A NOZZLE FLOW FOR A COMPRESSIBLE FLUID

by
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Thesis submitted to the Faculty of the Graduate School of the University of Maryland in partial fulfillment of the requirements for the degree of Doctor of Philosophy

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ACKNOWLEDGMENT

I wish to express here my sincere thanks to Professor M.H. Martin for his patient and skillful guidance in the preparation of this thesis.

William R. Thickstun, Jr.

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1. Introduction. As was pointed out by Chaplygin [5] and Frankl [6] , the partial differential equation for the stream function ψ in steady plane flow of a compressible fluid takes the simple form

$$K(\sigma)\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \quad , \tag{1.1}$$

in the modified hodograph plane. Here θ is the angle of inclination of the streamline $\psi=$ const. and σ is defined in terms of the speed q by

$$\sigma = \int_q^{q_*} \frac{\rho}{q}\, dq \,, \quad \rho = \rho(q) \quad , \tag{1.2}$$

the function $\rho(q)$ for the density ρ, being determined by the equation of state.

If one seeks solutions of the form $\theta = \theta(\sigma, \psi)$, which defines ψ implicitly as a function of θ and σ, one finds that (1.1) is replaced [13] by

$$\theta_{\sigma\sigma} = \left(\frac{K(\sigma) + \theta_\sigma^2}{\theta_\psi}\right)_\psi \quad . \tag{1.3}$$

Moreover given a solution $\theta = \theta(\sigma, \psi)$ of (1.3), the flow in the physical plane (z -plane) is presented by

$$z = \int \varkappa e^{i\theta} \left\{ \frac{K + \theta_\sigma^2}{\theta_\psi}\, d\sigma + \left(\theta_\sigma + i\frac{\varkappa'}{\varkappa}\right) d\psi \right\} \quad , \tag{1.4}$$

where $\varkappa = \varkappa(\sigma)$ is any solution of

$$\varkappa'' = K(\sigma)\varkappa \,, \qquad ' = d/d\sigma \quad . \tag{1.5}$$

The function $K(\sigma)$ which arises in (1.1) is determined by the equation of state of the particular gas under consideration. Conversely following[1] [13] , given any function $K(\sigma)$, an equation of state is determined

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[1] See also [1], [2], [3], [11], [14], [16],

(not uniquely) as follows. For each solution $\varkappa = A\varkappa_1 + B\varkappa_2$ of (1.5), the density ρ, pressure p, and speed q, are defined as functions of σ by

(1.6) $\rho = \varkappa/\varkappa'$, $p = \int_0^\sigma \varkappa^{-2} d\sigma + C$, $q = 1/\varkappa$,

and the equation of state results when σ is eliminated from the first two equations above. Due to the presence of the arbitrary constants A,B,C, each function $K(\sigma)$ gives rise to a three parameter family of equations of state. By differentiating the first equation in (1.6) we see that ρ is a solution of the Ricatti equation

(1.7) $\rho' = 1 - K\rho^2$.

Conversely given any solution $\rho = \rho(\sigma)$ of (1.7), for which $\rho(0) \neq 0$, a solution to (1.5) is

(1.8) $\varkappa = \varkappa_* \exp\left(\int_0^\sigma \rho^{-1} d\sigma\right)$, $\varkappa_* = \varkappa(0)$.

In § 2 we study the equations of state for selected bilinear functions

$$K(\sigma) = \frac{a\sigma + b}{c\sigma + d}$$

to which we were led by the following question. Is there a solution of the form

(1.9) $\theta = \Psi_0(\psi) + \Psi_1(\psi)\sigma$

to (1.3)? In other words is there any flow in the physical plane for which the streamlines, in the modified hodograph plane, constitute a one-parameter family of non-parallel straight lines? In § 3 we show that for a flow of this nature to exist it is necessary that K be a bilinear function of σ.

In particular for the case $b=0$, $a=c=1$, $d=\ell$, equation (1.9) becomes

$$\theta = \sigma \tan\psi + \ell(\tan\psi - \psi) ,$$

and the straight lines $\psi =$ const. envelope a characteristic cycloid of (1.1) [Fig. 6] as indeed is the case in the work of Germain.[2] Using (1.4) the flow in the physical plane to which the above solution gives rise is discussed in detail in § 4, and is portrayed in Fig. 8a.

2. The Equation of State. For a polytropic gas ($p = k\rho^{\gamma}$) in the notation of Chaplygin [3], the function $\rho(q)$ becomes

(2.1) $\rho = \rho_0 (1-\tau)^{\beta}$, $\beta = 1/(\gamma - 1)$, $\tau = q^2/\hat{q}^2$,

where ρ_0 is the stagnation density, $\hat{q}$ the maximum speed, and γ is the ratio of specific heats. The function $\overline{K}(\sigma)$, defined by

(2.2) $\overline{K} = \dfrac{1-(2\beta+1)\tau}{\rho_0^2 (1-\tau)^{2\beta+1}}$, $\sigma = \rho_0 \displaystyle\int_{\tau}^{\tau_*} \frac{(1-\tau)^{\beta}}{2\tau} d\tau$,

where $\tau_* = q_*^2/\hat{q}^2$, and q_* is the critical speed, admits the power series expansion

$$\overline{K}(\sigma) = \frac{\gamma+1}{\rho_*^3} \sigma + \cdots , \quad \rho_* = \rho_0 (1-\tau_*)^{\beta} = \text{critical density.}$$

The graph of $K = \overline{K}(\sigma)$ is designated by $\overline{K}$ in Fig. 1a.

The simplest approximation to $\overline{K}$, the horizontal asymptote $K = 1/\rho_0^2$, arises when $\gamma = -1$ and the corresponding equations of state are those of the Kàrmàn-Tsien

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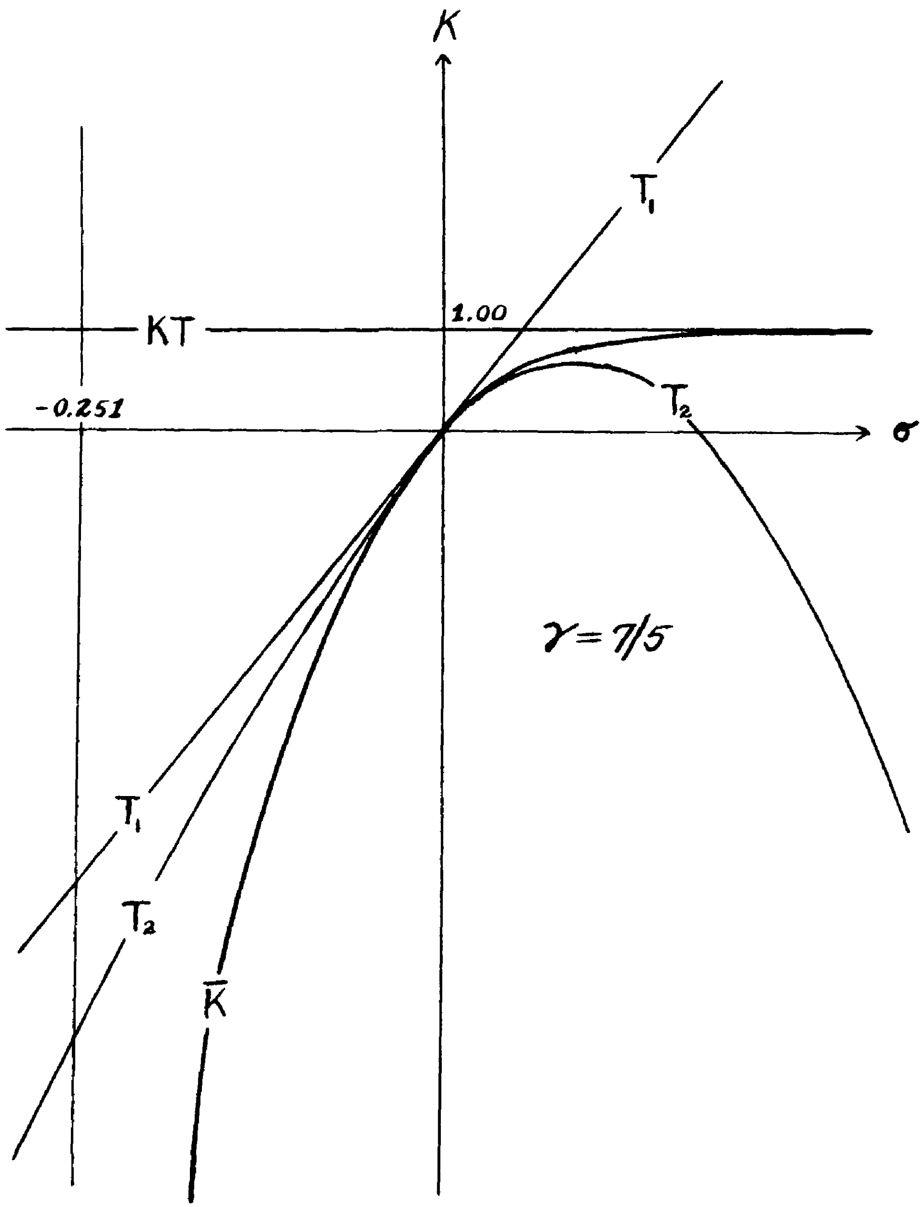
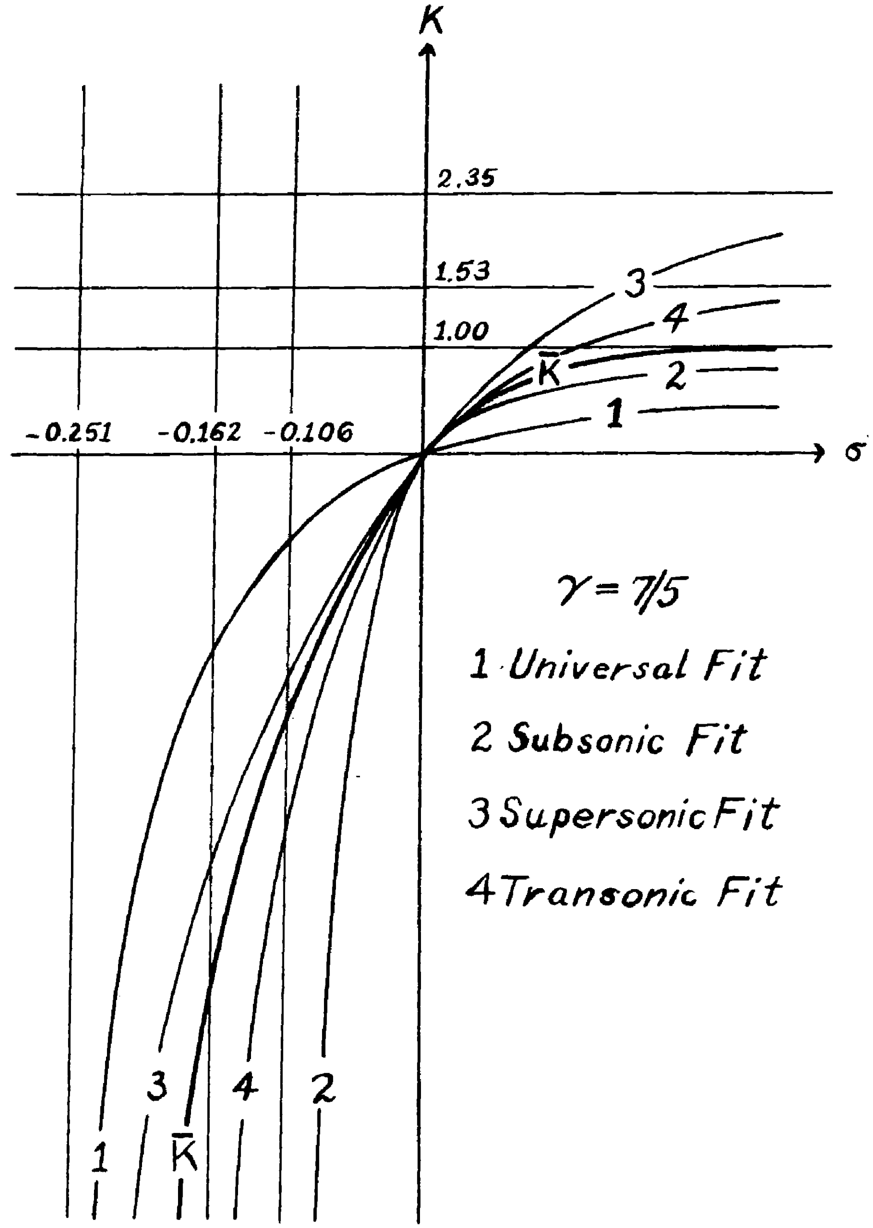
[2] (Unpublished) as stated by M.J.Lighthill in a recent visit to the Institute for Fluid Dynamics and Applied Mathematics at the University of Maryland.



