mu)) \subset F^i(\mu)$.

II. Construction of a functor $\Psi : \mathrm{TKVS}_X \to \mathrm{TKVB}_X$:

Given a family of doubly indexed subspaces $E^i(\mu)$ that correspond to $E = \bigoplus_{\chi \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$, we have a representation of $\mathbb{T}_I$ that can be extended to a representation φ_I of $\mathbb{T}$ by lemma 4.10. As the subspaces are decreasing, $f_{IJ}(tx_0) = \varphi_I(t)\varphi_J^{-1}(t)$ can be extended to $U_I \bigcap U_J$ by the first part of this proof. Therefore the cocycle f_{IJ} determines a topological Klyachko vector bundle $\mathscr{E} = \Psi(E, \{E^i(\mu)|i \in \Sigma^{(1)}, \mu \in \mathscr{R}\})$ over $X(\Delta)$. By lemma 4.6, $\mathscr{E}\,|_{U_I}$ does not depend on the choice of an extension of φ_I for each $I \in \Sigma^{(n)}$. That the bundle $\mathscr{E}$ does not depend on the choice of gradings E^I is exactly the same as Klyachko's original proof.

Suppose given a morphism $f : (E, \{E^i(\mu)\}) \to (F, \{F^i(\mu)\})$ in TKVS_X, then by lemma 4.6, we have a $\mathbb{T}$-equivariant morphism of vector bundles $f_I : U_I \times E \to U_I \times F$ for each $I \in \Sigma^{(n)}$. For any two $I, J \in \Sigma^{(n)}$, notice that f_I and f_J coincide as their restrictions to the dense subset $\mathbb{T} \bigcap U_I \bigcap U_J$ are the same. Hence they glue to a $\mathbb{T}$-equivariant morphism $\Psi(E, \{E^i(\mu)\}) \to \Psi(F, \{F^i(\mu)\})$.

We can quickly check that $\Psi(\Phi(\mathscr{E})) \simeq \mathscr{E}$ and $\Phi(\Psi(E, \{E^i(\mu)\})) \simeq (E, \{E^i(\mu)\})$. $\square$

4.3 Smooth Klyachko vector bundles

Definition 4.8. A smooth Klyachko complex vector bundle of rank r over an n-dimensional topological toric manifold $X(\Delta)$ associated with a topological toric fan $\Delta = (\Sigma, \beta)$ is a $\mathbb{T}$-equivariant smooth complex vector bundle $\mathscr{E} \to X(\Delta)$ whose restriction $\mathscr{E}|_{U_I}$ is equivariantly diffeomorphic to $U_I \times \mathbb{C}^r$ for all $I \in \Sigma$ and whose action map $\mathbb{T} \times \mathscr{E} \to \mathscr{E}$ is smooth.

Lemma 4.11. *A smooth character $\chi^\alpha : \mathbb{C}^* \to \mathbb{C}^*$ where $\alpha = (b + c\sqrt{-1}, v) \in \mathscr{R}$ can be smoothly extended to $\mathbb{C}$ iff $c = 0, b \in \mathbb{N}$ and $b \pm v \in 2\mathbb{N}$.*

Proof. Write i for $\sqrt{-1}$ and $g = \rho e^{i\theta} \in \mathbb{C}^* = x + yi$ where $\rho > 0, \theta \in [0, 2\pi), x, y \in \mathbb{R}$. The sufficiency is clear. We prove necessity below. Given $\chi^\alpha : \mathbb{C}^* \to \mathbb{C}^*$ that can be smoothly extended to $\mathbb{C}$ where $\alpha = (b + ci, v)$, by lemma 4.2, $b > 0$ or $b = c = v = 0$. In the latter case, obviously we have $c = 0, b \in \mathbb{N}$ and $b \pm v \in 2\mathbb{N}$. Suppose $b > 0$. Let $f(\rho) := \chi^\alpha|_{\mathbb{R}}(\rho) = \rho^{b+ci}$. We first prove that $b \in \mathbb{N}$. Suppose not, then there exists $n_0 \in \mathbb{N}$ such that $n_0 < b < n_0 + 1$ and $\frac{d^{n_0+1} f}{d\rho^{n_0+1}} = C_1 \rho^{b-n_0-1+ci}$ (C_1 is some non-zero constant) diverges as $\rho \to 0^+$, which contradicts that χ^α is smooth at 0. Therefore $b \in \mathbb{N}$. Next we prove that $c = 0$. Suppose $c \neq 0$, then $\frac{d^{b+1} f}{d\rho^{b+1}} = C_2 \cdot ci \cdot \rho^{-1+ci}$ (C_2 is some non-zero constant) diverges as $\rho \to 0^+$. Therefore $c = 0$. Now we prove that $b \pm v \in 2\mathbb{N}$. The proof below is based on the observation that for Puiseux series F, G in one indeterminate x with finitely many terms, $deg((\frac{F}{G})') \leq deg(\frac{F}{G})$. Here the degree of a Puiseux series with finitely many terms is the largest exponent whose corresponding coefficient is nonzero. We consider three cases: 1) $0 \leq v \leq b$, 2) $0 < b < v$, 3) $v < 0 < b$. In case 1, $\chi^\alpha(g) = g^v \rho^{b-v} = (\sum_{k=0}^{v} \binom{n}{k} x^k (yi)^{v-k})(x^2 + y^2)^{\frac{b-v}{2}}$. By separating the real and imaginary parts, we see that x^α is smooth if and only if $b - v \in 2\mathbb{N}$ if and only if $b \pm v \in 2\mathbb{N}$. In case 2, $\chi^\alpha(g) = \frac{g^v}{\rho^{v-b}}$.

Its real part is $\frac{F}{G}$ where $F(x,y)=\sum_{k=0}^{[\frac{v}{2}]}(-1)^k x^{v-2k}y^{2k}$ and $G(x,y)=(x^2+y^2)^{\frac{v-b}{2}}$. Since $degF=v, degG=v-b$, by the observation, $\lim_{\rho\to 0^+}\frac{\partial^{b+1}\frac{F}{G}}{\partial x^{b+1}}$ does not exist. In case 3, $\chi^\alpha(g)=\rho^{b+v}\overline{g}^{-v}$. By a similar reason as in case 1, $\chi^\alpha(g)$ is smooth if and only $b+v\in 2\mathbb{N}$ if and only if $b\pm v\in 2\mathbb{N}$. Hence we proved the necessity. □

Definition 4.9. We can define a partial order $\geq_s$ on $\mathscr{R}$ as follows: For $\alpha=(b+c\sqrt{-1},v)\in\mathscr{R}$, we say $\alpha\geq_s 0=(0+0\sqrt{-1},0)$ if $c=0, b\in\mathbb{N}, b\pm v\in 2\mathbb{N}$. For $\alpha,\beta\in\mathscr{R}$, we say $\alpha\geq_s\beta$ if $\alpha-\beta\geq_s 0$.

Proposition 4.12. *$\geq_s$ is a partial order.*

Proof. Reflexivity and anti-symmetry are clear. Transitivity: Suppose $\alpha_i=(b_i+c_i\sqrt{-1},v_i)\in\mathscr{R}, i=1,2,3$ and $\alpha_1\geq_c\alpha_2,\alpha_2\geq_c\alpha_3$, i.e. $b_1-b_2, b_2-b_3\in\mathbb{N},(b_1-b_2)\pm(v_1-v_2),(b_2-b_3)\pm(v_2-v_3)\in 2\mathbb{N}$, then by adding these expressions, we have $b_1-b_3\in\mathbb{N},(b_1-b_3)\pm(v_1-v_3)\in 2\mathbb{N}$. □

Lemma 4.13. *If $I\in\Sigma$, then $\chi^\alpha:\mathbb{T}\to\mathbb{C}^*,\alpha\in\mathscr{R}^n$ can be smoothly extended to U_I iff $\langle\alpha,\beta_i\rangle\geq_s 0\ \ \forall i\in I$.*

Proof. The proof is the same as lemma 4.5. □

Lemma 4.14. *Let $X=U_I$ be an affine chart of a topological tori manifold corresponding to a cone $I\in\Sigma^{(n)}$.*

i) Define a family of subspaces of a complex $\mathbb{T}$-module E by

$$E^i(\mu)=\bigoplus_{\langle\chi,i\rangle\geq_s\mu}E_{[\chi]}, i\in I,$$

where $E_\chi\subset E$ is the isotypical component of the smooth character $\chi\in\mathscr{R}^n$.If F is another $\mathbb{T}$-module, we define $F^i(\mu)$ similarly. Then the space of equivariant homomorphisms $\mathrm{Hom}_{\mathbb{T}}(X\times E,X\times F)$ is canonically isomorphic to the space of order preserving linear operators $\varphi:E\to F$

ii) Two smooth Klyachko vector bundles $X \times E$ and $X \times F$ are isomorphic if and only if the restrictions of the smooth representations are isomorphic $E\mid_{\mathbb{T}_I} \simeq F\mid_{\mathbb{T}_I}$.

Proof. The proof is same as the one for lemma 4.6 except that we replace lemma 4.5 by lemma 4.13. □

Definition 4.10. For a topological toric manifold X, the category SKVB_X is the category whose objects are smooth Klyachko complex vector bundles over X. A morphism $f : \mathscr{E} \to \mathscr{F}$ in SKVB_X is a morphism of smooth complex vector bundles over X that is $\mathbb{T}$-equivariant.

Definition 4.11. For a topological toric manifold X, the category SKVS_X is the category whose objects are complex vector spaces E with a poset P of complex subspaces $E^i(\mu)$ of E indexed by $\mu \in \mathscr{R}, i \in \Sigma^{(1)}$ which satisfy the following compatibility condition:
For any $I \in \Sigma^{(\mathrm{n})}$, there exists a $\mathscr{R}^n/\beta_I^{\perp}$-grading

$$E = \bigoplus_{[\chi] \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$$

for which

$$E^i(\mu) = \sum_{\langle \chi, \beta_i \rangle \geq_s \mu} E_{[\chi]}, \ \forall i \in I.$$

A morphism $f : (E, \{E^i(\mu)\}) \to (F, \{F^i(\mu)\})$ in SKVS_X is a linear transformation $f : E \to F$ such that $f(E^i(\mu)) \subset F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$. f is an isomorphism if $E \to F$ is an isomorphism of complex vector spaces. f is a monomorphism if $E \to F$ is a monomorphism of complex vector spaces. f is an epimorphism if $E \to F$ is an epimorphism of complex vector spaces and $f(E^i(\mu)) = F^i(\mu)$ for all $i \in \Sigma^{(1)}$ and $\mu \in \mathscr{R}$.

Following the proofs in Section 3, we can prove:

Theorem 4.15. *The category* SKVB_X *of smooth Klyachko vector bundles over the topological toric manifold* $X = X(\Delta)$ *is equivalent to the category* SKVS_X.

Corollary 4.16. *Let* $\Phi : \mathrm{SKVB}_X \to \mathrm{SKVS}_X$ *be an equivalence in the proof of Theorem 4.7, then a morphism* $f : \mathscr{E} \to \mathscr{F}$ *in* SKVB_X *is a monomorphism (resp. epimorphism, resp. isomorphism) if and only if* $\Phi(f)$ *is a monomorphism (resp. epimorphism, resp. isomorphism).*

4.4 Holomorphic Klyachko vector bundles

Definition 4.12. Let $X(\Delta)$ be a topological toric manifold with a $\mathbb{T}$-equivariant complex structure. A holomorphic Klyachko vector bundle of rank r over $X(\Delta)$ is a $\mathbb{T}$-equivariant holomorphic complex vector bundle $\mathscr{E} \to X(\Delta)$ whose restriction $\mathscr{E}|_{U_I}$ is equivariantly biholomorphic to $U_I \times \mathbb{C}^r$ for all $I \in \Sigma$ and whose action maps $\mathbb{T} \times \mathscr{E} \to \mathscr{E}$ and $\mathbb{T} \times X(\Delta) \to X(\Delta)$ are holomorphic.

Define $diag(\mathbb{Z}) := \{(v,v) \in \mathscr{R} | v \in \mathbb{Z}\}$ and $diag(\mathbb{Z}^n) = \{(\alpha, \cdots, \alpha) \in \mathscr{R}^n | \alpha \in diag(\mathbb{Z})\}$.

Lemma 4.17. *If a smooth character* $\chi^\alpha : \mathbb{T} \to \mathbb{C}^*$ *is holomorphic for* $\alpha \in \mathscr{R}$, *then* $\alpha \in diag(\mathbb{Z})$.

Proof. Let $\alpha = (b+ci, v) \in \mathscr{R}$ and $f(\rho,\theta) = \rho^{b+ci} e^{i\theta v} = \rho^b e^{ci\ln\rho + i\theta v} = U(\rho,\theta) + iV(\rho,\theta)$ where $U(\rho,\theta) = \rho^b \cos(c\ln\rho + \theta v)$ and $V(\rho,\theta) = \rho^b \sin(c\ln\rho + \theta v)$. Because χ^α is holomorphic, U, V must satisfy the Cauchy-Riemann equations

$$\frac{\partial U}{\partial \rho} = \frac{1}{\rho}\frac{\partial U}{\partial \theta},$$
$$\frac{\partial V}{\partial \rho} = -\frac{1}{\rho}\frac{\partial U}{\partial \theta}.$$

By simple computation,

$$
\begin{aligned}
\frac{\partial U}{\partial \rho} &= b\rho^{b-1}\cos(w) - \rho^{b-1}c\sin(w),\\
\frac{1}{\rho}\frac{\partial U}{\partial \theta} &= \rho^{b-1}v\cos(w),\\
\frac{\partial V}{\partial \rho} &= b\rho^{b-1}\sin(w) + \rho^{b-1}c\cos(w),\\
-\frac{1}{\rho}\frac{\partial U}{\partial \theta} &= \rho^{b-1}v\sin(w).
\end{aligned}
$$

where $w = c\ln\rho + \theta v$.

