ABSTRACT

Title of dissertation: Essays on Asset Pricing
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This dissertation consists of three essays. The first essay is titled "Speculative Dynamics I: Imperfect Competition, and the Implications for High Frequency Trading". In this essay, I analyze the nature of imperfect competition among informed traders who continuously generate and exploit private information about a risky asset's liquidation value which follows either a mean reverting process or random walk. I find the following results: (i) The combined trading of multiple informed traders is much more aggressive than the monopolistic trader in Chau and Vayanos (2008). (ii) The equilibrium price is even more revealing of the informed trader's private information. (iii) Market depth improves as the number of informed traders increases. (iv) In the limit of continuous trading, market is strong form efficient while aggregate profits of the informed traders remain bounded away from zero, in sharp contrast to the corresponding results in Holden and Subrahmanyam (1992), and Foster and Viswanathan (1993). (vi) Informed traders' inventories follows a Brownian motion, therefore enabling them to contribute significantly to total trading volume and price variance. These results shed light on empirical findings regarding high frequency traders by helping explain why they remain profitable despite aggressive competition with each other, why their trading

volume is very high, to what extent they improve efficiency, and through what mechanism they improve liquidity.

The second essay is titled "Speculative Dynamics II: Asymmetric Informed Traders". In this essay, I study the strategic interaction between hierarchical duopolistic informed traders who continuously generate and exploit private information about a risky asset's liquidation value, which follows either a mean reverting process or random walk. I find the following results: (i) Both traders duopolize the private information they both observe and the more informed trader monopolizes the additional exclusive private information. (ii) The common private information is incorporated into prices more efficiently than the monopolistic private information. (iii) In the limit of continuous trading, both traders' inventories based on their shared information follow Brownian motions. (iv) The trader with less superior information has more contribution to the trading volume and price volatility when the frequency of trading is sufficiently high. (v) As trading becomes more frequent, the less informed trader's expected profits may fall but converges to a strictly positive constant in the limit.

The third essay is titled "Real Options and Product Differentiation". In this essay, I develop a continuous time real investment model in an oligopoly industry where the products are heterogenous. Although the heterogenous products assumption can lower each firm's incentive to exercise the growth options prematurely, the preemption strategy is still profitable.

Essays on Asset Pricing

by

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Dedication

To my family.

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Chapter 1

Speculative Dynamics I: Imperfect Competition, and the Implications for High Frequency Trading

1.1 Introduction

Fama (1970) suggests that in a strong form efficient market, price reflects all public and private information. Will it be possible for traders with superior private information to earn strictly positive expected profits in a strong-form efficient market? Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) find that this is not possible and instead they reach a Bertrand-like result[1]. The intuition is that if there are two or more strategic traders receiving the same private information, each trader tries to preempt the others with the result that information is revealed almost instantaneously and each trader's expected profits quickly dissipate to zero in the limit of continuous trading. Surprisingly, Chau and Vayanos (2008) find that positive expected profits are possible while the market is strong form efficient. A monopolistic informed trader privately observes the flow of information and chooses to trade aggressively on her information to push the price towards her valuation of the asset[2]. In the limit, the information asymmetry disappears but the insider's profits

[1]The informed traders in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) submit market orders to the market maker instead of a demand schedule. The intuition is similar to that of Back and Paulsen (2009), in which each firm tries to preempt other firms by investing earlier with the result that the value of growth options equal to zero and the outcome is competitive.

[2]According to Chau and Vayanos (2008), the monopolist is "impatient" for three reasons: (1) time discounting, (2) public revelation of information, (3) mean-reversion of profitability.

converge to a positive constant. A similar result can also be found in Hellwig (1982) in which competitive informed traders submit market order conditioning on past prices. Even as the time interval between trades converges to zero, insiders can make positive returns while pushing the price to an arbitrarily closed to the efficient value. The difference among these models depends on the arrival of private information. Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) follow the assumptions in Kyle (1985) in which private information is one-shot and the asset's value is fixed. The private information is received when the trading session starts and there is a predetermined date when information is publicly announced and the asset is liquidated. In Chau and Vayanos (2008), the informed trader receives new information repeatedly, the fundamental value of the asset is stochastic, and trading takes place over an infinite horizon.

It is reasonable to believe that in actual financial markets, there are several informationaly large strategic investors consistently generating new information and trading to profit from the information. In this paper, I study the nature of imperfect competition among those traders. One of the purposes is to examine how imperfect competition and repeated arrivals of private information affect market efficiency, market liquidity, trading volume, price volatility, and the profitability of informed traders, especially relative to the monopolistic case in Chau and Vayanos' model and the oligoplistic case in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993). The other purpose of this paper is to shed light on high frequency trading by examining the properties of equilibrium in the limit of continuous trading. The paper can explain why high frequency traders[3] remain profitable

[3]According to Chlistalla (2011), there are three types of high frequency traders: (i) electronic market

despite aggressive competition with one another, why they contribute a significant portion to trading volume and price volatility, and to what extent do those traders improve market efficiency and add to market liquidity.

In the model, there is a riskless bond and a risky asset. The liquidation value of the risky asset which follows a stochastic process can only be observed by informed traders. Like the "market order" model in Kyle (1985), multiple identically informed strategic traders and exogenous liquidity traders execute batched market orders against competitive risk neutral market makers. The informed traders receive new information each period and trading takes place until the asset is liquidated at a random date. I prove that there exists a unique linear equilibrium, obtain the closed form solution up to a set of nonlinear equations, and derive analytical forms when trading becomes continuous. Not surprisingly, the combined trading of multiple informed traders is more aggressive than the monopolistic trader in Chau and Vayanos, the equilibrium price is even more revealing of the informed trader's private information, and market depth improves as the number of informed traders increases. Oligopolistic imperfect competition makes the informed traders trade more aggressively than a monopolist, thus improving market efficiency and increasing aggregate trading volume. The effects of imperfect competition on market depth is slightly more difficult to interpret since it has two opposite effects. One the one hand, with increasing competition, initially the net order flow will contain more information relative to the noise trading, and therefore the adverse selection is more severe and market depth is worse. On the other hand, as market becomes more efficient, there is less private information contained in the order flow, thus

making; (ii) statistical arbitrage strategies; (iii) liquidity detection.

improving market depth. In the stationary state, I show that the second effect dominates and imperfect competition among informed traders makes market deeper, thus improving liquidity.

Surprisingly, the model uncovers some important but unexpected results in the limit as the time interval between trades goes to zero. The first result is that the variance of the private information held by informed traders goes to zero at a rate proportional to the time interval between rounds of trading. This is much faster than the corresponding strong from efficiency result in the Chau and Vayanos' model, where the convergence rate is proportional to the square root of the time interval between rounds of trading. The second result is that the aggregate profits of the informed traders remain bounded away from zero. As the number of informed traders increases, their aggregate profits fall, tending to zero only as the number of informed traders becomes large. To be more specific, the aggregate profits near continuous trading is inversely proportional to the square root of the number of informed traders. The result has the flavor of Cournot competition, not the flavor of Bertrand competition found in the one-shot private information model of Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993). The third result concerns volume and volatility. The trading volume[4] of informed traders over Δt is of order $\sqrt{\Delta t}$, the same magnitude as the trading volume of liquidity traders. This implies that in the limit informed traders make a non-negligible contribution to total trading volume and thus price volatility, and the fraction converges to one as the number of informed traders becomes large. The result is novel since in almost all

[4]Trading volume is not well defined in continuous time Kyle's model since total variation of Brownian motion over any finite time is infinite.

the continuous time market microstructure model with price impact (Kyle's model and many extensions), the contributions to trading volume and price volatility by informed traders is negligible compared to the contributions of liquidity traders.

The results near continuous trading can help explain some empirical findings regarding high frequency trading. In recent years, financial markets have witnessed rapid growth in high frequency trading[5], made possible by the evolution of technology. High frequency traders are a subset of algorithmic traders. Those traders apply mathematical algorithms to either public or private statistical information, and they use fast computers to implement the algorithms, transmitting orders in a few milliseconds or less. High frequency traders contribute significantly to trading volume[6]. Despite aggressive competition with one another, high frequency traders remain profitable[7]. Kirilenko et al. (2011) document that high frequency traders are consistently profitable. They even turned a profit on the day on May 6, 2010 Flash Crash.

I focus on the type of high frequency traders who are pursuing low latency statistical arbitrage strategies. According to Chlistalla (2011), these traders "seek to correlate between assets and try to profit from the imbalance in these correlations". It might not be appropriate to label those traders as "informed" if one defines information as corporate news on "merger and acquisition decisions" or "content of earnings announcements". However,

[5]As pointed out by Duhigg in Stock Traders Find Speed Pays, in Milliseconds (*New York Times*, July 23, 2009), "Average daily volume has soared by 164 percent since 2005, according to data from NYSE. ..., stock exchanges say that a handful of high-frequency traders now account for more than half of all trades."

[6]In "The Real Story of Trading Software Espionage" (*AdvancedTrading.com*, July 10, 2009), Iati mentions that "High-frequency trading firms, which represent approximately 2% of the trading firms operating in the U.S. markets today, account for 73% of all U.S. equity trading volume."

[7]Iati's article states that "TABB group estimates that annual aggregate profits of low-latency arbitrage strategies exceed $21 billion, spread out among the few hundred firms that deploy them."

if those traders are faster and better than average market participants in gathering and processing information on order flows and price movements on the security and any other correlated securities to generate private signals which they can profit from, it is reasonable to call them "informed"[8]. At first glance, it appears that Chau and Vayanos (2008) can be a good model for those high frequency statistical arbitrageurs. Although their model has good insights on market efficiency and profitability, it does not explain trading volume and market liquidity. The fraction of trading volume of the monopolistic informed trader in Chau and Vayanos is essentially zero in the limit of continuous trading, and the market becomes less liquid as the frequency of trading increases. Their model does not address the question of what happens to high frequency traders' profits if they compete very aggressively with each other. By focusing on imperfect competition in a non-cooperative duopolistic setting, I am able to show that high frequency traders incorporate private signals into prices much faster than in Chau and Vayanos' model, high frequency traders add to market liquidity by competing aggressively with each other, they contribute significantly to trading volume, and most important of all, those traders remain profitable despite exploiting from the same information set and implementing the same algorithms.

After the flash crash of May 6, 2010, a recent policy proposal suggests that batching orders less frequently can reduce the participation rate and profits of high frequency traders and improve market depth. My model suggests that such a regulation would have the opposite effect of reducing liquidity. If each high frequency trader has to pay an entry cost

[8]Hendershott and Riordan (2011b) find that the market orders by high frequency traders have information advantage.

and operating cost, the number of high frequency traders might decrease with less frequent order batching, with the result that less competition will lead to market being less liquid.

This paper belongs to the literature on strategic trading with asymmetric information. In the pioneering work of Kyle (1985), a monopolistic insider uses liquidity traders as camouflage, reveals her private information gradually, and exploits her monopoly power over time when facing a competitive risk neutral market maker. In the subsequent extensions by Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993), due to the imperfect competition among identically informed traders, almost all private information is revealed only after a few trading rounds. Foster and Viswanathan (1994) replace homogenous private information with a hierarchical information structure to study learning among strategic informed traders. Foster and Viswanathan (1996) relax the assumption even further to allow for a more general correlation structure among the signals received by multiple informed traders. They show the initial correlation among the signals has a strong effect on the trading strategies and informativeness of prices. Traders initially compete aggressively on the common part of the private information and later play a "waiting" game by making smaller bets and trying to infer private information exclusive to others. Back et al. (2000) solve a similar problem in continuous time and derive a closed-form solution.

The traders in my model exploit their private signals via market orders. Rosu (2009) directly models the limit order book. He also finds a similar prediction that higher competition causes smaller price impact. However, competition in his model is measured by how fast traders arrive in the market whereas in my model, competition is measured by the number of informed traders. In his model, traders trade for liquidity reasons and there is no asymmetric information, whereas the motive for trade in my model comes from private information.

Martinez and Rosu (2011) study a very similar problem. They tackle the problem directly in setup similar to that of Back (1992) and focus on non-stationary equilibrium. In order to generate linear equilibrium in continuous time, they assume an informed trader to have uncertainty aversion regarding the level of the asset value (Informed traders care more about the change in the value of asset than the level) and impose a technological constraint on the market maker. My paper, on the other hand, does not require such assumptions and focuses on stationary equilibrium.

In my model, high frequency traders are risk neutral. Therefore, the model cannot explain the phenomenons that high frequency traders reverse their inventories frequently, and move in and out of short-term positions very quickly. Future work may explain pattern of mean-reverting inventories by making the traders risk averse instead of risk neutral.

The paper is organized as follows. In Section II, I describe the model, solve the linear equilibrium, prove its uniqueness. Section III characterizes the equilibrium near continuous trading. Section IV shows some comparative statics results and derives empirical implications. Section V concludes.

1.2 The Model

Assumption 1: Securities

I consider an economy with a single consumption good. There are a riskless bond with zero interest rate and a non-dividend paying risky asset with a liquidation value v_n which evolves stochastically. Trading takes place from $t = 0$ to $t = +\infty$ at the discrete points t_n ($t_n = nh$), until the risky asset is liquidated where h is the time interval between the auctions. At the end of each period, there is a probability $1 - \exp(-rh) = rh + o(h^2)$ that the risky asset is liquidated. I further assume the riskless bonds are in perfectly elastic supply. The liquidation value v_n follows a mean-reverting process or random walk:

$$v_n - v_{n-1} = \kappa(\bar{v} - v_{n-1})h + \epsilon_{v,n}. \tag{1.1}$$

In the above specification, κ determines the adjustment speed of the liquidation value v_n to its long run fixed target $\bar{v}$. κ is assumed to be greater than or equal to zero such that the prices are stationary. If $\kappa = 0$, then v_n follows a random walk. The innovation $\epsilon_{v,n}$ is independently and normally distributed with mean zero and variance $\sigma_v^2 h$.

Assumption 2: Market Participants and Information Structure

The risk neutral market participants consist of a competitive market maker, M (M is a positive integer) informed strategic traders, and a number of liquidity traders. The informed traders are each assumed to be able to perfectly observe the liquidation value v_n at each period. Conditional on that the asset has not yet been liquidated at the beginning of the nth period, $I_n^i \equiv \{p_\tau, v_\tau | 0 \leq \tau \leq n\}$ is each informed trader's information set at $t = nh$, and $I_n^m = \{p_\tau | \tau \leq n\}$ is the market makers' information set.

At each period, both the informed traders and liquidity traders submit market orders to the market maker. The liquidity traders' order is denoted by u_n, which is normally distributed with mean zero and variance $\sigma_u^2 h$. I further assume u_n is uncorrelated with $\epsilon_{v,n}$. I denote the market order submitted by the jth informed trader at the nth period $(t = nh)$ by $x_{j,n}$. In equilibrium, I should have $x_{1,n} = ... = x_{M,n} = x_n$ because of symmetry argument.

Assumption 3: Timing of events

I assume at the nth period, the informed traders and the noise traders submit their demands before new information arrives. After submitting their market orders, the informed traders observe ϵ_n and thus v_n. The market maker observes the total order imbalance $y_n = \sum_{j=1}^{M} x_{j,n} + u_n$, then sets the price p_n equal to the expected value of the asset based on the history of order flows, and takes the other side of the trade. At the end of the period, there is a probability $(1 - \exp(-rh))$ that the liquidation value v_n is public announced, the risky asset is liquidated and investors profits are realized.

Pricing

Since the market maker is assumed to be competitive and risk neutral, therefore, at period n she sets the price p_n equal to the expected value of the asset after she receives the total batched market order $y_n = x_{1,n} + ... + x_{M,n} + u_n$. Therefore,

$$p_n = E[\sum_{n'=n}^{+\infty}(1 - \exp(-rh)) \exp(-r(n' - n)h) v_{n'} | I_{n-1}^m, y_n], \tag{1.2}$$

where $(1 - \exp(-rh)) \exp(-r(n' - n)h)$ is the probability that the asset is liquidated at the end of the n'th period.

Lemma 1.2.1: The price p_n is a linear function of the market maker's expectation of the current liquidation value of the risky asset $E(v_n|I_n^m)$:

$$p_n = \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}E(v_n|I_n^m) + \frac{\kappa h \bar{v}}{1-\exp(-rh)(1-\kappa h)}. \tag{1.3}$$

Proof: See Appendix A.

Optimization

Suppose the risky asset were liquidated at a random future date νh. Given that the asset has not been liquidated at $nh < \nu h$, the j^{th} informed trader's profits that accrue to her from period n should equal to the difference between the value of her position ($\sum_{n\le\tau\le\nu} v_\nu x_{j,\tau}$) and the cost of this position ($\sum_{n\le\tau\le\nu} p_\tau x_{j,\tau}$):

$$\begin{aligned}\pi_{j,n} &= \sum_{n\le\tau\le\nu}(v_\nu - p_\tau)x_{j,\tau} \qquad (1.4)\\ &= \sum_{n'=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)(\sum_{\tau=n}^{n'} x_{j,\tau}(v_{n'}-p_\tau))\end{aligned}$$

Since informed traders are risk neutral, at the nth period, the jth informed trader tries to maximize her expected trading profits:

$$\max_{x_{j,n'\ge n}} E[\sum_{n'=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)(\sum_{\tau=n}^{n'} x_{j,\tau}(v_{n'}-p_\tau))|I_{n-1}^i]. \tag{1.5}$$

Lemma 1.2.2: The jth informed trader's objective function can be written as:

$$\max_{x_{j,n'\geq n}} E[\sum_{n'=n}^{+\infty} \exp(-r(n'-n)h)x_{j,n'}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n'}+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n'})|I_{n-1}^{i}]. \tag{1.6}$$

Proof: See Appendix A.

1.2.1 Equilibrium Concept

The equilibrium concept in this paper is similar to the previous literature. I follow Foster and Viswanathan (1996) closely here and let $X_j = (x_{j,1}, ..., x_{j,\nu})$ (for each j) and $P = (p_1, ..., p_\nu)$ represent the strategy functions were the asset liquidated at νh. A Bayesian Nash equilibrium of the trading game is a $M+1$ vector of strategies $(X_1, ..., X_M, P)$ such that:

1. For any $j = 1, ..., M$ and all $n = 1, ..., \nu$, I have for $X'_j = (x'_{j,1}, ..., x'_{j,\nu})$

$$E[\pi_{j,n}(X_1, ..., X_j, ..., X_M, P)|I_{n-1}^{i}] \geq E[\pi_{j,n}(X_1, ..., X'_j, ..., X_M, P)|I_{n-1}^{i}] \tag{1.7}$$

2. For all $n = 1, ..., \nu$, I have

$$p_n = \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}E(v_n|I_n^m) + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}. \tag{1.8}$$

Therefore, the market maker sets the price equal to the expected value of the risky asset conditional on her information set inferred from the order flow. Each risk neutral informed trader, taking as given the price process set by the market maker and the strategies of other informed traders, submits market orders to maximize the expected profits taking into account the effect on the price.

I restrict attention to stationary linear Markov eqilibrium. In order to set the price p_n which takes a linear form in Equation (1.3), the market maker has to solve the inference problem about v_n. I then conjecture that informed trader j's optimal strategy at period n is to submit market orders which depend linearly on the pricing error defined as the difference between v_{n-1} and the market maker's conditional estimate $\hat{v}_{n-1} = E(v_{n-1}|I^m_{n-1})$, i.e.,

$$x_{j,n} = \beta_j(v_{n-1} - \hat{v}_{n-1}), \tag{1.9}$$

to maximize her expected profits.

1.2.2 The Market Maker's Inference Problem

To solve the market maker's inference problem, I use Kalman filtering. Conjecture that at the end of the $(n-1)$th period, the market maker believe v_{n-1} to be normally distributed with mean $\hat{v}_{n-1}$ and variance Σ_v[9]. Then, at the nth period, after observing the order imbalance y_n, the market maker updates her belief about v_{n-1} in the form of

$$\begin{aligned} v_{n-1} &= E(v_{n-1}|I_{n-1}) + \frac{\lambda}{1-\kappa h} y_n + \eta_n \qquad (1.10) \\ &= \hat{v}_{n-1} + \frac{\lambda}{1-\kappa h}(X_n + u_n) + \eta_n, \end{aligned}$$

[9] Σ_v is strictly greater than $\sigma_v^2 h$ since the informed traders observe $\epsilon_{v,n-1}$ after they submit the market order at the $(n-1)$th period.

where λ is the inference parameter for the market maker to be derived next. Since v_n follows a mean-reverting process (random walk if $\kappa = 0$) in Equation (1.1), v_n has the following expression:

$$v_n = (1 - \kappa h)\hat{v}_{n-1} + \kappa h \bar{v} + \lambda y_n + (1 - \kappa h)\eta_n + \epsilon_{v,n}. \tag{1.11}$$

Therefore, the market maker's posterior belief about v_n is normally distributed with mean

$$E(v_n|I_n^m) = \hat{v}_n = (1 - \kappa h)\hat{v}_{n-1} + \kappa h \bar{v} + \lambda y_n \tag{1.12}$$

and variance

$$var(v_n|I_n^m) = var((1 - \kappa h)\eta_n + \epsilon_{v,n}) = (1 - \kappa h)^2 var(\eta_n) + \sigma_v^2 h. \tag{1.13}$$

Stationary condition requires that $var(v_n|I_n^m) = \Sigma_v$.

Lemma 1.2.3: Given the trading strategy of the informed traders defined in equation (1.9), the market maker's inference parameter λ is given by

$$\lambda = \frac{(1 - \kappa h)\Sigma_v M \beta}{M^2 \beta^2 \Sigma_v + \sigma_u^2 h}, \tag{1.14}$$

and the variance of the market maker's belief on v_n satisfies the equation

$$\frac{(1 - \kappa h)^2 \Sigma_v \sigma_u^2 h}{M^2 \beta^2 \Sigma_v + \sigma_u^2 h} + \sigma_v^2 h = \Sigma_v. \tag{1.15}$$

Proof: See Appendix A.

1.2.3 The Informed Traders' Optimization Problem

At the beginning of each period, each informed trader submits market orders given the price process generated by the market maker to maximize the present value of the expected profits scaled by $\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}$. The j^{th} informed trader's optimization problem becomes

$$E[\sum_{n'=n}^{+\infty} x_{j,n'}(v_{n'} - \hat{v}_{n'})e^{-r(n'-n)h}|I^i_{n-1}]. \tag{1.16}$$

In the optimization, the informed trader takes account of how his trading and his estimate of the trading by other market makers influence the market price, which equal to market makers estimate $\hat{v}_n$. I conjecture that the value function for the j^{th} informed trader is quadratic with respect to $v_{n-1} - \hat{v}_{n-1}$ and takes the form of

$$V_{j,n} = B(v_{n-1} - \hat{v}_{n-1})^2 + C. \tag{1.17}$$

Later, I will prove the quadratic form is sustained and valid in Lemma 1.2.4. The value function must satisfy the following Bellman equation:

$$V(v_{n-1}, \hat{v}_{n-1}) = max_{x_n} E[x_n(v_n - \hat{v}_n) + e^{-rh}V(v_n, \hat{v}_n)|I^i_{n-1}]. \tag{1.18}$$

The solution of the above Bellman equation is provided in the following theorem:

Lemma 1.2.4: Given the price process set by the market maker, each informed trader's strategy (equation (1.9)) is characterized by a trading intensity parameter β, given by

$$\beta = \frac{(1 - 2e^{-rh}B\lambda)(1 - \kappa h)}{\lambda(M(1 - 2e^{-rh}B\lambda) + 1)}. \tag{1.19}$$

Equation (1.18) has a quadratic solution of the form $V(v_{n-1}, \hat{v}_{n-1}) = B(v_{n-1} - \hat{v}_{n-1})^2 + C$ where B and C satisfy the following set of equations:

$$B = \frac{(1 - \kappa h)^2(1 - e^{-rh}B\lambda)}{\lambda(1 + M - 2Me^{-rh}B\lambda)^2} \tag{1.20}$$

and

$$C = \frac{e^{-rh}B(\lambda^2\sigma_u^2 + \sigma_v^2)h}{1 - e^{-rh}}. \tag{1.21}$$

Proof: See Appendix A.

1.2.4 Equilibrium

Proposition 1.2.1: There exists a unique linear Markovian equilibrium characterized by five parameters λ, Σ_v, β, B and C which satisfy the system of five nonlinear equations: (1.14), (1.15), (1.19), (1.20) and (1.21). The expressions for Σ_v, β and λ are given by

$$\Sigma_v = \frac{\sigma_v^2 h}{1 - \frac{(1-\kappa h)^2}{M(1-2q_1)+1}}, \tag{1.22}$$

$$\beta = \sqrt{\frac{(1-2q_1)(1-\frac{(1-\kappa h)^2}{M(1-2q_1)+1})}{M}}\frac{\sigma_u}{\sigma_v}, \tag{1.23}$$

and

$$\lambda = \frac{(1-\kappa h)\sqrt{M}\sigma_v\sqrt{\frac{(1-2q_1(1+M(1-2q_1)))}{1+M(1-2q_1)-(1-\kappa h)^2}}}{M\sigma_u(1-2q_1)+\sigma_u}, \tag{1.24}$$

where

$$\begin{aligned} q_1 &= \frac{M+1}{3M} \qquad (1.25)\\ &- \frac{1}{6M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2-((M+1)^2-3Z)^3}}\\ &- \frac{1}{6M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2+((M+1)^2-3Z)^3}} \end{aligned}$$

and $Z = e^{-rh}(1-\kappa h)^2 < 1$.

Proof: See Appendix A.

1.3 Asymptotic Properties of Equilibrium in the Limit of Continuous Trading

1.3.1 Research Questions

It is important to study how imperfect competition affects the properties of equilibrium, especially when the frequency of trading becomes very high approaching to the limit of continuous trading. Firstly, it is intuitively to believe that increasing competition will make the already "impatient" monopolistic trader in Chau and Vayanos (2008) even more "impatient".

The aggregate trading of multiple informed traders competing strategically should be more aggressive than in the monopolist case. Chau and Vayanos find that when the time interval between rounds of trading is small, the variance of private information not incorporated into price Σ_v is proportional to $\sqrt{h}$. How does the more aggressive trading by informed traders improve the convergent rate? Can the rate be of the same or higher order than $\sqrt{h}$?

Secondly, does more competition improve market depth or not? As shown in the non-stationary setup in Holden and Subrahmanyam (1992), imperfect competition has two opposite effects. The market is less liquid in the beginning of the trading sessions since the net order flow contains more private information relative to the noise. After most information is revealed in the remaining of trading sessions, the market becomes very deep since price is already very efficient and there is less information asymmetry between the informed traders and market maker. It would be interesting to know which effect dominates in the stationary state.

Thirdly, Foster and Viswanathan (1993) show that identically informed traders' profits converge to zero in a continuous trading limit since by trading more frequently the traders have more opportunities to preempt each other. In their model and the model of Holden and Subrahmanyam (1992), private information is one-shot and the liquidation value of the risky asset is fixed. It is important to examine whether this Bertrand-like result still holds in the model where private information arrives repeatedly and the value of the asset is stochastic. In the steady state of this model, the informed traders earn zero profits if and only if the price impact is zero[10]. The market maker cannot set price impact to zero because there is

[10]This is because the market maker sets price efficiently and informed traders benefit when the liquidity

always new information coming in and the market maker has to learn from the net order flow. As long as the price impact is strictly positive in the limit, informed traders' profits remain bounded away from zero.

