INVESTIGATION OF VANISHING OF A HORIZON FOR BIANCHI TYPE IX (THE MIXMASTER) UNIVERSE

by

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ABSTRACT

TITLE OF THESIS: INVESTIGATION OF VANISHING OF A HORIZON FOR BIANCHI TYPE IX (THE MIXMASTER) UNIVERSE

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In this dissertation, the generic, non-rotating, homogeneous closed model universe (the "Mixmaster Universe", Bianchi Type IX) is studied to gain some insight into how the broad-scale homogeneity of the universe may have been produced at very early times. We begin our discussion by sketching the development of relativistic cosmology until the last decade. In the second chapter we discuss particle horizons in the Robertson-Walker models. These standard models of the universe possess particle horizons. Thus, only a finite part of such a universe could have been causally connected; while the isotropy of 2.7^0K microwave radiation implies the universe to be homogeneous on a much larger scale than the size of the horizon. The third chapter discusses in detail the evolution of the Mixmaster Universe near the singularity using the Hamiltonian techniques developed by Misner for these models. At a fixed time (or volume) epoch Ω_0, a Mixmaster Universe is specified by initial conditions β_+, β_- (shape anisotropy) and p_+, p_- (expansion rate anisotropy). In the fourth chapter we derive the equations for rays of high-frequency sound waves and light waves. When these equations are applied in the Mixmaster Universe, we find that for certain subsets of initial conditions, some of these sound rays and light rays would circumnavigate the corresponding universes in certain directions. Our results for light rays parallel those of Doroshkevich and Novikov, however we use entirely different methods (Hamiltonian methods) for treating the Einstein equations.

In the last chapter the evolution of the Mixmaster Universe is shown equivalent to a geodesic flow within a bounded region of the Lobatchewsky plane. The boundary shape makes this flow ergodic. The ergodicity is proved by invoking a certain group of conformal transformations, G, which makes this flow of broken geodesics on the Lobatchewsky plane, D, into a continuous one on D/G. The Einstein equations in this problem lead to a natural measure on initial conditions related to $\beta_\pm$, $p_\pm$. The measure of the circumnavigation sets depends upon the epoch and it goes to zero as the volume of the universe shrinks to zero. Finally, we compute the probability for circumnavigation along any one axis of the universe. It turns out to be roughly 1% for an empty universe and it decreases to 0.02% for realistic models containing radiation and matter in them.

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CHAPTER I.

Formulation of Cosmology

The cosmological problem within the framework of general relativity consisted, for a long time, in finding a simple model of the physical universe which is a solution to Einstein's equations. The simplification was involved in idealizing the crudely observed symmetries of homogeneity and isotropy of the universe and extrapolating them to all times in the past. The investigation of world-models without any postulate of homogeneity was first done by Raychaudhuri (1955) who obtained an equation which showed how the expansion of the universe was influenced by shear and rotation. Heckmann and Schücking (1959) classified several world-models by studying the four-vector u^μ of flow of incoherent matter. They also found homogeneous but anisotropic solutions for universes filled with pressureless fluid. Analytic solutions were also found for stress-energy tensors, corresponding to a pressureless fluid with magnetic field, or to a radiation fluid with $p = \frac{1}{3}\rho$ by taking a specialized form for the anisotropy. Misner (1968) studied a general class of homogeneous, anisotropic solutions to the Einstein's equations in which the pressure anisotropies were included, and proved that a considerable number of those solutions would evolve into the present day universe. Belinskii and Khalatnikov (1969), Thorne (1967), Doroshekevich, Zeldovich, and Novikov (1967) analysed cosmological models with pronounced anisotropy at an early stage of cosmological expansion. This new way of studying cosmology where the models considered looked very different in the distant past, exploited the richness of Einstein's equations and gave better insight into many problems -- the problem of particle horizons being one of them. Before we turn to the study of horizons in this new light, let us sketch briefly the development of relativistic cosmology until the last decade and the status of horizons in those cosmologies.

Building upon the observational work of earlier astronomers (Curtis, Shapley, Slipher) and greatly extending the investigation himself, Hubble (1924) in the early twenties proved the extragalactic nature of the Andromeda Nebula, discovered the expansion of the universe in the late twenties (1929), and showed in 1936 that the large scale distribution of galaxies was homogeneous and isotropic to the telescope limit. These three crucial discoveries set the stage for cosmology for years to come. It was immediately noticed that the models speculated by Friedman (1924) fitted very well with the above observations. Friedman was able to find solutions to the Einstein equations for the metrics

$$ds^2 = - c^2dt^2 + R^2(t) \left\{ \frac{dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)}{[1 + (k/4)r^2]^2} \right\} \tag{1.1}$$

with $k = +1$ and -1; the corresponding space metrics describing a three-sphere and a three-hyperboloid respectively. The Einstein equations, $G_{\mu\nu} - \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$ for the above metrics reduce to

$$\frac{1}{c^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \frac{\Lambda}{3} = \frac{8\pi G}{3c^4}\rho \tag{1.2}$$

and

$$\frac{2\ddot{R}}{c^2 R} + \frac{1}{c^2}\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} - \Lambda = - \frac{8\pi G}{c^4} p \tag{1.3}$$

where Λ is the cosmological constant and ρ, p are respectively the energy density and pressure as observed by the comoving observers.

Robertson (1929) extended this class of solutions by including the case where the 3-space geometry is Euclidean. The metric for that case is

$$ds^2 = - dt^2 + R^2 (t) \left[dx^2+dy^2+dz^2\right] \tag{1.4}$$

which has the same Einstein equations as the two Friedman cases but with $k=0$. Robertson and later on Walker (1936) showed that these three cases ($k=0$, ± 1) are the only solutions satisfying the "Cosmological Principle." The cosmological principle which was first explicitly stated by Milne (1931), demands that the universe be homogeneous and isotropic. The homogeneity of the universe means that through each event in the universe, there passes a spacelike hypersurface whose events are physically indistinguishable from each other. While the isotropy means that a comoving observer cannot distinguish one of his space directions from the others. Thus, if one restricts to Robertson-Walker models for the universe, theoretical cosmology reduces to the study of equations (1.2) and (1.3). Through the equations of motion for the fluid, $T^{\mu\nu}{}_{;\nu} = 0$ we could obtain p and ρ in terms of $R(t)$ and an appropriate equation of state would give a relationship between p and ρ. (Note, however that a solution to equation (1·2) is also a solution to equation (1.3) by virtue of one of the Bianchii identities provided it is not a static solution. Einstein in the pre-Hubble days was trying to find a static, closed model for the universe. With $k=1$, $\dot{R}=0$, no realistic fluid can satisfy equation (1.3) unless $\Lambda\neq 0$. Thus the need to introduce the cosmological constant Λ in the Einstein's equations persisted till Hubble observed the general recession of the galaxies. The various parameters k, Λ, the energy density, ρ, the size of the universe now R_0 appearing in the theoretical solution are directly or indirectly connected with the observed astronomical parameters like the average density now, the "Hubble constant," $H_0=(\dot{R}/R)_0$, and the deceleration parameter, $q_0 = -(\frac{\ddot{R}}{R_0})\frac{1}{H_0^2}$. As the observations for these astronomical parameters get better, the range of values for the theoretical parameters gets smaller. However, the observations are still not precise enough to distinguish between various Robertson-Walker models ($k=0, \pm 1$) or to predict the value of Λ. (See Sandage (1970) for the review of the measurements of the Hubble constant and the deceleration parameter.)

The Robertson-Walker models have a singularity at a finite time in the past when the energy density and the curvature become infinite. To avoid the singularity Bondi and Gold (1948) proposed a temporally-invariant or steady-state model universe. It is based on the so called perfect cosmological principle, i.e. the universe presents the same large scale view at all times. The postulates of homogeneity and equivalence of all world-points completely determine the metric. Indeed, the metric clearly must be of the Robertson-Walker type and furthermore, the expansion factor in the Robertson-Walker metric must be such that the Hubble constant is independent of the epoch, i.e.

$$H = \dot{R}/R = \text{constant}$$

$$\text{i.e. } R = e^{Ht} \; .$$

The constancy of curvature of the space-like sections t = constant requires $k=0$ and the metric becomes

$$ds^2 = -dt^2 + e^{2Ht}\,(dx^2+dy^2+dz^2) \; . \tag{1.5}$$

The metric is of the de Sitter type and is a solution to the Einstein equations, for vanishing energy-density and $\Lambda>0$. To rectify the situation, Hoyle (1949) modified Einstein's equations to read

$$G_{\mu\nu} = + \frac{8\pi G}{C^4}\,(T_{\mu\nu} - C_{\mu\nu}), \tag{1.6}$$

where $C_{\mu\nu}$ is the matter creation term. Till a few years ago, observations were going on two fronts to decide between steady-state model and the "big-bang" Robertson-Walker models as a correct description of the universe. In the steady-state theory all matter must be continuously accelerated, whereas galaxies in Friedman models must decelerate. The deviation from a linear expansion law between distance and red-shift would provide the necessary information on the change of motion with time. A second approach has been to look for evolutionary differences in galaxies in the great clusters. The steady-state model requires that galaxies must be formed during all epochs. On the other hand, in Robertson-Walker models, most galaxies are formed at a unique epoch. Thus integral properties such as color, and mass to light ratio for an adequate sample of galaxies would show dispersion in the steady state theory owing to the age difference of their stellar content. Although, these observations did not provide very clear cut answers, they favored evolving Robertson-Walker models. The discovery by Penzias and Wilson (1965) of the cosmic microwave background changed the situation considerably. There was no satisfactory way in which the steady-state theory could explain the 2.7^0K black-body spectrum of the background radiation while the presence of relic thermal radiation is to be expected if we trace the expansion of the universe back to a highly contracted, hot phase of the universe. Curiously, long before the discovery of Penzias and Wilson and its explanation by Dicke etc. (1965), Gamow and his colleagues (1949) had predicted a primeval fireball and had even predicted its present temperature somewhat larger than 2.7^0K. The isotropy measurements of the black-body radiation gave an impetus to the study of anisotropic cosmologies which was already started by Heckmann and Schücking (1959) and Zel'dovich (1965). The precise measurement (0.2%) [see Partridge (1969)] in its isotropy put much more stringent limits on the anisotropy of cosmological models than those derived from the isotropy of the Hubble red-shifts. Zel'dovich (1956) and

Thorne (1966) studied the anisotropic cosmologies in view of the possibility of a primordial magnetic field. Hawking and Tayler (1966) and Thorne (1967) investigated the influence of anisotropy on the primordial helium production. A number of people studied exact solutions for anisotropic cosmologies. Misner (1968) studied a large class of homogeneous, but anisotropic universes and found that just above 10^{10} ^{0}K, neutrino viscosity was very efficient in reducing the anisotropy. This led Misner (1967) to formulate the principle of "Chaotic Cosmology." The philosophy of chaotic cosmology was based on the existence of singularity proof of Hawking and Penrose (1970). Assuming that the singularity demanded by Einstein's equations is of the Friedman type, i.e. a singularity involving infinite densities a finite proper time in the past, the idea of chaotic cosmology is to start with arbitrary initial conditions near the singularity and try to prove that they would develop into the present day universe. Furthermore, one could try to formulate chaos in terms of maximum symmetry breaking near the singularity, or that the universe goes through almost all irregular stages near the singularity. The first part of chaotic cosmology which consists of proving the cosmological principle has a great esthetic appeal while the second part could give rise to important consequences.

In this thesis, we will try to characterize the "chaotic" behavior of the Bianchi Type IX (generic, homogenous, anisotropic models) cosmologies near the singularity. The Hamiltonian methods applied to these models by Misner (1969) describe the evolution of these universes (called the Mixmaster Universe) in terms of a particle motion in a potential well. The evolution is given completely by specifying the initial conditions β_+, β_- (shape anisotropy) and p_+, p_- (expansion rate anisotropy). We will show in the limit of infinite potential approximation that the evolution of the Mixmaster Universe is equivalent to a geodesic flow within a bounded region of the Lobatchewsky plane. It will be shown that the boundary shape is such that, as the universe goes towards the singularity, the collisions with the equipotential walls bounding the region make the geodesic flow ergodic. Thus the chaos of the Mixmaster Universe will be characterized by the ergodic nature of the geodesic flow, or more specifically, it means that if we start with a well-defined state of the universe (given by certain value of $\beta_\pm$; $p_\pm$ at certain Ω_0), as it evolves towards the singularity, it goes through almost all possible anisotropic stages. We will study the consequence of ergodicity on the problem of horizons. For certain subsets of initial conditions, some null-geodesics proceed to circumnavigate the corresponding universe. We will see that the Einstein equations lead to a natural measure on the initial conditions ($\beta_\pm$, $p_\pm$) in this problem. However, the measure of the circumnavigation sets depends upon the epoch, so that the ergodicity does not guarantee that a typical solution will evolve through such a set. Relying on the mixture property of the ergodicity, we will compute the probability for circumnavigation along any axis. It will turn out to be very small but finite.

CHAPTER II.

Particle Horizons

Horizons are frontiers between observable and unobservable events. The first systematic study of visual horizons was done by Rindler (1956). He defined two types of horizons. The first, called an event-horizon, for an observer A is a surface in space-time which divides all events into two non-empty classes: those that are observable by A and those that are forever unobservable by A. The de Sitter universe can be seen to possess an event-horizon. The other type of horizon is called a particle horizon for a given comoving observer A at some time t_o. It is a surface in the space-like hypersurface $t = t_o$ which divides all comoving observers into two non-empty classes: those that have already been observable by A at time t_o and those that have not. Penrose (1968) defines particle horizons differently. In his definition, the particle horizon of an observer γ (see Fig. 1) separates events I from which a particle with world line γ can be observed, from events II from which the particle cannot be observed. Note that P and Q will form particle horizon in Rindler's definition. For spatially homogeneous universes where the three-space sections are symmetric, the intersection of Penrose's particle horizon with the surface t = constant would give the Rindler particle horizon. We will restrict ourselves exclusively to the Rindler's definition of particle horizon. As an example, consider Robertson-Walker model in the early times when the universe was radiation-dominated. The radiation energy-density is given by

$$\rho_r R^4 = \rho_{ro} R_o^{\ 4} \tag{2.1}$$

where ρ_{ro} is the energy-density now and R_o is the present radius of the universe. Ignoring the curvature term and cosmological constant term in comparison with the radiation term in equation (2.1) we obtain

$$\frac{1}{R^2}\left(\frac{dR}{dt}\right)^2 = \frac{8\pi G}{3c^2}\,\frac{\rho_{ro} R_o^{\ 4}}{R^4}$$

or, integrating,

$$\frac{R}{R_o} = \left(\frac{32\pi G}{3c^2}\,\rho_{ro}\right)^{1/4} \sqrt{t}_{\sec} \; . \tag{2.2}$$

The structure of the particle horizon at time t for the observer at $r = 0$ will be given by studying the propagation of null lines during the time interval o to t:

$$\int_0^t \frac{cdt}{R(t)} = \int_0^r \frac{dr}{(1 + \frac{k}{4}\, r^2)} \equiv \chi(r)$$

$\chi(r) = r$ for $k = 0$, while $\chi(r) = 2 \tan^{-1} r$ for $k = +1$. Thus the radius of the particle horizon at time t is given by

$$\chi(r) = \frac{c}{R_o\left(\frac{32\pi G}{3c^2}\,\rho_{ro}\right)^{1/4}}\; 2\sqrt{t}_{\sec} \; . \tag{2.3}$$

Taking $\rho_{ro} = 5 \times 10^{-13}$ ergs/cm^3 corresponding to the black-body radiation temperature 2.7°K and $R_o = 10^{28}$ cm, we obtain the coordinate size for the horizon at time t to be

$$\chi(r) = 10^{-8} \sqrt{t/1 \sec} \; .$$

For the present matter-density $\rho_m = 10^{-30}$ gm/cm^3, the universe remains radiation dominated until $T_r \simeq 3{,}000°$K which corresponds to the age of the universe $t \simeq 10^{13}$ sec. Thus, during the radiation dominated phase, the light can travel the coordinate distance of 3×10^{-2} which is a small fraction of the range of χ going from 0 to 2π for the closed universe. To get a better idea, let us compute the mass of the baryons in a typical horizon. Consider the k = 0 case, for which the proper radius of the horizon at time t will be given by

$$r_H = R(t)r,$$

where r is the maximum distance that light could have traveled. Substituting for R(t) from (2.2) and r from (2.3) we obtain

$$r_H = 2ct \quad . \tag{2.4}$$

Writing $\rho_r = aT^4$, equations (2.2) and (2.1) would give the following relationship between temperature T and the age of the universe, t:

$$\frac{1}{T^2} = \left(\frac{32\pi G}{3c^2}\, a\right)^{1/2} t \quad . \tag{2.5}$$

Thus the total baryon mass in a horizon will be given by

$$M_H = \frac{4\pi}{3}\, \rho_m\, r_H^{\,3} \tag{2.6}$$

where ρ_m, the matter density is given by

$$\rho_m R^3 = \rho_{mo} R_o^{\,3}$$

or

$$\rho_m = \rho_{mo} \left(\frac{T}{T_o}\right)^3 \tag{2.7}$$

where ρ_{mo} is the matter density now and T_o is the present radiation temperature. Substituting (2.4) and (2.7) into (2.5) and using (2.5) we get

$$M_H = \frac{4\pi}{3} \rho_{mo} \left(\frac{T}{T_o}\right)^3 8c^3 \frac{1}{T^6} \frac{1}{\left(\frac{32\pi G}{3c^2} a\right)^{3/2}} , \tag{2.8}$$

