ABSTRACT

Title of Thesis: Regular Homomorphisms of Minimal Sets

Peter Stephen Shoenfeld, Doctor of Philosophy, 1974

Thesis directed by: Professor Joseph Auslander

The classification of minimal sets is a central theme in abstract topological dynamics. Recently this work has been strengthened and extended by consideration of homomorphisms.

Background material is presented in Chapter I. Given a flow on a compact Hausdorff space, the action extends naturally to the space of closed subsets, taken with the Hausdorff topology. These hyperspaces are discussed and used to give a new characterization of almost periodic homomorphisms.

Regular minimal sets may be described as minimal subsets of enveloping semigroups. Regular homomorphisms are defined in Chapter II by extending this notion to homomorphisms with minimal range. Several characterizations are obtained.

In Chapter III, some additional results on homomorphisms are obtained by relativizing enveloping semigroup notions.

In Veech's paper on point distal flows, hyperspaces are used to associate an almost one-to-one homomorphism with a given homomorphism of metric minimal sets. In Chapter IV, a non-metric generalization of this construction is studied in detail using the new notion of a highly proximal homomorphism. An abstract characterization is obtained, involving only the abstract properties of homomorphisms. A strengthened version of the Veech Structure Theorem for point distal flows is

proved.

In Chapter V, the work in the earlier chapters is applied to the study of homomorphisms for which the almost periodic elements of the associated hyperspace are all finite. In the metric case, this is equivalent to having at least one fiber finite. Strong results are obtained by first assuming regularity, and then assuming that the relative proximal relation is closed as well.

REGULAR HOMOMORPHISMS OF MINIMAL SETS

by
Peter Stephen Shoenfeld

Dissertation submitted to the Faculty of the Graduate School
of the University of Maryland in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
1974

ACKNOWLEDGEMENTS

I wish to thank Joseph Auslander for his patience, encouragement, and many valuable suggestions. I wish to thank my wife Janet for her understanding and support. Special thanks are also due to James DeVoe, my Section Chief at the National Bureau of Standards, for his encouragement and interest in the writing of this thesis. Finally, I wish to thank my father who first interested me in mathematics many years ago.

TABLE OF CONTENTS

Chapter		Page
I.	INTRODUCTION	1
	1. Preliminaries	1
	2. Hyperspaces	10
II.	REGULAR HOMOMORPHISMS	19
	1. Construction of the Regularizer	19
	2. Abstract Characterizations	22
	3. Automorphism Groups	33
	4. An Algebraic Characterization	39
	5. Products and Admissible Properties	41
III.	RELATIVE ENVELOPING SEMIGROUPS	51
IV.	X^* AND Y^*	57
	1. Open and Highly Proximal Homomorphisms	57
	2. Construction and Basic Properties of X^* and Y^*	67
	3. An Abstract Characterization	74
	4. Non-Metric Veech Structure Theorem	81
V.	GENERALIZED ALMOST FINITE HOMOMORPHISMS	89
	1. Definition and Basic Properties	89
	2. Regular Generalized Almost Finite Homomorphisms	94
	SELECTED BIBLIOGRAPHY	108

CHAPTER I

INTRODUCTION

1. Preliminaries.

This section is primarily a review of some of the material in [7].

A *transformation group*, or *flow*, (X,T), will consist of a jointly continuous action of the topological group T on the compact Hausdorff space X. The group T, with identity e, is assumed to be topologically discrete and will remain fixed throughout this paper, so we may write X instead of (X,T). If T acts on X via the map $F\colon X \times T \longrightarrow T$, we will write xt to denote $F(x,t)$.

A *point transitive flow*, (X,x) consists of a flow X with a distinguished point x which has dense orbit, i.e., $\overline{xT} = X$. A flow is said to be *minimal* if every point has dense orbit, or, equivalently, if it contains no proper closed invariant subset. Minimal flows are also referred to as *minimal sets*. Every flow contains minimal sets.

A *homomorphism*, or *extension*, of flows is a continuous, equivariant map. A homomorphism of point transitive flows is a homomorphism which preserves the distinguished point. A homomorphism whose range is minimal is always onto, and a homomorphism whose domain is point transitive is determined by its value at a single point.

A point $x \in X$ is said to be *almost periodic* if, given any neighborhood U of x, the set $A = \{t \in T \mid xt \in U\}$ is syndetic, i.e., there exists a compact set $K \subset T$ such that $AK = T$. A point is almost periodic if and only if its orbit closure is minimal.

The compact Hausdorff space X carries a natural uniformity whose indices are the neighborhoods of the diagonal in $X \times X$. Two points $x, x' \in X$ are said to be <u>proximal</u> if, given any index α, there exists a $t \in T$ such that $(x,x')t \in \alpha$. The set of proximal pairs in X is called the <u>proximal relation</u>. X is said to be <u>distal</u> if the proximal relation equals the diagonal and is said to be <u>proximal</u> if the proximal relation equals $X \times X$.

The points $x, x' \in X$ are said to be <u>regionally proximal</u> if there exist nets, $\langle x_n \rangle$ and $\langle x_n' \rangle$ in X, a net $\langle t_n \rangle$ in T, and a point $x'' \in X$ such that

$$x_n \longrightarrow x \qquad x_n t_n \longrightarrow x''$$
$$x_n' \longrightarrow x' \qquad x_n' t_n \longrightarrow x''$$

The relation thus determined on X is called the <u>regionally proximal relation</u>. X is said to be (uniformly) <u>almost periodic</u> if the regionally proximal relation equals the diagonal. X is almost periodic if and only if T determines an equicontinuous family of homeomorphisms of X.

A homomorphism $\pi: X \longrightarrow Y$ determines a closed, invariant, equivalence relation on X, called R_π where

$$R_\pi = \{(x,x') \in X \times X \mid \pi(x) = \pi(x')\}.$$

Conversely, a closed, invariant equivalence relation R on

X determines an epimorphism $\pi: X \to X/R$. Given $\pi: X \to Y$ we define the relative (to π) proximal relation on X to be the intersection of R_π with the proximal relation on X. A pair of points (x,x') in X are said to be relatively regionally proximal, or to belong to the relative (to π) regionally proximal relation if there exist nets $\langle x_n \rangle$ and $\langle x'_n \rangle$ in X, a net $\langle t_n \rangle$ in T, and a point $x'' \in X$ such that each $(x_n, x'_n) \in R_\pi$ and

$$\begin{array}{ll} x_n \longrightarrow x & x_n t_n \longrightarrow x'' \\ x'_n \longrightarrow x' & x'_n t_n \longrightarrow x'' \end{array}$$

Note that this is not the same as intersecting the regionally proximal relation with R_π. The homomorphism $\pi: X \to Y$ is said to be distal if the relative proximal relation equals the diagonal, proximal if the relative proximal relation equals R_π, and almost periodic if the relative regionally proximal relation equals the diagonal. Note that these definitions correspond to our definitions for the absolute case when applied to the unique homomorphism $\pi: X \to 1$, where 1 is the one-point transformation group.

Given a family $\{X_i\}$ of flows we define the product $\prod_i \{X_i\}$ in the natural way; i.e., $(x_1, x_2, \cdots)t = (x_1 t, x_2 t, \cdots)$. Given a family $\{(X_i, x_i)\}$ of point transitive flows, we define its product to be the point transitive flow $(\langle x_i \rangle, \overline{\langle x_i \rangle T}) \subset \prod_i \{X_i\}$. Given a family of homomorphisms $\{\pi_i : X_i \to Y\}$, where Y is fixed, we define the product $\prod_i (\pi_i : X_i \to Y)$ to be the homomorphism

$\pi\colon X \longrightarrow Y$ where $X = \{\langle x_i \rangle \in \prod_i \{X_i\} \mid \pi_i(x_i)$ the same for all $i\}$ and π is defined by $\pi(\langle x_i \rangle) = y$ if and only if $\pi_i(x_i) = y$ for each i. It is easy to see that X is a transformation group and that π is a homomorphism. $\pi\colon X \longrightarrow Y$, as constructed, is also called the <u>pullback</u> of the family $\{\pi_i\colon X_i \longrightarrow Y\}$. These are all true products in the sense of category theory.

Given a transformation group (X,T), we may regard T, or an appropriate quotient thereof, as a set of self-homeomorphisms of X. We define $E(X)$, the <u>enveloping semigroup</u> of X to be the closure of T in X^X, taken with the product topology. $E(X)$ is at once a transformation group and a sub-semigroup of X^X. Viewing $E(X)$ as a collection of functions, it has a natural action on X. We let xp denote the point in X thus obtained from $x \in X$ and $p \in E(X)$. Some important properties of $E(X)$ are summarized in the following lemma.

<u>Lemma 1.1.1</u>. <u>Suppose</u> $E(X)$ <u>is the enveloping semigroup of</u> X. <u>Then</u>

(*i*) <u>Left multiplication by an element of</u> $E(X)$ <u>is continuous on</u> $E(X)$.

(*ii*) <u>Right multiplication by an element of</u> T <u>is continuous on</u> $E(X)$.

(*iii*) <u>The identity</u>, e, <u>in</u> $E(X)$ <u>has dense orbit</u>.

(*iv*) <u>The maps</u> $\theta_x\colon E(X) \longrightarrow X$ <u>defined by</u> $\theta_x(p) = xp$ <u>are homomorphisms with range</u> $\overline{xT}$.

(*v*) *Given an epimorphism* $\pi: X \longrightarrow Y$ *there exists a unique epimorphism* $\overline{\pi}: E(X) \longrightarrow E(Y)$ *such that for each* $x \in X$, $\pi \circ \theta_x = \theta_{\pi(x)} \circ \overline{\pi}$.

(*vi*) *For any set* I, $E(X^I)$, $E(X)$, *and* $E(E(X))$ *are all isomorphic, both as semigroups and as point transitive transformation groups with base point* e.

If E is some enveloping semigroup, and there exists a homomorphism $\theta: (E,e) \longrightarrow (E(X), e)$ we say that E is an *enveloping semigroup for* X. If such a homomorphism exists, it must be unique, and, given $x \in X$ and $p \in E$ we may write xp to mean $x\theta(p)$ unambiguously. We make this definition to evade confusion concerning which enveloping semigroup we're talking about in situations where such a distinction is irrelevant. Loosely speaking, any enveloping semigroup for X acts on X in the same way that $E(X)$ does.

Lemma 1.1.2. *If* (X,x) *and* (Y,y) *are point transitive flows, and* E *is an enveloping semigroup for both* X *and* Y, *there exists a (unique) homomorphism* $\psi: (X,x) \longrightarrow (Y,y)$ *if and only if* $xp = xq$ *for* $p,q \in E$ *implies* $yp = yq$.

The minimal right ideals of $E(X)$, considered as a semigroup, coincide with the minimal sets of $E(X)$, considered as a transformation group. Given a minimal right ideal I in some enveloping semigroup, we will let

$J(I)$ denote the set of idempotent elements in I.

Lemma 1.1.3. *Let E be an enveloping semigroup for X, with minimal right ideals I, I', I'', etc. Then*

(*i*) $J(I)$ *is non-empty, for each minimal right ideal* I.

(*ii*) $up = p$ *whenever* $p \in I$ *and* $u \in J(I)$.

(*iii*) Iu *is a group with identity* u *for each* $u \in J(I)$.

(*iv*) *The collection* $\{Iu \mid u \in J(I)\}$ *partitions* I.

(*v*) *Given* $u \in J(I)$, *there exists a unique* $u' \in J(I')$ *such that* $uu' = u$ *and* $u'u = u'$. *We say that* u *and* u' *are equivalent idempotents and the relation thus defined is actually an equivalence relation.*

(*vi*) *If* u *and* u' *are equivalent idempotents in* I *and* I', *respectively,* (I,u) *and* (I',u') *are isomorphic as point-transitive transformation groups.*

(*vii*) *Given* $x \in X$, *the following conditions are equivalent:*

(a) x *is an almost periodic point;*

(b) $\overline{xT} = xI$;

(c) $x = xu$ *for some* $u \in J(I)$;

(*viii*) *Given* $x \in X$ *and a minimal subset* K *of* $\overline{xT}$, *there exists a minimal right ideal* I' *such that* $K = xI'$.

Given an enveloping semi-group, E, for X, and an element $u \in E$ we say that u is a *minimal idempotent* if $u \in J(I)$ for some minimal right ideal I in E. We have the following characterization of the proximal relation:

Lemma 1.1.4. *Suppose E is an enveloping semigroup for X. Then for any points $x,x' \in X$ (i) and (ii) are equivalent:*

(*i*) *x and x' are proximal.*

(*ii*) *There exists a minimal right ideal I in E such that $xp = x'p$ for every $p \in I$.*

Moreover, if X is minimal, (i) and (ii) are equivalent to:

(*iii*) *There exists a minimal idempotent u such that $x'u = x$.*

It is also easy to see that if $x,x' \in X$ and the pair (x,x') is both proximal and almost periodic, then $x = x'$.

Let βT denote the Stone-Cěch compactification of T. Since T is discrete, βT may be regarded as the set of ultrafilters on T. βT is a compact Hausdorff space with the discrete space T a dense subset and any map from T to a compact Hausdorff space extends uniquely to βT. Regarding right multiplication by a fixed element of T as a map from T to βT, we may extend that map to βT. This defines a point-transitive action of T on βT, with base point e. For $p \in \beta T$, $L_p(t) = pt$ then defines another map from T to βT which we extend to βT to get a left-continuous semigroup structure on βT. It may be shown that $(\beta T,e)$ is the essentially unique universally repelling object in the category of point-transitive transformation groups with acting group T. From this it follows that βT is its own enveloping semigroup. Hence the minimal sets in βT are all isomorphic and any one of these may be regarded as a universal minimal set, i.e., as an essentially unique, universally repelling object in the category of minimal transformation groups. We will single out one of these minimal sets and call it M. It is also clear that βT is an enveloping semigroup for X, whenever X is a transformation group with acting group T.

The set of idempotent elements in M, regarding M as a semigroup, will be denoted by J. Given a point y in some transformation group Y, we will let $J(y)$ denote $\{u \in J \mid yu = y\}$.

Given a compact Hausdorff space X, $C(X)$ denotes the Banach algebra of real-valued continuous functions on X with the sup norm. Given two such spaces X and Y there is a bijective correspondence between the set of continuous surjections $\psi: X \longrightarrow Y$ and the set of monomorphisms $\psi^*: C(Y) \longrightarrow C(X)$ such that $\psi^*(f)(x) = f \circ \psi(x)$ for each $x \in X$ and $f \in C(Y)$. This gives rise to a correspondence between point transitive flow and the so called T-subalgebras of βT which we explicate below.

Given $f \in C(\beta T)$ and $t \in T$ we define the function $tf \in C(\beta T)$ by $(tf)(x) = f(xt)$ for all $x \in \beta T$. Given $p \in \beta T$ we define $fp \in C(\beta T)$ by $(fp)(x) = f(px)$ for all $x \in \beta T$. A subalgebra, $\mathcal{A}$ of $C(\beta T)$ is called a

T-subalgebra if it is norm closed and if $tf \in \mathcal{A}$ whenever $f \in \mathcal{A}$ and $t \in T$.

Given a point transitive flow (X,x), there is a canonical homomorphism $\psi\colon (\beta T,e) \longrightarrow (X,x)$. We define the T-subalgebra, $\mathcal{A}$, associated with (X,x) by $\mathcal{A} = \psi^*(C(X))$. Note that if (X,x) and (X',x') are isomorphic they give rise to the same associated T-subalgebra.

Next we will see how to construct a point transitive flow $(|\mathcal{A}|, e|\mathcal{A})$ from a given T-subalgebra $\mathcal{A}$ in such a way that if $\mathcal{A}$ is associated with (X,x), then (X,x) and $(|\mathcal{A}|, e|a)$ are isomorphic.

If $p \in \beta T$ we may define an endomorphism of $C(\beta T)$, which we also call p, by $p(f) = fp$. Given a T-subalgebra $\mathcal{A}$, we let $|\mathcal{A}| = \{p|\mathcal{A}\colon p \in \beta T\}$ and define $(p|\mathcal{A})t = pt|\mathcal{A}$ for $t \in T$, where $p|\mathcal{A}$ denotes the restriction to $\mathcal{A}$ of the endomorphism given by p, so that $p|\mathcal{A} = q|\mathcal{A}$ iff $fp = fq$ for all $f \in \mathcal{A}$. $|\mathcal{A}|$ gets its topology as a quotient of βT. Using the Stone-Weierstrauss Theorem, it may be shown that (X,x) and $(|\mathcal{A}|, e|\mathcal{A})$ are isomorphic.

The following lemma is used to characterize minimality algebraically.

<u>Lemma 1.1.5</u>. <u>Suppose the algebra</u> $\mathcal{A}$ <u>is associated with the point transitive transformation group</u> (X,x) <u>and suppose</u> $u \in J$. <u>Then</u> $xu = u$ <u>if and only if</u> $fu = f$ <u>for all</u> $f \in \mathcal{A}$.