K
T1
1.00
KT
-0.251
T2
σ
γ=7/5
T1
T2
K̄

FIG.1a



K
2.35
1.53
1.00
3
4
K̄
2
1
-0.251
-0.162
-0.106
σ
γ=7/5
1 Universal Fit
2 Subsonic Fit
3 Supersonic Fit
4 Transonic Fit
3
4
2
1
K̄

FIG.1b

gases.[3] This horizontal asymptote is labeled KT in Fig. 1a.

The curve $\bar{K}$ is approximated by the tangent line, $K = \sigma$, at the origin [T_1 in Fig. 1a] if one sets $\rho_*^3 = \gamma + 1$. Equation (1.1) becomes Tricomi's equation and the three-parameter family of equations of state constitute the Tricomi gases. By adjusting the values of the arbitrary constants A,B,C, it is possible [13] to bring the speed, density, and pressure of a Tricomi gas at $\sigma = 0$ into agreement with the sonic values of these quantities for a polytropic gas and to show that the graphs of the equations of states of the two gases have contact of second order [13] at the critical point(ρ_*, p_*).

Recently Tierney [15] has replaced the tangent line to $\bar{K}$ by the osculating parabola [T_2 in Fig. 1a] and finds contact of the third order between the two curves representing the equation of state at the sonic point.

(i) The Function K(σ). Good qualitative agreement between K and $\bar{K}$ over the whole range of speeds is obtained by using the hyperbola,

$$K = \frac{k\sigma}{\sigma + l} \tag{2.3}$$

as an approximation to $\bar{K}$. If one takes $k = 1/\rho_0^2$, $l = -\hat{\sigma} > 0$,

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[3] This is readily seen from (1.5) and (1.6) by carrying out the prescribed operations. Thus for $\rho_0 = 1$, $K(\sigma) \equiv 1$, (1.5) gives

$$x = a\cosh(\sigma - b), \text{ whence } \rho = \coth(\sigma - b),\ p = C + a^{-2}[\tanh(\sigma - b) - \tanh b].$$

On eliminating σ, one has the form $p = c + 1/a^2\rho$, which is the equation of state for the Karman-Tsien gases.

where $\hat{\sigma}$ is the value of σ in (2.2) for $\tau = 1$; the hyperbola will have the same asymptotes as $\bar{K}$. We shall refer to this as the <u>universal fit</u>. [Fig. 1b] The slope at the origin is $K'(0)=-1/\rho_0\hat{\sigma}$, instead of $\bar{K}'(0)=(\gamma+1)/\rho_*^3$ and when $\gamma = 7/5$ the former is less than the latter.[4] Alternately we may, in order to have better transonic agreement, take $k/\ell=(\gamma+1)/\rho_*^3$ to insure that K is tangent to $\bar{K}$ at the origin. Two cases arise according to whether one takes $k=1/\rho_0^2$, to obtain the same horizontal asymptote as $\bar{K}$ (<u>subsonic fit</u>) [Fig. 1b] ; or whether one takes $\ell=-\hat{\sigma}$ to obtain the same vertical asymptote (<u>supersonic fit</u>) [Fig. 1b] . One could also of course give up agreement with either asymptote and choose K to have the same curvature as $\bar{K}$ at the origin (<u>transonic fit</u>) [Fig. 1b] The last three approximations are compared with K in a recent report of Chang [2] .

To form some idea of the magnitudes of k and ℓ corresponding to the various fits let us take $\rho_0=1, \gamma=7/5$, then[4] from (2.2), $\hat{\sigma}=-0.2513$, $\rho_*=0.6339$, whence

Universal fit:	$k=1$,	$\ell=0.2513$
Subsonic fit:	$k=1$,	$\ell=0.1060$
Supersonic fit:	$k=2.393$,	$\ell=0.2513$
Transonic:	$k=1.530$,	$\ell=0.1624$

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4 For $\gamma=7/5$, (2.2) can be integrated [2] obtaining $\sigma/\rho_0=-0.2513-(1-\tau)^{1/2}\left[1+\frac{1}{3}(1-\tau)+\frac{1}{5}(1-\tau)^2\right]+\tanh^{-1}(1-\tau)^{1/2}$; whence for $\tau=1$, $\sigma=\hat{\sigma}=-0.2513$. Thus for $\rho_0=1$, one has $K'(0)=3.98$, while $K'(0)=9.42$.

ii) The Functions $\mathcal{H}(\sigma)$, $\rho(\sigma)$. From now on we shall take $\rho_0 = 1$, and concentrate on the universal fit. Consequently $k=1$ in (2.3) which with (1.5) becomes

$$K = \frac{\sigma}{\sigma+\ell} \quad , \quad \mathcal{H}'' = \frac{\sigma}{\sigma+\ell}\mathcal{H} \quad , \qquad 0<\ell<1 \; . \tag{2.4}$$

For $\mathcal{H}>0$ the solution curves of (2.4) are concave upward for $\sigma>0$ and concave downward for $-\ell<\sigma<0$. In particular this holds for the solution curve, $\mathcal{H}=\mathcal{H}(\sigma)$, determined by the initial conditions

$$\mathcal{H}(0) = 1/q_* = \mathcal{H}_* \quad , \quad \mathcal{H}'(0) = 1/\rho_* q_* = \mathcal{H}'_* \tag{2.5}$$

We remark in passing that the solutions of (2.4) are expressible by Whittaker's [2], [17] functions $W_{\frac{\ell}{2},\frac{1}{2}}(2\sigma+2\ell)$. We shall need only the property[5]

$$\lim_{\sigma\to\infty} \frac{\sigma+\ell}{\mathcal{H}(\sigma)} = 0 \; . \tag{2.6}$$

To study the particular solution with prescribed initial values (2.5) we begin with (1.7) and interchange the role of dependent and independent variables to obtain

$$\frac{d\sigma}{d\rho} = \frac{\sigma+\ell}{\ell+(1-\rho^2)\sigma} \quad . \tag{2.7}$$

The straight line $\sigma=-\ell$ is obviously a solution, and we confine our attention to the quarter plane $\rho \geqq 0$, $\sigma \geqq -\ell$. This quarter plane is divided into two regions by the vertical isocline $\ell+(1-\rho^2)\sigma=0$, below which the solutions $\sigma(\rho)$ are monotonic increasing functions of σ and above which they are monotonic decreasing functions.

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[5] See p-343 in [17].

On the latter curve, $d\sigma/d\rho = \infty$. On the boundary $\rho = 0$; $(\sigma \neq -\ell)$ we have $d\sigma/d\rho = 1$, while the point $(0,-\ell)$ itself is a singular point of the differential equation.