Taking $\rho_{mo} = 10^{-30}$ gm/cm^3, $T_o = 2.7°K$ and $T = 3,000°K$, we compute the mass of the horizon

$$M_H \simeq 4 \times 10^{49} \text{ gm} = 10^{16} M_\odot$$

while the mass of the observable universe is roughly 10^{54}gm. Thus we find that in these standard evolutionary models of the universe, only a small fraction of the universe could have had communication between its various parts. As a result, no physical mechanism would make the properties of the two parts of the universe uniform when these two parts of the universe lie outside of each other's horizon. But the universe is observed to be homogeneous on a much larger scale than the size of the horizon. The 2.7°K microwave relic radiation gives the deepest look into the universe. The microwave photons give us direct information about the nature of the universe at the time they last interacted with matter. The main interaction with matter is the Thomson scattering by free electrons. The radiation first ceases to interact with matter when the plasma recombines as the temperature of the universe drops to $\simeq$ 4000°K (see Peebles, 1968). The photons now travel freely until the epoch when the intergalactic plasma is turned on. To find out when the presently measured microwave photons scattered last with the intergalactic plasma, we consider the optical depth τ ($e^{-\tau}$ is the attenuation factor for the photon intensity) obtained by Bahcall and Salpeter (1965) for Thomson scattering in the universe with Hubble constant $H_o = (10^{10}\text{ years})^{-1}$. Taking the deceleration parameter $q_o = \frac{1}{2}$ and a dense intergalactic plasma with electron density $n_{eo} = 10^{-5}\text{cm}^{-3}$, we obtain the optical depth to be unity for a distance corresponding to a red-shift of $z = 7$. Thus the isotropy (see Partridge, 1969) of the present day observed 2.7°K microwave radiation implies that the universe has been expanding isotropically at least since $z = 7$. And the inhomogeneity in the microwave radiation would be washed out only over a horizon size at $z = 7$. Let us compute the relevant horizon size for $k = 0$ Robertson-Walker model, the metric for which is

$$ds^2 = -c^2dt^2 + R^2(t)\,[dx^2 + dy^2 + dz^2]$$

$$= R^2(t)\,[-\frac{c^2dt^2}{R^2(t)} + dx^2 + dy^2 + dz^2]\,.$$

Defining a new time variable η by

$$d\eta = \frac{cdt}{R(t)}$$

we obtain a conformally flat metric:

$$ds^2 = R^2(\eta)\,[-d\eta^2 + dx^2 + dy^2 + dz^2]\,. \tag{2.9}$$

When the universe is radiation dominated $R(t) \propto \sqrt{t}$ in which case $\eta \propto \sqrt{t}$

giving us

$$R(\eta) \propto \eta \quad . \tag{2.10}$$

When the universe is matter dominated, $R(t) \propto t^{2/3}$ giving us $\eta \propto t^{1/3}$; or

$$R(\eta) \propto \eta^2 \quad . \tag{2.11}$$

Suppose that the microwave photons which were scattered last at $\eta_{scatt.}$ (corresponding to $z = 7$) by intergalactic plasma at P and Q (see Fig. 2) are just observable now at $\eta_{obs.}$ Then setting $ds^2 = 0$ in equation (2.9) we get

$$OP = OQ = \eta_{obs.} - \eta_{scatt.} \tag{2.12}$$

while $\eta_{scatt.}$ is related to the epoch $z = 7$ by the red-shift formula

$$1 + z = \frac{R(\eta_{obs.})}{R(\eta_{scatt.})} \quad . \tag{2.13}$$

Since the universe is matter dominated during this relevant era, we can use (2.11) to obtain

$$\left(\frac{\eta_{obs.}}{\eta_{scatt.}}\right)^2 = 8 \quad . \tag{2.14}$$

Thus we are seeing photons of exactly the same temperature (up to 0.3%) which were scattered by plasma at P and Q at $\eta = \eta_{scatt.}$ But light propagation between regions around P and Q before the time of microwave photon scattering is possible only if

$$PQ < \eta_{scatt.} \quad . \tag{2.15}$$

Thus, the maximal angular separation (θ_m) between the plasma regions which could have had prior causal communication is given by

$$\sin \theta_m/2 = \frac{PR}{OP} = \frac{\eta_{scatt.}}{2(\eta_{obs.} - \eta_{scatt.})}$$

$$= \frac{1}{2\left[\left(\frac{\eta_{obs.}}{\eta_{scatt.}}\right) - 1\right]} = \frac{1}{2(\sqrt{8} - 1)}$$

or,

$$\theta_m = 31.6° \quad .$$

So in Robertson-Walker models of the universe, the inhomogeneities in the temperature of microwave radiation would persist over regions separated by angle $\theta > \theta_m$ at $z = 7$. While we observe the microwave radiation to have exactly the same temperature over widely different angles in the sky as opposed to the expected variations over regions separated by angle ϕ (= PO'Q) > 22°.

Thus the isotropic models of the universe have an unpleasant feature in terms of the existence of the particle horizons. Misner (1969a) first pointed out the possibility of absence of horizons in a more general, anisotropic, Bianchi Type IX model of the universe. That the structure of horizons is quite different in anisotropic models can be seen in Kasner solutions (Kasner, 1921) of Einstein equations, whose importance as models for the initial singularity was first demonstrated by Lifshitz and Khalatnikov (1963). These solutions correspond to the gravitational field in empty, homogeneous Euclidean space and the metric for these solutions is given by

$$ds^2 = - dt^2 + t^{2P1} dx^2 + t^{2P2} dy^2 + t^{2P3} dz^2 \qquad (2.16)$$

where p_1, p_2, p_3 are three arbitrary numbers satisfying the relations:

$$p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2 . \tag{2.17}$$

Therefore, only one of those numbers is independent. So a single real parameter u (first used by Landau and Lifshitz, 1962) can represent p_1, p_2, p_3 as

$$p_1 = \frac{-u}{1+u+u^2}, \qquad p_2 = \frac{u+1}{1+u+u^2}, \qquad p_3 = \frac{u(u+1)}{1+u+u^2} \tag{2.18}$$

Consider a particular Kasner solution with $p_1 = 1$ and $p_2 = p_3 = 0$. Then the metric is

$$ds^2 = -dt^2 + t^2dx^2 + dy^2 + dz^2 . \tag{2.19}$$

In terms of the new time variable $\eta = \ln t$, the metric reduces to

$$ds^2 = e^{2\eta}(-d\eta^2 + dx^2) + dy^2 + dz^2 .$$

Thus the distance Δx covered by the light signal propagating in the x-direction is

$$\Delta x = \Delta\eta .$$

Assuming that the metric has the form (2.19) for all t from t = 0, then η takes on all real values from $\eta = -\infty$. Thus at any given time η, any finite distance along the x-axis can be covered by a light signal in the time available since the initial singularity. Thus the particle horizons are washed out along the x-axis by causal propagation. In the next chapter, we will describe the behavior of Bianchi Type IX model as investigated by Misner (1969b), Belinskii, Lifshitz, Khalatnikov (1970). They find out that this model closely approximates Kasner solutions during certain periods of its evolution. Here we will investigate that subset of Type IX solutions which, during their evolution, approximate those Kasner solutions which open particle horizons in one spatial direction. This subset of solutions (called the "circumnavigation solutions") will be characterised by the initial conditions β_+, β_- (shape anisotropy) and p_+, p_- (expansion rate anisotropy). We will see how the Einstein equations lead to a natural measure on the initial conditions $(\beta_\pm, p_\pm)$ in this problem, enabling us to compute the probability for a typical Bianchi Type IX solution to have no horizon along any one axis.

CHAPTER III

The Non-rotating Bianchi Type IX Model

The metric for the generic, non-rotating, closed homogeneous cosmological model of Type IX can be written as

$$ds^2 = - dt^2 + \frac{1}{4} R^2 (e^{2\beta})_{ij} \sigma^i \sigma^j \quad . \tag{3.1}$$

Here R and β_{ij} are functions of time t only. The matrix β_{ij} is diagonal and traceless and measures the anisotropy of the universe. The σ^i are three independent differential forms which remain invariant under the Bianchi Type IX homoegenity group. Thus the σ^i satisfy the following relation

$$d\sigma_i = \varepsilon_{ijk} \sigma_j \wedge \sigma_k \tag{3.2}$$

and they can be represented as

$$\begin{aligned} \sigma_1 &= \sin\psi d\theta - \cos\psi \sin\theta d\phi \\ \sigma_2 &= \cos\psi d\theta + \sin\psi \sin\theta d\phi \\ \sigma_3 &= - (d\psi + \cos\theta d\phi) \end{aligned} \tag{3.3}$$

where $\psi\theta\phi$ are the Euler angles with $0 \le \psi < 4\pi$, $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$. The numerical factor of 1/4 in Equation (3.1) is chosen so that when $\beta_{ij} = 0$ the space part of the metric is just the standard metric for a 3-sphere of radius R. Thus for $\beta = 0$, one obtains the closed Robertson-Walker cosmological model which is both homogeneous and isotropic. We will measure the volume of the universe (proportional to $\sqrt{g} = \frac{R^3}{8}$) by a parameter Ω defined as

$$R = (2/3\pi)^{1/2}\, e^{-\Omega} \quad .$$

For the two independent anisotropy parameters choose

$$\beta_+ = (\beta_1 + \beta_2)/2$$

and $$\beta_- = (\beta_1 - \beta_2)/2\sqrt{3} \quad .$$

Thus the variables Ω, β_+ and β_- characterize the three-geometries. To study the dynamics of these universes, the most elegant method is to obtain the Hamiltonian action functional which Misner,[1969] did by adapting the Hamiltonian formalism developed by Arnowitt, Deser and Misner [1962]. The idea of the Hamiltonian formalism is to cast the variational principle for Einstein's equations which is $\delta I = 0$ with

$$I = (16\pi)^{-1} \int {}^4R(-{}^4g)^{1/2}\, d^4x \quad , \tag{3.4}$$

into a canonical form. Arnowitt, Deser and Misner [1962] show that the variational principle can be put in the form

$$I = (16\pi)^{-1} \int \pi^{ik}\, \dot{g}_{ik}\, d^4x \tag{3.5}$$

where not all π^{ik} and g_{ik} are independent as they satisfy the constraint equations:

$$g\, {}^3R + \frac{1}{2}(\pi^k{}_k)^2 - \pi^{ik}\pi_{ik} = 0 \quad . \tag{3.6}$$

$$-2\pi^{ik}{}_{|k} = 0 \quad . \tag{3.7}$$

The quantity ${}^3R$ is the curvature scalar formed from the spatial metric g_{ij}, "|" indicates the covariant derivative using this metric, and spatial

(Latin) indices are raised and lowered using g^{ij} and g_{ij}. The equations (3.6) and (3.7) are nothing but the Einstein's initial value equations $G^{o\mu} = 0$, up to certain multiplicative factors. The action integral (3.5) for our metric (3.1) would be

$$I = (16\pi)^{-1} \int \pi^{ij} \, dg_{ij} \wedge \sigma_1 \wedge \sigma_2 \wedge \sigma_3$$
$$= \pi \int \pi^{ij} \, dg_{ij} \quad ,$$

where

$$g_{ij} = \frac{1}{4} R^2 (e^{2\beta})_{ij} = \frac{1}{6\pi} e^{-2\Omega} (e^{2\beta})_{ij} \quad .$$

Thus, $dg_{ij} = -2g_{ij} \, d \; + 2g_{ik} \, d \;_{kj}$.

So

$$I = (2\pi) \int [\pi_k{}^j \, d\beta_{kj} - \pi^k{}_k \, d\Omega] \quad . \tag{3.8}$$

Since β_{kj} is traceless, we separate out the traceless part of $\pi_k{}^j$ by defining

$$p_k{}^j = (2\pi)(\pi_k{}^j - \frac{1}{3} \delta_k{}^j \pi^i{}_i) \quad . \tag{3.9}$$

Since (3.8) is almost in a canonical form, taking Ω as a time coordinate, we see that $(2\pi)\pi^k{}_k$ will be the Hamiltonian so we write

$$H = (2\pi) \, \pi^k{}_k \quad . \tag{3.10}$$

Thus the action integral now reduces to

$$I = \int [p_k{}^j \, d\beta_{kj} - H d\Omega] \quad .$$

Parameterizing $p^j{}_k$ in the same way as β_{kj}, namely

$$6 \, p^j{}_k = \text{diag.}(p_+ + p_- \sqrt{3}, \; p_+ - p_- \sqrt{3}, \; -2p_+)$$

we obtain

$$I = \int (p_+ \, d\beta_+ + p_- d\beta_- - H d\Omega) \quad . \tag{3.11}$$

Thus we have obtained the standard canonical form provided $\beta_\pm$, $p_\pm$ and Ω can be varied independently and H is expressed as a function of $\beta_\pm$, $p_\pm$ and Ω. To see this note that the constraint equation (3.7) is identically satisfied in any case and the only remaining constraint equation (3.6 reduces to

$$\left(\frac{R^3}{8}\right)^2 {}^3R + \frac{1}{2}(\pi^k{}_k)^2 - \left(\frac{1}{2\pi} p^i{}_k + \frac{1}{3}\delta^i{}_k \pi^\ell{}_\ell\right) x$$

$$\left(\frac{1}{2\Pi} p^k{}_i + \frac{1}{3}\delta^k{}_i \, \pi^m{}_m\right) = 0$$

or

$$\frac{R^6}{64} {}^3R + \frac{1}{2}(\pi^k{}_k)^2 - \frac{1}{3}(\pi^\ell{}_\ell)^2 - \frac{1}{4\pi^2} \, p^i{}_k \, p^k{}_i = 0 \quad .$$

Substituting $p^i{}_k \, p^k{}_i = \frac{1}{6}(p_+{}^2 + p_-{}^2)$, $\pi^k{}_k = \frac{1}{2\pi} H$, and ${}^3R = \frac{6}{R^2}(1-V)$ and writing $R = \frac{2}{3\pi} e^{-\Omega}$, we can solve the above equation for H, to obtain

$$H = [p_+{}^2 + p_-{}^2 + e^{-4\Omega}(V-1)]^{1/2} \tag{3.12}$$

where $$V(\beta) = \frac{1}{3} \text{trace } (e^{4\beta} - 2e^{-2\beta} + 1)$$

$$= \frac{2}{3} e^{4\beta_+}(\cosh 4\sqrt{3}\beta_- - 1) + 1$$

$$- \frac{4}{3} e^{-2\beta_+} \cosh 2\sqrt{3}\beta_- + \frac{1}{3} e^{-8\beta_+} \quad . \tag{3.13}$$

Thus, with Ω as the choice for the independent, coordinate time variable, the state of the universe at any epoch Ω, is given by the field amplitudes β_+ and β_- and their congugate momenta p_+ and p_-. And the evolution of the universe is given by the Hamilton's equations:

$$\frac{d\beta_\pm}{d\Omega} = \frac{\partial H}{\partial p_\pm} , \frac{dp_\pm}{d\Omega} = - \frac{\partial H}{\partial \beta_\pm} , \frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} . \tag{3.14}$$

In terms of the Ω time, the beginning of the universe appears at $\Omega = + \infty$. The proper time of comoving observers is related to Ω by (see Misner [1969b])

$$dt = \sqrt{\frac{2}{3\pi}} \, \frac{1}{H} \, e^{-3\Omega} \, d\Omega . \tag{3.15}$$

Thus the non-rotating Bianchi Type IX problem is reduced to a Hamiltonian problem corresponding to a particle moving in two dimensions in a time dependent potential. The "anisotropy potential" $V(\beta)$ is positive definite with

$$V \simeq 8 \, (\beta_+^2 + \beta_-^2) \quad \text{near } \beta = 0 . \tag{3.16}$$

The equipotentials near the origin of the $\beta_+\beta_-$ plane are closed curves for $V < 1$. The potential walls rise steeply away from $\beta = 0$, with the equipotentials forming equilateral triangles in the $\beta_+\beta_-$ plane. [See Figure 1.] The potential has reflection symmetry in the β_+ - axis and one side of the equilateral triangle cuts the negative β_+ - axis. The corners of the triangle are not closed however, but have channels leading off to infinity. These channels narrow exponentially and the deviation from the straight-sided traingular shape takes up a very small part of the total equipotential. For example, near the positvie β_+ - axis, for large β_+, $V(\beta)$ has the form

$$V(\beta) \simeq 1 + 16\beta_-^2 \, e^{4\beta_+} . \tag{3.17}$$

In directions opposite the three corners, the potential rises exponentially for large distances from $\beta = 0$. Along the negative β_+ - axis, for example V has the value

$$V(\beta) \approx \frac{1}{3} e^{-8\beta_+} \quad \text{for} \quad \beta_+ << -1 \quad . \tag{3.18}$$

Thus the equipotential contours for $V > 1$ consist of three disjointed curves, each of which runs off to infinity at the channels. When not running off to infinity, the contours are approxmiately straight lines. Thus, the evolution of the universe is given by the motion of the system point $\beta \equiv (\beta_+,\beta_-)$ as a function of the time coordinate Ω, moving in a time-dependent potential well. In the first approximation near the singularity for $\Omega \to \infty$, we can neglect the potential term $e^{-4\Omega}(V-1)$ in the Hamiltonian [Eqn. (3.12)] to give

$$H = [p_+^{\,2} + p_-^{\,2}]^{1/2}. \tag{3.19}$$

The Hamilton's equations, then give $p_\pm$, H as constants of motion and $\frac{d\beta_\pm}{d\Omega} = \frac{p_\pm}{H}$ Thus the universe point moves with velocity $\beta' \equiv d\beta/d\Omega = \{(d\beta_+/d\Omega)^2 + (d\beta_-/d\Omega)^2\}^{1/2}$ of unit magnitude in straight lines. The approximation used would fail when $V(\beta)$ becomes sufficiently large. The limiting equipotential (the potential "wall") is one which would make β' to go to zero. Using the Hamilton's equations

$$\frac{d\beta_+}{d\Omega} = \frac{\partial H}{\partial p_\pm} = p_\pm/H \tag{3.20}$$

one can rewrite Equation (3.12) as

$$1 = \beta'^{\,2}_+ + \beta'^{\,2}_- + H^{-2}\, e^{-4\Omega}\, V, \text{ for } V >> 1 \tag{3.21}$$

so that the condition $\beta' = 0$ gives

$$V(\beta_{wall}) = H^2 e^{4\Omega} \quad .$$

Substituting the asymptotic form for the potential for $\beta_+ << -1$ from Equation (3.18), we obtain

$$\beta_+ \simeq \beta_{wall} = -\frac{1}{2}\Omega - \frac{1}{8}\ln(3H^2) \quad . \tag{3.22}$$