Lastly, in almost all dynamic market microstructure models with price impact, the trading volume contributed by the informed traders is negligible compared to the trading volume of liquidity traders when trading is continuous. For the same reason, the fraction of price volatility contributed by informed traders is also zero. Even though the monopolistic trader in Chau and Vayanos trades very aggressively, his trading volume over a short interval h is of order $h^{\frac{3}{4}}$[11], which is much smaller than $\sqrt{h}$ contributed by the liquidity trader over the same short period. In the presence of imperfect competition, the aggregate trading of informed traders should be more aggressive relative to the monopolistic case. But it remains to be shown whether the results will change qualitatively such that the trading volume of the informed traders is comparable to the trading volume of liquidity traders. If so, informed traders will also contribute significantly to price volatility as well.

To answer the above questions, I derive the asymptotic properties of the equilibrium near continuous trading in the following theorem.

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trader moves price away from its efficient value.

[11]Since trading intensity parameter $\beta \sim \sqrt{h}$ and pricing error is of order $h^{\frac{1}{4}}$, the trading volume over one period is of order $h^{\frac{3}{4}}$.

1.3.2 Asymptotic Properties of Equilibrium

Proposition 1.2.2: In the limit of continuous trading ($h \to 0$), the asymptotic behaviors of Σ_v, β, λ, B and C are given by:

$$\lim_{h \to 0} \frac{\Sigma_v}{h} = \frac{\sigma_v^2(1 + M(1 - 2q_0))}{M(1 - 2q_0)} \tag{1.26}$$

$$\lim_{h \to 0} \beta = \frac{\sigma_u(1 - 2q_0)}{\sigma_v \sqrt{1 + M(1 - 2q_0)}} \tag{1.27}$$

$$\lim_{h \to 0} \lambda = \frac{\sigma_v}{\sigma_u} \sqrt{\frac{1}{1 + M(1 - 2q_0)}} \tag{1.28}$$

$$\lim_{h \to 0} B = \frac{q_0 \sigma_u}{\sigma_v} \sqrt{1 + M(1 - 2q_0)} \tag{1.29}$$

$$\lim_{h \to 0} C = \frac{q_0 \sigma_u \sigma_v}{r} \left(\sqrt{1 + M(1 - 2q_0)} + \sqrt{\frac{1}{1 + M(1 - 2q_0)}} \right). \tag{1.30}$$

where

$$\begin{aligned} q_0 &= \frac{M+1}{3M} \\ &- \frac{1}{6M} \sqrt[3]{(M+1)^3 - 18M - 9 + \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2 + 2M - 2)^3}} \\ &- \frac{1}{6M} \sqrt[3]{(M+1)^3 - 18M - 9 - \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2 + 2M - 2)^3}}. \end{aligned} \tag{1.31}$$

Proof: See Appendix A.

From the above theorem, the parameter which measures the uncertainty of the market maker about the underlying profitability of the risky asset, Σ_v, converges to 0 when h goes to 0. The value Σ_v is also the variance of private information not incorporated into price at each period. The notation $\Sigma_v \sim h$ means that all private information is reflected in the price, information asymmetry disappears, and the market is strong form efficient when trading is continuous. Although we reach the same conclusion regarding market efficiency as in Chau and Vayanos' model, there is still some difference in how efficient the price becomes or how different the convergence rate is for small h. Since $\Sigma_v \sim \sqrt{h}$ for the case of a monopolistic informed trader and $\Sigma_v \sim h$ for our imperfectly competitive case, we have $\frac{\Sigma_v(M=1)}{\Sigma_v(M>1)} \sim \frac{1}{\sqrt{h}}$ with the ratio converging to infinity when $h \to 0$. Therefore, the equilibrium price with $M \geq 2$ informed traders is even more revealing of the informed traders' private information than the monopolist case. We should also expect that trading intensity is qualitatively different. In Chau and Vayanos, when there is only one monopolistic informed trader, $\beta \sim \sqrt{h}$. However, β is of order 1 when the market is populated with multiple informed traders. This implies that imperfect competition makes traders trade much more aggressively and bring information into the price much more quickly. Therefore, market makers learn more from the order flows and set more efficient price.

Next, I examine the trading volume contributed by informed traders to check whether it is comparable to the trading volume of liquidity traders in continuous trading. One can tell that β is of order 1 and $|v_{n-1} - \hat{v}_{n-1}| \sim \sqrt{h}$. Then over one trading period, the absolute aggregate trading volume of an informed trader $|x_n| = \beta|v_{n-1} - \hat{v}_{n-1}|$ is of order $\sqrt{h}$. Since the trading volume contributed by liquidity trader is of the same order $|u_n| = \sigma_u\sqrt{h}$, it follows that the informed traders generate a non-negligible fraction of total trading volume because the ratio $\frac{|X_n|}{|u_n|}$ converges to a positive constant bounded away from zero. The next theorem derives the fraction of trading volume contributed by informed traders.

Proposition 1.2.3: In the continuous trading limit ($h \to 0$), define ξ_M to be the fraction of trading volume contributed by the informed traders. The value of ξ_M can be expressed as

$$\xi_M = \frac{M\sqrt{1-2q_0}}{M\sqrt{1-2q_0}+1} \tag{1.32}$$

which depends only on the number of informed traders.

In most dynamic models with price impact, the informed traders' trade does not have a diffusion component which contributes to volatility in the limit of continuous trading since their fraction of trading volume is zero. In my model, as illustrated above, informed traders contribute significantly to trading volume, and thus they should contribute significantly to price volatility as well. It is trivial to write Δp_n as

$$\begin{aligned} \Delta p_n &= p_n - p_{n-1} = \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}\Delta \hat{v}_n \\ &= \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}\lambda(M\beta(v_{n-1}-\hat{v}_{n-1})+u_n). \end{aligned} \tag{1.33}$$

The price variance can therefore be written as

$$\frac{Var(\Delta p_n)}{h} = (\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)})^2 \frac{\lambda^2(M^2\beta^2\Sigma_v+\sigma_u^2 h)}{h}. \tag{1.34}$$

The next theorem illustrates the contribution of price variance by informed traders and liquidity traders.

Proposition 1.2.4: In the continuous trading limit ($h \to 0$), the price variance $\lim_{h\to 0} \frac{Var(\Delta p_n)}{h}$ can be decomposed into two components: (i) a contribution from informed traders given by $(\frac{r}{r+\kappa})^2\beta^2\lambda^2\frac{\sigma_v^2(1+M(1-2q_0))}{M(1-2q_0)}$; (ii) a contribution from liquidity traders given by $(\frac{r}{r+\kappa})^2\lambda^2\sigma_u^2$. The total price variance which is the sum of these two components, is $(\frac{r}{r+\kappa})^2\sigma_v^2$, independent of the number of traders.

Proof: (i) and (ii) are trivial to prove. To prove the last point regarding total price variance, observe that

$$\begin{aligned}
\lambda^2(M^2\beta^2\Sigma_v + \sigma_u^2) &= \frac{\sigma_v^2}{\sigma_u^2}\frac{1}{1+M(1-2q_0)}\left(\frac{M^2(1-2q_0)^2}{1+M(1-2q_0)}\frac{\sigma_u^2}{\sigma_v^2}\frac{\sigma_v^2(1+M(1-2q_0))}{M(1-2q_0)}\right. \\
&+ \left.\sigma_u^2\right) \\
&= \frac{\sigma_v^2}{\sigma_u^2}\frac{1}{1+M(1-2q_0)}(M(1-2q_0)\sigma_u^2 + \sigma_u^2) \\
&= \sigma_v^2.
\end{aligned} \tag{1.35}$$

Finally, I examine each informed trader's profitability. The expected profits can be written as

$$\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}V(v_{n-1},\hat{v}_{n-1}) = \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}(B(v_{n-1}-\hat{v}_{n-1})^2+C). \tag{1.36}$$

The term $(v_{n-1}-\hat{v}_{n-1})^2$ converges to 0 when $h \to 0$. But $\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}C$ converges to a positive constant from Proposition 1.2.2. Hence, competition makes the aggregate profits fall, but it does not drive profits to zero. The results are in sharp contrast with the ones found in Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) although sharing a similar result in terms of market efficiency.

1.3.3 Properties of the Perfectly Competitive Equilibrium

I examine another class of asymptotic results by taking the limit as M goes to infinity. It is easy to verify that when $M \to +\infty$, $\lim_{M\to\infty} M^2 q_0 = 1$. Then substituting $\frac{1}{M^2}$ for q_0 in Proposition 1.2.2, I can derive the properties of the perfectly competitive equilibrium in the limit of continuous trading in the next theorem.

Proposition 1.2.5: In the perfectly competitive case (i.e., when the number of traders goes to infinity), the asymptotic properties of the equilibrium now becomes:

$$\lim_{h\to 0, M\to +\infty} \frac{\Sigma_v}{h} = \sigma_v^2 \tag{1.37}$$

$$\lim_{h\to 0, M\to +\infty} \sqrt{M}\beta = \frac{\sigma_u}{\sigma_v} \tag{1.38}$$

$$\lim_{h\to 0, M\to +\infty} \sqrt{M}\lambda = \frac{\sigma_v}{\sigma_u} \tag{1.39}$$

$$\lim_{h\to 0, M\to +\infty} M^{\frac{3}{2}} B = \frac{\sigma_u}{\sigma_v} \tag{1.40}$$

$$\lim_{h\to 0, M\to +\infty} M^{\frac{3}{2}} C = \frac{\sigma_u \sigma_v}{r} \tag{1.41}$$

$$\lim_{M\to +\infty} \xi_M = 1 \tag{1.42}$$

.

Since $\sigma_v^2 h$ is the variance of new private information the informed traders learn at each period, $\lim_{h\to 0, M\to +\infty} \frac{\Sigma_v}{h} = \sigma_v^2$ implies that in the perfectly competitive case, there is no information left on the table. The result on β suggests that although each individual trader's trading intensity can be infinitesimally small, the aggregate trading intensity can be very large as the number of traders increases. The results on λ and C suggest that as the number of informed traders increases, market depth improves, but aggregate profits fall, tending to zero only as the number of informed traders becomes large. The result on ξ_M, the fraction of trading volume from informed traders, can be arbitrarily close to 1 as the number of traders is large enough.

1.4 Numerical Illustrations and Comparative Statics

In what follows, I numerically illustrate how information structure and imperfect competition among informed traders affect market efficiency, market liquidity, trading volume, price volatility, and expected profits of the informed traders. I also provide some empirical implications.

Since M is the number of informed traders in the market, the model reduces to the monopolist case if M is set to be 1. Most of the comparative static analysis is concerned with the effect of changing M. One issue concerns the comparison between the duopolist case $M = 2$ and monopolist case $M = 1$. I find that adding just one more informed trader to the monopolist case will change the asymptotic properties of equilibrium in Chau and Vayanos' model qualitatively. Another issue concerns within the oligopolistic situation. As we increase M from $M = 2$ to large values, the competition among informed traders increases. I study how changing the intensity of competition affects the properties of equilibrium.

Market Efficiency

I set the parameters such that $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$. Σ_v measures the market maker's uncertainty about the liquidation value of the risk asset. It is therefore a measure of the efficiency of price, with a smaller value corresponding to a more efficient price. $\Sigma_v = 0$ corresponds to the scenario where information asymmetry vanishes and the market is strong-form efficient. If increasing the number of imperfect competitors makes traders willing to incorporate more private information into the price, the price should become more informative and we therefore expect a smaller Σ_v as we increase M. As trading becomes more frequent (h is smaller), the noncooperative setting results in a more aggressive competition, making the already "impatient" informed traders even more "impatient". To illustrate this intuition, I show how Σ_v varies with h and M in Figure 1.1(A).

As shown in the figure, Σ_v monotonically decreases with h for the monopolist case and for the oligopolist cases when $M = 2$, 3 and 10. The value of Σ_v declines more rapidly for $M \geq 2$ than for $M = 1$. If we fix h and vary only the number of informed traders, Σ_v is found to be inversely related with the number of informed traders M. This confirms the previous intuition that increasing competition makes the market more efficient.

Next I examine the asymptotic properties of Σ_v. According to Chau and Vayanos (2008), $\Sigma_v \sim \sqrt{h}$ as $h \to 0$ in the monopolist case $M = 1$. I prove in Proposition 1.2.2 that $\Sigma_v \sim h$ in the oligopolist case $M \geq 2$. In Figure 1.1(B), I show how the scaled value of Σ_v varies with h for different M. I scale Σ_v by $\sqrt{h}$ for $M = 1$ and h for $M \geq 1$. From the figure, $\frac{\Sigma_h}{\sqrt{h}}$ approaches to a constant for $M = 1$, confirming the asymptotic result obtained for the monopolistic trader. When $M \geq 2$, $\frac{\Sigma_v}{h}$ converges to a positive constant confirming the asymptotic property of Σ_v obtained in Proposition 1.2.2. Since the ratio $\frac{\Sigma_v(M=1)}{\Sigma_v(M\geq 2)}\frac{\sqrt{h}}{h} \to \infty$ as h converges to zero, private information is revealed much more quickly and price becomes more efficient when there are multiple traders in the market.

Figure 1.2 illustrates how the degree of competition affects the asymptotic properties of Σ_v. When M is large enough, $\lim_{h\to 0}\frac{\Sigma_v}{\Sigma_v^2 h}$ can be very close to 1. Since Σ_v is always greater than $\sigma_v^2 h$, in the limit as $M \to \infty$, we have the result of strong-form efficiency, consistent with intuition based on perfect competition.

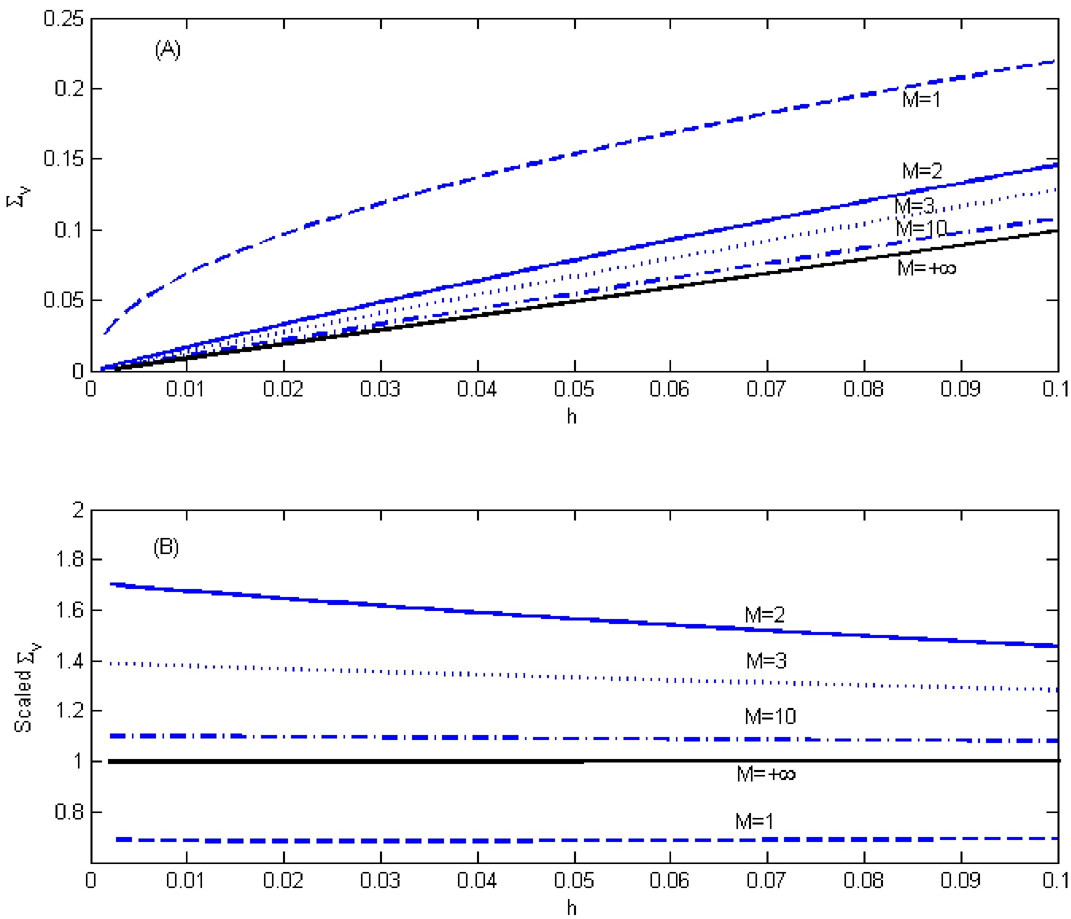

Figure 1.1: (A) Σ_v as a function of h for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. (B) Scaled Σ_v as a function of h. Σ_v is scaled by $\sqrt{h}$ for $M = 1$, and by h for $M \geq 2$. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

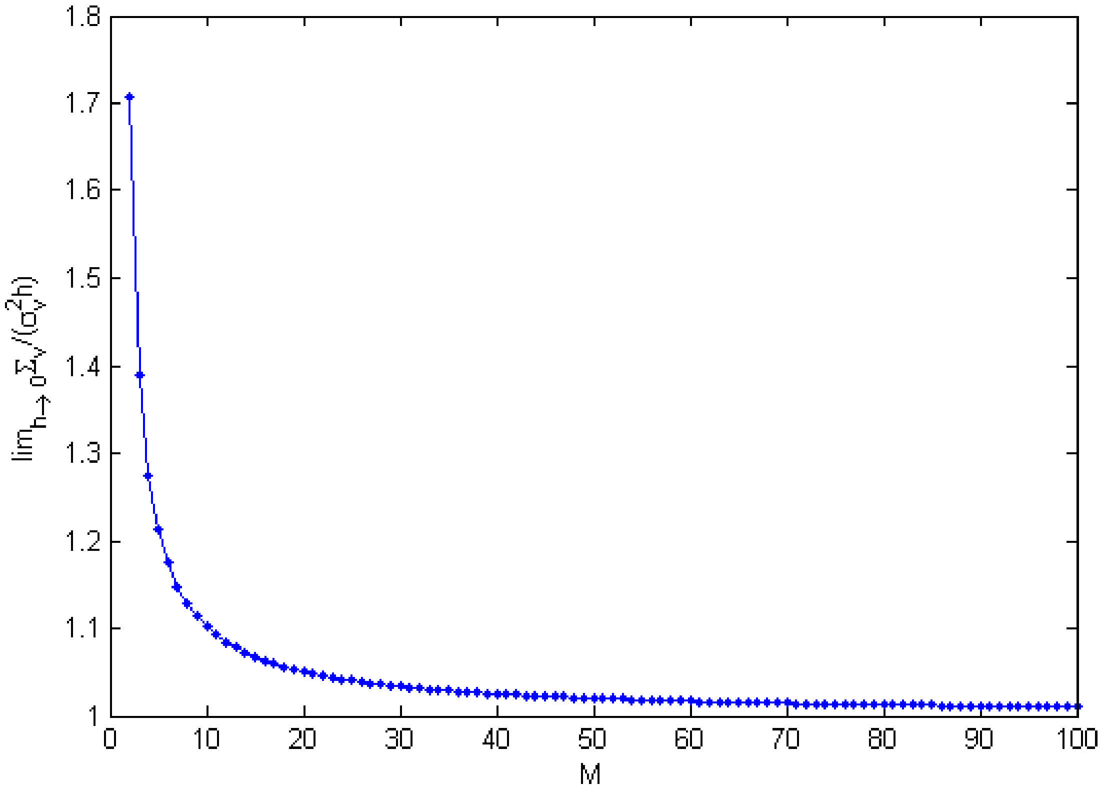

Figure 1.2: $\lim_{h\to 0} \frac{\Sigma_v}{\sigma_v^2 h}$ as a function of the number of informed traders M.

Trading Intensity β and Expected Quantity of Informed Trading

I have shown that, with increasing competition, informed traders reveal more information through their trading. Intuitively, one should expect that the aggregate trading intensity should be higher, and the fraction of trading volume contributed by informed traders should be higher when M increases. I demonstrate numerically how each trader's trading intensity β and aggregate trading volume generated by the informed traders per period $E(|X_n|)$ vary with h, respectively, for the cases when $M = 1$, 2 ,3 and 10 in Figure 1.3.

As can be seen from the figure, when we compare monopoly $M = 1$ with duopoly $M = 2$, the increased competition between the two traders induces each duopolist to choose a higher

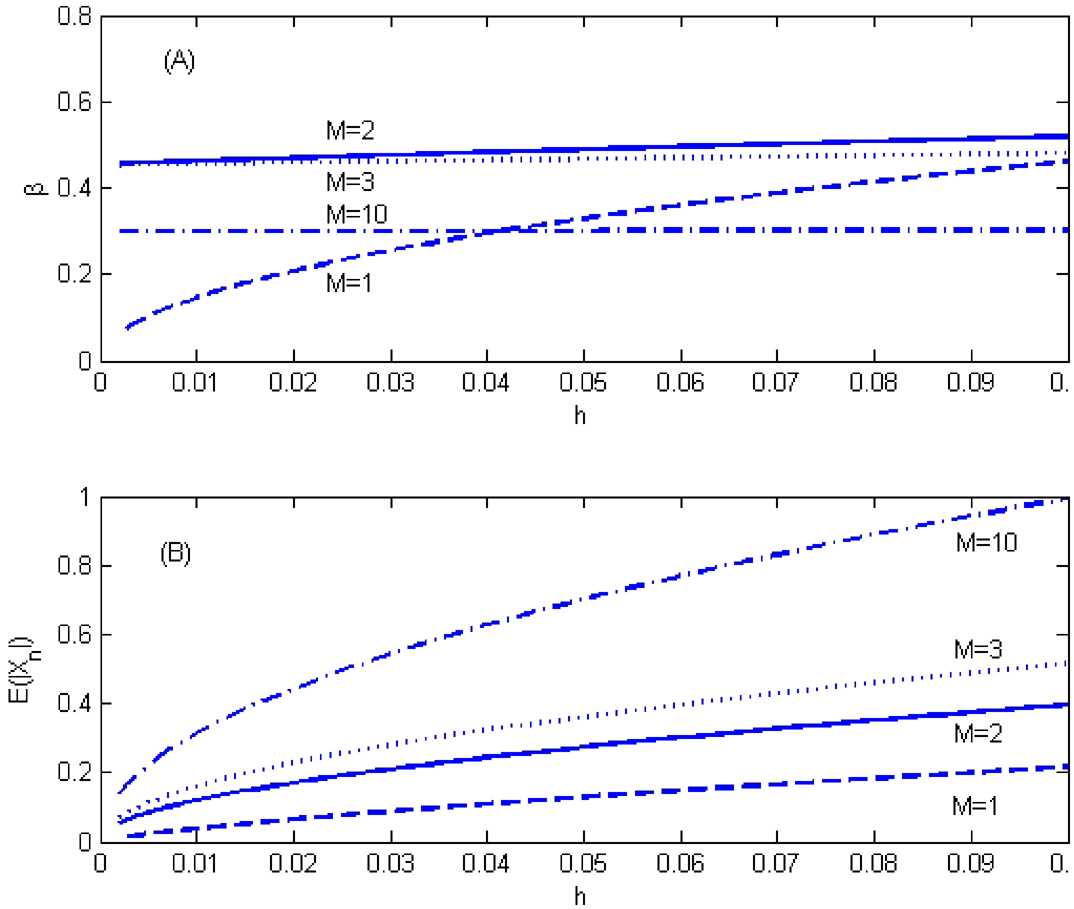

Figure 1.3: (A) The trading intensity parameter β as a function of h for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. (B) The aggregate expected trading volume per period $E(X_n|)$ as a function of h for $M = 1$, $M = 2$, $M = 3$ and $M = 10$. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

trading intensity β as shown in Figure 1.3(A) when h is small. But the result reverses sign as we continue adding more informed traders when h is small. For $M \geq 2$, the more the informed traders, the less trading intensity from each individual trader. Each trader's optimal strategy is to exploit less the investment opportunity determined by the difference between the true valuation of the asset and the price set by the market maker. Consequently, each trader may actually trades less intensely as more traders become informed. In aggregate, competition does make the traders behave more aggressively since the aggregate trading intensity $M\beta$ increases as competition becomes more intensive. In other words, when h is small, for $M \geq 2$, the value of β is decreasing in M while the value of $M\beta$ is increasing in M. Figure 1.4 illustrates how the asymptotic trading intensity $\lim_{h\to 0}\beta$ and aggregate trading intensity $M\lim_{h\to 0}\beta$ vary with the number of informed traders. Although each individual trader trades less intensely, the aggregate trading intensity monotonically increases with M.

As shown in Figure 1.3(B), the expected aggregate quantity of informed trading per period $E(|X_n|)$ monotonically increases with the number of traders M. Therefore, although each individual trader tends to submit lower demand when the market becomes more competitive, the aggregate trading volume which contains private information increases, conveying more information to the market maker.

The asymptotic properties of β and $E(|X_n|)$ can be inferred from Figure 1.3. In the monopolist case, β is of order $\sqrt{h}$ and $E(|X_n|) = \beta\sqrt{\Sigma_v}$ is of order $h^{\frac{3}{4}}$. In the imperfectly competitive case, β converge to a positive constant and $E(|X_n|) \propto h^{\frac{1}{2}}$. Intuitively, as trading becomes more frequent, there is less liquidity trading at each period to provide camouflage.

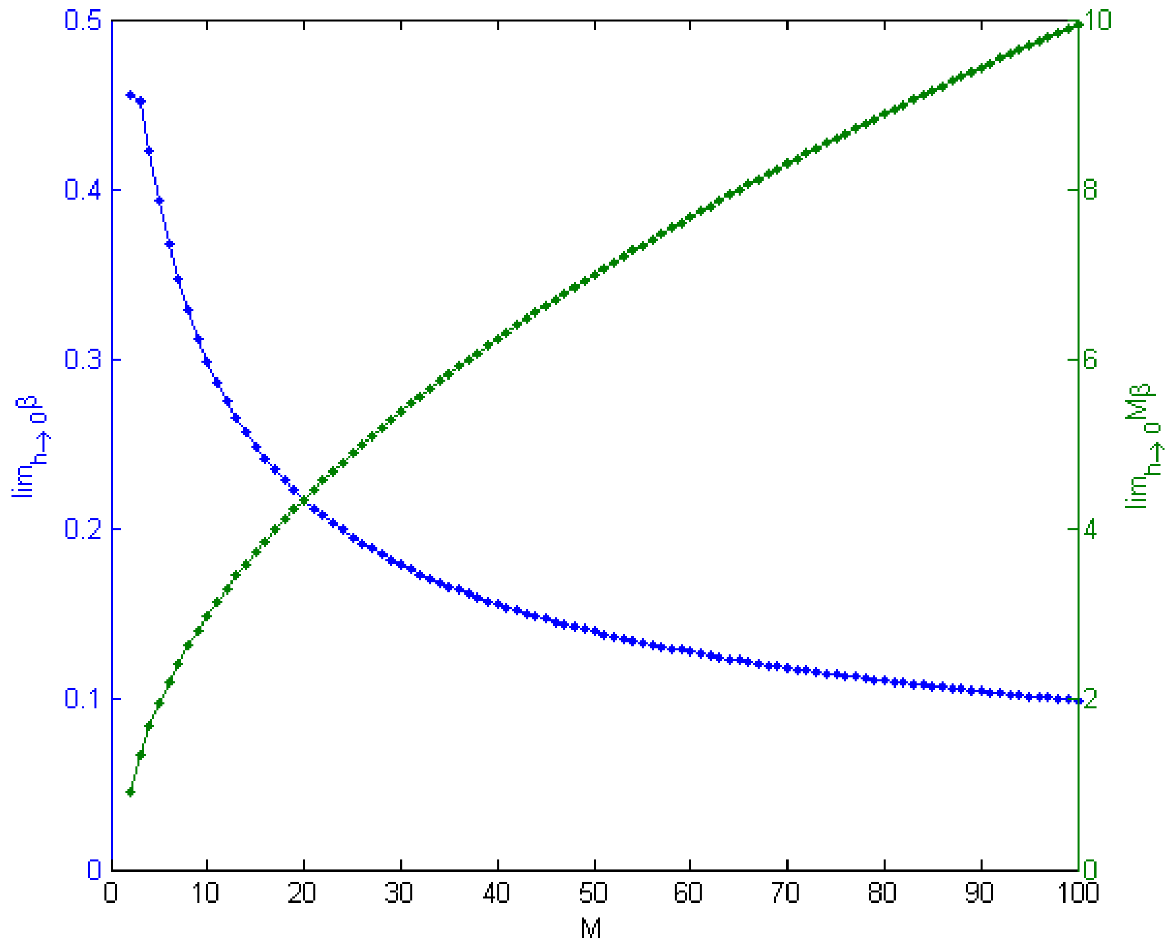

Figure 1.4: $\lim_{h\to 0}\beta$ and $\lim_{h\to 0} M\beta$ as functions of the number of informed traders.