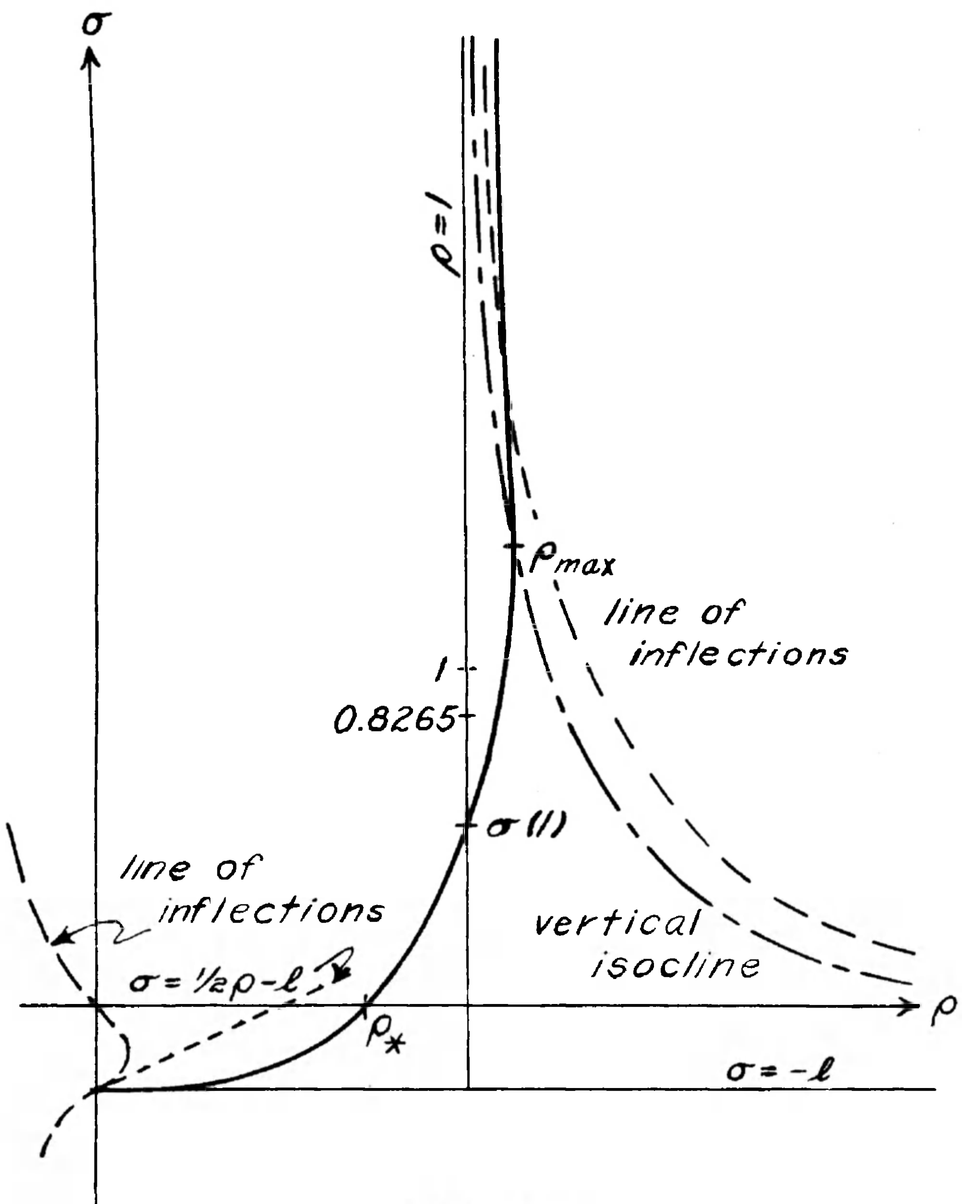
Each solution curve crosses the segment $0 < \rho \leqq 2\ell$ of the line $\sigma = \rho/2 - \ell$ with slope $d\sigma/d\rho = [1+\rho(2\ell-\rho)]^{-1} > 1/2$, i.e. with slope greater than that of the line itself. Thus every solution curve starting on the ρ-axis with $\rho > 2\ell$ when followed back to the left, therefore, can not cross the above segment of the line $\sigma = \rho/2 - \ell$; but neither can it cross the boundary line $\sigma = -\ell$ $(\rho > 0)$, since this is itself a solution curve. Hence necessarily any solution curve with initial values $\sigma = 0$, $\rho = \rho_* > 2\ell$ must pass through the singular point $(0,-\ell)$. In particular this applies when $\gamma = 7/5$, since $\rho_* > 2\ell$ follows from the numerical results in § 21.

To study the nature of the solution curves in the neighborhood of the singular point, we observe that for the slopes of the solution curves on a parabola $\sigma + \ell = \alpha\rho^2$, $(\alpha > 0)$, we have, for sufficiently small $\rho > 0$,

$$\frac{d\sigma}{d\rho} = \frac{\alpha}{\alpha(1-\rho^2)+\ell} > \frac{\alpha}{\alpha+\ell} > 2\alpha\rho .$$

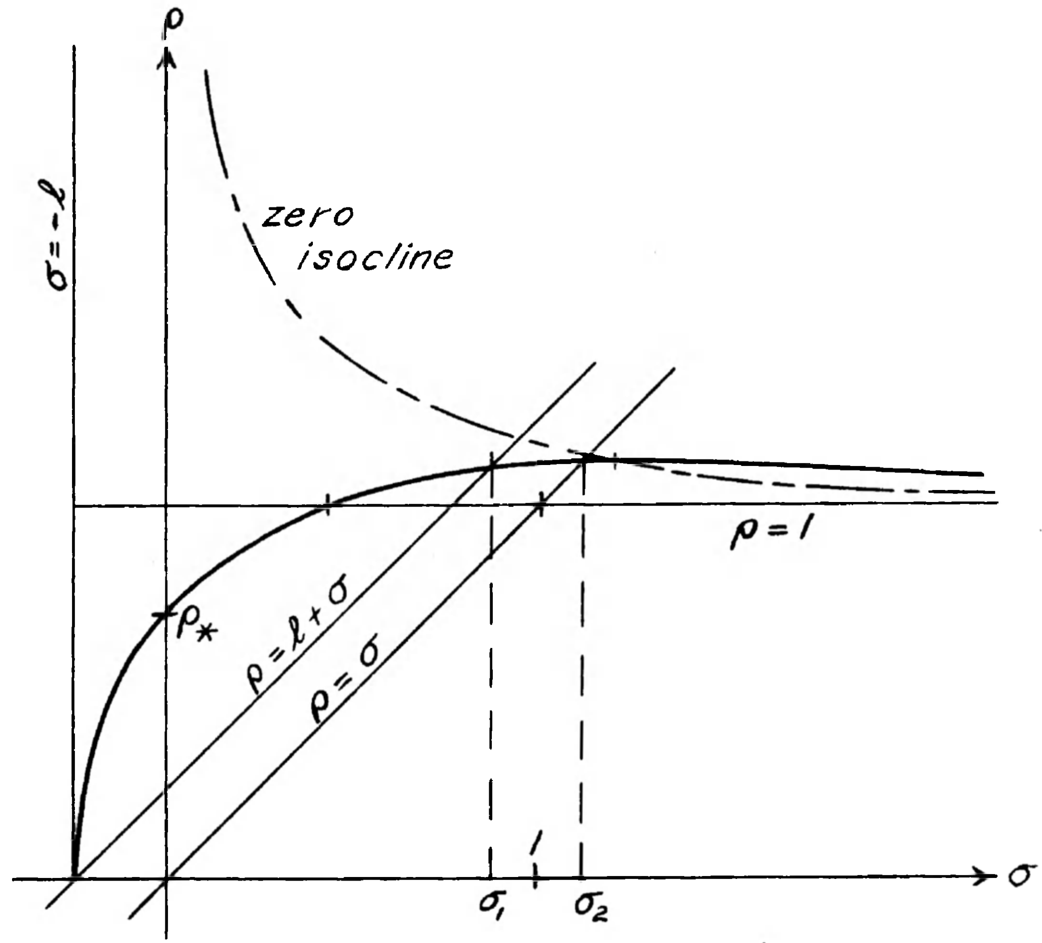
Thus, sufficiently near the singular point. for every solution curve $\sigma = \sigma(\rho)$ which meets the parabola, we must have since the slope of the parabola is $2\alpha\rho$, that $0 < \sigma(\rho) + \ell < \alpha\rho^2$, i.e. $\sigma(\rho) + \ell = O(\rho^2)$.

For the particular solution $\sigma = \sigma(\rho)$, $\sigma(\rho_*) = 0$, in which we are interested, the solution curve starts at the singular point and rises monotonely as ρ increases from 0 to ρ_*.



σ
ρ=1
ρmax
line of inflections
1
0.8265
σ(1)
line of inflections
σ = ½ρ-ℓ
vertical isocline
ρ
ρ*
σ = -ℓ

FIG. 2a



ρ
σ = -ℓ
zero isocline
ρ=1
ρ*
ρ=ℓ+σ
ρ=σ
σ
σ1
1
σ2

FIG. 2b

Thereafter $\sigma(\rho) > 0$ and

(2.8) $\quad 0 < d\sigma/d\rho \leqq (\sigma+\ell)/\ell \;, \quad (\rho_* < \rho \leqq 1).$

Hence the solution curve must cross the line $\rho = 1$ for some finite σ and continue rising until it crosses the vertical isocline; after which $d\sigma/d\rho$ becomes negative, so the value of ρ for which this occurs is a maximum [Fig. 2a] . Thereafter as ρ decreases, σ increases indefinitely.

