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The Uniqueness Question of Discrete Wavelet Maxima Representation

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The Uniqueness Question of Discrete Wavelet Maxima Representation.

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Abstract

In this paper, we analyze the discrete wavelet maxima representation from the reconstruction point of view. Assuming finite data length and using the finite dimensional linear space approach we present necessary and sufficient conditions for the given representation to be unique. The algorithm which tests for uniqueness is shown. A general form of a solution to the reconstruction problem is described. The above results are valid for any bank of linear filters where the outputs are sampled at extreme values. For illustration we show two wavelet transform examples. The first is the common unique-representation case. The second is an interesting example of family of sequences which have the same maxima representation. That is, we show an example of a non-unique discrete wavelet maxima representation.

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1 Introduction.

Traditionally, multiscale edges are determined either as extrema of Gaussian-filtered signals [6] or as zero-crossings of signals convolved with the Laplacian of a Gaussian (see e.g. [1] for a comprehensive review). S. Mallat in series of papers [4, 3, 2] introduced zero-crossings and extrema of wavelet transform as multiscale edge representation. Although [2] points out some advantages of maxima representation of signals, a comparison between approaches using extrema and those utilizing zero crossing has not been completed yet. In both methods, there are still important open problems, e.g.: stability, uniqueness, structure of a reconstruction set. Mallat and Zhong [2] show accurate numerical reconstruction results from maxima representation which seem to verify Marr's conjecture [5] about possible completeness and stability of multiscale edge representation. In [2], as in many others works in this area, the basic approach was developed using continuous variables. In the continuous framework, analytic tools to investigate reconstruction results are not as yet available. In this paper, **the problem of reconstruction from a discrete extrema representation is addressed.** Finite data length is assumed. A model, used in this paper, generalizes that at [2]. Model assumptions are satisfied by any bank of linear filters, provided the outputs are sampled at extreme values.

Section 2 defines discrete extrema representation, and describes necessary and sufficient conditions for the uniqueness of this representation. Morever, the general solution for the reconstruction from a given maxima representation is shown. The general conditions for uniqueness are used in Section 3 to develop an algorithm which tests for uniqueness. Section 4 describes two examples. For the first example, the uniqueness is showed, and the reconstruction from the maxima representation is given. The second example is even more interesting, since to the best of the author's knowledge, this is the first time different sequences with the same maxima representation are shown. For this particular example, the exact set of all sequences satisfying the given maxima representation is described.

2 Definitions and the main results

Let us start with a precise definition of a discrete extrema representation, which is a generalization of wavelet maxima representation used in [2]. Consider $\mathcal{L}$, a linear space of real, finite sequences of length N:

$$\mathcal{L} = \{f : f = \{f_n\}_{n \in I_N} : f_n \in \Re\}$$

where $I_N = \{-\frac{N}{2}+1, -\frac{N}{2}+2, \ldots, -1, 0, 1, \ldots, \frac{N}{2}\}$. N is assumed to be even. Let $W_1, W_2, \ldots, W_J, S_J$ be linear operators on $\mathcal{L}$. For the sequence $W_j f \quad (f \in \mathcal{L}, \quad j = 1, 2, \ldots, J)$ we define $X_j f$ and $N_j f$ as sets of maxima and minima points, respectively.

$$X_j f = \{k : W_j f(k+1) \leq W_j f(k) \ \ and \ \ W_j f(k-1) \leq W_j f(k) \ \ k \in I_N\}$$

$$N_j f = \{k : W_j f(k+1) \geq W_j f(k) \ \ and \ \ W_j f(k-1) \geq W_j f(k) \ \ k \in I_N\}$$

To avoid boundary problems, we assume an N-periodic extension of the finite sequences.

M denotes the operator of extrema representation, and is defined as follows:

$$Mf = \{\{X_j f, N_j f, \{W_j f(n)\}_{n \in X_j f \cup N_j f}\}_{j=1}^{J}, S_J f\} \qquad \forall f \in \mathcal{L}$$

In the sequel, following [2], we shall call Mf maxima representation as well. Generally speaking, M is a nonlinear operator and its analysis is not easy. Our approach is to separate the linear and the non-linear components. The determination of the extrema points sets is highly non-linear. However, for the given $X_j f, N_j f$ the remaining data are obtained by linear operation of sampling linear operator outputs at fixed points. This simple observation is the motivation to consider Mf as consists of two parts: the sampling information and the maxima information. The sampling information is the sequence $S_J f$ and the values of $W_j f$ at the points $X_j f \cup N_j f$. The maxima information is the fact that the elements of $X_j f$ and $N_j f$ are local maxima and minima of $W_j f$.

The space, $\mathcal{N}$, called zero information space , is very important for our results. $\mathcal{N}$ is defined as:

$$\mathcal{N} = \{p \in \mathcal{L} : S_J p = 0 \ \ and \ \ W_j p(n) = 0 \ \ \forall j \in \{1, 2, \ldots, J\} \ \ \forall n \in X_j f \cup N_j f\}$$

The following lemma states that uniqueness depends only on the sampling information.

Lemma 1 *Given a maxima representation, Mf, there exists a unique sequence $f \in \mathcal{L}$ which has this maxima representation if and only if the zero sequence is the only sequence belonging to $\mathcal{N}$.*

See Appendix A for a proof.

From the above lemma one can make the following observations.

Conclusion 1 *The uniqueness of discrete maxima representation depends on the sets $X_j f$, $N_j f$ $j = 1, 2, \ldots, J$ and does not depend on the particular extreme values of $W_j f$.*

Conclusion 2 *If the maxima representation is unique then the sampling information gives a unique characterization as well.*

2.1 A general solution for the reconstruction problem

Consider the reconstruction problem, i.e. the question about the structure of the reconstruction set, $\Gamma = \{\gamma \in \mathcal{L} : M\gamma = Mf\}$ for a given maxima representation Mf.

The reconstruction problem can be solved in two stages. First the sampling information implies a set of linear equations on elements of a sequence γ. Then the maxima information yields a set of linear inequalities on elements of a sequence γ. By straightforward analysis one can obtain the following result.