The Ω dependence of H is given by the Hamilton equation

$$\frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -\frac{2}{H}e^{-4\Omega}(V-1) = -\frac{2}{H}H^2(1-\beta'^2)$$

or

$$\frac{d\ \ln H^2}{d\Omega} = -4(1-\beta'^2) \quad . \tag{3.23}$$

Thus, when the system point β is well inside the potential walls, $\beta' \simeq 1$ and H is nearly constant. From Equation (3.22) we then conclude that the sides of the limiting equipotential triangle move outward at velocity $\left(\frac{d\beta_{wall}}{d\Omega}\right)\Big| = \frac{1}{2}$ when the system point is moving with unit velocity inside the triangle. If the system point is moving in a straight line making an angle θ with β_+ - axis, then $d\beta_+/d\Omega = \cos\theta$ and $d\beta_-/d\Omega = \sin\theta$. Substituting $\beta_+ = (\cos\theta)\Omega + \text{const}$ and $\beta_- = (\sin\theta)\Omega + \text{const.}$ in the metric (3.1) we obtain

$$\begin{aligned} ds^2 &= -dt^2 + \frac{1}{6\pi}\Big[e^{2(\beta_+ + \sqrt{3}\beta_-)-2\Omega}\sigma^{1\,2} + e^{2(\beta_+ - \sqrt{3}\beta_-)-2\Omega}\sigma^{2\,2} \\ &\quad + e^{-4\beta_+ -2\Omega}\sigma^{3\,2}\Big] \\ &= -dt^2 + \frac{1}{6\pi}\Big[\alpha_1 e^{\Omega(2\cos\theta + \sqrt{3}\sin\theta - 2)}\sigma^{1\,2} \\ &\quad + \alpha_2 e^{\Omega(2\cos\theta - \sqrt{3}\sin\theta - 2)}\sigma^{2\,2} + \alpha_3 e^{\Omega(-4\cos\theta - 2)}\sigma^{3\,2}\Big] \end{aligned}$$

where α_1, α_2, α_3 are some constants. Since H is constant when the system is moving freely inside the potential walls, Equation (3.15) can be integrated to give

$$t \ \alpha \ e^{-3\Omega} \quad .$$

Thus, the metric reduces to

$$ds^2 = -dt^2 + \frac{1}{6\pi}\left[A_1\, t^{2p_1}\, \sigma^{1^2} + A_2\, t^{2p_2}\, \sigma^{2^2} + A_3\, t^{2p_3}\, \sigma^{3^2}\right] \qquad (3.24)$$

where A_1, A_2, A_3 are some constants and

$$\left.\begin{aligned} p_1 &= -\frac{1}{3}\cos\theta - \frac{1}{\sqrt{3}}\sin\theta + \frac{1}{3} \\ p_2 &= -\frac{1}{3}\cos\theta + \frac{1}{\sqrt{3}}\sin\theta + \frac{1}{3} \\ p_3 &= \frac{2}{3}\cos\theta + \frac{1}{3} \quad . \end{aligned}\right\} \qquad (3.25)$$

These coefficients satisfy $p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2$ which are just the relations which Kasner coefficients satisfy. Belinski, Khalatnikov, Lifshitz [1970] call these metrics generalized Kasner solutions. Thus the universe has a Kasner epoch when the system point β is moving freely. This epoch, then can be parameterized by "u". From equation (2.18) substituting for p_3 in (3.25), we get

$$\frac{u(u+1)}{1+u+u^2} = \frac{2}{3}\cos\theta + \frac{1}{3}$$

or

$$\frac{d\beta_+}{d\Omega} = \cos\theta = \frac{u^2+u-1/2)}{u^2+u+1} \quad .$$

Similarly, we would get (3.26)

$$\frac{d\beta_-}{d\Omega} = \sin\theta = \frac{\sqrt{3}(u+1/2}{u^2+u+1}$$

Thus the evolution of the Bianchi Type IX universe can be described as a series of Kasner epochs - the system point β moving in a certain direction characterized by a certain value of the parameter u, with unit velocity untill it comes close to one of the equipotential walls (which are moving more slowly) and feels the potential, and would then bounce off the wall and would follow freely a different Kasner epoch until it reaches the next wall. The Kasner solution which does not have a horizon for causal propagation along one axis, say the z-axis, has indices $p_1 = p_2 = 0$ and $p_3 = 1$ which corresponds to $u = \infty$ [see Equation (2.18)]. $u = \infty$ corresponds to $\frac{d\beta_+}{d\Omega} = 1$ and $\frac{d\beta_-}{d\Omega} = 0$, i.e. the system point is moving horizontally, parallel to the β_+ -axis in the positive direction. Note that the inclined potential walls are also moving with unit-velocity in the positive β_+ direction. Thus, this epoch would correspond to the system point following one of the inclined potential walls. In the next chapter we will investigate how closely a Bianchi Type IX solution would have to approximate this particular Kasner epoch for it to lead to a vanishing horizon along one axis. The Kasner solutions corresponding to horizon vanishing along the other two axes are given by $p_1 = p_3 = 0$, $p_2 = 1$ and $p_2 = p_3 = 0$, $p_1 = 1$ respectively: those are parameterized by $u = 0$ and $u = -1$ and correspond to the system point moving parallel to the other two corner axes and moving in the same directions as those corners.

As the system point bounces back and forth between the equipotential walls, let us note the physical significance of various paths as they relate to $V(\beta)$. Consider a particular path when the system point is close to β_+-axis and following the corner ($|\beta_-| << 1$, $\beta_+ \to \infty$). This corresponds to a "pancake" singularity - a relative compression of the 3-axis with the other two axis approximately equal. With $\beta_- = 0$, it corresponds to the Taub Vacuum solution with the lengths in the 1 and 2-axes remaining finite and equal. When the system point is near the potential walls, say for example the vertical wall which corresponds to $\beta_+ \to -\infty$ and $\beta_- < \sqrt{3}\,\beta_+$, it corresponds to a "cigar" singularity with stretching of the 3-axis relative to the other; the other two sides of the triangular equipotentials correspond to preferential stretching of the other two axes.

CHAPTER IV

Propagation of Light and High-frequency Sound Waves

We want to investigate possible modes of communication in the Bianchi Type IX model. In the Appendix A, we obtain an equation for the propagation of high- frequency sound waves. It is a geodesic-like equation for sound rays describing a wave-packet moving in the ψ-direction. The equation reads

$$\frac{d\psi}{dt} = \sqrt{6\pi}\, V_s\, e^{\Omega}\, e^{2\beta_+} \quad , \tag{4.1}$$

where $V_s = (\frac{\partial p}{\partial \varepsilon})_s^{1/2}$ is the sound velocity. Setting $V_s = 1$ one would get the law of propagation for light going in the ψ-direction. Or, independently, one could get it from the geodesic equation $D_{\vec{V}} \vec{V} = 0$ as done in Appendix B. For the light going in the ψ-direction, setting $V^2 = V^3 = 0$ (the components of the tangent vector to the geodesic in the other two principal directions) in Equation (B 1.6) and using the null condition $(V^o)^2 - (V^o)^2 = 0$, one obtains

$$\frac{d\psi}{dt} = \sqrt{6\pi}\, e^{\Omega}\, e^{2\beta_+} \quad . \tag{4.2}$$

We want to study the above equations for large u (asymptotically $u = \infty$) epoch when the universe approximates a generalized Kasner solution with $p_1 = p_2 = 0$, $p_3 = 1$ which is that Kasner solution for which light propagates round the universe in the ψ-direction. The $u = \infty$ have two distinct cases - the axial and the non- axial ones. In the axial case, the system point is close to the β_+-axis and is running towards the corner, while the corner is moving outwards with unit velocity. It is more convenient to describe the motion of the system point in terms of β_o defined as

$$\beta_o \equiv \beta_+ - \Omega \quad . \tag{4.3}$$

β_o is then a measure of the horizontal distance from the corners. β_o is negative and it decreases as the universe feels the potential in the channel. The solution to Einstein's equations for this epoch can be described as

$$\beta_- = Z_o \left(\frac{2e^{2\beta_o}}{K}\right) \qquad \text{[See Appendix A, Eqn. (4.8).]}$$

where K is a small constant of motion, and Z_o is a spherical Bessel function. For large values of the argument, it reduces to

$$\beta_- = \frac{K}{\pi} e^{-\beta_o} \sin\left(\frac{2e^{2\beta_o}}{K} - \pi/4\right)$$

which corresponds to the adiabatic harmonic oscillator given by Misner (1969). Substituting this solution in the equations for sound rays, we find that high-frequency sound waves can go round the universe in the ψ-direction during seven cycles of β_-; where we have taken $V_s = \frac{1}{\sqrt{3}}$ corresponding to $p = \frac{1}{3}\varepsilon$ for radiation fluid. While the light ray would do the same thing during four cycles of β_-. The result corresponds to the one quoted by Doroshekevich and Novikov (1970) which reads

$$N_e = \frac{1}{4} N_m \quad ,$$

where N_e = number of times light goes round in the ψ-direction and N_m = number of maxima for β_-. Note that β_- is related to Q in their notation: $Q = e^{4\sqrt{3}\beta_-}$. Note also that the argument of the Bessel function for β_- is decreasing, since β_o is negative and decreasing. For small argument, we can write β_- as

$$\beta_- \simeq ax + b\log x \quad , \quad \text{where } x = \frac{2e^{2\beta_o}}{K} \quad .$$

Thus as x decreases, β_- increases and soon the approximation for $|\beta_-| \ll 1$ would fail. At this stage we can consider the system point to have come out of the channel.

Next consider the more general case, the off-axial case with $\beta_- > 1$ and u very large. Suppose that the system has bounced back off the vertical wall (at $\Omega = \Omega_b$) and is going towards the inclined wall with large u. The equipotential wall is moving outward with velocity 1/2 while the system point has velocity $\frac{1}{2} + \frac{3}{2u}$ towards the wall. Thus the system point will catch up with the wall and will experience its potential for a long time before getting bounced off. In the Appendix A, we compute the changes in various, relevant quantities as the system point enters and leaves the region where it experiences the potential of the wall. When we study the equation for propagation of sound rays during this epoch, we find the estimate of how large u should have been when it first bounced off the vertical wall, sothat a horizon is washed out during the next collision with the inclined wall. The relevant equation [Appendix A, Eqn. (3.7)] reads as

$$u > \frac{\sqrt{3}}{2V_s} 4\pi\, e^{-6(\beta_+)_{wall}}$$

$(\beta_+)_{wall}$ is negative; it goes like $(\beta_+)_{wall} \simeq -\frac{\Omega}{2} + \alpha$ where α is a constant. Thus, we find that at any epoch, Ω, there exist small sectors around the lines parallel to β_+-axis such that when the system point is running along these sectors at Ω, a horizon is removed in the ψ-direction. The angular extent of these sectors $(\theta = \sqrt{3}/u)$ depends upon Ω and it decreases to zero as Ω goes to ∞. Therefore, at each epoch Ω, we obtain a subset out of the set of all possible initial conditions $(\beta_+, \beta_-, \theta)$ such that the solutions corresponding to the initial data specified by this subset don't have a horizon in the ψ-direction. The totality of all such subsets will be called a circumnavigation set at that epoch. In the next chapter, we will show how to assign a set theoretical measure on these initial conditions.

Let us now follow the system point after the first bounce with the inclined wall. As shown in the Appendix A (first shown by Belinskii and Khalatnikov (1969)) u changes from u_i to $u_f = - u_i$, i.e. $\frac{d\beta_-}{d\Omega} < 0$. So the system point follows a Kasner epoch of straight line motion in the β_- plane till it comes close to the second inclined wall. A similar analysis as done in the Appendix A would show that a collision with this inclined wall changes u to $-u -2$, or the value of the parameter u now becomes $u-2$ and the system point leads back towards the first inclined wall. The process repeats itself till u becomes less than unity in which case it stops rattling back and forth between the inclined walls and heads towards the vertical wall and would start rattling back and forth in one of the other two corners. The above description can be described very elegantly by the Lifshitz-Khalatnikov bounce law $u \rightarrow u - 1$ [1969]. Here we take our fundamental interval for u to be $1 \leq u \leq \infty$. Other values of u can be seen to represent the solutions corresponding to the values of u in the fundamental interval by simple relabeling of the axes. The operators permuting the Kasner exponents p_1, p_2, p_3 are:

$$P_{12} : u \rightarrow - (1 + u) \qquad : (p_1, p_2, p_3) \rightarrow (p_2, p_1, p_3)$$

$$P_{23} : u \rightarrow 1/u \qquad : (p_1, p_2, p_3) \rightarrow (p_1, p_3, p_2)$$

and

$$P_{31} : u \rightarrow - u/(1+4) \qquad : (p_1, p_2, p_3) \rightarrow (p_3, p_2, p_1) \quad .$$

Thus the bounce law : $u_f = - u_i$ given above can be transformed into $u_f = u_i - 1$ by the operation P_{23} B P_{12} P_{31}. The evolution of the Mixmaster universe, then consists of one Kasner epoch represented by u (≥ 1) replaced by another with $u - 1$. So it continues till the integral part of the initial value u is exhausted, i.e. until u becomes less than one.

The transformation p_{23}: $u \to 1/u$ will put u back in the fundamental interval and that will start another series of Kasner epochs. These successive series are called eras and the length of an era is given by the number of Kasner epochs it contains (see Belinski etc. (1970)). If the whole sequence begins by the number $u^o = k^o + x^o$, where k^o is the integral part of u^o, then the lengths k^1, k^2, k^3, ... of the successive eras are given by the numbers appearing in the expansion of x^o in an infinite continuous fraction

$$x^o = \cfrac{1}{k^1 + \cfrac{1}{k^2 + \cfrac{1}{k^3 + \cdots\cdots}}}$$

Belinskii etc. (1970 do an algebraic study of the above equation and prove that with increasing numbers of eras, the values of x approach a stationary distribution, i.e. if we start with a certain probabilistic distribution for x in the range (0,1) at some epoch Ω_o and follow the evolution of the system points towards the singularity, the values of x take on a stationary distribution asymptotically.

In the next chapter, we will use geometric methods to investigate the statistical properties of the Mixmaster universe. We will prove that the evolution of the universe is ergodic in a certain phase space related to β_+, β_- and u. Thus we prove certain statistical properties not only of the anisotropic expansion (which is related to u or x) but also of the anisotropy (β_+, β_-) of the universe.

CHAPTER V.

The Ergodic Behavior and the Horizon Problem

A. The Evolution of the Mixmaster Universe.

The evolution of the Mixmaster Universe is given completely by specifying its shape anisotropy "β" and the expansion rate anisotropy p_+, p_- at some epoch, Ω_o. As we follow its trajectory, the system point seems to wander about rattling back and forth between the equipotential walls and moving along various directions (direction specified by θ; $\tan\theta = p_+/p_-$). In this chapter, we will study the exact nature of this wandering motion. We will show that the evolution of the Mixmaster Universe is equivalent to a geodesic flow within a bounded region of the Lobatchewsky plane. We will find a certain group of transformations, G, which make this flow of broken geodesics on the Lobatchewsky plane, D, into a continuous one on D/G.

As we saw earlier, the variational principle for Einstein's equations $\delta I = 0$ with

$$I = \frac{1}{16\pi} \int R \sqrt{-g}\ d^4x$$

can be cast into a canonical form to obtain the Hamiltonian

$$H = [p_+{}^2 + p_-{}^2 + e^{-4\Omega}(V - 1)]^{1/2} \tag{5.1}$$

where p_+ and p_- are the momenta conjugate to the field amplitudes β_+ and β_- respectively.