If $\{\mathcal{A}_i\}$ is a family of T-subalgebras, $\cap_i \{\mathcal{A}_i\}$ is itself a T-subalgebra, and $\cup_i \{\mathcal{A}_i\}$ generates a T-subalgebra, which we denote by $\vee_i \{\mathcal{A}_i\}$.

Lemma 1.1.6. *Suppose we have a family of T-subalgebras* $\mathcal{A}_i$, *associated with point transitive flows* (X_i,x_i). *Then*

(*i*) *There exists a homomorphism* $\psi\colon (X_i,x_i) \longrightarrow (X_j,x_j)$ *if and only if* $\mathcal{A}_j \subset \mathcal{A}_i$.

(*ii*) $\vee_i \{\mathcal{A}_i\}$ *is associated with the product* $(X,\langle x_i\rangle)$, *where* X *is the orbit closure of* $\langle x_i\rangle$ *in* $\prod_i \{X_i\}$.

(*iii*) $\cap_i \{\mathcal{A}_i\}$ *is associated with the essentially unique, universally repelling object in the category of all point-transitive transformation groups which are homomorphic images of all the* (X_i,x_i), *i.e.*, (X,x) *is associated with* $\cap_i \{\mathcal{A}_i\}$ *if and only if there exist homomorphisms* $\psi_i\colon (X_i,x_i) \longrightarrow (X,x)$ *and there exists a homomorphism* $\psi\colon (X,x) \longrightarrow (X',x')$ *whenever* (X',x') *shares this property*.

The following lemma takes care of change of basepoint.

Lemma 1.1.7. *If the T-subalgebra* $\mathcal{A}$ *is associated with* (X,x) *and* $p \in \beta T$, *then* (X,xp) *is associated with the T-subalgebra* $\mathcal{A}p$, *where* $\mathcal{A}p = \{fp \mid f \in \mathcal{A}\}$.

2. Hyperspaces.

Given a compact Hausdorff space X, the hyperspace 2^X is the space of all closed, non-empty subsets of X with the Hausdorff topology. This can be described as follows: For each index α in the natural uniformity of X we define an index α' on 2^X by

$$\alpha' \equiv \{(A,B) \mid \alpha(a) \cap B \neq \phi \text{ and } \alpha(b) \cap A \neq \phi \text{ for each } a \in A \text{ and } b \in B\}.$$

The set of all such α' comprises a uniformity which yields a compact Hausdorff topology on 2^X. If X is metrizable so is 2^X. Convergence in 2^X may be described as follows: Suppose $A_i \longrightarrow A$. Then $x \in A$ if and only if there exist points $x_i \in A_i$ such that $x_j \longrightarrow x$ for some subnet $\langle x_j \rangle$ of $\langle x_i \rangle$. If X is a flow (with acting group T) then so is 2^X, where the action on 2^X is given by $At = \{xt \mid x \in A\}$. If $\pi: X \longrightarrow Y$ is a homomorphism we will also be interested in the sub-flow 2^π, where $2^\pi = \{A \in 2^X \mid A \subset \pi^{-1}(y) \text{ for some } y \in Y\}$. These flows were heavily used and systematically studied in [11].

X is naturally imbedded in 2^X, so any enveloping semigroup for 2^X is also an enveloping semigroup for X. If $A \in 2^X$ and p is an element of an enveloping semigroup for 2^X we use the notation $A \circ p$ to denote the action of p on A within 2^X and the notation Ap to denote the set $\{xp \mid x \in A\}$, which does not in general belong to 2^X. These two sets are not generally the same. However, we have the following:

Lemma 1.2.1. *Suppose A and B are elements of 2^X, p is an element of some enveloping semigroup for 2^X, $\langle t_n \rangle$ is a net in T such that $t_n \longrightarrow p$, and $t \in T$. Then:*

(*i*) $A \subset B \Rightarrow Ap \subset Bp$ *and* $A \circ p \subset B \circ p$.

(*ii*) $x \in A \circ p \Longleftrightarrow x_m t_m \longrightarrow x$, *where* $\langle x_m \rangle$ *is some net in* A *and* $\langle t_m \rangle$ *is some subnet of* $\langle t_n \rangle$.

(*iii*) $At = A \circ t$.

(*iv*) $Ap \subset A \circ p$.

(*v*) *If* A *is finite*, $A \circ p = Ap$.

Proof: (*i*) is entirely obvious.

(*ii*) follows from our description of convergence in 2^X.

(*iii*) is just the definition of the action of T on 2^X.

(*iv*) Suppose $x \in A$. Pick a net $\langle t_n \rangle$ in T with $t_n \longrightarrow p$, take $x_n = x$ for all n and apply (*ii*) noting that $x_n t_n \longrightarrow xp$.

(*v*) Suppose $A = \{x_1, x_2, \cdots, x_k\}$, with k finite and suppose $x \in A \circ p$. Then there exist nets $\langle t_m \rangle$ in T and $\langle x_m \rangle$ in A such that $t_m \longrightarrow p$ and $x_m t_m \longrightarrow x$. However, since A is finite, $\langle x_m \rangle$ must have a constant subnet, $\langle x_j \rangle$, with each $x_j = x_i$ for some fixed i, $1 \le i \le k$. Thus $x = \lim_j x_j t_j = \lim_j x_i t_j = x_i p$. Therefore $x \in Ap$. ||

We may use (*ii*) of the preceding lemma to define $A \circ p$ when $p \in \beta T$ and A is an arbitrary subset of X (not necessarily closed). The following lemma is clear.

Lemma 1.2.2. *Suppose* $A \subset X$ *and* $p \in \beta T$. *Then* $A \circ p = \bar{A} \circ p$.

The following facts will also prove useful:

Lemma 1.2.3. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism,* E *is an enveloping semigroup for* 2^X *and* I *is a minimal set in* E. *Then*

(*i*) $\pi(A)\circ p = \pi(A\circ p)$ *for all* $A \in 2^X$, $p \in E$.

(*ii*) $\pi^{-1}(B)\circ p \subset \pi^{-1}(B\circ p)$ *for all* $B \in 2^Y$, $p \in E$.

(*iii*) $\pi^{-1}(y)\circ p \subset \pi^{-1}(yp)$ *for all* $y \in Y$.

(*iv*) *Suppose* y *is an almost periodic point in* Y, $p \in I$, *and* $r \in I$. *Then* $\pi^{-1}(y)\circ(pr) = \pi^{-1}(yp)\circ r$.

Proof: (*i*) Suppose $y \in \pi(A)\circ p$. Then there exists nets $\langle y_n\rangle$ in $\pi(A)$ and $\langle t_n\rangle$ in T such that $t_n \longrightarrow p$ and $y_n t_n \longrightarrow y$. Choose a net $\langle x_n\rangle$ in A such that $\pi(x_n) = y_n$ for each n. Taking subnets if necessary we assume $x_n t_n \longrightarrow x$, for some $x \in X$. Then $x \in A\circ p$ and $\pi(x) = y$.

Now suppose $y \in \pi(A\circ p)$, so that $y = \pi(x)$ for some $x \in A\circ p$. Choose nets $\langle x_n\rangle$ in A and $\langle t_n\rangle$ in T so $t_n \longrightarrow p$ and $x_n t_n \longrightarrow x$. then $\pi(x_n)t_n \longrightarrow y$, so $y \in \pi(A)\circ p$.

(*ii*) This follows immediately from (*i*).

(*iii*) This follows from (*ii*) above and (*v*) of Lemma 1.2.1.

(*iv*) $\pi^{-1}(y)\circ(pr) = (\pi^{-1}(y)\circ p)\circ r \subset \pi^{-1}(yp)\circ r$ by (*iii*) and (*i*) of Lemma 1.2.1. By Lemma 1.1.3 we can find $u \in J(I)$ and $q \in I$ such that $yu = y$ and $pqu = qpu = u$. Then

$$\pi^{-1}(yp)\circ r \;=\; \pi^{-1}(yp)\circ(ur) \;=\; \pi^{-1}(yp)\circ(qpur)$$

$$= \pi^{-1}(yp)\circ(qupur) \subset \pi^{-1}(ypqu)\circ(pur)$$

(by (*iii*) and (*i*) of Lemma 1.2.1)

$$= \pi^{-1}(yu)\circ(pur) \;=\; \pi^{-1}(y)\circ(pr). \qquad ||$$

We will illustrate the power of hyperspace methods by characterizing almost periodicity of a homomorphism π of minimal sets in terms of the transformation group 2^{π}. A similar result was obtained earlier by Glasner, using different techniques.

We will need the following well-known lemma.

Lemma 1.2.4. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism of minimal sets,* S *is a dense subset of* R_{π}, *and* $x, x', x'' \in X$. *Then* x *and* x' *are relatively regionally proximal if and only if there exist nets* $\langle x_n \rangle$, $\langle x_n' \rangle$ *in* X, *and* $\langle t_n \rangle$ *in* T *such that each* $(x_n, x_n') \in S$ *and*

$$x_n \longrightarrow x \qquad x_n t_n \longrightarrow x''$$

$$x_n' \longrightarrow x' \qquad x_n' t_n \longrightarrow x''$$

Proof: The minimality allows us to specify x''. A uniform space argument shows that we can take the pairs (x_n, x_n') to be in S. $\qquad ||$

Given the homomorphism $\pi: X \longrightarrow Y$, we let $\bar{\pi}: 2^{\pi} \longrightarrow Y$ be the homomorphism defined by $\bar{\pi}(A) = y \Longleftrightarrow A \subset \pi^{-1}(y)$. We have the following lemma.

Lemma 1.2.5. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism with* Y *minimal and* $\overline{\pi}$ *distal and we have* $x, x' \in X$, $y \in Y$, $p \in \beta T$, *and nets* $\langle x_n \rangle$ *in* $\pi^{-1}(y)$ *and* $\langle t_n \rangle$ *in* T *such that* $x_n \longrightarrow x$, $t_n \longrightarrow p$, *and* $x_n t_n \longrightarrow x'$. *Then* $x' = xp$.

Proof: Let $A_n = \{x_{n'} \mid n' \geq n\}$ for each n. Pick $u \in J(y)$. Then $A \circ u = A$ for each closed $A \subset \pi^{-1}(y)$, since $\overline{\pi}$ is distal, and, in particular, $A_n \circ u = \overline{A}_n \circ u = \overline{A}_n$ for each n. Also $xu = u$. Clearly, $\bigcap_n \{\overline{A}_n\} = \{x\}$ and $x' \in \bigcap_n \{\overline{A}_n \circ p\}$. Pick $q \in M$ so $q = qu$ and $upq = qpu = u$. Now

$$\begin{aligned}\{x\} &= \bigcap_n \{\overline{A}_n\} = \bigcap_n \{\overline{A}_n \circ (upq)\} \\ &= \bigcap_n \{(\overline{A}_n \circ p) \circ q\} \supset (\bigcap_n \{\overline{A}_n \circ p\}) \circ q \supset \{x'\} \circ q = \{x'q\}.\end{aligned}$$

Therefore $x = x'q$. Pick $v \in J$ such that $upv = up$. Let $y' = \pi(x')$. Then $y' = \pi(x') = \pi(\lim x_n t_n) = \lim(\pi(x_n t_n)) = \lim yt_n = yp = yup = yupv = y'v$. We have $x' = x'v$ also, since $\overline{\pi}$ is distal. Finally, $x' = x'v = x'uv = x'qpuv = xpuv = xupv = xup = xp$. $||$

Theorem 1.2.6. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism of minimal sets. Then the following conditions are equivalent:*

(*i*) π *is almost periodic*.

(*ii*) *Each element of* 2^{π} *is almost periodic*.

(*iii*) $\overline{\pi}$ *is distal*.

(*iv*) $\overline{\pi}$ *is almost periodic*.

Proof *(iv)* $\Longrightarrow$ *(iii)* $\Longrightarrow$ *(ii)* is clear.

(i) $\Longrightarrow$ *(iv)*. Suppose A and A' are elements of 2^π which are relatively regionally proximal (relative to $\bar{\pi}$). Then there exists $B \in 2^\pi$ and nets $\langle A_n \rangle$ and $\langle A'_n \rangle$ in 2^π, $\langle y_n \rangle$ in Y, and $\langle t_n \rangle$ in T such that $\bar{\pi}(A_n) = \bar{\pi}(A'_n) = y_n$ for each n and

$$A_n \longrightarrow A \qquad A_n t_n \longrightarrow B$$
$$A'_n \longrightarrow A' \qquad A'_n t_n \longrightarrow B.$$

We must show $A = A'$. Consider an arbitrary $x \in A$. By symmetry, it will suffice to show $x \in A'$. Taking subnets if necessary, there exists $x_n \in A_n$, for each n, such that $x_n \longrightarrow x$. Taking a subnet again, $x_n t_n \longrightarrow x_0$ for some $x_0 \in B$. Taking subnets two more times, we can find points $x'_n \in A'_n$ such that $x'_n t_n \longrightarrow x_0$ and a point $x' \in A'$ such that $x'_n \longrightarrow x'$. We now have $\pi(x_n) = \pi(x'_n) = y_n$, for each n, and

$$x_n \longrightarrow x \qquad x_n t_n \longrightarrow x_0$$
$$x'_n \longrightarrow x' \qquad x'_n t_n \longrightarrow x_0 .$$

Since π is assumed almost periodic, we have $x = x'$. Therefore $x \in A'$ as required.

(ii) $\Longrightarrow$ *(iii)*. Consider $y \in Y$ and $u \in J(y)$. Given $A \in 2^\pi$ with $\bar{\pi}(A) = y$, it will suffice to show that $A \circ u = A$, since we'll then have that every pair of elements in $\bar{\pi}^{-1}(y)$ is almost periodic.

Clearly, the element-wise almost periodicity of 2^π implies that π is distal which in turn implies that $Bw = B$

whenever $B \subset \pi^{-1}(y)$ and $w \in J(y)$.

Thus

$$A = Au \subset A \circ u.$$

Since 2^{π} is element-wise almost periodic, we have $A \circ v = A$ for some $v \in J(y)$. Therefore

$$A \circ u = (A \circ u)v \subset (A \circ u) \circ v = A \circ (uv) = A \circ v = A$$

and we're done.

(*iii*) ⟹ (*i*). Suppose x and x' are relatively regionally proximal, with $\pi(x) = \pi(x') = y$. We must show $x = x'$. Now π is distal since $\bar{\pi}$ is, and π can be obtained by restricting $\bar{\pi}$. Therefore π is an open map [see 4]. Therefore, for every $y' \in Y$, there exists a net $\langle t_n \rangle$ in T such that $\pi^{-1}(y)t_n \longrightarrow \pi^{-1}(y')$ (see Lemma 4.1.1). As a result, the set $(\pi^{-1}(y) \times \pi^{-1}(y))T$ is dense in R_{π}. Applying Lemma 1.2.4 we see that there exists nets $\langle x_n \rangle$ and $\langle x_n' \rangle$ in $\pi^{-1}(y)$ and $\langle t_n \rangle$ and $\langle s_n \rangle$ in T such that

(a) $x_n t_n \longrightarrow x$ (b) $x_n t_n s_n \longrightarrow x$

(c) $x_n' t_n \longrightarrow x'$ (d) $x_n' t_n s_n \longrightarrow x$.

Taking subnets, we can find points x_1 and x_1' in $\pi^{-1}(y)$ and elements $p, r \in \beta T$ such that $t_n s_n \longrightarrow p$, $t_n \longrightarrow r$ and

(e) $x_n \longrightarrow x_1$ (f) $x_n' \longrightarrow x_1'$.

Applying Lemma 1.2.5 to (b) and (e), and (d) and (f) we get that $x_1 p = x = x_1' p$. Also $\pi(x_1) = \pi(x_1') = y$ and π is distal so $x_1 = x_1'$. Applying Lemma 1.2.5 to (a) and (e), and (c) and (f) gives $x = x_1 r$ and $x' = x_1' r$. Therefore $x = x_1 r = x_1' r = x'$, completing the proof. $||$

CHAPTER II

REGULAR HOMOMORPHISMS

Regular minimal sets were first studied by Auslander in [3]. A minimal set is said to be regular if it's isomorphic to a minimal right ideal in some enveloping semigroup. We call Z the regularizer of X if Z is isomorphic to the essentially unique minimal right ideal in the enveloping semigroup of X.

We extend these notions to homomorphisms.

1. Construction of the Regularizer.

In this section we will work with a fixed homomorphism $\pi: X \longrightarrow Y$, where Y is minimal. We will construct a minimal set N and a homomorphism $\bar{\pi}: N \longrightarrow Y$ which we call the regularizer of π.

Suppose $y \in Y$. Then $X^{\pi^{-1}(y)}$ is a transformation group whose elements are functions from $\pi^{-1}(y)$ to X.

