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Optimal Sensor Scheduling In Nonlinear Filtering of Diffusion Processes

by

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Abstract

We consider the nonlinear filtering problem of a vector diffusion process, when several noisy vector observations with possibly different dimension of their range space are available. At each time any number of these observations (or sensors) can be utilized in the signal processing performed by the nonlinear filter. The problem considered is the optimal selection of a schedule of these sensors from the available set, so as to optimally estimate a function of the state at the final time. Optimality is measured by a combined performance measure that allocates penalties for errors in estimation, for switching between sensor schedules and for running a sensor. The solution is obtained in the form of a system of quasi-variational inequalities in the space of solutions of certain Zakai equations.

1. Introduction

1.1. Motivation and preliminaries

The problem of nonlinear filtering of diffusion processes has received considerable attention in recent years; see the anthologies [1], [2], [3] for a review of important developments. In current studies as well as in related analyses of the partially observed stochastic control problem with such models [4], [5], a key role is played by the linear stochastic partial differential equation describing the evolution of the unnormalized conditional probability measure of the state process given the past of the observations, the so called Zakai equation.

A significant byproduct of these advances is the feasibility of analyzing complex signal processing problems, including adaptive and sensitivity studies, in an integrated, systematic manner, without heuristic or *adhoc* assumptions. A problem of interest in this area is the so called *sensor scheduling problem*. Roughly speaking this problem is concerned with the simultaneous selection (according to some performance measure) of a signal processing scheme *together* with the sensors that collect the data to be processed. Particular applications include multiple sensor platforms, distributed sensor networks, large scale systems. For example, in a multiple sensor platform, there is definite need for coordinating the data obtained from the various sensors which may include radar, infrared, sonar, etc. The data obtained from different sensors are of varying quality and a systematic way is needed for allocating confidence or basing decisions on data collected from different types of sensors. For example radar sensors are more accurate than infrared sensors for long range tracking while the opposite is true for short range tracking. In sensor networks one needs to coordinate data collected from a large number of sensors distributed over a large geographical area. Conflicts should be resolved and a preferred set of sensors must be selected, over finite (short) time intervals, and utilized in detection, estimation or control decisions. Similarly in large scale systems there is typically an attached information network with the objective of collecting data, processing them and making the results available to the many control agents for their decisions (actions). Again the need for coordinating this information in a systematic way is critical.

In such sensor scheduling problems the systematic utilization of sensors should be the result of optimizing reasonably defined performance measures. Clearly these performance measures shall include terms allocating penalties for errors in detection and/or estimation. But more importantly, they must include terms for costs associated with turning sensors on or off, and for switching from one sensor to another. Examples of such costs arising in practice abound. Turning on a radar sensor increases the detectability of the platform (since radars are active sensors) and this should be reflected as a switching cost. Deciding to use a more accurate, albeit more complex sensor, will require higher bandwidth communications and often more computational power allocated to that sensor. In distributed sensor networks it may mean the physical movement of a sensor carrying platform (such as a helicopter or airplane) to a particular geographical location. In large scale systems the utilization of several (often hundreds) sensors for decision making may provide better average performance but it certainly reduces the response speed of the system to changing conditions, and it increases computational and communication costs both in terms of hardware and software. The latter are obviously evident in large computer/communication networks. These running and swithching costs will depend often on the part of the state space occupied by the state vector, i. e. they will be functions of the state as well. For example sensors have different accuracy or noise characteristics when the state process takes values in different areas of the state space. Also there is cost associated with handling the transfer of information, or tracking record, when there are changes in the set of sensors used; and these costs often depend on the state process.

It is not our intent to provide an extensive description of applications here. Detailed descriptions of some of these problems can be found elsewhere; see for example [6], [7]. The underlying thread in all these problem areas is the existence of a variety of sensors, which provide data (for processing) including information of widely varying quality about parameters or variables of interest, for control, detection, estimation etc. Due to the complexity of these problems it is important to develop systematic conceptual, analytical and numerical methods for their study and to reduce reliance on ad hoc, heuristic methods as much as possible. The present paper is offered as a contribution in this direction. It provides a general methodology to this problem by reducing it to the analysis of a system of quasi-variational inequalities (see section 3 for details). Numerical methods will be described elsewhere [13].

The sensor scheduling problem is considered here in the context of non-linear filtering of diffusion processes, and is therefore applicable to detection problems with the same signal models. Modifications of the results apply to other situations including control. In the next section we present a somewhat heuristic definition of the problem, intended to describe the problem clearly, at an intuitive level. The intricasies of establishing this model in a rigorous mathematical fashion are given in section 2, and constitute one of the main contributions of the paper.

1.2. Preliminary description of the problem

The problem considered is as follows. A signal (or state) process $x(\cdot)$ is given, modelled by the diffusion

$$\begin{aligned} dx(t) &= f(x(t))dt + g(x(t))dw(t) \\ x(0) &= \xi \end{aligned} \tag{1.1}$$

in $I\!R^n$. We further consider M noisy observations of $x(\cdot)$, described by

$$\begin{aligned} dy^i(t) &= h^i(x(t))dt + R_i^{1/2}dv^i(t), \\ y^i(0) &= 0 \end{aligned} \tag{1.2}$$

with values in $I\!R^{d_i}$. Here $w(\cdot)$, $v^i(\cdot)$ are independent, standard, Wiener processes in $I\!R^n$, $I\!R^{d_i}$ respectively, and $R_i = R_i^T > 0$ are $d_i \times d_i$ matrices. Further mathematical details on the system (1.1), (1.2) will be given in section 2. Let us consider a finite time horizon $[O,T]$. To formulate the problem of determining an *optimal utilization schedule* for the available sensors, so as to *simultaneously minimize* the cost of errors in estimating a function of $x(\cdot)$ and the costs of using as well as of switching between various sensors, we need to specify these costs. To this end, let $c_i(x)$ denote the cost per unit time when using sensor i, and the state of the system is x; $k_{io}(x)$, $k_{oi}(x)$ denote the cost for turning off, respectively on, the ith sensor when the state of the system is x. The objective of the performed signal processing is to compute, at time T, an estimate $\hat{\phi}(T)$ of a given function $\phi(x(T))$ of the state. Penalties for errors in estimation are assesed according to the cost function

$$E\{c_e(\phi(x(T)) - \hat{\phi}(T))\} := E\{|\phi(x(T)) - \hat{\phi}(T)|^2\} \tag{1.3}$$

We shall comment briefly on more general estimation problems in section 4 of this paper. In particular the consideration of a quadratric $c_e(\cdot)$ is not a serious restriction.

We consider next, the set of all possible *sensor activation configurations*, denoted here by $\mathcal{N}$. An element $\nu \in \mathcal{N}$ is a *word* of length M from the alphabet $\{0,1\}$. If the ℓ^{th} position is occupied by an 1, the ℓth sensor is activated (used), if by a 0 the ℓ^{th} sensor is off. There are $N = 2^M$ elements in $\mathcal{N}$. A *schedule of sensors is then a piecewise constant function* $u(\cdot) : [O,T] \to \mathcal{N}$. We let $\tau_j \in [O,T]$ denote the instants of changing schedule; i. e. , the moments when at least one sensor is turned on or off. At such a switching moment, suppose the schedule before is characterized by $\nu \in \mathcal{N}$, and after by $\nu' \in \mathcal{N}$. Then the *switching cost associated with such a scheduling change* will be

$$\mathbf{k}_{\nu\nu'}(x) := \sum_{\{i\in\nu\}\{i\notin\nu'\}} k_{io}(x) + \sum_{\{j\notin\nu\}\{j\in\nu'\}} k_{oj}(x). \tag{1.4}$$

The *total running cost, associated* with schedule $\nu \in \mathcal{N}$ will be

$$\mathbf{c}_\nu(x) := \sum_{\{j\in\nu\}} c_j(x) \tag{1.5}$$

In (1.4), (1.5), *the symbol* $\{i \in \nu\}$ *denotes the set of all indices (from the set* $\{1,2,\ldots,M\}$ *which are occupied by an 1 in* ν (i. e. the indices corresponding to the sensors which are on); similarly the symbol $\{i \notin \nu\}$ denotes the set of indices corresponding to sensors that are off.

Using the above notation the available observations, *under sensor schedule* $u(\cdot)$ are described by

$$dy(t,u(t)) := h(x(t),u(t))dt + r(u(t))dv(t), \tag{1.6}$$

where it is apparent that the available observations depend explicitly on the sensor schedule $u(\cdot)$. In (1.6), for $x \in I\!R^n$, $\nu \in \mathcal{N}$,

$$h(x,\nu) := \begin{bmatrix} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix}, \tag{1.7}$$

a block column vector, where in standard notation

$$\chi_{\{\nu\}}(i) := \left\{ \begin{array}{ll} 1, & \text{if the } i^{th} \text{ position in the word } \nu \text{ is occupied by an 1} \\ 0, & \text{otherwise} \end{array} \right\} \tag{1.8}$$

Similarly for $\nu \in \mathcal{N}$

$$r(\nu) := \text{Block diagonal}\{R_i^{1/2}\chi_{\{\nu\}}(i)\}, \tag{1.9}$$

where R_i are the symmetric, positive matrices defined above. Finally

$$v(t) := \begin{bmatrix} v^1(t) \\ \vdots \\ v^M(t) \end{bmatrix} \tag{1.10}$$

is a higher dimensional standard Wiener process. In view of (1.7), for all $\nu \in \mathcal{N}$

$$h(\cdot,\nu) : \mathbb{R}^n \to \mathbb{R}^D, \tag{1.11}$$

while

$$r(\nu) : \mathbb{R}^D \to \mathbb{R}^D, \tag{1.12}$$

where

$$D = d_1 + d_2 + \cdots + d_M. \tag{1.13}$$

To make the notation clearer, consider the case M = 2, N = 4. Then $\mathcal{N} = \{00, 01, 10, 11\}$ and

$$\begin{aligned} h(x,00) &= \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ h(x,01) &= \begin{bmatrix} 0 \\ h^2(x) \end{bmatrix} \\ h(x,10) &= \begin{bmatrix} h^1(x) \\ 0 \end{bmatrix} \\ h(x,11) &= \begin{bmatrix} h^1(x) \\ h^2(x) \end{bmatrix}, \end{aligned} \tag{1.14}$$

while

$$\begin{aligned} r(00) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\ r(10) &= \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \\ r(01) &= \begin{bmatrix} 0 & 0 \\ 0 & R_2^{1/2} \end{bmatrix} \\ r(11) &= \begin{bmatrix} R_1^{1/2} & 0 \\ 0 & R_2^{1/2} \end{bmatrix} \end{aligned} \tag{1.15}$$

Clearly the dimension of the range space of $y(\cdot,\nu)$ is

$$D_\nu := \sum_{i=1}^{M} d_i \,\chi_{\{\nu\}}(i). \tag{1.16}$$

Of course for all ν, $y(t,\nu) \in I\!R^D$.