Let us rewrite the corresponding Hamilton's equations:

$$\frac{d\beta_+}{d\Omega} = p_+/H, \qquad \frac{d\beta_-}{d\Omega} = p_-/H$$

$$\frac{dp_+}{d\Omega} = \frac{-e^{-4\Omega}}{2H}\frac{\partial V}{\partial \beta_+}, \qquad \frac{dp_-}{d\Omega} = \frac{-e^{-4\Omega}}{2H}\frac{\partial V}{\partial \beta_-} \tag{5.2}$$

and

$$\frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega} = -\frac{2e^{-4\Omega}(V-1)}{H} .$$

Let us introduce a new independent variable, λ, defined by

$$d\lambda = \frac{d\Omega}{H} . \tag{5.3}$$

The Hamilton's equations can be written now as

$$\frac{d\beta_+}{d\lambda} = p_+, \quad \frac{d\beta_-}{d\lambda} = p_-, \quad \frac{d\Omega}{d\lambda} = -p_\Omega$$

$$\frac{dp_+}{d\lambda} = -\frac{1}{2}e^{-4\Omega}\frac{\partial V}{\partial \beta_+}, \quad \frac{dp_-}{d\lambda} = -\frac{1}{2}e^{-4\Omega}\frac{\partial V}{\partial \beta_-}, \quad \frac{dp_\Omega}{d\lambda} = 2e^{4\Omega}(V-1) \tag{5.4}$$

where

$$-p_\Omega \equiv H = [p_+^2 + p_-^2 + e^{-4\Omega}(V-1)]^{\frac{1}{2}} . \tag{5.5}$$

As Misner (1971) points out, one can see that equations (5.4) are just a new set of Hamilton's equations

$$\frac{dg^A}{d\lambda} = \frac{\partial K}{\partial p_A}, \quad \frac{dp_A}{d\lambda} = -\frac{\partial K}{\partial g^A} \tag{5.6}$$

with λ as the independent variable and

$$K = \frac{1}{2}[-p_\Omega^2 + p_+^2 + p_-^2 + e^{-4\Omega}(V-1)] \tag{5.7}$$

as the new Hamiltonian. Indices A, B, etc. are used to label coordinates in superspace, so $g^\Omega = \Omega$, $g^+ = \beta_+$, $g^- = \beta_-$. The variational principle can now be written as

$$0 = \delta \int [p_\Omega d\Omega + p_+ d\beta_+ + p_- d\beta_- - K d\lambda] \tag{5.8}$$

the Einstein's equations in addition giving the constraint $K = 0$. Consider now the behavior of the potential energy like term, R in the Hamiltonian:

$$\begin{aligned} R &\equiv e^{-4\Omega}[V(\beta_+, \beta_-) - 1] \\ &\sim \frac{1}{3} e^{-4\Omega - 8\beta_+} \text{ for } \beta_+ \to -\infty \quad . \end{aligned}$$

The condition that V be important is then given by

$$e^{-4(\Omega + 2\beta_+)} \approx 3H^2$$

or asymptotically, the "potential wall" defined by β_{wall} will be given by

$$\beta_{wall} = -\frac{1}{2}\Omega - \frac{1}{8}\ln(3H^2) \quad . \tag{5.9}$$

For large Ω approximation, we can ignore the variations in the second term, $\frac{1}{8}\ln(3H^2)$ (which would go as $\ln\ \Omega$). Thus the equipotential in the β-plane bounding the region in which the potential (space curvature) terms are significant is given by

$$\beta_{wall} = -\frac{1}{2}\Omega + \alpha \quad . \tag{5.10}$$

Consider the following set of transformations (see Misner, 1971) which would make the above walls stationary in the new coordinates:

$$\Omega - 2\alpha = e^{t} \cosh\xi$$
$$\beta_{+} = e^{t} \sinh\xi \cos\phi \tag{5.11}$$
$$\beta_{-} = e^{t} \sinh\xi \sin\phi \,.$$

The equipotential walls now will be given by

$$\tanh \xi = -\frac{1}{2} \sec \phi \tag{5.12}$$

and the other two are obtained by replacing $\phi \rightarrow \phi \pm \frac{2\pi}{3}$ in the above formulae.

Substituting the new canonical coordinates into the action, the variational principle reads now

$$0 = \delta \int (p_t dt + p_\xi d\xi + p_\phi d\phi - K\, d\lambda) \tag{5.13}$$

where

$$p_t = e^{t}(p_{+} \sinh\xi\cos\phi + p_{-} \sinh\xi\sin\phi + p_{\Omega}\cosh\xi)$$
$$p_\xi = e^{t}(p_{+} \cosh\xi\cos\phi + p_{-} \cosh\xi\sin\phi + p \sinh\xi)$$
$$p_\phi = e^{t}(- p_{+} \sinh\xi\sin\phi + p_{-} \sinh\xi\cosh\phi) \,.$$

In terms of these new canonical variables, the Hamiltonian, K, is given by

$$2K = (- p_t^{\,2} + p_\xi^{\,2} + \frac{p_\phi^{\,2}}{\sinh^2\xi}) e^{-2t} + R(\xi, \phi) \,. \tag{5.14}$$

The factor e^{-2t} can be removed by defining a new variable $d\lambda' = e^{-2t}d\lambda$, to give the new Hamiltonian

$$2K' = -p_t^{\,2} + p_\xi^{\,2} + \frac{p_\phi^{\,2}}{\sinh^2\xi} + R(\xi, \phi)e^{2t} \quad .$$

In the asymptotic limit, taking $R(\xi, \phi)$ to be zero inside the equipotential walls and $+\infty$ outside, we find that the evolution of the Mixmaster Universe inside the equipotential walls is governed by the stationary Hamiltonian,

$$K' = \frac{1}{2}\left(-p_t^{\,2} + p_\xi^{\,2} + \frac{p_\phi^{\,2}}{\sinh^2\xi}\right) \quad . \tag{5.15}$$

Note that for a particle Hamiltonian

$$K = \frac{1}{2}\, g^{\mu\nu} p_\mu p_\nu$$

the motion of the particle is along the geodesics of the manifold possessing the metric $g_{\mu\nu}$. Thus, the evolution of the Mixmaster Universe as projected on the two-dimensional space ξ, ϕ will be given by a geodesic flow on the Riemannian manifold with the metric

$$(d\ell)^2 = d\xi^2 + \sinh^2\xi(d\phi)^2$$

inside the region bounded by the curves given by (5.12). By making a transformation $\sinh\xi = 2r/(1 - r^2)$, we obtain the metric on the Lobatchewsky plane which is a unit disc with constant negative curvature. The metric for it can be written in the cartesian coordinates as

$$(d\ell)^2 = \frac{4(dx^2 + dy^2)}{(1 - x^2 - y^2)^2} \quad . \tag{5.16}$$

The geodesics on this manifold are circular arcs, orthogonal to the circle $r = 1$. The equipotential walls are now given by the three circular arcs as shown in Fig. 4. These arcs bound the fundamental region in which the system point moves along geodesics (circular arcs) until it hits the wall and is then bounced back along another geodesic and so it goes on. The "bounce law" would be governed by the shape of the border.

A typical solution to Einstein's equations would be given by a mesh of broken geodesics. We will show that the solution can be represented by a continuous geodesic curve under a suitable set of transformations. Consider the following transformation which takes one wall (given by a circular arc) into a vertical diameter as shown in Fig. 5:

$$A : z \to \frac{z + (2 - \sqrt{3})}{(2 - \sqrt{3})z + 1} \tag{2.12}$$

It is a conformal transformation, thus preserving the metric. The bounce law at any point on the diameter AB is very simple as it is governed by two constants of motion, the Hamiltonian

$$H' = \frac{(1 - x^2 - y^2)^2}{4} (p_x{}^2 + p_y{}^2) \tag{2.18}$$

and the vertical component of the momentum p_y. Thus $\frac{dy}{dx} = p_y/p_x$ has the same value before and after the bounce. Hence, the transformation $z \to -\bar{z}$ would transform the reflected geodesic onto a continued part of the incident geodesic. By applying A^{-1} now we recover the fundamental region. Figure 6 illustrates how the transformation $R_1 = A^{-1}RA$ applied to a continuous part of the incident geodesic outside the fundamental region gives the proper reflected geodesic inside the fundamental region. Similarly, one could construct transformations R_2 and R_3 for the other two walls or they can be generated from R and $S = e^{\frac{2\pi}{3} i}$. Figure 7 finally shows how a repeated application of these transformations would give a representation of a broken mesh of geodesic lines by a continuous one. Thus the evolution of our model is now given by a continuous geodesic flow on D/G; where D is the unit disc with the metric given by (5.16), and G is the group generated by R_1, R_2 and R_3 or by R_1 and S. To get an orientable manifold, consider a subgroup F in G defined by

$$F = \{R_{i_1}, R_{i_2} \ldots, R_{i_n}\} \text{where } n \text{ is even} . \qquad (5.17)$$

Thus, considering points which are congruent under G as identical, we obtain a closed, orientable, two-dimensional Riemannian manifold of constant negative curvature. We will denote it by M_F ($\equiv$ D/F). These groups (called Fuchsian groups) have been extensively studied (see Hedlund, [1939] and references therein). Some of the important properties of these groups for the relevance of geodesic flow on M_F are the following:

Corresponding to any such group F there exists a normal fundamental region R. This is a simply connected region bounded by arcs of hyperbolic lines (circular arcs perpendicular to the circle $|z| = 1$) which are congruent in pairs, such that no two interior points are congruent and any point of D is congruent to some point within or on the boundary of R. The region R for our group F is ABCP in Figure 8.

The group F given by (5.17) is a Fuchsian group of the first kind which is characterised by the fact that the fundamental region does not contain an interval of $|z| = 1$. The geodesics on M_F are determined uniquely by specifying a point (x, y) in the fundamental region and a direction $\theta(0 \leq \theta < 2\pi)$ The set of these elements $[(x, y, \theta)]$ forms a Hausdorff space, denoted by Ω_F. Measure in Ω_F is defined by the integral

$$\int\int\int \frac{4dxdyd\theta}{(1 - x^2 - y^2)^2} \quad . \tag{5.18}$$

To prove certain results for the geodesic flow on M_F, the property that plays a crucial role is that the area of Ω_F is finite. Hopf (1936) and Hedlund (1939) have proved that the geodesic flow on M_F is ergodic in the metrically transitive sense, i.e., if we start with any arbitrary measurable set M_o of Ω_F (points of M_o denote initial conditions of the Mixmaster Universe at some time Ω_o), the corresponding universes given by the geodesics specified by M_o will have non-zero intersection with any measurable set M of Ω_F. The other important result (again proved by Hedlund [1939]) says that the geodesic flow on M_F is a "mixture", i.e., any measurable set M of positive measure tends, with increasing or decreasing time, to occupy a definite fractional part of any other measurable set M*, and the fraction is simply the fractional part of Ω_F which M occupies. This can be written as

$$\lim_{t \to \pm\infty} m(M_t \cdot M^*) = \frac{(mM)\ (mM^*)}{m\Omega_F}$$

where mM denotes the measure of M. The mixture property tells us that the sets in Ω_F tend towards homogeneous distribution. Thus we obtain the result that in whatever range of anisotropy shape (specified by a certain set of x and y values) and anisotropic expansion rates (specified by a certain range of θ) the universe might have started near the singularity, we will expect it to have a uniform distribution over Ω_F -- far away from the singularity. This then characterises the "chaos" of the universe. If we start with a well-defined state of the universe and let it evolve towards the singularity, we find that it goes through almost all possible anisotropic stages.

B. <u>The Probability for the Vanishing of a Horizon.</u>

As seen earlier, the light can circumnavigate the universe in the ψ-direction if

$$\frac{\frac{2}{3}\, e^{2\beta_+ - 2\Omega}\, u}{H} \geq 4\pi, \text{ for large } u \quad . \tag{5.19}$$

In the asymptotic limit approximations,

$$H = \frac{1}{\sqrt{3}} e^{-4\alpha}$$

where the position of the equipotential wall is given by

$$(\beta_+)_{wall} = -\frac{1}{2}\Omega + \alpha \quad .$$

Thus, the circumnavigation sets are given by small sectors of angle

$$\delta\theta \approx \frac{\sqrt{3}}{u} = \frac{1}{2\pi} e^{4\alpha} e^{2\beta_+ - 2\Omega} \quad . \tag{5.20}$$

When converted into the new coordinates, the angle is given by

$$\delta\theta \approx \frac{1}{2\pi} e^{-2e^{t}\left[\frac{1 + x^2 + y^2 - 2x}{1 - x^2 - y^2}\right]} \quad . \tag{5.21}$$

The geodesic flow is given by the Hamiltonian

$$H'^2 = \frac{(1 - x^2 - y^2)^2}{4} [p_x^{\,2} + p_y^{\,2}] \quad .$$

Writing $p_x = \dfrac{2H'\cos\theta}{(1 - x^2 - y^2)}$ and $p_y = \dfrac{2H'\sin\theta}{(1 - x^2 - y^2)}$, the invariant volume element would be given by

$$dx\wedge dy\wedge dp_x\wedge dp_y = \frac{4\ dx\wedge dy\wedge H'dH'\wedge d\theta}{(1 - x^2 - y^2)^2}$$

Since the motion of the system point is ergodic, let us choose a stationary microcanonical ensemble distribution in the phase-space x, y, p_x, p_y. If N is the total number of solutions, then the probability distribution function will be given by

$$\rho(x, y, \theta, H') = \frac{N}{8\pi h'} \frac{\delta(H' - h')}{(\text{Area})} \tag{5.22}$$

where the area, A, is given by

$$A = \iint_{\substack{\text{fundamental}\\ \text{region}}} \frac{dxdy}{(1 - x^2 - y^2)^2} \quad .$$

The probability for a typical solution to lie in the circumnavigation set at time t will then be given by

$$\frac{4}{N} \int \frac{\rho(x, y, \theta, H')dxdyd\theta dH'H'}{(1 - x^2 - y^2)^2} \tag{5.23}$$

integrated over the circumnavigation set. Note that these sets given by (5.21) decrease exponentially as t (or Ω) increases. So it is a fairly good estimate to compute the above probability at the minimum possible value of t. The $t_{min.}$ will be calculated so that our approximations are valid until that epoch. It turns out to be very close to the turn-around (the maximum expansion) epoch for an empty universe. For a more realistic model (i.e., radiation and matter containing universe) the $t_{min.}$ epoch, corresponding to the radiation energy being as dominant as the anisotropic energy, is much earlier and thus reduces the probability for a horizon vanishing considerably.

To see the validity of approximations, note that $H' = p_t$ is a constant given by the value of p_t from (5.13). Thus

$$H' = H(\Omega - 2\alpha) - \beta_+ p_+ - \beta_- p_-$$

When the system point is colliding with the vertical wall, $p_\pm \approx 0$ and the position of the wall is given by

$$H^2 = \frac{1}{3} e^{-4\Omega} e^{-8\beta_+} \quad .$$

Combining these two equations, we obtain

$$(\beta_+)_{wall} = -\frac{1}{2}\Omega + \frac{1}{4}\log(\Omega - 2\alpha) - \frac{1}{4}\log(\sqrt{3}\,H') \quad . \tag{5.24}$$

Writing $\alpha = -\frac{1}{4}\log(\sqrt{3}\,H')$ we see that taking

$$(\beta_+)_{wall} = -\frac{1}{2}\Omega + \alpha$$

is a good approximation provided

$$\left|\frac{1}{4}\log(\Omega - 2\alpha)\right| < \left|-\frac{1}{2}\Omega + \alpha\right| \tag{5.25}$$

or

$$|\Omega - 2\alpha| > \frac{1}{2} \quad .$$

We now substitute the probability distribution function from equation (5.22) into (5.23) and integrate over the circumnavigation set given by (5.21). Taking the value of t at $t_{min.}$ i.e., $e^{t_{min.}} = \frac{1}{2}$, the probability for a typical solution not to have any horizon in the ψ-direction is then obtained as

$$P = \frac{N}{4\pi^2 NA} \iint \frac{dxdy}{(1 - x^2 - y^2)^2} e^{-2(\frac{1}{2})\left(\frac{1 + x^2 + y^2 - 2x}{1 - x^2 - y^2}\right)} \quad .$$

Since the factor $\left(\frac{1 + x^2 + y^2 - 2x}{1 - x^2 - y^2}\right)$ is always less than unity in the fundamental region, a lower limit on the probability is given by

$$P = \frac{1}{4\pi^2}\frac{1}{e} \approx 1\% \quad .$$

Let us now consider the effect of observed radiation on the evolution of the Mixmaster Universe. The Hamiltonian would be modified to read

$$H^2 = p_+{}^2 + p_-{}^2 - g\,{}^3R + \frac{8\pi G}{c^4} g\, T^{00} \quad .$$

The radiation energy density T^{00} is given by

$$T^{00} R^4 = (T^{00})_o R_o{}^4$$

where $(T^{00})_o$ is the energy density now and R_o is the present radius of the universe. The radius of the universe is given by

$$R = \frac{1}{2} R_o e^{-\Omega} \quad .$$

By taking the length factor R_o to be the present radius of the universe, we get Ω to be equal to 0.70 for the present epoch. Taking $(T^{oo})_o = 5 \times 10^{-13}$ ergs/cm^3 corresponding to the black-body radiation temperature 2.7°K and $R_o = 10^{28}$cm., we obtain

$$T^{oo} = 16 \times e^{4\Omega} \times 5 \times 10^{-13} \quad .$$

Substituting the expressions for ^3R and T^{oo} in the Hamiltonian we obtain

$$H^2 = p_+{}^2 + p_-{}^2 - \frac{1}{64} R_o{}^6 \; e^{-6\Omega} \frac{6}{R^2} (1 - V) + \frac{8\pi G}{c^4} \frac{R_o{}^6}{64} e^{-6\Omega} \; T^{oo}$$

$$= p_+{}^2 + p_-{}^2 + \frac{3}{8} R_o{}^4 \; e^{-4\Omega}(V - 1) + \frac{8\pi G}{c^4} \frac{R_o{}^6}{4} e^{-2\Omega} \; 5 \times 10^{-13} .$$

Writing $V = \frac{1}{3} e^{-8\beta+} = \frac{1}{3} e^{4\Omega-8\alpha}$ and calling $\Omega = \Omega_a$ for the epoch when the radiation energy is as important as the anisotropy-energy, we get

$$\frac{1}{8} e^{-8\alpha} = \frac{8\pi G}{c^4} \frac{5}{4} R_o{}^2 \; e^{-2\Omega_a} \times \; 10^{-13}$$

$$= \frac{10^{-27}}{9 \times 10^{20}} \quad \frac{5}{4} 10^{56} \times 10^{-13} \times e^{-2\Omega_a}$$

or

$$e^{-8\alpha + 2\Omega_a} = 10^{-4}$$

$$\text{i.e.} \qquad \begin{aligned} \Omega_a &= 4\alpha - 2 \log 10 \\ &= 4\alpha - 4.6 \end{aligned} \quad .$$

Thus, $t_{min.}$ is given by

$$e^{t_{min.}} = \Omega_a - 2\alpha = 2\alpha - 4.6 \quad .$$

So the probability for a horizon vanishing in the t-direction computed at $\Omega = \Omega_a$ will be given by

$$p(\alpha) = \frac{1}{4\pi^2} e^{-2(2\alpha - 4.6)} \approx 2 \times 10^2 e^{-4\alpha} \quad . \tag{5.26}$$

Note that a crude value for the estimate of minimum value of α will be given by assuming that the anisotropy decays as $\beta_+ = -\frac{\Omega}{2} + \alpha$ and demanding that there should be no anisotropy left at $T = 3{,}000°K$, which corresponds to the radius $R_1 = 2 \times 10^{22}$cm. and correspondingly $\Omega \approx 7$, giving $\alpha_{min.} = 3.5$. Substituting, the minimum value of α in equation (5.26) we obtain the probability for the vanishing of a horizon to be 0.02%.