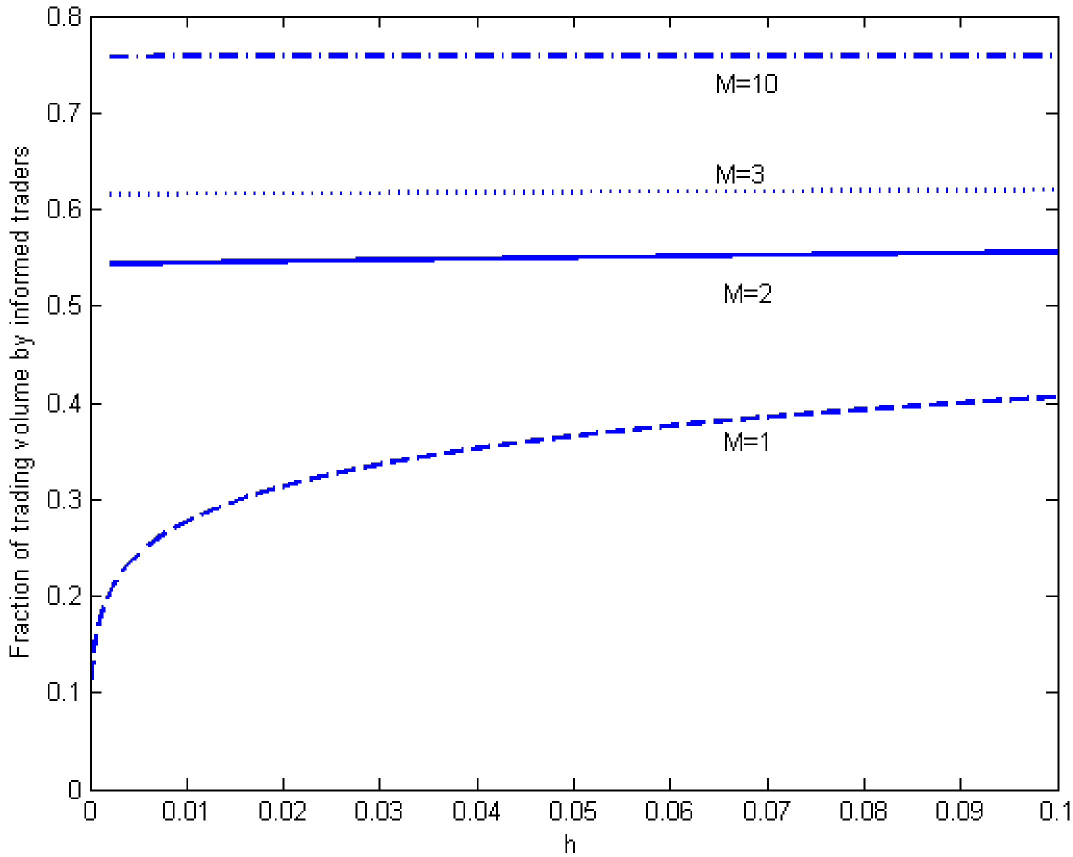

Figure 1.5: Fraction of trading volume of informed traders as a function of h for $M = 1, 2, 3$ and 10. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$..

Therefore, the informed traders trade less intensely and scale back the trading volume at each period. It can also be noted that in the monopolist case, the insider generates a negligible fraction of total trading volume, whereas in the imperfectly competitive case, the total aggregate volume submitted by the informed traders is comparable to the volume by the liquidity traders. The results are illustrated in Figure 1.5. When $M = 1$, the ratio quickly converges to zero when h is small. But the ratios converge to positive constants when $M > 1$ and become 1 when $M \to +\infty$.

Price variance

Since the aggregate trading volume is comparable to the trading volume of the liquidity traders, following similar argument and Proposition 1.2.3, the informed traders' contribution to the total price variance is also non-negligible. Figure 1.6 illustrates how the total price variance (blue lines) and its contribution by the informed traders (red lines) varies with the time interval between rounds of trading and the number of informed traders.

In the monopolist case, although total price variance increases when h is small, the contribution by the monopolistic trader converges to zero. The liquidity trader therefore contributes almost all of the price volatility near continuous trading. In the imperfectly competitive case, not only do informed traders contribute significantly to the total price variance near continuous trading, but the ratio increases as the number of informed traders increases. My numerical calculation make it reasonable to believe that the ratio converges to 1 when $M \to \infty$.

Market Liquidity and Profitability

Stationarity requires that the price impact λ is a time independent constant. Figure 1.7 illustrates the effect of competition on λ. If we fix h and increase the number of informed traders, λ declines accordingly. This is because in a steady state, more competition decreases information asymmetry between informed traders and market maker, there is less adverse selection. When the number of informed traders is fixed and trading frequency increases, numerical calculations show that λ increases and converges to a positive constant. Two opposite effects occur as trading becomes more frequent. On the one hand, there is less

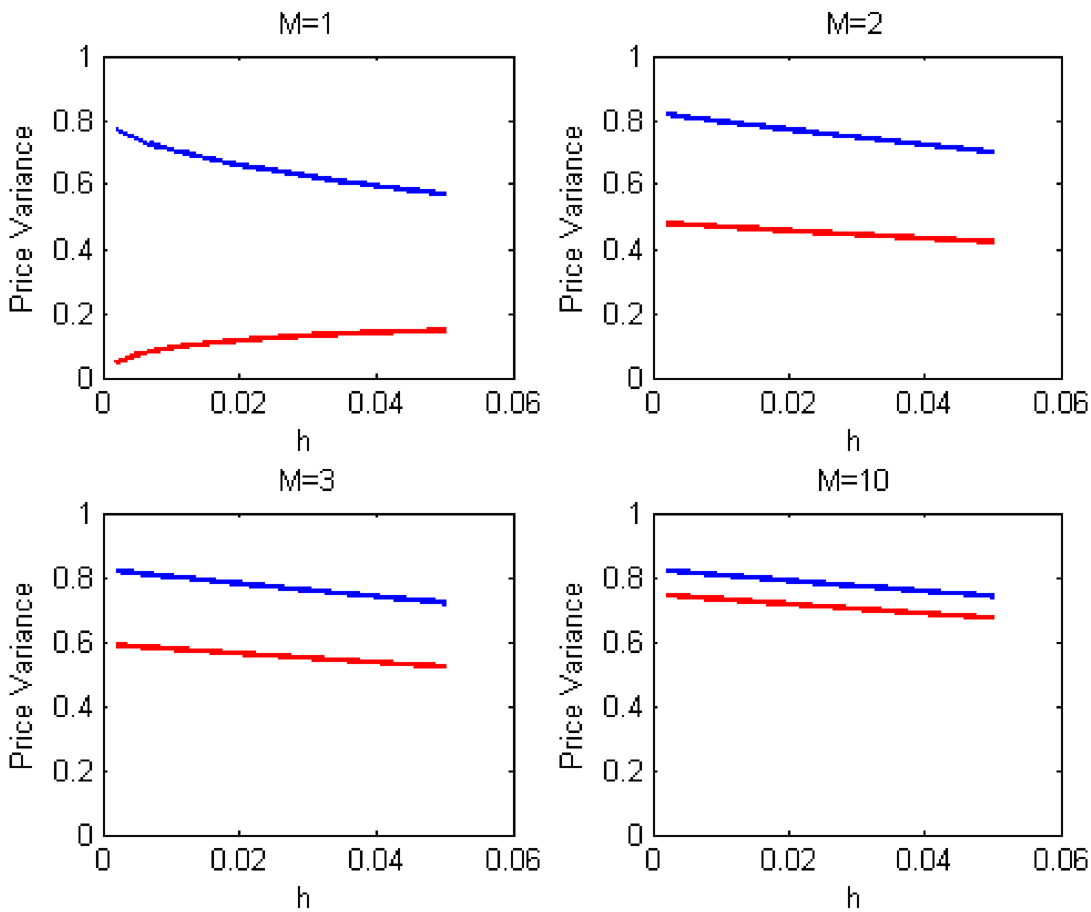

Figure 1.6: Total price variance (blue lines) and the contribution by the informed traders (red lines) as a function of h for $M = 1, 2, 3$ and 10. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$..

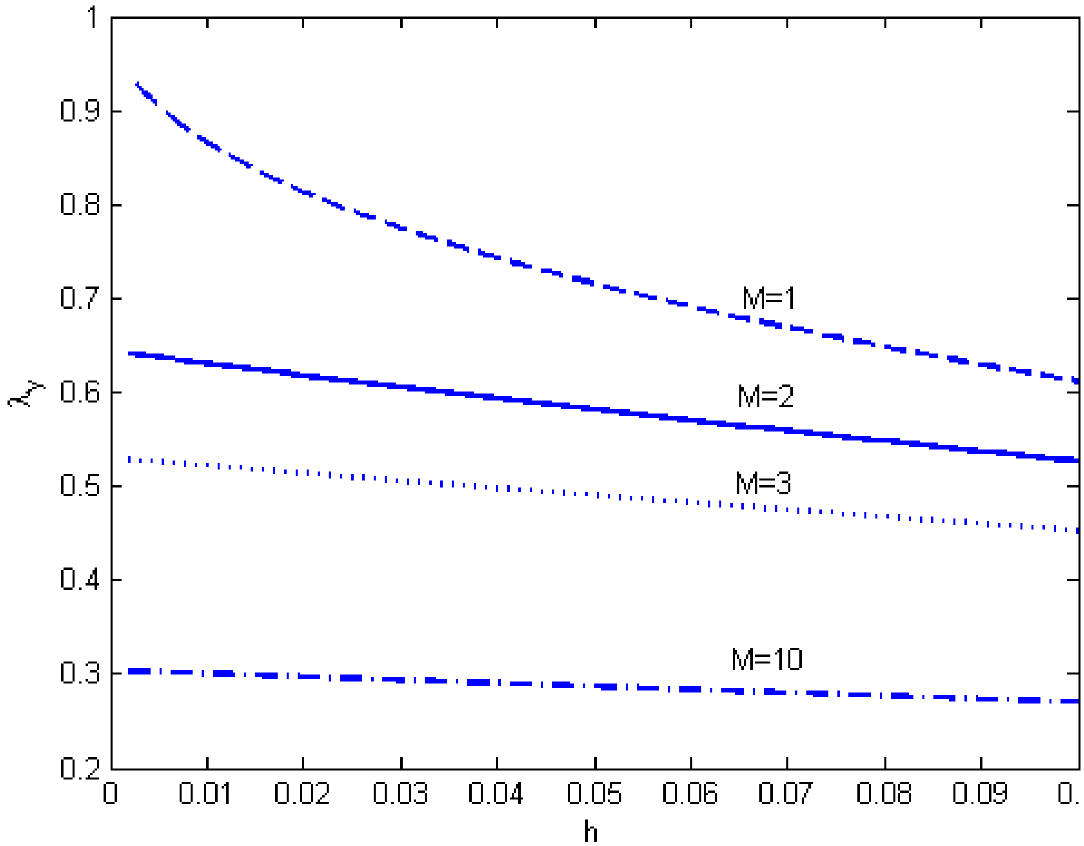

Figure 1.7: The market liquidity parameter λ as a function of h for $M = 1, 2, 3$ and 10. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

liquidity trading over each period and the adverse selection problem is more severe. On the other hand, as market becomes more efficient, there is less information asymmetry between the informed traders and the market maker, and the adverse selection problem is less severe. It can be shown from the figure that the first effect dominates with the adverse selection parameter λ a decreasing function of h.

The fact that λ remains strictly positive in the continuous trading limit ensures that informed traders make strictly positive expected profits. This is because informed traders benefit from the liquidity traders. Their profits are higher when price impact is higher and when liquidity traders are able to move the price further away from the efficient value giving informed traders opportunities to trade. To give a more rigorous explanation, remember that profit margin per share is Σ_v which is of the order $\sqrt{h}$ when h is small. The demand submit in period n in absolute term by each informed trader is proportional to $\sqrt{h}$. Then at each period, each informed trader earns a small expected profits in the order of h. The present value of the aggregate profits at any period n is proportional to $\sum_{k=n}^{\infty} e^{-r(k-n)h}h$, which is finite when $h \to 0$. Hence, our model predicts that each informed trader can still earn positive expected profits in the continuous trading limit. But imperfect competition does make each informed trader worse off. To demonstrate the effect of competition on expected profits, we plot in Figure 1.8 the aggregate expected profits of informed traders of informed traders as a function of h for different M. When $M \geq 2$, the aggregate profits monotonically decrease with M for fixed h and converge to positive constants as $h \to 0$.

Figure 1.9 illustrates how the asymptotic price impact $\lim_{h \to 0} \lambda$ varies with the number of informed traders. $\lim_{h \to 0} \lambda$ monotonically decreases with M. When the number of traders is large enough, the market can be infinitely deep with $\lim_{h \to 0} \lambda$ very close to zero. Therefore, the aggregate profits of informed traders also converges to zero as $M \to \infty$.

Empirical Implications

Hendershott and Riordan (2011b) find that high frequency traders' marketable orders have information advantage. Those traders trade in the direction of permanent price changes and in the opposite direction of transitory pricing errors. Therefore, it is reasonable to believe

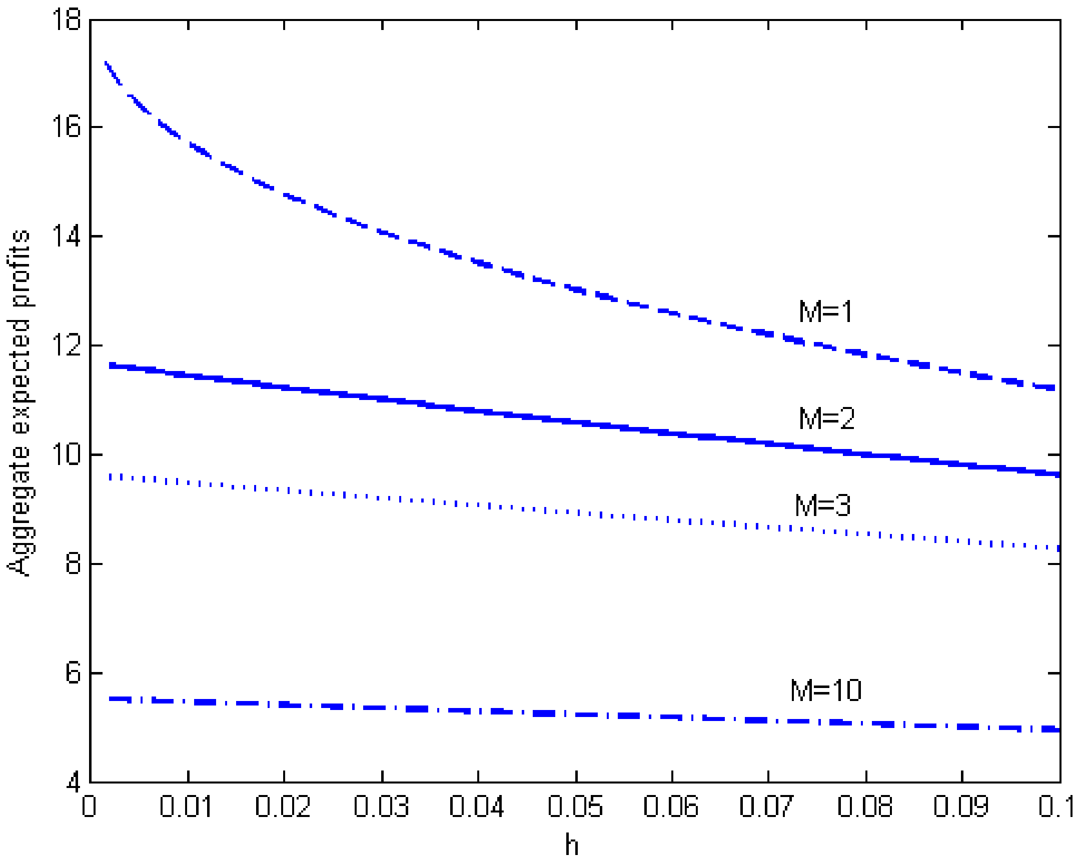

Figure 1.8: Aggregate expected profits as a function of h for $M = 1, 2, 3$ and 10. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

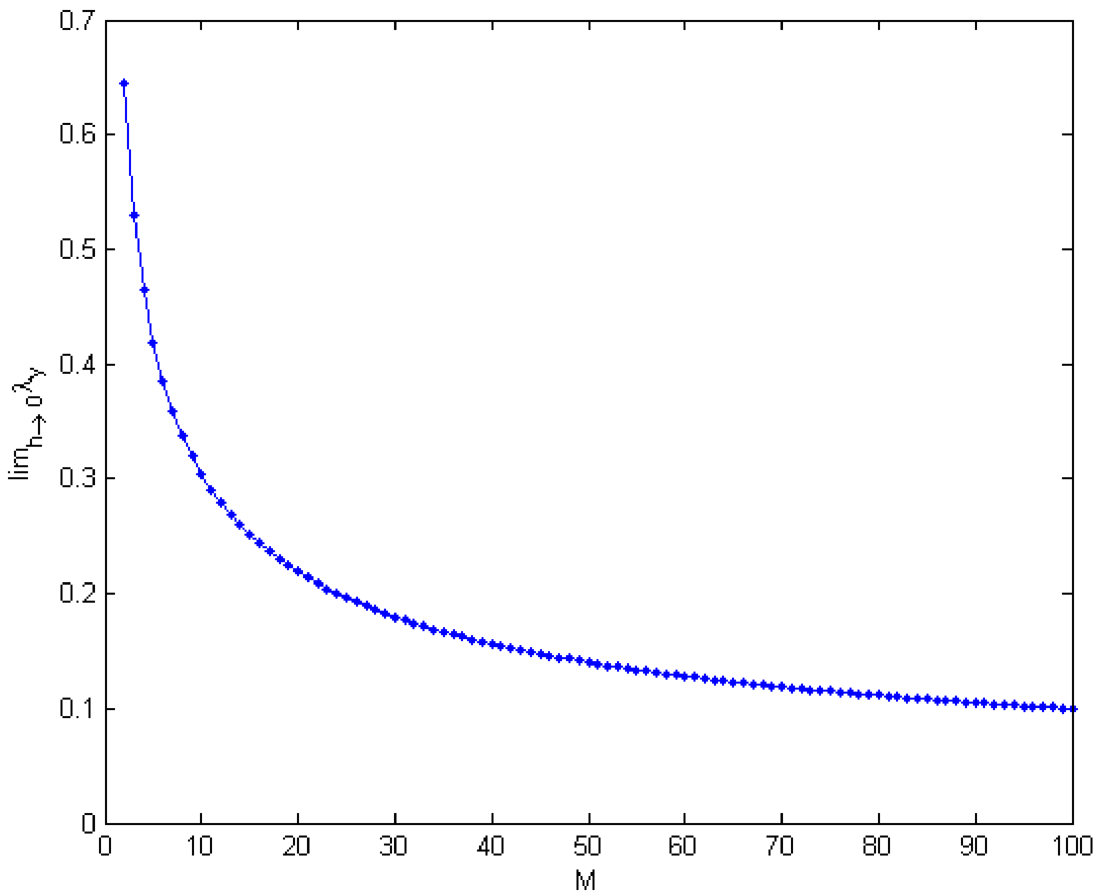

Figure 1.9: $\lim_{h\to 0}\lambda$ as a function of the number of informed traders.

that the high frequency traders who demand liquidity are "informed" traders. They may not possess corporate inside information but are better and quicker than the average market participants in gathering and processing information from massive market wide data to generate private signals. The informed traders in my model can be treated as high frequency traders when they are able to trade their information quickly (i.e, the frequency of trading is very high). The unique results from my model in terms of efficiency, liquidity, trading volume, price volatility, and expected profits can help us better understand the effects of high frequency traders on financial markets than previous literature on strategic trading.

Hendershott et al. (2011), Hendershott and Riordan (2011a), Hendershott and Riordan (2011b) and Brogaard (2010) provide evidences showing that the existence of high frequency traders are beneficial to the market. Those traders provide more efficient quotes and the growth of high frequency traders accompanies improvements in market liquidity, no matter whether the traders are liquidity providers or liquidity demanders. In addition, high frequency traders contribute significantly to trading volume. For example, Hendershott and Riordan (2011b) estimate that high frequency traders initiating trades are responsible for roughly 43% of trading volume in large stocks from a unique dataset from Nasdaq. High frequency traders are found to implement highly correlated strategies but remain to be highly profitable despite aggressive competition with one another.

At first glance, Chau and Vayanos (2008), Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) seem to be good candidates to be used to understand high frequency trading. All three models can predict that high frequency trading make price more efficient, although traders in those models are "impatient" to reveal their superior information very quickly for different reasons. However, all three models cannot explain trading volume. The fraction of trading volume contributed by informed traders is essentially zero in continuous trading limit. Chau and Vayanos (2008) shows that market is thinner when trading is more frequent, and Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) show that market can be infinitely illiquid when trading starts. Their predictions cannot perfectly explain the empirical fact that market liquidity increases with more participation of high frequency traders. In addition, Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993) predict that profits of informed traders quickly converge to zero when there is imperfect competition. Zero expected profits are inconsistent with the profitability of the imperfectly competitive high frequency traders. Therefore, both models cannot explain why high frequency traders can survive in the long run in an imperfect competitive environment.

My model combines the two important features: (i) imperfect competition and (ii) continuous arrival of new private information. As the number of high frequency traders increases due to ease of entry: (i) information is incorporated into prices more quickly and therefore speeds up price discovery; (ii) high frequency traders participate in an even larger fraction of total trading volume and price volatility, (iii) high frequency traders add to market liquidity by competing aggressively with one another, and (iv) high frequency traders should make profits which are bounded away from zero (like Cournot and not like Bertrand competition) unless the number of high-frequency traders is very large.

1.5 Conclusion

In this paper, I analyze how imperfect competition among informed traders affects market efficiency, liquidity, trading volume and the profitability of informed traders. The combined trading of multiple informed traders is more aggressive than the monopolistic trader, the equilibrium price is even more revealing of the informed traders' private information, and market depth improves as the number of informed traders increases. In the continuous trading limit, the variance of private information held by informed traders goes to zero at a rate proportional to the time interval between rounds of trading. This is much faster than the corresponding strong from efficiency result in the Chau and Vayanos model, where the convergence rate is proportional to the square root of the time interval. In addition, in the limit as the time interval between rounds of trading goes to zero, the aggregate profits of the informed traders remain bounded away from zero and they contribute significantly to the total trading volume and price volatility.

If high frequency traders are "informed" in a sense that they are able to generate profitable private signals consistently by processing information from order flows and price movements of securities across market, then this model provides a reasonable characterization of those traders. My results suggest that the entry of more high frequency traders improves market efficiency by incorporating information more quickly into price, improves market liquidity by lowering price impact, and increases the fraction of trading volume from high frequency traders. But those traders remain profitable despite exploiting the same information set and implementing similar algorithms.

Future research can extend the results of this paper in two directions. First, explain why high frequency traders quickly reverse their inventories, we may add risk aversion. Second, the assumption that traders are identically informed is too strong. The assumption does not allow the more realistic scenario in which the informed traders learn from each other. Li (2012) extends this paper by introducing a hierarchical information structure in which there is one strictly better informed trader and one less informed trader. Kyle et al. (2012) further relax the assumption in this paper even further to allow for a more general correlation among streams of private information in which each trader has to forecast the forecasts of other traders.

Chapter 2

Speculative Dynamics II: Asymmetric Traders

2.1 Introduction

In a financial market, there are a few large institutional traders who actively spend resources generating information and competing aggressively with one another. These informed traders are found to follow positively correlated strategies. Li (2011) provides a theoretical framework that characterize the equilibrium behavior of those traders. If their information sets and strategies are perfectly correlated, adding more informed traders can speed up price discovery and improve market depth. They contribute significantly to trading volume and price variance. In the seminal paper of Kyle (1985) and many extensions, the informed trader's contributions to the total trading volume and price variance are negligible to the contributions by the liquidity trader in continuous trading. This is because that the informed trader's strategy is continuous (of order dt) and liquidity trading is a Brownian motion (of order $\sqrt{dt}$). and manage to earn a profit strictly bounded away from zero even when the frequency trading is very high near continuous trading. This is in sharp contrast to the strategic trading models with one-shot private information. In the one shot private information setup of Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993), multiple informed traders' profits vanish due to imperfect competition in continuous trading.

In this paper, I relax the assumption in Li (2011) of identically informed traders. This assumption ignores the possibility that informed traders produce different information. If their signals are imperfectly correlated, some traders have information that others don't possess. The inference problem becomes more complicated relative to the scenarios of one monopolistic informed trader or multiple identically informed traders. On the one hand, informed traders who do not know all the private information will infer any additional information they do not have from past order flows and price movements, like uninformed market makers. On the other hand, informed traders will treat their available information differently: they trade cautiously on any exclusive information they may possess to exploit their monopoly power, and they trade aggressively on information they share.

By allowing the signals observed by the informed traders to differ, especially when new information arrives repeatedly, makes the problem very difficult to solve analytically. One has to overcome the infinite regress problem arising in a dynamic model when traders try to infer each other's information. In this paper, we bypass the problem by focusing on a hierarchical duopolistic setting. To be more specific, we consider an economy with a risky asset which is going to be liquidated at a random date. The liquidation value is not fixed, but rather evolves in a stochastic way and is only perfectly observed by an informed trader. The other trader, however, observes a stream of noisy signals of the liquidation value. These noisy signals are also observed by the first trader. Therefore, the second trader is less informed and the first trader is strictly better informed. Both traders submit marketable orders to the market maker along with liquidity traders at each trading period. The risk neutral market maker cannot see each individual trader's order but rather observes the batched order. She then sets the price equal to the expected value of the asset by solving an inference problem, and takes the other side of the trade. The history of order imbalances provides information not only to the market maker, but to the less informed trader as well. Like the market maker, the less informed trader imperfectly infers the more informed trader's additional information from the history of order flows by subtracting her own contribution. In the meanwhile, she has to compete with the more informed trader concerning any private information they both observe.