Following established terminology (c.f. [9]) we see that a sensor scheduling strategy is defined by an increasing sequence of switching times $\tau_j \in [O,T]$ and the corresponding sequence $\nu_j \in \mathcal{N}$ of sensor activation configurations. We shall denote such a strategy by $u(\cdot)$, where

$$u(t) = \nu_j, \ \ t \in [\tau_j, \tau_{j+1}); \quad j = 1, 2, \ldots \tag{1.17}$$

As stated earlier we are interested in the *simultaneous* minimization of costs due to estimation errors as well as sensor scheduling. We shall therefore consider *joint estimation and sensor scheduling strategies*. Such a strategy consists of two parts: the sensor scheduling strategy u (see (1.17)) and the estimator $\hat{\phi}$. The set of admissible strategies U_{ad} is the customary set of strategies adapted to the sequence of σ-algebras

$$\mathcal{F}_t^{y(\cdot,u(\cdot))} := \sigma\{y(s,u(\cdot)), \ s \leq t\}. \tag{1.18}$$

That is, we consider *strict sense* admissible controls in the sense of [4]. For the problem under investigation this last statement must be interpreted very carefully. First, we have indicated in (1.18), that the available past observation data information σ-algebra depends (as is evident from (1.6) - (1.9)) very strongly on the sensor schedule $u(\cdot)$. This dependence is non-standard, as here the dimension of the observation vector and the noise covariance change drastically at each switching time τ_i. In standard stochastic control formulations [4], [5], the dependence of y on $u(\cdot)$ is much more implicit. This is a difficult part of the formulation here, since it prevents us from using Girsanov transformations in a straightforward manner. Secondly (1.18) means that the switching times τ_i and the variables ν_i, which define $u(\cdot)$, must be adapted to the filtration $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, *which depends essentially* on the values of τ_i and ν_i! Finally (1.18) also means that $\hat{\phi}(T)$ must be measurable with respect to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$. We shall describe a rigorous mathematical construction of such a model in section 2.

Given such a strategy the corresponding cost is

$$\begin{aligned} J(u(\cdot),\hat{\phi}) \ &:= \ E\{|\phi(x(T)) - \hat{\phi}(T)|^2 && (1.19)\\ &+ \int_0^T c(x(t),u(t))dt && (1.20)\\ &+ \sum_j k(x(t),u(\tau_{j-1}),u(\tau_j))\}. && (1.21) \end{aligned}$$

Here for $x \in I\!R^n$, $\nu, \nu' \in \mathcal{N}$

$$c(x,\nu) := \mathbf{c}_\nu(x), \tag{1.22}$$

(c.f. Eq. (1.5)), and

$$k(x,\nu,\nu') = \mathbf{k}_{\nu,\nu'}(x), \tag{1.23}$$

(c.f. Eq. (1.4)).

The optimal sensor scheduling in nonlinear filtering is thus formulated as the determination of a strategy achieving

$$\inf_{u(\cdot),\hat{\phi}} J(u(\cdot),\hat{\phi}) \tag{1.24}$$

among all admissible strategies.

To simplify the notation a little, let us order the elements of $\mathcal{N}$ according to the numbers they represent in binary form. For example in the case M = 2, N = 4 we replace $\mathcal{N} = \{00, 01, 10, 11\}$ by the set of integers {1, 2, 3, 4}. That is the one-one correspondence between $\mathcal{N}$ and $\{1, 2, \ldots, N\}$ is described by

$$\begin{aligned} \nu &\longmapsto \text{(integer represented by } \nu) + 1 \\ k &\longmapsto \text{binary representation of } (k-1). \end{aligned} \tag{1.25}$$

So in the sequel of the paper we replace all the ν, ν' in equations (1.4) - (1.23) by the corresponding integers from {1, 2, ..., N }.

The structure of the paper is as follows. In section 2 a precise mathematical formulation is given and the corresponding stochastic control problem is precisely defined. In section 3 the set of quasi-variational inequalities solving the problem is derived. In section 4 we offer some comments and discussion for extensions, further developments and computational methods.

2. The Stochastic Control Formulation

2.1. Setting of the model

Let $(\Omega, \mathcal{A}, P)$ be a complete probability space, on which a filtration $\mathcal{F}_t$ is given, $\mathcal{A} = \mathcal{F}_\infty$. Let $w(\cdot)$ and $z(\cdot)$ be two independent, standard $\mathcal{F}_t$-Wiener processes with values in $I\!R^n$ and $I\!R^D$ respectively, carried by this probability space. On the same space we consider also an $I\!R^n$-valued random variable ξ, independent of $w(\cdot), z(\cdot)$, and with probability distribution function π_0.

We consider the Itô equation (1.1), where $f(\cdot)$ is $I\!R^n$-valued, bounded and Lipschitz, while $g(\cdot)$ is $I\!R^{n\times n}$-valued, bounded and Lipschitz. Letting $a = \frac{1}{2}gg^T$, we assume $a > \alpha I_n$, where $\alpha > 0$ and I_n is the $n \times n$ identity matrix. The Lipschitz property is unnecessary and can be easily removed using Girsanov's transformation (i.e. consider weak solutions of (1.1)) [8]. It is assumed here to simplify the technicalities not related with the main issues of the paper. Under these assumptions (1.1) has a strong solution with well known properties [8]. Note that *under P, $z(\cdot)$ is independent of $x(\cdot)$*.

Consider next functions $h^i(\cdot)$, $i = 1, \ldots, M$, from $I\!R^n$ into $I\!R^{d_i}$, which are bounded and Hölder continuous. We shall denote by L the infinitesimal generator of the Markov process $x(\cdot)$

$$L := \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} f_i(x)\frac{\partial}{\partial x_i} \tag{2.1}$$

or in divergence form

$$L := \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x)\frac{\partial}{\partial x_j} - \sum_{i=1}^{n} a_i(x)\frac{\partial}{\partial x_i} \tag{2.1a}$$

where

$$a_i(x) := -f_i(x) + \sum_{j=1}^{n} \frac{\partial a_{ij}(x)}{\partial x_j} \tag{2.1b}$$

Let us next consider an *impulsive control* defined as follows. There is a sequence $\tau_1 < \tau_2 \ldots < \tau_k < \ldots$ of increasing $\mathcal{F}_t$-stopping times. To each time τ_i we attach an $\mathcal{F}_{\tau_i}$-measurable random variable u_i with values in the set of integers { 1, 2, ..., N } [1]. We define

$$u(t) = u_i, \quad \tau_i \leq t < \tau_{i+1}, \quad i = 0, 1, 2, \ldots \tag{2.2}$$

and set $\tau_0 = 0$. We require that

$$\tau_i \uparrow T \text{ as } i \uparrow \infty, \tag{2.3}$$

while $\tau_k = T$ is possible for some finite k.

Let ν_i be the element of $\mathcal{N}$, corresponding to u_i via (1.25).

Then define

$$h(x, u(t)) := h(x, \nu_i), \qquad \tau_i \leq t < \tau_{i+1}, \tag{2.4}$$

where $h(x, \nu)$ is defined by (1.7), in terms of the given functions $h^i(\cdot)$. Clearly $h(\cdot, u(t))$ maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. Define also

$$r(u(t)) := r(\nu_i), \qquad \tau_i \leq t < \tau_{i+1}, \tag{2.5}$$

where $r(\cdot)$ is defined by (1.9), in terms of the given matrices R_i, $i = 1, 2, \ldots, M$. Clearly $r(u(t))$ maps $\mathbb{R}^D$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ but it is *singular*. Next we define $\tilde{h}(x, \nu)$ to be the vector valued function

$$\tilde{h}(x, \nu) := \begin{bmatrix} R_1^{-1/2} h^1(x) \chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1/2} h^i(x) \chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1/2} h^M(x) \chi_{\{\nu\}}(M) \end{bmatrix} \tag{2.6}$$

with $\chi_{\{\nu\}}(i)$ defined as in (1.8). Let

$$\tilde{h}(x, u(t)) \;\; := \;\; \tilde{h}(x, \nu_i), \qquad \tau_i \leq t < \tau_{i+1}. \tag{2.7}$$

Clearly $\tilde{h}(\cdot, u(t))$ maps $\mathbb{R}^n$ into $\mathbb{R}^D$ for all sensor schedules $u(\cdot)$ and is obviously bounded and Hölder continuous in x. We shall refer to $u(\cdot)$ as the *impulsive control*. As we shall see, it describes essentially the decision to select at a sequence of decision times one of the functions $h(\cdot, k)$, $k \in \{1, 2, \ldots, N\}$. This is the precise mathematical implementation of the sensor selection decision described in the introduction.

To see that indeed this is the case, we can, with the above preparation, use Girsanov's measure transformation method. Let us then consider the process

$$\varsigma(t) = \exp\{\int_0^t \tilde{h}(x(s), u(s))^T dz(s) - \frac{1}{2}\int_0^t \|\tilde{h}(x(s), u(s))\|^2 ds\} \tag{2.8}$$

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[1] Recall that $N = 2^M$ and the binary representation of each integer 1, 2, ..., N determines a sensor activation configuration by (1.25).

where ^{T} denotes transpose, $\|\cdot\|$ is the $I\!R^D$ norm. Note that the process $u(t)$ is adapted to $\mathcal{F}_t$. Then since $x(\cdot)$ is adapted to $\mathcal{F}_t^w \subset \mathcal{F}_t$ and $u(\cdot)$ is *cadlag* [8], (2.8) is well defined. Moreover since $\tilde{h}$ is bounded, by Girsanov's theorem [8], [14], $\varsigma(\cdot)$ is an $\mathcal{F}_t$-martingale. We can thus define a change of probability measure

$$\frac{dP^{u(\cdot)}}{dP}\bigg|_{\mathcal{F}_t} = \varsigma(t) \tag{2.9}$$

and consider the process

$$v(t) = z(t) - \int_0^t \tilde{h}(x(s), u(s))ds. \tag{2.10}$$

By Girsanov's theorem [8], [14], under the probability measure $P^{u(\cdot)}$ on $(\Omega, \mathcal{A})$, $v(\cdot)$ is a standard $\mathcal{F}_t$-Wiener process with values in $I\!R^D$. Furthermore, by the independence of $w(\cdot)$ and $z(\cdot)$, $w(\cdot)$ remains a standard $I\!R^n$-valued, $\mathcal{F}_t$-Wiener process which is independent of $v(\cdot)$. Finally ξ remains independent of $w(\cdot)$, $v(\cdot)$ while keeping its probability law, denoted by π_0. Thus $x(\cdot)$ also retains its probability law under $P^{u(\cdot)}$.

To relate this construction, i.e. (2.2) - (2.10) with the M noisy observations (sensors) loosely described in the introduction (c.f. in particular eq. (1.6)), observe that (2.10) can be written as

$$r(u(t))dz(t) = h(x(t), u(t))dt + r(u(t))dv(t) \tag{2.11}$$

in view of (1.7), (1.9), (2.4), (2.5), (2.6) and (2.7). Indeed

$$\begin{aligned} r(u(t))\tilde{h}(x,(u(t))) &= \begin{bmatrix} R_1^{1/2}\chi_{\{\nu_i\}}(1) & 0 & 0 \\ 0 & R_2^{1/2}\chi_{\{\nu_i\}}(2) & 0 \\ & \ddots & \\ 0 & 0 & R_M^{1/2}\chi_{\{\nu_i\}}(M) \end{bmatrix} \begin{bmatrix} R_1^{-1/2}h^1(x)\chi_{\{\nu_i\}}(1) \\ R_2^{-1/2}h^2(x)\chi_{\{\nu_i\}}(2) \\ \vdots \\ R_M^{-1/2}h^M(x)\chi_{\{\nu_i\}}(M) \end{bmatrix} \\ &= h(x,\nu_i), \qquad \tau_i \le t < \tau_{i+1}. \end{aligned} \tag{2.12}$$

To give a precise meaning to (1.2), or (1.6), let us introduce the process

$$y(t, u(t)) := y^{\nu_i}(t), \qquad \tau_i \le t < \tau_{i+1} \tag{2.13}$$

where

$$dy^{\nu_i}(t) := r(\nu_i)\,dz(t) = h(x(t),\nu_i)\,dt + r(\nu_i)\,dv(t). \tag{2.14}$$

It is clear that if we select $u(t) = \nu$, $\forall t$, where ν has 0 everywhere except for one 1 in the i^{th} location, then (1.2) results. It is also rather plain that $y^\nu(t) \in I\!R^{D_\nu}$ and that in this case the Wiener process $r(\nu)v(\cdot)$ is also D_ν-dimensional (see (1.16) for the definition of D_ν). The process $y^{\nu_i}(t)$ represents exactly the observation which is available in $[\tau_i, \tau_{i+1})$.