Theorem 1 *Let γ_p be a sequence which fulfils the samples conditions: $S_J\gamma_p = S_J f$ and $W_j\gamma_p(n) = W_j f(n)$ $\forall n \in X_j f \cup N_j f$. The sequences $\{p_l\}_{l=1}^{L}$ are a basis for the zero information space, $\mathcal{N}$.
Let $DW_j\gamma_p(n) = W_j\gamma_p(n+1) - W_j\gamma_p(n)$
and $DW_j p_i(n) = W_j p_i(n+1) - W_j p_i(n)$. Then a sequence γ satisfies the given maxima representation (i.e $\gamma \in \Gamma$) if and only if it can be written as:*

$$\gamma = \gamma_p + \sum_{l=1}^{L} \alpha_l p_l \tag{1}$$

where the coefficients $(\alpha_1, \alpha_2, \ldots, \alpha_L)$ satisfy the set of linear inequalities of the form:

$$(-1)^{k_j(n)} \sum_{i=1}^{L} \alpha_i DW_j p_i(n) < -(-1)^{k_j(n)} DW_j\gamma_p(n) \tag{2}$$

$\forall n$ such that $n \in \mathcal{L}\backslash(X_j f \cup N_j f)$ or $(n+1) \in \mathcal{L}\backslash(X_j f \cup N_j f)$, $\forall j \in \{1, 2, \ldots, J\}$, $k_j(n) = 2$ if n+1 is on the right of a maximum (the largest point from $X_j f \cup N_j f$, which is smaller then n+1, is a maximum. Otherwise $k_j(n) = 1$.

Now, one can continue to make the following observations:

Conclusion 3 *The reconstruction set, as a solution to a combination of linear equations and strong linear inequalities, is open, convex and its boundary is an intersection of hyperplanes.*

The set Γ can be empty, then either γ_p does not exists or there is no solution to the set of inequalities (2). The unique solution corresponds to the case where $\mathcal{N} = \{0\}$, and the sequence γ_p is feasible in the sense that it provides the required extrema.

Conclusion 4 *If the maxima representation is not unique, the sampling information defines an unbounded linear space. The maxima information stabilizes the solution in the sense that it bounds this space and limits the reconstruction error.*

3 Test for uniqueness

This section is aimed to develop an algorithm which can test for uniqueness of a given discrete wavelet maxima representation. We start with a short introduction to discrete dyadic wavelet transforms.

For a given finite sequence f, its discrete dyadic wavelet transform (with J levels) is defined as a set of sequences:

$$\{\{W_j f\}_{j=1,2,\ldots,J}, S_J f\}$$

computed by the following recursion formula:

$$W_{j+1}f = S_j f * g_j$$

$$S_{j+1}f = S_j f * h_j$$

for $j = 0, 1, 2, \ldots, J-1$, with $S_0 f = f$. The symbol $*$ denotes discrete (N periodic) convolution operator , and the sequences h_j and g_j are obtained as impulse responses of filters whose transfer functions are $H(2^j w)$ and $G(2^j w)$, respectively.

Two cases were described in [2]. One corresponds to a cubic spline wavelet with:

$$H(w) = (cos(\frac{w}{2}))^4$$

and the second corresponds to a Haar wavelet with:

$$H(w) = exp(-i\frac{w}{2})cos(\frac{w}{2}).$$

G(w) is chosen to satisfy:

$$\mid G(w) \mid^2 + \mid H(w) \mid^2 = 1.$$

For the cubic spline wavelet, following [2], the transfer function $G(w)$ was chosen as:

$$G(w) = (1 - (cos(\frac{w}{2}))^8)^{\frac{1}{2}} exp(-\frac{iw}{2} - \frac{\pi}{2}).$$

Sampling at $w = \frac{2\pi n}{N}$ for $n \in I_N$ gives the Discrete Fourier Transforms (DFT's) $\widehat{h}$ and $\widehat{g}$:

$$\widehat{h}(n) = (cos(\frac{\pi n}{N}))^4 \quad \forall n \in I_n \tag{3}$$

$$\widehat{g}(n) = (1 - (cos(\frac{\pi n}{N}))^8)^{\frac{1}{2}} exp(-\frac{i\pi n}{N} - \frac{\pi}{2}) \quad \forall n \in I_n \tag{4}$$

Eventually sequences h and g are obtained by the inverse DFT.

$$h(n) = \frac{1}{N} \sum_{k \in I_N} \widehat{h}(k) exp(\frac{i\pi kn}{N}) \tag{5}$$

$$g(n) = \frac{1}{N} \sum_{k \in I_N} \widehat{g}(k) exp(\frac{i\pi kn}{N}) \tag{6}$$

For N=256, the following numerical results were obtained:

$$\begin{array}{rlrl} h(0) & = 0.375 & g(1) = -g(0) & = 0.5907 \\ h(1) = h(-1) & = 0.250 & g(2) = -g(-1) & = 0.1107 \\ h(2) = h(-2) & = 0.0625 & g(3) = -g(-2) & = 0.0145 \end{array}$$

There are small differences with the coefficients which appears in [2], for larger N (e.g N=4096) one can exactly get the numbers given in [2].

Let us rewrite the definition of discrete dyadic wavelet transform, in the Discrete Fourier Transform terms. First define:

$$\widehat{h}_j(n) = \widehat{h}(2^j k \mid modN)$$

$$\widehat{g}_j(n) = \widehat{g}(2^j k \mid modN)$$

Remark: The above modulo function is nonstandard in the sense that it should take values in I_N.

The symbol $\widehat{}$ will indicate the DFT of the appropriate sequence, e.g. $\widehat{W_j} f$ denotes the DFT of the sequence $W_j f$.

Since the periodic convolution corresponds to the multiplication of DFT's, from the definition of dyadic wavelet transform one can get:

$$\widehat{W_j} f(n) = \widehat{g}_{j-1}(n)\widehat{h}_{j-2}(n), \ldots, \widehat{h}_1(n)\widehat{h}(n)\widehat{f}(n) \ \ \forall n \in I_N \tag{7}$$

$$\widehat{S}_j f(n) = \widehat{h}_{j-1}(n)\widehat{h}_{j-2}(n), \ldots, \widehat{h}_1(n)\widehat{h}(n)\widehat{f}(n) \ \ \forall n \in I_N. \tag{8}$$

Following lemma 1, section 2, uniqueness is related to the size of the space $\mathcal{N}$, the zero information space. The space $\mathcal{N}$ is an intersection of the null space of the operator S_J and the space of $p \in \mathcal{L}$ such that:

$$W_j p(n) = 0 \ \ j = 1, 2, \ldots, , J \ \ \forall n \in X_j f \cup N_j f. \tag{9}$$

Therefore, first we will find a basis for a null space of S_J, and then we will consider linear equations induced by (9).