Definition 2.1.1. Define $z_y \in X^{\pi^{-1}(y)}$ by $z_y(x) = x$ for all $x \in \pi^{-1}(y)$. Let $E(\pi,y)$ be the orbit closure of z_y, i.e., $E(\pi,y) = \overline{z_y T} \subset X^{\pi^{-1}(y)}$.

Note that if Y is a singleton $\{y\}$, $E(\pi,y)$ is just the enveloping semigroup of X, considered as a transformation group.

Definition 2.1.2. Let $\bar{\pi}_y: E(\pi,y) \longrightarrow Y$ be the unique homomorphism with $\bar{\pi}_y(z_y) = y$.

We will show that the minimal sets of $E(\pi,y)$ are isomorphic and independent of the choice of y. N will be defined to be the essentially unique minimal set thus defined.

Lemma 2.1.3. *Suppose* $y \in Y$. *Then* $E(X)$ *is an enveloping semigroup for* $E(\pi,y)$.

Proof: This follows from the fact that $E(\pi,y) \subset X^{\pi^{-1}(y)}$ and $E(X^{\pi^{-1}(y)}) \simeq E(X)$. $||$

Lemma 2.1.4. *Suppose that* $y \in Y$ *and that* N *and* N' *are minimal subsets of* $E(\pi,y)$. *Then there is an isomorphism* $\varphi\colon N \longrightarrow N'$ *such that* $(\bar{\pi}_y|N')\circ\varphi = \bar{\pi}_y|N$.

Proof: By *(viii)* of Lemma 1.1.3 we can find minimal ideals I and I' in $E(X)$ such that $N = z_y I$ and $N' = z_y I'$. We choose a minimal idempotent u in $J(I)$ so that $yu = y$. Let u' be an equivalent idempotent in I'. Then $yu' = yuu' = yu = y$. We wish to define a homomorphism $\varphi\colon N \longrightarrow N'$ with $\varphi(z_y u) = z_y u'$. By Lemma 1.1.2, we need only check that given elements p and q in $E(X)$ with $(z_y u)p = (z_y u)q$, we also have $(z_y u')p = (z_y u')q$. Consider such p and q. Since $z_y up = z_y uq$, we have $xup = xuq$ for all $x \in \pi^{-1}(y)$. For $x \in \pi^{-1}(y)$, we have $xu' \in \pi^{-1}(y)$ also, since $yu' = y$. Thus $xu'p = xu'up = xu'uq = xu'q$ for all $x \in \pi^{-1}(y)$. Thus $z_y u'p = z_y u'q$ and φ is well-defined. Reversing this argument, there exists $\psi\colon N' \to N$ with $\psi(z_y u') = z_y u$. Since a homomorphism of minimal sets is determined by its value at one point, this proves that φ

is an isomorphism. $\bar{\pi}_y \circ \varphi(z_y u) = y = \bar{\pi}_y(z_y u')$ so $(\bar{\pi}_y | N') \circ \varphi = \bar{\pi}_y | N$. ||

<u>Lemma 2.1.5</u>. <u>For any two points $y, y' \in Y$, there exist minimal sets $N \in E(\pi,y)$ and $N' \in E(\pi,y')$ and an isomorphism $\psi: N' \longrightarrow N$ such that $(\bar{\pi}_y | N) \circ \psi = \bar{\pi}_{y'} | N'$.</u>

<u>Proof</u>: Let I be a minimal ideal in $E(X)$ again and pick minimal idempotents $u,u' \in J(I)$ such that $yu = y$ and $y'u' = y'$. Since Y is minimal we can find $p \in I$ with $yp = y'$. Then Range $(z_y p) = \pi^{-1}(y)p \subset \pi^{-1}(yp) =$ Range $(z_{y'})$. Thus, given elements $q,r \in E(X)$ with $z_{y'}q = z_{y'}r$, it follows that $z_y pq = z_y pr$. Therefore we can define a homomorphism $\psi: E(\pi,y') \longrightarrow E(\pi,y)$ with $\psi(z_{y'}) = z_y p$. $\bar{\pi}_y \circ \psi(z_{y'}) = \bar{\pi}_y(z_y p) = yp = y' = \bar{\pi}_{y'}(z_{y'})$ so $\bar{\pi}_y \circ \psi = \bar{\pi}_{y'}$.

We now let $N' = z_{y'}I$ and $N = \psi(N')$, recalling that a homomorphic image of a minimal set is necessarily minimal. If we can show that $\psi | N'$ is one-to-one we'll have the desired isomorphism. Suppose $\psi(z_{y'}r) = \psi(z_{y'}s)$ for some $r,s \in I$. Then $z_y pr = z_y ps$ and we must show $z_{y'}r = z_{y'}s$. In other words, we need $xr = xs$, for all $x \in \pi^{-1}(y')$. We pick $q \in Iu$ so that $q = qu$ and $pq = qpu = u$. Consider $x \in \pi^{-1}(y')$. $\pi(xq) = y'q = ypq = yu = y$. Thus $xq \in \pi^{-1}(y)$ and we have $xqpr = xqps$, since $z_y pr = z_y ps$. Finally, $xr = xur = xqpur = xqpr = xqps = xqpus = xus = xs$. ||

<u>Theorem 2.1.6</u>. <u>Suppose N and N' are minimal subsets of $E(\pi,y)$ and $E(\pi,y')$ respectively. Then there exists an isomorphism $\varphi: N \longrightarrow N'$ such that $(\bar{\pi}_{y'} | N') \circ \varphi = \bar{\pi}_y | N$.</u>

$$\begin{tikzcd} N \arrow[rr, "\varphi"] \arrow[dr, "\bar{\pi}_y|N"'] & & N' \arrow[dl, "\bar{\pi}_{y'}|N'"] \\ & Y & \end{tikzcd}$$

Proof: Just combine the preceding two lemmas.

If $\psi: W \to Z$ and $\psi': W' \to Z$ are transformation group homomorphisms with the same range, we say that ψ and ψ' are isomorphic if there exists a transformation group isomorphism $\theta: W' \to W$ such that $\psi \circ \theta = \psi'$. Thus Theorem 2.1.6 defines an essentially (up to isomorphism) unique homomorphism which we call $\bar{\pi}: N \to Y$.

Definition 2.1.7. Given a homomorphism $\pi: X \to Y$, with Y minimal, we call the homomorphism $\bar{\pi}: N \to Y$ the regularizer of π and we say that $\bar{\pi}$ is a regular homomorphism.

Thus a homomorphism is regular if and only if it's isomorphic to the regularizer of some homomorphism.

2. Abstract Characterizations.

In this section we obtain an abstract characterization of the regularizer from which other characterizations follow. In this section $\pi: X \to Y$ is a fixed homomorphism with Y minimal once again and $\bar{\pi}: N \to Y$ is the regularizer of π as constructed in the previous section. Recall that M is the universal minimal set.

Definition 2.2.1. We say that a homomorphism $\psi: Z \longrightarrow Y$, Z and Y minimal, is regular with respect to $\pi: X \longrightarrow Y$ if, given any pair of homomorphisms $\gamma: M \longrightarrow X$ and $\delta: M \longrightarrow Z$ with $\pi \circ \gamma = \psi \circ \delta$ there exists a homomorphism $\theta: Z \longrightarrow X$ with $\theta \circ \delta = \gamma$ and $\pi \circ \theta = \psi$.

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ X \arrow[dr, "\pi"'] & & Z \arrow[ll, "\theta"'] \arrow[dl, "\psi"] \\ & Y & \end{tikzcd}$$

It will turn out that $\bar{\pi}: N \longrightarrow Y$ can be characterized as the unique "least" homomorphism which is regular with respect to $\pi: X \longrightarrow Y$. To obtain this result we first make some observations about $\bar{\pi}$. To model $\bar{\pi}$, we choose $y \in Y$ and let N be some minimal subset of $E(\pi, y)$. We take $\bar{\pi} = \bar{\pi}_y | N$. We may vary the choice of y and N to suit our convenience since all choices are isomorphic.

Lemma 2.2.2. Suppose γ and γ' are homomorphisms from M into N with $\bar{\pi} \circ \gamma = \bar{\pi} \circ \gamma'$. Then there exists an automorphism $\theta: N \longrightarrow N$ with $\theta \circ \gamma' = \gamma$ and $\bar{\pi} \circ \theta = \bar{\pi}$

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[dr, "\gamma'"] & \\ N \arrow[dr, "\bar{\pi}"'] & & N \arrow[ll, "\theta"'] \arrow[dl, "\bar{\pi}"] \\ & Y & \end{tikzcd}$$

Proof: Regarding M as a minimal ideal in βT we may represent N by $N = z_y M$. Let u be a minimal idempotent in J. We can find elements p and p' in M such that

$z_y p = \gamma(u)$ and $z_y p' = \gamma'(u)$ since N is minimal. Then $z_y pu = \gamma(u)u = \gamma(uu) = \gamma(u) = z_y p$. Similarly, $z_y p'u = z_y p'$. Also $yp = \overline{\pi}(z_y p) = \overline{\pi}\circ\gamma(u) = \overline{\pi}\circ\gamma'(u) = \overline{\pi}(z_y p') = \overline{\pi}(z_y)p' = yp'$. We pick another minimal idempotent $v \in J(y)$ and pick $q \in Mv$ so $q = qv$ and $pq = qpv = v$. Now we'll show that the functions $z_y p$ and $z_y p'$ have the same range. Consider $x \in \mathrm{Range}(z_y)$, recalling that $\mathrm{Range}(z_y) = \pi^{-1}(y)$. Let $x' = xp'q$. Then we have $\pi(x') = \pi(xp'q) = yp'q = ypq = yv = y$, so $x' \in \mathrm{Range}(z_y)$ also. We have $z_y p' = \gamma'(u) = \gamma'(uu) = \gamma'(u)u = z_y p'u$ and similarly $z_y p = z_y pu$. Therefore $xpu = xp$ and $x'pu = x'p$. Now $xp' = xp'u = xp'vu = xp'qpvu = x'pvu = x'pu = x'p$. Thus $\mathrm{Range}(z_y p') \subset \mathrm{Range}(z_y p)$. Similarly $\mathrm{Range}(z_y p) \subset \mathrm{Range}(z_y p')$.

We define θ so $\theta(z_y p') = z_y p$. It is clear from the above that $z_y p'r = z_y p's \Longrightarrow z_y pr = z_y ps$ for any $r,s \in \beta T$. Thus θ is a well-defined homomorphism. We can reverse this to get θ^{-1}, so θ is actually an automorphism.

Finally $\theta\circ\gamma'(u) = \theta(z_y p') = z_y p = \gamma(u)$, so $\theta\circ\gamma' = \gamma$. ||

It will soon be seen that the property ascribed to $\overline{\pi}\colon N \longrightarrow Y$ in the preceding lemma characterizes regularity.

The next lemma concerns factoring homomorphisms from M into X through N.

<u>Lemma 2.2.3</u>. <u>Given a homomorphism</u> $\gamma\colon M \longrightarrow X$, <u>there exist homomorphisms</u> $\varkappa\colon M \longrightarrow N$ <u>and</u> $\lambda\colon N \longrightarrow X$ <u>such that</u> $\lambda\circ\varkappa = \gamma$, $\pi\circ\lambda = \overline{\pi}$, and $\overline{\pi}\circ\varkappa = \pi\circ\gamma$

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[dr, "\varkappa"] & \\ X \arrow[dr, "\pi"'] & & N \arrow[ll, "\lambda"] \arrow[dl, "\bar{\pi}"] \\ & Y & \end{tikzcd}$$

Proof: Pick some minimal idempotent $u \in J$. Let $x = \gamma(u)$ and $y = \pi(x)$. We model N by taking $N = z_y M$. We now take $\varkappa(u) = z_y u$ and $\lambda(z_y u) = x$. Consider $p, q \in \beta T$. If $up = uq$, then $z_y up = z_y uq$. If $z_y up = z_y uq$ then $xp = xq$ since $x \in \text{Range}(z_y u)$, M and N are both minimal, so $\varkappa$ and λ are well-defined homomorphisms, chosen to make the diagram commute. ||

Proposition 2.2.4. *$\bar{\pi}$ is regular with respect to π.*

Proof: Given homomorphisms γ and δ as shown in the diagram below we must find a homomorphism θ to complete the diagram

$$\begin{tikzcd} & M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\ X \arrow[dr, "\pi"'] & & N \arrow[ll, "\theta"] \arrow[dl, "\bar{\pi}"] \\ & Y & \end{tikzcd}$$

We apply Lemma 2.2.3 to find homomorphisms $\varkappa\colon M \longrightarrow N$ and $\lambda\colon N \longrightarrow X$ such that $\lambda \circ \varkappa = \gamma$, $\pi \circ \lambda = \bar{\pi}$, and $\bar{\pi} \circ \varkappa = \pi \circ \gamma$. We then apply Lemma 2.2.2 to obtain an automorphism φ with $\varphi \circ \delta = \varkappa$ and $\bar{\pi} \circ \varphi = \bar{\pi}$. We then take $\theta = \lambda \circ \varphi$ and we're done. The complete picture is shown below.

$$\begin{tikzcd}
& M \arrow[dl, "\gamma"'] \arrow[d, "\varkappa"] \arrow[dr, "\delta"] & \\
X \arrow[dr, "\pi"'] & N \arrow[l, "\lambda"] \arrow[d, "\bar{\pi}"] & N \arrow[l, "\varphi"'] \arrow[dl, "\bar{\pi}"] \\
& Y &
\end{tikzcd}$$

||

Definition 2.2.5. Given a homomorphism $\psi\colon W \longrightarrow Z$ we say that ψ is *coalescent* if every endomorphism $\theta\colon W \longrightarrow W$ with $\psi \circ \theta = \psi$ is actually an automorphism.

This is a generalization of the notion of coalescence of transformation groups.

Proposition 2.2.6. *$\bar{\pi}\colon N \longrightarrow Y$ is coalescent.*

Proof: This is an immediate consequence of Lemma 2.2.2.

||

We are now in a position to prove the main result of this section.

Theorem 2.2.7. *Suppose a homomorphism $\pi'\colon N' \longrightarrow Y$ is regular with respect to $\pi\colon X \longrightarrow Y$. Then there exists a homomorphism $\psi\colon N' \longrightarrow N$ such that $\bar{\pi} \circ \psi = \pi'$. Moreover, any other homomorphism onto Y which is regular with respect to π and which has this additional property is isomorphic to $\bar{\pi}\colon N \longrightarrow Y$.*

Proof: We represent N by taking $N = z_y M$. Pick $u \in J(y)$. Define $\gamma\colon M \longrightarrow N$ by $\gamma(u) = z_y u$. Pick an element $r \in \pi'^{-1}(y) \subset N'$ and define $\delta\colon M \longrightarrow N'$ by $\delta(u) = ru$. Now $\pi' \circ \delta(u) = \pi'(ru) = yu = \bar{\pi}(z_y u) = \bar{\pi} \circ \gamma(u)$. Thus $\pi' \circ \delta = \bar{\pi} \circ \gamma$.

Consider any $x \in \pi^{-1}(y)$. We define $\theta_x\colon N \longrightarrow X$ by

$\theta_x(z_y u) = xu$. This is well-defined since $xu \in \text{Range}(z_y u)$. Now $\pi \circ \theta_x \circ \gamma(u) = \pi \circ \theta_x(z_y u) = \pi(xu) = y = \pi'(ru) = \pi' \circ \delta(u)$. Thus $\pi \circ (\theta_x \circ \gamma) = \pi' \circ \delta$. By the regularity of π' with respect to π there exists a homomorphism $\varphi_x\colon N' \longrightarrow X$ with $\varphi_x \circ \delta = (\theta_x \circ \gamma)$ and $\pi \circ \varphi_x = \pi'$.

The situation so far is shown by the following diagram. We still wish to construct $\psi\colon N' \longrightarrow N$.

$$\begin{tikzcd}
& M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
N \arrow[dr, "\theta_x"] \arrow[ddr, "\overline{\pi}"'] & & N' \arrow[ll, "\psi"'] \arrow[dl, "\varphi_x"'] \arrow[ddl, "\pi'"] \\
& X \arrow[d, "\pi"] & \\
& Y &
\end{tikzcd}$$

We'll now see that for any $m, m' \in M$, $\delta(m) = \delta(m') \Longrightarrow \gamma(m) = \gamma(m')$. Observe that $\gamma(m) = \gamma(m') \Longleftrightarrow z_y m = z_y m' \Longleftrightarrow xm = xm'$ for all $x \in \pi^{-1}(y)$. Consider an arbitrary $x \in \pi^{-1}(y)$. We have $\delta(m) = \delta(m') \Longrightarrow \varphi_x \circ \delta(m) = \varphi_x \circ \delta(m') \Longrightarrow \theta_x \circ \gamma(m) = \theta_x \circ \gamma(m') \Longrightarrow \theta_x(z_y m) = \theta_x(z_y m') \Longrightarrow xm = xm'$.