The next issue that we wish to clarify relates to the measurability question that we discussed in section 1.2, after eq. (1.18). For any $u(\cdot)$, given the construction of $y(\cdot, u(\cdot))$, above we can now consider $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ as defined by (1.18). We shall say that $u(\cdot)$ is *admissible*, denoted $u \in U_{ad}$, if $u(t)$ is $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ measurable, $t > 0$, where $\mathcal{F}_t^{y(\cdot,u(\cdot))}$ is constructed as above. This more precisely means that the τ_i are $\mathcal{F}_t^{y(\cdot,u(\cdot))}$-stopping times or that

$$\{\tau_i < t\} \subset \mathcal{F}_t^{y(\cdot,u(\cdot))} \tag{2.15}$$

and that

$$\nu_i \in \mathcal{F}_{\tau_i}^{y(\cdot,u(\cdot))}. \tag{2.16}$$

Note that since $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t$ for any sensor schedule $u(\cdot)$ adapted to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, if τ_i are $\mathcal{F}_t^{y(\cdot,u(\cdot))}$-stopping times they are also $\mathcal{F}_t$-stopping times, and the above construction (2.8) - (2.14) is still valid. The implication of (2.15), (2.16) is that one should check *that an optimizing strategy, obtained by some procedure, must satisfy the admissibility conditions.* Clearly U_{ad} is nonempty as strategies $u(t) = \nu, \quad t \in [O, T]$, obviously are admissible. Also strategies with fixed switchings are also admissible. Note that for an admissible control $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t^z$.

We have thus established in this section the precise mathematical models of nonlinear filtering problems where selection of sensors is possible. In particular we have succeeded in circumventing the subtleties associated with the definition of admissible sensor schedules discussed in section 1.2.[2]

2.2. The optimization problem

For the dynamical system described in 2.1, we consider now the cost functional (1.19) where the underlying probability measure is $P^{u(\cdot)}$. As indicated in the introduction, the general problem where the function ϕ will be in a nice class, e.g., bounded C^2, or polynomial, or C^∞ can be treated along identical lines. To simplify the notation we have chosen to formulate the problem for $\phi(x) = x$. The technical difficulties for this case are identical to the ones in the more general cases discussed above, particularly since this $\phi(\cdot)$ is unbounded on $\mathbb{R}^n$. For this choice the selection of the optimal estimator $\hat{\phi}(T)$ is the conditional mean

$$\hat{\phi}(T) = E^{u(\cdot)}\{x(T) \mid \mathcal{F}_T^{y(\cdot,u(\cdot))}\}, \tag{2.17}$$

where $E^{u(\cdot)}$ denotes expectation with respect to $P^{u(\cdot)}$. Let $\mu(u,t)$ denote the conditional probability measure of $x(t)$, given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, on $\mathbb{R}^n$. It is convenient to express (2.17) as a vector valued functional of $\mu(u,t)$

$$\hat{\phi}(T) = \Phi(\mu(u,T)) = \int_{\mathbb{R}^n} x d\mu(u,T). \tag{2.18}$$

We shall further assume that the running and switching cost functions $\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot)$, $i, j \in \{1, \dots, N\}$, introduced in (1.4) and (1.5) have the following regularity

$$\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot) \text{ are in } C_b(\mathbb{R}^n) \text{ (i. e. bounded and continuous)} \tag{2.19}$$

[2]Since $r(u(t))$ is a singular matrix, this stage is more delicate than in standard stochastic control theory, where $\mathcal{F}_t^z$ would suffice.

As a result of this simple transformation we can rewrite the cost as a function of the impulsive control $u(\cdot)$ only (i.e. the selection of $\hat{\phi}(\cdot)$ has been eliminated):

$$\begin{aligned} J(u(\cdot)) &= E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2+\int_0^T c(x(t),u(t))dt \\ &\quad +\sum_{j=1}^{\infty} k(x(\tau_j),u(\tau_{j-1}),u(\tau_j))\chi_{\tau_j<T}\}, \end{aligned} \tag{2.20}$$

where $\chi_{\tau_i<T}$ is the characteristic function of the Ω-set $\{\omega;\ \tau_i(\omega)<T\}$. We further assume that the switching costs are uniformly bounded below

$$k(x,i,j)\geq k_o,\ \ x\in I\!R^n,\ i,\ j\in\{1,\ldots,N\} \tag{2.21}$$

with k_0 a positive constant. Note that as a consequence of (2.20) if for some admissible $u(\cdot)$ with positive probability, the number of times $\tau_i<T$ is infinite, then the cost $J(u(\cdot))$ will be infinite. Therefore for T finite the optimal policy will exhibit a finite number of sensor switchings.

The optimal sensor selection problem can now be stated precisely as the optimization problem

P : Find an admissible impulsive control $u^*(\cdot)$ such that

$$J(u^*(\cdot))=\inf_{u(\cdot)\in U_{ad}} J(u(\cdot)), \tag{2.22}$$

where U_{ad} are all impulsive control strategies adapted to $\mathcal{F}^{y(\cdot,u(\cdot))}$, or equivalently satisfying (2.15), (2.16). Problem P is a *non-standard* stochastic control problem of a partially observed diffusion.

2.3. The equivalent fully observed problem

In this section we transform the problem of section 2.2, to a fully observed stochastic control problem, by introducing appropriate Zakai equations. As is customary in the theory of nonlinear filtering [1], [2], [3], [4], let us introduce the operator

$$p(u(\cdot),t)(\psi)=E\{\varsigma(t)\psi(x(t))\mid \mathcal{F}_t^{y(\cdot,u(\cdot))}\} \tag{2.23}$$

for each impulsive control $u(\cdot)$. The notation is chosen so as to emphasize the dependence on $u(\cdot)$, which is due to the dependence of $\varsigma(\cdot)$ on $u(\cdot)$ as introduced in eq. (2.8).[3] The operator (2.23) maps the set of Borel bounded functions on $I\!R^n$, into the set of real valued stochastic processes adapted to $\mathcal{F}_t^{y(\cdot,u(\cdot))}$. Note that $p(u(\cdot),t)$ can be viewed as a positive finite measure on $I\!R^n$. It is the *unnormalized conditional probability measure* of $x(t)$ given $\mathcal{F}_t^{y(\cdot,u(\cdot))}$, [1], [2].

[3]But the expectation is with respect to P and not $P^{u(\cdot)}$.

With the help of these measures we can rewrite the various cost terms in (2.20) as follows:

$$\begin{aligned} E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2\} &= E\{\varsigma(T)\|x(T)-\Phi(\mu(u,T))\|^2\} \\ &= E\{p(u(\cdot),T)(\theta)\}, \end{aligned} \tag{2.24}$$

where

$$\theta(x) := \|x - \frac{p(u(\cdot),T)(\chi)}{p(u(\cdot),T)(\mathbb{1})}\|^2, \tag{2.25}$$

with χ representing the function $\chi(x) := x$ and $\mathbb{1}$ the function $\mathbb{1}(x) := 1,\ x \in I\!R^n$. A straightforward computation implies that

$$E^{u(\cdot)}\{\|x(T)-\Phi(\mu(u,T))\|^2\} = E\{\Psi(p(u(\cdot),T))\} \tag{2.26}$$

where Ψ is the functional on finite measures on $I\!R^n$ defined by

$$\Psi(\mu) = \mu(\chi^2) - \frac{\|\mu(\chi)\|^2}{\mu(\mathbb{1})} \tag{2.27}$$

where $\chi^2(x) = \|x\|^2,\ x \in I\!R^n$, and μ is any finite measure on $I\!R^n$ such that the quantities $\mu(\chi^2)$ and $\mu(\chi)$ make sense.

Next

$$\begin{aligned} E^{u(\cdot)}\{\int_0^T c(x(t),u(t))dt\} &= E\{\varsigma(T)\int_0^T c(x(t),u(t))dt\} \\ &= E\{\int_0^T E\{\varsigma(T)c(x(t),u(t))\,|\mathcal{F}_t\}dt\} \\ &= E\{\int_0^T E\{\varsigma(T)\,|\mathcal{F}_t\}c(x(t),u(t))dt\} \\ &= E\{\int_0^T \varsigma(t)c(x(t),u(t))dt\}, \end{aligned} \tag{2.28}$$

because $x(t), u(t)$ are measurable with respect to $\mathcal{F}_t$ and $\varsigma(\cdot)$ is an $\mathcal{F}_t$-martingale. Now define a map C with values in $C_b(I\!R^n)$ via

$$C(u_i) := \mathbf{c}_{u_i}(\cdot),\ u_i \in \{1,2,\ldots,N\}. \tag{2.29}$$

Then in view of (2.29), (2.23), we can rewrite (2.28) as

$$\begin{aligned} E^{u(\cdot)}\{\int_o^T c(x(t),u(t))dt\} &= E\{\int_o^T E\{\varsigma(t)c(x(t),u(t))|\mathcal{F}_t^{y(\cdot,u(\cdot)}\}dt\} \\ &= E\{\int_o^T p(u(\cdot),t)(C(u(t))dt\}. \end{aligned} \tag{2.30}$$

Finally

$$\begin{aligned} E^{u(\cdot)}\{k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} &= E\{\varsigma(\tau_i)k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} \\ &= E\{E\{\varsigma(\tau_i)k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\,|\mathcal{F}_{\tau_i}^{y(\cdot,u(\cdot)}\}\} \\ &= E\{p(u(\cdot),\tau_i)(K(u(\tau_{i-1}),u(\tau_i)))\chi_{\tau_i<T}\}. \end{aligned} \tag{2.31}$$

Here we have introduced the function K with values in $C_b(I\!R^n)$, via

$$K(u_i, u_j) = \mathbf{k}_{u_i,u_j}(\cdot), \quad u_i,\ u_j \in \{1,2,\ldots,N\}, \tag{2.32}$$

and we utilized the admissibility of $u(\cdot)$. Note that in the simpler case where $\mathbf{c}_i(\cdot), \mathbf{k}_{ij}(\cdot)$, $i,\ j \in \{1,2,\ldots,N\}$ are constant independent of x, (2.30) simplifies to

$$E^{u(\cdot)}\{\int_0^T c(x(t),u(t))dt\} = E\{\int_0^T p(u(\cdot),t)(\mathbb{1})\mathbf{c}_{u(t)}dt\} \tag{2.33}$$

and (2.31) simplifies to

$$E^{u(\cdot)}\{k(x(\tau_i),u(\tau_{i-1}),u(\tau_i))\chi_{\tau_i<T}\} = E\{\mathbf{k}_{u_{i-1},u_i}\chi_{\tau_i<T}p(u(\cdot),\tau_i)(\mathbb{1})\}. \tag{2.34}$$

Utilizing (2.26), (2.30), (2.31) we can rewrite the cost corresponding to policy $u(\cdot)$, given in (2.20), as follows

$$\begin{aligned} J(u(\cdot)) &= E\{\Psi(p(u(\cdot),T)) + \int_0^T p(u(\cdot),t)(C(u(t)))dt \\ &+ \sum_{i=1}^{\infty} p(u(\cdot),\tau_i)(K(u_{i-1},u_i))\chi_{\tau_i<T}\}. \end{aligned} \tag{2.35}$$

In (2.35) we have succeeded in displaying the cost as a functional of the unnormalized conditional measure $p(u(\cdot),\cdot)$ which is the "information" state of the equivalent fully observed stochastic control problem. To complete this transformation we need to derive the evolution equation for $p(u(\cdot),\cdot)$, i.e. the Zakai equation. We turn into this problem next and derive a weak form of the Zakai equation for $p(u(\cdot),\cdot)$ in the following lemma. Here $C_b^{2,1}$ denotes the space of all functions $\psi(x,t)$ on $I\!R^n \times I\!R$ which are bounded, continuous together with their first and second derivatives with respect to x, and first derivatives with respect to t.