After simplification, we get

$$(b-v)\cos(w) = c\sin(w),$$

$$(b-v)\sin(w) = -c\cos(w).$$

Therefore $(b-v)^2 = -c^2$, i.e. $b = v \in \mathbb{Z}, c = 0$, i.e. $\alpha \in diag(\mathbb{Z})$. $\square$

Theorem 4.18. *A holomorphic Klyachko vector bundle $\mathscr{E}$ over a complex topological toric manifold $X = X(\Delta)$ is $\mathbb{T}$-equivariantly biholomorphic to a toric vector bundle over a toric variety.*

Proof. In the proof of theorem 1.2 in [9], Ishida showed that for a topological toric manifold X with $\mathbb{T}$-equivariant complex structure, the equivariant smooth charts $\varphi_I : U_I \to V(\bigoplus_{i\in I}\chi^{\alpha_i^I})$ are actually equivariant holomorphic charts. Since the $\mathbb{T}$-action is holomorphic, the map

$$
\begin{aligned}
\mathbb{T} &\longrightarrow V(\bigoplus_{i\in I}\chi^{\alpha_i^I})\\
g &\longmapsto (\chi^{\alpha_i^I}(g)z_i)_{i\in I}
\end{aligned}
$$

is holomorphic for each $I \in \Sigma^{(n)}, g \in \mathbb{T}, z \in V(\bigoplus_{i\in I} \chi^{\alpha_i^I})$, in particular

$$\begin{aligned} \mathbb{T} &\longrightarrow \mathbb{C}^* \\ g &\longmapsto \chi^{\alpha_i^I}(g) \end{aligned}$$

is holomorphic for all $I \in \Sigma^{(n)}, i \in I$. By lemma 4.17, $\alpha_i^I \in diag(\mathbb{Z}^n)$. Hence $\beta_i \in diag(\mathbb{Z}^n)$, i.e. the topological toric fan Δ comes from an ordinary fan $\overline{\Sigma}$, i.e. X is a toric variety. Now fix a point x_0 in the open dense torus and let E be the fiber of $\mathscr{E}$ over x_0. For each $I \in \Sigma^{(n)}$, $\mathscr{E}|_{U_I}$ is $\mathbb{T}$-equivariantly biholomorphic to $U_I \times E$ and the $\mathbb{T}$-module E decomposes into weight spaces

$$E = \bigoplus_{\chi} E_{\chi}.$$

By lemma 4.17, each χ appearing in the decomposition must be algebraic. In the proof of theorem 4.7, we showed that the transition functions f_{IJ} of $\mathscr{E}$ is determined by the weighs χ, hence f_{IJ} are algebraic and $\mathscr{E}$ is exactly a toric vector bundle. □

4.5 Examples

In this section, we will recall the construction of L_i, prove that $\bigoplus_{i=1}^m L_i$ is a smooth Klyachko vector bundle.

Construction of L_i: Let $\pi_i : (\mathbb{C}^*)^m \to \mathbb{C}^*$ be the projection onto the i-th factor of $(\mathbb{C}^*)^m$ and denote by $V(\pi_i)$ the complex one-dimensional representation space of π_i. Let $(\mathbb{C}^*)^m$ act on $U(\Sigma) \times V(\pi_i)$ by

$$(g_1, \cdots, g_m)((z_1, \cdots, z_m), w) = ((g_1 z_1, \cdots, g_m z_m), g_i w)$$

for $(g_1,\cdots,g_m)\in(\mathbb{C}^*)^m,(z_1,\cdots,z_m)\in U(\Sigma),w\in V(\pi_i)$. The $(\mathbb{C}^*)^m$-equivariant complex line bundle L_i is

$$L_i:(U(\Sigma)\times V(\pi_i))/\mathrm{Ker}\,\lambda\to U(\Sigma)/\mathrm{Ker}\,\lambda=X(\Delta).$$

Equivariant trivializations: $X(\Delta)=\bigcup_{I\in\Sigma^{(n)}}U_I$. For each $I\in\Sigma^{(n)}$, on U_I, depending on whether $i\in I$, we have two types of equivariant trivializations.

When $i\in I$, the equivariant trivialization

$$\psi_I:(U(I)\times V(\pi_i))/\mathrm{Ker}\,\lambda\to U_I\times\mathbb{C}$$

is given by

$$[(z,w)]\mapsto([z],w\cdot\prod_{k\notin I}z_k^{\langle\alpha_i^I,\beta_k\rangle}).\tag{4.2}$$

When $i\notin I$, the equivariant trivialization

$$\psi_I:(U(I)\times V(\pi_i))/\mathrm{Ker}\,\lambda\to U_I\times\mathbb{C}$$

is given by

$$[(z,w)]\mapsto([z],\frac{w}{z_i}).\tag{4.3}$$

Injectivity of ψ_I: Suppose $\psi_I([z,w])=\psi_I([z',w'])$.

Case 1: $i\in I.[z]=[z'],w\cdot\prod_{k\notin I}z_k^{\langle\alpha_i^I,\beta_k\rangle}=w'\cdot\prod_{k\notin I}z_k'^{\langle\alpha_i^I,\beta_k\rangle}$, therefore $z_k'=hz_k$ for some $h\in\mathrm{Ker}\,\lambda$ and $w'=w\prod_{k\notin I}h^{\langle\alpha_i^I,\beta_k\rangle}=w\cdot h$ where the last equality follow from 3.3. Hence $[z,w]=[z',w']$.

Case 2: $i\notin I.[z]=[z'],\frac{w}{z_i}=\frac{w'}{z_i'}$, therefore $z_k'=z_k$ for some $h\in\mathrm{Ker}\,\lambda$ and $w'=\frac{z_i'}{z_i}w=w\cdot h$.

Hence $[z,w]=[z',w']$.

Surjectivity of ψ_I: Suppose $(z,c)\in U_I\times\mathbb{C}$.

Case 1: $i\in I$. Let $w=c\cdot\prod_{k\notin I} z_k^{-\langle\alpha_i^I,\beta_k\rangle}$, then $\psi_I([z,w])=(z,c)$.

Case 2: $i\notin I$. Let $w=c\cdot z_i$, then $\psi_I([z,w])=(z,c)$.

Equivariance of ψ_I :

Case 1: $i\in I$. Let $\mathbb{T}$ act on $U_I\times\mathbb{C}$ via

$$g\cdot(z,c)=(g\cdot z,\chi^{\alpha_i^I}(g)c), g\in\mathbb{T},(z,c)\in U_I\times\mathbb{C}.$$

Given $g\in\mathbb{T},([z],c)\in U_I\times\mathbb{C}$, let $h=(h_1,\cdots,h_m)\in(\mathbb{C}^*)^m$ where

$$h_k=\begin{cases}\chi^{\alpha_k^I}(g) & \text{for } k\in I,\\ 1 & \text{for } k\notin I.\end{cases}$$

Then $\lambda(h)=g$.

$$\begin{aligned}\psi_I(g\cdot[z,w]) &= \psi_I([(h_kz_k)_{k\in I},h_iw])\\ &=(g\cdot[z],h_iw\prod_{k\notin I}(h_kz_k)^{\langle\alpha_i^I,\beta_k\rangle})\\ &=(g\cdot[z],h_iw\prod_{k\notin I}z_k^{\langle\alpha_i^I,\beta_k\rangle})\\ &=g\cdot\psi_I([z,w]).\end{aligned}$$

Case 2: $i \notin I$. Let $\mathbb{T}$ act on $U_I \times \mathbb{C}$ via

$$g \cdot ([z], c) = ([g \cdot z], c), g \in \mathbb{T}, (z, c) \in U_I \times \mathbb{C}.$$

Given $g \in \mathbb{T}, (z, c) \in U_I \times \mathbb{C}$, choose h as above,

$$\begin{aligned}
\psi_I(g \cdot [z, w]) &= \psi_I([(h_k z_k)_{k \in I}, h_i w]) \\
&= (g \cdot [z], \frac{h_i w}{h_i z_i}) \\
&= (g \cdot [z], \frac{w}{z_i}) \\
&= g \cdot \psi_I([z, w]).
\end{aligned}$$

Transition functions g_{IJ} of L_i: Given $[z] \in U_I \bigcap U_J, I, J \in \Sigma^{(n)}$, there are three cases:

Case 1: $i \in I \bigcap J$.

$$\psi_I \circ \psi_J^{-1}([z], c) = ([z], c \frac{\prod_{k \notin I} z_k^{\langle \alpha_i^I, \beta_k \rangle}}{\prod_{k \notin J} z_k^{\langle \alpha_i^J, \beta_k \rangle}}),$$

therefore

$$g_{IJ}([z]) = \frac{\prod_{k \notin I} z_k^{\langle \alpha_i^I, \beta_k \rangle}}{\prod_{k \notin J} z_k^{\langle \alpha_i^J, \beta_k \rangle}}.$$

Case 2: $i \in I, i \notin J$.

$$\psi_I \circ \psi_J^{-1}([z], c) = ([z], z_i \prod_{k \notin I} z_k^{\langle \alpha_i^I, \beta_k \rangle} c).$$

Therefore

$$g_{IJ}([z]) = z_i \prod_{k \notin I} z_k^{\langle \alpha_i^I, \beta_k \rangle}.$$

Case 3: $i \notin I, i \in J$. This is essentially the same as case 2.

Hence L_i is a smooth Klyachko vector bundle.

Gradings of L_i: Let x_0 be the identity of $\mathbb{T} \subset X(\Delta)$ and $E_i \simeq \mathbb{C}$ the fiber of L_i over x_0. For $I \in \Sigma^{(n)}$,

$$E_i = \bigoplus_{[\chi] \in \mathscr{R}^n / \beta_I^{\perp}} E_{i[\chi]} = \begin{cases} E_{i_{[\chi^{\alpha_i^I}]}} & i \in I, \\ E_{i_{[\chi^0]}} & i \notin I. \end{cases}$$

Chapter 5: Complex equivariant vector bundles

5.1 Complex equivariant line bundles

Theorem 5.1. *Let X be the $2n$-dimensional topological toric manifold associated with a topological fan Δ and $\pi : \mathscr{E} \to X$ a topological $\mathbb{T}$-equivariant complex line bundle, then $\mathscr{E}$ is a topological Klyachko line bundle.*

Proof. Given $I \in \Sigma^{(n)}$, we will give a $\mathbb{T}$-equivariant trivialization of $\mathscr{E}|_{U_I}$. Let x_0 be the unique point fixed by $\mathbb{T}$ in U_I. Let

$$G := (S^1)^n < \mathbb{T}, \qquad \chi : \mathbb{T} \to \mathrm{GL}_1(\mathscr{E}(x_0)) = \mathbb{C}^*$$

be respectively the maximal compact subgroup and the representation of $\mathbb{T}$ on the fiber $\mathscr{E}(x_0)$ of $\mathscr{E}$ over x_0.

Step 1: We first prove that there is a G-equivariant trivialization of $\mathscr{E}$

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{C}$$

for some G-invariant open subset $U \subset U_I$ containing x_0.

Let $\phi : \pi^{-1}(W) \to W \times \mathbb{C}$ be a trivialization of $\mathscr{E}$ (not necessarily G-equivariant) over an open neighborhood W of x_0. Let $\langle\ ,\ \rangle$ be an arbitrary Riemannian metric on X. Because

G is compact, it admits a Haar measure μ, i.e. a probability measure satisfying

$$\int_G f(kh)\, d\mu(h) = \int_G f(hk)\, d\mu(h) = \int_G f(h)\, d\mu(h)$$

for all $k, h \in G$ and $f \in L^1(G, \mu)$. Let $\langle\ ,\ \rangle_G$ be the G-invariant Riemannian metric on X defined by

$$\langle v, w\rangle_G := \int_G \langle d\rho_h(v), d\rho_h(w)\rangle\, d\mu(h) \qquad \forall\, p \in X,\ v, w \in T_pX,$$

where $\rho_h : X \to X$ is the action of $h \in G$ on X. Let $d(-,-)$ be the distance function associated $\langle\ ,\ \rangle_G$. Define U to be the G-invariant open ball

$$B(x_0, \varepsilon) := \{x \in W \mid d(x, x_0) < \varepsilon\} \subset W$$

for some $\varepsilon > 0$. Define a G-action on $U \times \mathbb{C}$ via

$$g \cdot (u, v) = (g \cdot u, \chi(g)v) \text{ for } (u, v) \in U \times \mathbb{C},$$

where the action on the first factor is the restriction of the $\mathbb{T}$-action on X. Let $\phi_u : \mathscr{E}(u) \to \mathbb{C}$ be the linear isomorphism induced from ϕ for each $u \in U$. Define

$$\begin{aligned} \varphi : \pi^{-1}(U) &\longrightarrow U \times \mathbb{C} \\ e &\longmapsto (u, \overline{\phi}_u(e)), \end{aligned}$$

where

$$\overline{\phi}_u(e) = \int_G \chi(g^{-1})\phi_{g\cdot u}(g \cdot e)\, d\mu(g) \text{ with } u = \pi(e).$$

φ is clearly an isomorphism of bundles. We claim that φ is G-equivariant. To see this, we need to show that

$$\varphi(h\cdot e) = (h\cdot u, \overline{\phi}_{h\cdot u}(h\cdot e)) = (h\cdot u, \chi(h)\overline{\phi}_u(e)) = h\cdot\varphi(e) \qquad \forall\, h\in G,\ e\in\pi^{-1}(U).$$

On the one hand,

$$\begin{aligned}
\overline{\phi}_{h\cdot u}(h\cdot e) &= \int_G \chi(g^{-1})\phi_{gh\cdot u}(gh\cdot e)\, d\mu(g) \\
&= \int_G \chi((kh^{-1})^{-1})\phi_{k\cdot u}(k\cdot e)\, d\mu(kh^{-1}) \qquad (k := gh) \\
&= \int_G \chi(hk^{-1})\phi_{k\cdot u}(k\cdot e)\, d\mu(k) \qquad \text{(by the invariance of } \mu\text{).}
\end{aligned}$$

On the other hand,

$$\chi(h)\overline{\phi}_u(e) = \chi(h)\int_G \chi(k^{-1})\phi_{k\cdot u}(k\cdot e)\, d\mu(k).$$

Comparing the last two equalities, we prove the claim.