We now define $\psi\colon N' \longrightarrow N$ by $\psi(ru) = z_y u$. Suppose $rup = ruq$ for some p and q in βT. Then $\delta(up) = rup = ruq = \delta(uq)$. Thus $z_y up = \gamma(up) = \gamma(uq) = z_y uq$. Therefore ψ is well-defined. Also $\overline{\pi} \circ \psi(ru) = \overline{\pi}(z_y u) = yu = \pi'(ru)$ so $\overline{\pi} \circ \psi = \pi'$.

Finally, suppose $\pi''\colon N'' \longrightarrow Y$ is another homomorphism which is regular with respect to $\pi\colon X \longrightarrow Y$ and which has the property just established for $\overline{\pi}\colon N \longrightarrow Y$. We wish to find an isomorphism $\alpha\colon N'' \longrightarrow N$ with $\bar{\pi}\circ\alpha = \pi''$. Since both $\bar{\pi}$ and π'' are regular with respect to π and both have the property shown above we can find homomorphisms $\alpha\colon N'' \longrightarrow N$ and $\beta\colon N \longrightarrow N''$ such that $\bar{\pi}\circ\alpha = \pi''$ and $\pi''\circ\beta = \bar{\pi}$. Since $\bar{\pi}$ is coalescent, $\alpha\circ\beta$ is an automorphism of N. Therefore α is one-to-one and hence an isomorphism. $||$

Using the language of category theory, we can describe $\bar{\pi}\colon N \longrightarrow Y$ as the unique universally attracting object in the category of homomorphisms which are regular with respect to $\pi\colon X \longrightarrow Y$.

In [3], Auslander obtained several different characterizations of regular minimal sets. These extend to homomorphisms.

Proposition 2.2.8. *Given a homomorphism* $\pi\colon X \longrightarrow Y$ *with* X *and* Y *minimal, the following statements are equivalent:*

(*i*) π *is regular.*

(*ii*) π *is regular with respect to itself.*

(*iii*) π *is its own regularizer.*

(*iv*) *For any two points* $x, x' \in X$ *with* $\pi(x) = \pi(x')$ *there exists an endomorphism* $\theta\colon X \longrightarrow X$ *such that* $\theta(x)$ *and* x' *are proximal and* $\pi\circ\theta = \theta$.

(*v*) *For any two points* $x, x' \in X$ *with* (x, x') *almost periodic and* $\pi(x) = \pi(x')$ *there exists an endomorphism* $\theta\colon X \longrightarrow X$ *such that* $\theta(x) = x'$ *and* $\pi\circ\theta = \theta$.

Proof: $(i) \Longrightarrow (v)$. Since (x,x') almost periodic, we can find an idempotent $u \in J$ with $(x,x')u = (x,x')$. We define homomorphisms $\gamma\colon M \longrightarrow X$ and $\gamma'\colon M \longrightarrow X$ by $\gamma(u) = x$ and $\gamma'(u) = x'$. Applying Lemma 2.2.2 then yields the required $\theta\colon X \longrightarrow X$.

$(v) \Longrightarrow (ii)$. Given homomorphisms $\gamma\colon M \longrightarrow X$ and $\delta\colon M \longrightarrow X$ with $\pi \circ \gamma = \pi \circ \delta$ we must find $\theta\colon X \longrightarrow X$ with $\theta \circ \delta = \gamma$ and $\pi \circ \theta = \theta$. Pick $u \in J$. Let $x = \gamma(u)$ and $x' = \delta(u)$. Then $(x,x')u = (x,x')$ so (x,x') is almost periodic. Applying (v) yields the necessary homomorphism.

$(ii) \Longrightarrow (iii)$. Suppose $\pi'\colon X' \longrightarrow Y$ is some other homomorphism which is regular with respect to π. Pick an arbitrary $y \in Y$ and $u \in J(y)$. Pick $x \in \pi^{-1}(y)$ and $x' \in \pi'^{-1}(y)$ arbitrarily. We can define homomorphisms $\gamma\colon M \longrightarrow X$ and $\delta\colon M \longrightarrow X'$ with $\pi' \circ \delta = \pi \circ \gamma$ by $\gamma(u) = xu$ and $\delta(u) = x'u$. By regularity of π' with respect to X we can find a homomorphism $\theta\colon X' \longrightarrow X$ such that $\pi \circ \theta = \pi'$ and $\theta \circ \delta = \gamma$. Thus π attracts any other homomorphism which is regular with respect to π. Therefore, by Theorem 2.2.7, π is its own regularizer.

$(iii) \Longrightarrow (i)$. By definition.

$(iv) \Longrightarrow (v)$. If a pair of points is both proximal and almost periodic, the two points are identical.

$(v) \Longrightarrow (iv)$. Suppose $x,x' \in X$ and $\pi(x) = \pi(x')$. Pick $u \in J(x)$. Let $x'' = x'u$. Then $x''u = (x'u)u = x'u$ so x'' and x' are proximal. $(x,x'')u = (x,x'')$ so (x,x'') is almost periodic. Applying (v) then yields the required endomorphism. ||

Since regular homomorphisms are coalescent, "endomorphism" may be replaced by "automorphism" in (*iv*) and (*v*).

A subset of a transformation group is called "almost periodic" if it's the range of an almost periodic element in an appropriate product space. This is not the same as being an almost periodic element in a hyperspace. Condition (*v*) of Proposition 2.2.8 tells us that the fibers of regular homomorphisms are partitioned by their maximal almost periodic subsets, which are of the form $\pi^{-1}(y)u$ where $y \in Y$ and $u \in J(y)$.

Corollary 2.2.9. *Suppose* $\pi: X \longrightarrow Y$ *is a regular homomorphism and* I *is a minimal right ideal in some enveloping semigroup for* X. *Suppose* $y \in Y$. *Then the collection of sets* $P = \{\pi^{-1}(y)u \mid u \in J(I)$ *and* $yu = y\}$ *partitions* $\pi^{-1}(y)$.

Proof: We'll say that two points in $\pi^{-1}(y)$ are P-related if they belong to some common member of P. We must show that this is an equivalence relation. Reflexivity follows from the fact that all points in X are almost periodic, X being minimal. Symmetry is obvious. Suppose (x,x') and (x',x'') are P-related. Then (x,x') and (x',x'') are almost periodic pairs in $\pi^{-1}(y)$. By (*v*) of Proposition 2.2.8, we can find automorphisms α and β such that $\alpha(x) = x'$, $\beta(x') = x''$, $\pi\circ\alpha = \pi$, and $\pi\circ\beta = \pi$. Then $\alpha\circ\beta$ is an automorphism also with $(\alpha\circ\beta)(x) = x''$ and $\pi\circ(\alpha\circ\beta) = \pi$. Pick an idempotent $u \in I$ such that $xu = x$.

Then $yu = y$ and $x''u = (\alpha \circ \beta)(x)u = (\alpha \circ \beta)(xu) = (\alpha \circ \beta)(x) = x''$. Thus (x,x'') is a P-related pair also and the P relation is transitive. ||

We can draw some additional conclusions from Proposition 2.2.8. Given a homomorphism $\psi\colon Z \longrightarrow W$, we define Aut ψ = $\{\theta \mid \theta$ an automorphism of Z and $\psi \circ \theta = \psi\}$. We say that ψ is a <u>group extension</u> if whenever $z, z' \in Z$ and $\psi(z) = \psi(z')$, there exists $\theta \in$ Aut ψ such that $\theta(z) = z'$.

<u>Corollary 2.2.10</u>. <u>A homomorphism of minimal sets is a group extension if and only if it's distal and regular.</u>

<u>Proof</u>: This follows from condition (v) of Proposition 2.2.8 and the fact that a homomorphism of minimal sets is distal if and only if the fibers are almost periodic sets [see 4]. ||

<u>Corollary 2.2.11</u>. <u>A proximal homomorphism of minimal sets is always regular.</u>

<u>Proof</u>: If a pair of points is both proximal and almost periodic, the two points are the same. Thus the identity automorphism connects any almost periodic pair in a common fiber and condition (v) of Proposition 2.2.8 applies. ||

<u>Proposition 2.2.12</u>. <u>Suppose $\pi\colon X \longrightarrow Y$ is a homomorphism of minimal sets. Then π is regular and almost periodic if and only if π is a group extension and Aut π admits a compact Hausdorff topology making it a topological group and making its action on X jointly continuous.</u>

Proof: $\Longleftarrow$ We've already shown that group extensions are always regular, so we just need to show that π is almost periodic. Suppose (x,x') is a relatively regionally proximal pair in X. Then there exist nets $\langle x_n \rangle$ and $\langle x'_n \rangle$ in X and $\langle t_n \rangle$ in T such that $\pi(x_n) = \pi(x'_n)$ for each n and:

$$x_n \longrightarrow x \qquad x_n t_n \longrightarrow x$$
$$x'_n \longrightarrow x' \qquad x'_n t_n \longrightarrow x.$$

Since π is a group extension, each $x'_n = \alpha_n(x_n)$ for some $\alpha_n \in \operatorname{Aut} \pi$. Thus we have

$$\text{(a)}\quad x_n \longrightarrow x \qquad \text{(c)}\quad x_n t_n \longrightarrow x$$
$$\text{(b)}\quad \alpha_n(x_n) \longrightarrow x' \qquad \text{(d)}\quad \alpha_n(x_n t_n) \longrightarrow x$$

Taking subnets if necessary, we can assume $\alpha_n \longrightarrow \alpha$ for some $\alpha \in \operatorname{Aut} \pi$, since $\operatorname{Aut} \pi$ is assumed compact Hausdorff. Applying the joint continuity to (a) and (b) gives $\alpha(x) = x'$ and (c) and (d) give $\alpha(x) = x$. Thus $x = x'$ and π is almost periodic.

$\Longrightarrow$ We know we have a group extension by Corollary 2.2.10. By the Ellis Joint Continuity Theorem [see 6] it will suffice to show that the action on X and the multiplication on $\operatorname{Aut} \pi$ are separately continuous.

First we'll show that given elements $x,x' \in X$ and a net $\langle \alpha_n \rangle$ in $\operatorname{Aut} \pi$ and $\alpha \in \operatorname{Aut} \pi$ we have

$$\alpha_n(x) \longrightarrow \alpha(x) \iff \alpha_n(x') \longrightarrow \alpha(x').$$

Assume $\alpha_n(x) \longrightarrow \alpha(x)$. It will suffice to show that some subnet of $\langle\alpha_n(x')\rangle$ converges to $\alpha(x')$ so we may assume that $\langle\alpha_n(x')\rangle$ converges. Pick $p \in \beta T$ with $xp = x'$ and a net $\langle t_m\rangle$ in T with $t_m \longrightarrow p$. Then

$$\lim_n \langle\alpha_n(x')\rangle = \lim_n \left\langle \lim_m \langle\alpha_n(x)t_m\rangle\right\rangle = \lim_j \langle\alpha_j(x)t_j\rangle$$

for some subnets $\langle\alpha_j\rangle$ of $\langle\alpha_n\rangle$ and $\langle t_j\rangle$ of $\langle t_m\rangle$. The almost periodicity of π implies that $\lim_j \langle\alpha_j(x)t_j\rangle = \alpha(x)p = \alpha(x')$ by Theorem 1.2.6 and Lemma 1.2.5. Thus $\alpha_n(x') \longrightarrow \alpha(x')$ as required.

Next we fix $x_0 \in X$ and $y_0 \in Y$ with $\pi(x_0) = y_0$. $\alpha \longmapsto \alpha(x_0)$ is a bijection from $\operatorname{Aut}\pi$ to $\pi^{-1}(y_0)$. We use this to topologize $\operatorname{Aut}\pi$ with the subspace topology of $\pi^{-1}(y_0)$ in X which is compact Hausdorff. We then have

$$\alpha_n \longrightarrow \alpha \iff \alpha_n(x_0) \longrightarrow x_0$$

If $x_n \longrightarrow x$ in X and $\alpha \in \operatorname{Aut}\pi$ we surely have $\alpha(x_n) \longrightarrow \alpha(x)$. If $\alpha_n \longrightarrow \alpha$ in $\operatorname{Aut}\pi$ and $x \in X$ we have $\alpha_n \longrightarrow \alpha \Longrightarrow \alpha_n(x_0) \longrightarrow \alpha(x_0) \Longrightarrow \alpha_n(x) \longrightarrow \alpha(x)$. Thus the action is separately continuous.

Suppose $\beta_n \longrightarrow \beta \in \operatorname{Aut}\pi$ and $\alpha \in \operatorname{Aut}\pi$. Then $\beta_n \longrightarrow \beta$ $\Longrightarrow \beta_n(x_0) \longrightarrow \beta(x_0) \Longrightarrow \beta_n(\alpha(x_0)) \longrightarrow \beta(\alpha(x_0)) \Longrightarrow \beta_n \circ \alpha \longrightarrow \beta \circ \alpha$. Also $\beta_n \longrightarrow \beta \Longrightarrow \beta_n(x_0) \longrightarrow \beta(x_0) \Longrightarrow \alpha(\beta_n(x_0)) \longrightarrow \alpha(\beta(x_0))$ $\Longrightarrow \alpha \circ \beta_n \longrightarrow \alpha \circ \beta$. Thus the multiplication in $\operatorname{Aut}\pi$ is separately continuous. ||

3. Automorphism Groups.

We let G be the automorphism group of the universal minimal set M. Given a homomorphism $\gamma: M \longrightarrow X$ we may define a subgroup $G(X,\gamma) = \{\alpha \in G \mid \gamma\circ\alpha = \gamma\}$. Varying γ while keeping X fixed yields conjugate subgroups. These groups and their quotients contain considerable information about minimal sets. They have been studied extensively by Ellis [7] and Auslander [4]. In particular, it's known that the normal subgroups of G correspond to regular minimal sets, up to proximal homomorphisms. We'll see that this extends to regular homomorphisms.

In this section, $\pi: X \longrightarrow Y$ is a fixed homomorphism of minimal sets and y is a fixed point in Y. We'll consider only those homomorphisms $\gamma: M \longrightarrow X$ such that $\pi\circ\gamma(m) = ym$ for all $m \in M$. Thus $\pi\circ\gamma$ is independent of the choice of γ and we may write $G(Y)$ instead of $G(Y,\pi\circ\gamma)$. We will also write $G(X)$ instead of $G(X,\gamma)$ when the choice of γ is irrelevant.

The following lemmas are from [4]:

Lemma 2.3.1. *Given homomorphisms* $\gamma: M \longrightarrow X$ *and* $\gamma': M \longrightarrow X$ *with* X *minimal, there exists an automorphism* β *of* M *such that* $\gamma'\circ\beta = \gamma$.

Lemma 2.3.2. *Suppose* X *and* Y *are minimal and we have homomorphisms* $\gamma: M \longrightarrow X$ *and* $\pi: X \longrightarrow Y$. *Then* π *is a proximal homomorphism if and only if* $G(X,\gamma) = G(Y,\pi\circ\gamma)$.

Proposition 2.3.3. *If* π *is regular, then* $G(X)$ *is a normal subgroup of* $G(Y)$ *and* $\mathrm{Aut}(\pi)$ *is naturally isomorphic to* $G(Y)/G(X)$.

Proof: Choose some $\gamma: M \longrightarrow X$ as indicated before and let $G(X) = G(X,\gamma)$. Suppose $\alpha \in G(X,\gamma)$ and $\beta \in G(Y)$. We must show that $\beta^{-1}\alpha\beta \in G(X,\gamma)$. By Proposition 2.2.8, there's a unique $\varphi \in \mathrm{Aut}(\pi)$ such that $\varphi \circ \gamma \circ \beta = \gamma$. Therefore $\gamma \circ (\beta^{-1}\alpha\beta) = \varphi \circ \gamma \circ \beta \circ (\beta^{-1}\alpha\beta) = \varphi \circ \gamma \circ \alpha \circ \beta = \varphi \circ \gamma \circ \beta = \gamma$. Thus we have $\beta^{-1}\alpha\beta \in G(X,\gamma)$ proving that $G(X,\gamma)$ is a normal subgroup.

As noted above, given $\beta \in G(Y)$ there exists a unique $\varphi \in \mathrm{Aut}\ \pi$ such that $\varphi \circ \gamma \circ \beta = \gamma$. Define $F: G(Y) \longrightarrow \mathrm{Aut}(\pi)$ by $F(\beta) = \varphi^{-1}$. We'll show that F is a group epimorphism and that $\ker F = G(X,\gamma)$. Consider β_1 and β_2 in $G(Y)$. Let $\beta_3 = \beta_1 \circ \beta_2$ and let $F(\beta_i) = \varphi_i$ for $i = 1,2$, and 3. Now $\varphi_2 \circ \varphi_1 \circ \gamma \circ \beta_1 \circ \beta_2 = \varphi_2 \circ \gamma \circ \beta_2 = \gamma$ so we must have $\varphi_2 \circ \varphi_1 = \varphi_3$. Therefore $\varphi_3^{-1} = \varphi_1^{-1}\varphi_2^{-1}$ and $F(\beta_1 \circ \beta_2) = F(\beta_3) = F(\beta_1) \circ F(\beta_2)$. Therefore F is a homomorphism.