From the equation (8), it seems that the easiest way to find the basis for the null space of S_J is to consider the exponential family, whose elements b_l are defined by their Discrete Fourier Transform as:

$$\widehat{b}_l(n) = \begin{cases} 1 & \text{if } l = n \\ 0 & \text{otherwise} \end{cases}$$

The sequences b_l are eigenfunctions of S_J (they are such for any linear, time invariant transformation) with eigenvalues:

$$\Lambda_S(l) = \widehat{h}_{J-1}(l)\widehat{h}_{J-2}(l), \ldots, \widehat{h}_1(l)\widehat{h}(l).$$

Therefore the basis for the null space of S is:

$$\{b_l \ : \Lambda_S(l) = 0\}$$

In the cubic spline wavelet case, the equation:

$$\widehat{h}(n) = 0 \quad n \in I_N$$

has only one solution $n = \frac{N}{2}$ (this is true also for the Haar basis case). Therefore, it can easily be shown that:

$$\Lambda_S(l) = 0 \quad for \quad l = \frac{kN}{2^J} \quad where \quad k = \pm 1, \pm 2, \ldots, \pm 2^{J-2}, 2^{J-1}.$$

Thus the dimension of this null space is $2^J - 1$. The above basis consists of complex sequences (the linear space is over a complex field). In order to have a real field one can define the following equivalent real basis :

$$p_{2l-1}(n) = cos(\frac{2\pi l n}{2^J}) \quad l = 1, 2, \ldots, 2^{J-1}$$

$$p_{2l}(n) = sin(\frac{2\pi l n}{2^J}) \quad l = 1, 2, \ldots, 2^{J-1} - 1.$$

A function p, which belongs to the null space of S_J can be written as:

$$p(n) = \sum_{l=1}^{2^J-1} \alpha_l p_l(n).$$

By the linearity properties we can write:

$$W_j p(n) = \sum_{l=1}^{2^J-1} \alpha_l W_j p_l(n).$$

Let us define the following vectors:

$$\underline{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{2^J-1})'.$$

$$\underline{\mathcal{W}}_j(n) = (W_j p_1(n), W_j p_2(n), \ldots, W_j p_{2^{J-1}-1})(n).$$

Using conditions from the previous section, every $n \in X_j f \cup N_j f$ introduces the following equation:

$$\underline{\mathcal{W}}_j(n) \cdot \underline{\alpha} = 0. \tag{10}$$

The symbol $\cdot$ denotes a scalar multiplication between the row $\underline{\mathcal{W}}_j(n)$ and the column $\underline{\alpha}$. In order to collect together equations (10) for different n's, we define the matrix $\mathcal{W}$ to consist of rows:

$$\underline{\mathcal{W}}_j(n) \quad j = 1, 2, \ldots, J \quad n \in X_j f \cup N_j f.$$

According to lemma 1, section 2, f has a unique maxima representation if and only if the only solution for:

$$\mathcal{W} \cdot \underline{\alpha} = 0 \tag{11}$$

is the vector $\underline{\alpha} = 0$. This condition is equivalent to:

$$rank(\mathcal{W}) = 2^J - 1.$$

Theorem 2 *The maxima representation Mf is unique if and only if the rank of the matrix $\mathcal{W}$ is $2^J - 1$.*

Conclusion 5 *If the number of the extrema points is less than $2^J - 1$, one can not get a unique representation. On the other hand, if the number of extrema points is greater than $2^J - 1$, there may be situations in which analysis of the $rank(\mathcal{W})$ can allow elimination of some extrema from the representation.*

4 Examples

During this work, we decomposed many signals and usually the number of the maxima points and their configuration were much above the requirements for the uniqueness. One of the reasons is that many maxima were obtained in the regions where the decomposed signal was close to zero. This phenomenon needs further treatment, nevertheless in examples we use signals with very clear maxima points.

Our first example illustrates the common case of a unique maxima representation. In addition to showing the uniqueness, we calculate the reconstructed sequence. The second example is for a non-unique representation. For this particular maxima representation we show, using Theorem 2, the exact family of all signals satisfying this representation.

4.1 Example 1

Let us assume N=256 and J=3 and consider the following sequence:

$$f(k) = sin(\frac{2\pi k}{256}(3(\frac{|k|}{N})^5 + 1)).$$

This sequence was chosen in order to exhibit different frequency components without evoking too many extreme points. Figure 1 shows its wavelet decomposition.

The maxima sets are as follows:

$$X_1 f \cup N_1 f = \{-127, -120, -109, -91, 0, 92, 110, 121\}$$

$$X_2 f \cup N_2 f = \{-127, -119, -109, -90, 1, 92, 111, 121\}$$

$$X_3 f \cup N_3 f = \{-126, -118, -107, -88, 2, 92, 111, 122\}$$

All together we have 24 extrema points, while the dimension of the null space of S_3 is 7. We can pick up 7 rows of the matrix $\mathcal{W}$ and check for regularity (non-singularity). We have chosen:

$$\mathcal{W}_s = (\mathcal{W}_1(0), \mathcal{W}_1(92), \mathcal{W}_1(110), \mathcal{W}_1(121), \mathcal{W}_2(1), \mathcal{W}_2(92), \mathcal{W}_3(2))$$