Step 2: Define the section $s : U \to \mathscr{E}|_U$ by

$$s(u) := \varphi^{-1}((u,1)) \qquad \forall\, u\in U.$$

We claim that s is a local G-eigenframe, i.e.

$$(g\cdot s)(u) = \chi(g)s(u) \qquad \forall\, g\in G,\ u\in U.$$

To see this, note that since φ is G-equivariant, we can write

$$\varphi(g\cdot(s(u))) = g\cdot\varphi(s(u)) = g\cdot(u,1) = (g\cdot u, \chi(g)).$$

Therefore,

$$g\cdot(s(u)) = \varphi^{-1}(g\cdot u, \chi(g)) = \chi(g)\varphi^{-1}(g\cdot u, 1) = \chi(g)s(g\cdot u),$$

and hence

$$(g\cdot s)(u) := g\cdot(s(g^{-1}\cdot u)) = \chi(g)(s(gg^{-1}\cdot u)) = \chi(g)s(u)$$

proving the claim.

Step 3: Decompose $t \in \mathbb{T} = (\mathbb{C}^*)^n$ as $t = r\cdot e^{i\theta}$, where

$$r := (r_1,\dots,r_n) \in \mathbb{R}^n_{>0}, \qquad e^{i\theta} := (e^{i\theta_1},\dots,e^{i\theta_n}) \in G.$$

Our goal is to extend the local G-eigenframe s from Step 2 to a $\mathbb{T}$-eigenframe

$$s : U_I \to \mathscr{E}|_{U_I}.$$

For this, we need to find an extension of s to U_I, such that

$$(t\cdot s)(x) := t\cdot(s(t^{-1}\cdot x)) = \chi^{\alpha}(t)s(x) \qquad \forall\, t \in \mathbb{T},\ x \in U_I.$$

for the continuous character $\chi^{\alpha} : \mathbb{T} \to \mathbb{C}^*$ corresponding to some $\alpha \in \mathscr{R}^n$, or equivalently

$$s(t\cdot x) = \chi^{\alpha}(t^{-1})t\cdot(s(x)) \qquad \forall\, t \in \mathbb{T},\ x \in U_I.$$

In the following, we will show that

$$s(t\cdot x)=\chi^{\alpha}(t^{-1})t\cdot(s(x)) \qquad \forall\, t\in\mathbb{T},\ x\in U \text{ such that } t\cdot x\in U. \tag{5.1}$$

Once this is proven, we can define $s:U_I\to\mathscr{E}$ by

$$s(t\cdot x):=\chi^{\alpha}(t^{-1})t\cdot(s(x)) \qquad \forall\, t\in\mathbb{T},\ x\in U$$

whether $t\cdot x\in U$ or not. Because $\mathbb{T}\cdot U=U_I$, this definition indeed extends the local G-eigenframe $s:U\to\mathscr{E}|_U$ to the $\mathbb{T}$-eigenframe $s:U_I\to\mathscr{E}$, as desired. This $\mathbb{T}$-eigenframe will then give a $\mathbb{T}$-equivariant trivialization of $\mathscr{E}|_{U_I}$ completing the proof of the theorem.

We now return to proving (5.1). The local section $s:U\to\mathscr{E}|_U$ from Step 2 satisfies

$$s(e^{i\theta}\cdot x)=\chi(e^{-i\theta})e^{i\theta}\cdot(s(x)) \qquad \forall\, e^{i\theta}\in G,\ x\in U. \tag{5.2}$$

Define $f:S:=\{(\rho,x)\in\mathbb{R}^n_{>0}\times U|x\in\rho\cdot U\}\to\mathbb{C}^*$ to be the continuous function satisfying

$$(r\cdot s)(x):=r\cdot(s(r^{-1}\cdot x))=f(r,x)s(x) \qquad \forall\, r\in\mathbb{R}^n_{>0},\ x\in U\bigcap(r\cdot U),$$

or equivalently,

$$s(r\cdot x)=f(r^{-1},x)r\cdot(s(x)) \qquad \forall\, r\in\mathbb{R}^n_{>0},\ x\in U\bigcap(r^{-1}\cdot U). \tag{5.3}$$

Given $r_1, r_2 \in \mathbb{R}^n_{>0}, x \in (r_1 \cdot U) \bigcap (r_2 \cdot U) \bigcap (r_1 \cdot r_2 \cdot U)$, we have

$$\begin{aligned} f(r_1 \cdot r_2, x)s(x) &= ((r_1 \cdot r_2) \cdot s)(x) \\ &= (r_1 \cdot (r_2 \cdot s))(x) \\ &= f(r_1, x)(r_2 \cdot s)(x) \\ &= f(r_1, x)f(r_2, x)s(x) \end{aligned}$$

and since $s(x) \neq 0$, we must have

$$f(r_1 \cdot r_2, x) = f(r_1, x)f(r_2, x). \tag{5.4}$$

Now for any $t = r \cdot e^{i\theta} \in \mathbb{T}$, $x \in U \bigcap (r^{-1} \cdot U)$, we have

$$\begin{aligned} s(t \cdot x) &= s(e^{i\theta} \cdot r \cdot x) \\ &= \chi(e^{-i\theta})e^{i\theta} \cdot (s(r \cdot x)) \\ &= \chi(e^{-i\theta})e^{i\theta} \cdot f(r^{-1}, x)r \cdot (s(x)) \qquad \text{(by (5.3))} \\ &= \chi(e^{-i\theta})f(r^{-1}, x)t \cdot (s(x)) \end{aligned} \tag{5.5}$$

and for $t = r \cdot e^{i\theta} \in \mathbb{T}$, $\rho \in \mathbb{R}^n_{>0}$, $x \in U \bigcap (r^{-1} \cdot U) \bigcap (r^{-1} \cdot \rho \cdot U)$,

$$\begin{aligned} f(\rho, t \cdot x)s(t \cdot x) &= (\rho \cdot s)(t \cdot x) \qquad \text{(by the definition of } f) \\ &= \rho \cdot (s(\rho^{-1} \cdot t \cdot x)) \\ &= \rho \cdot (\chi(e^{-i\theta})f(r^{-1} \cdot \rho, x)\rho^{-1} \cdot t \cdot (s(x))) \qquad \text{(by (5.5))} \\ &= f(\rho, x)\chi(e^{-i\theta})f(r^{-1}, x)t \cdot (s(x)) \qquad \text{(by (5.4)} \\ &= f(\rho, x)s(t \cdot x) \qquad \text{(by (5.5))}. \end{aligned}$$

We have shown that

$$f(\rho, t\cdot x) = f(\rho, x) \qquad \forall\, t\in\mathbb{T},\; x\in U\bigcap(\rho\cdot U) \text{ such that } t\cdot x\in U\bigcap(\rho\cdot U).$$

Given $x,y\in U\bigcap(\rho\cdot U)$, there exists a sequence of points $t_i\in\mathbb{T}$ such that

$$\lim_{i\to\infty} t_i\cdot x = y \quad \text{ and } \quad t_i\cdot x\in U\bigcap(\rho\cdot U).$$

By the continuity of f,

$$f(\rho,y) = f(\rho,(\lim_{i\to\infty} t_i)\cdot x) = \lim_{i\to\infty} f(\rho, t_i\cdot x) = f(\rho,x).$$

Thus $f(\rho,x)$ is independent of x, so we will denote it simply by $f(\rho)$. We can now rewrite (5.3) as

$$s(r\cdot x) = f(r^{-1})r\cdot(s(x)) \qquad \forall\, r\in\mathbb{R}^n_{>0},\; x\in U\bigcap(r^{-1}\cdot U). \tag{5.6}$$

Using (5.2) and (5.6), we see that (5.1) holds with the continuous $\mathbb{T}$-character

$$\chi^\alpha(r\cdot e^{i\theta}) := f(r)\chi(e^{i\theta}) \qquad \forall\, r\in\mathbb{R}^n_{>0},\; \theta\in\mathbb{R}^n$$

as claimed. □

5.2 General case

Theorem 5.2. *Let X be the $2n$-dimensional topological toric manifold associated with a topological fan Δ and $\pi:\mathscr{E}\to X$ a topological $\mathbb{T}$-equivariant complex vector bundle of* *rank k, then $\mathscr{E}$ is a topological Klyachko vector bundle.*

Proof. Given $I \in \Sigma^{(n)}$ and x_0 the unique point fixed by the action of $\mathbb{T}$ on U_I. Let $G = (S^1)^n$ be the maximal compact subgroup of $\mathbb{T}$.

Step 1: Following Step 1 of the proof of Theorem 5.1, we can find a G-invariant open ball $U = B(x_0, \varepsilon)$ over which there is a G-equivariant homeomorphism

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k.$$

Step 2: Define the section $s_i : U \to \mathscr{E}|_U$ by

$$s_i(u) := \varphi^{-1}((u, e_i)) \qquad \forall\, u \in U,$$

where e_i is the i-th standard basis element of $\mathbb{C}^k$, and define

$$s(u) := (s_1(u), \ldots, s_k(u))^T.$$

As in Step 2 of the proof of Theorem 5.1, we can check that $\{s_1, \ldots, s_k\}$ forms a local G-eigenframe, that is

$$(g \cdot s_i)(u) = \chi_i(g) s_i(u) \qquad \forall\, g \in G,\ u \in U$$

for some continuous characters $\chi_i : \mathbb{T} \to \mathbb{C}^*$. Equivalently,

$$s(e^{i\theta} \cdot x) = B(e^{-i\theta})(e^{i\theta} \cdot (s(x))) \qquad \forall\, e^{i\theta} := (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in G,\ x \in U, \tag{5.7}$$

where

$$B : G \to \mathrm{GL}(n,\mathbb{C}) \qquad g \mapsto \mathrm{diag}(\chi_1(g),\cdots,\chi_k(g)).$$

Step 3: Decompose $t \in \mathbb{T} = (\mathbb{C}^*)^n$ as $t = r \cdot e^{i\theta}$, where

$$r := (r_1,\ldots,r_n) \in \mathbb{R}^n_{>0}, \qquad e^{i\theta} := (e^{i\theta_1},\ldots,e^{i\theta_n}) \in G.$$

Define $A: S := \{(\rho,x) \in \mathbb{R}^n_{>0} \times U | x \in \rho \cdot U\} \to \mathrm{GL}(k,\mathbb{C})$ to be the continuous map satisfying

$$(r\cdot s)(x) := r\cdot(s(r^{-1}\cdot x)) = A(r,x)s(x) \qquad \forall\, r \in \mathbb{R}^n_{>0},\ x \in U \bigcap (r\cdot U),$$

or equivalently,

$$s(r\cdot x) = A(r^{-1},x)r\cdot(s(x)) \qquad \forall\, r \in \mathbb{R}^n_{>0},\ x \in U \bigcap (r^{-1}\cdot U). \tag{5.8}$$

By the same argument leading to (5.4), we can show that

$$A(r_1,x)A(r_2,x) = A(r_1\cdot r_2,x) = A(r_2\cdot r_1,x) = A(r_2,x)A(r_1,x) \tag{5.9}$$

for any $r_1,r_2 \in \mathbb{R}^n_{>0}$, $x \in (r_1\cdot U)\bigcap(r_2\cdot U)\bigcap(r_1\cdot r_2\cdot U)$ using that $\{s_1(x),\cdots,s_k(x)\}$ forms a $\mathbb{C}$-basis of the fiber $\mathscr{E}(x)$. One can similarly see that

$$A(r,x)B(e^{i\theta}) = B(e^{i\theta})A(r,x) \qquad \forall\, e^{i\theta} \in G,\ r \in \mathbb{R}^n_{>0},\ x \in U \bigcap (r\cdot U). \tag{5.10}$$

We also have that for any $t = r \cdot e^{i\theta} \in \mathbb{T}$, $x \in U \bigcap (r^{-1} \cdot U)$

$$\begin{aligned}
s(t \cdot x) &= s(e^{i\theta} \cdot r \cdot x) \\
&= B(e^{-i\theta})(e^{i\theta} \cdot (s(r \cdot x))) \qquad \text{(by (5.7))} \\
&= B(e^{-i\theta})(e^{i\theta} \cdot (A(r^{-1},x)r) \cdot (s(x))) \qquad \text{(by (5.8))} \\
&= B(e^{-i\theta})A(r^{-1},x)t \cdot (s(x)), \qquad (5.11)
\end{aligned}$$

and for any $t = r \cdot e^{i\theta} \in \mathbb{T}$, $\rho \in \mathbb{R}^n_{>0}$, $x \in U \bigcap (r^{-1} \cdot U) \bigcap (r^{-1} \cdot \rho \cdot U)$,

$$\begin{aligned}
A(\rho, t \cdot x)s(t \cdot x) &= (\rho \cdot s)(t \cdot x) \qquad \text{(by the definition of } A\text{)} \\
&= \rho \cdot (s(\rho^{-1} \cdot t \cdot x)) \\
&= \rho \cdot (B(e^{-i\theta})A(r^{-1} \cdot \rho, x)\rho^{-1} \cdot t \cdot (s(x))) \qquad \text{(by (5.11))} \\
&= A(\rho, x)B(e^{-i\theta})A(r^{-1}, x)t \cdot (s(x)) \qquad \text{(by (5.9), (5.10))} \\
&= A(\rho, x)s(t \cdot x) \qquad \text{(by (5.11))}.
\end{aligned}$$

Using that $\{s_1(t \cdot x), \cdots, s_k(t \cdot x)\}$ forms a $\mathbb{C}$-basis of the fiber $\mathscr{E}(t \cdot x)$, we have shown that

$$A(\rho, x) = A(\rho, t \cdot x) \qquad \forall\, t \in \mathbb{T},\ x \in U \bigcap (\rho \cdot U) \text{ such that } t \cdot x \in U \bigcap (\rho \cdot U).$$

By the same argument as in Step 3 of the proof of Theorem 5.3, the identity above leads us to

$$A(\rho, x) = A(\rho, y) \qquad \forall\, x, y \in U \bigcap (\rho \cdot U)$$

that is $A(\rho, x)$ is independent of x, so we will denote it simply by $A(\rho)$.