Lemma 2.1: *For any $\psi \in C_b^{2,1}$ we have the relation*

$$\begin{aligned} p(u(\cdot),t)(\tilde{\psi}(t)) &= \pi_0(\tilde{\psi}(0)) + \int_0^t p(u(\cdot),s)(\frac{\partial\tilde{\psi}}{\partial s} + L\tilde{\psi})ds \\ &+ \int_0^t \sum_{i=1}^{D} p(u(\cdot),s)(\tilde{H}_i(u(s))\tilde{\psi}(s))dz_i(s) \end{aligned} \tag{2.36}$$

where

$$\begin{aligned} \left[\tilde{H}_i(u(s))\phi\right](x) &:= \tilde{h}_i(x,u(s))\phi(x), \quad i=1,2,\ldots,D, \quad \phi \in C_b^2 \\ \tilde{\psi}(s)(x) &:= \psi(x,s), \end{aligned} \tag{2.37}$$

and $\tilde{h}_i$ is the i^{th} component of $\tilde{h}$ (see (2.6)).

Proof:

Let $\beta(\cdot) \in L^\infty(0,T;I\!R^D)$ given and consider the $\mathcal{F}_t$-martingale $\rho(t)$, defined by

$$d\rho(t) = \rho(t)\beta(t)^T dz(t), \quad \rho(0) = 1. \tag{2.38}$$

Recall that by definition of $\varsigma(t)$ (c.f. eq. (2.8))

$$d\varsigma(t) = \varsigma(t)\tilde{h}(x(t), u(t))^T dz(t), \qquad \varsigma(0) = 1. \tag{2.39}$$

Therefore by Itô's rule [8]

$$\begin{aligned} d(\varsigma(t)\rho(t)) &= \varsigma(t)\rho(t)[(\tilde{h}(x(t), u(t)) + \beta(t))^T dz(t) \\ &+ \tilde{h}^T(x(t), u(t))\beta(t)dt] \\ \varsigma(0)\rho(0) &= 1, \end{aligned} \tag{2.40}$$

and since $\psi \in C_b^{2,1}$

$$\begin{aligned} d\psi(x(t), t) &= (\frac{\partial \psi(x(t), t)}{\partial t} + L\psi(x(t), t))dt \\ &+ [\nabla\psi(x(t), t)]^T g(x(t))dw(t), \end{aligned} \tag{2.41}$$

where L is given in (2.1). Therefore suppressing some arguments for ease of notation

$$\begin{aligned} d[\psi(x(t), t)\ \varsigma(t)\rho(t)] &= \varsigma(t)\rho(t)[(\frac{\partial\psi}{\partial t} + L\psi + \tilde{h}^T\beta\psi)dt \\ &+ \nabla\psi^T g dw(t) + \psi(\tilde{h} + \beta)^T dz(t)]. \end{aligned} \tag{2.42}$$

In (2.41), (2.42) we used the notation $\nabla\psi = (\frac{\partial\psi}{\partial x_1}, \ldots, \frac{\partial\psi}{\partial x_n})^T$. Integrating (2.42), and taking expectations we deduce

$$E\{\psi(x(t), t)\varsigma(t)\rho(t)\} = \pi_0(\tilde{\psi}(0)) + E\{\int_0^t \varsigma(s)\rho(s)[\frac{\partial\psi}{\partial s} + L\psi + \tilde{h}^T\beta\psi]ds\}. \tag{2.43}$$

We can then write

$$\begin{aligned} E\{\int_0^t \varsigma(s)\rho(s)[\frac{\partial\psi}{\partial s} + L\psi]ds\} &= E\{\int_0^t E\{\rho(s)\varsigma(s)(\frac{\partial\psi}{\partial s} + L\psi) \mid \mathcal{F}_s^{y(\cdot,u(\cdot))}\}ds\} \\ &= E\{\int_0^t \rho(s)p(u(\cdot), s)(\frac{\partial\psi}{\partial s} + L\psi)ds\} \\ &= E\{\rho(t)\int_0^t p(u(\cdot), s)(\frac{\partial\psi}{\partial s} + L\psi)ds\} \end{aligned} \tag{2.44}$$

by virtue of the $\mathcal{F}_t$-martingale property of $\rho(\cdot)$. Similarly

$$\begin{aligned} & E\{\int_0^t \varsigma(s)\rho(s)\tilde{h}(x(s), u(s))^T\beta(s)\psi(x(s), s)ds\} \\ = \ & E\{\rho(t)\int_0^t \varsigma(s)\psi(x(s), s)\tilde{h}(x(s), u(s))^T dz(s)\} \\ = \ & E\{\rho(t)\int_0^t \sum_{i=1}^D p(u(\cdot), s)(\tilde{h}_i(\cdot, u(s))\psi(\cdot, s))dz_i(s)\}, \end{aligned} \tag{2.45}$$

where in the first equality we have used the representation $\rho(t) = 1 + \int_0^t \rho(s)\beta(s)^T dz(s)$, and the well known insomorphism between Itô stochastic integrals and L^2 [8]. Finally

$$E\{\psi(x(t), t)\varsigma(t)\rho(t)\} = E\{\rho(t)p(u(\cdot), t)(\tilde{\psi}(t))\}. \tag{2.46}$$

Using (2.44), (2.45), (2.46) in (2.43) we obtain

$$E\{\rho(t)[p(u(\cdot),t)(\tilde{\psi}(t)) - \pi_0(\tilde{\psi}(0)) - \int_0^t p(u(\cdot),s)(\frac{\partial\psi}{\partial s} + L\psi)ds$$
$$- \int_0^t \sum_{i=1}^{D} p(u(\cdot),s)(\tilde{H}_i(u(s))\tilde{\psi}(s))dz_i(s)]\} = 0. \tag{2.47}$$

We can replace in (2.47) $\rho(t)$ by a linear combination of such variables, with different β. The set of corresponding variables is dense in $L^2(\Omega, \mathcal{F}_t^z, P)$. However, the random variable in the brackets in the right hand side of (2.47) is clearly in $L^2(\Omega, \mathcal{F}_t^{y(\cdot,u(\cdot))}, P)$ and therefore in $L^2(\Omega, \mathcal{F}_t^z, P)$ since $\mathcal{F}_t^{y(\cdot,u(\cdot))} \subset \mathcal{F}_t^z$. Then (2.47) implies the result of the lemma (2.36).

As a remark we would like to note that the assumed nondegeneracy of $x(\cdot)$, implies that the solution of (2.36) is unique. This can be proved in general under our working hypotheses, for solutions which are measure-valued processes. Here we outline such a proof for the case when these conditional measures are absolutely continuous with respect to Lebesgue measure on $\mathbb{R}^n$; i.e., in the case unnormalized conditional densities exist. For this we need to assume in addition that

$$\pi_0 \textit{ has a density } p_0 \textit{ with respect to Lebesgue measure; } p_0 \in L^2(\mathbb{R}^n) \tag{2.48}$$

Let us denote by L^* the formal adjoint of L (see (2.1), (2.1a), (2.1b)):

$$L^* = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} a_{ij}(x) \frac{\partial}{\partial x_j} + \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i, \tag{2.49}$$

and consider the Hilbert space form of the Zakai equation [10]

$$\begin{aligned} dp &= L^* p dt + p\tilde{h}(\cdot, u(t))^T dz(t) \\ p(0) &= p_0. \end{aligned} \tag{2.50}$$

The function space in which the solution is sought is

$$L^2(\Omega, \mathcal{A}, P; C(0,T; L^2(\mathbb{R}^n))) \cap L^2_{\mathcal{F}^{y,(\cdot,u(\cdot))}}(0,T; H^1(\mathbb{R}^n)) \tag{2.51}$$

Here H^1 is the usual Sobolev space on $\mathbb{R}^n$ [11] and the subindex $\mathcal{F}^{y(\cdot,u(\cdot))}$ in the second L^2 space, denotes that the solution is adapted to the filtration $\mathcal{F}_t^{y(\cdot),u(\cdot))}$, $t \geq 0$. It follows from the results of E. Pardoux [11], that there exists a unique solution of (2.49) in the function space (2.50), under the assumptions made here. We can then establish the following.

Lemma 2.2: The following property holds

$$p(u(\cdot),t)(\psi) = (p(u(\cdot),t),\psi), \tag{2.52}$$

$\forall\psi$ in $L^2(\mathbb{R}^n)$ and bounded, where $(\cdot,\cdot)$ denotes inner product in $L^2(\mathbb{R}^n)$.