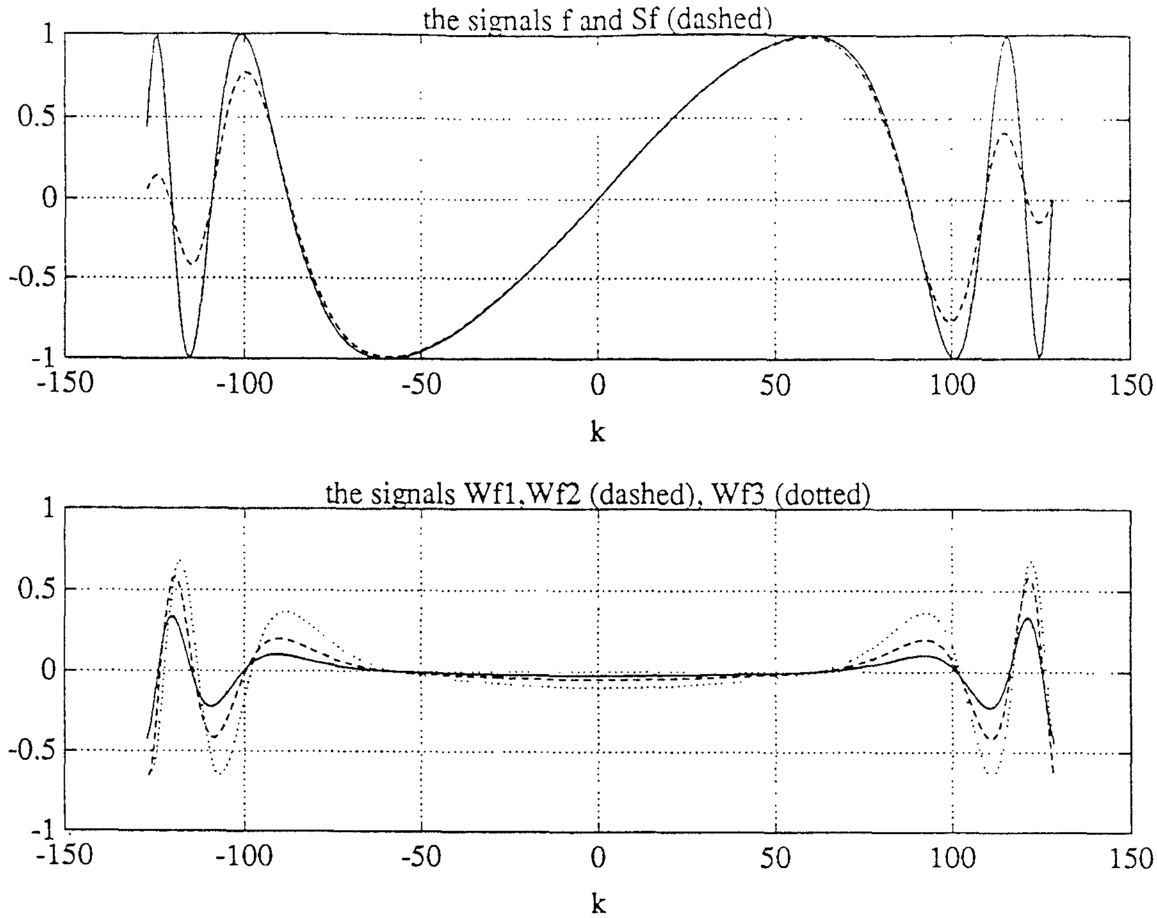

Figure 1: The signal f and its wavelet decomposition.

which is equal to:

−0.2621	−0.6328	−0.6847	−0.6847	−0.9237	−0.3826	−1.0000
0.2621	0.6328	−0.6847	−0.6847	0.9237	0.3826	−1.0000
−0.6328	0.2621	0.6847	0.6847	0.3826	−0.9237	−1.0000
0.2621	−0.6328	0.6847	−0.6847	0.9237	−0.3826	1.0000
−0.0000	−0.7054	−0.0000	−0.2500	0.0000	−0.0208	0
0.4988	0.4988	−0.2500	0.0000	0.0147	−0.0147	0
0.0000	−0.1821	0.0000	0.0000	0.0000	0.0054	0

This matrix is regular with an inverse:

−1.3151	−0.3815	0.5082	−1.1883	9.2922	2.7834	−20.2619
−0.0512	0.0000	0.0000	−0.0512	0.2804	−0.0000	−6.2206
−2.5894	−0.7229	1.0055	−2.3068	18.1422	1.5071	−50.1899
0.2889	−0.0000	−0.0000	0.2889	−5.5825	−0.0000	19.6132
0.5874	0.6496	−0.1442	1.0927	−6.7751	−0.7899	20.2882
−1.7392	0.0000	0.0000	−1.7392	9.5260	0.0000	−24.8083
1.0750	−0.0050	−0.6884	1.3816	−8.5990	−1.0318	20.9344

Conclusion: The maxima representation Mf of the sequence f(k), defined in this subsection is unique. Furthermore, essentially, one can give up 17 extrema points out of 24, and still have a unique maxima representation.

In this case, the reconstruction can be calculated as follows. Let us denote:

$$\hat{h}_t(n) = \hat{h}_{j-1}(n)\hat{h}_{j-2}(n), \ldots, \hat{h}_1(n)\hat{h}(n)$$

and define:

$$\hat{f}_s(n) = \begin{cases} \frac{\hat{S}_J f(n)}{\hat{h}_t(n)} & \hat{h}_t(n) \neq 0 \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

The sequence f_s is calculated from $\hat{f}_s$ by the inverse DFT.

Let us denote $\mathcal{W}_e$, the sampling operator at the chosen extreme points , i.e.:

$$\mathcal{W}_e f = (W_1 f(0), W_1 f(92), W_1 f(110), W_1 f(121), W_2 f(1), W_2 f(94), W_3 f(2))'$$

Then the reconstructed signal is given by:

$$f_r = f_s + \sum_{l=1}^{7} A(l) p_l$$

where the vector A is calculated by:

$$A = \mathcal{W}_s^{-1}(\mathcal{W}_e f - \mathcal{W}_e f_s).$$

Figure 2 describes $f - f_s$, the error from the deconvolution from $S_J f$ which is not large in this case. This error is harmonic, since it has to belong to the null space of S_J. The bottom part of the figure describes the error from the complete reconstruction, $f - f_r$ which is less then 10^{-7}.

One important remark is in order. The weak point in this reconstruction is the division by $\hat{h}_t(n)$ in the definition (12). Small values of $\hat{h}_t(n)$ introduce large sensitivity to numerical or approximation errors in $S_J f$. Our precise reconstruction results are due to floating point high numerical accuracy and due to a small number of levels. Other possible way to overcome this problem, is to reconstruct a signal using more maxima points than required by the uniqueness test.