Step 4: Let $P \in \mathrm{GL}(k, \mathbb{C})$ be chosen, so that the coordinates of

$$\tilde{s}(x_0) := P\, s(x_0)$$

form a $\mathbb{T}$-eigenbasis of the fiber $\mathscr{E}(x_0)$ (note that the coordinates of $s(x)$ form a G-eigenbasis of the fiber $\mathscr{E}(x_0)$). Define

$$\tilde{s}(x) := P\, s(x) \qquad \forall\, x \in U,$$

$$\tilde{A}(r) := P A(r) P^{-1}, \quad \tilde{B}(g) := P B(g) P^{-1} \qquad \forall\, r \in \mathbb{R}^n_{>0},\ g \in G.$$

By the choice of P, we know $\tilde{A}, \tilde{B}$ must be diagonal matrices. Moreover, by (5.11)

$$(t \cdot \tilde{s})(x) = \tilde{A}(r)\, \tilde{B}(e^{i\theta})\, \tilde{s}(x) \qquad \forall\, t = r \cdot e^{i\theta} \in \mathbb{T},\ x \in U \bigcap (r \cdot U),$$

or equivalently,

$$\tilde{s}(t \cdot x) = \tilde{A}(r^{-1})\, \tilde{B}(e^{-i\theta})\, t \cdot (\tilde{s}(x)) \qquad \forall\, t = r \cdot e^{i\theta} \in \mathbb{T}, x \in U \bigcap (r^{-1} \cdot U).$$

Extend $\tilde{s}$ to the section $\tilde{s} : U_I \to \mathscr{E}|_{U_I}$ by defining

$$\tilde{s}(t \cdot x) := \tilde{A}(r^{-1})\, \tilde{B}(e^{-i\theta})\, t \cdot (\tilde{s}(x)) \qquad \forall\, t = r \cdot e^{i\theta} \in \mathbb{T},\ x \in U.$$

Let $\tilde{\chi}_i(t)$ be the (i,i)-entry of the matrix $\tilde{A}(r)\, \tilde{B}(e^{i\theta})$ for $t = r \cdot e^{i\theta}$. Then the i-th coordinate $\tilde{s}_i$ of $\tilde{s}$ is a $\mathbb{T}$-eigensection over U_I with the eigenfunction being the continuous character $\tilde{\chi}_i$. Since $\{s_1, \ldots, s_k\}$ forms a G-eigenframe and $\tilde{s} = P\, s$, $\{\tilde{s}_1, \ldots, \tilde{s}_k\}$ must be a $\mathbb{T}$-eigenframe, and hence $\mathscr{E}|_{U_I}$ is $\mathbb{T}$-equivariantly homeomorphic to $U_I \times \mathbb{C}^k$ showing that $\mathscr{E}$ is a topological Klyachko vector bundle. $\square$

Theorem 5.3. *Let X be the $2n$-dimensional topological toric manifold associated with a* *topological fan Δ and $\pi : \mathscr{E} \to X$ a smooth $\mathbb{T}$-equivariant complex vector bundle, then $\mathscr{E}$ is a smooth Klyachko vector bundle.* $\square$

Proof. The proof is almost the same as that of Theorem 5.2. We keep the notations and point out the slight modifications below. Given $I \in \Sigma^{(n)}$, in step 1 we pick a G-invariant open ball $U = B(x_0, \varepsilon)$ over which there is a G-equivariant isomorphism

$$\varphi : \pi^{-1}(U) \to U \times \mathbb{C}^k.$$

Step 2 and step 3 are the same except that the sections s_i are smooth. In step 4, we construct a continuous $\mathbb{T}$-eigenframe $\{\tilde{s}_1, \ldots, \tilde{s}_k\}$ over U_I where the eigenfunction of $\tilde{s}_i$ is a continuous character $\tilde{\chi}_i$ as before. By Lemma 3.1, $\tilde{\chi}_i$ is smooth for all i, hence $\{\tilde{s}_1, \ldots, \tilde{s}_k\}$ is a smooth $\mathbb{T}$-eigenframe and so gives a smooth $\mathbb{T}$-equivariant trivialization of $\mathscr{E}|_{U_I}$. $\square$

Chapter 6: Real equivariant vector bundles

6.1 Equivalence of two categories of equivariant vector bundles

Definition 6.1. Let X be a topological toric manifold, define CCEVB_X to be the category whose objects are pairs $(\mathscr{E}, \tau)$ where $\mathscr{E}$ is a smooth complex equivariant vector bundle over X and $\tau : \mathscr{E} \to \mathscr{E}$ is a smooth map such that

1. τ is $\mathbb{T}$-equivariant,

2. $\tau^2 = \mathrm{Id}$,

3. for each $x \in X, \tau$ restricts to $\tau_x : \mathscr{E}_x \to \mathscr{E}_x$ and τ_x is $\mathbb{C}$ anti-linear.

For $(\mathscr{E}, \tau), (\mathscr{F}, \sigma) \in Ob(\mathrm{CCEVB}_X)$, a morphism $f : (\mathscr{E}, \tau) \to (\mathscr{F}, \sigma)$ is a $\mathbb{T}$-equivariant bundle morphism $\mathscr{E} \to \mathscr{F}$ such that $\sigma \circ f = f \circ \tau$.

Theorem 6.1. *Let X be a topological toric manifold,* REVB_X *the category of smooth real equivariant vector bundles over X. The map $\mathscr{C} : \mathrm{REVB}_X \to \mathrm{CCEVB}_X$ that sends a real equivariant $\mathscr{E} \to X$ to its complexification $\mathscr{E} \otimes_{\mathbb{R}} \mathbb{C} \to X$ with the conjugate map $\tau : \mathscr{E} \otimes_{\mathbb{R}} \mathbb{C} \to \mathscr{E} \otimes_{\mathbb{R}} \mathbb{C}$ defined via*

$$\tau(v \otimes z) := v \otimes \bar{z}$$

is an equivalence of categories.

Proof. Define $\mathscr{R} : \mathrm{CCEVB}_X \to \mathrm{REVB}_X$ as follows: for $(\mathscr{E}, \tau) \in \mathrm{Ob}(\mathrm{CCEVB}_X)$,

$$\mathscr{R}((\mathscr{E}, \tau)) = (\mathscr{E})^\tau := \{u \in \mathscr{E} | \tau(u) = u\}$$

with the induced $\mathbb{T}$-action.

Given $(\mathscr{E}, \tau) \in \mathrm{CCEVB}_X$, we prove that $(\mathscr{E})^\tau$ is a real equivariant vector bundle and $(\mathscr{E})^\tau \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathscr{E}$. For $I \in \Sigma^{(\mathrm{n})}$, by Theorem 5.2 we have a $\mathbb{T}$-equivariant isomorphism

$$\varphi : \mathscr{E}|_{U_I} \simeq U_I \times \mathbb{C}^k = \bigoplus_{\chi} (U_I \times \mathbb{C}_\chi)$$

where E is the fiber of $\mathscr{E}$ over a prescribed point x_0 in the open dense orbit of X and $\mathbb{C}_\chi \subset E$ is a weight space of complex dimension 1 corresponding to the character χ of $\mathbb{T}$. We denote by $\tilde{\tau}$ the conjugate transferred to $U_I \times \mathbb{C}^k$ via φ, so that we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}|_{U_I} & \xrightarrow{\varphi} & U_I \times \mathbb{C}^r \\ \Big\downarrow{\scriptstyle \tau} & & \Big\downarrow{\scriptstyle \tilde{\tau}} \\ \mathcal{E}|_{U_I} & \xrightarrow{\varphi} & U_I \times \mathbb{C}^r \end{array} .$$

Figure 6.1: Transferring of τ

First we prove that $(\mathscr{E})^\tau$ is a real vector bundle.

Let $e_1, \cdots, e_r$ be the pullback of the standard frame of $U_I \times \mathbb{C}^r$ on U_I by φ and for all $x \in U_I$ define

$$P_x := \frac{1}{2}(\mathrm{Id} + \tilde{\tau}_x),$$

$$S(x) = [P_x(e_1) \cdots P_x(e_r)\; P_x(ie_1) \cdots P_x(ie_r)] \in M_{2r \times 2r}(\mathbb{R}).$$

Notice that

$$\operatorname{Im} P_x = \{(x,v) | B(x)\overline{v} = v\} \simeq (\mathscr{E})^{\tau}_x,\ \mathbb{C}^r = \operatorname{Im} P_x \oplus i \operatorname{Im} P_x\ \forall x \in U_I. \tag{6.1}$$

Therefore $\dim_{\mathbb{R}} \operatorname{Im} P_x = r\ \forall x \in U_I$. Given $x_1 \in U_I$, we have $\operatorname{rank} S(x_1) = r$. Choose r linearly independent columns $c_1(x_1), \cdots, c_r(x_1)$ of $S(x_1)$. By continuity of the determinant function, there is a neighborhood $V \subset U_I$ containing x_1 such that $\{c_1(x) \cdots, c_r(x)\}$ is a local frame for $\mathscr{E}^{\tau}$. Therefore we have a local trivialization

$$\begin{aligned} V \times \mathbb{R}^r &\to \mathscr{E}^{\tau}|_V \\ (x, a_1, \cdots, a_r) &\mapsto \sum_{k=1}^{r} a_k \varphi^{-1}(x, c_k(x)) \end{aligned}$$

Moreover, $\{c_1(x), \cdots, c_r(x)\}$ is a subframe of the real frame $\{e_1, \cdots, e_r, ie_1, \cdots, ie_r\}$ of $\mathscr{E}$, hence $\mathscr{E}^{\tau}$ is a real vector sub-bundle of $\mathscr{E}$.

Next we prove that $\mathscr{E}^{\tau}$ is $\mathbb{T}$-stable. Given $(x,v) \in \varphi(\mathscr{E}^{\tau}), t \in \mathbb{T}$,

$$\tilde{\tau}_{tx}(A(t,x)v) = A(t,x)\tilde{\tau}_x(v) = A(t,x)v.$$

Hence $t \cdot (x,v) = A(t,x)v \in (\varphi(\mathscr{E}^{\tau}))_{tx}$. This proves that $\mathscr{E}^{\tau}$ is $\mathbb{T}$-stable and hence is a real $\mathbb{T}$-equivariant vector sub-bundle of $\mathscr{E}$. From the decomposition (6.1), $(\mathscr{E})^{\tau} \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathscr{E}$ follows. This proves that $\mathscr{C} \circ \mathscr{R} \simeq \mathrm{Id}$. It is clear that $\mathscr{R} \circ \mathscr{C} = \mathrm{Id}$. Hence $\mathscr{C}$ is an equivalence. □

Corollary 6.2. *Let $X = X(\Delta)$ be a topological toric manifold of dimension $2n$ and $\mathscr{E} \to X \in \mathrm{REVB}_X$, then for each $I \in \Sigma^{(n)}$, $\mathscr{E}|_{U_I}$ is $\mathbb{T}$-equivariant isomorphism to the direct sum of some real equivariant line bundles and some real equivariant plane bundles.*

Proof. Given a real $\mathbb{T}$-equivariant vector bundle $\mathscr{E}$ over X, let $\mathscr{C}(\mathscr{E}) = (\mathscr{E} \otimes_{\mathbb{R}} \mathbb{C}, \tau)$. Given

$I \in \Sigma^{(\mathrm{n})}$, by Theorem 5.2 we have a $\mathbb{T}$-equivariant isomorphism

$$\varphi : \mathscr{E} \otimes_{\mathbb{R}} \mathbb{C}|_{U_I} \simeq U_I \times \mathbb{C}^k = \bigoplus_{\chi \in \Lambda} L_\chi^{\oplus m(\chi)}$$

where E is the fiber of $\mathscr{E}$ over a prescribed point x_0 in the open dense orbit of X, $\mathbb{C}_\chi$ is a weight space of complex dimension 1 corresponding to the character χ of $\mathbb{T}$ and $L_\chi = U_I \times \mathbb{C}_\chi$. For each character χ of $\mathbb{T}$, define

$$\chi^*(t) := \overline{\chi(t)} \ \forall t \in \mathbb{T}.$$

For $t \in \mathbb{T}, v \in E_\chi$,

$$t \cdot \tau(v) = \tau(t \cdot v) = \tau(\chi(t)v) = \overline{\chi(t)}\tau(v).$$

Therefore $\tau(L_\chi) = L_{\chi^*}$. We partition the index set Λ into two subsets

$$\Lambda_1 := \{\chi \in \Lambda | \chi = \chi^*\}, \Lambda_2 := \{\chi \in \Lambda | \chi \neq \chi^*\}.$$

For $\chi \in \Lambda_1$, define $R_\chi := (L_\chi)^\tau$, then R_χ is a real equivariant line bundle and

$$L_\chi = R_\chi \bigoplus iR_\chi.$$

Given $\chi \in \Lambda_2$, we must have $m(\chi) = m(\chi^*)$ since $\mathscr{E} \otimes_{\mathbb{R}} \mathbb{C}$ is stable under τ. Define $R_\chi := (L_\chi \bigoplus L_{\chi^*})^\tau$. Let s be a complex eigenframe of L_χ, then $s^* := \tau(s)$ is a complex eigenframe of L_{χ^*} and $e^+ := s + s^*, e^- := i(s - s^*)$ is a real eigenframe of R_χ. This shows that R_χ is a real equivariant plane bundle over U_I. Hence

$$\mathscr{E}|_{U_I} = \bigoplus_{\chi \in \Lambda} R_\chi^{\oplus m(\chi)}$$

is a direct sum of real equivariant line bundles and real equivariant plane bundles as claimed.