To follow the solution curve further we return to (1.7). Denoting the value of σ for which the maximum ρ occurs by σ_m, one has for $\sigma > \sigma_m$, that ρ decreases monotonically to $\rho_0 \geqq 1$, [see Fig. 2b, in which the vertical isocline of Fig.2a is now the zero isocline] , since $\rho > (1+\ell/\sigma)^{1/2} > 1$ for $\sigma > \sigma_m$. If this were not true there would be a first value $\bar{\sigma} > \sigma_m$ for which $\rho = (1+\ell/\bar{\sigma})^{1/2}$; but at such a point $\rho'(\bar{\sigma}) = 0$. Hence $\rho \leqq (1+\ell/\sigma)^{1/2}$ in the neighborhood to the left of $\bar{\sigma}$, contradicting the assumption that $\bar{\sigma}$ is the first value after σ_m for which $\rho = (1+\ell/\sigma)^{1/2}$. Thus $\rho(\sigma) > 1$, monotone decreasing for $\sigma > \sigma_m$, must approach a limit $\rho_0 \geqq 1$, as σ tends to infinity. This limit must be unity since $\rho^2 \geqq 1+\nu$, $(\nu > 0)$ implies $\rho' \leqq 1-\sigma(1+\nu)/(\sigma+\ell)$, which inturn implies for any number μ, $0 < \mu < \nu$, that there is a value of σ beyond which $\rho' \leqq -\mu$, and ρ would thus eventually become less than one, contrary to our previous statement.

We now return to the discussion of the function $\varkappa(\sigma)$ determined by (2.5). From (1.8), since $\rho > 0$ for $\sigma > -\ell$, the function $\varkappa(\sigma)$ is monotone increasing. Moreover since we have

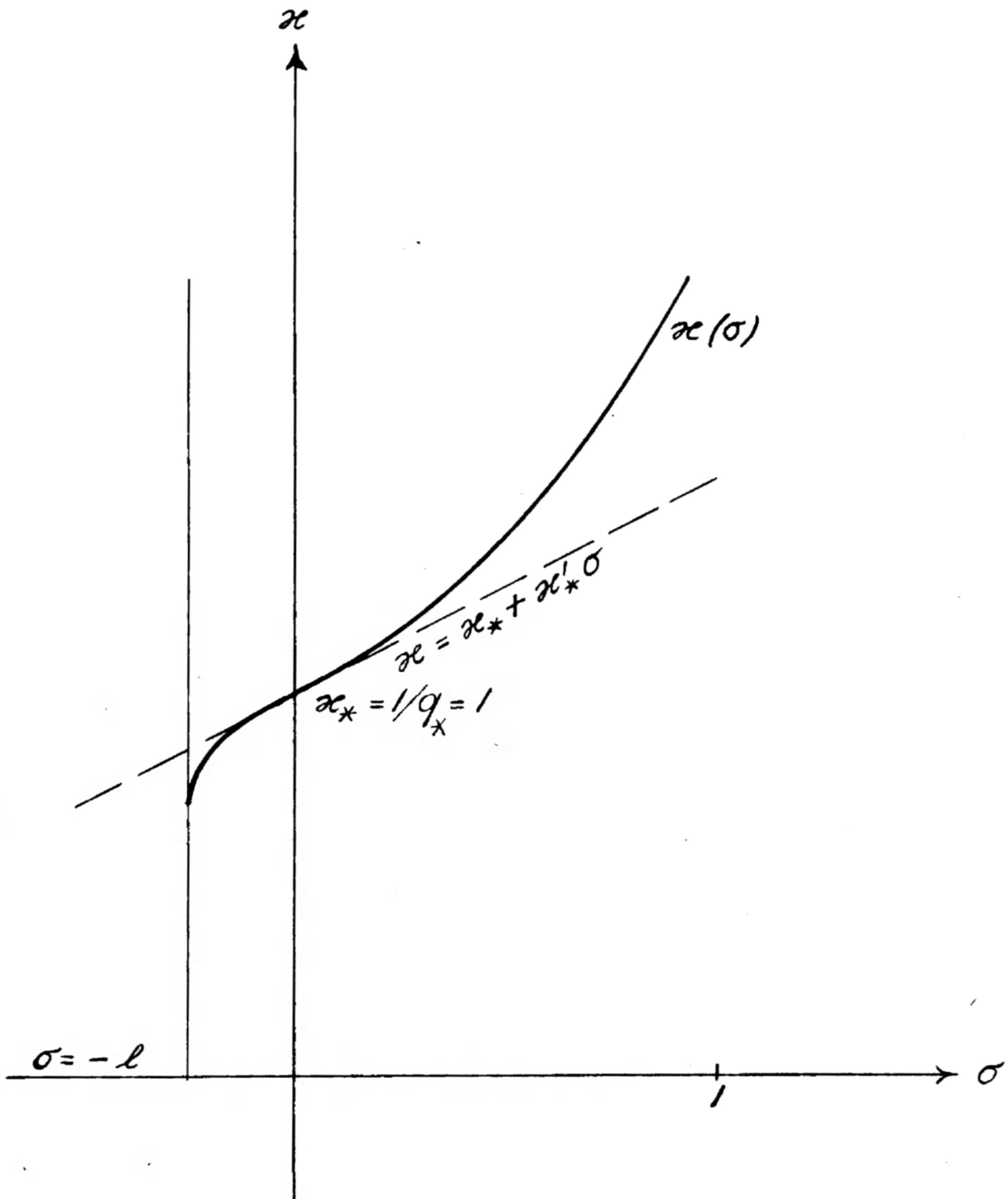

$\varkappa$
$\varkappa(\sigma)$
$\varkappa = \varkappa_* + \varkappa'_* \sigma$
$\varkappa_* = 1/q_* = 1$
$\sigma = -\ell$
1
σ

FIG. 3

seen above that $\sigma+\ell=O(\rho^2)$ as $\rho\to 0$, it follows that $\rho^{-1}=O[(\sigma+\ell)^{-1/2}]$ as $\sigma\to-\ell$. Consequently the integral $\int_0^\sigma \rho^{-1}d\sigma$ converges as $\sigma\to-\ell$. Thus $\varkappa(\sigma)$ approaches a finite positive number $\hat{\varkappa}$ as $\sigma\to-\ell$, and the graph of $\varkappa=\varkappa(\sigma)$ is shown in Fig. 3, with

(2.9) $\varkappa > \varkappa_* + \varkappa_*' \sigma$

holding for $\sigma>0$.

<u>iii The Function $p(\sigma)$.</u> We take $C=p_*$ in (1.6) and observe that $p(\sigma)$ is a monotone increasing function of σ which tends to a finite limit $\hat{p} \lesseqqgtr 0$ as $\sigma\to-\ell$, and to a finite limit $p_0>0$ as $\sigma\to+\infty$ in view of (2.9). The graph of the function $p(\sigma)$ is shown in Fig. 4 under the assumption that $\hat{p}>0$, whether this is actually true or whether one of the other possibilities $\hat{p}\leqq 0$ actually occurs is a question which our considerations leave unsettled.

The graph of the equation of state $p=p(\rho)$ itself is found by regarding the functions $\rho=\rho(\sigma)$ and $p=p(\sigma)$ as defining $p=p(\rho)$ parametrically. If we compare the graphs of these functions in Fig. 2b and Fig. 4, it is easy to see that the graph of $p=p(\rho)$ takes roughly the form[6] shown in Fig. 5.

<u>3. The Direction Function</u> $\theta(\sigma,\varphi)$. Integrating both sides of (1.3) with respect to φ one has

(3.1) $\theta_\varphi\left[\int\theta_{\sigma\sigma}\,d\varphi+\Sigma(\sigma)\right]=K(\sigma)+\theta_\sigma^2$,

where $\Sigma(\sigma)$ is an arbitrary function. Now if we seek solutions of the form (1.9) we find on substituting in (3.1)

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6 $d^2p/d\rho^2$ evaluated at $\rho=\rho_*$ is negative for the universal fit only.