Proof:

By slight abuse of notation we use the same symbol to denote the conditional unormalized measure *and* density (whenever the latter exists). Let us prove inductively that

$$p(u(\cdot),\tau_i \vee (t\wedge \tau_{i+1}))(\psi) = (p(u(\cdot),\tau_i \vee (t\wedge\tau_{i+1})),\psi), \tag{2.53}$$

where the left hand side notation refers to the measure appearing in (2.36), while the right hand side notation to the solution of (2.50), which is uniquely defined. Suppose then that (2.53) holds for i-1, and therefore in particular

$$p(u(\cdot),\tau_i)(\psi) = (p(u(\cdot),\tau_i),\psi), \forall\psi. \tag{2.54}$$

Consider now the solution η of

$$\begin{aligned} \frac{\partial\eta}{\partial s} + L\eta &= -\eta\tilde{h}(\cdot,u(s))^T\beta(s), \qquad s\in(\tau_i,\tau_i\vee(t\wedge\tau_{i+1})) \\ \eta(x,\tau_i\vee(t\wedge\tau_{i+1})) &= \psi(x) \end{aligned} \tag{2.55}$$

where $\psi\in C_0^\infty(I\!R^n)$ and β is a smooth deterministic function with values in $I\!R^D$. From the assumptions on f, g and h^i (it is here that we use the assumed Hõlder continuity of h^i), we can assert that the solution of (2.55) belongs to $C_b^{2,1}(I\!R^n\times(\tau_i,\tau_i\vee(t\wedge\tau_{i+1})))$, for any sample ω, [11]. Therefore (2.36) implies (using (2.55))

$$\begin{aligned} p(u(\cdot),\tau_i\vee(t\wedge\tau_{i+1}))(\psi) &= p(u(\cdot),\tau_i)(\tilde{\eta}(\tau_i)) \\ &- \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}\sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\tilde{\eta}(s))\beta_j(s)ds \\ &+ \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}\sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\tilde{\eta}(s))dz_j(s), \end{aligned} \tag{2.56}$$

where $\tilde{H}_j$ is as defined in lemma 2.1, and $\tilde{\eta}(s)(x) := \eta(x,s)$. Therefore by Itô's rule

$$\begin{aligned} p(u(\cdot),\tau_i\vee(t\wedge\tau_{i+1}))(\psi)\rho(\tau_i\vee(t\wedge\tau_{i+1})) &= p(u(\cdot),\tau_i)(\tilde{\eta}(\tau_i))\rho(\tau_i) \\ &+ \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}[\rho(s)\sum_{j=1}^D p(u(\cdot),s)(\tilde{H}_j(u(s))\tilde{\eta}(s)) \\ &+ \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}\rho(s)\sum_{j=1}^D p(u(\cdot),s)(\tilde{\eta}(s))\beta_j(s)]dz_j(s). \end{aligned} \tag{2.57}$$

Hence

$$\begin{aligned} &E\{p(u(\cdot),\tau_i\vee(t\wedge\tau_{i+1}))(\psi)\rho(\tau_i\vee(t\wedge\tau_{i+1}))\} \\ &= E\{p(u(\cdot),\tau_i)(\tilde{\eta}(\tau_i))\rho(\tau_i)\}. \end{aligned} \tag{2.58}$$

On the other hand from (2.50) and (2.55) we obtain

$$\begin{aligned} (p(u(\cdot),\tau_i\vee(t\wedge\tau_{i+1})),\psi) &= (p(u(\cdot),\tau_i),\tilde{\eta}(\tau_i)) \\ &+ \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}\sum_{j=1}^D(p(u(\cdot),s)\tilde{h}_j(\cdot,u(s)),\tilde{\eta}(s))dz_j(s) \\ &- \int_{\tau_i}^{\tau_i\vee(t\wedge\tau_{i+1})}\sum_{j=1}^D(p(u(\cdot),s),\tilde{H}_j(u(s))\tilde{\eta}(s)\beta_j(s)ds, \end{aligned} \tag{2.59}$$

and thus also

$$E\{(p(u(\cdot),\tau_i \vee (t \wedge \tau_{i+1})),\psi)\rho(\tau_i \vee (t \wedge \tau_{i+1}\} = E\{(p(u(\cdot),\tau_i),\tilde{\eta}(\tau_i))\rho(\tau_i\}. \tag{2.60}$$

But from the inductive hypothesis (2.54), the right hand sides of (2.58) and (2.60) are equal. Hence the left hand sides coincide. Varying β, we easily deduce that (2.53) holds, at least for $\psi \in C_0^\infty(I\!R^n)$, which is sufficient to conclude the proof of the lemma.

With this result we can rewrite the cost (2.35) as follows

$$\begin{aligned} J(u(\cdot)) &= E\{\Psi(p(u(\cdot),T)) + \int_0^T (p(u(\cdot),t),C(u(t)))dt \\ &+ \sum_{i=1}^{\infty} \chi_{\tau_i<T}(p(u(\cdot),\tau_i),K(u_{i-1},u_i))\} \end{aligned} \tag{2.61}$$

where (see (2.27))

$$\Psi(p(u(\cdot),T)) = (p(u(\cdot),T),\chi^2) - \frac{\|(p(u(\cdot),T),\chi)\|^2}{(p(u(\cdot),T),1\!\!1)}. \tag{2.62}$$

Since the expression (2.62) involves unbounded functions we have to show that it makes sense.

At this point it is useful to introduce a weighted Hilbert space in order to express $\Psi(p(u(\cdot),T))$ in a more convenient form. To this end let

$$\mu(x) = 1 + \| x \|^4 \tag{2.63}$$

and $L^2(I\!R^n;\mu)$ denotes the space of functions φ such that $\varphi\mu \in L^2(I\!R^n)$. Define in a similar way the space $L^1(I\!R^n;\mu)$. From the discussion of existence and uniqueness of solutions of (2.50) in the functional space (2.51) and if

$$p_0 \in L^2(I\!R^n;\mu) \cap L^1(I\!R^n;\mu),$$

it is easy to check that (2.50), under the assumptions made in section 2.1, has a unique solution in the space

$$L^2(\Omega,\mathcal{A},P;C(0,T;L^2(I\!R^n;\mu) \cap L^1(I\!R^n;\mu))) \cap L^2(0,T;H^1(I\!R^n;\mu)) \tag{2.64}$$

where $H^1(I\!R^n;\mu)$ is the obvious modification of $H^1(I\!R^n)$. This justifies that the quantities arising in (2.62) have a meaning.

We note that $J(u(\cdot))$ is indexed implicitly (we do not include this in our notation) by π_0 (or p_0) and $u(0) = j$, $j \in \{1,\ldots,N\}$ which is determistic since it is $\mathcal{F}_0^z$-measurable, by construction.

We close this section by rewriting the dynamics (2.50), in terms of the originally given observation nonlinearities h^i, and with forcing inputs the processes $y^i(\cdot)$ introduced in (2.13), (2.14). In view of (2.5), (2.6), (2.7), (2.13), (2.14) we have

$$\tilde{h}(\cdot,u(t))^T dz(t) = \sum_{j=1}^{M} h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j)R_j^{-1/2}dz_j(t), \tau_i \le t < \tau_{i+1}$$

(where we have written $z = [z_1, z_2, \dots z_M]^T$)

$$
\begin{aligned}
&= \sum_{j=1}^{M} h^{j^T}(\cdot)\chi_{\{\nu_i\}}(j) R_j^{-1} R_j^{1/2} \chi_{\{\nu_i\}}(j) dz_j(t),\ \tau_i \leq t < \tau_{i+1} \\
&= \delta(\cdot, \nu_i)^T dy(t, \nu_i),\ \tau_i \leq t < \tau_{i+1} \\
&=: \delta(\cdot, u(t))^T dy(t, u(t)),
\end{aligned}
$$

where

$$
\delta(x,\nu) = \begin{bmatrix} R_1^{-1} h^1(x)\chi_{\{\nu\}}(1) \\ \vdots \\ R_i^{-1} h^i(x)\chi_{\{\nu\}}(i) \\ \vdots \\ R_M^{-1} h^M(x)\chi_{\{\nu\}}(M) \end{bmatrix} \tag{2.65}
$$

Therefore the system dynamics (2.50) can be written equivalently

$$
\begin{aligned}
dp(u(\cdot),t) &= L^* p(u(\cdot),t)dt + p(u(\cdot),t)\delta(\cdot,u(t))^T dy(t,u(\cdot)) \\
p(u(\cdot),0) &= p_0,
\end{aligned} \tag{2.66}
$$

where $y(t, u(t))$ is defined in (2.13), (2.14). This makes precise the construction of a Zakai equation driven by "controlled" observations alluded to in the introduction. It also becomes now clear that the spaces described by (2.51), (2.64) are the appropriate ones as far as solutions of (2.50) or (2.66) are concerned.

3. The solution of the optimization problem

3.1. Setting up a system of quasi-variational inequalities

Let us consider the Banach space $H = L^2(I\!R^n; \mu) \cap L^1(I\!R^n; \mu)$ and the metric space H^+ of positive elements of H. Let

$$
\begin{aligned}
\mathcal{B} &:= \text{space of Borel measurable, bounded functions on} H^+ \\
\mathcal{C} &:= \text{space of uniformly continuous, bounded functions on} H^+.
\end{aligned} \tag{3.1}
$$

Let us now define semigroups $\Phi_j(t)$ on $\mathcal{B}$ or $\mathcal{C}$ as follows. Consider (2.50) with fixed schedule $u(t) = j$, and let p_j denote the corresponding density $p(\cdot, j)$. Then for $j \in \{1, 2, \dots, N\}$

$$
dp_j = L^* p_j\, dt + p_j \tilde{h}^{j^T} dz(t),\ p_j(0) = \pi, \tag{3.2}
$$

where

$$
\tilde{h}^j := \tilde{h}(\cdot, j). \tag{3.3}
$$

We set

$$
\Phi_j(t)(F)(\pi) = E\{F(p_{j,\pi}(t))\},\quad F \in \mathcal{B} \text{ or } \mathcal{C}, \tag{3.4}
$$

where $p_{j,\pi}$ indicates the solution of (3.2) with initial value π. It is easy to see that Φ_j is a semigroup since $p_j(t)$ is a Markov process with values in H^+. It is also useful to introduce the subspaces $\mathcal{B}_1$ and $\mathcal{C}_1$ of functions such that

$$\|F\|_1 \sup_{\pi\in H^+} \frac{|F(\pi)|}{1+\|\pi\|_\mu} < \infty \tag{3.5}$$

where $\| \ \pi \ \|_\mu = \| \ \pi \ \|_{L^1(R^n;\mu)}$. The spaces $\mathcal{B}_1$ and $\mathcal{C}_1$ are also Banach spaces. They are needed, because we shall encounter functionals with linear growth in the cost function (2.61). To simplify the statement and analysis of the quasi-variational inequalities that solve the optimization problem considered here, we give the details for the case N =2 only in the sequel. We shall insert remarks to indicate how the results should be modified for the general case. Let us introduce the notation

$$\begin{aligned} C_i &:= C(i,\cdot), \quad i=1,2, \\ K_1 &:= K(1,2) \\ K_2 &:= K(2,1). \end{aligned} \tag{3.6}$$

Since C_1, C_2, K_1, K_2 are bounded functions, one can utilize them to define elements of $\mathcal{C}_1$ via (for example)

$$C_1(\pi) = (C_1, \pi) \tag{3.7}$$

where a slight abuse of notation, in denoting the functional and the function by the same symbol, has been allowed. Similarly the functional on H^+

$$\Psi(\pi) = (\pi, \chi^2) - \frac{\|(\pi,\chi)\|^2}{(\pi, \mathbb{1})} \tag{3.8}$$

belongs to $\mathcal{C}_1$ since it is positive and

$$\Psi(\pi) \le (\pi,\chi^2) \le \|\pi\|_\mu. \tag{3.9}$$

Consider now the set of functionals $U_1(\pi,t)$, $U_2(\pi,t)$ such that

$$\begin{aligned} U_1, U_2 &\in C(0,T;\mathcal{C}_1) \\ U_1(\cdot,t) \ge 0, & \quad U_2(\cdot,t) \ge 0 \\ U_1(\pi,T) &= U_2(\pi,T) = \Psi(\pi) \\ U_1(\pi,t) &\le \Phi_1(s-t)U_1(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\ U_2(\pi,t) &\le \Phi_2(s-t)U_2(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\ \forall s &\ge t \\ U_1(\pi,t) &\le K_1(\pi) + U_2(\pi,t) \\ U_2(\pi,t) &\le K_2(\pi) + U_1(\pi,t). \end{aligned} \tag{3.10}$$

In the sequel we will occasionally use the notation $U_i(s)(\pi) = U_i(\pi,s), \quad i=1,2.$

3.2. Existence of a maximum element

We shall refer to (3.10) as the system of quasi-variational inequalities (QVI). Our first objective is to prove the following.