Thus, we conclude that there is a very small but finite probability for a horizon vanishing (in one direction) before the photons get decoupled from the matter.

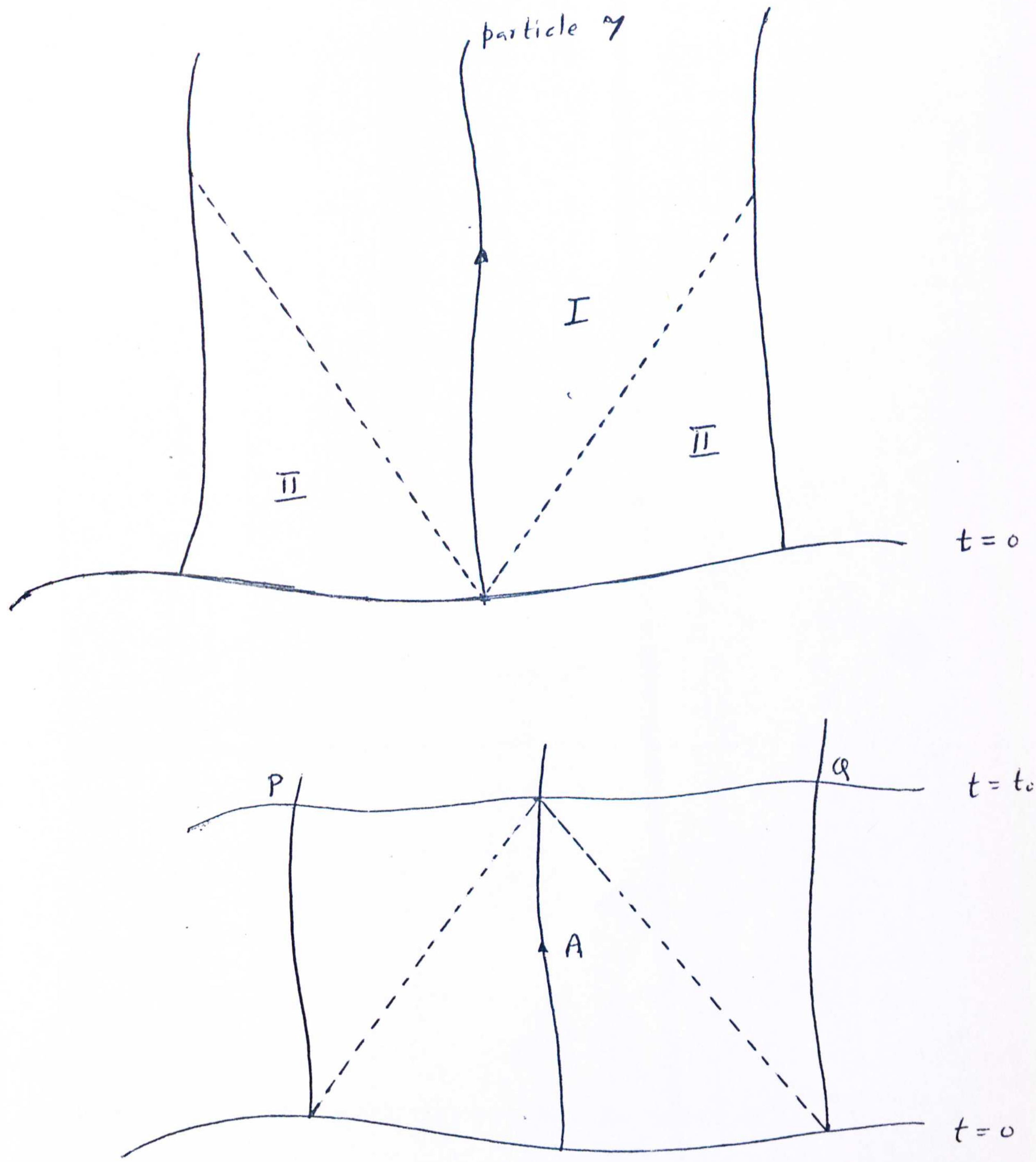

FIGURE 1. Description of particle horizons.

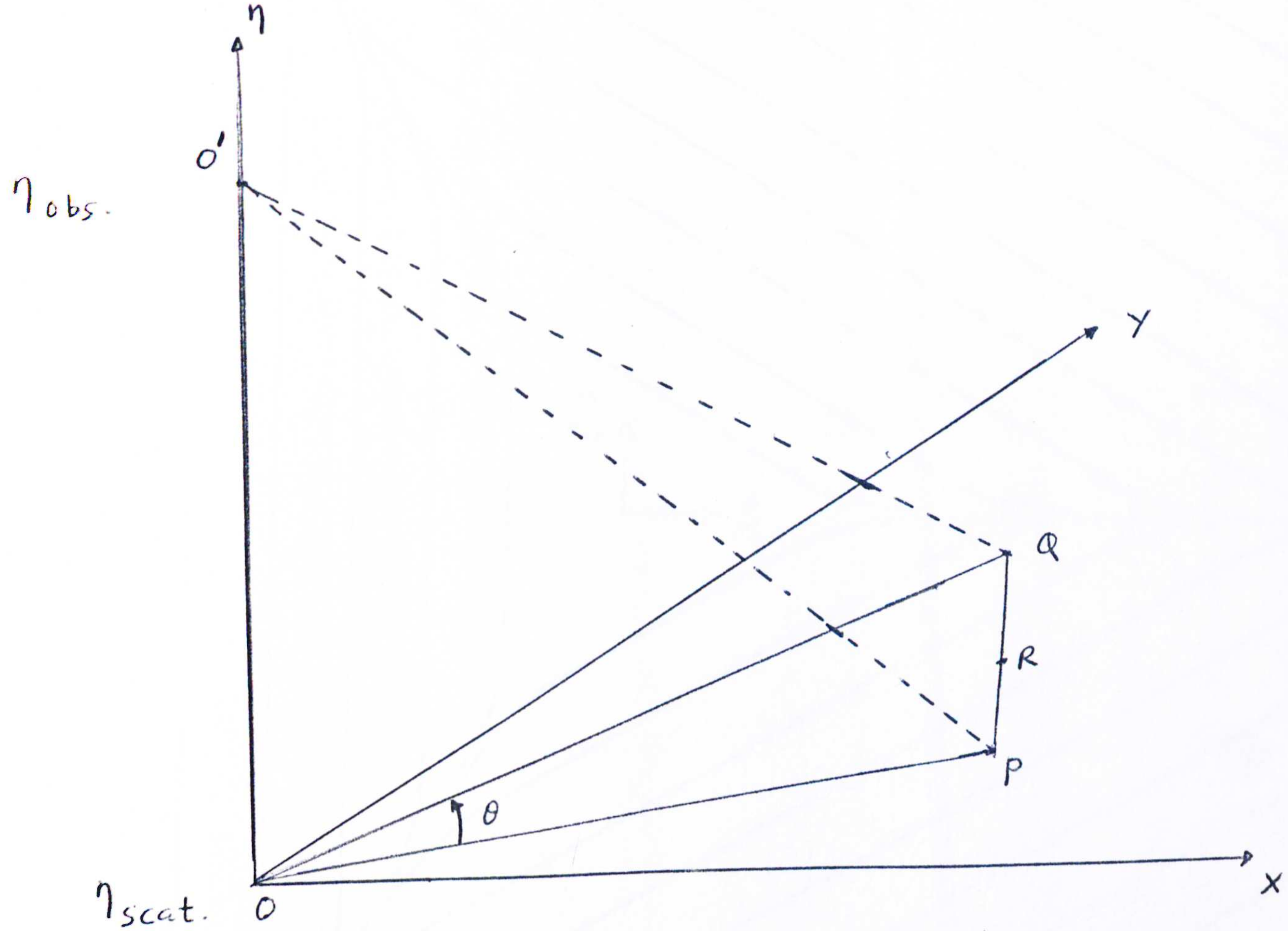

FIGURE 2. The last scattering of microwave radiation.

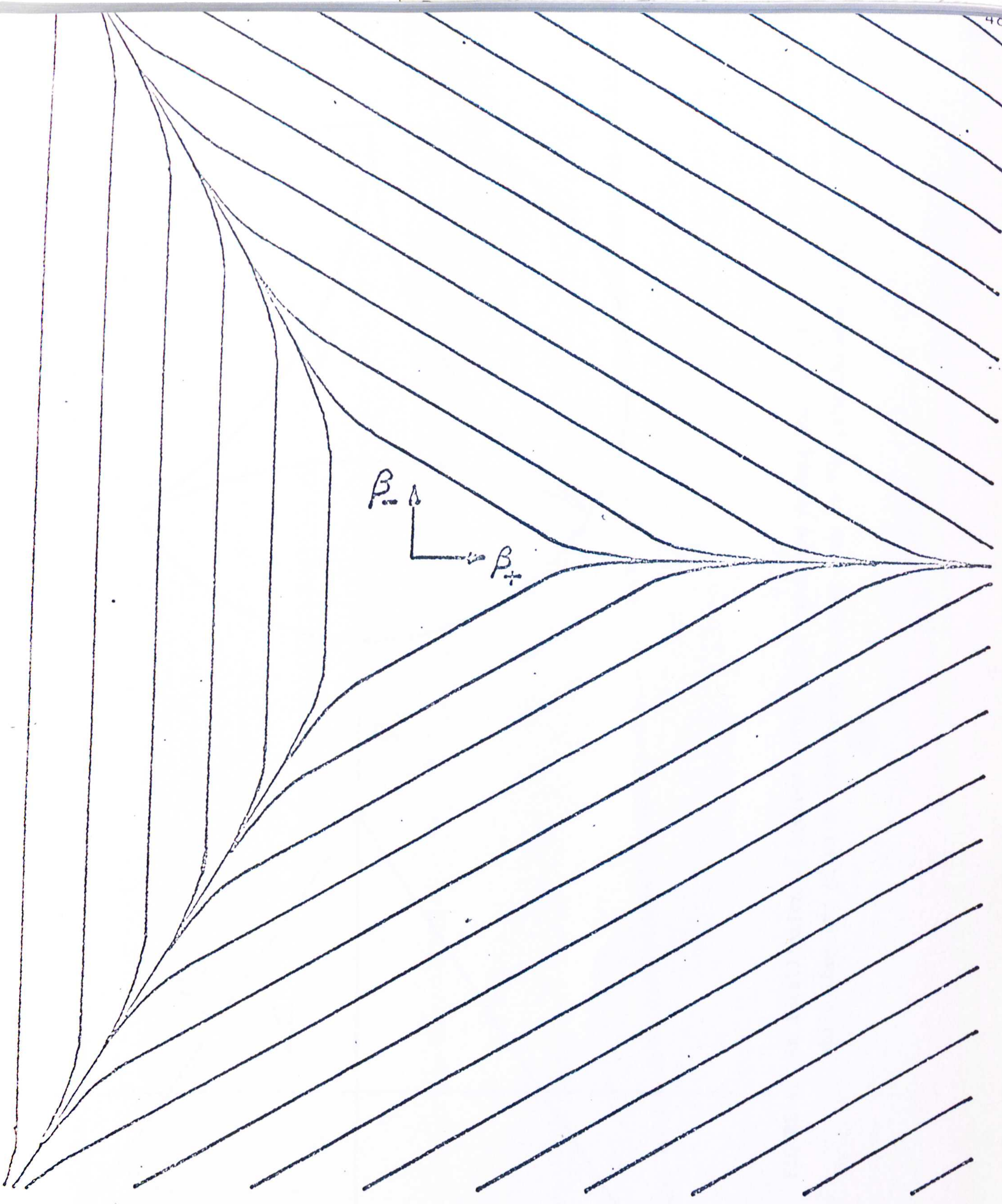

FIGURE 3. The equipotentials of $V(\beta_+, \beta_-)$ for large "β". (Figure courtesy of C. Misner.)

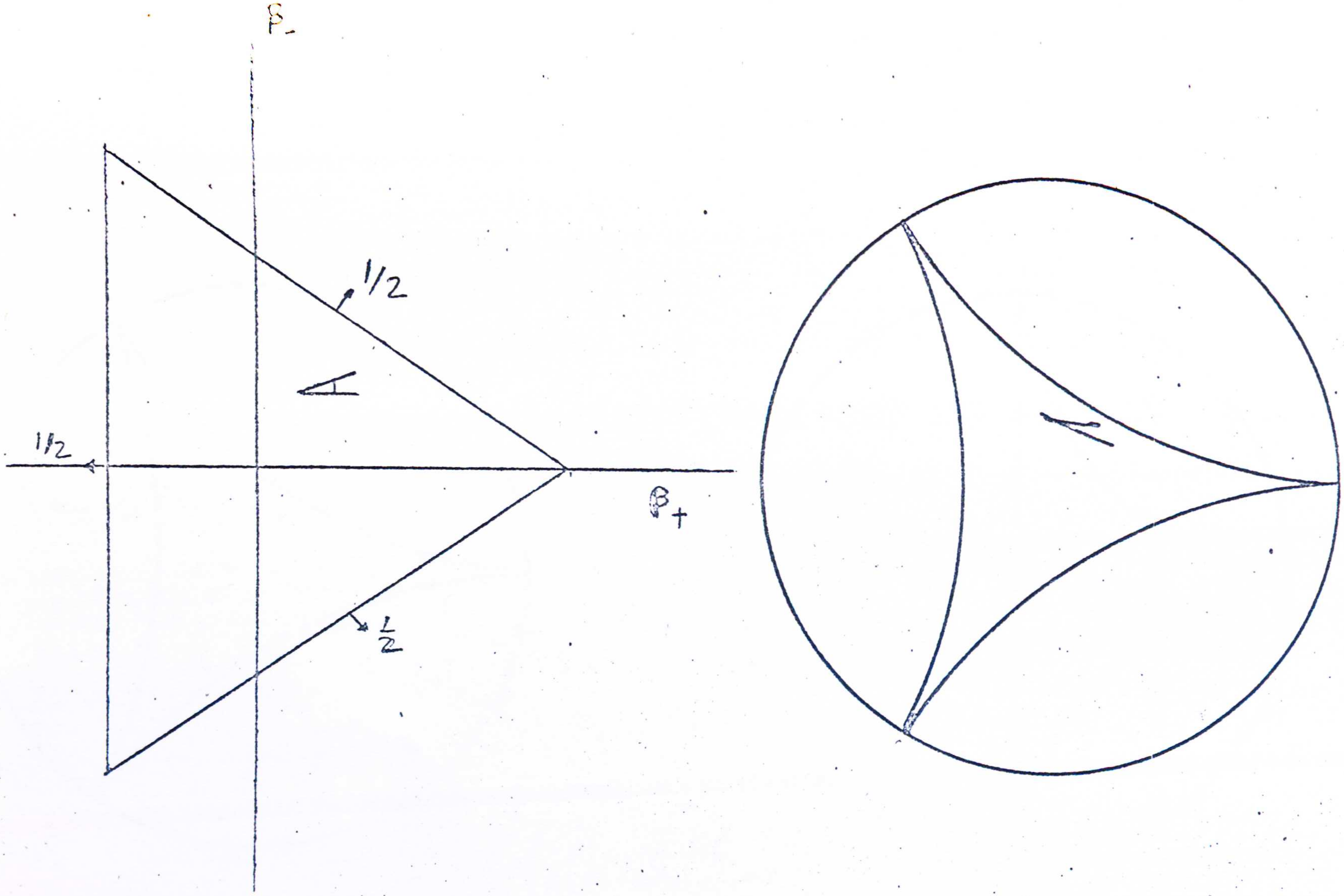

FIGURE 4. An idealization of equipotential walls which are moving out in $\beta_+\beta_-$ plane while they are given by stationary circular arcs in the Lobatchewshy plane.