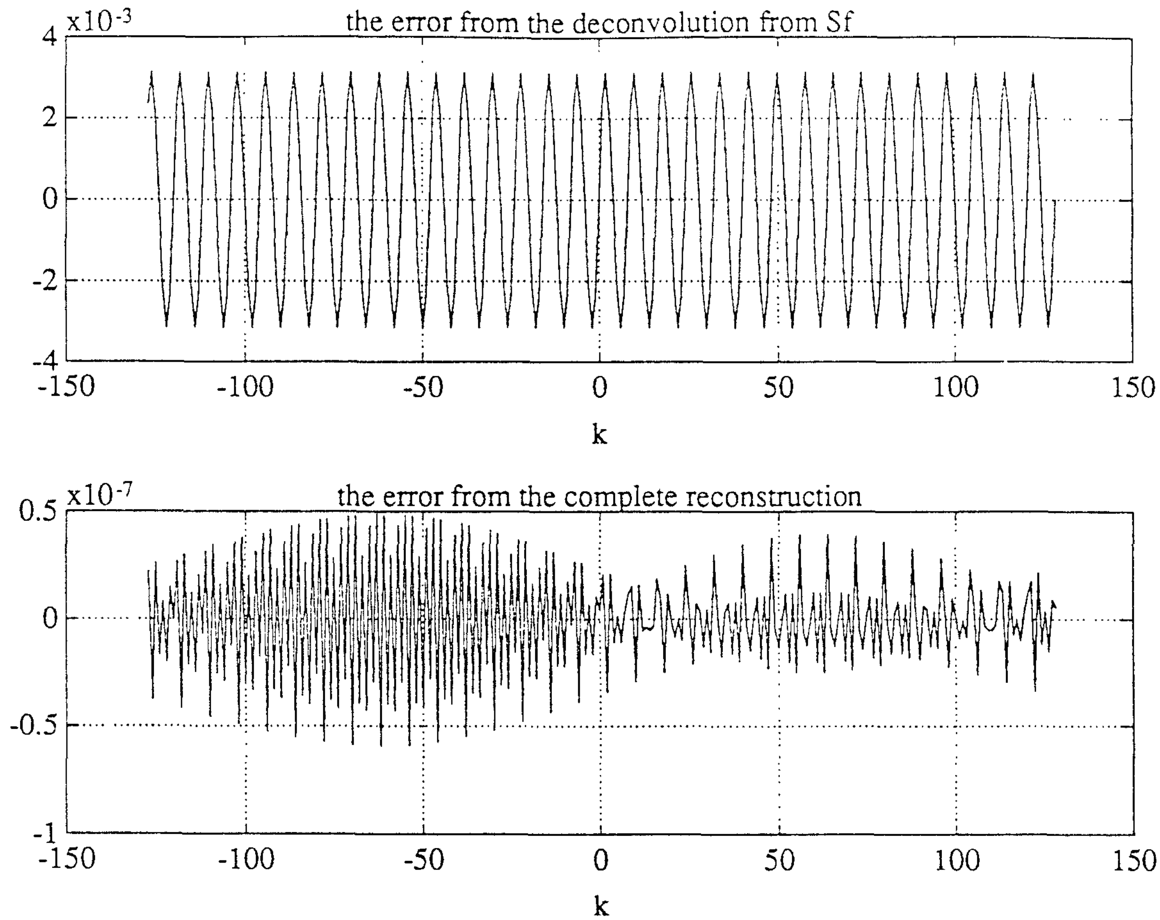

Figure 2: The reconstruction errors.

4.2 Example 2

In this subsection we try to find a sequence which gives no unique maxima representation. Consider:

$$p_p(k) = cos(\frac{2\pi k}{256} + \frac{\pi}{6}).$$

This sequence is from the null space of S_3. Figure 3 describes its wavelet decomposition($Sp_p = 0$ is omitted). Here at every level we have 64 extrema points, they appear in regular distances. The set of the extrema is given below:

$$X_1 f = \{2 + 8 * k : k \;\; integer\}$$

$$N_1 f = \{6 + 8 * k : k \;\; integer\}$$

$$X_2 f = \{2 + 8 * k : k \;\; integer\}$$

$$N_2 f = \{6 + 8 * k : k \;\; integer\}$$

$$X_3 f = \{3 + 8 * k : k \;\; integer\}$$

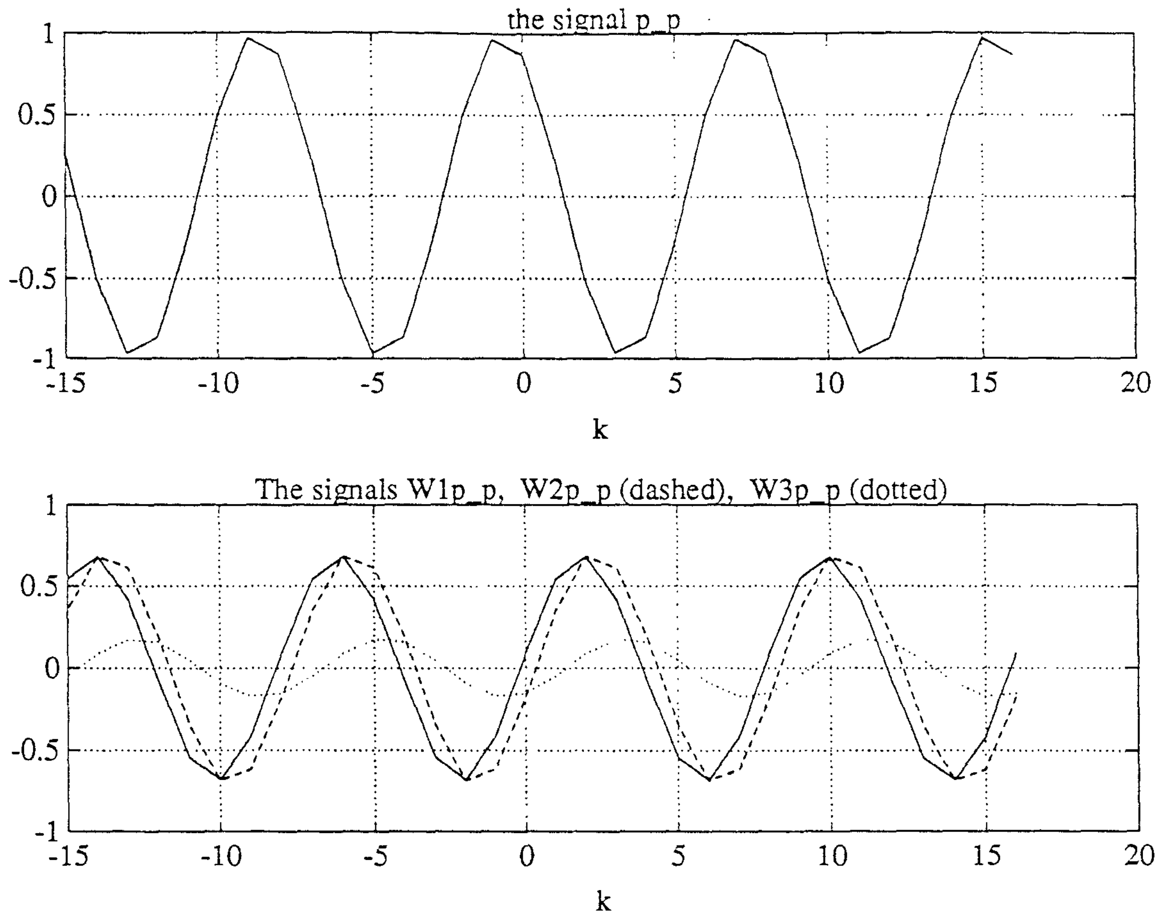

Figure 3: The signal $p_p(k)$ and its wavelet decomposition.

$$N_3 f = \{7 + 8 * k : k \ \ integer\}.$$

The basis for the null space of S_3 is:

$$\begin{aligned} p_1(n) &= cos(\tfrac{2\pi n}{8}) & p_2(n) &= sin(\tfrac{2\pi n}{8}) \\ p_3(n) &= cos(\tfrac{4\pi n}{8}) & p_4(n) &= sin(\tfrac{4\pi n}{8}) \\ p_5(n) &= cos(\tfrac{6\pi n}{8}) & p_6(n) &= sin(\tfrac{6\pi n}{8}) \\ p_7(n) &= cos(\tfrac{8\pi n}{8}) \end{aligned}$$

All the functions p_i are 8-periodic. Linear operator preserves this periodicity. Therefore the rows $\mathcal{W}_j(n)$ are also 8-periodic in the sense:

$$\mathcal{W}_j(n) = \mathcal{W}_j(n+8) \quad n \in I_N \quad j = 1, 2, 3.$$

Thus every level contributes only 2 different rows to the matrix $\mathcal{W}$, and there are only 6 different rows in $\mathcal{W}$! The ultimate conclusion is that in this case the maxima representation is **not unique**.