Consider $\beta \in G(Y)$. $F(\beta) = \text{identity} \iff \text{identity} \circ \gamma \circ \beta = \gamma \iff \gamma \circ \beta = \gamma \iff \beta \in G(X,\gamma)$. Thus $\ker F = G(X,\gamma)$. Finally, consider $\varphi \in \mathrm{Aut}\ \pi$. By Lemma 2.3.1, there is an automorphism β of M such that $\varphi^{-1} \circ \gamma \circ \beta = \gamma$. Also $\pi \circ \gamma \circ \beta = \pi \circ \varphi^{-1} \circ \gamma \circ \beta = \pi \circ \gamma$ so $\beta \in G(Y)$. Thus $F(\beta) = (\varphi^{-1})^{-1} = \varphi$. Therefore F is onto.||

To prove a partial converse we need the following technical lemma:

Lemma 2.3.4. Suppose we have homomorphisms of minimal sets $\gamma: M \longrightarrow X$ and $\pi: X \longrightarrow Y$. Suppose $u \in J$, $x_0 = \gamma(u)$, and $y = \pi(x_0)$. We represent the regularizer of π by taking $N = z_y M$. We define a homomorphism $P: N \longrightarrow X$ by

$P(z_y u) = x_0 u$. *Suppose* $\varphi \in \operatorname{Aut} \bar{\pi}$ *and* $P \circ \psi \circ \varphi = P \circ \psi$ *for every* $\psi \in \operatorname{Aut} \bar{\pi}$. *Then* φ *is the identity automorphism on* N.

Proof: It will suffice to show that $\varphi(z_y u) = z_y u$. Pick $r \in M$ such that $\varphi(z_y u) = z_y r$. Let $r' = uru$. Then $\varphi(z_y u) = \varphi(z_y u)u = z_y ru = z_y uru = z_y r'$ also. $\operatorname{Range}(z_y) = \pi^{-1}(y)$, so we must show that $xr' = xu$ for all $x \in \pi^{-1}(y)$. Consider such an x. Pick $p \in M$ so that $x = x_0 p$. We wish to define $\psi \in \operatorname{Aut}(\bar{\pi})$ by $\psi(z_y u) = z_y pu$. Suppose $x' \in \operatorname{Range}(z_y p)$. Then $x' = x''p$ for some $x'' \in \pi^{-1}(y)$. Therefore $\pi(x') = \pi(x''p) = yp = \pi(x_0)p = \pi(x) = y$. $\operatorname{Range}(z_y) = \pi^{-1}(y)$. Thus $\operatorname{Range}(z_y p) \subset \operatorname{Range}(z_y)$ and ψ is well-defined. Also $\bar{\pi} \circ \psi(z_y u) = \bar{\pi}(z_y pu) = ypu = y = \bar{\pi}(z_y u)$ so $\psi \in \operatorname{Aut}(\bar{\pi})$.

Finally, we see that $P \circ \psi \circ \varphi = P \circ \psi \Longrightarrow P \circ \psi \circ \varphi(z_y u) = P \circ \psi(z_y u) \Longrightarrow P \circ \psi(z_y r') = P \circ \psi(z_y u) \Longrightarrow P(z_y pr') = P(z_y pu) \Longrightarrow P(z_y upr') = P(z_y upu) \Longrightarrow x_0 upr' = x_0 upu \Longrightarrow x_0 pr' = x_0 pu \Longrightarrow xr' = xu$ and we're done. $||$

We can now prove the following partial converse to Proposition 2.3.3.

Proposition 2.3.5. *Suppose* $\bar{\pi}: N \longrightarrow Y$ *is the regularizer of the homomorphism of minimal sets* $\pi: X \longrightarrow Y$. *If* $G(X)$ *is a normal subgroup of* $G(Y)$ *there is a proximal homomorphism* $P: N \longrightarrow X$ *with* $\pi \circ P = \bar{\pi}$.

Proof: Pick an idempotent $u \in J$, a homomorphism

$\gamma\colon M \longrightarrow X$, and $x_0 \in X$ such that $\gamma(u) = x_0$ and $\pi(x_0) = y$. Represent N by $N = z_y M$. Define a homomorphism $\delta\colon M \longrightarrow N$ by $\delta(u) = z_y u$. Define $P\colon N \longrightarrow X$ by $P(z_y u) = x_0 u = x_0$. We then have the following commutative diagram:

$$\begin{tikzcd}
& M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
X \arrow[dr, "\pi"'] & & N \arrow[ll, "P"'] \arrow[dl, "\overline{\pi}"] \\
& Y &
\end{tikzcd}$$

It will suffice to show $G(N,\delta) = G(X,\gamma)$, by Lemma 2.3.2. It's clear that $G(N,\delta) \subset G(X,\gamma)$. We'll show that $G(X,\gamma) \subset G(N,\delta)$. Consider $\alpha \in G(X,\gamma)$. We must show that $\alpha \in G(N,\delta)$, i.e., that $\delta\circ\alpha = \delta$. We have $\overline{\pi}\circ\delta\circ\alpha = \pi\circ\gamma\circ\alpha = \overline{\pi}\circ\delta$ so there exists an automorphism $\varphi \in \mathrm{Aut}(\overline{\pi})$ with $\varphi\circ\delta = \delta\circ\alpha$, by the regularity of $\overline{\pi}$ and Proposition 2.2.8. If we can show that φ is the identity we'll have $\delta\circ\alpha = \delta$ and we'll be done.

Consider any $\psi \in \mathrm{Aut}(\overline{\pi})$. By Lemma 2.3.1, we can find $\beta \in G$ such that $\psi\circ\delta = \delta\circ\beta$. This implies that $\psi^{-1}\circ\delta = \delta\circ\beta^{-1}$ as well. Also, $\overline{\pi}\circ\delta\circ\beta = \overline{\pi}\circ\psi\circ\delta = \overline{\pi}\circ\delta \Longrightarrow \pi\circ\gamma\circ\beta = \pi\circ\gamma$ so $\beta \in G(Y)$. Now $\beta\circ\alpha\circ\beta^{-1} \in G(X,\gamma)$ by normality. Thus $P\circ\psi\circ\varphi\circ\psi^{-1}\circ\delta = P\circ\psi\circ\varphi\circ\delta\circ\beta^{-1} = P\circ\psi\circ\delta\circ\alpha\circ\beta^{-1} = P\circ\delta\circ\beta\circ\alpha\circ\beta^{-1} = \gamma\circ\beta\circ\alpha\circ\beta^{-1} = \gamma = P\circ\delta$. Since δ is onto, we have $P\circ\psi\circ\varphi\circ\psi^{-1} = P$ which implies $P\circ\psi\circ\varphi = P\circ\psi$. Applying Lemma 2.3.4, we see that φ is the identity and we're done. ||

We conclude this section by computing the group associated with the regularizer of a homomorphism of minimal sets. This turns out to be the largest subgroup of $G(X)$ which is normal in $G(Y)$.

Proposition 2.3.6. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism of minimal sets, with regularizer* $\overline{\pi}: N \longrightarrow Y$. *Then* $G(N) = \cap \{\beta G(X)\beta^{-1} \mid \beta \in G(Y)\}$.

Proof: We choose $\gamma: M \longrightarrow X$, $\delta: M \longrightarrow N$, and $P: N \longrightarrow X$ as in the proof of the preceding proposition. We regard $G(X)$ as $G(X,\gamma)$ and $G(N)$ as $G(N,\delta)$. Clearly $G(N,\delta) \subset G(X,\gamma) \subset G(Y)$ and we have $G(N,\delta)$ a normal subgroup of $G(Y)$ by Proposition 2.3.3. Therefore $G(N,\delta) \subset \beta G(X,\gamma)\beta^{-1}$ for each $\beta \in G(Y)$. We still must show $\cap \{\beta G(X,\gamma)\beta^{-1} \mid \beta \in G(Y)\} \subset G(N,\delta)$. Consider $\alpha \in \cap \{\beta G(X,\gamma)\beta^{-1} \mid \beta \in G(Y)\}$. We wish to show $\alpha \in G(N,\delta)$. Let $F: G(Y) \longrightarrow \mathrm{Aut}(\overline{\pi})$ be the group homomorphism defined in the proof of Proposition 2.3.3. By definition, $F(\alpha)^{-1} \circ \delta \circ \alpha = \delta$ so $F(\alpha) \circ \delta = \delta \circ \alpha$. Since F is onto, every element of $\mathrm{Aut}(\overline{\pi})$ can be represented as $F(\theta)$, for some $\theta \in G(Y)$. Consider an arbitrary $F(\theta) \in \mathrm{Aut}(\overline{\pi})$. We have $F(\theta) \circ \delta = \delta \circ \theta$ and $F(\theta)^{-1} \circ \delta = \delta \circ \theta^{-1}$. Since $\alpha \in \cap \{\beta G(X,\gamma)\beta^{-1} \mid \beta \in G(Y)\}$ and $\theta \in G(Y)$, there exists $\lambda \in G(X,\gamma)$ such that $\alpha = \theta^{-1} \circ \lambda \circ \theta$. Thus $\alpha \circ \theta^{-1} = \theta^{-1} \circ \lambda$. We then have $P \circ F(\theta) \circ F(\alpha) \circ F(\theta)^{-1} \circ \delta = P \circ F(\theta) \circ F(\alpha) \circ \delta \circ \theta^{-1} = P \circ F(\theta) \circ \delta \circ \alpha \circ \theta^{-1} = P \circ \delta \circ \theta \circ \alpha \circ \theta^{-1} = \gamma \circ \theta \circ \alpha \circ \theta^{-1} = \gamma \circ \theta \circ \theta^{-1} \circ \lambda = \gamma \circ \lambda = \gamma = P \circ \delta$. δ is onto, so we've shown that $P \circ F(\theta) \circ F(\alpha) \circ F(\theta)^{-1} = P$, i.e., $P \circ F(\theta) \circ F(\alpha) = P \circ F(\theta)$. Applying Lemma 2.3.4, we get that $F(\alpha)$ is the identity. Thus $\alpha \in \ker F = G(N,\delta)$. ||

4. An Algebraic Characterization.

In this section we fix an idempotent $u \in J$ and consider only point-transitive transformation groups whose base point is left fixed by u. Such transformation groups are necessarily minimal. There is a natural isomorphism between G, the automorphism group of M, and the group Mu, with the element $\alpha \in G$ going into the element $\alpha(u)$ in Mu. We will use this isomorphism in deliberately confusing the two groups. We will write $G(X,x)$ to mean $G(X,\gamma)$ where $\gamma\colon M \longrightarrow X$ is defined by $\gamma(u) = x$.

Suppose we have a homomorphism $\pi\colon (X,x) \longrightarrow (Y,y)$. Recall that we can represent the regularizer as $\overline{\pi}\colon (N,z_y u) \longrightarrow (Y,y)$ where $z_y \in X^{\pi^{-1}(y)}$ and $z_y(x) = x$ for all $x \in \pi^{-1}(y)$. The following proposition describes N algebraically.

Proposition 2.4.1. Suppose the T-subalebras $\mathcal{A}$ and $\mathcal{B}$ are associated with (X,x) and (Y,y) respectively. Then $(N,z_y u)$ is associated with the T-subalgebra $\bigvee \{\mathcal{A}\alpha \mid \alpha \in G(Y,y)\}$.

Proof: We first observe that $\pi^{-1}(y)u = \{x\alpha \mid \alpha \in G(Y,y)\}$. We have $\gamma\colon M \longrightarrow X$ with $\gamma(u) = x$ and $G(Y,y) = G(Y,\pi\circ\gamma)$. If $\alpha \in G(Y,y)$, $\pi(x\alpha) = \pi\circ\gamma\circ\alpha(u) = \pi\circ\gamma(u) = y$ and $x\alpha u = x\alpha$ so $x\alpha \in \pi^{-1}(y)u$. Suppose $x' \in \pi^{-1}(y)u$. Since (X,x) is point-transitive, $xp = x'$ for some $p \in \beta T$. Since $xu = x$ and $x'u = x'$ we have $xupu = x'$. We can take $\alpha = upu$ and we have $\alpha \in G(Y,y)$ as required.

Since $z_y u$ consists of the elements of $\pi^{-1}(y)u$, "strung out" in a product space and N is the orbit closure of $z_y u$ in that product space, the result follows by applying (*ii*) of Lemma 1.1.6 and Lemma 1.1.7. ||

Corollary 2.4.2. Suppose again that the T-subalgebras $\mathcal{A}$ and $\mathcal{B}$ are associated with (X,x) and (Y,y) respectively. Then $\pi: (X,x) \longrightarrow (Y,y)$ is regular if and only if $\mathcal{A}\alpha \subset \mathcal{A}$ for all $\alpha \in G(Y,y)$.

Proof: Since a homomorphism is regular if and only if it is its own regularizer, Proposition 2.4.1 tells us that π is regular if and only if $\mathcal{A} = \bigvee \{\mathcal{A}\alpha \mid \alpha \in G(Y,y)\}$. However, since we're assuming $xu = x$, we always have $\mathcal{A} = \mathcal{A}u \subset \bigvee \{\mathcal{A}\alpha \mid \alpha \in G(Y,y)\}$. The result follows. ||

In [7], occasional mention and use is made of algebras like those discussed here. Much of the development in this chapter could be carried out in the algebraic context, and certain arguments would simplify. However, this would have several disadvantages. Firstly, in discussing $\pi: X \longrightarrow Y$, we would have to assume both X and Y minimal, while really it's only the minimality of Y which is essential. Secondly, we'd have to carry unwanted base points and we'd have to repeatedly re-prove that the choice of base point doesn't matter. Finally, such an approach would be less intuitive.

5. Products and Admissible Properties.

In this section we establish a result connecting regularizers and products of homomorphisms and use it to show that the regularizer depends only on the minimal sets involved. Admissible properties were studied in [3] where it is shown that the class of regular minimal sets coincides with the class of P-universal minimal sets for admissible properties P. We extend this result to homomorphisms.

Given a homomorphism $\pi: X \longrightarrow Y$, with Y minimal, we let $K(\pi)$ denote the class of homomorphisms which are regular with respect to π. We have seen that the regularizer of π is the essentially unique universally attracting object in $K(\pi)$.

We'll consider a family of homomorphisms $\pi_i: X_i \longrightarrow Y$, with Y minimal. Let $\bar{\pi}_i: N_i \longrightarrow Y$ be the regularizer of $\pi_i: X_i \longrightarrow Y$, for each i. Let $\bar{\pi}: N \longrightarrow Y$ be the product of the homomorphisms $\bar{\pi}_i: N_i \longrightarrow Y$ so $N = \{\langle n_i \rangle \mid n_i \in N$ and $\bar{\pi}_i(n_i)$ the same for all $i\}$. Let $\tilde{N}$ be a minimal set in N and $\tilde{\pi}$ be the restriction of $\bar{\pi}$ to $\tilde{N}$.

<u>Proposition 2.5.1</u>. <u>$\tilde{\pi}: \tilde{N} \longrightarrow Y$ is the essentially unique, universally attracting object in $\cap_i K(\pi_i)$. Also $\tilde{\pi}$ is regular</u>.

<u>Proof</u>: First we observe that $\tilde{\pi} \in K(\pi_i)$ for any given i. Suppose we have homomorphisms $\gamma: M \longrightarrow X_i$ and $\delta: M \longrightarrow \tilde{N}$ with $\tilde{\pi} \circ \delta = \pi_i \circ \gamma$. We must find $\theta: \tilde{N} \longrightarrow X_i$ completing the diagram

$$\begin{tikzcd}
& M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
X_i \arrow[dr, "\pi_i"'] & & N \arrow[ll, "\theta"'] \arrow[dl, "\tilde{\pi}"] \\
& Y &
\end{tikzcd} .$$

Letting $P_i\colon \tilde{N} \longrightarrow N_i$ be the projection we have a diagram

$$\begin{tikzcd}
& M \arrow[dl, "\gamma"'] \arrow[dr, "\delta"] & \\
X_i \arrow[dr, "\pi_i"'] & N_i \arrow[l, "\varphi"] \arrow[d, "\bar{\pi}_i"] & \tilde{N} \arrow[l, "P_i"'] \arrow[dl, "\tilde{\pi}"] \\
& Y &
\end{tikzcd}$$

where φ is the homomorphism whose existence is guaranteed by the regularity of $\bar{\pi}_i$ with respect to π_i. We just take $\theta = \varphi \circ P_i$.