There is another problem which is associated with the existence of linear equilibrium in continuous trading when traders have perfectly correlated signals. Holden and Subrahmanyam (1992) and Back et al. (2000) find that linear equilibrium does not exist when informed traders are identically informed in the limit of continuous trading. This is because risk neutral informed traders try to preempt each other causing the private information to be revealed instantaneously, with trading intensity and price impact becoming infinite large at the start of the trades (at $t = 0^+$). Through the remaining of the trading sessions ($t > 0$), there is no information asymmetry between the informed traders and therefore market is infinitely deep. It is reasonable to believe that in a hierarchical private information setting of Foster and Viswanathan (1994) where both asymmetric informed traders share a common noisy signal about the value of a risky asset to be liquidated at a predetermined date, there is no linear equilibrium when trading is continuous[1]. In an economy where an risky asset's liquidation value is stochastic and private information arrives repeatedly, Li (2011)

[1]What should happen when trading is continuous is that both traders would trade on the common private information so aggressively such that the common private information incorporated into price almost instantaneously when trading starts and the more informed trader is able to monopolize the additional exclusive information through the remaining trading session like the informed speculator in Kyle (1985).

has found that stationary linear equilibrium does exist in the limit of continuous trading with identically informed traders.

In this paper, we study the equilibrium in an economy with duopolistic traders endowed with hierarchical information sets and a risk neutral market maker. Our model differs from Foster and Viswanathan (1994) in several aspects. In Foster and Viswanathan's model, all the private information is known at time zero, the liquidation value of the asset is fixed before trading starts, and there is a predetermined date when the asset is liquidated. In this paper we have more realistic assumptions: both traders receive continuous flows of new information, the fundamental value of the risky asset is stochastic, and the time for the risky asset to be liquidated is random, following a geometric distribution. In their paper, Foster and Viswanathan characterize the equilibria numerically and focus on time series properties of equilibrium in an economy with only four trading periods. Because of the above argument on the existence of equilibria in continuous trading, their paper does not address the economic properties of equilibrium when trading periods are shortened to continuous limit. In this paper, by making different assumptions and focusing on stationary equilibrium, we are able to derive the equilibria in a closed form, able to derive economic propertis like to market efficiency, market liquidity, and profitability.

By analyzing the properties of the equilibrium, the overall private information held by both traders can be decomposed into two orthogonal components: a "common" component of the private information known to both traders and "monopolistic" component known only to the more informed trader. Consistent with the results found in Chau and Vayanos

(2008) and Li (2011), the variances of both pieces of information converge to zero in the continuous trading limit. However, the rates of how the two components of information are incorporated into prices are quite different. When h is small (denote h to be the time interval between trades), the variance of common component of private information not incorporated into prices is proportional to h, while the variance of the monopolistic component of private information not incorporated into prices is proportional to the square root of h. This suggests that the common private information is incorporated more efficiently than the additional monopolistic information.

The degree of competition is measured by the precision of the information stream that is shared by both traders. Increasing the precision ρ can have two effects on market depth. On the one hand, there will be more common private information and less monopolistic information at each period and therefore more information will be incorporated into prices conditional on the total private information. Since the market orders carry more information, market maker should set a higher price impact. On the other hand, over time price should become more efficient, there should be less information asymmetry among the market participants, and the market should be more liquid. It can be shown in the stationary state, the second effect dominates, thus the more precise the shared private information, the more liquid the market becomes. Following the same logic, the inference parameter which measures the sensitivity of belief to order flow for the less informed trader monotonically also decreases with ρ.

In almost all dynamics models with price impact, the contribution to the trading volume or price variance from informed trading is negligible. This is because in the continuous time limit as the interval between trades h goes to zero, the value of informed trading is of order h, whereas the value of liquidity or noise trading strategy is a Brownian motion of order $\sqrt{h}$. Li (2011) shows that with imperfect competition and continuous arrival of new private information, the informed traders' strategy can be a Brownian motion in the limit as $h \to 0$. Therefore, informed traders can contribute significantly to the total trading volume and add a volatility component to the price formation process. In this model, since imperfectly competitive informed traders share a stream of signals, their strategies on the common private information also become Brownian motions as $h \to 0$ and therefore the contributions to the total trading volume and price variance by the two traders should be non-negligible in the limit.

A somewhat surprising result is that, contrary to conventional wisdom, when the frequency of trading is sufficiently high, the less informed trader may contribute a more significant fraction to total trading volume and price variance than the more informed trader. A trader may rationally trade more because her information is "worse". The intuition for why this happens is that the less informed trader trades more aggressively on the common private information, and the difference in trading on the common information may dominate the trading volume in the additional private information by the more informed trader. The more informed trader's strategy on her additional private information is of order $h^{\frac{3}{4}}$ which is negligible compared to $\sqrt{h}$ when h is small enough.

Although not formally stated in Foster and Viswanathan (1994), by following the same spirit of Holden and Subrahmanyam (1992) and Foster and Viswanathan (1993), when the time interval between trades goes to zero, the informed traders compete so aggressively based on shared information that this information is incorporated into prices almost instantaneously and the profits of the less-informed trader go to zero. This paper generates an opposite result in terms of the profitability of the less informed trader. When the interval between rounds of trading is small, both the profits of the more informed and less informed traders remain bounded away from zero, with the profits of the more informed trader much greater than the profits of the less informed when the signals are not highly correlated.

The plan of the paper is as follows. In Section II, I describe the model. The equilibrium and its asymptotic properties are presented in Section III. Section IV shows some comparative statics and empirical implications. Section V concludes.

2.2 The Model

2.2.1 Economy

Assumption 1: Securities

Trading takes place from $t = 0$ to $t = +\infty$ at the discrete points t_n ($t_n = nh$) through time where h is the time interval between the auctions. There are a riskless bond with zero interest rate and a non-dividend paying risky asset with a stochastic liquidation value v_n. At the end of each period, there is a probability $1 - \exp(-rh)$ that the risky asset is liquidated. I further assume the riskless bond is in perfect elastic supply. The liquidation value v_n follows a mean-reverting process or random walk:

$$v_n - v_{n-1} = \kappa(\bar{v} - v_{n-1})h + \epsilon_{v,n}. \tag{2.1}$$

In the above specification, κ determines the adjustment speed of the liquidation value v_n to its long run fixed target $\bar{v}$; κ is assumed to be greater than or equal to zero implying that the prices are stationary[2]; and the innovation $\epsilon_{v,n}$ is independently and normally distributed with mean zero and variance $\sigma_v^2 h$.

Assumption 2: Market Participants and Information Structure

The risk neutral market participants consist of a competitive market maker, two informed strategic traders, and a number of liquidity traders. The two strategic traders, however, are asymmetrically informed with a hierarchical information structure such that one trader is less informed and the other is better informed. Therefore, their information sets are hierarchical. At each period, the less informed trader observes a noisy signal s_n about $\epsilon_{v,n}$ (the innovation term in equation (2.1)) in the form

$$s_n = \rho\epsilon_{v,n} + \sqrt{1-\rho^2}e_n \tag{2.2}$$

where $0 \leq \rho \leq 1$ and $e_n \sim N(0, \sigma_v^2 h)$ is normally distributed and uncorrelated with other shocks in the economy. The better informed trader observes v_n perfectly but also observes the noisy signal s_n each trading period. The variable ρ measures the level of precision of the noisy signal received by both traders. When $\rho = 0$, the less informed trader is completely uninformed. When $\rho = 1$, both traders are identically and perfectly informed.

[2] If $\kappa = 0$, then v_n follows a random walk.

$I_n^i = \{p_\tau, v_\tau, s_\tau | \tau \le n\}$ is the more informed trader's information set at $t = nh$, $I_n^l = \{p_\tau, s_\tau | \tau \le n\}$ is the less informed trader's information set, and $I_n^m = \{p_\tau | \tau \le n\}$ is the market maker's information set. Clearly, we have $I_n^i \supseteqq I_n^l \supseteqq I_n^m$ and the information sets are nested.

Assumption 3: Timing of Events

At the beginning of the nth period, the informed traders and the liquidity traders submit their demands before new information arrives. Let z_n denote the less informed trader's market order and x_n denote the more informed trader's market order. After the market orders are submitted, both informed traders observe s_n but only the more informed perfectly observes v_n. The market maker observes the total order flow $y_n = x_n + z_n + u_n$, then sets the price p_n equal to the expected value of the asset based on the history of orders flows, and then clears the market.

Pricing

Since the market maker is assumed to be competitive and risk neutral, at period n she sets the price p_n equal to the expected value of the asset after she receives the total batched market order $y_n = x_n + z_n + u_n$. Therefore,

$$p_n = E[\sum_{n'=n}^{+\infty} (1 - \exp(-rh)) \exp(-r(n' - n)h) v_{n'} | I_n^m], \tag{2.3}$$

where $(1 - \exp(-rh)) \exp(-r(n' - n)h)$ is the probability that the asset is liquidated at the end of the n'th period.

Lemma 2.2.1: The price p_n is the following linear function of the market maker's expectation of the current liquidation value of the risky asset $E(v_n|I_n^m)$:

$$p_n = \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}E(v_n|I_n^m) + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}. \tag{2.4}$$

Proof: See Appendix B.

Although the market maker observes the total order flow, she cannot distinguish the contributions made by the more informed trader, less informed trader and the liquidity trader. The less informed trader, however, by observing the price set by the market maker, can infer the sum of the orders by the more informed trader and liquidity trader, $x_n + u_n$, therefore enabling her to maker a sharper inference about v_n than the market maker. Similar to the pricing function of Lemma 2.2.1, the less informed trader's valuation of the risky asset at the nth period equals to the expected value conditional on her information which is denoted by l_n:

$$\begin{aligned} l_n &= E[\sum_{n'=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)v_{n'}|I_n^l] \qquad (2.5) \\ &= \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}E(v_n|I_n^l) + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}. \end{aligned}$$

Optimization

Since informed traders are risk neutral, then at the nth period, the better informed trader tries to maximize her expected trading profits:

$$\max_{x_{n'\geq n}} E[\sum_{\tau=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)(\sum_{\tau=n}^{n'}x_\tau(v_{n'}-p_\tau))|I_{n-1}^i]. \tag{2.6}$$

Lemma 2.2.2: The better informed trader's objective function can be written as:

$$\max_{x_{n'\geq n}} E[\sum_{n'=n}^{+\infty} \exp(-r(n'-n)h)x_{n'}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n'}+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n'})|I^i_{n-1}]. \tag{2.7}$$

Proof: See Appendix B.

Following a similar argument, the objective function of the less informed trader can be expressed as:

$$\begin{aligned}\max_{z_{n'\geq n}} \quad & E[\sum_{n'=n}^{+\infty} \exp(-r(n'-n)h)z_{n'}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n'} \\ + \quad & \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n'})|I^l_{n-1}].\end{aligned} \tag{2.8}$$

2.2.2 Equilibrium Strategies and Value Functions

Suppose that conditional on information up to period $n-1$, the market maker believes that v_{n-1} is normally distributed with mean $\hat{v}_{n-1}$ and variance $\Sigma_{v,n-1}$, while the less informed trader believes the mean to be v^*_{n-1} and variance to be $\Lambda_{v,n-1}$. Then, $\Sigma_{v,n-1}$ measures the total variance of private information not incorporated into prices at the $(n-1)$th period. The value of $\Lambda_{v,n-1}$ measures the variance of private information withheld exclusively by the better informed trader, and therefore $\Omega_{v,n-1} = \Sigma_{v,n-1} - \Lambda_{v,n-1}$ measures the variance of private information withheld by both informed traders.

I hypothesize that the more informed insider submits demand

$$x_n = \beta(v_{n-1} - v^*_{n-1}) + \gamma(v^*_{n-1} - \hat{v}_{n-1}) \tag{2.9}$$

to maximize the present value of her expected profits scaled by $\frac{(1-\exp(-rh))}{(1-\exp(-rh)(1-\kappa h))}$ which is quadratic with respect to $v_{n-1} - v^*_{n-1}$ and $v^*_{n-1} - \hat{v}_{n-1}$:

$$V^i(v_{n-1}-v^*_{n-1}, v^*_{n-1}-\hat{v}_{n-1}) = A(v_{n-1}-v^*_{n-1})^2+B(v^*_{n-1}-\hat{v}_{n-1})^2+C(v_{n-1}-v^*_{n-1})(v^*_{n-1}-\hat{v}_{n-1})+E. \tag{2.10}$$

The strategy and value function not only depend on v_{n-1} but also depend on v^*_{n-1} since the more informed trader understands that her action will be closely followed by the less informed trader as well as the market maker. The less informed insider submits demand

$$z_n = \theta(v^*_{n-1} - \hat{v}_{n-1}) \tag{2.11}$$

to maximize the present value of her expected profits scaled by $\frac{(1-\exp(-rh))}{(1-\exp(-rh)(1-\kappa h))}$, which is quadratic with respect to $v^*_{n-1} - \hat{v}_{n-1}$:

$$V^l(v^*_{n-1} - \hat{v}_{n-1}) = F(v^*_{n-1} - \hat{v}_{n-1})^2 + G. \tag{2.12}$$

Note that the order submission strategy of the more informed trader (Equation 2.9) can be decomposed into two parts. The first term is proportional to the difference between the perfect signal she receives and the information she shares with the other trader with intensity

β. The second term is proportional to the difference between the information she shares and the estimation of v_{n-1} by the market maker with intensity γ. The less informed trader, bases her trade on the difference between the information she shares and market maker's estimation on v_{n-1} with intensity θ.

2.3 Inference Problems, Optimizations and The Equilibrium

2.3.1 The Market Maker's Inference Problem

The market maker uses the total order flow y_n, together with her prior belief on v_{n-1} to form the posterior belief about v_{n-1}:

$$\begin{aligned} v_{n-1} &= E(v_{n-1}|I^m_{n-1}) + \frac{\lambda}{1-\kappa h}(x_n + z_n + u_n) + \eta_n \\ &= \hat{v}_{n-1} + \frac{\lambda}{1-\kappa h}(x_n + z_n + u_n) + \eta_n. \end{aligned} \tag{2.13}$$

She believes that v_{n-1} is normally distributed with mean $E(v_{n-1}|I^m_n)$ in the form

$$E(v_{n-1}|I^m_n) = \hat{v}_{n-1} + \frac{\lambda}{1-\kappa h} y_n \tag{2.14}$$

and variance $var(\eta_n)$. Then according to the stochastic process of v_n (equation (2.1)), she believes that $v_n \sim N(\hat{v}_n, var((1-\kappa h)\eta_n + \epsilon_{v,n}))$ is normally distributed with conditional mean

$$E(v_n|I^m_n) = \hat{v}_n = (1-\kappa h)\hat{v}_{n-1} + \lambda y_n + \kappa h \bar{v} \tag{2.15}$$

and conditional variance

$$var(v_n|I_n^m) = (1-\kappa h)^2 var(\eta_n) + \sigma_v^2 h. \tag{2.16}$$

In the next theorem, we derive the explicit form of the inference parameter λ and set up nonlinear equation which can be solved for Σ_v.

Lemma 2.3.1: Given the trading strategies of the informed traders defined in equations (2.9) and (2.11), the market maker's inference parameter λ is given by

$$\lambda = \frac{(1-\kappa h)(\beta\Lambda_v + (\gamma+\theta)\Omega_v)}{\beta^2\Lambda_v + (\gamma+\theta)^2\Omega_v + \sigma_u^2 h}. \tag{2.17}$$

The uncertainty of the market maker's belief on the risky asset's liquidation value Σ_v satisfies the following nonlinear equation:

$$\Sigma_v = (1-\kappa h)^2(\Sigma_v - \frac{(\beta\Lambda_v + (\gamma+\theta)\Omega_v)^2}{\beta^2\Lambda_v + (\gamma+\theta)^2\Omega_v + \sigma_u^2 h}) + \sigma_v^2 h. \tag{2.18}$$

Proof: See Appendix B.

2.3.2 The Less Informed Trader's Inference Problem

In this subsection, we solve the inference problem of the less informed trader. At period n, the less informed insider can infer the realization of $x_n + u_n$ since the trader needs only to subtract her contribution z_n from the total order flow y_n. Because $x_n + u_n$ contains private information which is exclusive to the more informed trader about v_{n-1}, the less informed trader can update her belief about v_{n-1} after observing p_n:

$$\begin{aligned} v_{n-1} &= v_{n-1}^{*} + \frac{\phi}{1-\kappa h}(x_n + u_n - \gamma(v_{n-1}^{*} - \hat{v}_{n-1})) + \varepsilon_n \\ &= v_{n-1}^{*} + \frac{\phi}{1-\kappa h}(x_n + u_n - \gamma(v_{n-1}^{*} - \hat{v}_{n-1})) + \varepsilon_n \end{aligned} \tag{2.19}$$

with the conditional mean

$$E(v_{n-1}|I_n^l) = v_{n-1}^{*} + \frac{\phi}{1-\kappa h}(x_n + u_n - \gamma(v_{n-1}^{*} - \hat{v}_{n-1})) \tag{2.20}$$

and conditional variance $var(\varepsilon_n)$.

When forming the posterior belief about v_n, the trader needs to take into account of the signal s_n. Hence, the conditional mean and conditional variance of v_n are given by

$$v_n^{*} = E(v_n|I_n^l) = (1-\kappa h)v_{n-1}^{*} + \kappa h\bar{v} + \phi(x_n + u_n - \gamma(v_{n-1}^{*} - \hat{v}_{n-1})) + \rho s_n. \tag{2.21}$$

and

$$var(v_n|I_n^l) = (1-\kappa h)^2 var(\varepsilon_n) + (1-\rho^2)\sigma_v^2 h \tag{2.22}$$

respectively.

In the following theorem, we derive the explicit form of the less informed trader's inference parameter ϕ and setup the nonlinear equation for Λ_v.

Lemma 2.3.2: Given the more informed trader's strategy defined in equation (2.9) and the price process determined by the market maker in equation (2.15), the inference parameters of the less informed trader ϕ is given by

$$\frac{\phi}{1-\kappa h}=\frac{\beta\Lambda_v}{\beta^2\Lambda_v+\sigma_u^2 h}. \tag{2.23}$$

The trader's uncertainty about the liquidation value of the risky asset Λ_v satisfies the following nonlinear equation:

$$\Lambda=(1-\kappa h)^2\frac{\Lambda_v^2\sigma_u^2 h}{\beta^2\Lambda_v+\sigma_u^2 h}+(1-\rho^2)\sigma_v^2 h. \tag{2.24}$$

Proof: see Appendix B.

2.3.3 The More Informed Trader's Optimization Problem

At period n, the more informed trader submits demand x_n to maximize the the expected profits scaled by $\frac{1-\exp(-rh)}{1-\exp(-rh)(1-\kappa h)}$

$$E[\sum_{n'=n}^{+\infty}x_{n'}(v_{n'}-\hat{v}_{n'})e^{-r(n'-n)h}|I_{n-1}^i]. \tag{2.25}$$

We have conjectured that the trader's value function $V_1(v_{n-1},v_{n-1}^*,\hat{v}_{n-1})$ in period n, evaluated at after submitting the market order x_n, is a quadratic function of $v_{n-1}-v_{n-1}^*$ and $v_{n-1}^*-\hat{v}_{n-1}$, as shown in equation (2.10). In equilibrium, $V^i(v_{n-1},v_{n-1}^*,\hat{v}_{n-1})$ must satisfy the following Bellman equation

$$V^i(v_{n-1}, v^*_{n-1}, \hat{v}_{n-1}) = max_{x_n} E[(v_n - \hat{v}_n)x_n + e^{-rh}V^i(v_n, v^*_n, \hat{v}_n)|I^i_{n-1}]. \tag{2.26}$$

Lemma 2.3.3: Given the price process determined by the market maker and the strategy of the less informed trader, the trading strategy of the more informed trader is characterized by:

$$\beta = (2\tau)^{-1}[1 - \kappa h - 2e^{-rh}A\phi(1 - \kappa h) + e^{-rh}C(\phi - \lambda)(1 - \kappa h)] \tag{2.27}$$

and

$$\begin{aligned} \gamma &= (2\tau)^{-1}[1 - \kappa h - \lambda\theta - 2e^{-rh}A\phi\gamma \\ &+ 2e^{-rh}B(\phi - \lambda)(1 - \kappa h - \phi\gamma - \lambda\theta) \\ &+ e^{-rh}C(\phi - \lambda)\phi\gamma - e^{-rh}C\phi(1 - \kappa h - \phi_x\gamma - \lambda\theta)] \end{aligned} \tag{2.28}$$

where

$$\tau = \lambda - e^{-rh}(A\phi^2 + B(\phi - \lambda)^2 - C\phi(\phi - \lambda)) \tag{2.29}$$

The Bellman equation (2.26) has the solution of quadratic form $A(v_{n-1} - v^*_{n-1})^2 + B(v^*_{n-1} - \hat{v}_{n-1})^2 + C(v_{n-1} - v^*_{n-1})(v^*_{n-1} - \hat{v}_{n-1}) + E$, with A, B, C and E characterized by the following set of nonlinear equations:

$$\begin{aligned} A &= \beta(1-\kappa h-\lambda\beta)+e^{-rh}A(1-\kappa h-\phi\beta)^2+(\phi-\lambda)\beta]^2 \\ &+ e^{-rh}C(1-\kappa h-\phi\beta)(\phi-\lambda)\beta, \end{aligned} \tag{2.30}$$

$$B=\gamma(1-\kappa h-\lambda(\gamma+\theta))+e^{-rh}B[1-\kappa h-\lambda\gamma-\lambda\theta]^2, \tag{2.31}$$

$$\begin{aligned} C &= \beta(1-\kappa h-\lambda(\gamma+\theta))+\gamma(1-\kappa h-\lambda\beta) \\ &+ 2e^{-rh}B(\phi-\lambda)\beta[1-\kappa h-\lambda\gamma-\lambda\theta] \\ &+ e^{-rh}C(1-\kappa h-\phi\beta)[1-\kappa h-\lambda\gamma-\lambda\theta], \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} E &= e^{-rh}[A(\phi^2\sigma_u^2h+(1-\rho^2)\sigma_v^2h) \\ &+ B((\phi-\lambda)^2\sigma_u^2h+\rho^2\sigma_v^2h) \\ &- C(\phi(\phi-\lambda)\sigma_u^2h)+E]. \end{aligned} \tag{2.33}$$

Proof: See Appendix B.

2.3.4 The Less Informed Trader's Optimization Problem

Having solved the dynamic programming of the more informed trader, we then focus on the less informed trader's optimization problem. Conditional on her most recent information set, she submits market order z_n at period n to maximize the present value of expected profits scaled by $\frac{1-\exp(-rh)}{1-(1-\exp(-rh))(1-\kappa h)}$,

$$E[\sum_{n'=n}^{+\infty} z_{n'}(v_{n'} - \hat{v}_{n'})e^{-r(n'-n)h}|I_{n-1}^l]. \tag{2.34}$$

The value function of the less informed trader is assumed to be quadratic in $v_{n-1}^* - \hat{v}_{n-1}$ and solves the following Bellman equation in equilibrium:

$$V^l(v_{n-1}^*, \hat{v}_{n-1}) = max_{z_n} E[z_n(v_n - \hat{v}_n) + e^{-rh}V^l(v_n^* - \hat{v}_n)|I_{n-1}^l]. \tag{2.35}$$

The following theorem characterizes the trading strategy of the less informed trader and the solution for the Bellman equation.

Lemma 2.3.4: Given the trading strategy of the more informed trader and the pricing rule determined by the market maker, the optimal strategy of the less informed trader is characterized by

$$\theta = \frac{(1 - 2e^{-rh}\lambda F)(1 - \kappa h - \lambda\gamma)}{2\lambda(1 - e^{-rh}\lambda F)}. \tag{2.36}$$

The Bellman equation (2.35) has the solution in a quadratic form $F(v_{n-1}^* - \hat{v}_{g-1})^2 + G$, with

F and G characterized by the following set of nonlinear equations:

$$F = \theta(1 - \kappa h - \lambda\gamma - \lambda\theta) + e^{-rh}F(1 - \kappa h - \lambda\gamma - \lambda\theta)^2 \tag{2.37}$$

and

$$\begin{aligned} G &= e^{-rh}F[(\phi-\lambda)^2\beta^2\Lambda_v + (\phi-\lambda)^2\sigma_u^2 h \\ &+ \rho^2\sigma_v^2 h] + e^{-rh}G. \end{aligned} \tag{2.38}$$

Proof: See Appendix B.

2.3.5 Equilibrium and Trading in the Continuous Time Limit

Having solved the inference problems for the market maker and the less informed trader and the optimization problems for both informed traders, in the next theorem we establish the necessary and sufficient conditions for the stationary equilibrium.

Proposition 1: The linear stationary equilibrium of the model with hierarchical information structure is characterized by 13 parameters (λ, ϕ, Σ_v, Λ_v, β, γ, θ, A, B, C, E, F and G) which are solutions to the system of equations (2.17 - 2.18), (2.23 - 2.24), (2.27 - 2.34) and (2.36 - 2.39).

We are mostly concerned with the asymptotic properties of the equilibria in the limit of continuous trading, i.e., $h \to 0$. When there is only one monopolistic informed insider, Chau and Vayanos prove that the variance of information not incorporated into prices at each trading period is proportional to $\sqrt{h}$ and the trading intensity is of order $\sqrt{h}$. By contrast, in an economy with multiple identical informed traders, the variance of private information left at each period is proportional to h and the trading intensity converges to a strictly positive constant. In this paper, since the total private information can be decomposed into two orthogonal components, we should expect that the variance of common private information not incorporated into prices Ω_v at each period is proportional to h and the variance of additional private information Λ_v left at each period converges to zero at rate of order of $\sqrt{h}$. The intensity at which the more informed trader trades, based on her additional private information, β, should converge to zero at the order $\sqrt{h}$. The intensities at which both traders trade, based on their common information, γ and θ, should converge to positive constants. The next theorem establishes the asymptotic behavior of the equilibrium.

Proposition 2: When the time interval between rounds of trading converges to zero, we have

$$\lim_{h \to 0} \lambda = \lambda_0 - a\sqrt{h} \tag{2.39}$$

$$\lim_{h \to 0} \phi = \phi_0 - a\sqrt{h} \tag{2.40}$$

$$\lim_{h \to 0} \frac{\beta}{\sqrt{h}} = b \tag{2.41}$$

$$\lim_{h \to 0} \gamma = \gamma_0 \tag{2.42}$$

$$\lim_{h \to 0} \theta = \theta_0 \tag{2.43}$$

$$\lim_{h \to 0} \Sigma_v = Oh + L\sqrt{h} \tag{2.44}$$

$$\lim_{h \to 0} \Lambda_v = L\sqrt{h} \tag{2.45}$$

and

$$\lim_{h\to 0} A = A_0 \quad \lim_{h\to 0} B = B_0 \quad \lim_{h\to 0} C = C_0 \tag{2.46}$$
$$\lim_{h\to 0} E = E_0 \quad \lim_{h\to 0} F = F_0 \quad \lim_{h\to 0} G = G_0,$$

where λ_0, ϕ_0, a, b, γ_0, θ_0, L, O, A_0, B_0, C_0, E_0, F_0 and G_0 satisfy the following set of nonlinear equations:

$$\lambda_0 = \frac{(\gamma_0 + \theta_0)O + bL}{(\gamma_0 + \theta_0)^2 O + \sigma_u^2}, \tag{2.47}$$

$$\phi_0 = \frac{bL}{\sigma_u^2}, \tag{2.48}$$

$$b - \frac{A_0 a}{\lambda_0 - \phi_0 + A_0\phi_0^2 - B_0(\phi_0 - \lambda_0)^2}, \tag{2.49}$$

$$\gamma_0 = \frac{1 - \lambda\theta_0 - 2A_0\phi\gamma_0 + 2B(\phi - \lambda)(1 - \phi\gamma_0 - \lambda\theta) + C(\phi - \lambda)\phi\gamma_0 - C\phi(1 - \phi\gamma_0 - \lambda\theta_0)}{2\tau_0}, \tag{2.50}$$

$$\theta_0 = \frac{(1 - 2\lambda F_0)(1 - \lambda\gamma_0)}{2\lambda(1 - \lambda F)}, \tag{2.51}$$

$$F = \theta_0(1 - \lambda(\gamma_0 + \theta_0)) + F_0(1 - \lambda(\gamma_0 + \theta_0))^2, \tag{2.52}$$

$$B_0 = \gamma_0(1 - \lambda(\gamma_0 + \theta_0)) + B_0(1 - \lambda(\gamma + \theta))^2, \tag{2.53}$$

$$1 - 2A_0\phi_0 - C_0(\phi_0 - \lambda_0) = 0, \tag{2.54}$$

$$-\lambda b^2 - (r + 2\kappa)A_0 + 2A_0 ab + A\phi^2 b^2 - \phi b^2 C_0(\phi - \lambda) = 0, \tag{2.55}$$

$$E = \frac{A(\phi^2\sigma_u^2 + (1-\rho^2)\sigma_v^2) + B((\phi-\lambda)^2\sigma_u^2 + \rho^2\sigma_v^2) - C(\phi(\phi-\lambda)\sigma_u^2)}{r}, \tag{2.56}$$

$$G = \frac{F(\phi-\lambda)^2\sigma_u^2 + \rho^2\sigma_v^2}{r} \tag{2.57}$$

2.4 Properties of Equilibriums and Comparative Statics

In what follows, we illustrate how information structure, the strategic interaction between the two informed traders, and increasing trading frequency affect market efficiency, liquidity, trading strategies, trading volume, and expected profits of the informed traders.