Theorem 3.1. *We assume that the conditions on the data f, g, h^i introduced in section 2.1 hold. Then the set of functionals U_1, U_2 satisfying (3.10) is non-empty and has a maximum element, in the sense that if $\tilde{U}_1, \tilde{U}_2$ denotes this maximum element and U_1, U_2 satisfies (3.10), then*

$\tilde{U}_1 \geq U_1, \tilde{U}_2 \geq U_2.$

The proof will be carried out in several steps. In fact there is some difficulty due to the functional $\Psi(\pi)$. We shall modify it in order to assume that

$$0 \leq \Psi(\pi) \leq \overline{\Psi}(\pi, \mathbb{1}) \tag{3.11}$$

where $\overline{\Psi}$ is a constant. We shall prove the theorem with the additional assumption (3.11), prove the probabilistic interpretation, i.e. the connection with the infimum of (2.61). The probabilistic formula will be next used in an approximation procedure. We can approximate for instance the functional Ψ defined by (3.8) in the following way. Set

$$\Psi_n(\pi) = \int \frac{\pi \|x\|^2}{1 + \frac{\|x\|^2}{n}} dx - \frac{\|\int \frac{\pi x}{(1+\frac{\|x\|^2}{n})^{1/2}} dx\|^2}{\int \pi dx} \tag{3.12}$$

which clearly satisfies (3.11) with $\overline{\Psi} = n$.

Proof of Theorem 3.1 under the assumption (3.11). The set of functionals satisfying (3.10) is a subset of $\mathcal{B}_1$ or $\mathcal{C}_1$ defined in (3.5). However for this subset the norm (3.5) is unnecessarily restrictive. For those functionals it is sufficient to set

$$\begin{aligned} \tilde{H} &= L^2(\mathbb{R}^n) \cap L^1(\mathbb{R}^n) \\ \tilde{H}^+ &= \text{set of positive elements of } \tilde{H} \end{aligned} \tag{3.13}$$

and to consider $\tilde{\mathcal{B}}_1, \tilde{\mathcal{C}}_1$ the space of Borel or continuous functionals on $\tilde{H}^+$ such that

$$\|F\|_1 = \sup_{\pi \in \tilde{H}^+} \frac{|F(\pi)|}{1 + (\pi, \mathbb{1})} < \infty \tag{3.14}$$

We shall then study the system (3.10) with $\mathcal{C}_1$ replaced by $\tilde{\mathcal{C}}_1$. Let us notice that

$$H^+ \subset \tilde{H}^+$$

and if one considers a functional F in $\tilde{\mathcal{B}}_1$ or $\tilde{\mathcal{C}}_1$, its restriction to H^+ belongs to $\mathcal{B}_1$ or $\mathcal{C}_1$; the injection

$$F \to \text{restriction of } F \text{ to } H^+$$

is continuous from $\tilde{\mathcal{B}}_1$ or $\tilde{\mathcal{C}}_1$ to $\mathcal{B}_1$ or $\mathcal{C}_1$. Therefore replacing in (3.10) $\mathcal{C}_1$ by $\tilde{\mathcal{C}}_1$ gives a stronger result.

We shall in the proof omit to write the symbol $\sim$ and write $\mathcal{B}_1, C_1$ instead of $\tilde{\mathcal{B}}_1, \tilde{C}_1, H^+$ instead of $\tilde{H}^+$, the norm $\| \, \|_1$ is then given by (3.14).

The proof is then an adaptation of the methods of Bensoussan-Lions [9] to the present case in order to take into account the fact that we use C_1 instead of C.

First note that

$$\|\Phi_1(t)\|_{\mathcal{L}(C_1;C_1)} \leq 1 \tag{3.15}$$

where $\mathcal{L}(C_1;C_1)$ is the space of linear continuous operators from C_1 into itself. Indeed we have

$$\begin{aligned} \frac{|\Phi_1(t)(F)(\pi)|}{1+(\pi,\mathbb{1})} &= \frac{|E\{F(p_{1,\pi}(t))\}|}{1+(\pi,\mathbb{1})} \\ &\leq \|F\|_1 \frac{(1+E(p_{1,\pi}(t),\mathbb{1}))}{1+(\pi,\mathbb{1})} \\ &= \|F\|_1 \end{aligned}$$

since from (3.2)

$$E(p_{1,\pi}(t),\mathbb{1}) = (\pi,\mathbb{1}) \tag{3.16}$$

Therefore

$$\|\Phi_1(t)(F)\|_1 \leq \|F\|_1 \tag{3.17}$$

which implies (3.15).

Note also that a solution of (3.10) will satisfy

$$U_1(\pi,t) \leq \Phi_1(T-t)U_1(\pi,T) + \int_t^T \Phi_1(\lambda - t)C_1(\pi)d\lambda \tag{3.18}$$

and due to positivity, we also have

$$\|U_1(t)\|_1 \leq \|U_1(T)\|_1 + \|C_1\|_1(T-t) \leq \overline{\Psi} + \|C_1\|(T-t) \tag{3.19}$$

where $\|C_1\| = \sup_x C_1(x)$.

As it is customary in the study of QVI we begin with the corresponding obstacle problem,

$$\begin{aligned} U_1, U_2 &\in C(0,T;C_1) \\ U_1(\cdot,t) &\geq 0, U_2(\cdot,t) \geq 0 \\ U_1(\pi,T) &= U_2(\pi,T) = \Psi(\pi) \\ U_1(\pi,t) &\leq \Phi_1(s-t)U_1(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\ U_2(\pi,t) &\leq \Phi_2(s-t)U_2(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\ \forall s &\geq t \\ U_1(\pi,t) &\leq K_1(\pi) + \varsigma_2(\pi,t) \\ U_2(\pi,t) &\leq K_2(\pi) + \varsigma_1(\pi,t) \end{aligned} \tag{3.20}$$

where we assume that

$$\begin{aligned} \varsigma_1, \varsigma_2 &\in C(0,T;C_1) \\ \varsigma_1(\pi,t) \geq 0, &\qquad \varsigma_2(\pi,t) \geq 0 \\ \varsigma_1(\pi,T), \varsigma_2(\pi,T) &\geq \Psi(\pi). \end{aligned} \tag{3.21}$$

We then have the following.

Proposition 3.1: *For ς_1, ς_2 as in (3.21) the set of U_1, U_2 satisfying (3.20) is not empty and has a maximum element.*

It is clear that for ς_1, ς_2 given, the system of inequalities (3.20) can be decoupled and U_1, U_2 can be considered separately. Let us then omit indices momentarily and consider

$$\begin{array}{rcl} U & \in & C(0,T;C_1) \\ U(\cdot,t) & \geq & 0 \\ U(\pi,T) & = & \Psi(\pi) \\ U(\pi,t) & \leq & \Phi(s-t)U(\pi,s) + \int_t^s \Phi(\lambda - t)C(\pi)d\lambda \\ \forall s & \geq & t \\ U(\pi,t) & \leq & \varsigma(t) \end{array} \tag{3.22}$$

where ς stands for instance, for $K_1(\pi) + \varsigma_2(\pi,t)$. To prove proposition 3.1, it suffices to show that (3.22) has a maximum element. This can be done by the penalty method. So we look for U_ϵ solving

$$\begin{array}{rcl} U_\epsilon(t) & = & \Phi(s-t)U_\epsilon(s) + \int_t^s \Phi(\lambda - t)[C(\pi) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda \\ \text{for} & & t \leq s \leq T \\ U_\epsilon(T)(\pi) & = & \Psi(\pi) \\ U_\epsilon & \in & C(0,T;C_1) \\ U_\epsilon(\cdot,t) & \geq & 0. \end{array} \tag{3.23}$$

We can then assert

Lemma 3.1 *There is a unique solution of* (3.23).

Proof: Notice that (3.23) is equivalent to

$$U_\epsilon(t) = \Phi(T-t)U_\epsilon(T) + \int_t^T \Phi(\lambda - t)[C(\pi) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda \tag{3.24}$$

and also to

$$\begin{array}{rcl} U_\epsilon(t) & = & e^{-\frac{1}{\epsilon}(T-t)}\Phi(T-t)\Psi(\pi) + \int_t^T e^{-\frac{1}{\epsilon}(\lambda - t)}\Phi(\lambda - t) \\ & & [C(\pi) + \frac{1}{\epsilon}U_\epsilon(\lambda) - \frac{1}{\epsilon}(U_\epsilon(\lambda) - \varsigma(\lambda))^+]d\lambda \end{array} \tag{3.25}$$

Let us define the transformation T_ϵ of $C(0,T;C_1)$ into itself using the right hand side of (3.25). Then the latter can be written as a fixed point equation

$$U_\epsilon = T_\epsilon U_\epsilon \tag{3.26}$$

Using (3.11) and (3.15) one can show precisely as in Bensoussan-Lions [9, p.488] that some power of T_ϵ is a contraction. Hence the result of the lemma follows.

One then can also prove as in [9, pp.489 - 490], that if $\epsilon \leq \epsilon', \|U_\epsilon\|_1 \leq K$, then $0 \leq U_\epsilon \leq U_{\epsilon'}$. As in [9, pp.494 - 495] one then shows that as $\epsilon \downarrow 0, U_\epsilon \downarrow U$ which is the maximum element of (3.22). The convergence takes place in $C(0,T;\mathcal{C}_1)$. This establishes Proposition 3.1.

We can then proceed with the

Proof of Theorem 3.1: (Continuation)

Let us consider the map H mapping $C(0,T;\mathcal{C}_1) \times C(0,T;\mathcal{C}_1)$ into itself defined by

$$H(\varsigma_1, \varsigma_2) = (U_1, U_2) \tag{3.27}$$

where the right hand side represents the maximum element of (3.20). Let now

$$\begin{aligned} U_1^o(\pi,t) &= \Phi_1(T-t)\Psi(\pi) + \int_t^T \Phi_1(\lambda - t)C_1(\pi)d\lambda \\ U_2^o(\pi,t) &= \Phi_2(T-t)\Psi(\pi) + \int_t^T \Phi_2(\lambda - t)C_2(\pi)d\lambda \end{aligned} \tag{3.28}$$

Consider $\varsigma_i(t), \xi_i(t), i = 1,2$ such that

$$0 \leq \varsigma_i(t) \leq \xi_i(t) \leq U_i^o(t), i = 1,2, \tag{3.29}$$

and

$$\xi_i(t) - \varsigma_i(t) \leq \gamma \xi_i(t),\ \gamma \in [0,1]. \tag{3.30}$$

Then we have

$$0 \leq H(\xi_1,\xi_2) - H(\varsigma_1,\varsigma_2) \leq \gamma(1-\gamma')H(\xi_1,\xi_2) \tag{3.31}$$

where

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \max(\|C_1\|, \|C_2\|)T} \tag{3.32}$$

Indeed, setting

$$\kappa = 1 - \gamma(1-\gamma') \tag{3.33}$$

we have to prove that

$$\kappa H(\xi_1,\xi_2) \leq H(\varsigma_1,\varsigma_2). \tag{3.34}$$

Let us set

$$\begin{aligned} (U_1, U_2) &= H(\varsigma_1,\varsigma_2) \\ (\tilde{U}_1, \tilde{U}_2) &= H(\xi_1,\xi_2). \end{aligned} \tag{3.35}$$