Now suppose $\pi'\colon N' \longrightarrow Y$ is also in $\bigcap_i K(\pi_i)$. Pick a point $y \in Y$ and an idempotent $u \in J(y)$. Pick $\tilde{n} \in \tilde{N}$ and $n' \in N'$ such that $\pi'(n') = \tilde{\pi}(\tilde{n}) = y$, $\tilde{n}u = \tilde{n}$, and $n'u = n$. It will suffice to show that, given p and q in βT, $n'p = n'q \Longrightarrow \tilde{n}p = \tilde{n}q$ as this will insure the existence of a homomorphism $\alpha\colon N' \longrightarrow N$ such that $\alpha(n') = n$. Now we can write $\tilde{n} = \langle n_i \rangle$ where $n_i \in N_i$ for each i, $n_i u = n_i$ and $\bar{\pi}_i(n_i) = y$. Thus we need only show $n_i p = n_i q$ for each i. We define $z_i\colon \pi_i^{-1}(y) \longrightarrow X_i$ by $z_i(x) = x$ for all $x \in \pi_i^{-1}(y)$. Then we may represent N_i as $z_i M$ and, by regularity, there exists an automorphism β_i of N_i such that $\beta_i(z_i u) = n_i$, for each i. Therefore it suffices to show that $z_i up = z_i uq$, i.e., that $xup = xuq$ for all $x \in \pi_i^{-1}(y)$.

Given $x \in \pi_i^{-1}(y)$, we can define homomorphisms $\varkappa: M \longrightarrow X_i$ and $\lambda: M \longrightarrow N'$ such that $\varkappa(u) = xu$, and $\lambda(u) = n'$. We have a diagram

$$\begin{tikzcd} & M \arrow[dl, "\varkappa"'] \arrow[dr, "\lambda"] & \\ X_i \arrow[dr, "\pi_i"'] & & N' \arrow[ll, "\theta"] \arrow[dl, "\pi'"] \\ & Y & \end{tikzcd}$$

since $\pi' \in K(\pi_i)$ implies the existence of θ. Thus $xup = \theta(n'p) = \theta(n'q) = xuq$.

Next we see that $\tilde{\pi}$ is regular. Suppose (n,n') is an almost periodic pair in $\tilde{\pi}^{-1}(y)$ with $n = \langle n_i \rangle$ and $n' = \langle n_i' \rangle$. Then (n_i, n_i') is an almost periodic pair in $\bar{\pi}_i^{-1}(y)$, for each i, and we can use the regularity of the homomorphisms $\bar{\pi}_i$ to get the necessary automorphism.

Since $\tilde{\pi}: \tilde{N} \longrightarrow Y$ is regular, it is also coalescent. Thus, if there is another universally attracting object in $\bigcap_i K(\pi_i)$, it must be isomorphic to $\tilde{\pi}$. ||

Next we see that the regularizer of $\pi: X \longrightarrow Y$, where X is not necesarily minimal, depends only on the restrictions of π to minimal subsets of X.

Given $\pi: X \longrightarrow Y$, with Y minimal, let $\{W_i\}$ be the collection of minimal subsets of X and let $\pi_i = \pi | W_i$, for each i. Let $\bar{\pi}: N \longrightarrow Y$ be the regularizer of π and let $\tilde{\pi}: \tilde{N} \longrightarrow Y$ be the universally attracting object in $\bigcap_i K(\pi_i)$.

Proposition 2.5.2. *$\tilde{\pi}$ and $\bar{\pi}$ are isomorphic; i.e., there exists an isomorphism $\theta: \tilde{N} \longrightarrow N$ such that $\bar{\pi} \circ \theta = \tilde{\pi}$.*

Proof: Since $\bar{\pi}$ is the essentially unique universally attracting object in $K(\pi)$ it suffices to show that $\bar{\pi} \in \bigcap_i K(\pi_i)$ and $\tilde{\pi} \in K(\pi)$.

Suppose we're given $\alpha: M \longrightarrow W_i$ and $\beta: N \longrightarrow Y$ such that $\bar{\pi} \circ \beta = \pi_i \circ \alpha$. To show $\bar{\pi} \in K(\pi_i)$ we must find $\gamma: N \longrightarrow W_i$ completing the diagram below

$$\begin{tikzcd}
& M \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\
W_i \arrow[dr, "\pi_i"'] & & N \arrow[ll, "\gamma"'] \arrow[dl, "\bar{\pi}"] \\
& Y &
\end{tikzcd}$$

However, the existence of γ is guaranteed since $W_i \subset X$, $\pi_i = \pi | W_i$, and $\bar{\pi} \in K(\pi)$. Similarly, given $\delta: M \longrightarrow X$ and $\lambda: M \longrightarrow \tilde{N}$ with $\pi \circ \delta = \tilde{\pi} \circ \lambda$ we must find $\varphi: \tilde{N} \longrightarrow X$ completing the diagram

$$\begin{tikzcd}
& M \arrow[dl, "\delta"'] \arrow[dr, "\lambda"] & \\
X \arrow[dr, "\pi"'] & & \tilde{N} \arrow[ll, "\varphi"'] \arrow[dl, "\tilde{\pi}"] \\
& Y &
\end{tikzcd}$$

to show $\tilde{N} \in K(\pi)$. We can do this, since $\delta(M)$ must be one of the sets W_i, $\pi_i = \pi | W_i$, and $\tilde{\pi} \in K(\pi_i)$. $||$

Definition 2.5.3. Given a property P of homomorphisms and a minimal set Y, we say that P is *Y-admissible* if

(*i*) Y has a P-extension.

(*ii*) Given a family of P-extensions $\pi_i: X_i \longrightarrow Y$, the homomorphisms obtained by restricting the product $\pi: X \longrightarrow Y$ to minimal subsets of X also have property P (here $X = \{\langle x_i \rangle \mid \pi_i(x_i)$ the same for all $i\}$). We say that P is admissible if it is Y-admissible for every minimal set Y.

Definition 2.5.4. If P is a property of homomorphisms, we say that $\pi: X \longrightarrow Y$ is a P-universal extension of Y if it is universally repelling in the category of P-extensions of Y.

Proposition 2.5.5. *A homomorphism $\pi: X \longrightarrow Y$, with Y minimal, is regular if and only if it's a P-universal extension, for some Y-admissible property P.*

Proof: $\Rightarrow$Define a homomorphism $\psi: Z \longrightarrow Y$ to have property P if and only if there exists a homomorphism $\delta: X \longrightarrow Z$ with $\psi \circ \delta = \pi$. π is P-universal by construction, but we still must show that P is Y-admissible. Suppose we have families of homomorphisms $\psi_i: W_i \longrightarrow Y$ and $\delta_i: X \longrightarrow W_i$ with $\psi_i \circ \delta_i = \pi$ for each i. Let $\psi: W \longrightarrow Y$ be the product and suppose $w = \langle w_i \rangle$ is an almost periodic point in W. It will suffice to find a homomorphism $\gamma: X \longrightarrow W$ such that $\gamma(x') = w$ for some $x' \in \pi^{-1}(y)$ and $\psi \circ \gamma = \pi$. Let $y = \psi(w)$ and pick an idempotent $u \in J(w)$. Pick $x' \in \pi^{-1}(y)$ so $x'u = x'$. For each i, pick $x_i \in \delta_i^{-1}(w_i)$ so $x_i'u = x_i$. $\pi(x_i) = \pi(x')$ and (x', x_i) is an almost periodic pair, for each i, so there exist automorphisms θ_i on X such that $\theta_i(x') = x_i$. Define $\gamma: X \longrightarrow W$ by $\gamma(x') = \langle \delta_i \circ \theta_i(x') \rangle$.

Then $\gamma(x') = w$.

$\Leftarrow$ Let $\{\pi_i : X_i \longrightarrow Y\}$ be the family of P-extensions of Y and let $\tilde{\pi}: \tilde{N} \longrightarrow Y$ be the essentially unique, universally attracting object in $\cap_i K(\pi_i)$. Then the Y-admissibility of P and the method of construction of $\tilde{\pi}$ implies that $\tilde{\pi}$ is a P-extension. The construction also tells us that, for each i, there exists a homomorphism $\delta_i : \tilde{N} \longrightarrow X_i$ with $\pi_i \circ \delta = \tilde{\pi}$ so $\tilde{\pi}$ is P-universal, $\tilde{\pi}$ is regular, hence coalescent, and this implies that it is isomorphic to any other P-universal extension of Y. ||

<u>Corollary 2.5.6</u>. *If P is a Y-admissible property and Y is minimal, there exists an essentially unique P-universal extension of Y.*

<u>Proof</u>: Let $\{\pi_i : X_i \longrightarrow Y\}$ be the family of P-extensions of Y and let $\tilde{\pi}: \tilde{N} \longrightarrow Y$ again be the essentially unique, universally attracting object in $K(\pi_i)$. Then $\tilde{\pi}$ is the desired object, by the proof of the preceding proposition. ||

Next we show that distal, almost periodic, and proximal are all admissible properties of homomorphisms. Since the identity homomorphism is always distal, almost periodic, and proximal it will suffice to show that condition (*ii*) of Definition 2.5.3 is satisfied in each case.

<u>Proposition 2.5.7</u>. *Distal and almost periodic are admissible properties of homomorphisms.*

<u>Proof</u>: Assume we have a family of distal (almost periodic)

homomorphisms $\pi_i\colon X_i \longrightarrow Y$ with product $\pi\colon X \longrightarrow Y$ and projections $P_i\colon X \longrightarrow X_i$. Then if (x,x') is a relatively proximal (regionally proximal) pair in X each $(P_i(x), P_i(x'))$ is relatively proximal (regionally proximal) in X_i and the result follows. ||

The universal distal and almost periodic extensions of minimal sets are used extensively in [7].

<u>Lemma 2.5.8</u>. <u>Suppose</u> $\pi\colon X \longrightarrow Y$ <u>is a homomorphism of minimal sets. Then</u> π <u>is proximal if and only if</u> $\pi^{-1}(y)u$ <u>is a singleton whenever</u> $y \in Y$ <u>and</u> $u \in J(y)$.

<u>Proof</u>: $\Longrightarrow$ Suppose we have $y \in Y$, $u \in J(y)$ and $\pi(x) = \pi(x') = y$. Then the pair $(xu,x'u)$ is proximal and almost periodic. Therefore $xu = xu'$.
$\Longleftarrow$ Obvious. ||

<u>Proposition 2.5.9</u>. <u>Proximal is an admissible property of homomorphisms</u>.

<u>Proof</u>: Suppose Y is minimal and we have a family $\pi_i\colon X_i \longrightarrow Y$ of proximal homomorphisms and suppose Z is a minimal subset of the product $\pi\colon X \longrightarrow Y$. Let $P_i\colon Z \longrightarrow X_i$ denote the projections and let $W_i = P_i(Z)$, a minimal set. Suppose x and x' are in Z and $\pi(x) = \pi(x') = y$. Pick a minimal idempotent $u \in J(y)$. By the preceding lemma, we just need $xu = x'u$. Let $x = \langle x_i \rangle$ and $x' = \langle x_i' \rangle$. Applying the lemma to the homomorphisms $\pi_i | W_i$ we get $x_i u = x_i' u$ for each i, completing the proof. ||

We will also show that a homomorphism of minimal sets is distal (almost periodic) if and only if its regularizer is. The next two lemmas are well-known.

Lemma 2.5.10. *Given a diagram of minimal sets,*

$$\begin{tikzcd} X \arrow[dd, "\pi"'] \arrow[dr, "\lambda"] & \\ & W \arrow[dl, "\varphi"] \\ Y & \end{tikzcd}$$

π *is distal if and only if both* λ *and* φ *are distal.*

Proof: It is clear from the definition that λ and φ distal implies π distal and that π distal implies λ distal. Proposition 5.22 of [7] says that, since X and W are both minimal, λ carries the proximal relation on X onto the proximal relation on W. From this it follows that π distal implies λ distal. ||

Lemma 2.5.11. *If we have a diagram of minimal sets*

$$\begin{tikzcd} X \arrow[dd, "\pi"'] \arrow[dr, "\lambda"] & \\ & W \arrow[dl, "\varphi"] \\ Y & \end{tikzcd}$$

with π *almost periodic, then* φ *is almost periodic.*

Proof: By Theorem 1.2.6, 2^{π} is element-wise almost periodic and it will suffice to show that 2^{φ} is element-wise almost periodic. Suppose $A \in 2^{\varphi}$. Then $\lambda^{-1}(A) \in 2^{\pi}$ and

$\lambda^{-1}(A) = (\lambda^{-1}(A))\circ u$ for some $u \in J$. Then

$$A\circ u = (\lambda(\lambda^{-1}(A)))\circ u = \lambda((\lambda^{-1}(A))\circ u) = \lambda(\lambda^{-1}(A)) = A$$

and A is an almost periodic element. $||$

Proposition 2.5.12. *A homomorphism of minimal sets is distal (almost periodic) if and only if its regularizer is.*

Proof: Given $\pi: X \longrightarrow Y$, with X and Y minimal and π distal (almost periodic) the regularizer is also, by its method of construction, since distal (almost periodic) is an admissible property.

If $\overline{\pi}: N \longrightarrow Y$ is the regularizer of $\pi: X \longrightarrow Y$ and $\overline{\pi}$ is distal (almost periodic) then so is π by Lemmas 2.10 and 2.11, since $\overline{\pi}$ factors through π. $||$

Remark. The notion of a "universal object" used here and elsewhere in the literature of topological dynamics differs from standard usage. Generally an object Y in a category K is called "universally attracting" if, given any object X in K, there exists a unique morphism from X to Y. Such objects are automatically unique in the sense that given two such there is a unique isomorphism between them.

We call the object Y in the category K universally attracting if, given an object X in K, there exists some (not necessarily unique) homomorphism from X to Y. The uniqueness of a universal object in this sense remains to be proved. We generally do this by using a "coalescence" property (every endomorphism is an automorphism).

CHAPTER III

RELATIVE ENVELOPING SEMIGROUPS

The enveloping semi-group $E(X)$ of a transformation group, X, is at once both a semigroup and a transformation group. Both of these structures carry important dynamical information about X. Given a homomorphism $\pi: X \longrightarrow Y$, with Y minimal, we have constructed transformation groups $E(\pi,y)$, for each $y \in Y$ which in some ways generalize the transformation group structure of $E(X)$. This analogy is pursued further in this chapter. However, $E(\pi,y)$ has no semigroup structure. We will define another object associated with the homomorphism π, called $S(\pi,y)$, for each $y \in Y$, which is a semigroup. Some of the information in the semigroup structure of $E(X)$ will generalize to $S(\pi,y)$. Generally, the properties of $E(X)$ split in two directions when relativized, with the transformation group properties going to $E(\pi,y)$ and the semigroup properties going to $S(\pi,y)$.

Once again, in this chapter we'll be dealing with a fixed homomorphism $\pi: X \longrightarrow Y$, with Y minimal.

Let $P_\pi(y) = \{(x,x') \mid x,x'$ proximal in X and $\pi(x) = \pi(x') = y\}$, for each $y \in Y$. The following theorem generalizes a result of Ellis [7].

Theorem 3.1. *Let $y \in Y$. Then $P_\pi(y)$ is an equivalence relation if and only if $E(\pi,y)$ contains just one minimal set.*

Proof: $\Longrightarrow$ Suppose we have $y \in Y$ with $P_\pi(y)$ an equivalence relation. Recall that each minimal set in $E(\pi,y)$ is of the form $z_y I$, for some minimal right ideal I in $E(X)$. Suppose I and I' are minimal right ideals in $E(X)$; we'll show that $z_y I = z_y I'$. Pick equivalent idempotents u and u' in $J(I)$ and $J(I')$ respectively, such that $yu = y = yu'$. Consider $x \in \pi^{-1}(y)$; it will suffice to show that $xu = xu'$. We have $(x,xu) \in P_\pi(y)$ and $(x,xu') \in P_\pi(y)$. Therefore $(xu,xu') \in P_\pi(y)$ by hypothesis. Since xu and xu' are proximal, there exists an idempotent u'' in some minimal right ideal I'' in $E(X)$ such that u'' is equivalent to u and u' and $xuu'' = xu'u''$. We have $xu = xuu'' = xu'u'' = xu'$.