The different rows of $\mathcal{W}$ are as follows:

0.6328	−0.2621	0.6847	0.6847	−0.3826	0.9237	−1.0000
−0.6328	0.2621	0.6847	0.6847	0.3826	−0.9237	−1.0000
0.4988	−0.4988	0.2500	0.0000	0.0147	0.0147	0
−0.4988	0.4988	0.2500	−0.0000	−0.0147	−0.0147	0
0.1288	−0.1288	0.0000	0.0000	−0.0038	−0.0038	0
−0.1288	0.1288	−0.0000	0.0000	0.0038	0.0038	0

The rank of this matrix is only 5. One can observe that the last two rows are dependent. Consider:

$$\mathcal{W}_s = \mathcal{W}(1:5, [2, 3, 5, 6, 7]),$$

the submatrix of the elements from the rows 1,2,3,4,5 and the columns 2,3,5,6,7. It is a regular matrix, with an inverse matrix not having large elements. Therefore we will use it for the representation of the general sequence satisfying $S_3 f = 0$ and giving zero samples at extreme points (i.e. from the space $\mathcal{N}$). Let us assume a_1, a_2 to be free coefficients corresponding to the sequences p_1 and p_4. Let, α_1^p and α_2^p be defined as:

$$\alpha_1^p = -\mathcal{W}_s^{-1} \cdot \mathcal{W}^{c1}$$

$$\alpha_2^p = -\mathcal{W}_s^{-1} \cdot \mathcal{W}^{c4}$$

where $\mathcal{W}^{c1}$ and $\mathcal{W}^{c4}$ are the first and the fourth columns of the first five rows of $\mathcal{W}$. The space $\mathcal{N}$ is spanned by the two following functions:

$$pw_1 = p_1 + \alpha_1^p(1)p_2 + \alpha_1^p(2)p_3 + \alpha_1^p(3)p_5 + \alpha_1^p(4)p_6 + \alpha_1^p(5)p_7$$

$$pw_2 = p_4 + \alpha_2^p(1)p_2 + \alpha_2^p(2)p_3 + \alpha_2^p(3)p_5 + \alpha_2^p(4)p_6 + \alpha_2^p(5)p_7.$$

Therefore, the general form of the solution is :

$$p = p_p + a_1 pw_1 + a_2 pw_2.$$

Conditions for $W_j p$ to be monotonic between its extrema introduce 24 linear inequalities in a_1 and a_2. Elementary analysis of this system leads to the following seven dominant inequalities:

$$\begin{array}{lcccr}
1.0000a_1 & + & 1.3059a_2 & > & -0.4330 \\
1.0000a_1 & - & 1.3059a_2 & > & -0.4330 \\
0.0000a_1 & + & 1.0000a_2 & > & -0.1849 \\
0.0000a_1 & + & 1.0000a_2 & < & 0.1849 \\
1.0000a_1 & + & 0.3574a_2 & < & 0.1007 \\
1.0000a_1 & - & 0.3574a_2 & < & 0.1007 \\
1.0000a_1 & + & 0.0000a_2 & < & 0.0991
\end{array}$$

Figure 4 shows the boundary of the set $\mathcal{A}$. All the points inside the boundary satisfy the above set of inequalities.

In order to visualize this result, let us define three sequences. The first is the original p_p and the next two are defined as:

$$p_{pa} = p_p - 0.4pw_1$$

$$p_{pb} = p_p - 0.2pw_1 + 0.16pw_2.$$

Figure 5 and Figure 6 show these sequences and their wavelet transforms. From the graphs one can indeed see that all have the same discrete wavelet maxima representation.

These examples consist of the functions from the null space of S_3. From the definition of wavelet transform one can easily see that, for every function p from this space $W_j p = 0 \;\; \forall j > 3$. Therefore these functions have the same wavelet maxima representation for any $J \geq 3$. **Conclusion: the discrete wavelet maxima representation for the cubic spline wavelet with $J \geq 3$ and $N = 256$ is not always unique.** It is interesting to check how this example behave with different N's, but has not yet been investigated.

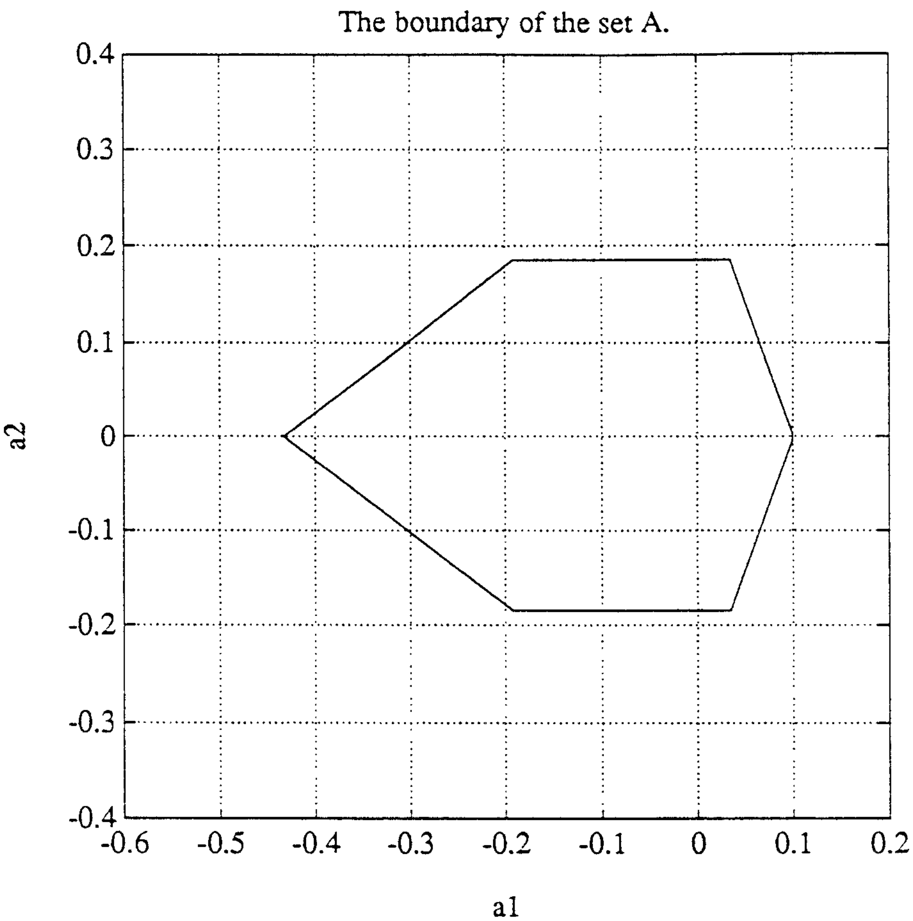

Figure 4: The boundary of the set $\mathcal{A}$ on the plane (a_1, a_2)