□

6.2 Classification of real equivariant vector bundles

Definition 6.2. For a topological toric manifold X, let FVSC_X be the category whose objects are pairs (E,τ) where E is a complex vector space and $\tau : E \to E$ is an anti-linear involution with a poset P of complex subspaces $E^i(\mu)$ of E indexed by $\mu \in \mathscr{R}, i \in \Sigma^{(1)}$ which satisfy the following compatibility conditions:

1. For any $I \in \Sigma$, there exists a $\mathscr{R}^n/\beta_I^{\perp}$-grading

$$E = \bigoplus_{[\chi] \in \mathscr{R}^n/\beta_I^{\perp}} E_{[\chi]}$$

for which

$$E^i(\mu) = \sum_{\langle \chi, \beta_i \rangle \geq_s \mu} E_{[\chi]}, \ \ \forall i \in I.$$

2. $\tau(E_\chi) = E_{\chi^*}$ for all $[\chi] \in \mathscr{R}^n/\beta_I^{\perp}$ where $\chi^*(t) = \overline{\chi(t)}$ for all $t \in \mathbb{T}$.

Given $(E,\tau),(F,\sigma) \in \mathrm{FVSC}_X$, a morphism $f : (E,\tau) \to (F,\sigma)$ is a linear transformation $f : E \to F$ between the complex vector spaces E and F such that

$$f \circ \tau = \sigma \circ f.$$

Theorem 6.3. *The category* CCEVB_X *is equivalent to* FVSC_X.

Proof. Given $(\mathscr{E},\tau) \in \mathrm{CCEVB}_X$, we can associate a filtered complex vector space $\Phi(\mathscr{E},\tau) = (E, E^i(\mu))$ satisfying the compatibility condition 1 where E is the fiber of $\mathscr{E}$ over a prescribed point x_0 in the open dense orbit of X. When restricted to x_0, $\tau_{x_0} : E \to E$ is a conjugate map and the proof of 6.2 shows that τ satisfies the compatibility condition 2.

Given a morphism $f : (\mathscr{E}, \tau) \to (\mathscr{F}, \sigma)$, we can associate a morphism between filtered complex vector spaces $\Phi(f) : (E, E^i(\mu)) \to (F, F^i(\mu))$. By definition, $f \circ \tau = \sigma \circ f$, therefore $\Phi(f) \circ \tau = \sigma \circ \Phi(f)$.

To construct a quasi-inverse to Φ, we relate τ_x and τ_y for all $x, y \in X$. Over each U_I,

$$\mathscr{E}|_{U_I} = U_I \times E = \bigoplus_{\chi} U_I \times E_\chi.$$

On U_I, τ_{x_0} is the direct sum of $\tau_{x_0} : E_\chi \to E_{\chi^*}$ where χ runs over all χ appearing in the decomposition above. For $t \in \mathbb{T}, v \in E_\chi$,

$$\overline{\chi(t)}\tau_{tx_0}(v) = \tau_{tx_0}(\chi(t)v) = \tau_{tx_0}(t \cdot v) = t \cdot \tau_{x_0}(v) = \chi^*(t)\tau_{x_0}(v) = \overline{\chi(t)}\tau_{x_0}(v),$$

therefore

$$\tau_{tx_0}(v) = \tau_{x_0}(v).$$

By continuity of τ,

$$\tau_x = \tau_{x_0} \ \forall x \in X.$$

Given a filtered complex vector space with conjugate $(E, E^i(\mu), \tau)$ satisfying the compatibility conditions, we can construct a smooth complex $\mathbb{T}$-equivariant vector bundle $\mathscr{E}$ over X. By the analysis above, we define $\tau : \mathscr{E} \to \mathscr{E}$ by $\tau_x = \tau : \mathscr{E}_x = E \to \mathscr{E}_x = E$ for all $x \in X$. Define $\Psi(E, E^i(\mu), \tau) = (\mathscr{E}, \tau)$. It is clear that $\tau : \mathscr{E} \to \mathscr{E}$ is an involution that is $\mathbb{C}$ anti-linear in fibers. For $x \in X$, pick a neighborhood U_I of x, then $\mathscr{E}|_{U_I} \simeq U_I \times E$. For $v \in (\mathscr{E}_x)_\chi = E_\chi$,

$$\tau_{tx}(tv) = \tau(tv) = \tau(\chi(t)v) = \chi^*(t)\tau(v) = t \cdot \tau_x(v).$$

Hence $\tau : \mathscr{E} \to \mathscr{E}$ is $\mathbb{T}$-equivariant.

Given a morphism $f : (E, E^i(\mu), \tau) \to (F, F^i(\mu), \sigma)$ in FVSC_X, there is a smooth $\mathbb{T}$-equivariant bundle morphism $\Psi(f) : \mathscr{E} \to \mathscr{F}$. Given $t \in \mathbb{T}$, we choose a neighborhood U_I containing tx_0. Over U_I, the bundles $\mathscr{E}, \mathscr{F}$ are all $\mathbb{T}$-equivariantly trivial and $\Psi(f)|_{U_I} : \mathscr{E}|_{U_I} \to \mathscr{F}|_{U_I}$ is given by $U_I \times E \to U_I \times F$ sending (x,v) to $(x, f(v))$ for all $(x,v) \in U_I \times E$. Then

$$\sigma_{tx_0}(f_{tx_0}(v)) = \sigma_{tx_0}(f_{x_0}(v)) = \sigma_{x_0}(f_{x_0}(v)) = f_{x_0}(\sigma_{x_0}(v)) = f_{tx_0}(\tau_{tx_0}(v)).$$

By continuity of σ_x and τ_x with respect to x, we have

$$\sigma_x \circ f_x(v) = f_x \circ \tau_x(v) \; \forall x \in U_I.$$

Hence $\Psi(f)$ is a morphism in CCEVB_X. It is clear that Φ and Ψ are quasi-inverses to each other. $\square$

Corollary 6.4. *The category* REVB_X *is equivalent to* FVSC_X.

Chapter 7: Hirzebruch-Riemann-Roch theorem

7.1 t-divisors

7.1.1 t-divisors and t-line bundles

Definition 7.1. Denote $V(\{i\})$ by D_i for each $i \in \Sigma^{(1)}$. A t-Weil divisor D is a formal linear combination $D = \sum_{i=1}^{m} a_i D_i, a_i \in \mathscr{R}$. The restriction $D|_U$ of D to an invariant subset $U \subset X$ is

$$D|_U := \sum_i a_i D_i$$

where i runs over $i \in \Sigma^{(1)}$ such that $U \bigcap V(\{i\}) \neq \emptyset$. For a smooth character $\chi \in \mathscr{R}^n$, define its t-divisor to be

$$\operatorname{div}(\chi) := \sum_{i=1}^{m} \langle \chi, \beta_i \rangle D_i.$$

Any t-Weil divisor which is equal to the divisor of a smooth character is called a principal t-divisor. Two t-Weil divisors D, D' are linearly equivalent, written $D \sim D'$, if $D - D'$ is a principal t-divisor. A t-Cartier divisor D is a t-Weil divisor such that

$$D|_{U_I} = \operatorname{div}(-\chi_I) \text{ for all } I \in \Sigma^{(n)}$$

and we call $\{\chi_I | I \in \Sigma^{(n)}\}$ the t-Cartier data of D. We denote by

$$\mathrm{Div}_t(X) := \bigoplus_{i=1}^{m} \mathscr{R} D_i$$

the group of t-Weil divisors under the operation of $+$.

Remark. In toric geometry, the restriction $D|_U$ of a Weil divisor $D = \sum_\rho a_\rho D_\rho$ is defined to be $D|_U := \sum_i a_\rho D_\rho|_U$. We introduce t-divisors as a convenient language to describe smooth t-line bundles, so we treat D_i as abstract symbols and do not define $D_i|_U$ to be the subspace $V(\{i\}) \bigcap U$. According to our definition, $D_i|_U = D_i$ when $U \bigcap V(\{i\}) \neq \emptyset$.

Theorem 7.1. *Every t-Weil divisor $D = \sum_{i=1}^m a_i D_i$ is a t-Cartier divisor.*

Proof. Let $\beta_i = (\beta_i^1, \cdots, \beta_i^n)$. Given a t-Weil divisor $D = \sum_{i=1}^m a_i D_i$ with $a_i \in \mathscr{R}$ and $I = \{i_1, \cdots, i_n\} \in \Sigma^{(n)}$, let

$$B_I = \begin{bmatrix} \beta_{i_1}^1 & \beta_{i_2}^1 & \cdots & \beta_{i_n}^1 \\ \beta_{i_1}^2 & \beta_{i_2}^2 & \cdots & \beta_{i_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_{i_1}^n & \beta_{i_2}^n & \cdots & \beta_{i_n}^n \end{bmatrix}, \qquad \chi_I = -(a_{i_1}, \cdots, a_{i_n}) \cdot B_I^{-1},$$

then $\langle -x_I, \beta_i \rangle = a_i$ for all $i \in \sigma$. Hence $\{\chi_I | I \in \Sigma^{(n)}\}$ is the t-Cartier data of D. $\square$

In the following, we will simply call a t-Weil (t-Cartier) divisor D a t-divisor.

Definition 7.2. We call a complex smooth $\mathbb{T}$-equivariant line a t-line bundle. For any topological toric manifold X, we define the equivariant Picard group of $X, \mathrm{Pic}^{\mathbb{T}} X$, to be the group of $\mathbb{T}$-equivariant isomorphism classes of t-line bundles on X, under the operation of $\otimes$.

For a t-divisor D with t-Cartier data $\{\chi_I | I \in \Sigma^{(n)}\}$, we can define a t-line bundle L_D over X as follows: Let $x_0 = [(1, \cdots, 1)] \in X = U(\Sigma)/\mathrm{Ker}\,\lambda$. For each $I \in \Sigma^{(n)}$, let $L_{-\chi_I}$ be the trivial t-line bundle over U_I where $\mathbb{T}$ acts via

$$t(x,v) := (tx, \chi_I^{-1}(t)v) \text{ for all } t \in \mathbb{T}, (x,v) \in U_I \times \mathbb{C}.$$

For $I, J \in \Sigma^{(n)}$, by Lemma 4.11 the function $g_{IJ}(tx_0) = \frac{\chi_J(t)}{\chi_I(t)}$ can be smoothly extended from $\mathbb{T}$ to $g_{IJ} : U_{I \bigcap J} = U_I \bigcap U_J \to \mathbb{C}^*$ and for $I, J, K \in \Sigma^{(n)}$, $g_{IJ} g_{JK} = g_{IK}$. Therefore we can glue $L_{-\chi_I}$ and $L_{-\chi_J}$ via the isomorphism

$$\varphi_{IJ} : U_J \times \mathbb{C} \supset (U_I \bigcap U_J) \times \mathbb{C} \to (U_I \bigcap U_J) \times \mathbb{C} \subset U_I \times \mathbb{C}$$

defined by

$$\varphi_{IJ}(x,c) := (x, g_{IJ}(x)c) \text{ for } (x,c) \in U_{I \bigcap J} \times \mathbb{C}.$$

Theorem 7.2. *The map*

$$f : \mathrm{Div}_t(X) \to \mathrm{Pic}^{\mathbb{T}}(X)$$
$$D \mapsto L_D.$$

is an isomorphism of abelian groups.

Proof. It is clear that f is a homomorphism of abelian groups.
Injectivity of f: Given a t-divisor D with t-Cartier data $\{\chi_I\}$ such that L_D is equivariantly isomorphic to $X \times \mathbb{C}$ where on the RHS $\mathbb{T}$ acts trivially on the fibers. For each $I \in \Sigma^{(\mathrm{n})}$, we have $L_D|_{U_I} \simeq U_I \times \mathbb{C}_{-\chi_I}$ where on the RHS $\mathbb{T}$ acts via the smooth character χ_I^{-1}. Therefore we have $\mathbb{T}$-equivariant isomorphism $U_I \times \mathbb{C} \simeq U_I \times \mathbb{C}_{-\chi_I}$. Hence $\chi_I = 1$ for all $I \in \Sigma^{(\mathrm{n})}$ and $D = 0$.

Surjectivity of f: Given $L \in \mathrm{Pic}^{\mathbb{T}}$, by Theorem 5.3, $L|_{U_I}$ is equivariantly isomorphic to $L_{-\chi_I}$ for each $I \in \Sigma^{(n)}$, hence $L \simeq L_D$ where D is the t-divisor with t-Cartier data $\{\chi_I | I \in \Sigma^{(n)}\}$. □

Definition 7.3. If U is an invariant subset of X and D is a t-divisor on X, then we define

$$L_{(D|_U)} := (L_D)|_U.$$

7.1.2 First Chern class of t-divisors

We denote $c_1(L_D) \in H^2(X,\mathbb{Z})$ by $c_1(D)$ for simplicity. We identify $v \in \mathbb{Z}$ with $(v,v) \in \mathbb{C} \times \mathbb{Z}$, so z^v is the same as $z^{(v,v)}$ for $z \in \mathbb{C}^*$.