The variable ρ measures the precision of the noisy signal. Intuitively, it also measures the degree of information asymmetry and hence the degree of imperfect competition between the two informed traders. The higher the value of ρ, the more precise the stream of noisy signals observed by both traders, the more information shared by both traders and less private information exclusively held by the more informed trader. To illustrate the effect of imperfect competition and the strategic interaction between the two traders, I compare the results of when $0 \leq \rho \leq 1$ with two benchmark cases. When $\rho = 1$, both traders are equally informed and there is no information asymmetry between them; when $\rho = 0$, the less informed trader is completely uninformed and the more informed trader is a monopolist on the perfect signal she receives.

Market Efficiency

Since the information sets are hierarchical, I can decompose the overall private information held by the informed traders $v_n - \hat{v}_n$ into two orthogonal components: the private information known to both traders $v_n^* - \hat{v}_n$ and the information exclusive to the more informed trader $v_n - v_n^*$. It is natural to expect that both traders act like duopolists on the common component of the information, and the more informed trader is able to monopolize the information exclusive to herself.

Following the intuitions, we should expect that Ω_v, which measures the market maker's uncertainty about the first component, is of order h when h is small and Λ_v, which measures the variance about the second component, is of order $\sqrt{h}$[3]. Therefore, the shared private information should be incorporated into prices in a much higher rate than the monopolistic private information. Because $\Sigma_v = \Omega_v + \Lambda_v$ and the $\sqrt{h}$ dominates h for small h, the variance of total private information not incorporated into price should be of order $\sqrt{h}$. Since $\sqrt{h} \to 0$ as $h \to 0$, the market approaches to strong-form efficiency in the continuous trading limit with no information asymmetry among the informed traders and market maker.

To illustrate the properties of market participants beliefs about the liquidation value v_n, we plot the variance Σ_v, Λ_v and Ω_v against h respectively, for the cases when $\rho = 0.7$, $\rho = 1$ and $\rho = 0$ in Figure 2.1. When $\rho = 0.7$, Σ_v is always greater than Λ_v implying that the less informed trader is making a sharper inference about the liquidation value of the asset than the market maker. The difference between Σ_v and Λ_v is actually Ω_v which measures the variance of $v_n^* - \hat{v}_n$. As h becomes smaller, Σ_v and Λ_v also decrease converging to zero and so does their difference, Ω_v.

We then compare the results when $\rho = 0.7$ to the benchmark cases when $\rho = 1$ and $\rho = 0$. The stream of "noisy" signal received by the less informed trader becomes perfect

[3]In the monopolist case, Chau and Vayanos (2008) prove that the private information not incorporated into price is of order $\sqrt{h}$ when h is small. In the oligopolist case in Li (2011), the author proves that the convergence rate is proportional to h.

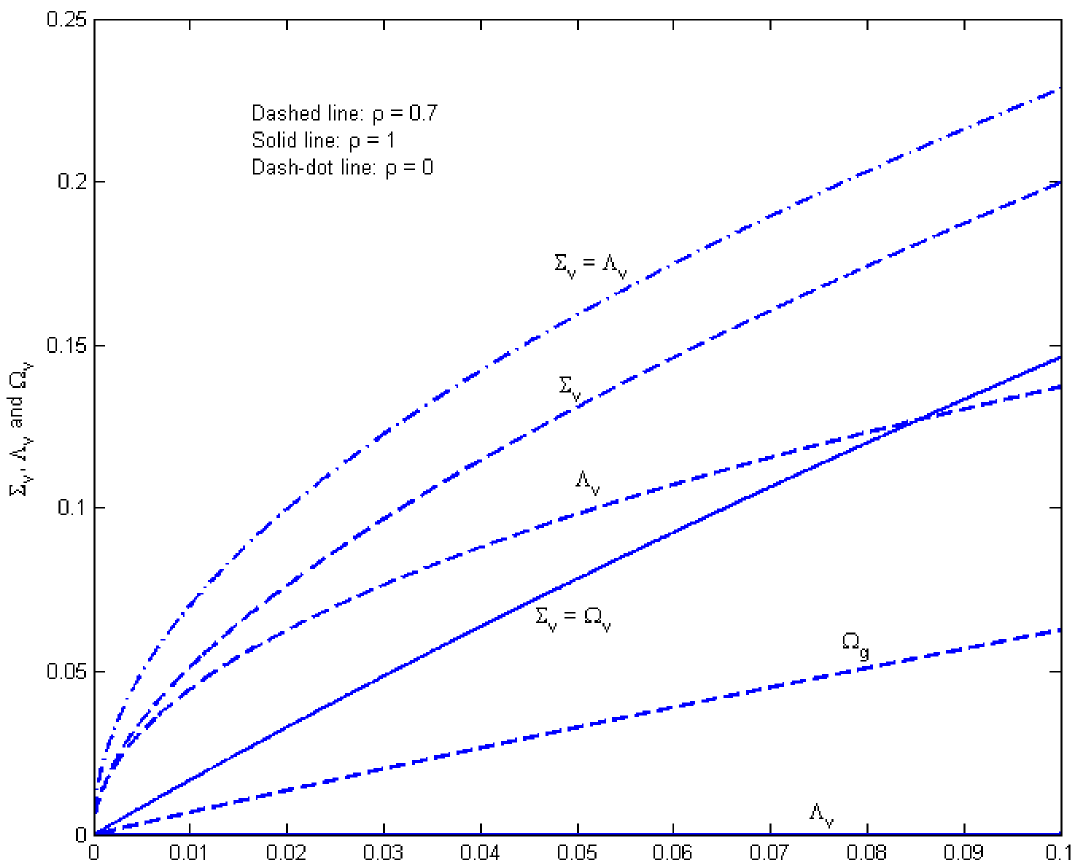

Figure 2.1: Σ_v, Λ_v and Ω_v as functions of h for the cases when $\rho = 0$, 0.7 and 1. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

and she is as informed as the other trader if $\rho = 1$. As illustrated in Figure 2.1, Λ_v is always zero and $\Omega_v = \Sigma_v$. If $\rho = 0$, the fact that the less informed trader is completely uninformed and she can make an inference no better than the market maker leads to $\Sigma_v = \Omega_v$.

Trading Intensity Parameters: β, γ and θ

We have demonstrated that the more informed trader monopolizes any additional private information. Remember that β is the intensity with which the more informed trader trades on this information, β should be of order $\sqrt{h}$ when h is small. γ and θ are the intensities with which the traders trade on the common private information. From the asymptotic properties derived in Proposition 2, we expect γ and θ converge to positive constants when h is closed to zero.

We first examine how the trading intensity parameters vary with the time interval between trades and the precision of the stream of noisy signals. In Figure 2.2, we plot β, γ and θ against h for the cases when $\rho = 0.3$, $\rho = 0.9$ and $\rho = 1$. As h becomes smaller, so do β, γ and θ because the variance of noise trading is getting smaller at each trading period. The value of β converges to zero except at $\rho = 1$ when the "more" informed trader has no additional private information. The values of γ and θ converge to strictly positive values. In addition, θ is always greater than γ except at $\rho = 1$ implying that the less informed trader always acts more aggressively on their common private information than the more informed trader.

We next study the asymptotic properties of β, γ and θ. We let $b = \lim_{h\to 0} \frac{\beta}{\sqrt{h}}$, $\gamma_0 = \lim_{h\to 0} \gamma$ and $\theta_0 = \lim_{h\to 0} \theta$. Figure 2.3 illustrates how b, γ_0 and θ_0 vary with ρ. The value of

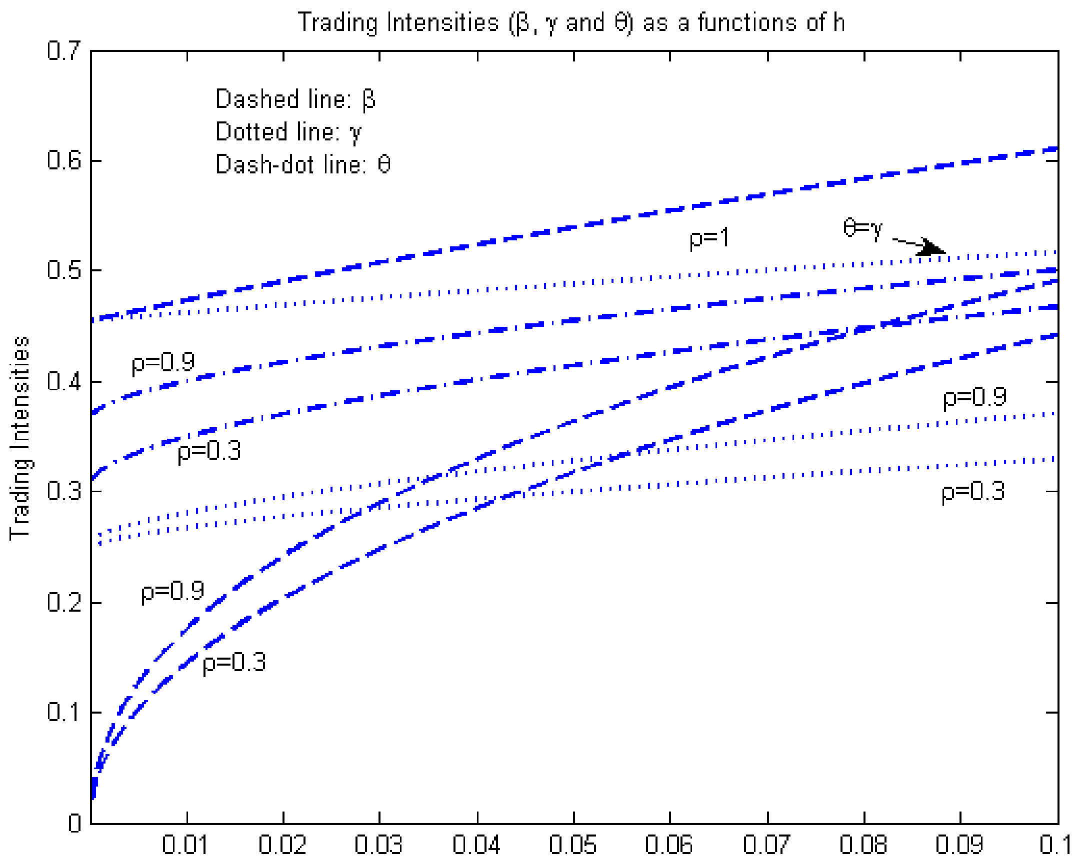

Figure 2.2: The trading intensity parameters (β, γ and θ) as functions of h for cases when $\rho = 0.3$, 0.9 and 1. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

b first increases with ρ slowly when ρ is small, but increases sharply with ρ when it is close to 1. Since $\lim_{h\to 0}\beta$ is found to be strictly positive at $\rho = 1$, we should expect $b \to +\infty$ as $\rho \to 1$. The result is consistent with the counterpart found in Chau and Vayanos (2008) in which b is infinity if there is no adverse selection between the monopolistic insider and the market maker.

Although the two traders act like competing duopolists, the asymptotic properties of trading intensities are quite different from the counterparts found in Li (2011). Li finds that the trading intensity can be sufficient large when the information asymmetry between the identically informed traders and the market maker is small enough in the limit of continuous trading, and that the trading intensity is inversely related to information asymmetry. One can tell from Figure 2.3(B) and (C) that γ_0 and θ_0 remain bounded at $\rho = 0$. γ_0 is found to be a non-monotonic function of ρ. γ_0 first monotonically decreases with ρ and later monotonically increases with ρ when ρ is close to 1. θ_0 is a monotonically increasing function for all values of ρ.

Inference Parameters: ϕ and λ

Li (2011) finds that more competition leads to a smaller price impact parameter λ and hence a more liquid market. The degree of competition is measured by the number of the traders in the market. In this paper, the number of traders is fixed to be two and the degree of competition is measured by the precision of the noisy signal both traders receive. We expect that the same conclusion still holds: as ρ increases, the market should become more liquid in the stationary state where the market maker is facing a less severe adverse selection.

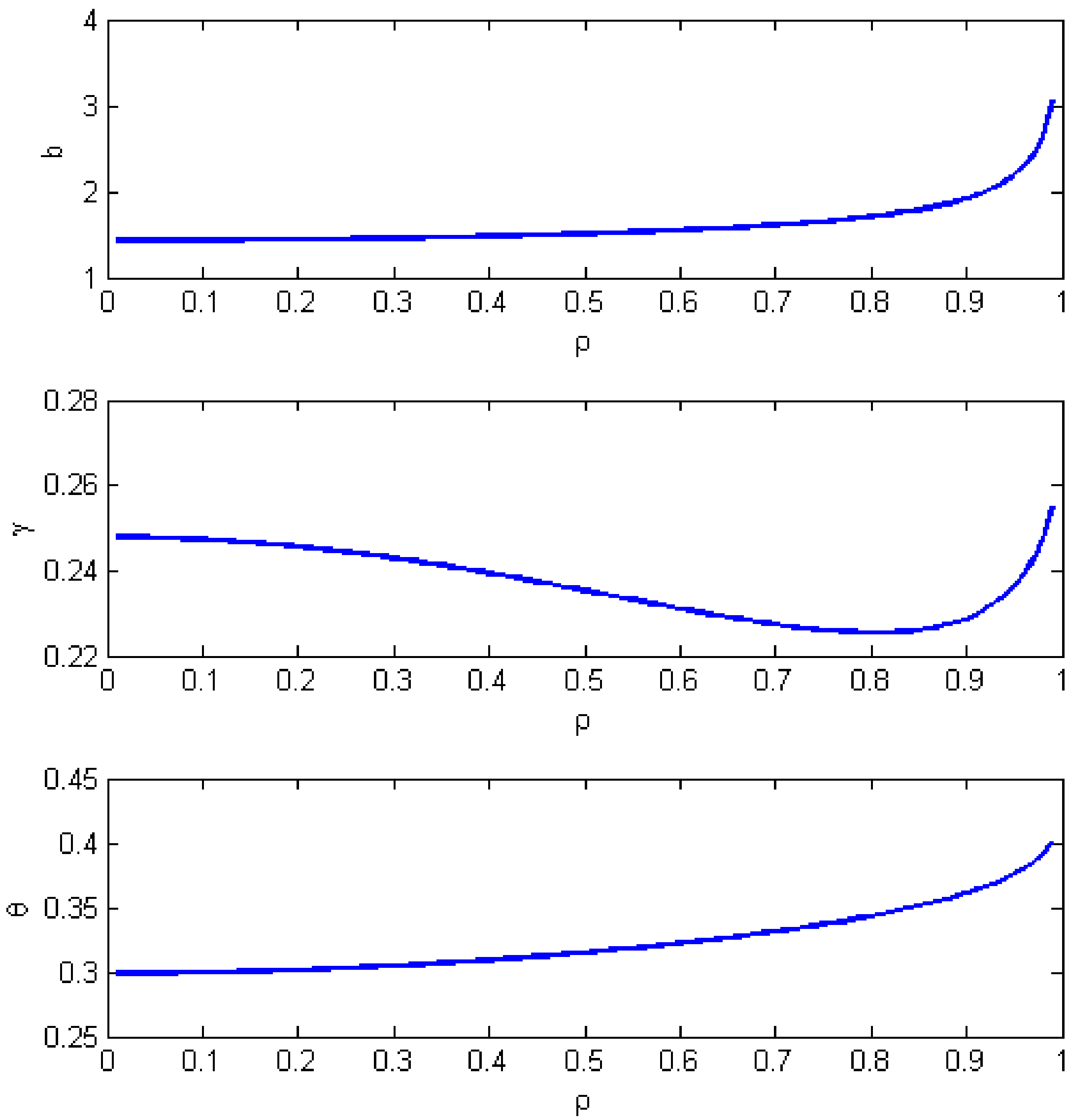

Figure 2.3: The asymptotic properties of trading intensity parameters (b, γ_0 and θ_0) as functions of h for cases when $\rho = 0.3$, 0.9 and 1. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

Figure 2.4(A) and (B) confirm the conjecture. For a fixed h, the price impact or the inference parameter for the market maker λ (inverse measure of market depth) monotonically decreases with ρ. The inference parameter for the less informed trader ϕ is also found to be negatively related to ρ since the trader becomes more informative about the value of the asset as the stream of signals she observes becomes more precise. When $\rho = 1$, ϕ strictly equals to zero since the "less" informed trader is perfectly informed and learns nothing from the history of order flows.

In Figure 2.4(C), we plot $\lambda_0 = \lim_{h \to 0} \lambda$ and $\phi_0 = \lim_{h \to 0} \phi$ against ρ. We finds that λ_0 is always greater than ϕ_0 except at $\rho = 0$. This is because the less informed trader is more informed than the market maker and learns less from the order flow.

Expected Quantity of Informed Trading

Intuition might suggest (incorrectly) that the better informed trader trade more than the less informed trader on average because she has more private information. However, since the less informed trader trades more intensely on the shared private information, the difference between how they trade the common information may dominate the trading volume by the more informed trader on her exclusive private information. We confirm the intuition in Figure 2.5. In Figure 2.5(A), we fix $\rho = 0.9$ and $\kappa = 1$. When h is relatively large, the contribution to the total trading volume by the more informed trader is greater than that by the less informed trader. The relation switches sign when h becomes smaller or the frequency of trading is higher. In Figure 2.5(B), we let $\rho = 0.6$ and $\kappa = 1$; in (C), we let $\rho = 0.6$ and $\kappa = 0$. We observe similar phenomena as in (A). But the frequency at which

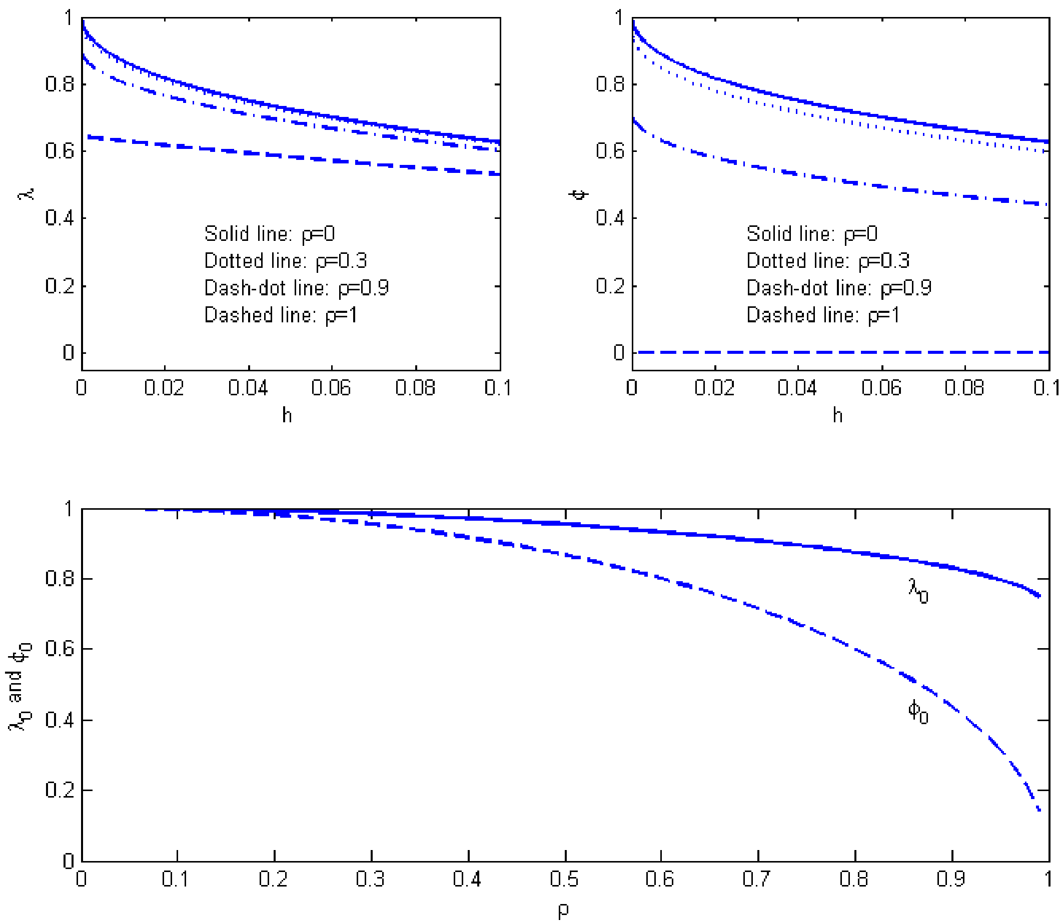

Figure 2.4: (A) and (B) The inference parameters ϕ, λ as functions of h for $\rho = 0$, 0.3, 0.9 and 1. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$. (C) λ_0 and ϕ_0 as functions of ρ.

the less informed trader starts to dominate the better informed trader in terms of trading volume becomes higher if we decrease ρ or κ.

Expected Profits

Although Foster and Viswanathan (1994) do not address the issue of each informed trader's expected profits in the limit of continuous trading, it is reasonable to believe that as the time interval between trades converges to zero, the informed traders compete so aggressively on shared private information that the information is reflected into prices instantaneously and the profits of the less informed trader go to zero. The better informed trader however, can earn strictly positive profits by trading the remaining monopolized information slowly. Li (2011) has shown that identically informed traders can earn strictly positive profits when trading is continuous if the information arrives repeatedly. It is reasonable to believe that in this model, despite both informed traders compete very aggressively on their shared stream of noisy signals, in the limit, the less informed trader can still earn a non-zero expected profits.

Figure 2.6 illustrates how each informed trader's expected profits vary with the time interval between trading and the precision of the flow of the noisy signals observed by both traders. We find that the less informed trader's expected profits increase when ρ becomes higher. Intuitively, the better the information received by the less informed trader, the more market power and hence the better investment opportunity she has. A more interesting result is that as h converges to zero, both the profits of the more and less informed traders remain strictly positive, with the profits of the more informed trader greater than the profits

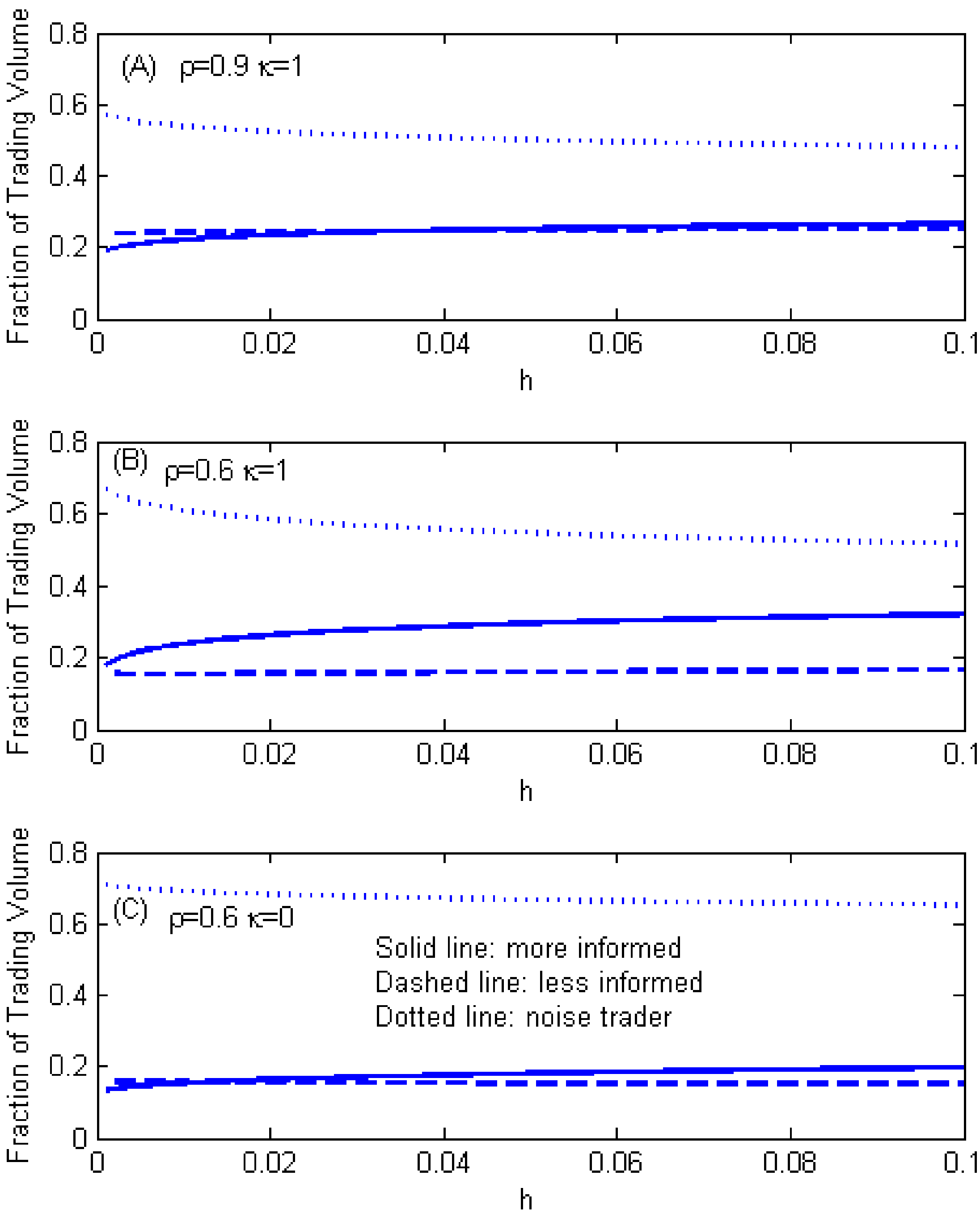

Figure 2.5: Contributions to the total trading volume. Parameter values: $\sigma_v = \sigma_u = 1$ and $r = 0.05$.

of the less informed trader. When $\rho = 1$, we reach to the benchmark case where both traders are equally informed. We should expect them to earn the same expected profits and it is confirmed in the Figure.