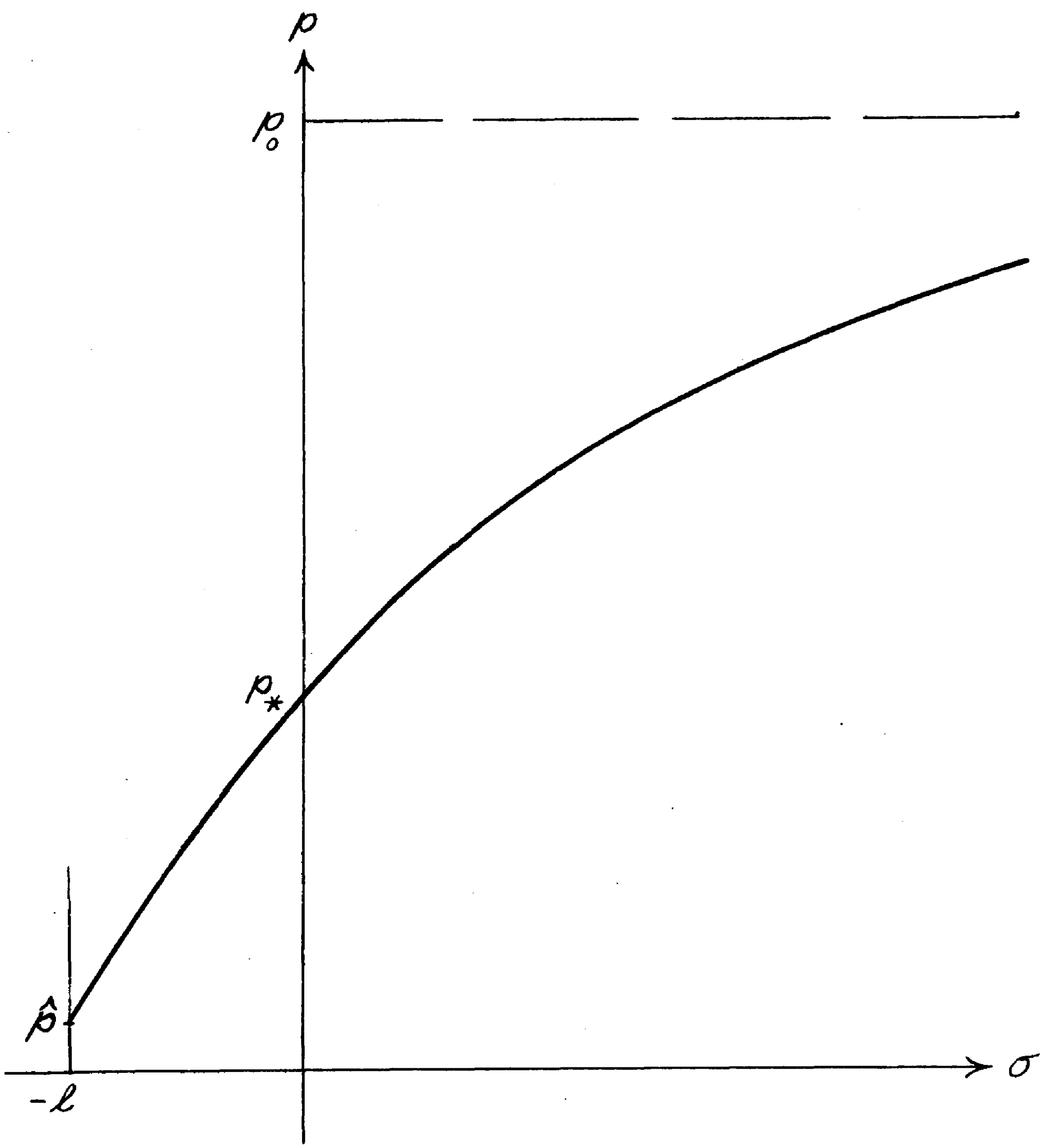

FIG. 4

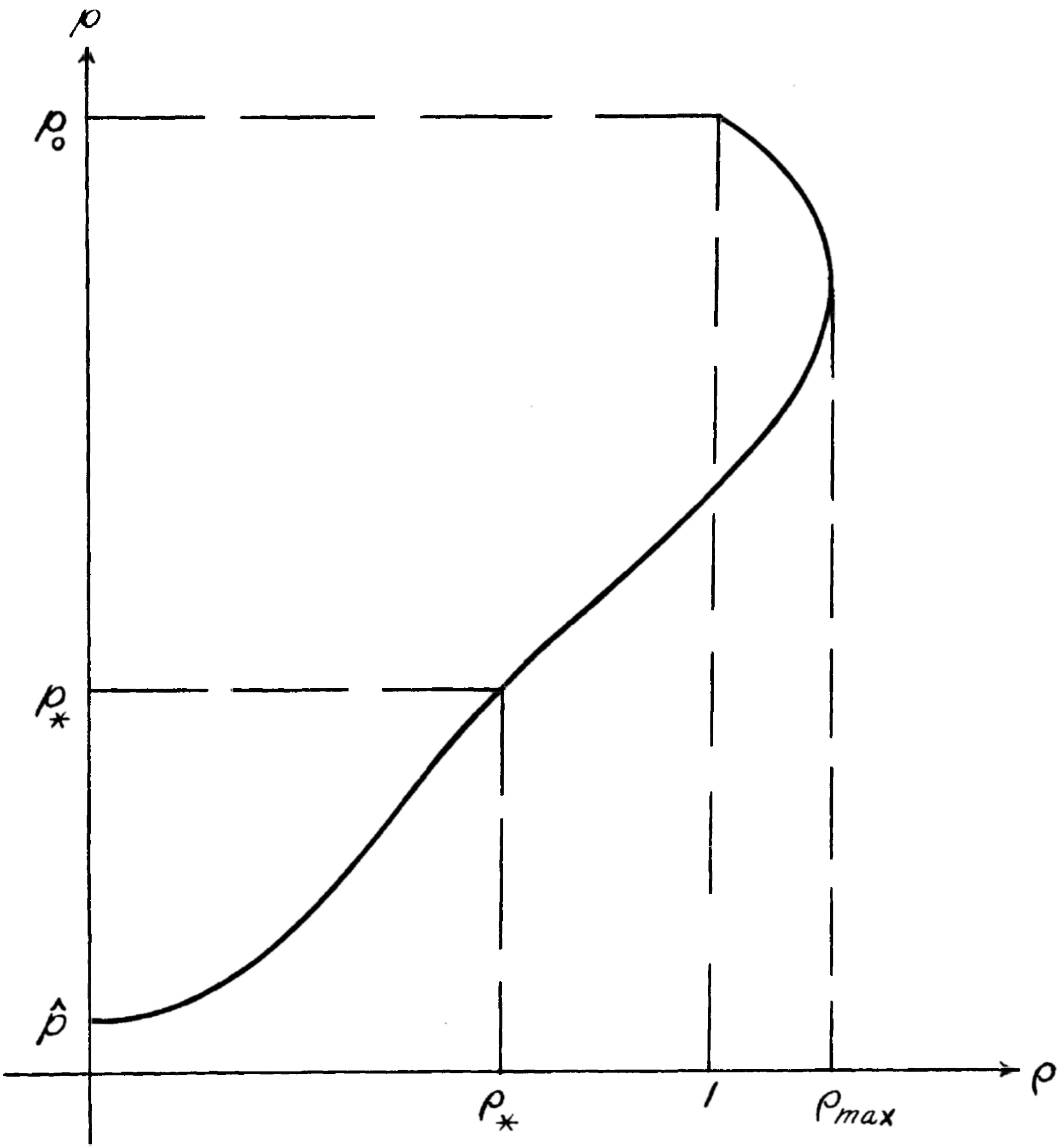

FIG. 5

that the identity

(3.2) $\quad \Sigma\,\Psi_0' + \sigma\Sigma\,\Psi_1' - K - \Psi_1^2 \equiv 0$

must hold. This identity has the form

(3.3) $\quad \sum_{i=1}^{i=4} F_i(\sigma)\, G_i(\psi) \equiv 0$

where $F_1=\Sigma$, $F_2=\sigma\Sigma$, $F_3=K$, $F_4=-1$; $G_1=\Psi_0'$, $G_2=\Psi_1'$, $G_3=-1$, $G_4=\Psi_1^2$

For an identity of this form to hold for the manifolds

$M:\ F_i=F_i(\sigma),\ N:\ G_i=G_i(\psi),\ (i=1,2,3,4),$

it is, according to a lemma of Martin [12] ,necessary and sufficient that $M\subset S_m$, $N\subset T_n$, where S_m, T_n are linear orthogonal subspaces defined by

$S_m:\ \sum_{\alpha=1}^{4} b_{k\alpha}F_\alpha=0,\ k=1,\dots,m;\ T_n:\ \sum_{\alpha=1}^{4} a_{i\alpha}G_\alpha=0,\ i=1,\dots,n;\ m+n=4,$

the matrices $A=\|a_{i\alpha}\|$, $B=\|b_{k\alpha}\|$ having ranks m,n respectively. The linear subspaces S_m , T_n are orthogonal if and only if the composite matrix C formed by taking the m rows of A followed by the n rows of B has the following property. The m-rowed minors in A are all proportional to their complimentary minors in B, the indices of the columns of A followed by the indices of the columns of B forming an even permutation of 1,2,3,4.

Applying the lemma to (3.2) the conditions on M,N are

(3.4) $$\begin{aligned} &b_{k1}\Sigma + b_{k2}\sigma\Sigma + b_{k3}K - b_{k4} = 0\ , \quad k=1,\dots,m \\ &a_{i1}\Psi_0' + a_{i2}\Psi_1' - a_{i3} + a_{i4}\Psi_1^2 = 0\ , \quad i=1,\dots,n\ . \end{aligned}$$

Now we notice that although (3.2) is of the form (3.3), we actually do not have four independent functions of σ and four independent functions of ψ , but only two independent functions (Σ ,K) of σ and two independent functions (Ψ_0, Ψ_1) of ψ . Thus one sees from (3.4) for m=4, n=0, and m=0, n=4, that the relations are inconsistent.

For the case m = 3, n = 1, the first set of equations (3.4) consists of three non-homogeneous linear equations for the three functions Σ , $\sigma\Sigma$, K. Hence Σ , $\sigma\Sigma$, and K must be constant. This can occur only if $\Sigma \equiv 0$. But then the identity (3.2) becomes $K + \Psi_1^{\,2} \equiv 0$. Thus one must have $\Psi_1 = -c_1$ = real const., $K = -c_1^2$. Whence $\theta(\sigma,\psi)$ becomes $\theta = \Psi_0(\psi) - c_1\sigma$ with $\Psi_0(\psi)$ an arbitrary function. This solution is seen to correspond to $\psi = f(\theta + c_1\sigma)$, the solution of (1.1) for $K = -c_1^2$, which is indeed the most general solution with constant K for which the ψ = const. curves are straight lines in the (θ,σ) -plane. They are seen to be a family of parallel straight lines.

Next we consider the case m = 1, n = 3, which gives three non-homogeneous linear equations for the three functions Ψ_0', Ψ_1', Ψ_1, which must therefore be constant; and indeed $\Psi_1' \equiv 0$.

Writing $\Psi_0' = c_0$, $\Psi_1 = c_1$, one then has

$$\theta = c_0(\psi - \psi_0) + c_1\sigma \ , \quad c_0 \neq 0$$

or alternatively,

(3.5) $\psi - \psi_0 = a\theta + b\sigma$, a, b = const.

But for this case the identity (3.2) reduces to

$$c_0\Sigma - K - c_1^2 \equiv 0 \ .$$

Thus the function $K(\sigma)$ is arbitrary, confirming the obvious fact that (1.1) always has a linear function for solution. As in the previous case the streamlines in the (σ, ψ)-plane constitute a family of parallel straight lines. It is to be observed that (3.5) is combined radial and circular flow described, for example, in Courant-Friedrichs [4, p.253] .