In §5 of [10], an equivariant complex line L_i was constructed for all $i \in \Sigma^{(1)}$. According to computations in §4.5, the weight characters of L_i are $\chi^{\alpha_i^I}$ for $I \in \Sigma^{(n)}$ containing i and χ^0 for $I \in \Sigma^{(n)}$ not containing i where α_i^I is an element in the basis $\{\alpha_k^I | k \in I\}$ dual to $\{\beta_k | k \in I\}$. Hence we have

Theorem 7.3.

$$L_{D_i} = L_i \textit{ for all } i \in \Sigma^{(1)}.$$

Theorem 7.4 ([10, Prop 8.3])**.** *Let X be an omnioriented topological toric manifold and let $\Delta(X) = (\Sigma(X), \beta(X))$ be the complete non-singular topological fan associated with X. We may assume that the vertex set $\Sigma^{(1)}(X)$ of $\Sigma(X)$ is $[m]$ and express $\beta(X)(i) = (b_i(X) + \sqrt{-1}c_i(X), v_i(X)) \in \mathbb{C}^n \times \mathbb{Z}^n$ for $i \in [m] = \Sigma^{(1)}(X)$. Then*

$$H^*(X;\mathbb{Z}) = \mathbb{Z}[\mu_1,\dots,\mu_m]/\mathscr{I}$$

where $\mu_i \in H^2(X;\mathbb{Z})$ and $\mathscr{I}$ is the ideal generated by the following two types of elements:

1. $\prod_{i\in I}\mu_i$ *for* $I\notin\Sigma(X)$*,*

2. $\sum_{i=1}^{m}\langle u, v_i(X)\rangle\mu_i$ *for any* $u\in\mathbb{Z}^n$*.*

Here μ_i *is actually the Poincaré dual of the t-divisor* D_i *in* X*.*

Theorem 7.5.

$$c_1(D_i)=\mu_i.$$

Proof. The section $(u, z_i(u))$ of the trivial bundle $U(\Sigma)\times\mathbb{C}\to U(\Sigma)$ descends to a section s_i of $L_i\to X$ where $z_i:\mathbb{C}^m\to\mathbb{C}$ is the $i-th$ coordinate function. The zero section $Z(s_i)$ of s_i is D_i. It is not difficult to see that s_i is transverse to $Z(s_i)$, hence $c_1(D_i)=\mu_i$. □

Theorem 7.6. *For a t-divisor* $D=\sum_{k=1}^{m}a_kD_k$*,*

$$L_D \text{ is non-equivariantly isomorphic to } \bigotimes_{k=1}^{m} L_k^{\otimes v(a_k)}$$

and consequently

$$c_1(D)=\sum_{k=1}^{m}v(a_k)\mu_k.$$

Proof. In this proof, we forget $\mathbb{T}$-actions on all the bundles involved. L_D is isomorphic to $\bigotimes_{k=1}^{m}L_{a_kD_k}$, so it suffices to prove that $L_{a_kD_k}$ is isomorphic to $L_k^{\otimes v(a_k)}$ for all k. Given k, suppose $a_k=(b+ci, v)$ and define

$$\begin{aligned} f:\mathbb{C}^*&\to\mathbb{C}^*\\ z&\mapsto z^{(b+ci,v)}. \end{aligned}$$

If the transition functions of $L:=L_k$ are g_{IJ}, then the transition functions of $L':=L_{a_kD_k}$ are $f\circ g_{IJ}$. We have the following pullback diagrams by the universal property of classifying spaces

$$\begin{tikzcd} L = \varphi_L^*\gamma \arrow[r] \arrow[d] & \gamma \arrow[d] \\ X \arrow[r, "\varphi_L"] & BU(1), \end{tikzcd} \qquad \begin{tikzcd} L' = (Bf \circ \varphi_L)^*\gamma \arrow[r] \arrow[d] & Bf^*\gamma \arrow[r] \arrow[d] & \gamma \arrow[d] \\ X \arrow[r, "\varphi_L"] & BU(1) \arrow[r, "Bf"] & BU(1) \end{tikzcd}$$

Figure 7.1: Pullback of γ

where $\gamma \to BU(1)$ is a universal line bundle, φ_L is a continuous map whose homotopy class is uniquely determined by L and Bf is a continuous whose homotopy class is the same as that of f. Notice that

$$f(z) = z^v \text{ for all } z \in S^1,$$

therefore the winding number of f is v and hence $Bf^*\gamma = \gamma^{\otimes v}, L' = L^{\otimes v}$.

$\square$

7.1.3 Equivariant cohomology classes of t-divisors

Recall the construction of a ring isomorphism s in §12.4 of [6]. The abelian group $M = \mathbb{Z}^n$ has the symmetric algebra

$$\mathrm{Sym}_{\mathbb{Z}}(M) = \mathbb{Z} \oplus M \oplus \mathrm{Sym}^2(M) \oplus \cdots.$$

For each $m \in M$, the map $\chi^m : \mathbb{T} \to \mathbb{C}^*$ induces $B\chi^m : B\mathbb{T} \to B\mathbb{C}^*$. At the level of cohomology groups, we obtain

$$\mathbb{Z} \cdot t \simeq H^2(B\mathbb{T}, \mathbb{Z}) \xrightarrow{(B\chi^\alpha)^*} H^2(B\mathbb{T}, \mathbb{Z}).$$

This defines a map

$$
\begin{aligned}
s : M &\to H^2(B\mathbb{T},\mathbb{Z}) \\
m &\mapsto (B\chi^m)^*(t)
\end{aligned}
$$

which induces a canonical isomorphism

$$s : \mathrm{Sym}_{\mathbb{Z}}(M) \simeq \Lambda := H^{\cdot}(B\mathbb{T},\mathbb{Z}).$$

We denote $\mathrm{Sym}_{\mathbb{Z}}(M) \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\mathrm{Sym}_{\mathbb{Q}}(M)$ and denote by

$$s : \mathrm{Sym}_{\mathbb{Q}}(M) \to (\Lambda)_{\mathbb{Q}} := H^{\cdot}(B\mathbb{T},\mathbb{Q})$$

the map obtained from the above s by passing to $\mathbb{Q}$ coefficients.

Consider $\mathscr{R}^n$ as an abelian group and it has the symmetric algebra

$$\mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) = \mathbb{Z} \oplus \mathscr{R}^n \oplus \mathrm{Sym}^2(\mathscr{R}^n) \oplus \cdots.$$

For each $\alpha \in \mathscr{R}^n$, the map $\chi^\alpha : \mathbb{T} \to \mathbb{C}^*$ induces $B\chi^\alpha : B\mathbb{T} \to B\mathbb{C}^*$. At the level of cohomology groups, we obtain

$$\mathbb{Z} \cdot t \simeq H^2(B\mathbb{C}^*,\mathbb{Z}) \xrightarrow{(B\chi^\alpha)^*} H^2(B\mathbb{T},\mathbb{Z}).$$

This defines a map

$$
\begin{aligned}
q : \mathscr{R}^n &\to H^2(B\mathbb{T},\mathbb{Z}) \\
\alpha &\mapsto (B\chi^\alpha)^*(t)
\end{aligned}
$$

which induces

$$q : \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) \to \Lambda := H^{\cdot}(B\mathbb{T}, \mathbb{Z}).$$

We denote $\mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) \otimes_{\mathbb{Z}} \mathbb{Q}$ by $\mathrm{Sym}_{\mathbb{Q}}(\mathscr{R}^n)$ and denote by

$$q : \mathrm{Sym}_{\mathbb{Q}}(\mathscr{R}^n) \to \Lambda_{\mathbb{Q}} := H^{\cdot}(B\mathbb{T}, \mathbb{Q})$$

the map obtained from the above p by passing to $\mathbb{Q}$ coefficients.

Definition 7.4. For a t-divisor D on a topological toric manifold X, let

$$(L_D)_{\mathbb{T}} := E\mathbb{T} \times_{\mathbb{T}} L_D \to E\mathbb{T} \times_{\mathbb{T}} X$$

be the line bundle induced by $L_D \to X$ and we define

$$[D]_{\mathbb{T}} := c_1((L_D)_{\mathbb{T}}) \in H^2(E\mathbb{T} \times_{\mathbb{T}} X, \mathbb{Z}) = H^{\cdot}_{\mathbb{T}}(X, \mathbb{Z}).$$

Definition 7.5. Let

$$i_X : X \hookrightarrow E\mathbb{T} \times_{\mathbb{T}} X$$

be the inclusion and

$$i_X^* : H^{\cdot}_{\mathbb{T}}(X, \mathbb{Q}) \to H^{\cdot}(X, \mathbb{Q})$$

the ring homomorphism induced by i_X.

Below is a t-divisor version of Prop 12.4.13 of [6]. The proof is exactly the same as in [6].

Theorem 7.7. *For any t-divisors D, D_1, D_2 on X we have:*

i. $[D_1+D_2]_{\mathbb{T}} = [D_1]_{\mathbb{T}} + [D_2]_{\mathbb{T}}$.

ii. $[\mathrm{div}(\chi^{\alpha})]_{\mathbb{T}} = -q(\alpha)\cdot \mathbf{1}$ *where* $\mathbf{1} \in H^0_{\mathbb{T}}(X,\mathbb{Q})$ *and* $q(\alpha) \in (\Lambda)_{\mathbb{Q}}$.

iii. $i^*[D]_{\mathbb{T}} = [D|_U]_{\mathbb{T}}$ *where* $i : U \to X$ *is the inclusion of an invariant open subset* U *of* X.

iv. $i_X^*[D]_{\mathbb{T}} = [D]$.

Definition 7.6. Recall that we defined $v : \mathscr{R}^n = \mathbb{C}^n \times \mathbb{Z}^n \to \mathbb{Z}^n$ to be the projection. v can be extended to a homomorphism of algebras

$$v : \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) \to \mathrm{Sym}_{\mathbb{Z}}(M).$$

Theorem 7.8. $q : \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) \to \Lambda_{\mathbb{T}}$ *is surjective and* $\mathrm{Ker}\, q = \{\alpha \in \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) | v(\alpha) = 0\}$.

Proof. Given $\alpha \in \mathscr{R}^n$, notice that $\chi^{\alpha} : \mathbb{T} \to \mathbb{C}^*$ is homotopic to $\chi^{v(\alpha)} : \mathbb{T} \to \mathbb{C}^*$, therefore

$$q = s \circ v : \mathscr{R}^n \to \Lambda.$$

This implies that

$$q = s \circ v.$$

q is surjective because v is surjective and s is an isomorphism of rings. $\alpha \in \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n)$ is in $\mathrm{Ker}\, q$ if and only if $v(\alpha) \in \mathrm{Ker}\, s = 0$. Hence $\mathrm{Ker}\, q = \{\alpha \in \mathrm{Sym}_{\mathbb{Z}}(\mathscr{R}^n) | v(\alpha) = 0\}$. □

7.2 $\mathscr{P}_D$ and $\overline{\int}_X$

In this section, we associate a multi-polytope $\mathscr{P}_D = (\Delta, \mathscr{F})$ with a t-divisor D.

7.2.1 Multi-polytope $\mathscr{P}_D$

Given a nonsingular complete topological fan $\Delta = (\Sigma, \beta)$, denote by A_I the matrix formed by $\{\beta_i\}_{i\in I}$ and define

$$C' : \Sigma^{(1)} \to N \cong \mathbb{Z}^n$$
$$i \mapsto q(\beta_i).$$

For $I \in \Sigma^{(\mathrm{n})}$, set

$$(w^+(I), w^-(I)) = \begin{cases} (1,0) \text{ if } |A_I| > 0, \\ (0,1) \text{ if } |A_I| < 0, \end{cases}$$

and $w(I) := w^+(I) - w^-(I)$. The multi-fan $(\Sigma, C', w^\pm)$ is denoted by Δ'.

Now we modify this to give another multi-fan.

For $I \in \Sigma^{(\mathrm{n})}$, set

$$C : \Sigma^{(1)} \to N \cong \mathbb{Z}^n$$
$$i \mapsto v_i := -q(\beta_i),$$

By abuse of notation, we denote the multi-fan $(\Sigma, C, w^\pm)$ also by Δ.

Lemma 7.9. *Δ is a complete non-singular multi-fan.*

Proof. Denote by d'_v the degree of Δ' and by d_v that of Δ. By Lemma 9.5 of [8], Δ' is a complete multi-fan. Notice that $d_v = d'_{-v}$ and $\Delta_K = (\Sigma_K, -C'_K, (-1)^n w'^{\pm}_K)$ for any $K \in \Sigma$, hence Δ is also a complete multi-fan. □

Remark. *While Δ' is used to establish completeness of Δ, all subsequent geometric results, including the Hirzebruch-Riemann-Roch theorem, pertain to the primary induced multifan*

Δ.

Definition 7.7. For a t-divisor D with t-Cartier data $\{\chi_I | I \in \Sigma^{(\mathrm{n})}\}$, define

$$\mathscr{F}_D : \Sigma^{(1)} \to \mathrm{HP}(N^*_{\mathbb{R}})$$
$$i \mapsto q(\chi_I) + v_i^{\perp}$$

where $v_i^{\perp} := \{v \in N^*_{\mathbb{R}} | \langle v, v_i \rangle = 0\}$. The multi-polytope associated with D is

$$\mathscr{P}_D := (\Delta, \mathscr{F}_D).$$

It is clear from the definition that $\mathscr{P}_D$ is a simple lattice multi-polytope.

Remark 2. Recall that $q : \mathscr{R}^n \to N^* = H^2(B\mathbb{T}, \mathbb{Z})$ can be factorized as $\mathscr{R}^n \xrightarrow{v} M = \mathbb{Z}^n \xrightarrow{s} N^*$. In practice when we draw $\mathscr{P}_D$, we identify $N^*_{\mathbb{R}}$ and $M_{\mathbb{R}}$ via s and identify $\mathscr{P}_D \subset N^*_{\mathbb{R}}$ and $s^{-1}(\mathscr{P}_D) \subset M_{\mathbb{R}}$.

Example 7.1. In this example, a vector $\beta = ((b_1 + ic_1, v_1), (b_2 + ic_2, v_2)) \in \mathscr{R}^2$ is identified with the matrix $\begin{pmatrix} b_1 & 0 \\ c_1 & v_1 \\ b_2 & 0 \\ c_2 & v_2 \end{pmatrix}$ and a dual vector $\alpha = ((b_1 + ic_1, v_1), (b_2 + ic_2, v_2))$ is identified with $\begin{pmatrix} b_1 & 0 & b_2 & 0 \\ c_1 & v_1 & c_2 & v_2 \end{pmatrix}$.

Let

$$\Sigma = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{1,2\}, \{2,0\}\}.$$

and the rays

$$\beta_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \beta_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

$\Delta = (\Sigma, \beta)$ is the topological fan of $X = \mathbb{CP}^2$. The multi-fan $\Delta = (\Sigma, C, w^{\pm})$ associated with the topological fan Δ is given

$$\begin{aligned} C : \Sigma^{(1)} &\to M_{\mathbb{R}} = \mathbb{R}^n \\ i &\mapsto v_i := -v(\beta_i), \end{aligned}$$

i.e. $v_0 = (1,1)^T, v_1 = (-1,0)^T, v_2 = (0,-1)^T$ and $(w^+(I), w^-(I)) = (1,0)$ for all $I \in \Sigma^{(2)}$. Let $D = D_0 + D_1 + D_2$ be the anticanonical divisor of X, then $\mathscr{P}_D$ is the polytope with vertices $(-1,3), (3,-1), (-1,1)$. The inward normal fan of $\mathscr{P}_D$ is the usual toric fan of $\mathbb{CP}^2$ with rays $(1,0), (0,1), (-1,-1)$.