$\Longleftarrow$ Suppose $E(\pi,y)$ contains just one minimal set. Consider points x, x', x'' in X such that $(x,x') \in P_\pi(y)$ and $(x',x'') \in P_\pi(y)$. Then there exist minimal right ideals I and I' in $E(X)$ such that $xp = x'p$ for all $p \in I$ and $x'q = x''q$ for all $q \in I'$. By hypothesis $z_y I = z_y I'$ so we can pick $p \in I$ and $q \in I'$ so $z_y p = z_y q$. Then $xq = xp = x'p = x'q = x''q$. Thus x and x'' are proximal and $(x,x'') \in P_\pi(y)$. $P_\pi(y)$ is obviously reflexive and symmetric and we've shown $P_\pi(y)$ transitive so we have that $P_\pi(y)$ is an equivalence relation. ||

Corollary 3.2. *The relative (to π) proximal relation is an equivalence relation if and only if $E(\pi,y)$ contains just one minimal set, for all $y \in Y$.*

Examples of Markley [14] show that there are homomorphisms $\pi\colon X \to Y$ such that $P_\pi(y)$ is an equivalence relation for some points $y \in Y$ and not for others. Let C be a minimal subset of the circle under the action of an intransitive homeomorphism without periodic points. Then C is a Cantor set and there exists a homomorphism $\varphi\colon C \to Y$, where Y is a circle with an irrational rotation, such that each fiber is either a singleton or a doubly asymptotic pair of endpoints of an interval excluded from C. A distal homomorphism $\psi\colon X \to C$ is obtained by taking a "finite skew product" of C and the discrete space $\{1,\cdots,n\}$ under the action of the permutation group S_n. Take $\pi = \varphi\circ\psi$. Then if $y \in Y$ is the image of a doubly asymptotic pair in C, $P_\pi(y)$ will not generally be an equivalence relation but if $\varphi^{-1}(y)$ is a singleton, then $P_\pi(y)$ is the diagonal.

Some further consequences of Theorem 3.1 will be useful later.

<u>Proposition 3.3</u>. <u>If the relative (to π) proximal relation is closed it is also an equivalence relation</u>.

<u>Proof</u>: Take $y \in Y$. We need only show that $E(\pi,y)$ contains just one minimal set. Consider such minimal sets z_yI and z_yI' where I and I' are minimal right ideals in $E(X)$. Pick equivalent idempotents u and u' in $J(I)$ and $J(I')$, respectively, such that $yu = yu' = y$. Consider $x \in \pi^{-1}(y)$ and pick a net $\langle t_n\rangle$ in T such that $t_n \to u'$. We have x and xu relatively proximal; hence xt_n and

xut_n are relatively proximal for each n; hence xu' and xuu' are relatively proximal, since that relation is closed. However, (xu', xuu') is an almost periodic pair. Therefore $xu' = xuu' = xu$. Since x was an arbitrary point in $\pi^{-1}(y)$, we have $z_y u = z_y u'$. Therefore $z_y I = z_y I'$. ||

Proposition 3.4. *If both X and Y are minimal then the relative proximal relation is closed for $\pi: X \longrightarrow Y$ if and only if $\pi = \gamma \circ \lambda$ where λ is a proximal homomorphism and γ is a distal homomorphism.*

Proof: $\Longrightarrow$ Let P_π denote the relative proximal relation. P_π is certainly invariant, hence it is a closed invariant equivalence relation and $P_\pi \subset R_\pi$. Letting $W = X/P_\pi$ we have uniquely defined homomorphisms $\lambda: X \longrightarrow W$ and $\gamma: W \longrightarrow Y$ such that $\gamma \circ \lambda = \pi$. Clearly λ is proximal. Proposition 5.22 of [7] says that since X and W are minimal, λ carries the proximal relation on X onto the proximal relation on W. Suppose w and w' are proximal and $\gamma(w) = \gamma(w')$. Then there exist $x, x' \in X$ which are proximal and such that $\lambda(x) = w$ and $\lambda(x') = w'$. We have $\pi(x) = \gamma \circ \lambda(x) = \gamma(w) = \gamma(w') = \gamma \circ \lambda(x') = \pi(x')$ so (x,x') is in the relative (to π) proximal relation. Therefore $\lambda(x) = \lambda(x')$ and $w = w'$. Therefore γ is a distal homomorphism.

$\Longleftarrow$ Suppose we have a net $\langle (x_n, x_n') \rangle$ in X^2 with each (x_n, x_n') relatively proximal and $(x,x') \in X^2$ such that $(x_n, x_n') \longrightarrow (x,x')$. Then, for each n, $\lambda(x_n)$ and $\lambda(x_n')$

are proximal, by Proposition 5.22 of [7] and $\gamma \circ \lambda(x_n) = \gamma \circ \lambda(x'_n)$. Since γ is distal, we have $\lambda(x_n) = \lambda(x'_n)$ for each n. Therefore $\lambda(x) = \lambda(x')$ and x and x' are relatively proximal since λ is a proximal homomorphism. ||

Next we define the semigroups $S(\pi,y)$ associated with a homomorphism $\pi: X \longrightarrow Y$. For $y \in Y$, let $E_y = \{p \in \beta T \mid yp = y\}$ and define an equivalence relation $\sim$ on E_y by $p \sim q$ if and only if $xp = xq$ for all $x \in \pi^{-1}(y)$. For $p,q,r,s \in E_y$, we have that $p \sim q$ and $r \sim s$ implies $pr \sim qs$. Thus the quotient $E_y/\sim$ carries a semigroup structure.

Definition 3.5. Given a homomorphism $\pi: X \longrightarrow Y$ and $y \in Y$, the semigroup $S(\pi,y)$ is defined by $S(\pi,y) = E_y/\sim$.

Proposition 3.6. *Suppose* $\pi: X \longrightarrow Y$ *is a homomorphism of minimal sets. Then* π *is distal if and only if* $S(\pi,y)$ *is a group, for each* $y \in Y$. *Moreover, in this case, each* $S(\pi,y)$ *is isomorphic to* Aut $\overline{\pi}$, *where* $\overline{\pi}: N \longrightarrow Y$ *is the regularizer of* π.

Proof: Given an element $p \in E_y$, for some y, we'll let $\tilde{p}$ denote the equivalence class of p in $S(\pi,y)$. Suppose $S(\pi,y)$ is a group, $x,x' \in \pi^{-1}(y)$, and x and x' are proximal. Then there exists a minimal idempotent $u \in \beta T$ such that $x = xu = x'u$. Then $u \in E_y$ and $\tilde{u} \in S(\pi,y)$. By hypothesis, we can find $p \in E_y$ such that $\tilde{p} = \tilde{u}^{-1}$. Then $x' = x'up = xup = x$. Thus, if each $S(\pi,y)$ is a group, π must be distal.

Now suppose π is distal. Consider $y \in Y$ and $u \in J(y)$. We have $\pi^{-1}(y) = \pi^{-1}(y)u$ so we may represent $\overline{\pi}: N \longrightarrow Y$ by taking $N = \overline{z_y uT} = \overline{z_y T}$. Define a function $F: E_y \longrightarrow \mathrm{Aut}\,\overline{\pi}$ by $F(p) = \alpha$ if and only if $\alpha(z_y) = z_y p$. F is well-defined since endomorphisms of minimal sets are determined by their value at one point and since $\mathrm{Aut}\,\overline{\pi}$ is transitive on the fibers of $\overline{\pi}$. F is onto, since $\overline{z_y T} = N$. Also, for $p,q \in E_y$ we have

$$F(p) = F(q) \Longleftrightarrow z_y p = z_y q \Longleftrightarrow p \sim q.$$

Thus F induces a bijection $\tilde{F}: S(\pi,y) \longrightarrow \mathrm{Aut}\,\overline{\pi}$. Consider $p \in E_y$ and suppose $F(p) = \alpha$. There exists an automorphism $L_{up}: M \longrightarrow M$ defined by $L_{up}(m) = upm$ for $m \in M$. We can define a homomorphism $\lambda: M \longrightarrow N$ by $\lambda(u) = z_y$. Then $\overline{\pi}\circ\lambda = \overline{\pi}\circ\lambda\circ L_{up}$ and, by the regularity of $\overline{\pi}$, there exists an $\alpha' \in \mathrm{Aut}\,\overline{\pi}$ such that $\alpha'\circ\lambda = \lambda\circ L_{up}$. Now $\alpha'(z_y) = \alpha'\circ\lambda(u) = \lambda\circ L_{up}(u) = \lambda(upu) = z_y pu = z_y p = \alpha(z_y)$. Thus $\alpha' = \alpha$ and $F(p)\circ\lambda = \lambda\circ L_{up}$. Now consider $q \in E_y$ and suppose $F(q) = \beta$. Then $F(pq)(z_y) = F(upuq)(z_y) = F(upuq)\circ\lambda(u) = \lambda\circ L_{upuq}(u) = \lambda\circ L_{up}\circ L_{uq}(u) = \alpha\circ\lambda\circ L_{uq}(u) = \alpha\circ\beta\circ\lambda(u) = \alpha\circ\beta(z_y)$. Thus $\tilde{F}(\tilde{p}\tilde{q}) = F(pq) = \alpha\circ\beta$ when $\tilde{F}(\tilde{p}) = \alpha$ and $\tilde{F}(\tilde{q}) = \beta$. Thus $\tilde{F}$ is an isomorphism and $S(\pi,y)$ is a group. ||

CHAPTER IV

X^* AND Y^*

Given a homomorphism $\pi\colon X \longrightarrow Y$ of metric minimal sets, Veech, in [16] constructed a diagram

$$\begin{tikzcd} X^* \arrow[r, "\pi^*"] \arrow[d, "\delta"'] & Y^* \arrow[d, "\gamma"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

where the homomorphism π^* is open and the homomorphisms δ and γ are almost one-to-one. He used this to obtain his well-known structure theorem for point distal minimal sets. Ellis used a similar, but different construction to extend Veech's theorem to point-distal, quasi-separable homomorphisms in [8]. McMahon and Wu studied a non-metric version of the original Veech construction in [13], obtaining a partial generalization.

Here we obtain a complete generalization of this construction in the non-metric case. The notion of a "highly-proximal" homomorphism, recently formulated by Auslander, is essential and is studied in some detail. Certain uniqueness results are obtained. Our results are applied to strengthen Ellis' version of the Veech Structure Theorem. These results are also applied to almost finite extensions in Chapter V.

1. Open and Highly Proximal Homomorphisms.

We call a homomorphism open if it's an open map. Since we always work with compact Hausdorff spaces we may formulate openness quite easily in terms of nets. Thus a homomorphism $\pi: X \longrightarrow Y$ is open at a point $x \in X$ if given any net $\langle y_i \rangle$ in Y with $y_i \longrightarrow \pi(x)$ there exists a net $\langle x_j \rangle$ in X with $x_j \longrightarrow x$ and $\langle \pi(x_j) \rangle$ a subnet of $\langle y_i \rangle$. The following lemma puts this in terms of the enveloping semigroup action on 2^{π}.

Lemma 4.1.1. *Suppose* Y *minimal and* $y_0 \in Y$. *Then* $\pi: X \longrightarrow Y$ *is open at all points of* $\pi^{-1}(y_0)$ *if and only if* $\pi^{-1}(y) \circ p = \pi^{-1}(y_0)$ *whenever* $y \in Y$ *and* $p \in \beta T$ *such that* $yp = y_0$.

Proof: $\Longrightarrow$ Consider y and p with $yp = y_0$. Surely $\pi^{-1}(y) \circ p \subset \pi^{-1}(yp) = \pi^{-1}(y_0)$. Consider $x_0 \in \pi^{-1}(y_0)$. Pick a net $\langle t_i \rangle$ in T such that $t_i \longrightarrow p$. Then $yt_i \longrightarrow y_0$. Taking a subnet if necessary, we can find $\langle x_i \rangle$ in X such that $x_i \longrightarrow x_0$ and with each $\pi(x_i) = yt_i$. Thus we have $(x_i t_i^{-1}) t_i \longrightarrow x_0$ and each $x_i t_i^{-1} \in \pi^{-1}(y)$. Therefore $x_0 \in \pi^{-1}(y) \circ p$.

$\Longleftarrow$ Suppose we have y_0 and a net $\langle y_i \rangle$ in Y with $y_i \longrightarrow y_0$ and $x_0 \in \pi^{-1}(y_0)$. For each i we can find $p_i \in \beta T$ such that $y_i = y_0 p_i$. Taking a subnet if necessary, we can assume $p_i \longrightarrow p$ for some $p \in \beta T$. Then $y_0 p = \lim y_0 p_i = y_0$ so, by hypothesis, $\lim \pi^{-1}(y_0) \circ p_i = \pi^{-1}(y_0) \circ p = \pi^{-1}(y_0)$. Thus, for each i, we can pick $x_i \in \pi^{-1}(y_0) \circ p_i \subset \pi^{-1}(y_i)$ so that $x_i \longrightarrow x_0$. Therefore π is open at x_0. $||$

It is clear from the proof that we can replace βT by M in the above lemma. It is also clear that π is an open homomorphism if and only if $\pi^{-1}(y)\circ p = \pi^{-1}(yp)$ for any $y \in Y$ and $p \in \beta T$.

The following lemma is also useful:

Lemma 4.1.2. *Given a homomorphism* $\pi\colon X \longrightarrow Y$ *with* Y *minimal, the following are equivalent.*

(i) π *is open.*

(ii) *For every* $y \in Y$ *and* $p \in \beta T$, $\pi^{-1}(y)\circ p = \pi^{-1}(yp)$.

(iii) *For every* $y \in Y$ *and* $p \in M$, $\pi^{-1}(y)\circ p = \pi^{-1}(yp)$.

(iv) *For some* $y \in Y$ *and every* $p \in M$, $\pi^{-1}(y)\circ p = \pi^{-1}(yp)$.

Proof: We only need show (iv) $\Longrightarrow$ (iii). Suppose (iv) holds for some $y \in Y$. Pick $y' \in Y$ and $p \in M$ arbitrarily. We must show $\pi^{-1}(y')\circ p = \pi^{-1}(y'p)$. Pick $q \in M$ such that $yq = y'$ and $v \in J$ such that $pv = p$. Then $\pi^{-1}(y')\circ p = \pi^{-1}(y'p)\circ v$ (by Lemma 1.2.3) $= \pi^{-1}(yqp)\circ v = (\pi^{-1}(y)\circ qp)\circ v$ (by (iv)) $= \pi^{-1}(yqpv)$ (by (iv) again) $= \pi^{-1}(y'p)$. ||

Suppose X and Y are compact Hausdorff spaces. A map $\varphi\colon Y \longrightarrow 2^X$ is said to be *upper semi-continuous* if $\{y \in Y \mid \varphi(y) \subset U\}$ is open whenever U is an open subset of X. The following proposition is often quoted without proof. We provide a proof.

Proposition 4.1.3. *Suppose* Y *is a compact Hausdorff space,* X *is compact metric, and the map* $\varphi\colon Y \longrightarrow 2^X$ *is upper semi-continuous. Then* φ *is continuous on a dense,* G_δ, *subset of* Y.

Proof: For each $A \in 2^X$ and for each positive integer i, let $L_i(A)$ equal the minimum number of open balls of radius $1/i$ required to cover A (A is always compact). The following facts are easy to see:

(a) Suppose $A, B \in 2^X$ with A a proper subset of B. Then $L_i(A) < L_i(B)$ for some i.

(b) Suppose we have a net $\langle A_n \rangle$ in 2^X with $A_n \longrightarrow A$. Then for each i, $L_i(A_n) \leq L_i(A)$ for sufficiently large n.

Now suppose O is an open subset of Y. We'll show that φ is continuous on a G_δ subset of O. We define a sequence of sets

$$O = O_0 \supset P_0 \supset O_1 \supset P_1 \supset \cdots \supset O_i \supset P_i \supset \cdots$$

as follows:

Let $O_0 = O$. Let P_i be a closed subset of O_i which has non-empty interior and let $O_{i+1} = \{y \in \operatorname{Int} P_i \mid L_i(\varphi(y)) = m(i)\}$ where $m(i) = \min \{L_i(\varphi(y)) \mid y \in \operatorname{Int} P_i\}$, for each i.

To insure that everything is well-defined we must verify that O_i open implies O_{i+1} open. Suppose O_i is open and $y \in O_{i+1}$. Then there exist open balls $B_1, \cdots, B_{m(i)}$, each of radius $1/i$ such that $\varphi(y) \subset B_1 \cup \cdots \cup B_{m(i)}$. By the upper semi-continuity of φ, y has an open neighborhood Q with $\varphi(y') \subset B_1 \cup \cdots \cup B_{m(i)}$ for all $y' \in Q$. Then $Q \cap \operatorname{Int} P_i$ is an open neighborhood of y which is contained in O_{i+1}. Thus O_{i+1} is open.

We have $\cap \{O_i\} = \cap \{P_i\} \neq \phi$ by compactness. We'll show φ to be continuous on $\cap\{O_i\}$. Consider $y \in \cap\{O_i\}$ and a net $\langle y_n \rangle$ in Y with $y_n \longrightarrow y$ and $\varphi(y_n) \longrightarrow A$, for some $A \in 2^X$. We must show that $A = \varphi(y)$. The upper semi-continuity of φ directly implies $A \subset \varphi(y)$. In light of (a), it then suffices to show $L_i(A) \geq L_i(\varphi(y))$ for all i. Consider some $i > 0$. For some N_1 we have $y_n \in O_{i+1}$ for all $n \geq N_1$, since O_{i+1} is a neighborhood of y. By (b), there exists an N_2 such that $L_i(\varphi(y_n)) \leq L_i(A)$ whenever $n \geq N_2$. Now pick an $n \geq N_1, N_2$. Then $L_i(\varphi(y_n)) = L_i(\varphi(y)) = m(i)$ since both y and y_n are in O_{i+1}. Finally, we have $L_i(A) \geq L_i(\varphi(y_n)) = L_i(\varphi(y))$ and we're done. ||

This proposition is the key to some of those results in topological dynamics which hold for metric spaces only. We will just use the following corollary.