$$d\ell^2 = \frac{4|dz|^2}{(1-|z|^2)^2} \;;$$

$$A: z \to \frac{z + (2-\sqrt{3})}{(2-\sqrt{3})z + 1}$$

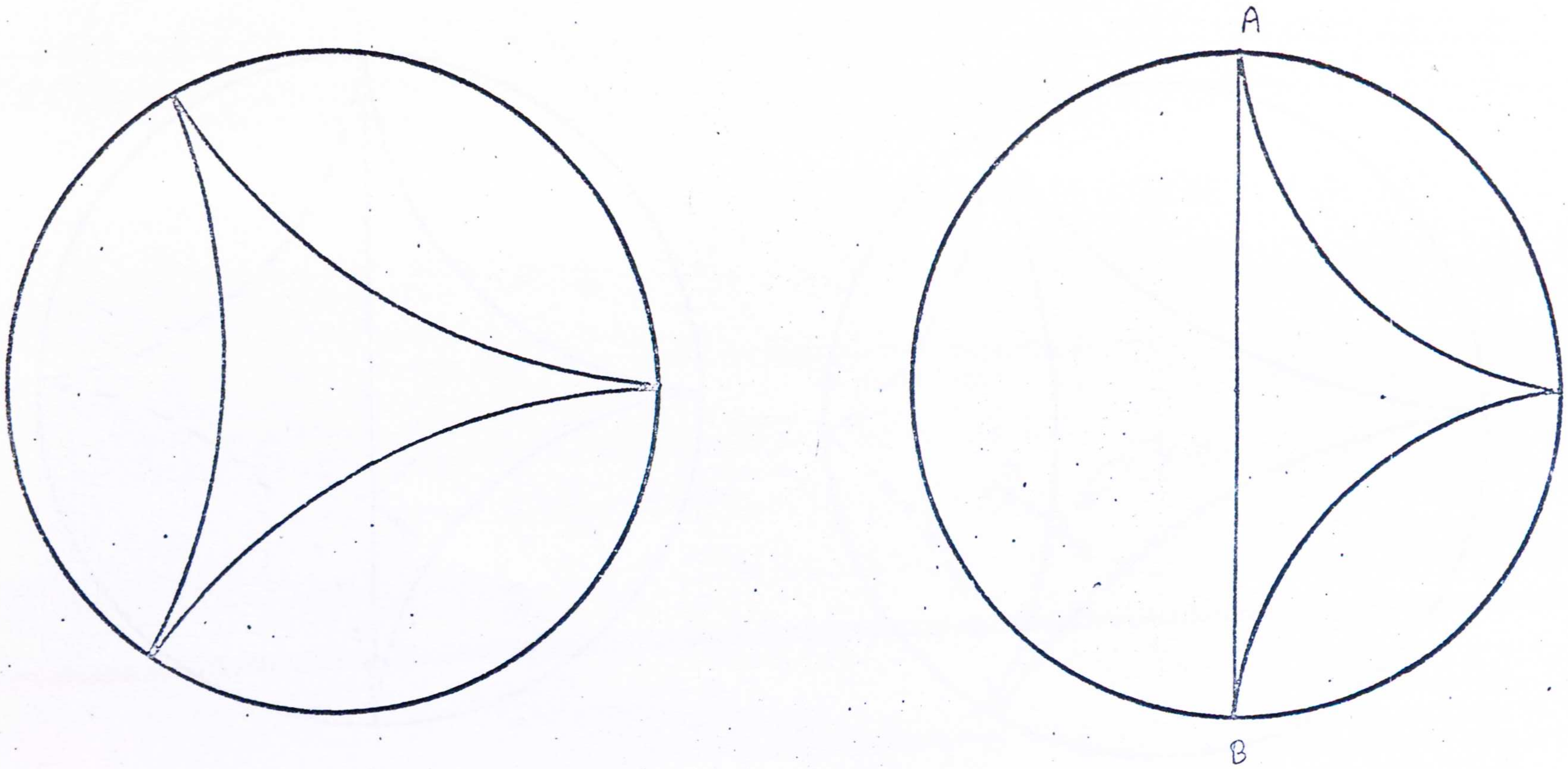

FIGURE 5. A is the transformation which takes the equipotential wall represented by a circular arc into a diameter.

$R: z \to -\bar{z}$ $\qquad$ $A^{-1}: z \to \dfrac{-z + (2-\sqrt{3})}{(2-\sqrt{3})z - 1}$

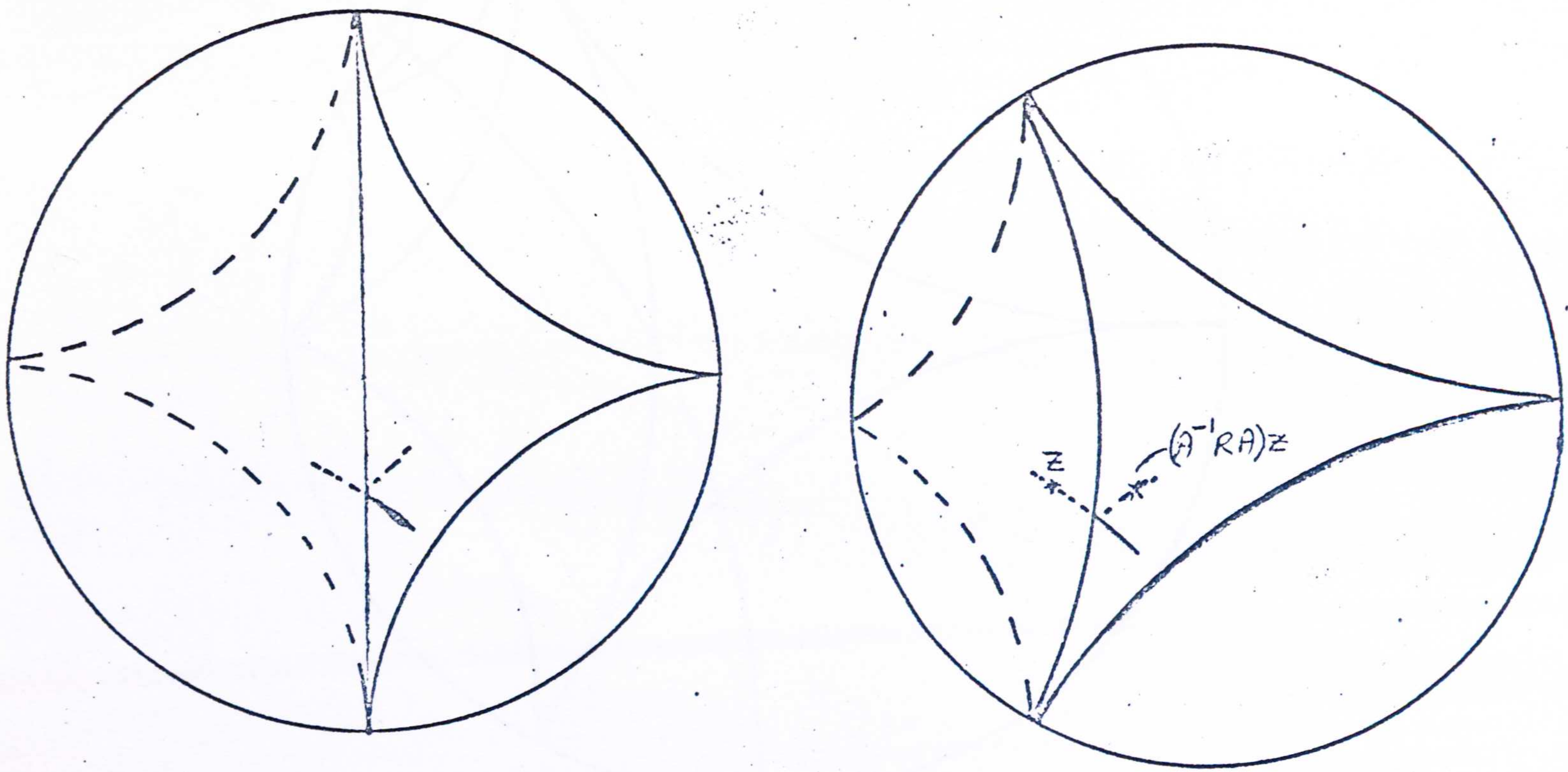

FIGURE 6. The effect of the transformation $A^{-1}RA$ is to map the reflected geodesic into a continued part of the incident trajectory.

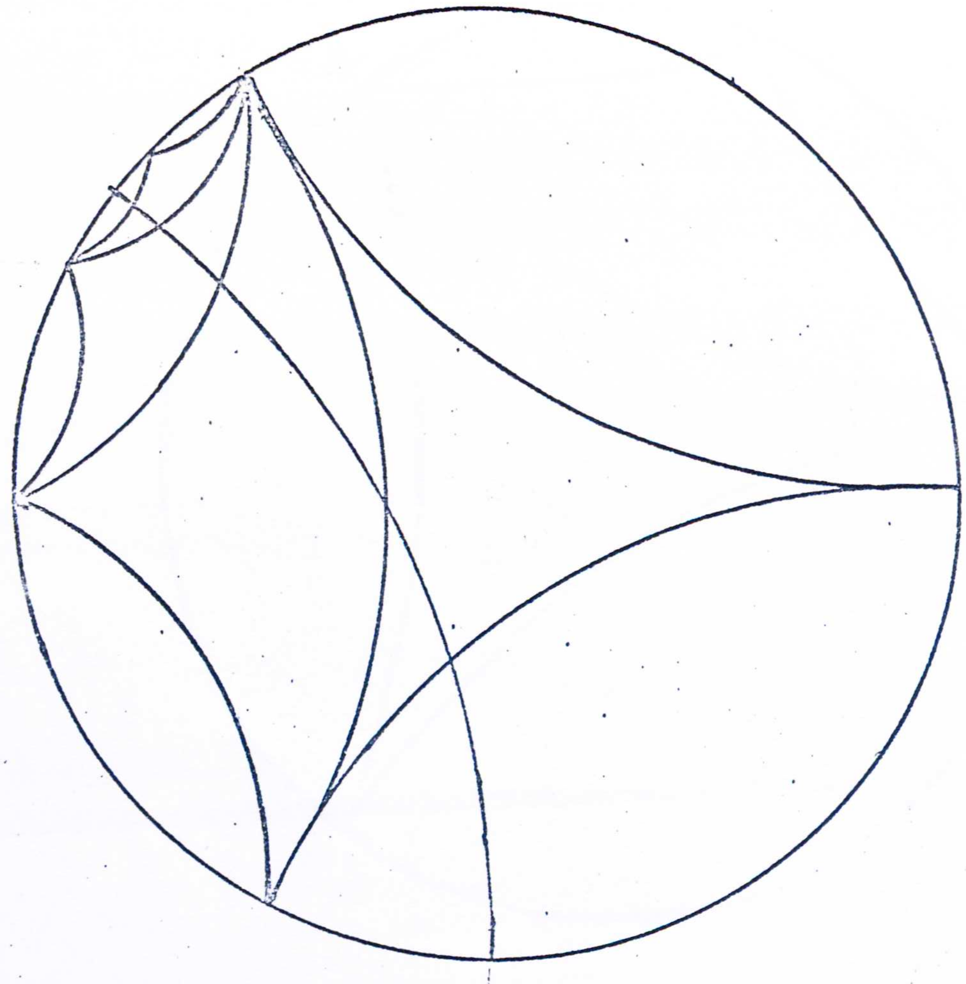

FIGURE 7. A repeated application of the transformations $S = A^{-1}RA$ makes a typical solution of the Mixmaster Universe evolve along a continuous geodesic.

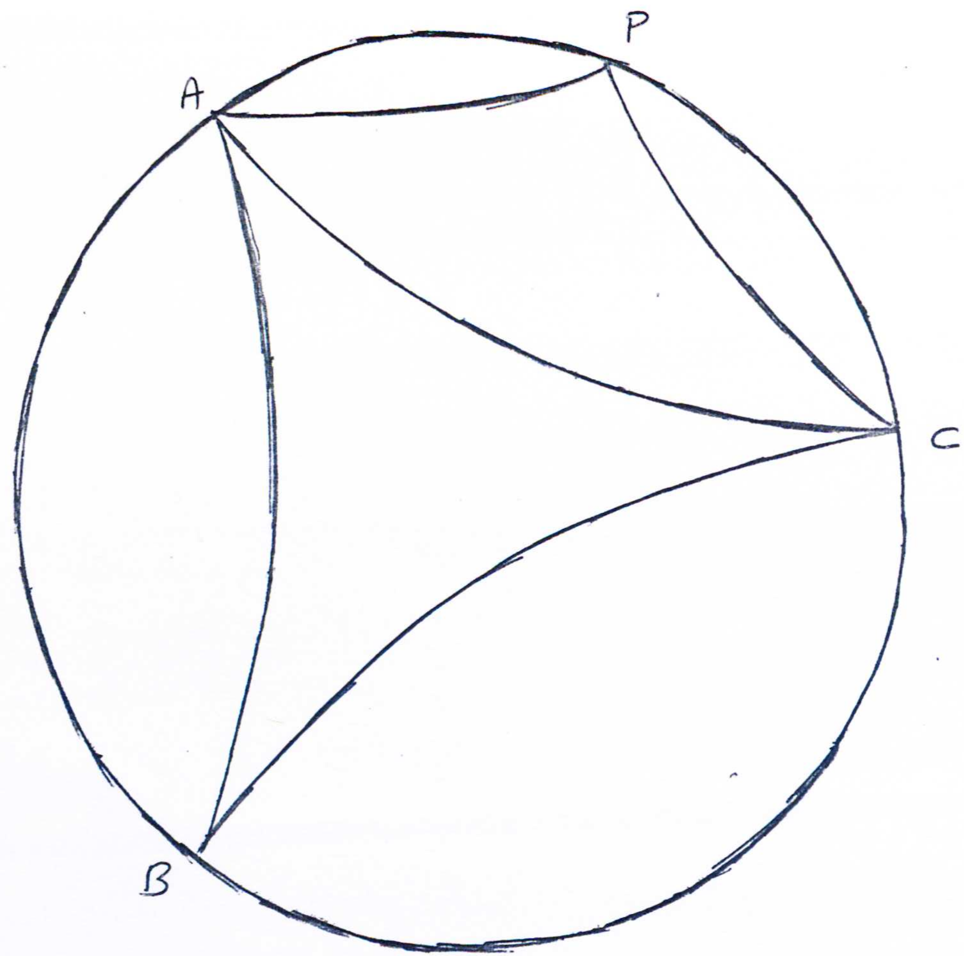

FIGURE 8. The area bounded by arcs AB, BC, CP and PA represents the normal fundamental region of the group, F.

APPENDIX A

High-frequency Sound Waves To Eliminate A Horizon In The Mixmaster Universe

ABSTRACT

From the linear wave equation for small amplitude sound waves in a curved spacetime, there is derived a geodesic-like differential equation for sound rays to describe the motion of wave packets. These equations are applied in the generic, non-rotating, homogeneous closed model universe (the "Mixmaster Universe", Bianchi Type IX). As for light rays [described by Doroshkevich and Novikov (DN)], these sound rays can circumnavigate the universe near the singularity to remove particle horizons only for a small class of these models and in special directions. Although these results parallel those of DN, different Hamiltonian methods are used for treating the Einstein equations.

1. Introduction

The present day universe can be described very well by the Robertson-Walker cosmological models. The extrapolation of these models for the early times of the universe gives rise to the problem of particle horizons.[1] A particle horizon at a particular epoch bounds each finite part of the universe which could have been spanned by a causal signal during the time available since the initial singularity. Since the Robertson-Walker models possess particle horizons, only a finite part of such universe could have been causally connected. Thus, we are faced with the observation of the microwave background radiation having precisely ($\leq 0.2\%$) the same temperature[2] in widely different directions even though the regions of plasma which scattered the radiation last had no prior causal relationship. The Robertson-Walker models therefore are too simplified to describe the early phase of the universe. Here we would consider a more general model of the universe - the non-rotating Bianchi Type IX model. It has a very different singularity behavior,[3,4] but it could evolve into the closed Robertson-Walker model at the present epoch. Misner[3] first pointed out the possibility of mixing by light in these models.

Doroshkevich and Novikov[5] quote the results of their investigation of the propagation of light in the Mixmaster Universe. Doroshkevich, Lukash and Novikov[6] in a recent preprint apply these results for finding the likelihood of horizon vanishing and find it to be very low. Our results are in substantial agreement with theirs. In a future paper, we will show how our formulation and treatment of the problem gives us a natural probabalistic estimate for horizon vanishing. Here we will derive the equations for rays of high-frequency sound waves in these generic models and study their behavior in a certain class of solutions to Einstein's equations. The Hamiltonian methods which we use to obtain information about the relevant solutions to Einstein's equations are quite different from the ones employed by Belinski etc.[7] or Doroshkenich and Novikov.[5] Also we do not reject the application of our calculations to epochs where quantum effects could enter. We look forward to calculations in which quantum effects might be included and would meaningfully modify the interpretation of these small perturbations.