In the remaining case $m=2$, $n=2$, (3.4) may be written

$$(3.6)\quad a_{11}\Psi_0' + a_{12}\Psi_1' = a_{13} - a_{14}\Psi_1^2 \qquad (b_{11}+b_{12}\sigma)\Sigma + b_{13}K = b_{14}$$
$$a_{21}\Psi_0' + a_{22}\Psi_1' = a_{23} - a_{24}\Psi_1^2 \qquad (b_{21}+b_{22}\sigma)\Sigma + b_{23}K = b_{24}\ .$$

Introducing the minors

$$A_{ik} = \begin{vmatrix} a_{1i} & a_{1k} \\ a_{2i} & a_{2k} \end{vmatrix} \quad , \quad B_{ik} = \begin{vmatrix} b_{1i} & b_{1k} \\ b_{2i} & b_{2k} \end{vmatrix} \quad ,$$

and solving (3.6) one obtains

$$(3.7)\quad B_{34}\Psi_0' = -B_{14} - B_{13}\Psi_1^2 \quad , \quad \Sigma = B_{43}(B_{13}+B_{23}\sigma)^{-1}$$
$$B_{34}\Psi_1' = B_{42} - B_{23}\Psi_1^2 \qquad K = (B_{14}+B_{24}\sigma)(B_{13}+B_{23}\sigma)^{-1} ,$$

where the $A_{ik}'s$ in the first two equalities have been eliminated by virtue of the relations

$$\frac{A_{12}}{B_{34}} = \frac{A_{13}}{B_{42}} = \frac{A_{14}}{B_{23}} = \frac{A_{23}}{B_{14}} = \frac{A_{42}}{B_{13}} = \frac{A_{34}}{B_{12}} \ .$$

If $B_{34} \neq 0$ the first two equations determine $\Psi_0(\varphi)$, $\Psi_1(\varphi)$. the case $B_{34}=0$ may be disregarded since it implies $B_{13}=B_{23}=B_{14}=B_{24}=0$, which together with $B_{12}\neq 0$ imply $b_{13}=b_{23}=b_{14}=b_{24}=0$, whereupon the second set of equations in (3.6) reduces to $(b_{11}+b_{12}\sigma)\Sigma=0$, $(b_{21}+b_{12}\sigma)\Sigma=0$, to imply $\Sigma\equiv 0$, which has been treated under the case $m=3$, $n=1$ above. We observe, therefore, that the following theorem has been proved.

THEOREM <u>The only flows in the physical plane for which the streamlines in the modified hodograph plane constitute a one-parameter family of non-parallel straight lines are those for which the function $K(\sigma)$ in Chaplygin's equation $K(\sigma)\psi_{\theta\theta}+\psi_{\sigma\sigma}=0$ is a bilinear function</u>

$$K = \frac{a\sigma+b}{c\sigma+d} \ .$$

The special case $K(\sigma)=\sigma$, Tricomi's equation, included in this theorem has been discussed[7] in [13] .

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[7] One obtains the solution noted by Falkovitch [5] namely $\theta=\frac{1}{3}\varphi^3+\sigma\varphi$, which gives a flow similar to that given by (3.9) See Fig. 8b, for comparison.

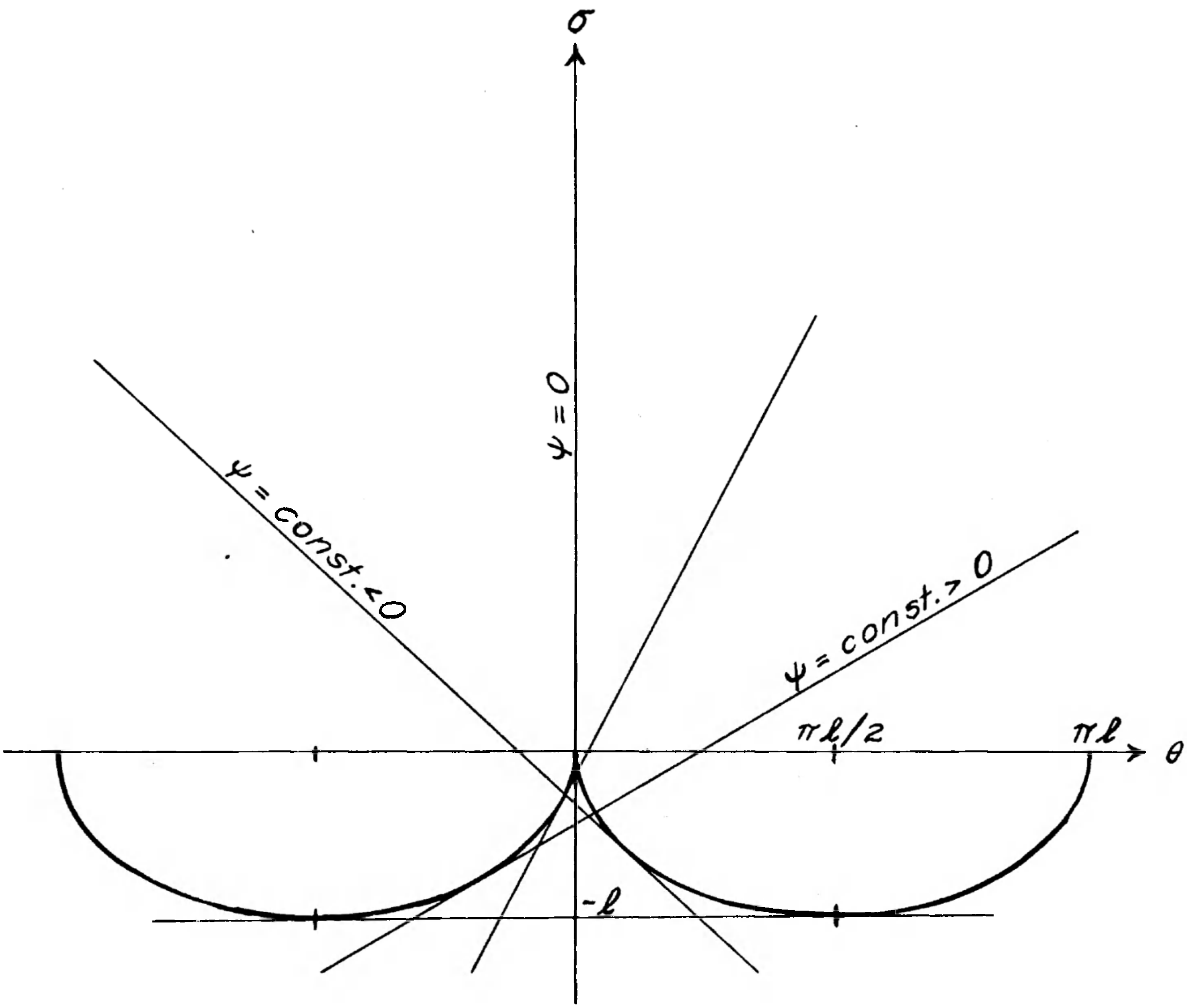


FIG. 6

To bring K into the form (2.4) we set

$$a = B_{24} = 1, \quad b = B_{14} = 0, \quad c = B_{23} = 1, \quad d = B_{13} = \ell$$

so that (1.1) becomes

$$\frac{\sigma}{\sigma+\ell}\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0 \tag{3.8}$$

and equations (3.7) for Ψ_0, Ψ_1 reduce to

$$\Psi_0' = \ell\Psi_1^2, \quad \Psi_1' = 1 + \Psi_1^2$$

provided we set $B_{34} = -1$. When these equations are integrated to determine Ψ_0, Ψ_1, substitution in (1.9) yields the solution [See Fig.6]

$$\theta = \sigma\tan\varphi + \ell(\tan\varphi - \varphi)\,. \tag{3.9}$$

4. The Flow in the Physical Plane. For the solution (3.9) the mapping (1.4) of the (σ, φ)-plane [Fig. 7][8] upon the physical plane reduces to

$$z = \int \varkappa(\sigma)\, e^{i\theta}\left\{\frac{d\sigma}{\sigma+\ell} + \left(\tan\varphi + i/\rho(\sigma)\right)d\varphi\right\}\,. \tag{4.1}$$

This transformation carries the horizontal lines $\varphi =$ const. (vertical lines $\sigma =$ const.) of the (σ, φ)-plane into the streamlines(isovels) in the physical plane.