We need then to show that

$$\kappa\tilde{U}_1 \leq U_1, \qquad \kappa\tilde{U}_2 \leq U_2. \tag{3.36}$$

If we can establish that

$$\begin{aligned} \kappa K_1(\pi) + \kappa\xi_2(\pi,t) &\leq K_1(\pi) + \varsigma_2(\pi,t) \\ \kappa K_2(\pi) + \kappa\xi_1(\pi,t) &\leq K_2(\pi) + \varsigma_1(\pi,t), \end{aligned} \tag{3.37}$$

then (3.36) is implied by the monotonicity properties of Variational Inequalities. But

$$\xi_2(\pi,t)(1-\gamma) \leq \varsigma_2(\pi,t), \tag{3.38}$$

hence it is enough to establish that

$$\begin{aligned} \kappa K_1(\pi) + \kappa \xi_2(\pi,t) &\leq K_1(\pi) + (1-\gamma)\,\xi_2(\pi,t) \\ \kappa K_2(\pi) + \kappa \xi_1(\pi,t) &\leq K_2(\pi) + (1-\gamma)\xi_1(\pi,t) \end{aligned} \tag{3.39}$$

The first of (3.39) will be satisfied if

$$[\kappa - (1-\gamma)]\xi_2(\pi,t) \leq (1-\kappa)K_1(\pi) \tag{3.40}$$

or if

$$\gamma'\xi_2(\pi,t) \leq (1-\gamma')K_1(\pi). \tag{3.41}$$

But observe that

$$\xi_2(\pi,t) \leq U_2^0(\pi,t) \leq (\overline{\Psi} + \|C_2\|T)\,(\pi,\mathbb{1})$$

So it is enough to choose γ' so that

$$\gamma'(\overline{\Psi} + \|C_2\|T)(\pi,\mathbb{1}) \leq (1-\gamma')k_0(\pi,\mathbb{1}) \tag{3.42}$$

where k_0 is the uniform lower bound (2.21), since $K_1(\pi) \geq k_0(\pi,\mathbb{1})$. This last inequality requires

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \|C_2\|T} \tag{3.43}$$

In an identical fashion, the second of (3.39) will be satisfied if

$$\gamma' \leq \frac{k_0}{k_0 + \overline{\Psi} + \|C_1\|T}. \tag{3.44}$$

So both of (3.39) will be satisfied if we choose γ' according to (3.32). The proof of the theorem then proceeds via the standard iteration

$$(U_1^{n+1}, U_2^{n+1}) = H\,(U_1^n, U_2^n) \tag{3.45}$$

as in [9, pp.512 - 514].

Remark: The extension of this result to the general case $N \neq 2$ is straightforward. The system (3.10) has N functionals $U_1, \ldots, U_N$. Everything in (3.10) is the same except for the last two inequalities which are replaced by

$$U_i(\pi,t) \leq \min_{\substack{j\neq i \\ j=1,\ldots,N}} (K_{ij}(\pi) + U_j(\pi,t)), \qquad i = 1,\ldots,N \tag{3.46}$$

One again introduces the system (3.20) where the last two inequalities are replaced by

$$U_i(\pi,t) \leq \min_{\substack{j\neq i \\ j=1,\dots,N}} (K_{i\,j}(\pi) + \varsigma_j(\pi,t)), \qquad i=1,\dots,N \tag{3.47}$$

where $\varsigma_i \in C(0,T;C_1)$, and satisfy the remainder of (3.21). One then establishes the analog of Proposition 3.1 by penalization. The analog of Theorem 3.1 is established by introducing a map H mapping $C(0,T;C_1)^N$ into itself defined by

$$H(\varsigma_1,\varsigma_2,\dots,\varsigma_N) = (U_1,U_2\dots,U_N)$$

where the right hand side is the maximum element of the analog of (3.20).

3.3. Existence of an admissible sensor schedule

Our objective in this section is to show that the maximum element U_1, U_2 of the QVI (3.10) provides the value function for the optimization problem (2.61), (2.66) when the assumption (3.11) holds. Furthermore we want to show how an admissible optimal sensor schedule is determined once the pair U_1, U_2 is known.

We shall prove that

$$U_i(\pi,0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J(u(\cdot)), \qquad i=1,2 \tag{3.48}$$

where $\pi \in H^+$ satisfies $(\pi,\mathbb{1}) = 1$. An optimal schedule will be constructed as follows. Suppose, to fix ideas that $i=1$. Then define

$$\tau_1^* = \inf_{t\leq T}\{U_1(p_1(t),t) = K_1(p_1(t)) + U_2(p_1(t),t)\} \tag{3.49}$$

where again $p_i(t)$ is the solution of (3.2). We write

$$p^*(t) = p_1(t), \qquad t\in[0,\tau_1^*]. \tag{3.50}$$

Next define

$$\tau_2^* = \inf_{\tau_1^*\leq t\leq T}\{U_2(p_2(t),t), = K_2(p_2(t)) + U_1(p_2(t),t)\} \tag{3.51}$$

In (3.51), it must be kept in mind that $p_2(t)$ represents the solution of (3.2) with j=2, starting at τ_1^* with value $p_1(\tau_1^*)$. We then define

$$p^*(t) = p_2(t), \qquad t\in[\tau_1^*,\tau_2^*] \tag{3.52}$$

Note that, unless $\tau_1^* = T$,

$$\tau_2^* > \tau_1^*, \tag{3.53}$$

otherwise

$$\begin{aligned} U_1(p_1(\tau_1^*),\tau_1^*) &= K_1(p_1(\tau_1^*)) + U_2(p_1(\tau_1^*),\tau_1^*) \\ U_2(p_1(\tau_1^*),\tau_1^*) &= K_2(p_1(\tau_1^*)) + U_1(p_1(\tau_1^*),\tau_1^*) \end{aligned} \tag{3.54}$$

which is impossible since

$$K_1(p_1(\tau_1^*)) > 0, K_2(p_1(\tau_1^*)) > 0 \qquad a.s. \tag{3.55}$$

Similarly one proceeds to construct a sequence of $\tau_1^* < \tau_2^* < \tau_3^* < \ldots$ and the process $p^*(\cdot)$. We can then prove the following.

Theorem 3.2. *With the same assumptions as in Theorem 3.1 and in addition assuming that (3.11) holds; the sequence of stopping times $\tau_1^*, \tau_2^*, \ldots$ defines an optimal admissible sensor schedule.*

Proof: Considering (3.10) as a VI with obstacle ς_2, ς_1, we can write from the definition of τ_1^*

$$\begin{aligned} U_1(\pi,0) &= E\{U_1(p_1(\tau_1^*),\tau_1^*) \\ &+ \int_0^{\tau_1^*} C_1(p_1(\lambda))d\lambda\}. \end{aligned} \tag{3.56}$$

This can be established by utilizing the penalization (3.23), along similar lines as in [9, pp. 578 - 587]. Then

$$\begin{aligned} E\{U_1(p_1(\tau_1^*),\tau_1^*)\} &= E\{U_1(p^*(\tau_1^*),\tau_1^*\} \\ &= E\{\Psi(p^*(T))\chi_{\tau_1^*=T}\} \\ &+ E\{U_1(p^*(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\}. \end{aligned}$$

Substituting back in (3.56) and using the definition of τ_1^* in (3.49) we obtain

$$\begin{aligned} U_1(\pi,0) &= E\Big\{\Psi(p^*(T))\chi_{\tau_1^*=T} + \int_0^{\tau_1^*} C_1(p^*(\lambda))d\lambda \\ &\quad +K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} + U_2(p^*(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\Big\} \end{aligned} \tag{3.57}$$

Furthermore, again by employing penalization one can show that

$$\begin{aligned} E\{U_2(p^*(\tau_1),\tau_1^*)\} = E\{U_2(p_2(\tau_1^*),\tau_1^*)\} &= E\{U_2(p_2(\tau_2^*),\tau_2^*) \\ &+ \int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))d\lambda\}. \end{aligned} \tag{3.58}$$

This implies

$$\begin{aligned} E\{U_2(p_2(\tau_1^*),\tau_1^*)\chi_{\tau_1^*<T}\} &= E\{U_2(p_2(\tau_2^*),\tau_2^*)\chi_{\tau_1^*<T} \\ &+ \chi_{\tau_1^*<T}\int_{\tau_1^*}^{\tau_2^*} C_2(p_2(\lambda))d\lambda\}. \end{aligned} \tag{3.59}$$

Next

$$
\begin{aligned}
E\{U_2(p_2(\tau_2^*),\tau_2^*)\chi_{\tau_1^*<T}\} &= E\{\Psi(p^*(T))\chi_{\tau_1^*<T,\tau_2^*=T}\} \\
&+ E\{U_2(p^*(\tau_2^*),\tau_2^*)\chi_{\tau_2^*<T}\}.
\end{aligned}
$$

Substituting back in (3.57) and using the definition of τ_2^* in (3.51) we obtain

$$
\begin{aligned}
U_1(\pi,0) &= E\{\Psi(p^*(T))\chi_{\tau_2^*=T} + K_1(p^*(\tau_1^*))\chi_{\tau_1^*<T} \\
&+K_2(p^*(\tau_2^*))\chi_{\tau_2^*<T} + \int_0^{\tau_1^*} C_1(p^*(\lambda))d\lambda \\
&+\int_{\tau_1^*}^{\tau_2^*} C_2(p^*(\lambda))d\lambda + U_1(p^*(\tau_2^*),\tau_2^*)\chi_{\tau_2^*<T}\}
\end{aligned}
\tag{3.60}
$$

Proceeding in a similar fashion, and collecting results we can write:

$$
\begin{aligned}
U_1(\pi,0) &= E\{\Psi(p^*(T))\chi_{\tau_n^*=T} \\
&+ \sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T} \\
&+ \sum_{i=0}^{n-1} \chi_{\tau_{i+1}^*<T}\int_{\tau_i^*}^{\tau_{i+1}^*} C_{i+1}(p^*(\lambda))d\lambda\}, \\
&+ U_{n+1}(p^*(\tau_n^*),\tau_n^*)\chi_{\tau_n^*<T}
\end{aligned}
\tag{3.61}
$$

where we used the notation

$$
\begin{aligned}
K_i &= \begin{cases} K_1, & \text{if i is odd} \\ K_2, & \text{if i is even} \end{cases} \\
C_i &= \begin{cases} C_1, & \text{if i is odd} \\ C_2, & \text{if i is even} \end{cases} \\
U_i &= \begin{cases} U_1, & \text{if i is odd} \\ U_2, & \text{if i is even.} \end{cases}
\end{aligned}
\tag{3.62}
$$

However, observe that necessarily $\tau_n^* = T$, for n large enough (random). Otherwise one has $\tau_n^* < T, \forall n$, on a set $\Omega_0 \subset \Omega$ of positive probability. But $\tau_n^* \uparrow \tau^* \leq T$ and

$$
(p^*(\tau_i^*),\mathbb{1}) \longrightarrow (p^*(\tau^*),\mathbb{1})
\tag{3.63}
$$

where (since $(\pi,\mathbb{1}) = 1$)

$$
(p^*(\tau^*),\mathbb{1}) = 1 + \int_0^{\tau^*} p^*\delta^T dy
\tag{3.64}
$$

(see (2.66)) and

$$
(p^*(\tau^*),\mathbb{1}) = E\{\varsigma(\tau^*)|\mathcal{F}_{\tau^*}^{y(\cdot,u^*)}\} > 0 \quad a.s.
\tag{3.65}
$$

where $\varsigma(\cdot)$ is the process introduced by (2.8). Therefore on Ω_0, as $n \to \infty$

$$
\sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T} \longrightarrow +\infty
\tag{3.66}
$$

and since Ω_0 has positive probability, as $n \to \infty$

$$E\{\sum_{i=1}^{n} K_i(p^*(\tau_i^*))\chi_{\tau_i^*<T}\} \longrightarrow \infty, \tag{3.67}$$

which contradicts (3.19).