2.5 Conclusion

In this paper, we examine how the correlation among flows of information received by informed traders affects the market efficiency, market liquidity, trading volume and their expected profits, especially in the limit of continuous trading. We consider a very special case in which duopolistic informed traders' information sets are nested.

We find the total private information can be decomposed into two components with each component incorporated into prices in a qualitatively different manner in the limit. The shared private information is incorporated into prices much more quickly than the information held exclusively by the more informed trader. One can find that the less informed trader acts more aggressively than the better informed trader. When the frequency of trading becomes high enough, the less informed trader who has worse information contributes more trading volume contrary to conventional wisdom. The profits for the less informed trader may fall as trading becomes more frequent, but converge to a positive constant in the limit as the time interval between trades goes to zero.

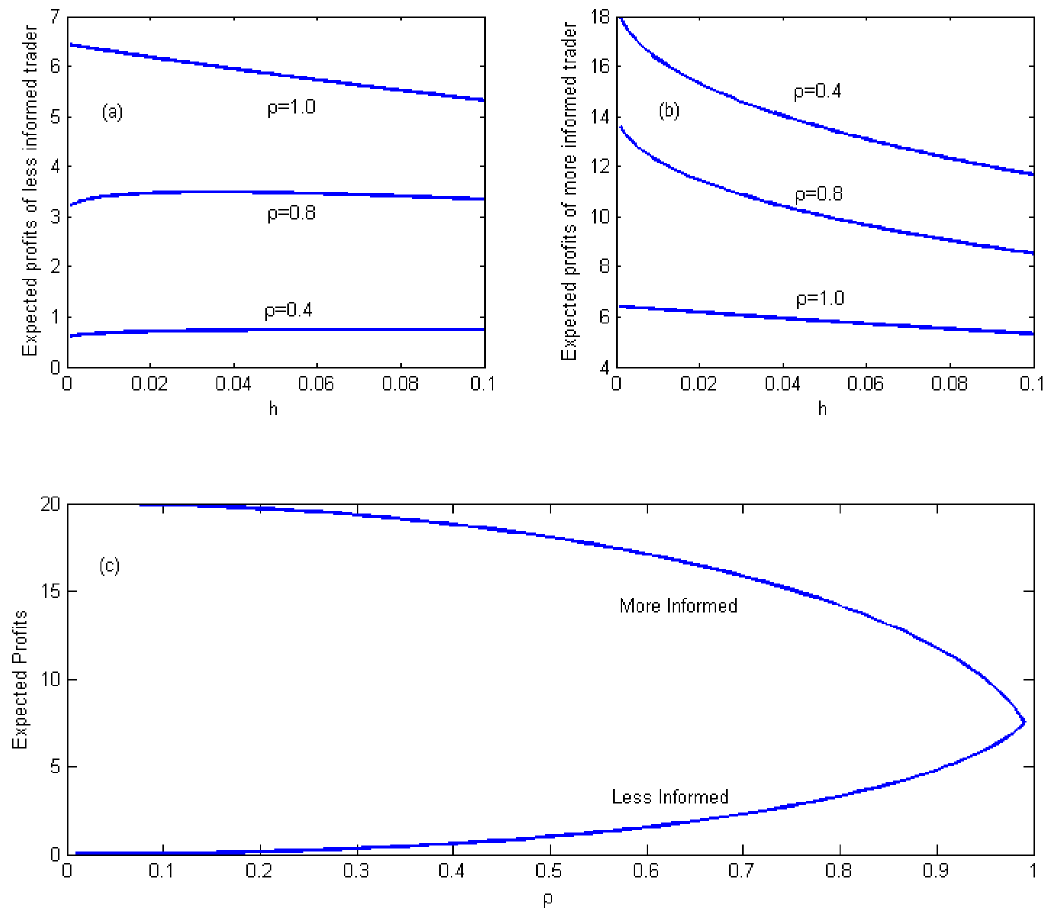

Figure 2.6: (A) and (B) The expected profits for each informed trader as functions of h for cases when $\rho = 0.4$, 0.8 and 1. (C) Expected profits for each trader as functions of ρ in the limit of continuous trading. Parameter values: $\sigma_v = \sigma_u = \kappa = 1$ and $r = 0.05$.

Chapter 3

Real Options and Product Differentiation

3.1 Introduction

Ever since the pioneering work of McDonald and Siegel (1986), the real options literature often either assumes the firm to be a monopolist (Pindyck (1988), He and Pindyck (1992)) or assumes the firm to be perfectly competitive (Dixit (1989)). Very little literature has modeled the imperfect competition among firms in exercising real options (option games). Smets (1993) provides the first approach to model real options game in a duopoly industry. In his model, the capital stock choice is discrete and there is a upper bound in the total industry capital stock. He also randomly picks a firm to be the leader by investing first (the other firm which invests later is called the follower) when time comes to make investment. The subsequent work that follow Smets's approach include Grenadier (1996) and Williams (1993) .

In the situation where the capital stock choice is continuous, implying that each firm can make arbitrarily small investment, the firms face a pre-emption problem. Each firm has incentive to preempt its competitors to prematurely invest to extract more rents. Grenadier (2002) claims to solve the equilibrium in an oligopoly industry by using a "myopic firm" approach. When a "myopic" firm is evaluating the optimal time to invest, it assumes that all other firms' capital stock is fixed forever. The equilibrium derived in his approach has lower growth options value relative to the monopolist case but is still positive. Such an equilibrium, however is not subgame perfect. It is an "open loop" equilibrium instead of a "closed loop" equilibrium, as pointed out by Back and Paulsen (2009). In an "open loop" equilibrium, each player in the game cannot observe the other players' actions. While in a "closed loop" equilibrium, all past play is common knowledge. In Grenadier (2002)'s setup, each firm can infer its opponent's capital stock from the price of the product. It can be shown that, by knowing its opponent's capital stock, each firm has incentive to preempt its opponent by investing earlier than the conjectured optimal time in Grenadier's equilibrium. As long as the value of the real option is positive, each firm will have incentive to invest till the real option value drops to zero. This means that the preemption never disappears as long as the growth options is positive. Back and Paulsen further conjecture that the equilibrium with pre-emption should be competitive in an oligopoly industry with elastic demand such that the real option value remains zero all the time .

In Grenadier (2002) and Back and Paulsen (2009), the products are homogenous (perfect substitute). This paper examines whether the incentive to preempt could diminish if we assume heterogenous products instead. By making such an assumption, each firm has some limited monopoly power over its product. I examine whether such monopoly power, although limited, could decrease the competition of exercising options to such an extent that preemption is no longer profitable and the value of growth options is still positive. However, we find that under the assumption of non-flexibility of capital stock, preemption is always profitable even though the products are imperfect substitute.

The plan of the paper is as follows. In Section II, we describe the model. We solve the equilibria using the "myopic" approach for the CES demand case in Section III and for the linear demand case in Section IV. We find the equilibrium is not subgame perfect in either case. Section V concludes.

3.2 Model Setup

3.2.1 Demand Function and Capacity Process

I make the following assumptions:

(1) There exits an industry composed of n firms each producing a single non-storable differentiated product. In section III, I use the CES (constant elasticity of substitution) demand function. The inverse demand function is given by

$$P_i = Y(t)(\sum_{j=1}^{n} q_j^{\rho})^{\theta-1} q_i^{\rho-1} \tag{3.1}$$

where $dY = \mu Y dt + \sigma Y dz$ is an exogenous shock process to demand. ρ measures the degree of differentiation, ranging from 0 for independent goods to 1 for perfect substitutes. The demand function is derived from a n-good industry with representative consumer with utility function given by $U(q) = (\sum_{i=1}^{n} q_i^{\rho})^{\theta}$ (Xavier Vives 2000). Another specification can be found in Dixit and Stiglitz (1977): $U(q) = (\sum_{i=1}^{n} q_i^{\rho})^{1/\rho}$.

In section 3.4 , I use the linear inverse demand function: $P_i = Y_t - \beta q_i - \gamma \sum_{j \neq i} q_j$.

(2) At time t, each firm produces $q_i(t) \leq K_i(t)$ units of output where K_i is the capital

stock of firm i. For simplicity, zero variable cost of production is assumed. The operating profit flow is denoted $\pi_i(Y(t), K_i(t), K_{-i}(t)) = P_i q_i$.

(3) At any point in time, each firm can invest in additional capacity to increase its output. Each unit of capacity costs I. I assume that K_i is a nondecreasing process implying completely irreversibility of capital stock. I also assume that there exists an investment strategy which is characterized by the trigger function $Y(K_i, K_{-i})$. Whenever $Y(t)$ rises to the trigger function, firm i increases its capacity level.

(4) The firm i is solving the following problem:

$$V^i[Y, K_i^*(t), K_{-i}^*(t)] = \max_{K_i} E\left(\int_0^\infty e^{-rt}\pi_i[Y(t), K_i(t), K_{-i}(t)]dt - \int_0^\infty e^{-rt} I dK_i\right) \tag{3.2}$$

3.2.2 Risk Neutral Measure

Following Carlson et al. (2004), let B_t denote the price of the riskless asset with dynamics $dB_t = rB_t dt$, where r is the risk-free interest rate, and let S_t be the risky asst with dynamics

$$dS_t = \eta S_t dt + \sigma S_t dz_t. \tag{3.3}$$

We use assets B and S to define a new probability measure (Q) under which $\hat{z}_t = z_t + \frac{\eta - r}{\sigma} t$ is a standard Brownian motion. Under this new measure, the demand dynamics becomes

$$dY_t = (r - \delta)Y_t dt + \sigma Y_t d\hat{z}_t \tag{3.4}$$

where $\delta = \eta - \mu > 0$.

3.3 Symmetric Open Loop Equilibrium for CES Demand Function

3.3.1 The Open Loop Equilibrium

I consider the case when there are two firms in the industry.

Proposition 3.1

1. In a symmetric open-loop equilibrium, the trigger strategy for firm i is given by

$$Y^* = \frac{I\delta}{2^{\theta-2}(1-\frac{1}{\lambda_1})(\rho\theta+\rho)K^{\rho\theta-1}} = \nu K^{1-\rho\theta} \tag{3.5}$$

with $\nu = \frac{I\delta}{2^{\theta-2}(1-\frac{1}{\lambda_1})(\rho\theta+\rho)}$ and $K = K_i = K_j$ to be each firm's capacity.

2. The firm i's value is given by

$$V_i(Y,K) = I(1 - \frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}(\frac{1}{\lambda_1(\rho\theta-1)+1})K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} + \frac{Y2^{\theta-1}K^{\rho\theta}}{\delta}. \tag{3.6}$$

The value of assets in place is

$$F_i(Y,K) = \frac{Y2^{\theta-1}K^{\rho\theta}}{\delta} + I\nu^{-\lambda_1}(\frac{1}{\lambda_1(\rho\theta-1)+1})K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} = J(K)Y + E(K)Y^{\lambda_1} \tag{3.7}$$

and the value of growth option is

$$G_i(Y,K) = -I\frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)}\nu^{-\lambda_1}(\frac{1}{\lambda_1(\rho\theta-1)+1})K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} = C(K)Y^{\lambda_1}. \tag{3.8}$$

Proof:

From standard arguments in the literature on investment with uncertainty, $V^i(Y, K_i, K_{-i})$ and $Y(K_i, K_{-i})$ are solutions to the following differential equation:

$$\frac{1}{2}\sigma^2 Y^2 V^i_{YY} + (r-\delta)YV^i_Y - rV^i + \pi_i(Y, K_i, K_{-i}) = 0, \tag{3.9}$$

with the following boundary conditions

$$\frac{\partial V^i}{\partial K_i}(Y(K_i, K_{-i}), K_i, K_{-i}) = I, \tag{3.10}$$

$$\frac{\partial^2 V^i}{\partial K_i \partial Y}(Y(K_i, K_{-i}, K_i, K_{-i}) = 0, \tag{3.11}$$

and

$$\frac{\partial V^i}{\partial K_{-i}}(Y(K_i, K_{-i}), K_i, K_{-i}) = 0. \tag{3.12}$$

Grenadier provides a simple approach to solve the equilibrium strategies without involving fixed point problem. He considers a myopic firm i that ignores all potential competitive exercise. The value of the myopic firm is denoted as $M^i(Y, K_i, K_{-i})$. Then, I will denote the myopic firm's marginal output by $m^i(K_i, K_{-i})$, with $m^i(Y, K_i, K_{-i}) = \frac{\partial M^i}{\partial K_i}$. It can be shown that m^i and $Y(K_i, K_{-i})$ satisfy the following differential equation:

$$\frac{1}{2}\sigma^2 m_{YY} + (r-\delta)m_Y - rm + \frac{\partial \pi}{\partial K_i} = 0 \tag{3.13}$$

subject to

$$m(Y(K_i, K_{-i}), K_i, K_{-i}) = I \tag{3.14}$$

and

$$\frac{\partial m}{\partial Y}(Y(K_i, K_{-i}), K_i, K_{-i}) = 0. \tag{3.15}$$

The function $m(Y, K_i, K_{-i})$ represents the value of a perpetual American call option, where the option has an exercise payoff of I, and a zero exercise price.

It can be easily verified that

$$\frac{\partial \pi}{\partial K_i} = Y(t)(K_i^\rho + K_{-i}^\rho)^{\theta-2}(\rho\theta K_i^{2\rho-1} + \rho K_i^{\rho-1} K_{-i}^\rho). \tag{3.16}$$

The solution for Equation 3.9 is

$$m = BY^{\lambda_1} + \frac{2^{\theta-2} Y(\rho\theta + \rho) K^{\rho\theta-1}}{\delta}. \tag{3.17}$$

From the boundary conditions, we have

$$BY^{*\lambda_1} + \frac{2^{\theta-2} Y^*(\rho\theta + \rho) K^{\rho\theta-1}}{\delta} = I \tag{3.18}$$

and

$$B\lambda_1 Y^{*\lambda_1 - 1} + \frac{2^{\theta-2}(\rho\theta + \rho) K^{\rho\theta-1}}{\delta} = 0. \tag{3.19}$$

Since firms are symmetric, we should have $K_i = K_j = K$, and the above equation becomes

$$\frac{\partial \pi}{\partial K_i} = Y(t)2^{\theta-2}\rho(\theta+1)K^{\theta\rho-1}. \tag{3.20}$$

Equation 3.19 can be reduced to

$$B = -\frac{2^{\theta-2}(\rho\theta+\rho)K^{\rho\theta-1}}{\lambda_1 Y^{*\lambda_1-1}\delta}. \tag{3.21}$$

Then, substituting the expression for B into 3.18, we have

$$-\frac{2^{\theta-2}(\rho\theta+\rho)K^{\rho\theta-1}Y^*}{\lambda_1\delta} + \frac{2^{\theta-2}Y^*(\rho\theta+\rho)K^{\rho\theta-1}}{\delta} = I, \tag{3.22}$$

which leads to the solution for Y^*,

$$Y^*(K) = \frac{I\delta}{2^{\theta-2}(1-\frac{1}{\lambda_1})(\rho\theta+\rho)K^{\rho\theta-1}} = \nu K^{1-\rho\theta}, \tag{3.23}$$

where $\nu = \frac{I\delta}{2^{\theta-2}(1-\frac{1}{\lambda_1})(\rho\theta+\rho)}$.

From a similar argument, we can also calculate the myopic trigger strategy when two firms have different capital stock in the beginning ($K_i \neq K_{-i}$).

$$Y^*(K_i, K_{-i}) = \frac{I\delta}{(1-1/\lambda_1)\rho(K_i^\rho + K_{-i}^\rho)^{\theta-2}(\theta K_i^{2\rho-1} + K_i^{\rho-1}K_{-i}^\rho)} \tag{3.24}$$

The value of the firm: assets in place and growth options.

Our next goal is to solve Equation 3.9 with the boundary conditions. Given firm i's instantaneous profit function $\pi_i = Y(K_i^\rho + K_j^\rho)^{\theta-1}K_i^\rho$, the solution can be written as follows

$$V_i = AY^{\lambda_1} + \frac{Y(K_i^\rho + K_j^\rho)^{\theta-1}K_i^\rho}{\delta} = A(K)Y^{\lambda_1} + \frac{Y2^{\theta-1}K^{\rho\theta}}{\delta}. \tag{3.25}$$

The subscript is ignored since in equilibrium $K_1 = K_2 = ... = K_n$.

From the boundary conditions, we have

$$\frac{\partial V}{\partial K}(Y^*, K) = I \tag{3.26}$$

which is equivalent to

$$A'(K)\nu^{\lambda_1}Y^{*\lambda_1} + \frac{Y^*2^{\theta-1}\rho\theta K^{\rho\theta-1}}{\delta} = I. \tag{3.27}$$

Solving Equation 3.27, we have

$$\begin{aligned} A'(K) &= (I - \frac{Y^*2^{\theta-1}\rho\theta K^{\rho\theta-1}}{\delta})Y^{*-\lambda_1} \qquad (3.28) \\ &= (I - \frac{\nu 2^{\theta-1}\rho\theta}{\delta})\nu^{-\lambda_1}K^{\lambda_1(\rho\theta-1)} = I(1 - \frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}K^{\lambda_1(\rho\theta-1)}. \end{aligned}$$

By integrating $A'(K)$, we find that

$$A(K) = -\int_K^\infty A'(k)dk = I(1 - \frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}\frac{1}{\lambda_1(\rho\theta-1)+1}K^{\lambda_1(\rho\theta-1)+1}. \tag{3.29}$$

Therefor, firm i's value can be expressed as

$$V_i = I(1 - \frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}\frac{1}{\lambda_1(\rho\theta-1)+1}K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} + \frac{Y2^{\theta-1}K^{\rho\theta}}{\delta}. \tag{3.30}$$

The value of assets in place is given by

$$F_i(K,Y) = \frac{Y2^{\theta-1}K^{\rho\theta}}{\delta} + I\nu^{-\lambda_1}(\frac{1}{\lambda_1(\rho\theta-1)+1})K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} = J(K)Y + E(K)Y^{\lambda_1}, \tag{3.31}$$

and the value of growth option is given by

$$G_i(K,Y) = -I\frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)}\nu^{-\lambda_1}(\frac{1}{\lambda_1(\rho\theta-1)+1})K^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} = C(K)Y^{\lambda_1}. \tag{3.32}$$

3.3.2 A Preemption Strategy

To determine whether there exists a preemption strategy, I follow the approach in Paulsen (2006). Suppose $K_{10} = K_{20} = K_0$. If both firms were to play the myopic strategy defined in Proposition 2, then both firms would always hold the same capacity and the expected present value of the future cash flow would be given by

$$V_i = I(1 - \frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}\frac{1}{\lambda_1(\rho\theta-1)+1}K_0^{\lambda_1(\rho\theta-1)+1}Y^{\lambda_1} + \frac{Y2^{\theta-1}K_0^{\rho\theta}}{\delta}. \tag{3.33}$$

Now consider the strategy for firm 1: invest $\overline{K} > K_0$ and keep this capacity constant until time τ when $K_2 = \overline{K}$, then play the symmetric trigger. We assume firm 2 plays the myopic trigger and time τ is the first passage time of $Y^*(\overline{K}, \overline{K})$.

From time τ on, both firms will hold the same capacities, thus firm 1's expected payoff is given by the value function $V(Y^*, \overline{K}, \overline{K})$ discounted by $e^{-r\tau}$. Up to time τ, firm 1's capacity is fixed and only K_2 increases. Firm 1's expected profit is thus

$$W(Y, \overline{K}, K_2) = E_{Y,K_2}[\int_0^\tau e^{-rt}(\pi(Y_t, \overline{K}, K_2) - rI\overline{K})dt + e^{-r\tau}V(Y_\tau, \overline{K}, \overline{K})]. \tag{3.34}$$

From the Feynman-Kac theorem, $W(Y, \overline{K}, K_2)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 Y^2 W_{YY} + (r-\delta)W_Y - rW + (\pi(Y, \overline{K}, K_2) - rI\overline{K}) = 0, \tag{3.35}$$

subject to the boundary conditions

$$W_{K_2}(Y(K_2, \overline{K}), \overline{K}, K_2) = 0, \tag{3.36}$$

$$W(0, \overline{K}, K_2) = -I\overline{K}, \tag{3.37}$$

and

$$W(Y(\overline{K}, \overline{K}), \overline{K}, \overline{K}) = V(Y(\overline{K}, \overline{K}), \overline{K}, \overline{K}). \tag{3.38}$$

The solution of the ODE is in the form of $W(Y, \bar{K}, K_2) = \frac{Y(\bar{K}^\rho + K_2^\rho)^{\theta-1}\bar{K}^\rho}{\delta} - I\bar{K} + A(\bar{K}, K_2)Y^{\lambda_1}$. From the boundary condition,

$$\frac{\partial W}{\partial K_2} = \frac{\rho(\theta-1)Y^*(K_2, \bar{K})(\bar{K}^\rho + K_2^\rho)^{\theta-2}\bar{K}^\rho K_2^{\rho-1}}{\delta} + A_{K_2}(\bar{K}, K_2)Y^{*\lambda_2}(K_2, \bar{K}) = 0, \tag{3.39}$$

we have

$$A_{K_2}(\bar{K}, K_2) = -\frac{\rho(\theta-1)(\bar{K}^\rho + K_2^\rho)^{\theta-2}\bar{K}^\rho K_2^{\rho-1}}{\delta Y^{*\lambda_1-1}(K_2, \bar{K})}. \tag{3.40}$$

Integrating $A_{K_2}(\bar{K}, K_2)$, we can find

$$A(\bar{K}, K_2) = A(\bar{K}) - \int_{K_2}^{\infty} A_{K_2}(\bar{K}, s)ds. \tag{3.41}$$

To find $A(\bar{K})$, we have to use the last boundary condition. First, we need to calculate $V(Y^*(\bar{K}, \bar{K}), \bar{K})$. From the result we have in section 3.3.1,

$$V_i = I(1-\frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}\frac{1}{\lambda_1(\rho\theta-1)+1}\bar{K}^{\lambda_1(\rho\theta-1)+1}Y^{*\lambda_1}(\bar{K})+\frac{Y^*(\bar{K})2^{\theta-1}\bar{K}^{\rho\theta}}{\delta}. \tag{3.42}$$

From the last boundary condition,

$$A(\bar{K}) = \int_{\bar{K}}^{\infty} A_{K_2}(\bar{K}, s)ds + I(1-\frac{2\lambda_1\theta}{(\lambda_1-1)(\theta+1)})\nu^{-\lambda_1}\frac{1}{\lambda_1(\rho\theta-1)+1}\bar{K}^{\lambda_1(\rho\theta-1)+1}. \tag{3.43}$$

Since the integrals cannot be derived in the closed forms except for a few parameter values, I evaluate the integrals numerically to find the expected payoff from the preemption strategy.

Some numerical examples.

First, we set $\sigma = 0.2$, $\delta = 0.12$, $r = 0.12$, $I = 1$, $\rho = 1$ and $\theta = 0.5$. One can tell from simple calculation (not shown here), $\lambda_1 = 3$. This is actually the set of parameters in the counter-example provided by Paulsen (2006), in which the products are perfect substitute.

We then plot $W(Y^*(1,1), \bar{K}, 1)$ vs. $\bar{K}$. It can be shown from Figure 3.1 that $W(Y(1,1), \bar{K}, 1)$ is increasing in $\bar{K}$ up to a value greater than 1 and then decreasing. Following the argument in Paulsen (2006), the symmetric trigger $Y^*(K_i, K_{-i})$ is not sub-game perfect, and it is profitable for the firm to exercise the growth option early to preempt the second firm.

We then keep θ fixed and decrease ρ from 1 to 0.5. Figure 3.2 shows the preempting value function $W(Y(1,1), \bar{K}, 1)$ as a function of $\bar{K}$. In addition, the preemption strategy still exists but the local maxima is less than the case when $\rho = 1$. I also tried other values of ρ and calculate $W(Y(1,1), \bar{K}, 1)$ numerically. As long as ρ is positive and less than 1, it is always profitable for one firm to deviate the conjectured strategy in Proposition 1 and preempt if the other firm plays the conjectured strategy.

If $\theta = 0$, then the utility of the representative customer is not well defined. However, we can define the form of logarithm as $U(q_1, q_2) \sim \log(q_1^\rho + q_2^\rho)$. Then the price can be written as $P_i(q_i, q_{-i}) = (q_i^\rho + q_{-i}^\rho)^{-1} q_i^{\rho-1}$. Fig. 3.3 shows the expected profit $W(Y(1,1), \bar{K}, 1)$ from the preemption strategy as a function of $\bar{K}$. Clearly, $W(Y(1,1), \bar{K}, 1)$ is monotonically decreasing when $\bar{K} > 1$. The myopic strategies form a Nash equilibrium. The result is not surprising since Heston and Loewenstein have already proved in the perfect substitute case that if the elasticity is one (when the representative customer has a utility in the form of $\log(q_i + q_{-i})$), each firm is indifferent between waiting and preempting.