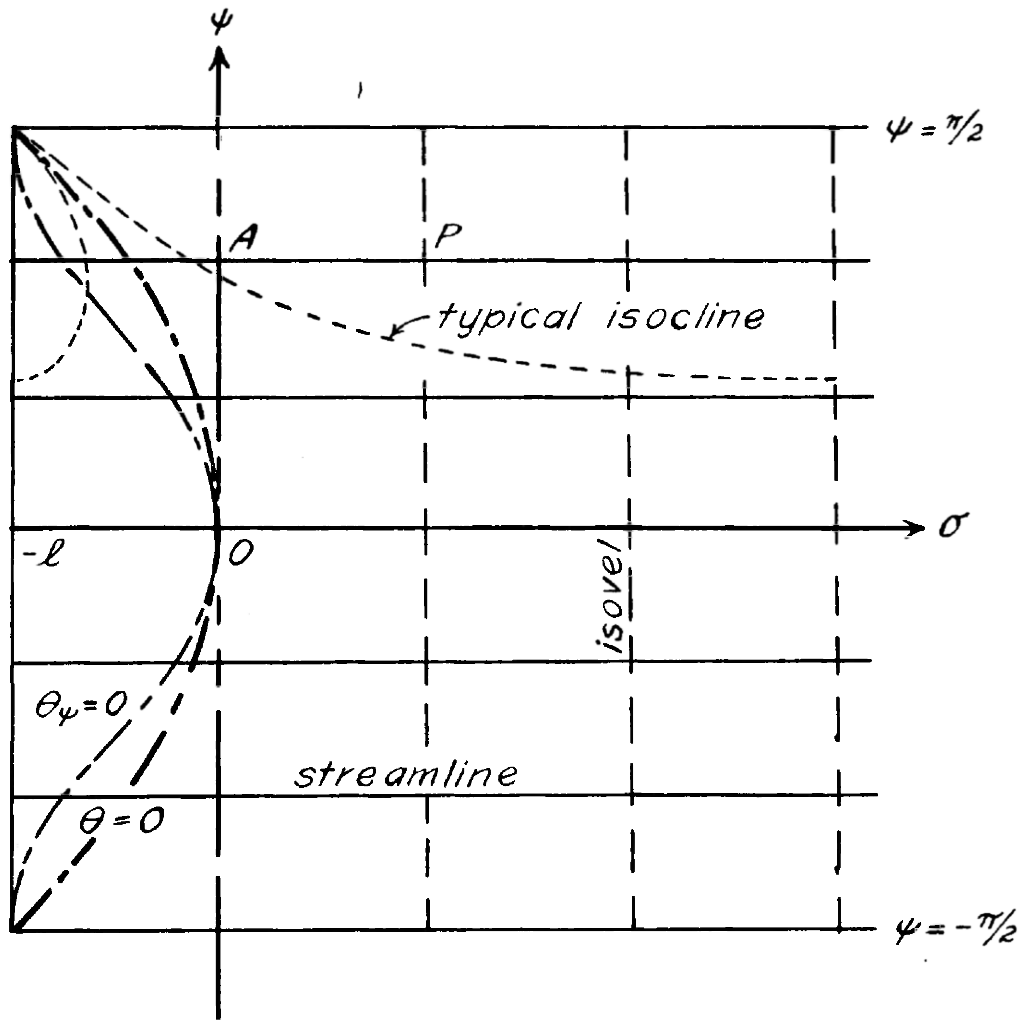
One observes that the Jacobian J of the mapping (4.1) of the (σ, φ)-plane upon the physical plane becomes

$$J = \varkappa\varkappa'(\sigma+\ell)^{-1} \tag{4.2}$$

which is seen to the positive and finite for all $\sigma > -\ell$. Thus no limiting line occurs for any speed $q < \hat{q}$.

---

[8] Notice from (3.9) that the zero isocline ($\theta = 0$) and the line of branch points ($\theta_\varphi = 0$) are given by $\sigma = \ell(\varphi\cot\varphi - 1)$, and $\sigma = -\ell\sin^2\varphi$ respectively.



ψ
$\psi = \pi/2$
A
P
typical isocline
σ
$-\ell$
0
isovel
$\theta_\psi = 0$
streamline
$\theta = 0$
$\psi = -\pi/2$

FIG. 7

For $\psi=0$ one finds from (3.9) that $\theta=0$ and from (4.2) that

$$z = x = \int_0^\sigma \frac{\varkappa(t)}{t+\ell}\,dt$$

associates to each value of σ a single value x. Since $\sigma\to-\ell$ implies $\varkappa(\sigma)\to\hat{\varkappa}>0$, it is clear that $\sigma\to-\ell$ implies $x\to-\infty$. On the other hand for $\sigma>0$ from (2.9), it is clear that $\sigma\to+\infty$ implies $x\to+\infty$. In short as σ increases from $-\ell$, the corresponding value of x increases monotonely from $-\infty$ to $+\infty$. Moreover since q, ρ, p are each known functions of σ the value for each of these quantities is assigned to each point of the x-axis.

A "clue" to the nature of the streamlines follows by considering their inclination $\theta=\theta(\sigma)$ from (3.9) for fixed ψ. To fix the ideas take ψ positive and less than $\pi/2$ (It is clear that the flow is symmetric about the x-axis since θ is an odd function of ψ for every σ.). Each streamline crosses the sonic line ($\sigma=0$) with the positive inclination $\ell(\tan\psi-\psi)$ and as σ increases from $-\ell$ to $+\infty$, the inclination θ increases linearly with σ from $-\ell\psi$ to $+\infty$.

For a more accurate description of the streamlines, we consider their curvature. From (4.1) one has

(4.3) $$s=\int_0^\sigma \frac{\varkappa(t)}{t+\ell}\,dt\,, \qquad K=\frac{\sigma+\ell}{\varkappa}\tan\psi\,,$$

where s denotes the arc length of a streamline measured from the sonic line, and also

$$\frac{dK}{ds}=\frac{\sigma+\ell}{\rho\varkappa^2}(\rho-\sigma-\ell)\tan\psi\,,\quad \sigma>-\ell$$

Referring to Fig.2b, it is clear from the properties of $\rho(\sigma)$ that there is exactly one value σ_1 such that

$$\rho(\sigma)\gtreqless\sigma+\ell \quad \text{as} \quad -\ell<\sigma\lesseqgtr\sigma_1\,.$$

consequently as σ ranges from $-\ell$ to $+\infty$, the curvature $\mathcal{K}$ increases monotonely from zero to a maximum on the isovel $\sigma = \sigma_1$ and thereafter decreases monotonely as $\sigma \to +\infty$. Thus regarding each streamline as consisting of two arcs $\sigma > \sigma_1$, $\sigma < \sigma_1$, one has [e.g.8, Cor.2.5.2] that each arc is simple (i.e. does not cross itself). Again according to Jackson [9, Theor.3.2] the arc $\sigma > \sigma_1$ is an outwinding spiral, whose distance from the origin in any direction is unbounded provided its curvature is strictly monotone decreasing to zero (by virtue of (2.6)), and the rate of change of radius of curvature with respect to inclination eventually exceeds a positive constant. Actually the rate of change of the radius curvature with respect to inclination equals

$$\frac{\varkappa}{\sigma+\ell}\left(\frac{1}{\rho} - \frac{1}{\sigma+\ell}\right)\cot^2\psi$$

and this obviously becomes infinite, from (2.6), as $\sigma \to +\infty$. The arc $\sigma < \sigma_1$ however, as $\sigma \to -\ell$, has monotone decreasing curvature and as the inclination decreases to $\theta = -\ell\psi$ as $\sigma \to -\ell$, can clearly wind outward at most through finitely many turns and then approaches the opposite limiting inclination.(Whether it approaches a linear asymptote is not decided by these simple considerations).

Let us now go on to the detailed discussion of the isovels. From (4.1) one has for the element of arc, s, inclination, φ, and curvature, $\mathcal{K}$:

$$\begin{aligned} s &= \varkappa\int\sqrt{\tan^2\psi + 1/\rho^2}\,d\psi \\ \varphi &= \theta(\sigma,\psi) + \arg(\tan\psi + i/\rho) \qquad (4.4) \\ \mathcal{K} &= \frac{\rho}{\varkappa}(1+\rho^2\tan^2\psi)^{-3/2}\left\{\rho^2(\sigma+\ell)\tan^4\psi + (\rho^2\sigma - \rho + \sigma + \ell)\tan^2\psi + (\sigma-\rho)\right\}. \end{aligned}$$

Notice for the inclination φ, if we introduce

(4.5) $\quad \cot\beta = \rho \tan\psi$,

that $\varphi = \theta + \beta$ and hence β is the angle between the streamline and the isovel through any point. Inserting $\psi = 0$ in (4.4) one has $\varphi = \beta = \pi/2$, thus every isovel is perpendicular to the $\psi = 0$ streamline (i.e. the x-axis). Now β is monotone in ψ and decreases to zero as $\psi \to \pi/2$, while from (3.9) one sees that θ increases monotonely to infinity for sufficiently large ψ. Moreover, for sufficiently large ψ, $\mathcal{K}$ is seen from (4.4) to increase monotonely to infinity. Hence from a sufficiently large value of on, the isovels according to Jackson [9, Theorem 6.1] are seen to be inwinding spirals, which wind in to a limit point as $\psi \to \pi/2$. One finds just as in the Tricomi case [13] and the case considered by Tierney [15] that every isovel winds in to the same limit point, $z_{\pi/2}$. This is an immediate consequence of the formula

$$z = z_* + e^{i\ell(\tan\psi - \psi)} \int_0^{\sigma} \frac{\varkappa}{\sigma + \ell}\, e^{i\sigma \tan\psi}\, d\sigma .$$

Here z_* is the point on the sonic line corresponding to A in Fig. 7, and the formula arises by evaluating the line integral (4.1) along the OAP. If we set $\sigma = \sigma_1 =$ const. and let ψ increase, z traces out an isovel and

$$\lim_{\psi \to \pi/2} \int_0^{\sigma_1} \frac{\varkappa}{\sigma + \ell}\, e^{i\sigma\tan\psi}\, d\sigma = 0\ , \quad (\sigma_1 > -\ell)$$

by Riemann's lemma.

Referring to the curvature formula (4.4)one sees that the initial curvature ($\psi = 0$) of the isovels is $\rho(\sigma - \rho)/\varkappa$. It is clear from the properties of $\rho(\sigma)$ [Fig. 2b] that there is exactly one value $\sigma_2 > \sigma_1$ such that

$$\rho(\sigma) \gtreqless \sigma \quad \text{as} \quad -\ell < \sigma \lesseqgtr \sigma_2 \ .$$

Thus all the isovels $\sigma < \sigma_2$ are concave toward the subsonic flow (including the sonic line $\sigma = 0$), and convex toward the subsonic flow for $\sigma > \sigma_2$.