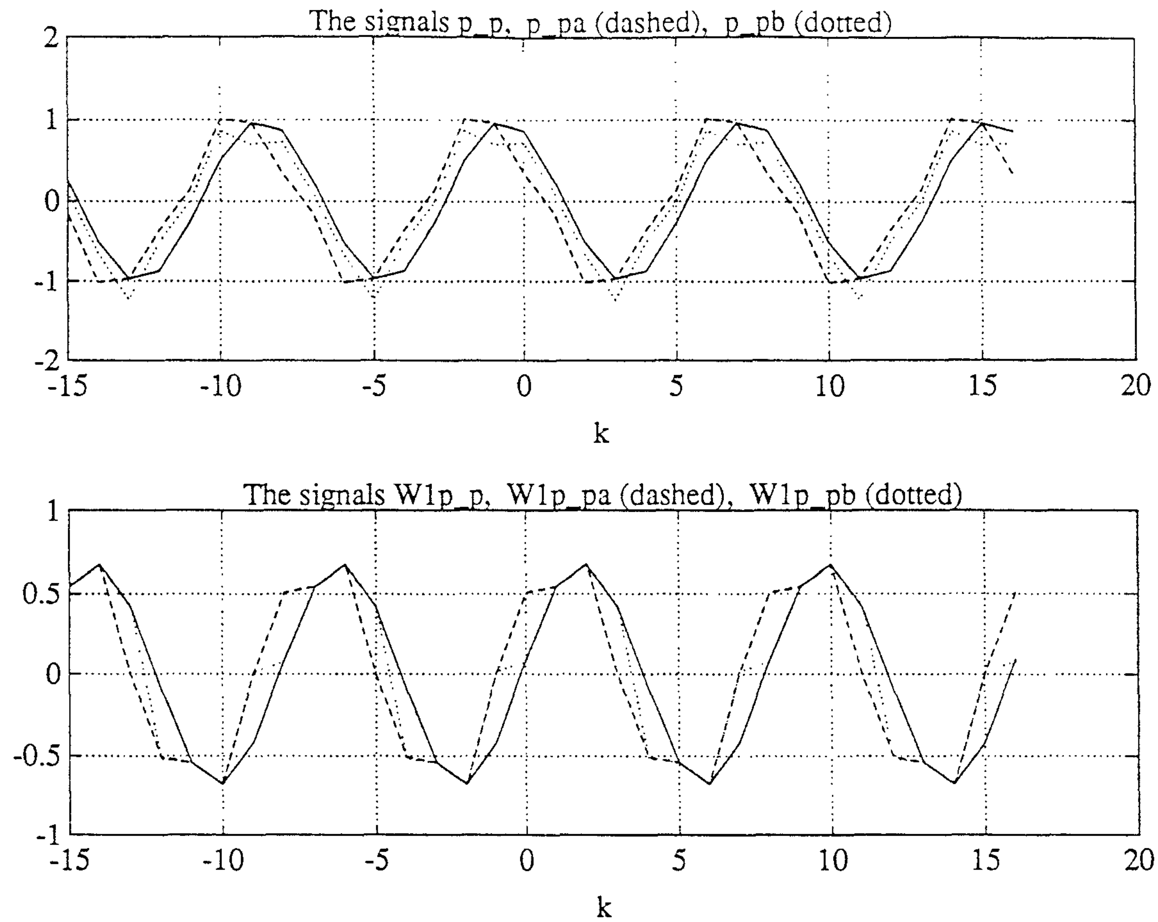

Figure 5: The signals and their first level wavelet transforms.

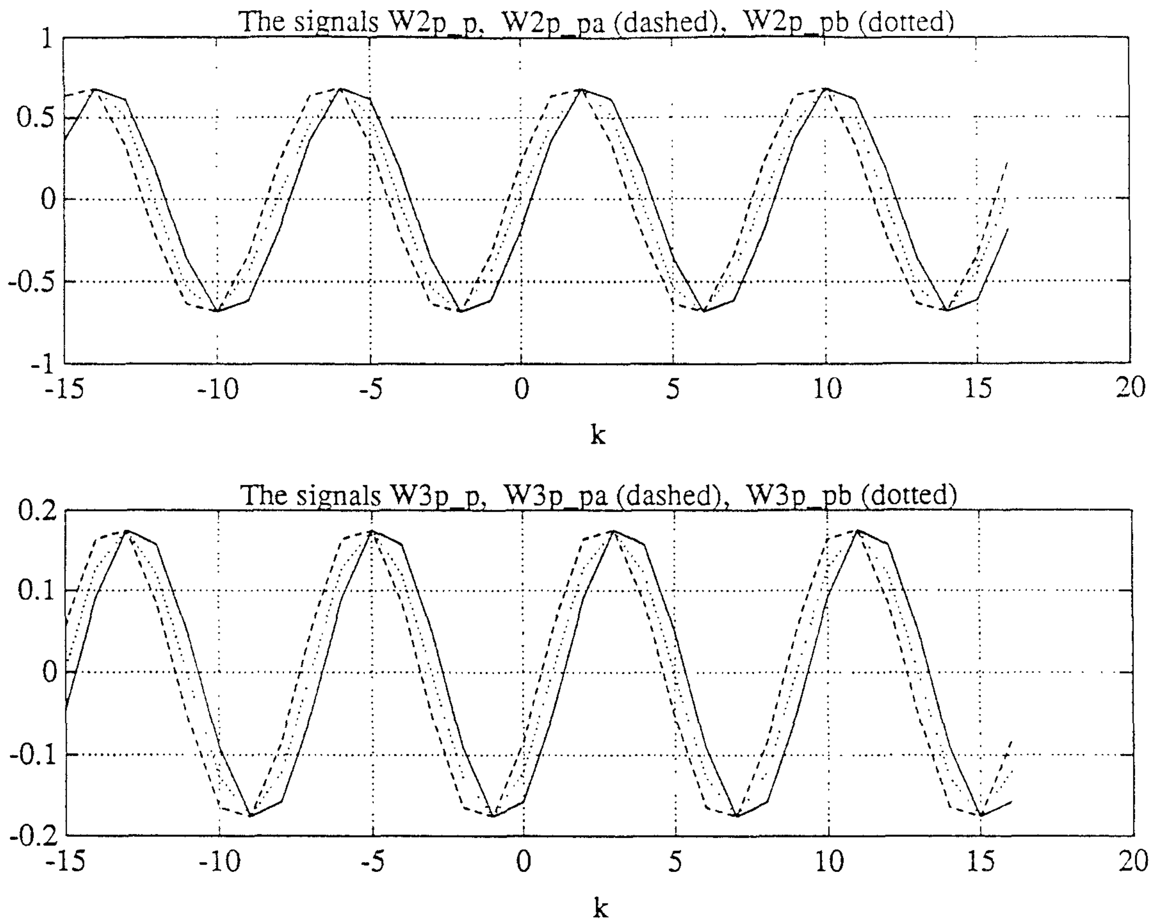

Figure 6: The second and the third level wavelet transforms.