We can thus assert that

$$\chi_{\tau_n^*=T} \longrightarrow 1 \qquad a.s. \tag{3.68}$$

In particular, it follows that the sequence $\tau_1^*, \tau_2^* \ldots$, defines an admissible schedule denoted by u^*. The corresponding state solution of (2.66), coincides with p^* and (3.61) implies

$$U_1(\pi,0) \geq J(u^*(\cdot)). \tag{3.69}$$

But by standard arguments, one checks that

$$U_1(\pi,0) \leq J(u(\cdot)), \quad \forall u(\cdot) \in U_{ad} \tag{3.70}$$

and therefore $u^*(\cdot)$ is indeed optimal.

3.4. The main result

We want now to get rid of (3.11) and consider the original functional Ψ in (3.8). Let us consider the approximation (3.12) Ψ_n of Ψ. To Ψ_n corresponds a system of QVI.

$$\begin{aligned}
U_1^n, U_2^n &\in C(0,T;\tilde{C}_1) \\
U_1^n, U_2^n &\geq 0 \\
U_1^n(\pi,T) &= U_2^n(\pi,T) = \Psi_n(\pi) \\
U_1^n(\pi,t) &\leq \Phi_1(s-t)U_1^n(\pi,s) + \int_t^s \Phi_1(\lambda-t)C_1(\pi)d\lambda \\
U_2^n(\pi,t) &\leq \Phi_2(s-t)U_2^n(\pi,s) + \int_t^s \Phi_2(\lambda-t)C_2(\pi)d\lambda \\
\forall s &\geq t \\
U_1^n(\pi,t) &\leq K_1(\pi) + U_2^n(\pi,t) \\
U_2^n(\pi,t) &\leq K_2(\pi) + U_1^n(\pi,t).
\end{aligned} \tag{3.71}$$

From Theorem 3.2, we can assert that

$$U_i^n(\pi,0) = \inf_{\substack{u(0)=i \\ p(0)=\pi}} J^n(u(\cdot)), \qquad i = 1,2 \tag{3.72}$$

where

$$\begin{aligned}
J^n(u(\cdot)) &= E\{\Psi^n(p(u(\cdot),T)) + \int_o^T (p(u(\cdot),t), C(u(t)))dt \\
&+ \sum_{i=1}^{\infty} \chi_{\tau_i<T}(p(u(\cdot),\tau_i), K(u_{i-1},u_i))\}.
\end{aligned} \tag{3.73}$$

Therefore we deduce that

$$J^n(u(\cdot)) - J(u(\cdot)) = E\{\Psi_n(p(u(\cdot),T)) - \Psi(p(u(\cdot),T))\} \tag{3.74}$$

and from (3.12) we deduce

$$\begin{aligned}|J^n(u(\cdot)) - J(u(\cdot))| \;\le\; & E\left\{\int \frac{p(u(\cdot),T)\|x\|^4}{n+\|x\|^2}dx\right\} \\ & +E\left\{\left(\int p(u(\cdot),T)x\left(1-\frac{1}{(1+\frac{\|x\|^2}{n})^{1/2}}\right)dx\right)^T\right. \\ & \left.\cdot\left(\int p(u(\cdot),T)x(1+\frac{1}{(1+\frac{\|x\|^2}{n})^{1/2}})dx\right)\times\frac{1}{\int p(u(\cdot),T)dx}\right\}\end{aligned} \tag{3.75}$$

But using the equation (2.50) yields (see (2.1a))

$$\begin{aligned}E\left\{\int\frac{p(u(\cdot),t)\|x\|^4}{n+\|x\|^2}dx\right\} = & E\left\{\int_o^t\int p(u(\cdot),s)(x)\left\{\frac{\partial a_{ij}}{\partial x_i}\frac{2\|x\|^2(2n+\|x\|^2)x_j}{(n+|x|^2)^2}\right.\right. \\ & +a_{ij}\left(\delta_{ij}\frac{2\|x\|^2(2n+\|x\|^2)}{(n+\|x\|^2)^2}+\frac{8x_ix_jn^2}{(n+\|x\|^2)^3}\right) \\ & \left.\left.- a_i\frac{2\|x\|^2(2n+\|x\|^2)x_i}{(n+\|x\|^2)^2}\right\}ds+\int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx\right\}\end{aligned}$$

where we employed the summation convention over repeated indices. Hence after majorizing conveniently

$$\begin{aligned}E\left\{\int\frac{p(u(\cdot),t)(x)\|x\|^4}{n+\|x\|^2}dx\right\} \le & \int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx \\ & + \Gamma\int_o^t E\left\{\int\frac{p(u(\cdot),s)(x)\|x\|^4}{n+\|x\|^2}dx\right\}ds+\frac{\Gamma t}{n}.\end{aligned} \tag{3.76}$$

We shall use capital Greek letters, $\Gamma, \Delta, \ldots$, to indicate constants in the following estimates. Finally we deduce

$$\begin{aligned}E\left\{\int\frac{p(u(\cdot),t)(x)\|x\|^4}{n+\|x\|^2}dx\right\} \le & \Gamma_t\left[\int\frac{\pi(x)\|x\|^4}{n+\|x\|^2}dx+\frac{1}{n}\right] \\ \le & \Gamma_t\left[\frac{1}{n}\int\pi(x)\|x\|^4dx+\frac{1}{n}\right].\end{aligned} \tag{3.77}$$

Next consider

$$\frac{p(u(\cdot),t)}{(p(u(\cdot),t),\mathbb{1})} = \sigma(u(\cdot),t)$$

which is the normalized conditional probability, measure and satisfies Kusner's equation

$$d(\sigma(t)(\varphi)) = \sigma(t)(L\varphi)dt + (\sigma(t)(\tilde{h}\varphi) - \sigma(t)(\varphi)\sigma(t)(\tilde{h}))\cdot(dz - \sigma(t)(\tilde{h})dt) \tag{3.78}$$

If we apply (3.78) with $\varphi = \|x\|^2 = \chi^2$, we obtain

$$\begin{aligned} dE\{\sigma(t)(\chi^2)\} &= E\{\sigma(t)(L\chi^2) - \sigma(t)(\tilde{h})[\sigma(t)(\tilde{h}\chi^2) - \sigma(t)(\chi^2)\sigma(\tilde{h})]\}\, dt \\ &\leq \Delta_0(1 + E\{\sigma(t)(\chi^2)\}). \end{aligned} \tag{3.79}$$

Finally

$$E\{\sigma(t)(\chi^2)\} \leq \Delta_t \int \pi(x)\|x\|^2 dx \tag{3.80}$$

But the 2nd term in (3.75) is

$$E\left\{\sigma(T)\left(\chi(1 + \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)^T (p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\right\}$$

$$\begin{aligned} &\leq \left[E\left\{\|\sigma(T)\left(\chi(1 + \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \left[E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \\ &\leq \Delta^1(E\{\sigma(T)(\chi^2)\})^{1/2}\left(E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right)^{1/2} \\ &\leq \Delta^2\left[E\left\{\|p(T)\left(\chi(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\|^2\right\}\right]^{1/2} \\ &= \Delta^3\left[E\left\{\sum_i\left(p(T)\left(\chi_i(1 - \frac{1}{(1+\frac{\chi^2}{n})^{1/2}})\right)\right)^2\right\}\right]^{1/2} \\ &\leq \Delta^3\left[E\left\{p(T)(\chi^2)p(T)\left(\frac{\chi^2}{n+\chi^2}\right)\right\}\right]^{1/2}. \end{aligned} \tag{3.81}$$

One easily checks that

$$\begin{aligned} E\left\{\left(p(T)(\chi^2)\right)^2\right\} &\leq \Delta^4 + \left(\int \pi(x)\|x\|^2 dx\right)^2 \leq \Delta^5 \\ dE\left\{|p(t)\left(\frac{\chi^2}{n+\chi^2}\right)|^2\right\} &\leq 2E\left\{p(t)\left(L\frac{\chi^2}{n+\chi^2}\right)p(t)(\frac{\chi^2}{n+\chi^2})\right\} dt \\ &+ \Delta^5 E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} dt \end{aligned}$$

But

$$L\frac{\chi^2}{n+\chi^2} \leq \frac{\Delta^6}{\sqrt{n}} \tag{3.82}$$

hence

$$dE\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} \leq \left[\Delta^5 E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|^2\right\} + \frac{\Delta^7}{n}\right] dt \tag{3.83}$$

which implies

$$E\left\{|p(t)(\frac{\chi^2}{n+\chi^2})|\right\} \leq \Theta_t\left[\frac{1}{n}+\left(\int\frac{\pi(x)\|x\|^2dx}{n+\|x\|^2}\right)^2\right] \leq \frac{\Theta_t}{n}(1+\int\pi(x)\|x\|^4dx) \tag{3.84}$$

Therefore, continuing from (3.81), the 2nd term in (3.75) is majorized by $\frac{\Gamma_0}{n^{1/4}}$. Collecting results (from (3.75), (3.77), (3.81), (3.84)) we can assert that

$$|J^n(u(\cdot)) - J(u(\cdot))| \leq \frac{\Delta}{n^{1/4}} \tag{3.85}$$

provided the initial distribution of $p(0)$, i.e. π satifies

$$\int\pi(x)\|x\|^4dx < \infty \tag{3.86}$$

The estimate in (3.85) is uniform with respect to n. Therefore

$$|U_i^n(\pi,0) - \inf_{\substack{u(0)=i\\p(0)=\pi}} J(u(\cdot))| \leq \frac{\Delta}{n^{1/4}} \tag{3.87}$$

In fact we can replace 0 by any $t \in [0,T]$ and consider the function

$$U_i(\pi,t) = \inf_{\substack{u(t)=i\\p(t)=\pi}} J_t(u(\cdot)) \tag{3.88}$$

where $J_t(u(\cdot))$ corresponds to a problem analogous to (2.50), (2.61) starting in t instead of 0. Therefore we have

$$|U_i^n(\pi,t) - U_i(\pi,t)| \leq \frac{\Delta}{n^{1/4}} \tag{3.89}$$

We have however to be careful to the fact that the constant in (3.89) depends on a bound on $\int\pi(x)\|x\|^4dx$. More precisely we have proved that

$$|U_i^n(\pi,t) - U_i(\pi,t)| \leq \frac{\Delta'}{n^{1/4}}(1+\int\pi(x)\|x\|^4dx) \tag{3.90}$$

where Δ' this time does not depend on π (assuming that π is a probability). It follows that

$$U_i^n(\pi,t) \longrightarrow U_i(\pi,t) \ \text{ in } \ C(0,T;\mathcal{C}_1). \tag{3.91}$$

Taking the limit in (3.71), we obtain that U_1, U_2 is a solution of (3.10) and moreover

$$U_i(\pi,0) = \inf_{\substack{u(0)=i\\p(0)=\pi}} J(u(\cdot)) \tag{3.92}$$

However by a probabilistic argument already used in section 3.3, any solution of (3.10) is smalller than the right hand side of (3.92). This completes the proof of Theorem 3.1, and also provides the same statement as in Theorem 3.2, without the assumption (3.11) and for our original Ψ given by (3.8).

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