Finally one observes that the inclination φ will be an extremum for the values of ψ for which $\mathcal{K}$ vanishes, hence for at most two values of ψ, ($0 \leqq \psi < \pi/2$). The isovels $\sigma < \sigma_2$, for which φ initially decreases, must have at least one minimum of inclination in order for φ to eventually increase to infinity, hence this minimum must be the only extremum. For an isovel $\sigma \geqq \sigma_2$, whose inclination first increases, we observe that if the form for $\mathcal{K}$ is positive definite, φ must increase monotonely to infinity. Clearly $\rho^2(\sigma + \ell)$ and $(\sigma - \rho)$ are positive. To show $(\rho^2\sigma - \rho + \sigma + \ell) > 0$ for $\sigma \geqq \sigma_2$, and hence that $\mathcal{K}$ is positive definite in ψ, it is sufficient to show that $\sigma_2 > 1$ and $\rho(\sigma_2) > 1$. Referring to (2.8) one sees that $\sigma(\rho) \leqq u(\rho)$ where $u = u(\rho)$ is defined by

$$\frac{du}{d\rho} = \frac{u + \ell}{\ell}, \quad u(\rho_*) = 0;$$

which has the solution

$$u = \ell\left(e^{(\rho - \rho_*)/\ell} - 1\right).$$

From this formula using the values $\rho_* = 0.6339$, $\ell = 0.2513$, one has $u(1) = 0.8265 < 1$. Hence $\sigma(1) < 1$. Thus in Fig. 2b, the line $\rho = \sigma$ must meet the curve $\rho = \rho(\sigma)$ at a point for which $\rho > 1$, i.e. $\rho(\sigma_2) > 1$. But this implies also that $\sigma_2 > 1$. Thus the isovels $\sigma > \sigma_2$ have no inflections. The features established by these results are shown in Fig. 8a, in which the flow is shown out to an arbitrary bounding streamline $\psi < \pi/2$.

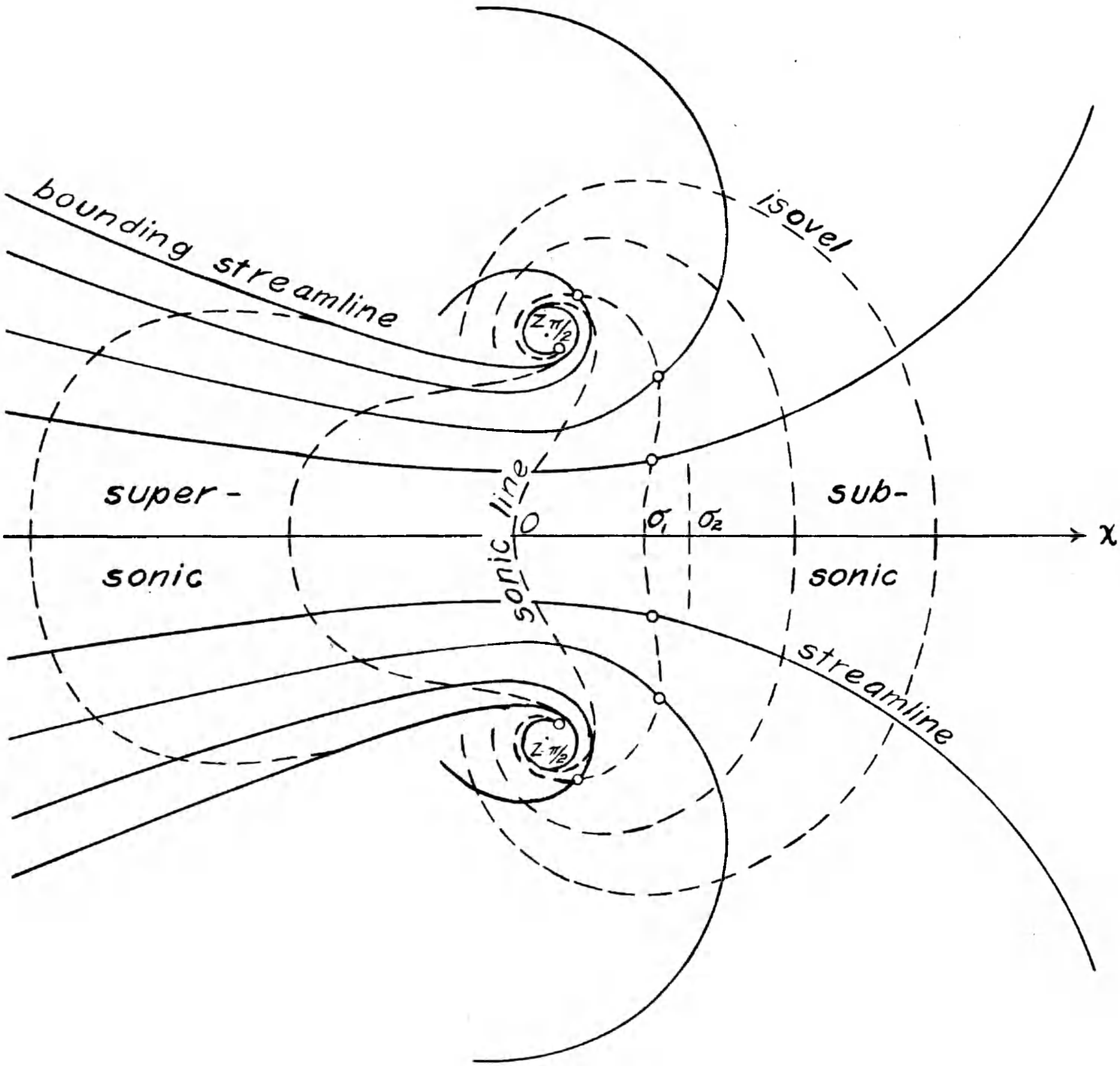
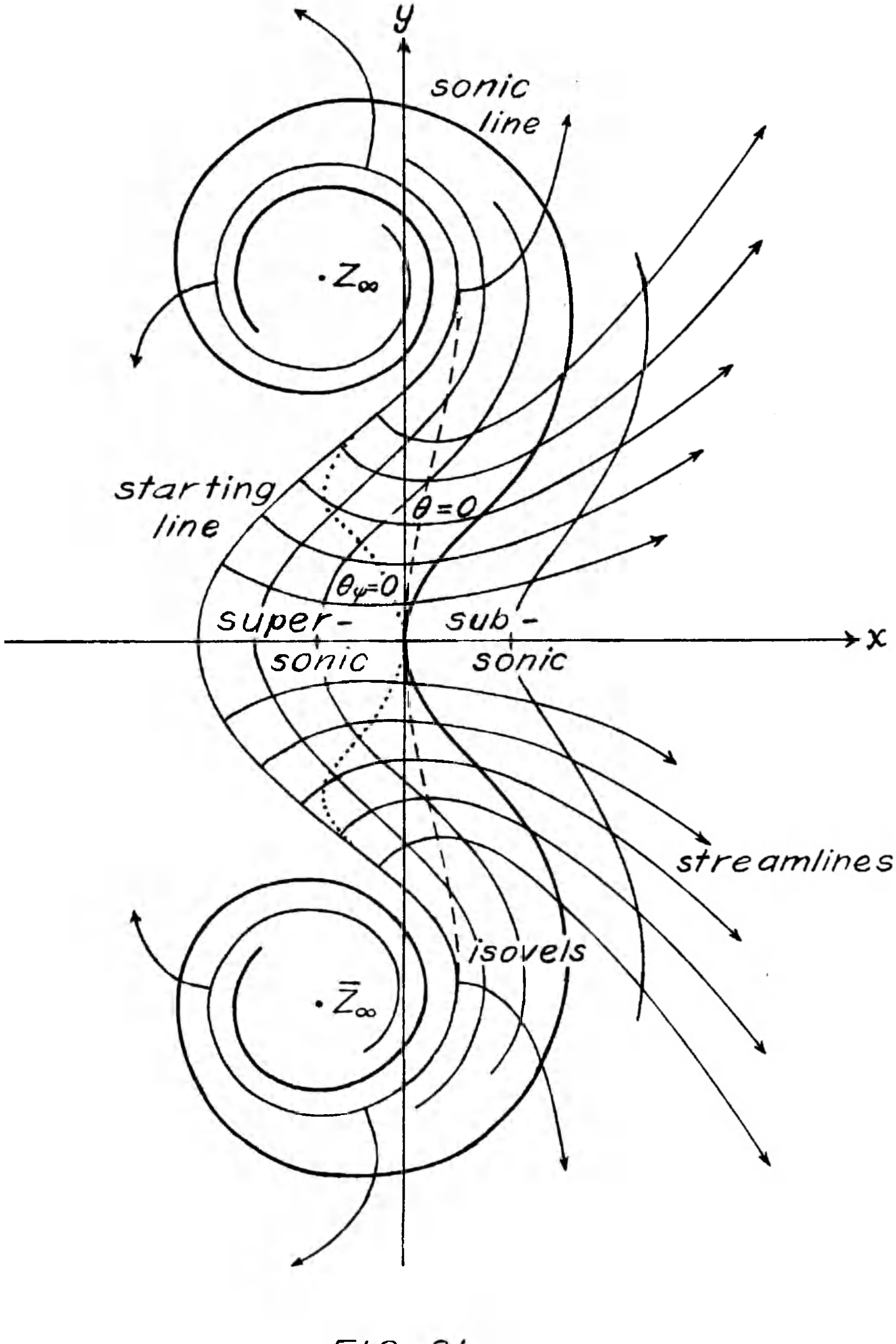

bounding streamline
isovel
super-
sonic
sonic line
0
σ_1
σ_2
sub-
sonic
x
streamline
$z_{\pi/2}$
$z_{-\pi/2}$

FIG. 8a



y
sonic
line
$\cdot Z_\infty$
starting
line
$\theta=0$
$\theta_\psi=0$
super-
sonic
sub-
sonic
x
streamlines
isovels
$\cdot \bar{Z}_\infty$

FIG. 8b

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