7.2.2 Generalized integrals

Given $I \in \Sigma^{(n)}$, let

$$i_I : \{\gamma_I\} \hookrightarrow X$$

be the inclusion map and let

$$p : X \to \{\mathrm{pt}\}$$

be the constant map. We have the following commutative diagram

$$
\begin{tikzcd}
\hat{H}^{\cdot}_{\mathbb{T}}(X,\mathbb{Q}) \arrow[dr, "i_I^*"] & \\
\hat{H}^{\cdot}_{\mathbb{T}}(\{\mathrm{pt}\},\mathbb{Q}) = \hat{\Lambda} \arrow[u, "p^*"] \arrow[r, "\simeq", "i_I^* \circ p^*"'] & \hat{H}^{\cdot}_{\mathbb{T}}(\{\gamma_I\},\mathbb{Q})
\end{tikzcd}
$$

Figure 7.2: Identification of the completion of two cohomology rings

We use the isomorphism $i_I^* \circ p^*$ to identify $\hat{H}^{\cdot}_{\mathbb{T}}(\{\mathrm{pt}\},\mathbb{Q}) = \hat{\Lambda}$ with $\hat{H}^{\cdot}_{\mathbb{T}}(\{\gamma_I\},\mathbb{Q})$ and write $i_I^*(\alpha) \in \hat{\Lambda}$ for $\alpha \in \hat{H}^{\cdot}_{\mathbb{T}}(X,\mathbb{Q})$.

Lemma 7.10. *Let X be a topological toric manifold of dimension $2n$ and $I \in \Sigma^{(n)}$, $j \in \Sigma^{(1)}$. Then :*

i. If $j \notin I$, then $i_I^*([D_j]_{\mathbb{T}}) = 0$.

ii. If $j \in I$, then $i_I^*([D_j]_{\mathbb{T}}) = -q(\alpha_j^I) \cdot \mathbf{1}$ where $D_j|_{U_I} = \mathrm{div}(\chi^{\alpha_j^I}), \mathbf{1} \in \hat{H}^{\cdot}_{\mathbb{T}}(\{\gamma_I\},\mathbb{Q})$.

Proof. Factor $i_I : \{\gamma_I\} \hookrightarrow X$ as $\{\gamma_I\} \hookrightarrow U_I \overset{j_I}{\hookrightarrow} X$. By Theorem 7.7 for any $i \in \Sigma^{(1)}$,

$$j_I^*([D_i]_{\mathbb{T}}) = [D_i|_{U_I}]_{\mathbb{T}}.$$

i. If $i \notin I$, then $i_I^*([D_j]_{\mathbb{T}}) = 0$ by definition of $D_i|_{U_I}$.
ii. If $i \in I$, then

$$j_I^*([D_i]_{\mathbb{T}}) = [D_i|_{U_I}]_{\mathbb{T}} = [\mathrm{div}(\chi_j^I)]_{\mathbb{T}} = -q(\alpha_j^I) \cdot \mathbf{1} \in \hat{H}^{\cdot}_{\mathbb{T}}(U_I,\mathbb{Q}).$$

The map $\hat{H}^{\cdot}_{\mathbb{T}}(U_I,\mathbb{Q}) \to \hat{H}^{\cdot}_{\mathbb{T}}(\{\gamma_I\},\mathbb{Q})$ is a homomorphism of $\hat{\Lambda}$-modules, hence $i_I^*([D_j]_{\mathbb{T}}) = -q(\chi_I) \cdot \mathbf{1}$.

$\square$

Lemma 7.11. *For $I \in \Sigma^{(\mathrm{n})}, i \in I$, we have $q(\alpha_i^I) \neq 0$.*

Proof. Given $I \in \Sigma^{(\mathrm{n})}$, let V be the $n \times n$ real matrix whose i-column is $v(\beta_i)$ and W the $n \times n$ real matrix whose i-row is $v(\alpha_i^I)$, then by the proof of Lemma 2.4 of [10] we have $W = V^{-1}$, hence $v(\alpha_i^I) \neq 0$ for all $i \in I$ and by Theorem 7.8, $q(\alpha_i^I) \neq 0$ for all $i \in I$. $\square$

Lemma 7.12. *For $I \in \Sigma^{(\mathrm{n})}$, the equivariant Euler class of the normal bundle N_I of the fixed point γ_I is*

$$e^{\mathbb{T}}(N_I) = \prod_{i \in I}(-q(\alpha_i^I)).$$

Proof. At each γ_I, its normal bundle $N_I \simeq (\bigoplus_{i \in I} L_i)|_{\gamma_I}$, therefore its equivariant Euler class

$$e^{\mathbb{T}}(N_I) = \prod_{i \in I} e^{\mathbb{T}}(L_{-\alpha_i^I}) = \prod_{i \in I} c_1^{\mathbb{T}}(L_{-\alpha_i^I}) = \prod_{i \in I}(-q(\alpha_i^I))$$

where the second equality follows from the local chart $U_I \simeq V(\bigoplus_{i \in I} \chi^{\alpha_i^I})$ and the third equality follows from Theorem 7.7 *ii*. $\square$

Let S be the multiplicative set consisting of nonzero homogeneous elements of positive degree in $\hat{\Lambda} = H^*(BG, \mathbb{Q}) = H^*(B\mathbb{T}, \mathbb{Q})$.

Theorem 7.13. *Let X be a topological toric manifold of dimension $2n$ and $\alpha \in \hat{H}_G^{\cdot}(X, \mathbb{Q})$, then*

$$\int_{X^{eq}} \alpha := p_!(\alpha) = \sum_{I \in \Sigma^{(\mathrm{n})}} \frac{i_I^*(\alpha)}{\prod_{i \in I}(-q(\alpha_i^I))} \in \hat{\Lambda}_S.$$

Proof. By Lemma 7.11 the right hand side of Theorem 7.13 is in $\hat{\Lambda}_S$. Denote by G the maximal compact subgroup $(S^1)^n$ of $\mathbb{T}$. For each $I \in \Sigma^{(\mathrm{n})}$, there is a unique G-fixed point γ_I in U_I. Since $\{U_I\}$ is an open cover of X, $F := \{\gamma_I | I \in \Sigma^{(\mathrm{n})}\}$ is the set of G-fixed points. At each γ_I, we have $e^{\mathbb{T}}(N_I) = \prod_{i \in I}(-q(\alpha_i^I))$ by Lemma 7.12, hence by the Atiyah-Bott localization formula, we have

$$\int_{X^{eq}} \alpha = \sum_{I \in \Sigma^{(\mathrm{n})}} \frac{i_I^*(\alpha)}{\prod_{i \in I}(-q(\alpha_i^I))} \in \hat{\Lambda}_S.$$

□

Definition 7.8. Let X be a topological toric manifold of dimension $2n$ and $\alpha \in \hat{H}_G^{\cdot}(X, \mathbb{Q})$, define

$$\overline{\int_{X^{eq}}} \alpha := \sum_{I \in \Sigma^{(\mathrm{n})}} \frac{w(I) i_I^*(\alpha)}{\prod_{i \in I}(-q(\alpha_i^I))} \in \hat{\Lambda}_S.$$

Theorem 7.14. *Let $X = X(\Delta)$ be a topological toric manifold of dimension $2n$, then*

$$H_{\mathbb{T}}^*(X, \mathbb{Z}) \cong \mathbb{Z}[\mu_1, \cdots, \mu_m]/(\mu_I | I \notin \Sigma).$$

Proof. This follows from Corollary 7.6 of [13] since X has vanishing odd degree cohomology groups. □

For a t-divisor D and the multi-fan Δ of $\mathscr{P}_D$, we have a ring isomorphism

$$f : H_{\mathbb{T}}^*(X, \mathbb{Z}) \to H_G^*(\Delta)$$

sending μ_i to x_i. Now we can freely use the results of §8 of [8].

Lemma 7.15. $\overline{\int}_{X^{eq}}(H_{\mathbb{T}}^*(X, \mathbb{Q})) \subset H^*(BG, \mathbb{Q})$.

Proof. This follows from Lemma 8.4 of [8] and the isomorphism f above. □

Let $i_{\mathrm{pt}}^* : \hat{\Lambda} \to \mathbb{Q}$ be the map that sends any element of positive degree to 0. Notice that $H^*(BG, \mathbb{Q})/H^2(B\mathbb{T}, \mathbb{Q})$ is isomorphic to $\mathbb{Q}$ and

$$i_X^* : H_{\mathbb{T}}^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q})$$

is the canonical projection by Lemma 7.7 *iv*. For $\overline{\alpha} \in H^*(X,\mathbb{Q})$, pick $\alpha \in H^*_{\mathbb{T}}(X,\mathbb{Q})$ such that $i_X^*(\alpha) = \overline{\alpha}$ and define

$$\overline{\int_X} \overline{\alpha} := i_{\mathrm{pt}}^*(\overline{\int_{X^{eq}}} \alpha) \in \mathbb{Q}. \tag{7.1}$$

The elements in $\operatorname{Ker} i_X^*$ have positive degree and are killed by i_{pt}^*, hence

$$\overline{\int_X} : H^*(X,\mathbb{Q}) \to \mathbb{Q}$$

is a well-defined ring homomorphism. By definition, we have the following commutative diagrams

$$\begin{tikzcd} H^*_{\mathbb{T}}(X,\mathbb{Q}) \arrow[r, "\overline{\int}_{X^{eq}}"] \arrow[d, "i_X^*"] & \Lambda \arrow[d, "i_{\mathrm{pt}}^*"] \\ H^*(X,\mathbb{Q}) \arrow[r, "\overline{\int}_X"'] & \mathbb{Q} \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \hat{H}^*_{\mathbb{T}}(X,\mathbb{Q}) \arrow[r, "\overline{\int}_{X^{eq}}"] \arrow[d, "i_X^*"] & \hat{\Lambda} \arrow[d, "i_{\mathrm{pt}}^*"] \\ \hat{H}^*(X,\mathbb{Q}) \arrow[r, "\overline{\int}_X"'] & \mathbb{Q} \end{tikzcd} .$$

Figure 7.3: Equivariant Integral

7.3 Td(X) and ch(D)

The tangent bundle TX of a topological toric manifold X is not a complex vector bundle, so its total Chern class $c(TX)$ is not defined and we cannot define the Todd class Td(X) using Chern roots as in [6]. We recall the Todd class of a simplicial toric variety to motivate our definition of the Todd class of a topological toric manifold X.

Let Σ' be a complete simplicial fan, $X_{\Sigma'}$ the associated toric variety and T the dense torus of X. The Todd class of $X_{\Sigma'}$ was given in [5]. We recall the definitions in [6] with a slight change of notations. Let u_ρ be the primitive generator of the ray $\rho \in \Sigma'(1)$ and

$$G = \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma'(1)} | \prod_\rho t_\rho^{\langle m, u_\rho \rangle} = 1 \text{ for all } m \in M\}.$$

For each $\rho \in \Sigma'(1)$,

$$\chi_\rho : (\mathbb{C}^*)^{\Sigma'(1)} \to \mathbb{C}^*$$

is the projection on the ρth factor. For each cone $\sigma \in \Sigma'$, let

$$\begin{aligned} G_\sigma &= \{g \in G | \chi_\rho(g) = 1 \text{ for all } \rho \notin \sigma(1)\} \\ &= \{(t_\rho) \in (\mathbb{C}^*)^{\Sigma(1)} | t_\rho = 1 \text{ for } \rho \notin \sigma(1), \prod_{\rho \in \sigma'(1)} t_\rho^{\langle m, u_\rho \rangle} = 1 \text{ for all } m \in M\}. \end{aligned}$$

Then we set

$$G_{\Sigma'} = \bigcup_{\sigma \in \Sigma} G_\sigma \subset G.$$

Theorem 7.16 ([5, Theorem 4.2], [6, Formula (13.3.18)])**.** *The equivariant Todd class of $X_{\Sigma'}$ is*

$$\mathrm{Td}^T(X_{\Sigma'}) = \sum_{g \in G_{\Sigma'}} \prod_{\rho \in \Sigma'(1)} \frac{[D_\rho]_T}{1 - \chi_\rho(g) e^{-[D_\rho]_T}}.$$

Below we define the equivariant Todd class of a topological toric manifold using notations in [8] and [10] and compare this definition with that of a torus manifold [8] in section 6. Let

$$G = \operatorname{Ker} \lambda = \{(h_1, \ldots, h_m) \in (\mathbb{C}^*)^m \mid \prod_{k=1}^m h_k^{\langle \alpha, \beta_k \rangle} = 1 \quad \text{for any } \alpha \in \mathscr{R}^n\}.$$

For each $i \in \Sigma^{(1)}$, let

$$\rho_i : (\mathbb{C}^*)^m \to \mathbb{C}^*$$

be the projection on the i-th factor. For any $I \in \Sigma$, let

$$\begin{aligned} G_I &= \{h \in G | \rho_i(h) = 1 \text{ for all } i \notin I\} \\ &= \{(h_i) \in (\mathbb{C}^*)^m | h_i = 1 \text{ for } i \notin I, \prod_{i \in I} h_i^{\langle \alpha, \beta_i \rangle} = 1 \text{ for all } \alpha \in \mathscr{R}^n\}. \end{aligned}$$

Set

$$G_\Sigma = \bigcup_{I \in \Sigma} G_I \subset G.$$

Notice that if $I \subset J \in \Sigma$, then $G_I \subset G_J$. For $I \in \Sigma^{(n)}$, by Lemma 4.1 of [10], $\mathrm{Ker}\,\lambda = \{(h_1, \ldots, h_m) \in (\mathbb{C}^*)^m \mid h_i \prod_{k \notin I} h_k^{\langle \alpha_i^I, \beta_k \rangle} = 1 \quad \text{for any } i \in I\}$, hence $G_I = \{(1, \cdots, 1)\}$ and $\rho_i(g) = 1$ for $g \in G_I$.