The metric of the Bianchi type IX for an anisotropic non-rotating universe can be written as

$$ds^2 = - dt^2 + (6\pi)^{-1} e^{-2\Omega}(e^{2\beta})_{ij} \sigma_i\sigma_j, \tag{A1.1}$$

where

$$\sigma_1 = \sin\psi d\theta - \cos\psi\sin\theta d\phi$$

$$\sigma_2 = \cos\psi d\theta + \sin\psi\sin\theta d\phi$$

and $$\sigma_3 = -(d\psi + \cos\theta d\phi)$$

satisfy $d\sigma_i = \frac{1}{2} \varepsilon_{ijk} \sigma_j \wedge \sigma_k$ and are differential forms on the three-sphere parameterized by Euler angles $\psi\theta\phi$ with $0 \leq \psi \leq 4\pi$, $0 \leq \theta \leq \pi$ and $0 \leq \phi \leq 2\pi$. The quantities Ω and β_{ij} depend only on time, with Ω determining the volume and β_{ij}, a diagonal traceless 3×3 matrix:

$$\beta = \mathrm{diag}\ (\beta_1, \beta_2, \beta_3)$$

governing the anisotropy (shape). Note that for $\beta_{ij} = 0$, this metric is one form for the positive curvature Robertson-Walker metric. As two independent shape parameters choose

$$\beta_+ = (\beta_1 + \beta_2)/2$$

and

$$\beta_- = (\beta_1 - \beta_2)/2\sqrt{3} \quad .$$

The variational principle for Einstein's equations $\delta I = 0$ with $I = (16\pi)^{-1} \int R\ (-g)^{1/2}\ d^4x$ can be cast into a canonical form to obtain[7] the Hamiltonian:

$$H = [p_+{}^2 + p_-{}^2 + e^{-4\Omega} (V-1)]^{1/2} \qquad (A1.2)$$

p_+ and p_- are the momenta conjugate to the field amplitudes β_+ and β_- respectively, with Ω as the choice for the independent (coordinate time) variable. An equation giving Ω as a function of the cosmic time t is

$$dt = -\sqrt{\frac{2}{3\pi}}\ \frac{1}{H}\ e^{-3\Omega}\ d\Omega \quad . \qquad (A1.3)$$

The "anisotropy potential" $V(\beta_+, \beta_-)$ arises due to the anisotropy of the curvature of the three-dimensional space sections of the universe. The potential walls rise steeply away from $\beta = 0$, with the equipotentials asymptotically forming equilateral triangles in the $\beta_+\ \beta_-$ plane as shown in Fig. 1. One of the three equivalent sides of the triangle is described by the asymptotic form

$$V \sim \frac{1}{3}\ e^{-8\beta+} \quad , \quad \beta_+ \to -\infty \qquad (A1.4)$$

which is valid in the sector $|\beta_-| < -\sqrt{3}\ \beta_+$. The corners of this triangular potential are flared open; for instance if $\beta_+ \to \infty$ with $|\beta_-| << 1$, one finds

$$V(\beta) \sim 16\beta_-{}^2\ e^{4\beta+} + 1. \qquad (A1.5)$$

The evolution of the universe is described by the motion of the system point $\beta \equiv (\beta_+, \beta_-)$ as a function of the time coordinate Ω. When β is well away from the potential walls, the universe point moves with velocity $\beta' \equiv \frac{d\beta}{d\Omega} = \left\{(\frac{d\beta_+}{d\Omega})^2 + (\frac{d\beta_-}{d\Omega})^2\right\}^{1/2}$ of unit magnitude in straight lines and it can be parameterized as

$$\frac{d\beta_+}{d\Omega} = \frac{u^2+u-1/2)}{u^2+u+1} \qquad \text{(A1.6)}$$

$$\frac{d\beta_-}{d\Omega} = \frac{\sqrt{3}(u+1/2}{u^2+u+1} \qquad \text{(A1.7)}$$

where the parameter u goes from $-\infty$ to ∞. The potential walls move outward with velocity (in the sense of $\frac{d\beta_{wall}}{d\Omega}$) one-half. The system point β would thus move in one direction with unit velocity till it comes close to one of the walls and feels the potential and would then bounce off the wall changing its direction. Furthermore, Belinskii and Khalatnikov[4] have shown that all solutions would come arbitrarily close to the values $u = -2, -1, -\frac{1}{2}, 0, 1, \infty$ after rattling back and forth between the walls. These values of u correspond to the system point moving parallel to the three corner axes.

When the system point is well inside the walls, the potential V can be neglected. But $V = 0$ just gives the Einstein equations $R_{\mu\nu} = 0$ for Bianchi type I. One finds,[3] then that these epochs parallel Kasner solutions using $\Omega = -\frac{1}{3}\log t + \text{constant}$ as the independent variable; the Kasner metric being given by

$$ds^2 = - dt^2 + R_o^2 \,(t^{2p_1}\, dx^2 + t^{2p_2}\, dy^2 + t^{2p_3}\, dz^2)$$

where the exponents p_1, p_2 and p_3 are connected by the following two relations:

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$

Thus the β point shifts from one Kasner-like model to another at each collision with a potential wall. For the Kasner solution with $p_1 = p_2 = 0$, $p_3 = 1$, there exist no horizons for causal propagation in the z-direction.[9] Similarly, there is absence of horizons in the other two directions for Kasner metrics with $p_1 = p_3 = 0$, $p_2 = 1$ and $p_2 = p_3 = 0$, $p_1 = 1$ respectively. This motivates us to study the epochs of Bianchi type IX model which approximate these Kasner-solutions for a long period of time. These epochs can be seen to be the ones when the system point is moving parallel to one of the axes of the equipotential triangle and is either running towards a corner or following an inclined wall. When the system point is running towards a corner on the β_+ -axis, the parameter "u" designating the direction of the velocity is asymptotically ∞ and β_+ is very large; $|\beta_-| << 1$ giving the universe a pancake shaped anisotropy corresponding to a relative compression of the 3-axis (ψ-axis) with the other two axis approximately equal. While near the inclined walls, say for $\beta_1 \to \infty$, the anisotropy is cigar shaped with the stretching of the 1-axis relative to the others. So we expect the null geodesics in the ψ-direction to go around the universe during the $u = \infty$ epochs.

In the next section we will derive the equations for the propagation of high-frequency sound waves and in the following sections we will study their behavior during the epochs when u is very large. It will be seen that there exist a set of initial conditions for which the special Kasner-like behavior persists long enough for these sound waves to go round the universe in the ψ-direction. This possibility of communication either by sound-waves or light rays along a certain direction during the evolution of a universe will be called the removal of horizon in that direction for that universe.

2. The Propagation of High-Frequency Sound Waves

Let $\bar{\varepsilon}$, $\bar{p}$ and $\bar{\underset{\sim}{u}}$ be the energy density, pressure and the four velocity of the fluid, and let ε', p' and $\underset{\sim}{u}'$ be the small amplitude, high-frequency perturbations on the above solution. The propagation of the disturbance is governed by the energy equation:

$$\varepsilon_{,\mu}\, u^{\mu} + (p + \varepsilon)u^{\mu}{}_{;\mu} = 0 \qquad \text{(A2.1)}$$

and the Euler equation:

$$(p + \varepsilon)u^{\mu}{}_{;\nu}\, u^{\nu} = -\,(g^{\mu\nu} + u^{\mu}u^{\nu})p_{,\nu} \quad . \qquad \text{(A2.2)}$$

Substituting $p = \bar{p} + p'$, $\varepsilon = \bar{\varepsilon} + \varepsilon'$ and $u = \bar{u} + u'$ in the equations (A2.1) and (A2.2) and linearising we obtain

$$\varepsilon'_{,\mu}\, \bar{u}^{\mu} + \varepsilon_{,\mu} u'^{\mu} + (\varepsilon' + p')\bar{u}^{\mu}{}_{;\mu} + (\bar{\varepsilon} + \bar{p})u'^{\mu}{}_{;\mu} = 0 \qquad \text{(A2.3)}$$

and

$$(\bar{\varepsilon} + \bar{p})(\bar{u}^{\mu}{}_{;\nu}u'^{\nu} + u'^{\mu}{}_{;\nu}\bar{u}^{\nu}) + (\varepsilon' + p')(\bar{u}^{\mu}{}_{;\nu}\bar{u}^{\nu})$$

$$= -\,(g^{\mu\nu} + \bar{u}^{\mu}\bar{u}^{\nu})p'_{,\nu} - u'^{\mu}\bar{u}^{\nu}\bar{p}_{,\nu} - \bar{u}^{\mu}u'^{\nu}\bar{p}_{,\nu} \quad . \qquad \text{(A2.4)}$$

Differentiating equations (A2.4) with respect to μ and substituting for $u'^{\mu}{}_{;\mu}$ from equation (A2.3) we get,

$$(g^{\mu\nu} + \bar{u}^{\mu}\bar{u}^{\nu})p'_{,\nu\mu} - \varepsilon'_{,\mu\nu}\, \bar{u}^{\mu}\bar{u}^{\nu} = F \qquad \text{(A2.5)}$$

where F is a scalar function which contains the high-frequency perturbations ε', p' and $\underset{\sim}{u}'$ only up to their first derivatives.

Writing $p' = Ae^{i\phi}$, where ϕ is a rapidly varying function and setting the dominant terms in the equation (A2.5) equal to zero, we obtain

$$(g^{\mu\nu} + \bar{u}^{\mu}\bar{u}^{\nu})\phi,_{\mu}\phi,_{\nu} - (\frac{\partial\varepsilon}{\partial p})_s \; \phi,_{\mu}\phi,_{\nu} \; \bar{u}^{\mu}\bar{u}^{\nu} = 0 \qquad (A2.6)$$

where $v_s = \sqrt{(\frac{\partial p}{\partial \varepsilon})_s}$ is the sound velocity. The equation(A2.6) is a Hamilton-Jacobi equation corresponding to

$$H = \frac{1}{2}(g^{\mu\nu} + \bar{u}^{\mu}\bar{u}^{\nu})p_{\mu}p_{\nu} - \frac{1}{v_s^2}\frac{1}{2}\bar{u}^{\mu}\bar{u}^{\nu}p_{\mu}p_{\nu} \qquad (A2.7)$$

as a particle Hamiltonian. To obtain the corresponding Lagrangian, we solve for p_{μ} from one set of Hamilton's equations:

$$\frac{dx^{\mu}}{d\lambda} = \frac{\partial H}{\partial p_{\mu}} = (g^{\mu\nu} + \bar{u}^{\mu}\bar{u}^{\nu})p_{\nu} - \frac{1}{v_s^2}\bar{u}^{\mu}\bar{u}^{\nu}p_{\nu} \qquad (A2.8)$$

where $x^{\mu} \equiv (t,\theta,\phi,\psi)$.

Noting that $\underset{\sim}{\bar{u}} = \frac{\partial}{\partial t}$ for comoving coordinates, we can invert (A2.8) to obtain

$$p_{\nu} = \frac{dx^{\mu}}{d\lambda}[g_{\mu\nu} + (1-v_s^2)\bar{u}_{\mu}\bar{u}_{\nu}] \quad . \qquad (A2.9)$$

Thus, we get the Lagrangian L as follows:

$$L = p_{\mu}\frac{dx^{\mu}}{d\lambda} - H$$

$$= \frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}[g_{\mu\nu} + (1-v_s^2)\bar{u}_{\mu}\bar{u}_{\nu}] - \frac{1}{2}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}[g_{\mu\nu} + (1-v_s^2)\bar{u}_{\mu}\bar{u}_{\nu}] \quad (A2.10)$$

$$= \frac{1}{2}\frac{dx^{\mu}}{d\lambda}\frac{dx^{\nu}}{d\lambda}[g_{\mu\nu} + (1-v_s^2)\bar{u}_{\mu}\bar{u}_{\nu}] \quad . \qquad (A2.11)$$

The propagation of rays is then given by the Lagrange's equations,

$$\frac{d}{d\lambda}[2\, g_{\mu\nu}\frac{dx^{\nu}}{d\lambda} + 2(1-v_s^2)\bar{u}_{\mu}\bar{u}_{\nu}\frac{dx^{\nu}}{d\lambda}] = \frac{dx^{\rho}}{d\lambda}\frac{dx^{\nu}}{d\lambda}\frac{\partial g_{\rho\nu}}{\partial x^{\mu}} \quad . \qquad (A2.12)$$

Consider a possible set of solutions with θ = constant and ϕ = constant. Then the Lagrange's equations reduce to

$$\frac{d}{d\lambda}[-2\frac{dt}{d\lambda} + 2(1-v_s^2)(\frac{dt}{d\lambda})] = (\frac{d\psi}{d\lambda})^2\frac{\partial g_{\psi\psi}}{\partial t} \qquad (A2.13)$$

$$\frac{\partial g_{\psi\psi}}{\partial\theta} \left(\frac{d\psi}{d\lambda} \right)^2 = 0 \tag{A2.14}$$

$$\frac{d}{d\lambda} \left[g_{\psi\phi} \frac{d\psi}{d\lambda} \right] = 0 \tag{A2.15}$$

$$\frac{d}{d\lambda} \left[g_{\psi\psi} \frac{d\psi}{d\lambda} \right] = \left(\frac{d\psi}{d\lambda} \right)^2 \frac{\partial g_{\psi\psi}}{\partial\psi} \quad . \tag{A2.16}$$

Since $g_{\psi\psi}$ is a function of t only, eqn.(A2. 14) is identically satisfied, while (A2.15) and (A2.16) reduce to

$$\frac{d}{d\lambda} \left[\cos\theta \quad e^{-2\Omega} \; e^{2\beta_3} \frac{d\psi}{d\lambda} \right] = 0 \tag{A2.17}$$

and

$$\frac{d}{d\lambda} \left[e^{-2\Omega} \; e^{2\beta_3} \frac{d\psi}{d\lambda} \right] = 0 \quad . \tag{A2.18}$$

For θ = constant,(A2. 17) reduces to (A2.18). So the Lagrange's eqns. now reduce to eqn. (A2.13) and eqn. (A2.8) which can be solved for $\frac{dt}{d\lambda}$ and $\frac{d\psi}{d\lambda}$.

Putting $H = 0$ in the equation (A2.10) we obtain

$$\frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \quad g_{\mu\nu} + (1-v_s^2) \; \bar{u}_\mu \; \bar{u}_\nu = 0$$

or

$$- \left(\frac{dt}{d\lambda} \right)^2 + g_{33} \left(\frac{d\psi}{d\lambda} \right)^2 + \left(\frac{dt}{d\lambda} \right)^2 (1-v_s^2) = 0 \tag{A2.19}$$

for θ = constant, ϕ = constant class of solutions. From eqn. (A2.19) we obtain

$$\frac{d\psi}{dt} = \frac{v_s}{\sqrt{g_{33}}} = \sqrt{6\pi} \quad v_s \; e^{\Omega} \; e^{-\beta_3} \quad . \tag{A2.20}$$

By putting $v_s = 1$, we get the law of propagation for light going in the ψ - direction.

3. The Removal of Horizons

Let us now study the behavior of the above high-frequency sound waves during the $u = \infty$ epochs. First consider the axial case when the system point is very close to the β_+ -axis and is running towards the corner. This is the case which Belinskii[7] etc call the case of small oscillations. The appropriate solution to Einstein's equations as derived in the next section is

$$\beta_- = Z_o\left(\frac{2e^{2\beta_o}}{K}\right) \qquad \text{[see (A4.8)]}$$

where Z_o is a Bessel function of order zero. K is a constant and β_o is defined as

$$\beta_o \equiv \beta_+ - \Omega \quad .$$

The variation of β_o is given by

$$\frac{d\beta_o}{d\Omega} = -\frac{K}{H} \qquad \text{[see (A4.3)]} \quad .$$

Writing $\beta_3 = -2\beta_+ = -2(\beta_o + \Omega)$ in equation (A2.20), we reexpress the equation of the sound-wave propagation in the ψ-direction as

$$\frac{d\psi}{dt} = \sqrt{6\pi}\ v_s\ e^{3\Omega}\ e^{2\beta_o} \quad .$$

Using equation (A1.3), the change in ψ can be given in terms of the variable Ω as

$$\frac{d\psi}{d\Omega} = \frac{d\psi}{dt}\frac{dt}{d\Omega} = -v_s\frac{2}{H}e^{2\beta_o} \quad . \qquad \text{(A3.1)}$$

Hence, the change in ψ along the sound wave between the epochs Ω_1 and Ω_2 is given by

$$\Delta\psi = \int_{\Omega_1}^{\Omega_2} d\psi = \int_{\Omega_1}^{\Omega_2} v_s \frac{2}{H} e^{2\beta_o} d\Omega$$

$$= v_s \int_{\Omega_1}^{\Omega_2} \frac{2}{H} e^{2\beta_o} \frac{d\Omega}{d\beta_o} d\beta_o \quad . \tag{A3.2}$$

Substituting equation (A4.3) in equation (A3.2), we obtain

$$\Delta\psi = -\frac{2}{K} v_s \int_{\Omega_1}^{\Omega_2} e^{2\beta_o} d\beta_o \tag{A3.3}$$

$$= \frac{1}{2} v_s \Delta \left(\frac{e^{2\beta_o}}{K}\right) \quad . \tag{A3.4}$$

Therefore, a change of $8\pi/v_s$ in $\frac{e^{2\beta_o}}{K}$ would give a change of 4π in ψ. Since $\beta_- = Z_o\left(\frac{2e^{\beta_o}}{K}\right)$ which for small K goes roughly as

$$\sqrt{\frac{K}{\pi e^{2\beta_o}}} \quad \cos\left(\frac{2e^{2\beta_o}}{K} - \pi/4\right) \quad \text{or}$$

$$\sqrt{\frac{K}{\pi e^{2\beta_o}}} \quad \sin\left(\frac{2e^{2\beta_o}}{K} - \pi/4\right) \quad ,$$

β_- would go through four cycles as its argument changes by 8π. Thus setting $v_s = 1$, we see that the light ray would circumnavigate the universe in the ψ-direction (i.e. ψ going from 0 to 4π) during four cycles of β_-. This corresponds to the DN result of $N_e = \frac{1}{4}N_m$. For radiation filled universes, the velocity v_s of the sound-wave propagation will be $1/\sqrt{3}$; as a result these waves would go round the universe in the ψ-direction during seven cycles of β_-. Similarly, when the system point is running towards the other two corners, the causal and the high - frequency sound wave influence would circumnavigate in the other two principal directions.