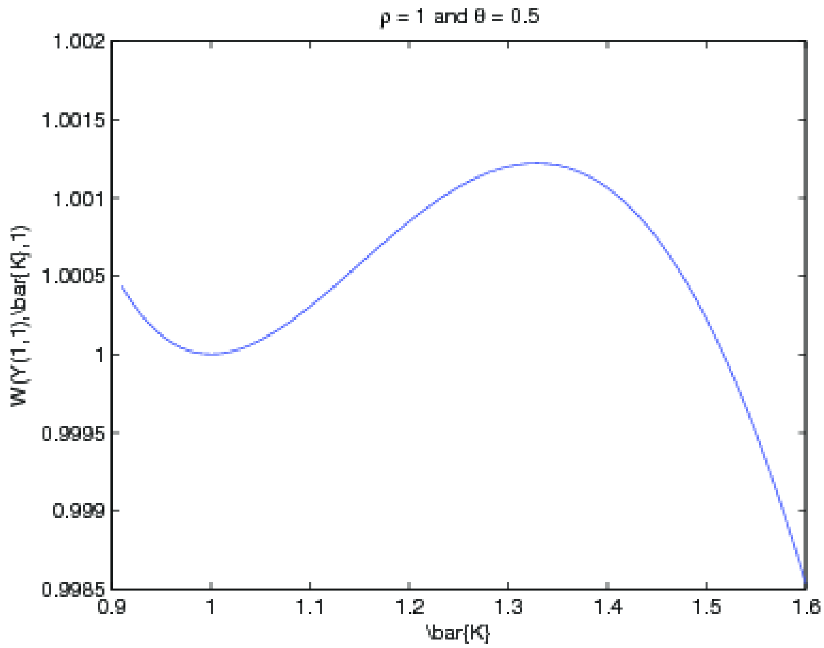

Figure 3.1: $\rho = 1$, $\theta = 0.5$

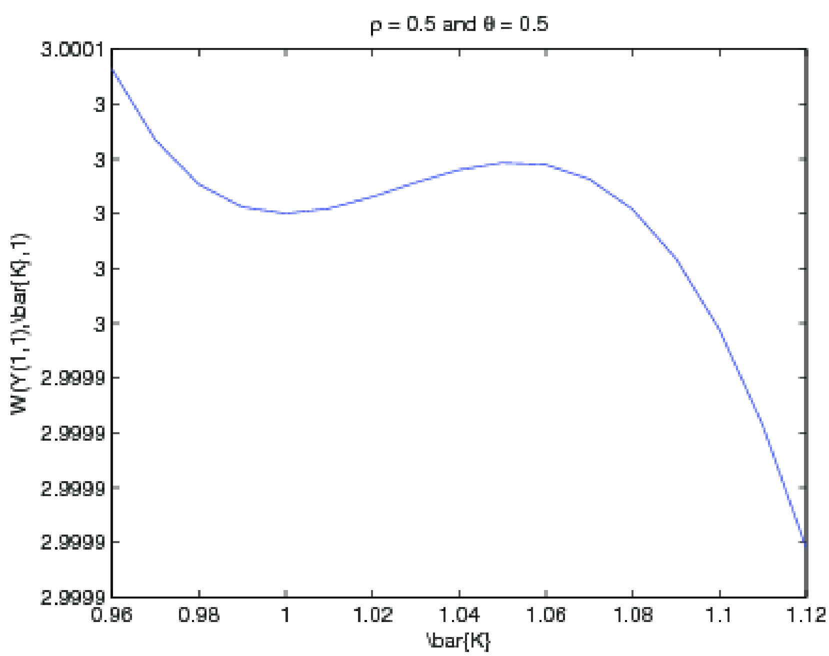

Figure 3.2: $\rho = 0.5$, $\theta = 0.5$

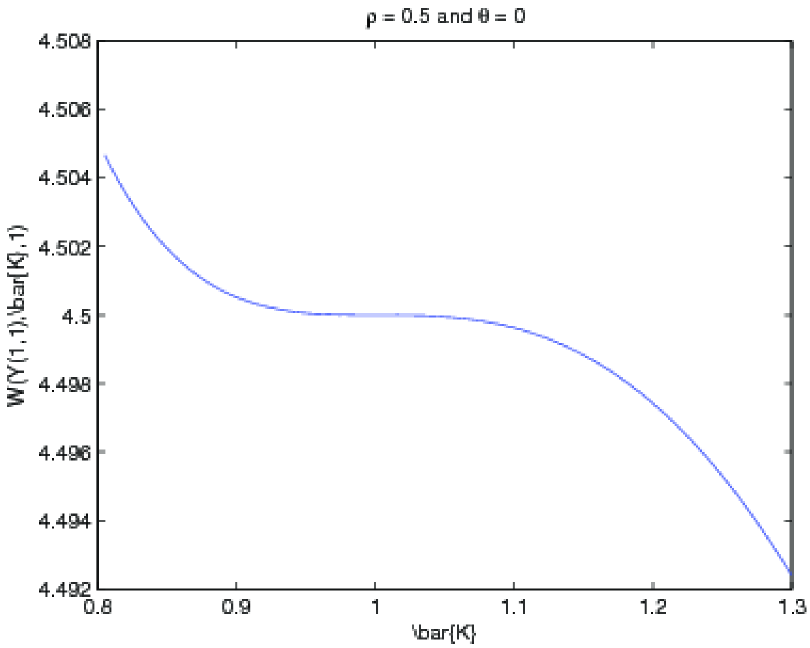

Figure 3.3: $\rho = 0.5$, $\theta = 0$ (equivalent to the case in which the representative customer has a log utility function.)

3.4 Linear Demand

3.4.1 The Open Loop Equilibrium

Now we study the case in which the inverse demand function is linear. The inverse demand function is given by

$$P_i = Y_t - \beta K_i - \gamma \sum_{j \neq i} K_j. \tag{3.44}$$

The value of ρ,$0 \leq \rho = \gamma/\beta \leq 1$ measures the degree of product differentiation. When $\rho = 1$, the goods are perfect substitute, when $\rho = 0$, the goods are independent, and when $\rho < 0$, the goods are complements. Baldursson (1998), Grenadier (2002) and Back and Paulsen (2009) all have studied the open loop equilibrium in an oligopoly setting with the assumption of linear demand. Our assumption is the same except that the products are not homogenous.

Again, we assume that there exist two firms in the industry. The instantaneous profit for firm i ($i = 1, 2$) can be expressed as

$$\pi_i = K_i(Y - \beta K_i - \gamma K_{-i}). \tag{3.45}$$

In the symmetric equilibrium, $K_i = K_{-i}$ and $\pi = K_i(Y - (\beta + \gamma)K_i)$.

Proposition 3.2: 1. The trigger strategy is given by

$$Y^*(K_i, K_{-i}) = \frac{\lambda_1 \delta(2\beta K_i + \gamma K_{-i} + rI)}{(\lambda_1 - 1)r}. \tag{3.46}$$

At equilibrium, $K_i = K_{-i}$ and we have

$$Y^*(K_i) = \frac{\lambda_1 \delta[(2\beta + \gamma)K_i + rI]}{(\lambda_1 - 1)r} \tag{3.47}$$

2. The value of marginal investment is given by

$$m(Y_t, K_i, K_{-i}) = \frac{Y_t}{\delta} - \frac{2\beta K_i + \gamma K_{-i}}{r} - \frac{I + \frac{2\beta K_i + \gamma K_j}{r}}{(\lambda_1 - 1)Y^{*\lambda_1}} Y_t^{\lambda_1}. \tag{3.48}$$

3. In the symmetric equilibrium, the value of firm i is given by

$$V(Y, K_i) = C(K_i)Y^{\lambda_1} + \frac{YK_i}{\delta} - \frac{(2\beta + \gamma)K_i^2}{r} \tag{3.49}$$

where

$$C(K_i) = -\int_{K_i}^{+\infty} \frac{I + \frac{2(2\beta+\gamma)K}{r} - \frac{Y^*(K)}{\delta}}{Y^{*\lambda_1}} dK = \frac{r}{\delta\lambda_1(2\beta+\gamma)} Y^{*-\lambda_1+1}(K_i) + \frac{r(\lambda_1 - 1)}{\delta\lambda_1(2\beta+\gamma)} \frac{\frac{2}{\delta\lambda_1} - \frac{1}{\delta}}{2-\lambda_1} Y^{*-\lambda_1+2}(K_i). \tag{3.50}$$

Proof:

We first calculate the marginal benefit

$$\frac{\partial \pi_i}{\partial K_i} = Y_t - 2\beta K_i - \gamma K_{-i}. \tag{3.51}$$

Let $m(Y_t, K_i, K_{-i})$ denote the value of the marginal investment. $m(Y_t, K_i, K_{-i})$ satisfies the differential equation,

$$\frac{1}{2}\sigma^2 Y^2 m_{YY} + (r-\delta) Y m_Y - rm + \frac{\partial \pi_i}{\partial K_i} = 0 \tag{3.52}$$

and is subject to the boundary conditions:

$$m(Y^*, K_i, K_j) = I, \tag{3.53}$$

$$\frac{\partial m}{\partial Y} = 0. \tag{3.54}$$

The solution of $m(Y_t, K_i, K_{-i})$ takes the form of

$$m(Y_t, K_i, K_{-i}) = \frac{Y_t}{\delta} - \frac{2\beta K_i + \gamma K_j}{r} + AY_t^{\lambda_1}. \tag{3.55}$$

From the boundary conditions (value matching and smooth-pasting), we have

$$\frac{Y_t^*}{\delta} - \frac{2\beta K_i + \gamma K_j}{r} + AY_t^{*\lambda_1} = I \tag{3.56}$$

and

$$\frac{1}{\delta} + \lambda_1 AY_t^{*\lambda_1 - 1} = 0. \tag{3.57}$$

By solving Equation 3.56 and 3.57, we find the trigger strategy $Y^*(K_i, K_{-i})$ and $m(Y_t, K_i, K_{-i})$ as follows

$$Y^*(K_i, K_{-i}) = \frac{\lambda_1}{\lambda_1 - 1}\delta\frac{2\beta K_i + \gamma K_{-i} + rI}{r} \tag{3.58}$$

and

$$m(Y_t, K_i, K_{-i}) = \frac{Y_t}{\delta} - \frac{2\beta K_i + \gamma K_{-i}}{r} - \frac{1}{\lambda_1 \delta Y^{*\lambda_1 - 1}} Y_t^{\lambda_1}. \tag{3.59}$$

We then need to find the value function of the firm $V_i(Y_t, K_i, K_{-i})$. The function V_i satisfies the following differential equation

$$\frac{1}{2}\sigma^2 \frac{\partial^2 V_i}{\partial Y^2} + (r - \delta)\frac{\partial V_i}{\partial Y} - rV_i + \pi = 0, \tag{3.60}$$

subject to

$$\frac{\partial V_i}{\partial K_i}(Y^*, K_i) = I, \tag{3.61}$$

with $\pi = K_i(Y - (\beta + \gamma)K_i) = YK_i - (\beta + \gamma)K_i^2$.

The solution of V_i can be expressed as

$$V_i = C * Y^{\lambda_1} + \frac{YK_i}{\delta} - \frac{(\beta+\gamma)K_i^2}{r}. \tag{3.62}$$

From the boundary condition, we have

$$C^{'}(K_i)Y^{*\lambda_1} + \frac{Y^*}{\delta} - \frac{2(\beta+\gamma)K_i}{r} = I. \tag{3.63}$$

Solving the above equation, we have

$$C^{'}(K_i) = \frac{I + \frac{2(\beta+\gamma)K_i}{r} - \frac{Y^*}{\delta}}{Y^{*\lambda_1}}. \tag{3.64}$$

From the fact that

$$\frac{2(\beta+\gamma)K_i}{r} = \frac{2(\beta+\gamma)}{2\beta+\gamma}(\frac{Y^*(\lambda_1-1)}{\delta\lambda_1} - I), \tag{3.65}$$

we can write $C^{'}(K_i)$ as

$$C^{'}(K_i) = \frac{-\frac{\gamma I}{2\beta+\gamma} + \frac{\gamma\lambda_1 - 2(\beta+\gamma)}{(2\beta+\gamma)\delta\lambda_1}Y^*}{Y^{*\lambda_1}}. \tag{3.66}$$

One can verify that

$$\int_{K_i}^{\infty} \frac{1}{Y^{*\lambda_1}} dK_i = \frac{r}{\lambda_1\delta(2\beta+\gamma)} Y^{*-\lambda_1+1} \tag{3.67}$$

and

$$\int_{K_i}^{\infty} \frac{1}{Y^{*\lambda_1-1}} dK_i = \frac{r(\lambda_1-1)}{\lambda_1\delta(\lambda_1-2)(2\beta+\gamma)} Y^{*-\lambda_1+2}. \tag{3.68}$$

We then can evaluate $C(K_i)$ by integrating $C^{'}(K_i)$

$$C(K_i) = -\int_{K_i}^{\infty} C^{'}(K)dK = +\frac{r\gamma I}{\delta\lambda_1(2\beta+\gamma)^2}Y^{*-\lambda_1+1}(K_i) - \frac{r(\lambda_1-1)(\gamma\lambda_1 - 2(\beta+\gamma))}{(\lambda_1-2)\delta^2\lambda_1^2(2\beta+\gamma)^2}Y^{*-\lambda_1+2}(K_i). \tag{3.69}$$

The growth option is then given by

$$G(Y, K_i) = (\frac{2r(\beta+\gamma)I}{\delta\lambda_1(2\beta+\gamma)^2}Y^{*-\lambda_1+1}(K_i) + \frac{r(\lambda_1-1)(2(\beta+\gamma)-\gamma\lambda_1)}{(\lambda_1-2)\delta^2\lambda_1^2(2\beta+\gamma)^2}Y^{*-\lambda_1+2}(K_i))Y^{\lambda_1}. \tag{3.70}$$

3.4.2 A Preemption Strategy

To determine whether there exists a preemption strategy, I again follow the approach in Paulsen (2006). Suppose $K_{10} = K_{20} = K_0$. If both firms were to play the myopic strategy defined in Proposition 2, then both firms would always hold the same capacity and the expected present value of the future cash flow would be given by

$$V(Y, K_0, K_0) = C(K_0)Y^{\lambda_1} + \frac{YK_0}{\delta} - \frac{(\beta+\gamma)K_0^2}{r}. \tag{3.71}$$

Now consider the strategy for firm 1: invest $\overline{K} > K_0$ and keep this capacity constant until time τ when $K_2 = \overline{K}$, then play the symmetric trigger. We assume firm 2 plays the myopic trigger and time τ is the first passage time of $Y^*(\overline{K}, \overline{K})$.

From time τ on, both firms will hold the same capacities, thus firm 1's expected payoff is given by the value function $V(Y^*, \overline{K}, \overline{K})$ discounted by $e^{-r\tau}$. Up to time τ, firm 1's capacity is fixed and only K_2 increases. Firm 1's expected profit is thus

$$W(Y,\overline{K},K_2) = E_{Y,K_2}[\int_0^{\tau} e^{-rt}(\pi(Y_t,\overline{K},K_2) - rI\overline{K})dt + e^{-r\tau}V(Y_\tau,\overline{K},\overline{K})]. \tag{3.72}$$

From the Feynman Kac theorem, $W(Y,\overline{K},K_2)$ satisfies the ODE

$$\frac{1}{2}\sigma^2 Y^2 W_{YY} + (r-\delta)W_Y - rW + (\pi(Y,\overline{K},K_2) - rI\overline{K}) = 0, \tag{3.73}$$

subject to the boundary conditions

$$W_{K_2}(Y(K_2,\overline{K}),\overline{K},K_2) = 0, \tag{3.74}$$

$$W(0,\overline{K},K_2) = -I\overline{K}, \tag{3.75}$$

and

$$W(Y(\overline{K},\overline{K}),\overline{K},\overline{K}) = V(Y(\overline{K},\overline{K}),\overline{K},\overline{K}). \tag{3.76}$$

The general the solution is in the form

$$W(Y,\overline{K},K_2) = \frac{\overline{K}Y}{\delta} - \frac{\overline{K}(\beta\overline{K}+\gamma K_2)}{r} - I\overline{K} + A(\overline{K},K_2)Y^{\lambda_1} \tag{3.77}$$

since $\pi(Y,\overline{K},K_2) = \overline{K}(Y - \beta\overline{K} - \gamma K_2)$.

From the value-matching condition, we have

$$-\frac{\overline{K}\gamma}{r} + A_{K_2}(\overline{K}, K_2)Y^{*\lambda_1}(K_2, \overline{K}) = 0. \tag{3.78}$$

Solving this equation, we have

$$A_{K_2}(\overline{K}, K_2) = \frac{\frac{\overline{K}\gamma}{r}}{Y^{*\lambda_1}(K_2, \overline{K})}. \tag{3.79}$$

Integrating $A_{K_2}(\overline{K}, K_2)$, we can find the expression for $A(\overline{K}, K_2)$,

$$A(\overline{K}, K_2) = A(\overline{K}) - \int_{K_2}^{\infty} A_{K_2}(\overline{K}, K)dK = A(\overline{K}) + \frac{\frac{\overline{K}\gamma}{r}}{2\beta\frac{\lambda_1\delta}{(\lambda_1-1)r}} \frac{1}{(\lambda_1 - 1)(2\beta K_2 + \gamma\overline{K} + rI)^{\lambda_1-1}}, \tag{3.80}$$

which can be simplified into

$$A(\overline{K}, K_2) = A(\overline{K}) + \frac{\overline{K}\gamma}{2\beta\lambda_1\delta} \frac{1}{(2\beta K_2 + \gamma\overline{K} + rI)^{\lambda_1-1}}. \tag{3.81}$$

From the last boundary condition, we can determine $A(\overline{K})$:

$$A(\overline{K}, \overline{K}) = A(\overline{K}) + \frac{\overline{K}\gamma}{2\beta\lambda_1\delta} \frac{1}{(2\beta\overline{K} + \gamma\overline{K} + rI)^{\lambda_1-1}} = C(\overline{K}). \tag{3.82}$$

Solving the equation, we find that

$$A(\overline{K}) = C(\overline{K}) - \frac{\overline{K}\gamma}{2\beta\lambda_1\delta} \frac{1}{(2\beta\overline{K} + \gamma\overline{K} + rI)^{\lambda_1-1}}. \tag{3.83}$$

Some numerical examples

I set $r = 0.12$, $\mu = 0$, $\sigma = 0.2$ and it is easy to find that $\lambda_1 = 3$ and $\lambda_2 = -1$. I first study the case in which $\beta = 2$, $\gamma = 1.5$ and the goods are imperfect substitute. In fig. 3.4, I plot $W(Y^*(1,1), \bar{K}, 1)$ as a function of $\bar{K}$. It is increasing in $\bar{K}$ up to a value around 1.2 and then decreasing. Hence the symmetric trigger strategy $Y^*(K_i, K_{-i})$ is not firm i's best response to firm $-i$'s strategy. I also tried smaller value of γ (not shown here). No matter how small γ is, as long as γ is positive, the function $W(Y^*(1,1), \bar{K}, 1)$ has a local maxima at some $\bar{K}$ which is greater than 1. This means that although product differentiation can decrease competition among firms, the open-loop equilibrium is still not sub-game perfect since pre-emption strategy exists for arbitrary positive γ values.

I then keep $\beta = 2$ and set $\gamma = 0$. This is the case in which goods are independent and firms are monopolists. As we can see from fig. 3.5, $W(Y^*(1,1), \bar{K}, 1)$ has a slope of zero at $\bar{K} = 1$ and is decreasing when $\bar{K} > 1$. This is expected since firms are monopolists and the equilibrium strategies do not depend on other firms' capacities.

Finally, I examine the case in which $\gamma < 0$ and the goods are complements. I set $\beta = 2$ and $\gamma = -1$. The value function $W(Y^*(1,1), \bar{K}, 1)$ of firm 1 preempting at $Y^*(1,1)$ is shown in fig. 3.6. Again, we find that $W(Y^*(1,1), \bar{K}, 1)$ has a local maxima at $\bar{K} > 1$. This illustrates that even for the case of complimentary goods, each firm has an incentive to preempt.

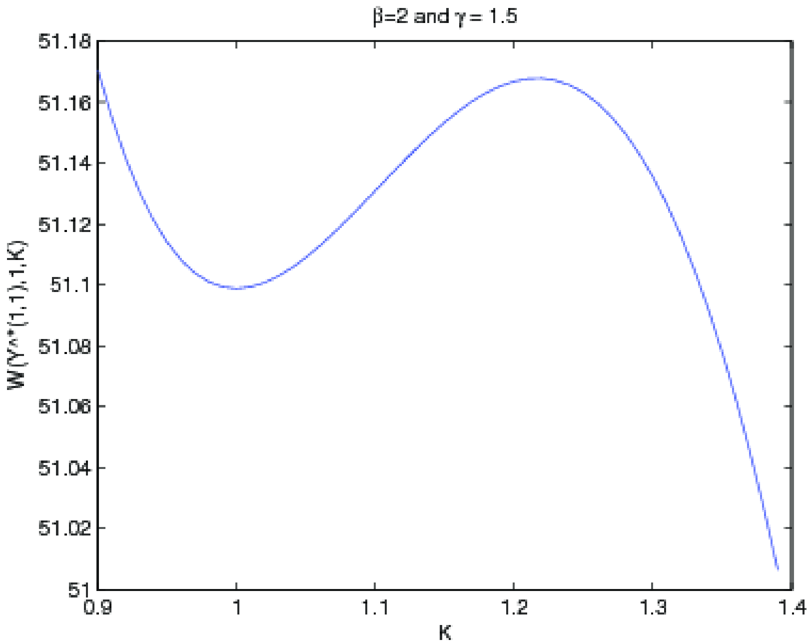

Figure 3.4: $\beta = 2$ and $\gamma = 1.5$

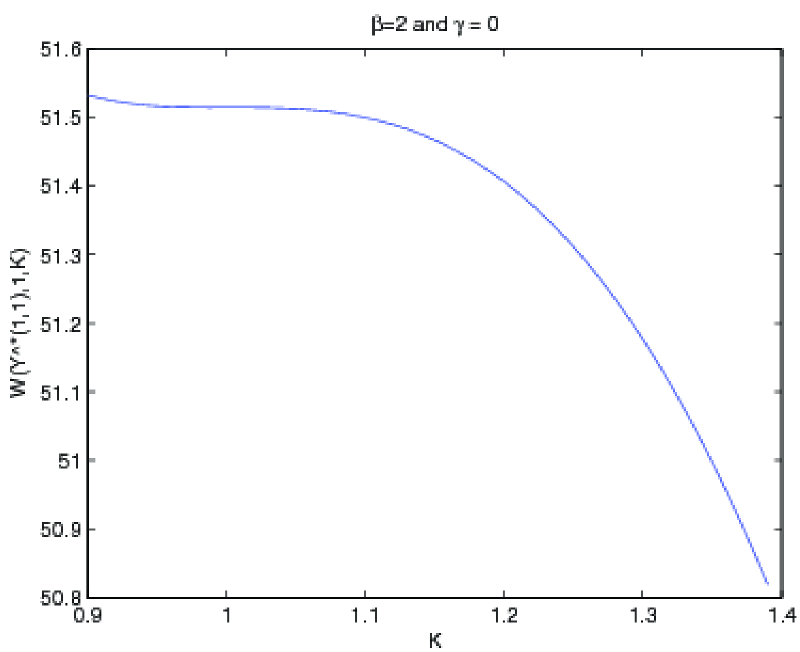

Figure 3.5: $\beta = 2$ and $\gamma = 0$

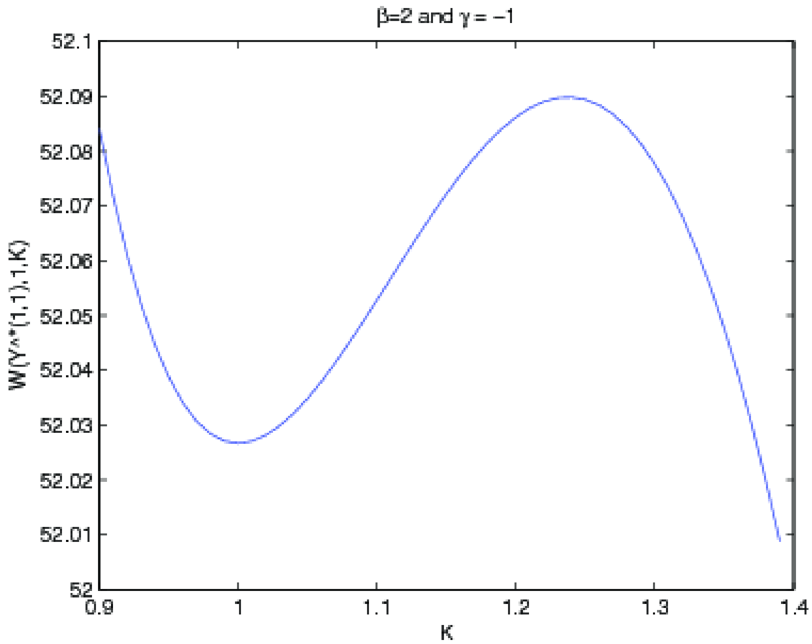

Figure 3.6: $\beta = 2$ and $\gamma = -1$

3.5 Conclusion

I develop a continuous time real option model in an oligopoly industry with heterogenous products. I find that although the heterogenous products assumption lowers the incentive for each firm to prematurely exercise the growth options, the preemption strategy is still profitable for each firm when the reverse demand function is either CES or linear with non-operating flexibility.

Chapter A

Appendix

Proof of Lemma 1.2.1: It is easy to verify that

$$v_{n'} = (1-\kappa h)^{n'-n} v_n + \kappa h \bar{v} \sum_{\tau=n}^{n'} (1-\kappa h)^{n'-n} + \sum_{\tau=n}^{n'} (1-\kappa h)^{n'-n} \epsilon_{v,\tau}. \tag{A.1}$$

Taking expectations in Equation A.1, we have

$$E(v_{n'}|I_n^m) = (1-\kappa h)^{n'-n} E(v_n|I_n^m) + \kappa h \bar{v}(1-(1-\kappa h)^{n'-n+1}). \tag{A.2}$$

Substituting into Equation 1.2, the price is equal to

$$\begin{aligned} p_n &= \sum_{n'=n}^{+\infty} (1-\exp(-rh)) \exp(-r(n'-n)h)((1-\kappa h)^{n'-n} E(v_n|I_n^m) \\ &+ \kappa h \bar{v}(1-(1-\kappa h)^{n'-n+1})) \\ &= \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)} E(v_n|I_n^m) + \frac{\kappa h \bar{v}}{1-\exp(-rh)(1-\kappa h)}. \end{aligned} \tag{A.3}$$

Proof of Lemma 1.2.2: First, one can find that

$$\sum_{n'=n}^{+\infty} \exp(-r(n'-n)h)(\sum_{\tau=n}^{n'} x_{j,\tau}(v_{n'}-p_\tau)) = \tag{A.4}$$

$$\begin{aligned}
& x_{j,n}\sum_{\tau=n}^{+\infty}(\exp(-r(\tau-n)h)(v_\tau-p_n)) \\
+ \quad & x_{j,n+1}\sum_{\tau=n+1}^{+\infty}(\exp(-r(\tau-n)h)(v_\tau-p_{n+1})) \\
+ \quad & x_{j,n+2}\sum_{\tau=n+1}^{+\infty}(\exp(-r(\tau-n)h)(v_\tau-p_{n+2}))+... \\
= \quad & x_{j,n}*(\sum_{\tau=n}^{+\infty}\exp(-r(\tau-n)h)v_\tau-\frac{p_n}{1-\exp(-rh)}) \\
+ \quad & x_{j,n+1}*(\sum_{\tau=n+1}^{+\infty}\exp(-r(\tau-n)h)v_\tau-\frac{\exp(-rh)p_n}{1-\exp(-rh)}) \\
+ \quad & x_{j,n+2}*(\sum_{\tau=n+2}^{+\infty}\exp(-r(\tau-n)h)v_\tau-\frac{\exp(-2rh)p_n}{1-\exp(-rh)})+...
\end{aligned} \tag{A.5}$$

Substituting Equation A.5 into Equation 1.5 and using the result in Equation A.1, we have

$$
\begin{aligned}
& E[\sum_{\tau=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)(\sum_{\tau=n}^{n'} x_{j,\tau}(v_{n'}-p_\tau))|I_{n-1}^i] = \qquad \text{(A.6)}\\
& E[x_{j,n}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_n + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)} - p_n)\\
+ \quad & \exp(-rh)x_{j,n+1}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n+1} + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)} - p_{n+1})\\
+ \quad & \exp(-2rh)x_{j,n+1}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n+1} + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)} - p_{n+2})\\
& ...|I_{n-1}^i]\\
= \quad & E[\sum_{n'=n}^{+\infty}\exp(-r(n'-n)h)(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n'} + \frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)} - p_{n'})|I_{n-1}^i].
\end{aligned}
$$

Proof of Lemma 1.2.3: We first compute each component of the covariance matrix of the vector (v_{n-1}, y_n) conditional on the market maker's information set I_{n-1}^m:

$$cov(y_n, v_{n-1}|I_{n-1}) = cov(M\beta(v_{n-1}-\hat{g}_{n-1}), v_{n-1}|I_{n-1}) = M\beta var(v_{n-1}|I_{n-1}) = M\beta\Sigma_v \quad \text{(A.7)}$$

$$var(y_n|I_{n-1}^m) = M^2\beta^2\Sigma_v + \sigma_u^2 h. \quad \text{(A.8)}$$

Then under the market maker's belief, we have the joint distribution of $(v_{n-1}, y_n)'$

$$\begin{pmatrix} v_{n-1} \\ y_n \end{pmatrix} \sim N(\quad \begin{pmatrix} \hat{v}_{n-1} \\ 0 \end{pmatrix}, \tag{A.9}$$

$$\begin{pmatrix} \Sigma_v & M\beta\Sigma_v \\ \Sigma_v \nu h & M\beta\Sigma_v \nu h \\ M\beta\Sigma_v & M^2\beta^2\Sigma_v + \sigma_u^2 h \end{pmatrix}).$$

Then applying the projection theorem, we have

$$\frac{\lambda}{1-\kappa h} = (M^2\beta^2\Sigma_v + \sigma_u^2 h)^{-1} M\beta\Sigma_v, \tag{A.10}$$

which can be reduced to equation (1.14).