Definition 7.9. The equivariant Todd class of a topological toric manifold X is

$$\mathrm{Td}^{\mathbb{T}}(X) := \sum_{g \in G_\Sigma} \prod_{i \in \Sigma^{(1)}} \frac{[D_i]_T}{1 - \rho_i(g) e^{-[D_i]_T}} = \prod_{i \in \Sigma^{(1)}} \frac{[D_i]_T}{1 - e^{-[D_i]_T}}.$$

The Todd class of X is

$$\mathrm{Td}(X) := i_X^*(\mathrm{Td}^{\mathbb{T}}(X)) = \prod_{i \in \Sigma^{(1)}} \frac{[D_i]}{1 - e^{-[D_i]}}.$$

Definition 7.10. The equivariant Chern character of a t-line bundle L_D corresponding to a t-divisor D is

$$\mathrm{ch}^{\mathbb{T}}(L_D) := e^{c_1^{\mathbb{T}}(D)}$$

and the Chern character character of L_D is

$$\mathrm{ch}(L_D) := e^{c_1(D)}.$$

It is clear that $i_X^*(\mathrm{ch}^{\mathbb{T}}(L_D)) = \mathrm{ch}(L_D)$ by definition.

7.4 HRR for topological toric manifold

In this section, we fix a t-divisor D with t-Cartier data $\{\chi_I | I \in \Sigma^{(\mathrm{n})}\}$.

Lemma 7.17. *For $I \in \Sigma^{(\mathrm{n})}, \alpha = ch^{\mathbb{T}}(D)\,\mathrm{Td}^{\mathbb{T}}(X)$,*

$$\frac{i_I^*(\alpha)}{\prod_{i\in I}(-q(\alpha_i^I))} = \frac{e^{q(\chi_I)}}{\prod_{i\in I}(1-e^{q(\alpha_i^I)})}.$$

Proof. We follow the proof of Theorem 13.3.1 of [6]. First

$$i_I^*([D]_{\mathbb{T}}) = q(\chi_I),$$

hence

$$i_I^*(\mathrm{ch}^{\mathbb{T}}(D)) = i_I^*(e^{[D]_{\mathbb{T}}}) = e^{i_I^*([D]_{\mathbb{T}})} = e^{q(\chi_I)}.$$

Next we compute $i_I^*(\mathrm{Td}^{\mathbb{T}}(X))$. If $i \notin I$, then

$$i_I^*(\frac{[D_i]}{1-e^{-[D_i]}}) = i_I^*(1+\frac{1}{2}[D_i]_{\mathbb{T}}+\cdots) = 1$$

by Lemma 7.10. If $i \in I$, then

$$i_I^*(\frac{[D_i]}{1-e^{-[D_i]}}) = \frac{i_I^*([D_i]_{\mathbb{T}})}{1-e^{-i_I^*([D_i]_{\mathbb{T}})}} = \frac{-q(\alpha_i^I)}{1-e^{q(\alpha_i^I)}}$$

and

$$i_I^*(\mathrm{Td}^{\mathbb{T}}(X)) = \prod_{i\in I}\frac{-q(\alpha_i^I)}{1-e^{q(\alpha_i^I)}} = (-1)^n\frac{q(\alpha_i^I)}{1-e^{q(\alpha_i^I)}}.$$

Hence

$$(-1)^n\frac{i_I^*(\alpha)}{\prod_{i\in I} q(\alpha_i^I)} = \frac{e^{q(\chi_I)}}{\prod_{i\in I}(1-e^{q(\alpha_i^I)})}.$$

□

Lemma 7.18. *For each $I \in \Sigma^{(\mathrm{n})}$, recall that $q(\chi_I)$ is the corresponding vertex of $\mathscr{P}_D$ and $\{u_i^I = -q(\alpha_i^I) | i \in I\}$ is the dual basis of $\{v_i = -q(\beta_i) | i \in I\}$ as in section 2. For any $v \in N$ such that $\langle u_i^I, v\rangle$ is a nonzero integer for any $I \in \Sigma^{(\mathrm{n})}$ and $i \in I$, we have*

$$\sum_{I \in \Sigma^{(\mathrm{n})}} \frac{w(I) t^{q(\chi_I)}}{\prod_{i \in I}(1 - t^{-u_i^I})} = \sum_{u \in N^*} \mathrm{DH}_{\mathscr{P}_{D+}}(u) t^u \in \mathbb{C}[N^*].$$

Proof. This is a special case of Corollary 7.4 of [8]. □

Theorem 7.19. *For a t-divisor D on a topological toric manifold $X = X(\Delta)$,*

$$\overline{\int_X} \mathrm{ch}(D)\,\mathrm{Td}(X) = \#\mathscr{P}_D.$$

Proof.

$$\begin{aligned}
\overline{\int_X} \mathrm{ch}(D)\,\mathrm{Td}(X) &= i_{\mathrm{pt}}^* \overline{\int_{X^{eq}}} \mathrm{ch}^{\mathbb{T}}(L_D)\,\mathrm{Td}^{\mathbb{T}}(X) \text{ (by diagram7.3)} \\
&= i_{\mathrm{pt}}^*\Big(\sum_{I \in \Sigma^{(\mathrm{n})}} \frac{w(I) i_I^*(\mathrm{ch}^{\mathbb{T}}(L_D)\,\mathrm{Td}^{\mathbb{T}}(X))}{\prod_{i \in I}(-q(\alpha_i^I))} \Big) \\
&= i_{\mathrm{pt}}^*\Big(\sum_{I \in \Sigma^{(\mathrm{n})}} \frac{w(I) e^{q(\chi_I)}}{\prod_{i \in I}(1 - e^{q(\alpha_i^I)})} \Big) \text{ (by Lemma7.17)} \\
&= i_{\mathrm{pt}}^*\Big(\sum_{u \in H^2(B\mathbb{T},\mathbb{Z})} \mathrm{DH}_{\mathscr{P}_{D+}}(u) e^u \Big) \text{ (by Lemma7.18)} \\
&= \sum_{u \in H^2(B\mathbb{T},\mathbb{Z})} \mathrm{DH}_{\mathscr{P}_{D+}}(u) \\
&= \#\mathscr{P}_D.
\end{aligned}$$

□

Appendix A: Equivariant Gysin maps

In this section, we present a compact $\mathbb{T}$-manifold version of Prop 13.A.9-A.11 of [6]. In [11], the equivariant Gysin map

$$f_! : h_G(M) \to h_G(N)$$

is constructed for all G-map $f : M \to N$ between smooth G-manifolds where G is a compact Lie group and $h_G()$ is an equivariant multiplicative cohomology theory. In this appendix, we take G to be the maximal compact subgroup $(S^1)^n$ of $\mathbb{T}$ and h_G the equivariant de Rham cohomology theory. Notice that $H^{\cdot}_{\mathbb{T}}(X,\mathbb{Z}) \simeq H^{\cdot}_{G}(X,\mathbb{Z})$ because $\mathbb{T}$ is homotopy equivalent to G.

Theorem A.1. *Assume $f : X \to Y$ is a proper $\mathbb{T}$-equivariant map between compact smooth $\mathbb{T}$-manifolds, then there is a $\mathbb{T}$-equivariant Gysin map $f_! : H^{\cdot}_{\mathbb{T}}(X,\mathbb{Q}) \to H^{\cdot+\dim Y-\dim X}_{\mathbb{T}}(Y,\mathbb{Q})$ satisfying the following properties:*

i. (functoriality) If $g : Y \to Z$ is another continuous $\mathbb{T}$-equivariant map between compact smooth $\mathbb{T}$-manifolds, then $(g\circ f)_! = g_! \circ f_!$.

ii. (projection formula) $f_!(f^*(\alpha)\cup\beta) = \alpha\cup f_!(\beta)$ for all $\alpha \in H^{\cdot}_{\mathbb{T}}(Y,\mathbb{Q}), \beta \in H^{\cdot}_{\mathbb{T}}(X,\mathbb{Q})$.

Proof. Cf. Theorem 6.1 of [11]. □

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Y \end{array}$$

Figure A.1: Cartesian Diagram

$$\begin{array}{ccc} H^{\cdot}_{\mathbb{T}}(X',\mathbb{Q}) & \xrightarrow{f'_!} & H^{\cdot}_{\mathbb{T}}(Y',\mathbb{Q}) \\ {\scriptstyle g'^*}\uparrow & & {\scriptstyle g^*}\uparrow \\ \hat{H}^{\cdot}_{\mathbb{T}}(X,\mathbb{Q}) & \xrightarrow{f_!} & H^{\cdot}_{\mathbb{T}}(Y,\mathbb{Q}). \end{array}$$

Figure A.2: Equivariant Gysin Maps

Theorem A.2. *Suppose we have a Cartesian diagram as above in the category of smooth $\mathbb{T}$-manifolds. Then we have a commutative diagram in equivariant cohomology*

Proof. For each $l \in \mathbb{N}$, define $E\mathbb{T}_l := (\mathbb{C}^{l+1}\backslash\{0\})^n$, $X_l := E\mathbb{T}_l \times_{\mathbb{T}} X$ and $f_l : X_l \to Y_l$. The diagram

$$\begin{array}{ccc} X'_l & \xrightarrow{f'_l} & Y'_l \\ \downarrow{\scriptstyle g'_l} & & \downarrow{\scriptstyle g_l} \\ X_l & \xrightarrow{f_l} & Y_l \end{array}$$

is a Cartesian diagram in the category of smooth manifolds with f_l and f'_l proper and smooth.

Now $g_l^* \circ f_{l!} = f'_{l!} \circ g'^*_l$ follows from the properties of usual Gysin maps. Hence $g^* \circ f_! =$

$$
\begin{tikzcd}
H^{\cdot}(X,\mathbb{Q}) \arrow[rr, "\int_X"] & & \mathbb{Q} \\
\hat{H}^{\cdot}_{\mathbb{T}}(X,\mathbb{Q}) \arrow[u, "i_X^*"] \arrow[r, "\int_{X^{eq}}"] & \hat{H}^{\cdot}_{\mathbb{T}}(\{\mathrm{pt}\},\mathbb{Q}). \arrow[ur, phantom] & \arrow[u, "i_{\mathrm{pt}}^*"']
\end{tikzcd}
$$

Figure A.3: Passing from Equivariant Integral to Ordinary Integral

$\varinjlim_{l\in\mathbb{N}} g_l^* \circ f_{l!} = \varinjlim_{l\in\mathbb{N}} f'_{l!} \circ g'^{*}_{l} = f'_! \circ g'^{*}$. □

Theorem A.3. *In the above situation, we have a commutative diagram*

Proof. Cf. Theorem 6.2 of [11]. □

Bibliography

[1] Michael F. Atiyah. K-Theory and Reality. *The Quarterly Journal of Mathematics*, 17(1):367–386, 1966.

[2] Victor V. Batyrev. Dual Polyhedra and Mirror Symmetry for Calabi–Yau Hypersurfaces in Toric Varieties. *Journal of Algebraic Geometry*, 3:493–535, 1994.

[3] Per Berglund and Tristan Hübsch. A generalized construction of Calabi–Yau models and mirror symmetry. *SciPost Phys.*, 4:009, 2018. pp. 1–30.

[4] Indranil Biswas, Arijit Dey, and Mainak Poddar. A classification of equivariant principal bundles over nonsingular toric varieties. *arXiv:1510.04014*, 2015.

[5] Michel Brion and Michèle Vergne. An equivariant Riemann–Roch theorem for complete, simplicial toric varieties. *Journal für die Reine und Angewandte Mathematik*, 482:67–92, 1997.

[6] David A. Cox, John B. Little, and Hal Schenck. *Toric Varieties*, volume 124 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2011.

[7] Roger Godement. *Topologie algébrique et théorie des faisceaux*, volume 1252 of *Actualités Scientifiques et Industrielles*. Hermann, Paris, 1958.

[8] Akio Hattori and Mikiya Masuda. Theory of multi-fans. *arXiv:math/0106229*, 2003.

[9] Hiroaki Ishida. Invariant stably complex structures on topological toric manifolds. *arXiv:1102.4673*, 2011.

[10] Hiroaki Ishida, Yukiko Fukukawa, and Mikiya Masuda. Topological toric manifolds. *arXiv:1012.1786*, 2012.

[11] Katsuo Kawakubo. *The Theory of Transformation Groups*. Oxford University Press, New York, 1991. Originally published in Japanese as *Henkan Gunron*, Iwanami Shoten, Tokyo, 1987.

[12] Alexander A. Klyachko. Equivariant bundles on toral varieties. *Math. USSR Izv.*, 35:337–375, 1990.

[13] Mikiya Masuda and Taras Panov. On the cohomology of torus manifolds. *Osaka Journal of Mathematics*, 43:711–746, 2006.

[14] David Mumford, John Fogarty, and Frances Kirwan. *Geometric Invariant Theory*. Springer-Verlag, Berlin, 3 edition, 1994.

[15] Tadao Oda. *Convex Bodies and Algebraic Geometry: An Introduction to the Theory of Toric Varieties*, volume 15 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3)*. Springer-Verlag, Berlin, 1988.

[16] Sam Payne. Moduli of toric vector bundles. *arXiv:0705.0410*, 2007.

[17] Markus Perling. *Resolutions and Moduli for Equivariant Sheaves over Toric Varieties*. PhD thesis, University of Kaiserslautern, 2003.

[18] Graeme Segal. Equivariant K-theory. *Publications Mathématiques de l'IHÉS*, 34:129–151, 1968.

[19] Tonny A. Springer. *Linear Algebraic Groups*, volume 9 of *Progress in Mathematics*. Birkhäuser, 2 edition, 2009.