Next consider the off-axial case with u very large and $\beta_- > 1$. The appropriate solution to Einstein's equations as derived in the next section again gives

$$\frac{d\beta_o}{d\Omega} = -K/H \qquad \text{[See (A4.11)]}$$

while total change in β_o during one bounce with the inclined potential wall for large u is given by

$$\Delta\beta_o = \frac{1}{2u_i} \qquad \text{[See (A4.17)]}$$

where u_i is the value of u before the bounce. The change in ψ along the high-frequency sound wave ray going in the ψ-direction is again given by equation (A3.3

$$\Delta\psi = \frac{2v_s}{K} \int e^{2\beta_o} d\beta_o \quad .$$

So during one collision with the wall, the change in ψ would be

$$\begin{aligned}\Delta\psi &= \frac{v_s}{K} (e^{2\beta_o})_i \, [1 - e^{2\Delta(\beta_o)}] \\ &= \frac{v_s}{K} (e^{2\beta_o})_i \, \frac{1}{u_i} \quad , \end{aligned} \qquad \text{(A 3.5)}$$

where the subscript i denotes the values of the variables before the collision. The value of the constant K can be obtained in terms of u_i and H_i from equation (A1.6):

$$\frac{d\beta_+}{d\Omega} = \frac{u^2+u-1/2}{u^2+u+1}$$

and the equation (A4.11)

$$\frac{d\beta_o}{d\Omega} = -\frac{K}{H} \quad .$$

Then $\Delta\psi$ is given in terms of the initial values as

$$\Delta\psi = V_s \frac{2}{3} \frac{u_i^2+u_i+1}{H_i} (e^{2\beta_o})_i \frac{1}{u_i}$$

$$= (\frac{2}{3} V_s \frac{e^{2\beta_o}}{H} u)_i \qquad \text{for large } u_i \quad . \qquad \text{(A3.6)}$$

As the system point evolves, consider the epoch when the system point had its first collision with the inclined wall for large u. So the system point has just bounced back off the vertical wall and is going towards the inclined wall at say $\Omega = \Omega_b$. The position of the potential wall is then given by

$$H^2 = e^{-4\Omega_b} V(\beta)$$

$$= e^{-4\Omega_b} \frac{1}{3} e^{-8(\beta_+)_{wall}}$$

$$= \frac{1}{3} e^{-12\Omega_b} e^{-8(\beta_o)_{wall}} \quad .$$

Substituting the expression for H in the equation (A3.6) and dropping the subscripts, we get

$$\Delta\psi = (V_s \frac{2}{3} e^{2\beta_o} u)/(\frac{1}{\sqrt{3}} e^{-6\Omega_b} e^{-4\beta_o})$$

$$= \frac{2}{\sqrt{3}} V_s \, e^{6(\beta_+)_{wall}} \quad . \qquad \text{(A3.7)}$$

Therefore, for all solutions for which at the beginning of the series of collisions with the inclined wall, the value of u is such that

$$u > \frac{\sqrt{3}}{2V_s} \, 4\pi \, e^{-6(\beta_+)_{wall}} \quad ,$$

then the high-frequency sound wave communication has an open channel in the ψ-direction. Since $(\beta_+)_{wall}$ is negative (it goes as: $\beta_+ \simeq -\frac{\Omega}{2}+$ constant), we find that there exist small sectors around the lines parallel to the β_+-axis such that when the system point is running along these sectors at Ω_b, a horizon is removed in the ψ-direction during the next bounce with the inclined potential wall. The angular extent of these sectors depends upon Ω and it goes to zero as Ω goes to ∞.

One concludes, therefore, that at each epoch , there exist certain subsets of initial conditions $[\beta_+, \beta_-; u(\Omega)]$, such that some rays of high-frequency sound waves and null-geodesics will proceed to circumnavigate the corresponding universe. It will be shown in a future publication that the universe point wanders about in a truly ergodic fashion and that by finding a measure on initial conditions, one can compute the probability for a typical solution to have no horizon along one axis.

4. $u = \infty$ SOLUTIONS OF EINSTEIN EQUATIONS

In this section we will derive the relevant information about $u = \infty$ solutions which we used in the last section. First consider the axial case when the system point is very close to one of the corner axes and is running towards the corner. For the corner on the β_+ -axis, the asymptotic form of the potential is

$$V(\beta) \sim 16\beta_-^{\,2}\, e^{4\beta_+} + 1; \quad \beta_+ \to \infty \text{ and } |\beta_-| \ll 1.$$

Then the Hamiltonian of the system is

$$H = [p_+^{\,2} + p_-^{\,2} + 16\beta_-^{\,2}\, e^{-4\Omega}\, e^{4\beta_+}]^{1/2} \ . \tag{A4.1}$$

To get a time-independent Hamiltonian, substitute

$$\beta_+ = \beta_0 + \Omega \text{ in the action integrand}$$

$$\omega = p_+\, d\beta_+ + p_- d\beta_- \; - H d\Omega \qquad \text{to give}$$

$$\omega = p_+ d\beta_0 + p_- d\beta_- - (H - p_+) d\Omega \ .$$

So the new Hamiltonian is

$$K = [p_+^{\,2} + p_-^{\,2} + 16\beta_-^{\,2}\, e^{4\beta_0}]^{1/2} - p_+ \ . \tag{A4.2}$$

And the corresponding Hamilton's equations give

$$\frac{d\beta_0}{d\Omega} = \frac{\partial K}{\partial p_+} = \frac{p_+}{K+p_+} - 1 = -\frac{K}{H} \tag{A4.3}$$

$$\frac{d\beta_-}{d\Omega} = \frac{\partial K}{\partial p_-} = \frac{p_-}{H} \tag{A4.4}$$

$$\frac{dp_+}{d\Omega} = \frac{-\partial K}{\partial \beta_0} = \frac{-\ 32\beta_-^{\ 2}}{H} e^{4\beta_0} \quad , \tag{A4.5}$$

$$\frac{dp_-}{d\Omega} = \frac{-\partial K}{\partial \beta_-} = \frac{-16\beta_- e^{4\beta_0}}{H} \tag{A4.6}$$

and

$$\frac{dK}{d\Omega} = \frac{\partial K}{\partial \Omega} = 0 \quad . \tag{A4.7}$$

Equation (A4.7) tells us that K is a constant while Eqns.(A 4.3) , (A4.4) and (A4.6) can be manipulated to give

$$\frac{d\beta_-}{d\beta_0} = -\frac{p_-}{K}$$

$$\frac{dp_-}{d\beta_0} = \frac{dp_-}{d\Omega} / \frac{d\beta_0}{d\Omega} = \frac{16\beta_- \ e^{4\beta_0}}{K} \quad .$$

Hence

$$\frac{d^2\beta_-}{d\beta_0^{\ 2}} = -\frac{1}{K}\frac{dp_-}{d\beta_0} = \frac{-16\beta_- e^{4\beta_0}}{K^{\ 2}}$$

or

$$\left(\frac{d^2\beta_-}{d\beta_0^{\ 2}}\right) + \left(\frac{16\ e^{4\beta_0}}{K^2}\right) \beta_- = 0$$ which has

the solution

$$\beta_- = Z_0 \left(\frac{2\ e^{2\beta_0}}{K}\right) \tag{A4.8}$$

where Z_0 is a Bessel function of order zero. Note from Eqns.(A4.1) and (A4.2) that K and H are strictly positive. Then from Eqn.(A4.3) β_0 is always decreasing, so Eqn.(A4.8) is valid starting from some initial value of β_0 until β_0 decreases to the point where the argument of the Bessel function gets small and β_- gets large contradicting the $|\beta_-| << 1$ assumption.

Next consider the off-axial case ($\beta_- > 1$). When the system point is almost parallel to the β_+ -axis (large u) and is following one of the inclined potential walls, the asymptotic form of the potential is

$$V(\beta) \sim \frac{1}{3} e^{4(\beta_+ + \sqrt{3}\beta_-)} \quad .$$

Then the Hamiltonian of the system is

$$H = [p_+{}^2 + p_-{}^2 + \frac{1}{3} e^{-4\Omega} e^{4(\beta_+ + \sqrt{3}\beta_-)}]^{1/2} \quad . \tag{A4.9}$$

Substituting $\beta_+ = \beta_0 + \Omega$ in the action, we get the time-independent Hamiltonian

$$K = [p_+{}^2 + p_-{}^2 + \frac{1}{3} e^{4\beta_0} e^{4\sqrt{3}\beta_-}]^{1/2} - p_+ \quad . \tag{A 4.10}$$

The Hamilton's equations give

$$\frac{d\beta_0}{d\Omega} = \frac{\partial K}{\partial p_+} = \frac{p_+}{K_+ + p_+} - 1 = -\frac{K}{H} \tag{A 4.11}$$

$$\frac{d\beta_-}{d\Omega} = \frac{\partial K}{\partial p_-} = \frac{p_-}{H} \tag{A 4.12}$$

$$\frac{dp_+}{d\,\Omega} = -\frac{\partial K}{\partial \beta_0} = -\frac{2}{3}\frac{e}{H}^{4(\beta_0 + \sqrt{3}\beta_-)} \tag{A 4.13}$$

$$\frac{dp_-}{d\Omega} = \frac{-\partial K}{\partial \beta_-} = \frac{-2}{\sqrt{3}}\frac{e}{H}^{4(\beta_0 + \sqrt{3}\beta_-)} \tag{A 4.14}$$

and

$$\frac{dK}{d\Omega} = \frac{\partial K}{\partial \Omega} = 0 \quad . \tag{A 4.15}$$

From Eqns. (A4.13) and (A414) we get

$$\sqrt{3}\,\frac{dp_+}{d\Omega} - \frac{dp_-}{d\Omega} = 0$$

or

$$\sqrt{3}\ p_+ - p_- = \text{constant} = \alpha, \text{ say.} \qquad \text{(A4.16)}$$

Substituting for p_+ and p_- in Eq. (A4.16) from Eqs(A 4.11) and(A4.12) we obtain

$$H\left(\sqrt{3}\,\frac{d\beta_+}{d\Omega} - \frac{d\beta_-}{d\Omega}\right) = \alpha \quad ,$$

Also from Eq. (A4. 15) $K = H - p_+ = H\left(1 - \frac{d\beta_+}{d\Omega}\right)$ is a constant. These two constants of motion enable us to find β'_+, β'_- after the bounce in terms of their values before. Let u_i and u_f be the values of the parameter u, characterising the velocities of the system point well before and well after the bounce. Then the constancy of $K = H(1 - \beta'_+)$ and $\sqrt{3}\ p_+ - p_- = H(\sqrt{3}\,\beta'_+ - \beta'_-)$ gives respectively.

$$H_i\left(1 - \frac{u_i^2 + u_i - \frac{1}{2}}{u_i^2 + u_i + 1}\right) = H_f\left(1 - \frac{u_f^2 + u_f - \frac{1}{2}}{u_f^2 + u_f + 1}\right)$$

and

$$H_i\left(\frac{u_i^2 - 1}{u_i^2 + u_i + 1}\right) = H_f\left(\frac{u_f^2 - 1}{u_f^2 + u_f + 1}\right) \quad ,$$

Hence,

$$\frac{H_f}{H_i} = \frac{u_f^2 + u_f + 1}{u_i^2 + u_i + 1}$$

and $u_f = - u_i$; where H_i and H_f are the values of H before and after the bounce, respectively.

During the collision with the wall,

$$H^2 = p_+^2 + p_-^2 + e^{-4\Omega}\frac{1}{3}\,e^{4(\beta_+ + \sqrt{3}\,\beta_-)}$$

$$= H^2\left(\frac{d\beta_+}{d\Omega}\right)^2 + H^2\left(\frac{d\beta_-}{d\Omega}\right)^2 + e^{-4\Omega}\,\frac{1}{3}\,e^{4(\beta_+ + \sqrt{3}\,\beta_-)} \; .$$

So the equation $\frac{dH}{d\Omega} = \frac{\partial H}{\partial \Omega}$ gives

$$H \frac{dH}{d\Omega} = -\frac{2}{3} e^{-4\Omega} e^{4(\beta_+ + \sqrt{3}\beta_-)} .$$

Using this result and solving for $\frac{d\beta_+}{d\Omega}$, $\frac{d\beta_-}{d\Omega}$ in terms of H, K and α, one obtains

$$H^2 = H^2(1 - \frac{K}{H})^2 + H^2[\sqrt{3}(1 - \frac{K}{H}) - \frac{\alpha}{H}]^2 - \frac{1}{2} H \frac{dH}{d\Omega} .$$

or

$$H \frac{dH}{d\Omega} = -6(H_i - H)(H - H_f) ;$$

hence

$$\frac{d\beta_o}{dH} = \frac{d\beta_o}{d\Omega} / \frac{dH}{d\Omega} = -\frac{K}{H} / \frac{dH}{d\Omega} = \frac{K}{6(H_i - H)(H - H_f)} .$$

A lower limit on the change in β_o during the collision can be computed as

$$\Delta(\beta_o) = (H_i - H_f) \text{ (minimum value of } \frac{d\beta_o}{dH}) .$$

The minimum value of $\frac{d\beta_o}{dH}$ is at that value of H, where $\frac{d^2\beta_o}{dH^2}$ vanishes; i.e., at

$$H^2 = (H_i + H_f)/2 .$$

Therefore, $\Delta(\beta_o) = \frac{2K}{3(H_i - H_f)}$.

But

$$(H_i - H_f) = \frac{2}{3} K [(u_i^2 + u_i + 1) - (u_f^2 + u_f + 1)]$$

$$= \frac{4}{3} K u_i ; \text{ since } u_f = -u_i .$$

Hence,

$$\Delta(\beta_o) = \frac{1}{2u_i} . \qquad \text{(A4.17)}$$

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APPENDIX B

1. The Geodesic Equations

As covariant basis vectors, we take the orthonormal tetrad:

$$
\begin{aligned}
\omega^0 &= dt \\
\omega^1 &= \frac{1}{\sqrt{6\pi}} e^{-\Omega} e^{\beta_1} \sigma_1 \\
\omega^2 &= \frac{1}{\sqrt{6\pi}} e^{-\Omega} e^{\beta_2} \sigma_2 \\
\omega^3 &= \frac{1}{\sqrt{6\pi}} e^{-\Omega} e^{\beta_3} \sigma_3 \ .
\end{aligned}
\tag{B1.1}
$$

The set of basis vectors dual to the ω^μ is

$$
\begin{aligned}
\vec{e}_0 &= \frac{\partial}{\partial t} \\
\vec{e}_1 &= \sqrt{6\pi}\, e^{\Omega}\, e^{-\beta_1}\left[\sin\psi \frac{\partial}{\partial\theta} - \frac{\cos\psi}{\sin\theta} \frac{\partial}{\partial\phi} + \cos\psi \cot\theta \frac{\partial}{\partial\psi}\right] \\
\vec{e}_2 &= \sqrt{6\pi}\, e^{\Omega}\, e^{-\beta_2}\left[\cos\psi \frac{\partial}{\partial\theta} + \frac{\sin\psi}{\sin\theta} \frac{\partial}{\partial\phi} - \sin\psi \cot\theta \frac{\partial}{\partial\psi}\right] \\
\vec{e}_3 &= -\sqrt{6\pi}\, e^{\Omega}\, e^{-\beta_3} \frac{\partial}{\partial\psi} \ .
\end{aligned}
\tag{B1.2}
$$

The connection forms $\omega^\mu{}_\nu$ as determined uniquely by $0 = dg_{\mu\nu} = \omega_{\mu\nu} + \omega_{\nu\mu}$ and $d\omega^\mu = -\omega^\mu{}_\nu \wedge \omega^\nu$ are

$$
\begin{aligned}
\omega^0{}_1 &= \omega^1{}_0 = (-\dot{\Omega} + \dot{\beta}_1)\omega^1 \\
\omega^0{}_2 &= \omega^2{}_0 = (-\dot{\Omega} + \dot{\beta}_2)\omega^2 \\
\omega^0{}_3 &= \omega^3{}_0 = (-\dot{\Omega} + \dot{\beta}_3)\omega^3
\end{aligned}
\tag{B1.3}
$$

and

$$\omega^1{}_2 = -\omega^2{}_1 = \frac{\sqrt{6\pi}}{2} e^{\Omega}(e^{2\beta_1} + e^{2\beta_2} - e^{2\beta_3})\omega^3$$

$$\omega^2{}_3 = -\omega^3{}_2 = \frac{\sqrt{6\pi}}{2} e^{\Omega}(e^{2\beta_2} + e^{2\beta_3} - e^{2\beta_1})\omega^1 \qquad \text{(B1.4)}$$

$$\omega^3{}_1 = -\omega^1{}_3 = \frac{\sqrt{6\pi}}{2} e^{\Omega}(e^{2\beta_3} + e^{2\beta_1} - e^{2\beta_2})\omega^2$$

where $\cdot$ denotes the differentiation with respect to t. If $\vec{v} = v^\mu \vec{e}_\mu = \frac{d}{d\lambda}$ is a tangent vector to a geodesic parametrized by λ, then

$$D_{\vec{v}}\, \vec{v} = 0$$

or

$$\frac{dv^\mu}{d\lambda} + v^\nu v^\rho \Gamma^\mu{}_{\nu\rho} = 0 \qquad \text{(B1.5)}$$

where $\Gamma^\mu{}_{\nu\rho}$ are the components of the connection forms, i.e.

$$\omega^\mu{}_\nu = \Gamma^\mu{}_{\nu\rho}\, \omega^\rho \quad .$$

Computing the values of $\Gamma^\mu{}_{\nu\rho}$ from (B1.3) and (B1.4) and writing $\frac{dv^\mu}{d\lambda} = v^0 \frac{dv^\mu}{dt} = v^0 \dot{v}^\mu$, the geodesic equations (B1.5) reduce to

$$v^0\dot{v}^0 + (v^1)^2(-\dot{\Omega}+\dot{\beta}_1) + (v^2)^2(-\dot{\Omega}+\dot{\beta}_2) + (v^3)^2(-\dot{\Omega}+\dot{\beta}_3) = 0$$

$$v^0\dot{v}^1 + v^0 v^1(-\dot{\Omega}+\dot{\beta}_1) + v^2 v^3 \sqrt{6\pi}\, e^{-\Omega}(e^{2\beta_2} - e^{2\beta_3}) = 0$$

$$v^0\dot{v}^2 + v^0 v^2(-\dot{\Omega}+\dot{\beta}_2) + v^3 v^1 \sqrt{6\pi}\, e^{-\Omega}(e^{2\beta_3} - e^{2\beta_1}) = 0 \qquad \text{(B1.6)}$$

and

$$v^0\dot{v}^3 + v^0 v^3(-\dot{\Omega}+\dot{\beta}_3) + v^1 v^2 \sqrt{6\pi}\, e^{-\Omega}(e^{2\beta_1} - e^{2\beta_2}) = 0$$

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