Applying the projection theorem again, we can derive the variance of η_n as

$$\begin{aligned} var(\eta_n) &= var(v_{n-1}|I_{n-1}) - \frac{\lambda}{1-\kappa h} M\beta\Sigma_v \\ &= \Sigma_v - \frac{\lambda}{1-\kappa h} M\beta\Sigma_v \\ &= \frac{\Sigma_v \sigma_u^2 h}{\Sigma_v M^2 \beta^2 + \sigma_u^2 h}. \end{aligned} \tag{A.11}$$

The uncertainty of market maker's posterior belief about g_n is given by

$$var(v_n|I_n^m) = var((1-\kappa h)\eta_n + \epsilon_{v,n}) = (1-\kappa h)^2 var(\eta_n) + \sigma_v^2 h. \tag{A.12}$$

By stationary condition, we must have $var(v_n|I_n^m) = \Sigma_v$ which leads to equation (1.15).

Proof of Lemma 1.2.4: From equations (1.1) and (1.12), market maker's estimation error on v_n is

$$v_n - \hat{v}_n = (1 - \kappa h)(v_{n-1} - \hat{v}_{n-1}) - \lambda y_n + \epsilon_{v,n}. \tag{A.13}$$

Substituting for $v_n - \hat{v}_n$ in equation (1.18), we find

$$\begin{aligned} V(v_{n-1}, \hat{v}_{n-1}) &= max_{x_{i,n}}(x_{i,n}((1-\kappa h)(v_{n-1} - \hat{v}_{n-1}) - \lambda(x_{i,n} + X_{i-,n})) \\ &+ e^{-rh}(B((1-\kappa h)(v_{n-1} - \hat{v}_{n-1}) - \lambda(x_{i,n} + X_{i-,n}))^2 \\ &+ \lambda^2\sigma_u^2 h + \sigma_v^2 h) + C)). \end{aligned} \tag{A.14}$$

The first order condition yields

$$x_{i,n} = \frac{(1 - 2e^{-rh}B\lambda)(1-\kappa h)(v_{n-1} - \hat{v}_{n-1}) + \lambda(2e^{-rh}B\lambda - 1)X_{i-,n}}{2\lambda(1 - e^{-rh}B\lambda)}. \tag{A.15}$$

The second order condition requires that

$$e^{-rh}B\lambda - 1 < 0. \tag{A.16}$$

Because of symmetry argument, the only possible equilibrium is one in which their strategies are identical. We should have $X_{i-,n} = (M-1)x_{i,n}$, which leads

$$x_{i,n} = \frac{(1 - 2e^{-rh}B\lambda)(1-\kappa h)}{\lambda(M(1 - 2e^{-rh}B\lambda) + 1)}(v_{n-1} - \hat{v}_{n-1}) = \beta(v_{n-1} - \hat{v}_{n-1}), \tag{A.17}$$

where

$$\beta = \frac{(1-2e^{-rh}B\lambda)(1-\kappa h)}{\lambda(M(1-2e^{-rh}B\lambda)+1)}. \tag{A.18}$$

Substituting for $x_{i,n}$ back in the Bellman equation and matching the $(v_{n-1} - \hat{v}_{n-1})^2$ term and constant term, we find

$$B = \beta[1-\kappa h - \lambda M\beta] + e^{-rh}B(1-\kappa h-\lambda M\beta)^2 \tag{A.19}$$

which can be reduced to

$$B = \frac{(1-\kappa h)^2(1-e^{-rh}B\lambda)}{\lambda(1+M-2Me^{-rh}B\lambda)^2}, \tag{A.20}$$

and

$$C = \frac{e^{-rh}B(\lambda^2\sigma_u^2+\sigma_v^2)h}{1-e^{-rh}}. \tag{A.21}$$

Proof of Proposition 1.2.1:

First, we define $q = e^{-rh}\lambda B$ and $Z = e^{-rh}(1-\kappa h)^2$. From equation (1.20), we have

$$f(q) = 4M^2q^3 - 4M(M+1)q^2 + ((M+1)^2+Z)q - Z = 0. \tag{A.22}$$

The cubic equation $f(q)$ has three real roots:

$$\begin{aligned} q_1 &= \frac{M+1}{3M} \\ &- \frac{1}{6M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2-((M+1)^2-3Z)^3}} \\ &- \frac{1}{6M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2+((M+1)^2-3Z)^3}} \end{aligned} \tag{A.23}$$

$$
\begin{aligned}
q_2 &= \frac{M+1}{3M} && \text{(A.24)}\\
&- \frac{1+i\sqrt{3}}{12M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2-((M+1)^2-3Z)^3}}\\
&- \frac{1-i\sqrt{3}}{12M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2+((M+1)^2-3Z)^3}}
\end{aligned}
$$

and

$$
\begin{aligned}
q_3 &= \frac{M+1}{3M} && \text{(A.25)}\\
&- \frac{1-i\sqrt{3}}{12M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2-((M+1)^2-3Z)^3}}\\
&- \frac{1+i\sqrt{3}}{12M}\sqrt[3]{(M+1)^3+(9-18M)Z+\sqrt{((M+1)^3+(9-18M)Z)^2+((M+1)^2-3Z)^3}}.
\end{aligned}
$$

It can be easily verified that for $M > 1$, we have

$$f(0) = -Z < 0, f(1/2) => 0, f(1) = (M-1)^2 > 0. \tag{A.26}$$

In addition, the quadratic function $f'(q) = 0$ has the two solutions

$$q_{4,5} = \frac{2(M+1) \pm \sqrt{4(M+1)^2 - 2((M+1)^2+Z)}}{2M}. \tag{A.27}$$

The solution with the positive sign

$$q_5 \in (\frac{1}{2}, 1). \tag{A.28}$$

Equations (A.26) and (A.28) imply that $0 < q_1 < \frac{1}{2}$ and $\frac{1}{2} < q_3 < q_4 < 1$.

From the second order condition (equation (A.16)) we have $q < 1$. In addition, from equation (1.19) any root that makes economically feasible must lie in the range $q \in (0, \frac{1}{2})$. The only possible solution is q_1. From equation (1.14) and equation (1.14), equation (1.19) can be rewritten as

$$\frac{\lambda\beta}{1-\kappa h} = \frac{\Sigma_v M\beta^2}{\Sigma_v M^2\beta^2 + \sigma_u^2 h} = \frac{1-2q_1}{1+M(1-2q_1)}. \tag{A.29}$$

From Equation A.29, one can find that

$$\beta = \sqrt{\frac{\sigma_u^2(1-2q_1)h}{\Sigma_v M}}. \tag{A.30}$$

Substituting Equation A.30 into Equation 1.15, we can find the expression for Σ_v

$$\Sigma_v = \frac{\sigma_v^2 h}{1 - \frac{(1-\kappa h)^2}{M(1-2q_1)+1}}. \tag{A.31}$$

Then, from Equations (A.30, 1.14), (A.24), (1.21) and the expression for q_1, we can derive the remaining parameters β, λ, B and C, respectively with the expressions for β and λ given by

$$\beta = \sqrt{\frac{(1-2q_1)(1-\frac{(1-\kappa h)^2}{M(1-2q_1)+1})}{M}}, \tag{A.32}$$

$$\lambda = \frac{(1-\kappa h)\sqrt{M}\sigma_v\sqrt{\frac{(1-2q_1(1+M(1-2q_1)))}{1+M(1-2q_1)-(1-\kappa h)^2}}}{M\sigma_u(1-2q_1)+\sigma_u}. \tag{A.33}$$

Proof of Proposition 1.2.2: We first define

$$S_v = \Sigma_v / h, \tag{A.34}$$

and

$$\begin{aligned} q_0 &= \frac{M+1}{3M} - \frac{1}{6M}\sqrt[3]{(M+1)^3 - 18M - 9 + \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2+2M-2)^3}} \\ &- \frac{1}{6M}\sqrt[3]{(M+1)^3 - 18M - 9 - \sqrt{((M+1)^3 - 18M - 9)^2 - (M^2+2M-2)^3}}. \end{aligned} \tag{A.35}$$

Then, as h approaches 0, equations (A.29) and (1.15) become

$$\frac{M^2 S_v \beta^2}{M^2 S_v \beta^2 + \sigma_u^2} = \frac{M(1-2q_0)}{1 + M(1-2q_0)} \tag{A.36}$$

$$\frac{S_v \sigma_u^2}{M^2 \beta^2 S_v + \sigma_u^2} + \Sigma_v^2 = S_v. \tag{A.37}$$

The set of the above nonlinear equations has the solution

$$S_v = \frac{\sigma_v^2 (1 + M(1-2q_0))}{M(1-2q_0)} \tag{A.38}$$

and

$$\beta = \frac{\sigma_u (1-2q_0)}{\sigma_v \sqrt{1 + M(1-2q_0)}}. \tag{A.39}$$

By the continuity argument, the limiting results of Σ_v and β become

$$\lim_{h\to 0}\frac{\Sigma_v}{h} = S_v = \frac{\sigma_v^2(1+M(1-2q_0))}{M(1-2q_0)} \tag{A.40}$$

and

$$\lim_{h\to 0}\beta = \frac{\sigma_u(1-2q_0)}{\sigma_v\sqrt{1+M(1-2q_0)}}. \tag{A.41}$$

Then, from equations (1.14), (A.28) and (1.21), we obtain the following asymptotic results for λ, B and C:

$$\lim_{h\to 0}\lambda = \frac{\sigma_v}{\sigma_u}\sqrt{\frac{1}{1+M(1-2q_0)}}, \tag{A.42}$$

$$\lim_{h\to 0}B = \frac{q_0\sigma_u}{\sigma_v}\sqrt{1+M(1-2q_0)}, \tag{A.43}$$

$$\lim_{h\to 0}C = q_0\sigma_u\sigma_v(\sqrt{1+M(1-2q_0)}+\sqrt{\frac{1}{1+M(1-2q_0)}}). \tag{A.44}$$

Chapter B

Appendix

Proof of Lemma 2.2.1: It is easy to verify that

$$v_{n'} = (1-\kappa h)^{n'-n} v_n + \kappa h \bar{v} \sum_{\tau=n}^{n'} (1-\kappa h)^{n'-n} + \sum_{\tau=n}^{n'} (1-\kappa h)^{n'-n} \epsilon_{v,\tau}. \tag{B.1}$$

Taking expectations in Equation B.1, we have

$$E(v_{n'}|I_n^m) = (1-\kappa h)^{n'-n} E(v_n|I_n^m) + \bar{v}(1-(1-\kappa h)^{n'-n+1}). \tag{B.2}$$

Substituting into Equation 2.3, the price is equal to

$$\begin{aligned} p_n &= \sum_{n'=n}^{+\infty} (1-\exp(-rh)) \exp(-r(n'-n)h)((1-\kappa h)^{n'-n} E(v_n|I_n^m) \\ &+ \kappa h \bar{v}(1-(1-\kappa h)^{n'-n+1})) \\ &= \frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)} E(v_n|I_n^m) + \frac{\kappa h \bar{v}}{1-\exp(-rh)(1-\kappa h)}. \end{aligned} \tag{B.3}$$

Proof of Lemma 2.2.2: First, one can find that

$$\sum_{n'=n}^{+\infty} \exp(-r(n'-n)h)(\sum_{\tau=n}^{n'} x_\tau(v_{n'}-p_\tau)) = \tag{B.4}$$

$$\begin{aligned}
& x_n \sum_{\tau=n}^{+\infty}(\exp(-r(\tau-n)h)(v_\tau - p_n)) \\
+ \quad & x_{n+1} \sum_{\tau=n+1}^{+\infty} (\exp(-r(\tau-n)h)(v_\tau - p_{n+1})) \\
+ \quad & x_{n+2} \sum_{\tau=n+1}^{+\infty} (\exp(-r(\tau-n)h)(v_\tau - p_{n+2})) + ... \\
= \quad & x_n * (\sum_{\tau=n}^{+\infty} \exp(-r(\tau-n)h)v_\tau - \frac{p_n}{1-\exp(-rh)}) \\
+ \quad & x_{n+1} * (\sum_{\tau=n+1}^{+\infty} \exp(-r(\tau-n)h)v_\tau - \frac{\exp(-rh)p_n}{1-\exp(-rh)}) \\
+ \quad & x_{n+2} * (\sum_{\tau=n+2}^{+\infty} \exp(-r(\tau-n)h)v_\tau - \frac{\exp(-2rh)p_n}{1-\exp(-rh)}) + ...
\end{aligned} \tag{B.5}$$

Substituting Equation B.5 into Equation 2.6 and using the result in Equation B.1, we have

$$
\begin{aligned}
& E[\sum_{\tau=n}^{+\infty}(1-\exp(-rh))\exp(-r(n'-n)h)(\sum_{\tau=n}^{n'}x_\tau(v_{n'}-p_\tau))|I_{n-1}^i] = \qquad\qquad (B.6)\\
& E[x_n(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_n+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_n) \\
+ \; & \exp(-rh)x_{n+1}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n+1}+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n+1}) \\
+ \; & \exp(-2rh)x_{n+1}(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n+1}+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n+2}) \\
& ...|I_{n-1}^i] \\
= \; & E[\sum_{n'=n}^{+\infty}\exp(-r(n'-n)h)(\frac{(1-\exp(-rh))}{1-\exp(-rh)(1-\kappa h)}v_{n'}+\frac{\kappa h\bar{v}}{1-\exp(-rh)(1-\kappa h)}-p_{n'})|I_{n-1}^i].
\end{aligned}
$$

Proof of Lemma 3.1: At period n, under the market maker's belief, we have the joint distribution of $(v_{n-1}, x_n+z_n+u_n)'$ conditional on her information set I_{n-1}^m

$$
\begin{pmatrix} v_{n-1} \\ x_n+z_n+u_n \end{pmatrix} \sim N(\begin{pmatrix} \hat{v}_{n-1} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_v & \beta\Lambda_v+(\gamma+\theta)\Omega_v \\ \beta\Lambda_v+(\gamma+\theta)\Omega_v & \beta^2\Lambda_v+(\gamma+\theta)^2\Omega_v+\sigma_u^2h \end{pmatrix}). \tag{B.7}
$$

Then applying the projection theorem on equation (2.1), we have

$$
\frac{\lambda}{1-\kappa h} = \frac{\beta\Lambda_v+(\gamma+\theta)\Lambda_v}{\beta^2\Lambda_v+(\gamma+\theta)^2\Omega_v+\sigma_u^2h}. \tag{B.8}
$$

Applying the projection theorem to find the conditional variance, we have

$$var(\eta_n) = \Sigma_v - \frac{(\beta\Lambda_v + (\gamma+\theta)\Lambda_v)^2}{\beta^2\Lambda_v + (\gamma+\theta)^2\Omega_v + \sigma_u^2 h}. \tag{B.9}$$

From equations (2.1), market maker's posterior belief about v_n is

$$v_n = (1-\kappa h)\hat{v}_{n-1} + \kappa h\bar{v} + \lambda(x_n + z_n + u_n) + (1-\kappa h)\eta_n + \epsilon_{v,n}. \tag{B.10}$$

The stationary condition requires that

$$\Sigma_v = (1-\kappa h)^2 var(\eta_n) + \sigma_v^2 h, \tag{B.11}$$

which leads to Equation (2.18).

Proof of Lemma 2.3.2: Under the less informed trader's belief, we have the joint distribution of $(v_{n-1}, x_n + u_n)'$ conditional on her information set I_n^l

$$\begin{pmatrix} v_{n-1} \\ x_n + u_n \end{pmatrix} \sim N\left(\begin{pmatrix} v_{n-1}^* \\ \gamma(v_{n-1}^* - \hat{v}_{n-1}) \end{pmatrix}, \begin{pmatrix} \Lambda_v & \beta\Lambda_v \\ \beta\Lambda_v & \beta^2\Lambda_v + \sigma_u^2 h \end{pmatrix}\right). \tag{B.12}$$

Applying the projection theorem on equation (2.20), we have

$$\frac{\phi_x}{1-\kappa h} = \frac{\beta\Lambda_v}{\beta^2\Lambda_v + \sigma_u^2 h} \tag{B.13}$$

and

$$var(\varepsilon_n) = \Lambda_v - \frac{\beta^2\Lambda_v^2}{\beta^2\Lambda_v + \sigma_u^2 h}. \tag{B.14}$$

Since the less informed insider observes a signal in the form $s_n = \rho\epsilon_{v,n} + \sqrt{1-\rho^2}e_n$, then under the less informed insider's belief,

$$\epsilon_{v,n} = \rho s_n + \eta_{s,t} \tag{B.15}$$

and

$$var(\eta_{s,t}) = (1-\rho^2)\sigma_v^2 h. \tag{B.16}$$

Thus, the less less informed trader's posterior belief about v_n is

$$v_n = (1-\kappa h)v_{n-1}^* + \kappa h\bar{v} + \phi(x_n + u_n - \gamma(v_{n-1}^* - \hat{v}_{n-1})) + (1-\kappa h)\varepsilon_n + \rho s_n + \eta_{s,n}. \tag{B.17}$$

The steady state condition requires that

$$\Lambda_v = (1-\kappa h)^2(\Lambda_v - \frac{\beta^2\Lambda_v^2}{\beta^2\Lambda_v + \sigma_u^2 h}) + (1-\rho^2)\sigma_v^2 h. \tag{B.18}$$

Proof of Lemma 2.3.3: Using the results in section 2.3.2 and 2.3.3, we first compute the less informed trader's estimation error of v_n at period n

$$
\begin{aligned}
v_n - v_n^* &= (1-\kappa h)v_{n-1} + \kappa h\bar{v} + \epsilon_{v,n} - [(1-\kappa h)v_{n-1}^* + \kappa h\bar{v} \\
&+ \phi(x_n + u_n - \gamma(v_{n-1}^* - \hat{v}_{n-1})) + \rho s_n] \\
&= [1-\kappa h](v_{n-1} - v_{n-1}^*) - \phi(x_n + u_n) \\
&+ \phi\gamma(v_{n-1}^* - \hat{v}_{n-1}) + \epsilon_{v,n} - \rho s_n,
\end{aligned} \tag{B.19}
$$

then compute the market maker's estimation error relative to the less informed trader's,

$$
\begin{aligned}
v_n^* - \hat{v}_n &= (1-\kappa h)v_{n-1}^* + \kappa h\bar{v} + \phi(x_n + u_n - \gamma(v_{n-1}^* - \hat{v}_{n-1})) + \rho s_n \\
&- [(1-\kappa h)\hat{v}_{n-1} + \kappa h\bar{v} + \lambda(x_n + z_n + u_n)] \\
&= (1-\kappa h - \phi\gamma)(v_{n-1}^* - \hat{v}_{n-1}) \\
&+ (\phi - \lambda)(x_n + u_n) - \lambda z_n + \rho s_n,
\end{aligned} \tag{B.20}
$$

and finally the market maker's estimation error

$$
v_n - \hat{v}_n = (1-\kappa h)(v_{n-1} - \hat{v}_{n-1}) - \lambda(x_n + z_n + u_n) + \epsilon_{v,n}. \tag{B.21}
$$

Substituting for the above three equations into the Bellman equation (2.26), the more informed trader solves the following problem:

$$
\begin{aligned}
V(v_n, v_n^*, \hat{v}_n) &= max_{x_n}\{x_n[(1-\kappa h)(v_{n-1}-\hat{v}_{n-1}) - \lambda(x_n+z_n)] \\
&+ e^{-rh}[A((1-\kappa h)(v_{n-1}-v_{n-1}^*) - \phi x_n + \phi_x \gamma(v_{n-1}^* - \hat{v}_{n-1}))^2 \\
&+ A(\phi^2\sigma_u^2 h + (1-\rho^2)\sigma_v^2 h) \\
&+ B((1-\kappa h - \phi\gamma)(v_{n-1}^* - \hat{v}_{n-1}) + (\phi-\lambda)x_n - \lambda z_n)^2 \\
&+ B((\phi-\lambda)^2\sigma_u^2 h + \rho^2\sigma_v^2 h) \\
&+ C((1-\kappa h)(v_{n-1}-v_{n-1}^*) - \phi x_n + \phi\gamma(v_{n-1}^* - \hat{v}_{n-1})) \\
&\times ((1-\kappa h - \phi\gamma)(v_{n-1}^* - \hat{v}_{n-1}) + (\phi-\lambda)x_n - \lambda z_n) \\
&- C\phi(\phi-\lambda)\sigma_u^2 h + E]\}.
\end{aligned} \tag{B.22}
$$

After simplification, the first order condition of the better informed trader's value function with respect to x_n is

$$
\begin{aligned}
2\tau x_n &= [1-\kappa h - 2e^{-rh}A\phi(1-\kappa h) \\
&+ e^{-rh}C(\phi-\lambda)(1-\kappa h)](v_{n-1}-v_{n-1}^*) \\
&+ [1-\kappa h - \lambda\theta - 2e^{-rh}A\phi\gamma \\
&+ 2e^{-rh}B(\phi-\lambda)(1-\kappa h - \phi\gamma - \lambda\theta) \\
&+ e^{-rh}C(\phi-\lambda)\phi\gamma - e^{-rh}C\phi(1-\kappa h - \phi\gamma - \lambda\theta)](v_{n-1}^* - \hat{v}_{n-1})
\end{aligned} \tag{B.23}
$$

and the second order condition is

$$\tau = \lambda - e^{-rh}(A\phi^2 + B(\phi-\lambda)^2 - C\phi(\phi-\lambda)) > 0. \tag{B.24}$$

Rewriting the first order condition leads to the expression $x_n = \beta(v_n - v^*_{n-1}) + \gamma(v^*_{n-1} - \hat{v}_{n-1})$ with the β and γ defined in equations (2.27) and (2.29).

Substituting x_n back into the Bellman equation, and by matching the coefficient of $(v_n - v^*_{n-1})$ term, the $(v_n - v^*_{n-1})(v^*_{n-1} - \hat{v}_{n-1})$, the $(v^*_{n-1} - \hat{v}_{n-1})^2$ term and the constant term, we are able to setup the nonlinear equations (2.31 to 2.34) that determine A, B, C and E.

Proof of Lemma 2.3.4: We first substitute for expressions of $v^*_n - \hat{v}_n$ and $v_n - \hat{v}_n$ found in previous proof into the Bellman equation (2.35). The less informed trader then solves the following optimization problem:

$$\begin{aligned} V_2(v^*_{n-1}, \hat{v}_{n-1}) &= max_{z_n}\{z_n[(1-\kappa h - \lambda\gamma)(v^*_{n-1} - \hat{v}_{n-1}) - \lambda z_n] \\ &+ e^{-rh}F[(1-\kappa h - \lambda\gamma)(v^*_{n-1} - \hat{v}_{n-1}) - \lambda z_n]^2 \\ &+ e^{-rh}F[(\phi-\lambda)^2\beta^2\Lambda_g + (\phi_x - \lambda)^2\sigma_u^2 h \\ &+ \rho^2\sigma_g^2 h] + e^{-rh}G\}. \end{aligned} \tag{B.25}$$

After simplification, the first order condition of the less informed trader's value function with respect to z_n is

$$z_n = \frac{(1-2e^{-rh}\lambda F)(1-\kappa h - \lambda\gamma)}{2\lambda(1-e^{-rh}\lambda F)}(v^*_{n-1} - \hat{v}_{n-1}) \tag{B.26}$$

and the second order condition is

$$\lambda(1 - e^{-rh}\lambda F) > 0. \tag{B.27}$$

Since the optimal strategy is assumed to be $z_n = \theta(v_n^* - \hat{v}_n)$ which is proportional to $v_n^* - \hat{v}_n$, we have

$$\theta = \frac{(1 - 2e^{-rh}\lambda F)(1 - \kappa h - \lambda\gamma)}{2\lambda(1 - e^{-rh}\lambda F)}. \tag{B.28}$$

Substituting for $z_n = \theta(v_n^* - \hat{v}_n)$ back into the objective function and matching the coefficients of the quadratic term $(v_{n-1}^* - \hat{v}_{n-1})^2$ and the constant term, we are able to setup the nonlinear equations (2.37) and (2.39) which determine F and G.

Proof of Proposition 2.3.2:

To study the asymptotic properties of the equilibrium for small h, we make the following assumptions:

$$\lambda \simeq \lambda_0 - a\sqrt{h}, \tag{B.29}$$

$$\phi_x \simeq \phi_0 - a\sqrt{h}, \tag{B.30}$$

$$\Omega_v \simeq Oh, \tag{B.31}$$

$$\Lambda_v \simeq L\sqrt{h}, \tag{B.32}$$

$$\Sigma_v \simeq L\sqrt{h} + Oh, \tag{B.33}$$

$$\beta \simeq b\sqrt{h}, \tag{B.34}$$

$$\gamma \simeq \gamma_0 \tag{B.35}$$

$$\theta \simeq \theta_0 \tag{B.36}$$

Substituting the asymptotic forms of λ, ϕ, Σ_v Λ_v, Ω_v, and β into equation 2.17, we have

$$\lambda = \frac{(\gamma_0 + \theta_0)O + bL}{(\gamma_0 + \theta_0)^2 O + \sigma_u^2}, \tag{B.37}$$

into Equation 2.23, we have

$$\phi = \frac{bL}{\sigma_u^2}, \tag{B.38}$$

into Equation 2.36, we have

$$\theta_0 = \frac{(1 - 2\lambda F_0)(1 - \lambda\gamma_0)}{2\lambda(1 - \lambda F)}, \tag{B.39}$$

into Equation 2.37, we have

$$F = \theta_0(1 - \lambda(\gamma_0 + \theta_0)) + F_0(1 - \lambda(\gamma_0 + \theta_0))^2, \tag{B.40}$$

$$B_0 = \gamma_0(1 - \lambda(\gamma_0 + \theta_0)) + B_0(1 - \lambda(\gamma + \theta))^2, \tag{B.41}$$

and into Equation 2.29, we have

$$\gamma_0 = \frac{1 - \lambda\theta_0 - 2A_0\phi\gamma_0 + 2B(\phi - \lambda)(1 - \phi\gamma_0 - \lambda\theta) + C(\phi - \lambda)\phi\gamma_0 - C\phi(1 - \phi\gamma_0 - \lambda\theta_0)}{2\tau_0}, \tag{B.42}$$

where $\tau_0 = \lambda - \phi + A_0\phi^2 - B_0(\phi - \lambda)^2$.

Substituting the asymptotic forms into Equation 2.27, the constant term should equal to zero which leads to

$$1 - 2A_0\phi_0 - C_0(\phi_0 - \lambda_0) = 0, \tag{B.43}$$

and the $\sqrt{h}$ term should equal to $b\sqrt{h}$, which leads to

$$b - \frac{A_0 a}{\lambda_0 - \phi_0 + A_0\phi_0^2 - B_0(\phi_0 - \lambda_0)^2} = 0. \tag{B.44}$$

We then substitute the asymptotic expressions of β, λ, ϕ_x into equation 2.31, match the coefficient of the $\sqrt{h}$ term and reach the following equation:

$$-\lambda b^2 - (r + 2\kappa)A_0 + 2A_0 ab + A\phi^2 b^2 - \phi b^2 C_0(\phi - \lambda) = 0. \tag{B.45}$$

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