5 Discussion

We have proposed a tool to analyze the problem of the reconstruction from the discrete maxima wavelet representation. The test for uniqueness and the general solution for the reconstruction problem are the major results. These results are not limited to the wavelet transform, but are also valid for any bank of linear filters where the outputs are sampled at extreme values. We believe that these tools can help to understand, develop, and evaluate better this kind of adaptive sampling technique.

The second part of the work dealt with particular cases of implementations. The test for uniqueness for N=256 and J=3 can easily be implemented. The complexity is linear in N, but exponential in J. Since in most practical application, the number of levels J is small (≈ 5), usually this algorithm can be implemented without sizable effort. In addition to answering "yes" or "no" to the uniqueness question, the algorithm can also give the number of redundant sampling points. This number may be a qualitative measure for reconstruction stability.

What is the importance of our counter-example to the uniqueness ? Certainly, it indicates that some carefulness is required. "Blind" use of this representation may lead to large reconstruction errors. In our opinion, the excellent stability properties of the representation justify making some effort to overcome this problem. When a non-unique representation is founded while using the uniqueness test, the size of the solution set can be checked and a few more sampling points can be added.

Although we hope that a new point of view was added to the analysis of the maxima representation problem, we do not claim that this "direct" approach provides stable and practical reconstruction method. The weakest part is the deconvolution of $S_J f$. Due to the very small magnitude of the eigenvalues of the operator S_J, this deconvolution is very sensitive to numerical errors. The situation is even worst if N or J increases. In the first place, one can replace very small eigenvalues by zeros, and enlarge the null space. Another possible approach is to deal directly with the space $\mathcal{N}$ (the intersection of the null space of S_J and the null space of the sampling operator at extreme points $\mathcal{W}_e$). In both cases, further investigation is required.

Another obstacle toward a practical applications is the model for sampling. A practical maxima representation would consider local maxima of the absolute value of a wavelet transform above some threshold [2]. Our model does not deal (yet) with this situation.

Some remarks on S. Mallat's conjecture, which deals with maxima representation in a continuous context are in order. One of the most powerful properties of the dyadic wavelet transform is that the discrete sequences $W_j f$ and $S_j f$ can be interpreted as samples at integers of continuous, well defined transformations of a continuous function f_c. S. Mallat defines extrema set points as the extreme points of continuous functions. In our approach, the maxima are calculated on the discrete sequence, after the function is sampled. Although every discrete extremum corresponds to a continuous one, there may be continuous maxima which are not detected from sampled data. Therefore a discrete maxima representation may contain less information than a continuous maxima representation. On the other hand, in many applications where the calculation complexity must be considered, the difference between these two representations might be hidden because only the sampled data is under consideration. Having many continuous maxima which are not detectable by a particular discrete sampling may exhibit a lack of stability with regard to sampling translation. It is clear that the correspondence between discrete and continuous maxima representation deserves further investigation. Besides the fact that discrete maxima representation is easier to analyze, it allows us to keep track on the performance of practical, discrete algorithms.

A The proof of lemma 1

First assume that the representation is not unique, i.e. $\exists g \neq f$ such that $Mg = Mf$. Then $p = f - g \neq 0$ belongs to $\mathcal{N}$.

Now assume that there exists p, a non zero sequence such that $p \in \mathcal{N}$. We will show slightly more than required, we claim that there exists a positive α_0 such that:

$$M(f + \alpha p) = Mf \quad \forall \alpha \quad 0 \leq \alpha < \alpha_0$$

From the hypothesis, by the linearity, it is clear that for every α:

$$W_j(f + \alpha p)(n) = W_j f(n) \quad \forall n \in X_j f \cup N_j f$$

and:

$$S_J(f + \alpha p) = S_J f$$

The only problem is to find α such that the maxima and the minima points sets are preserved, i.e.:

$$X_j(f + \alpha p) = X_j f$$

$$N_j(f + \alpha p) = N_j f.$$

The extrema are preserved if and only if the sequence $W_j(f + \alpha p)$ is monotonic between two consecutive extrema of $W_j f$ (increasing on the left of the maximum and the right of the minimum, and decreasing on the right of the maximum and the left of the minimum).

Let n_l, n'_l be two consecutive points from $X_j f \cup N_j f$. If

$$\mid n'_l - n_l \mid > 1$$

than $W_j f(n)$ is strongly monotonic between n_l and n'_l. Without loss of generality, we can assume that it is decreasing (for increasing, the treatment is symmetric). We are given:

$$W_j f(n+1) < W_j f(n) \quad n_l \leq n < n'_l \tag{13}$$

and want to find α such that $W_j(f + \alpha p)$ is strongly decreasing between n_l and n'_l. The condition:

$$W_j(f + \alpha p)(n+1) < W_j(f + \alpha p)(n)$$

is equivalent, by linearity, with:

$$\alpha[W_jp(n+1) - W_jp(n)] < W_jf(n) - W_jf(n+1) \quad n_l \leq n < n_l \qquad (14)$$

For $\alpha \geq 0$, the above condition is satisfied $\forall n$ such that:

$$W_jp(n+1) - W_jp(n) \leq 0.$$

In order to deal with remaining n's, the following boundary is defined:

$$\alpha_{j,l} = min\{\frac{W_jf(n) - W_jf(n+1)}{W_jp(n+1) - W_jp(n)} : n_l \leq n < n'_l \, and \, W_jp(n+1) - W_jp(n) > 0\}.$$

If the set under the minimum is empty, we define $\alpha_{j,l} = \infty$. Using inequality (13) we see that $\alpha_{j,l} > 0$.

Now it is very easy to show that condition (14) holds $\forall \alpha \; 0 \leq \alpha < \alpha_{j,l}$.

If $\mid n'_l - n_l \mid = 1$ there are no points between n_l and n'_l, so there is no problem in preserving the extrema sets in this interval.

The final step is to pass similarly through all scales j and through all points n_l from $X_jf \cup N_jf$. The crucial point is that the set of all admissible j's and l's is finite, therefore there exists $\alpha_0 > 0$ defined as:

$$\alpha_0 = \min_{j,l}\{\alpha_{j,l}\}.$$

and then we obtain:

$$M(f + \alpha p) = M(f) \quad \forall \alpha \; 0 \leq \alpha < \alpha_0 \qquad \square$$

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