<u>Corollary 4.1.4</u>. <u>Suppose</u> X <u>and</u> Y <u>are compact metric spaces and</u> $\psi: X \longrightarrow Y$ <u>is a continuous map</u>. <u>Then</u> ψ <u>is open at all points of</u> $\psi^{-1}(y)$ <u>for a dense</u>, G_δ <u>set of points</u> $y \in Y$.

<u>Proof</u>: ψ continuous implies $\psi^{-1}: Y \longrightarrow 2^X$ upper semi-continuous. For $y \in Y$, ψ^{-1} is continuous at y if and only if ψ is open at all points of $\psi^{-1}(y)$. ||

<u>Proposition 4.1.5</u>. <u>Suppose</u> $\pi: X \longrightarrow Y$ <u>is a homomorphism with</u> Y <u>minimal</u>. <u>Then the following are equivalent</u>:

(*i*) <u>All the almost periodic elements of</u> 2^π <u>are singletons</u>.

(*ii*) <u>For some</u> $y \in Y$ <u>and some net</u> $\langle t_n \rangle$ <u>in</u> T, $\lim \langle \pi^{-1}(y)t_n \rangle$ <u>is a singleton</u>.

(*iii*) <u>For some</u> $y \in Y$ <u>and some</u> $p \in \beta T$, $\pi^{-1}(y)\circ p$ <u>is a singleton</u>.

<u>Proof</u>: (*ii*) $\Longleftrightarrow$ (*iii*) is obvious as is (*i*) $\Longrightarrow$ (*iii*) since an element $A \in 2^{\pi}$ is almost periodic if and only if $A\circ u = A$ for some $u \in J$. We show (*iii*) $\Longrightarrow$ (*i*).

Suppose $y \in Y$, $p \in \beta T$ such that $\pi^{-1}(y)\circ p$ is a singleton. Pick $u \in J(y)$. Consider an almost periodic element $A \in 2^{\pi}$. Pick $y' \in Y$ and $u' \in J$ such that $A\circ u' = A$ and $A \subset \pi^{-1}(y')$. Pick $q \in M$ so $ypq = y'$. Using Lemma 1.2.3 we have

$$A = A\circ u' \subset \pi^{-1}(y')\circ u' = \pi^{-1}(yupq)\circ u'$$

$$= \pi^{-1}(y)\circ(upqu') \subset (\pi^{-1}(y)\circ p)\circ(qu')$$

and this last set is a singleton; consequently A is also. ||

<u>Definition 4.1.6</u>. We say that a homomorphism $\pi: X \longrightarrow Y$, with Y minimal is <u>highly proximal</u> if the three equivalent conditions of Proposition 4.1.5 are satisfied.

A simple argument shows that a highly proximal extension of a trivial minimal set is itself trivial. Suppose $\pi: X \longrightarrow 1$ is such an extension. Then $X\circ u$ is a singleton for $u \in J$. Pick a net $\langle t_n \rangle$ in T with $t_n \longrightarrow u$. Then $X\circ u = \lim_n Xt_n = \lim_n X = X$, so X is a singleton. This means that the notion of a highly proximal homomorphism is purely relative, being devoid of content in the absolute case.

A homomorphism is said to be almost one-to-one if some fiber is a singleton. Next we see that highly proximal is a non-metric generalization of almost one-to-one.

Proposition 4.1.7. *Suppose* $\pi\colon X \to Y$ *is a homomorphism, with* Y *minimal and* X *metrizable. Then* π *is highly proximal if and only if* π *is almost one-to-one.*

Proof: Almost one-to-one implies highly proximal in any case. Assume that π is highly proximal. X metrizable implies Y metrizable, so Corollary 4.1.4 applies and we can find $y \in Y$ with π open at all points of $\varphi^{-1}(y)$. Pick $u \in J(y)$. Then $\pi^{-1}(y) = \pi^{-1}(y) \circ u$, a singleton, by Lemma 4.1.1. ||

Certain results concerning almost one-to-one homomorphisms require a metrizability assumption which can be removed by substituting "highly proximal" for "almost one-to-one." Highly proximal is a nicer notion to work with since it is homogeneous in the sense that it can be defined without distinguishing a special set of points in the range space.

Next we'll see that the class of highly proximal homomorphisms is closed under composition and under certain kinds of products and inverse limits. The corresponding properties of almost one-to-one homomorphisms hold only if we remain within the category of pointed minimal sets.

Lemma 4.1.8. Suppose we have homomorphisms of minimal sets $\pi: X \longrightarrow Y$ and $\varphi: Y \longrightarrow Z$. Then $\varphi \circ \pi$ is highly proximal if and only if both π and φ are highly proximal.

Proof: $\Longrightarrow$ Assume $\varphi \circ \pi$ is highly proximal; pick $z \in Z$ and $u \in J(z)$. Pick x so $(\pi^{-1}(\varphi^{-1}(z)))\circ u = \{x\}$ and let $y = \pi(x)$. Then

$$\varphi^{-1}(z)\circ u = (\pi(\pi^{-1}(\varphi^{-1}(z))))\circ u = \pi((\pi^{-1}(\varphi^{-1}(z)))\circ u)$$

$$= \pi(\{x\}) = \{y\}$$

so φ is highly proximal. Also

$$\pi^{-1}(y)\circ u \subset (\pi^{-1}(\varphi^{-1}(z)))\circ u = \{x\}$$

so π is highly proximal.

$\Longleftarrow$ Assume both π and φ are highly proximal. Pick $z \in Z$ and $u \in J(z)$. Take $x \in X$ and $y \in Y$ so $\varphi^{-1}(z)\circ u = \{y\}$ and $\pi^{-1}(y)\circ u = \{x\}$. Let $\lambda = \varphi \circ \pi$. Then

$$\lambda^{-1}(z)\circ u = (\pi^{-1}(\varphi^{-1}(z)))\circ u \subset \pi^{-1}(\varphi^{-1}(z)\circ u) = \pi^{-1}(y)$$

so

$$\lambda^{-1}(z)\circ u = \lambda^{-1}(z)\circ u\circ u \subset \pi^{-1}(y)\circ u = \{x\}.$$

Thus λ is highly proximal. ||

Lemma 4.1.9. Highly proximal is an admissible property.

Proof: Consider a family $\pi_i: X_i \longrightarrow Y$ of highly proximal extensions of a fixed minimal set. Let W be some minimal set in the product $X = \{\langle x_i \rangle \in \prod_i \{X_i\} \mid \pi_i(x_i)$ the same for all $i\}$. Let $\pi: W \longrightarrow Y$ be the natural homomorphism. Suppose $y \in Y$ and $u \in J(y)$. There is an

$\bar{x}_i$ in each X_i such that $\{\bar{x}_i\} = \pi_i^{-1}(y) \circ u$. Suppose $\langle x_i \rangle \in \pi^{-1}(y) \circ u$. Then there is a net $\langle t_n \rangle$ in T with $t_n \longrightarrow u$ and an element $\langle x_i' \rangle \in W$ such that $x_i' t_n \longrightarrow x_i$ and $\pi_i(x_i') = y$ for each i. Therefore $x_i \in \pi_i^{-1}(y) \circ u$ for each i. Thus $\langle x_i \rangle = \langle \bar{x}_i \rangle$, and $\pi^{-1}(y) \circ u = \{\langle \bar{x}_i \rangle\}$. Thus π is highly proximal. ||

Suppose $\{(X_i, x_i) \mid i \in I\}$ is an inverse system of point transitive flows with homomorphisms $\gamma_{ij}: (X_i, x_i) \longrightarrow (X_j, x_j)$ whenever $i > j$ in the directed set I. Such a system has a unique inverse limit (X_∞, x_∞) where $x_\infty = \langle x_i \rangle$ and $X_\infty = \overline{\langle x_i \rangle T} \subset \prod_i \{X_i\}$. Given an inverse system of flows $\{X_i\}$ with homomorphisms γ_{ij} without base points it is possible to choose basepoints in a way that is consistent with the homomorphisms γ_{ij}. This follows from the fact that the inverse limit of an inverse system of compact Hausdorff spaces and continuous maps is necessarily non-empty [5].

Definition 4.1.10. Suppose $\{X_i \mid i \in I\}$ is an inverse system of minimal flows with homomorphisms $\gamma_{ij}: X_i \longrightarrow X_j$ whenever $i > j$. Suppose (X_∞, x_∞) is the inverse limit of the point transitive system $\{(X_i, x_i) \mid i \in I\}$ for some choice of base points $x_i \in X_i$ consistent with the homomorphisms γ_{ij}. We then say that X_∞ is a pointed inverse limit of the system $\{X_i \mid i \in I\}$.

Thus an inverse system of minimal sets $\{X_i \mid i \in I\}$ has at least one pointed inverse limit X_∞. In fact X_∞ can be chosen to be minimal; just let a fixed $u \in J$ act on one choice of base points $x_i \in X_i$ to obtain an almost periodic set of base points.

__Lemma 4.1.11__. __Suppose__ $\{\pi_i: X_i \to Y \mid i \in I\}$ __is a family of proximal homomorphisms of minimal sets. Let__ $X = \{\langle x_i \rangle \in \prod_i \{X_i\} \mid \pi_i(x_i)$ __the same for all__ $i\}$. __Then__ X __contains a unique minimal set.__

__Proof__: Suppose $\langle x_i \rangle$, $\langle x_i' \rangle$ are almost periodic points in X. Then there exist idempotents $u,u' \in J$ such that $\langle x_i \rangle = \langle x_i u \rangle$ and $\langle x_i' \rangle = \langle x_i' u' \rangle$. Fix a coordinate $j \in I$. We can pick $p \in M$ so $p = upu'$ and $x_j p = x_j'$ by the minimality of X_j. Now we'll show $x_i' = x_i p$ for an arbitrary coordinate $i \in I$. We have $\pi_i(x_i p) = \pi_j(x_j p) = \pi_j(x_j') = \pi_i(x_i')$. Therefore $(x_i p, x_i')$ is a proximal pair since π_i is a proximal homomorphism. Also $(x_i p, x_i')$ is almost periodic, since $(x_i p, x_i') = (x_i p, x_i')u'$. Thus $x_i p = x_i'$ and we have $\langle x_i' \rangle = \langle x_i \rangle p$. Therefore any two minimal sets in X are the same. ||

__Lemma 4.1.12__. __Suppose we have an inverse system of minimal sets__ $\{X_i \mid i \in I\}$ __with homomorphisms__ γ_{ij} __whenever__ $i > j$ __in the directed set__ I. __Suppose every__ γ_{ij} __is highly proximal and suppose the system has a least element__ X_0, __so that each__ X_i __is an extension of__ X_0. __Then the system has a unique minimal pointed inverse limit__ X_∞, __and the canonical projections__ $P_i: X_\infty \to X_i$ __are all highly proximal.__

__Proof__: Suppose X_∞ is a minimal pointed inverse limit.

Then $X_\infty \subset \{\langle x_i\rangle \in \prod_i \{X_i\} \mid \gamma_{i0}(x_i)$ the same for all $i\}$. Hence X_∞ must be the unique minimal subset of $\{\langle x_i\rangle \in \prod_i \{X_i\} \mid \gamma_{i0}(x_i)$ the same for all $i\}$ by Lemma 4.1.11.

The projection $P_0\colon X_\infty \longrightarrow X_0$ is highly proximal by Lemma 4.1.9 and the other projections $P_i\colon X_\infty \longrightarrow X_i$ are highly proximal by Lemma 4.1.8. ||

2. Construction and Basic Properties of X^* and Y^*

For the next few pages $\pi\colon X \longrightarrow Y$ will be a fixed homomorphism with Y minimal. We'll show that there is a unique minimal set contained in the orbit closure of a fiber in 2^π, and that this is independent of the fiber chosen.

Given a minimal set M in βT and $y \in Y$ we let $Y^*_{y,M} = \pi^{-1}(y)\circ M$.

<u>Proposition 4.2.1.</u> *Given minimal sets M and M' in βT and points $y_0, y_1 \in Y$, $Y^*_{y_0,M} = Y^*_{y_1,M'}$.*

<u>Proof</u>: First we show that $Y^*_{y_0,M} = Y^*_{y_0,M'}$. Pick equivalent idempotents $u \in M$ and $u' \in M'$ with $y_0 = y_0u = y_0u'$. $\pi^{-1}(y_0)\circ u \subset \pi^{-1}(y_0)$ so $\pi^{-1}(y_0)\circ u = \pi^{-1}(y_0)\circ(uu') = (\pi^{-1}(y_0)\circ u)\circ u' \subset \pi^{-1}(y_0)\circ u'$. Likewise $\pi^{-1}(y_0)\circ u' \subset \pi^{-1}(y_0)\circ u$ and we have $\pi^{-1}(y_0)\circ u = \pi^{-1}(y_0)\circ u'$ and $Y^*_{y_0,M} \cap Y^*_{y_0,M'}$ is non-empty. Since minimal subsets are disjoint or equal, we have $Y^*_{y_0,M} = Y^*_{y_0,M'}$.

Next we show that $Y^*_{y_0,M'} = Y^*_{y_1,M'}$. Once again it will suffice to show that the two minimal sets have a point in common. Pick $p \in M'$ such that $y_0 p = y_1$. Pick a minimal idempotent $u \in M'$ such that $pu = p$. Then, using Lemma 1.2.3,

$$\pi^{-1}(y_0)\circ p = \pi^{-1}(y_0)\circ(pu) = \pi^{-1}(y_0 p)\circ u = \pi^{-1}(y_1)\circ u.$$

We have $\pi^{-1}(y_0)\circ p \in Y^*_{y_0,M'}$ and $\pi^{-1}(y_1)\circ u \in Y^*_{y_1,M'}$. $\qquad ||$

We've now shown that $Y^*_{y,M}$ is independent of the choice of y and M. We call the resulting unique minimal set Y^*. We have a natural homomorphism $\gamma\colon Y^* \longrightarrow Y$ where for $A \in Y^*$, $\gamma(A) = y$ if and only if $A \subset \pi^{-1}(y)$.

Next we obtain a more convenient description of Y^*.

Proposition 4.2.2. $Y^* = \{\pi^{-1}(y)\circ u \mid y \in Y \text{ and } u \in J(y)\}$.

Proof: If $y \in Y$ and $u \in J(y)$, $\pi^{-1}(y)\circ u \in Y^*_{y,M}$ and $Y^*_{y,M} = Y^*$ by Proposition 4.2.1. Fix $y_0 \in Y$ and $u_0 \in J(y)$ and represent Y^* as $Y^*_{y_0,M}$. Consider an arbitrary $y^* \in Y^*$. Then $y^* = (\pi^{-1}(y_0)\circ u_0)\circ(pv)$ for some $p \in M$ and $v \in J$. Let $y = y_0 pv$. Then we have $y^* = (\pi^{-1}(y_0)\circ u_0)\circ(pv) = \pi^{-1}(y_0)\circ(pv) = \pi^{-1}(y_0 p)\circ v = \pi^{-1}(y)\circ v$ by Lemma 1.2.3. Thus $y^* \in \{\pi^{-1}(y)\circ u \mid y \in Y \text{ and } u \in J(y)\}$. $\qquad ||$

Proposition 4.2.3. The homomorphism $\gamma\colon Y^* \longrightarrow Y$ is highly proximal.

Proof: Consider $y \in Y$. We observe that the set $\gamma^{-1}(y)$ satisfies the hypothesis of Zorn's Lemma under downward inclusion. Suppose $\{A_i \mid i \in I\}$ is a chain in $\gamma^{-1}(y)$, and let $A_0 = \cap\{A_i \mid i \in I\}$. If we regard $\langle A_i\rangle_{i\in I}$

as a net directed by downward inclusion, we have $\lim_i \langle A_i \rangle = A_0$, so $A \in \gamma^{-1}(y)$ and A_0 is the required lower bound. Applying Zorn's Lemma, let B be a minimal element in $\gamma^{-1}(y)$. Pick $p \in M$ so $B = \pi^{-1}(y) \circ p$.

Now we'll show that $\gamma^{-1}(y) \circ p = \{B\}$. Suppose $A \in \gamma^{-1}(y) \circ p$. Then there exists nets $\langle t_i \rangle$ in T and $\langle A_i \rangle$ in $\gamma^{-1}(y)$ such that $t_i \longrightarrow p$ and $A_i t_i \longrightarrow A$. Each $A_i \subset \pi^{-1}(y)$ so $A = \lim A_i t_i \subset \lim \pi^{-1}(y) t_i = \pi^{-1}(y) \circ p = B$ and $\gamma(A) = y$. Thus $A = B$ by minimality.

Therefore γ is highly proximal. ||

<u>Corollary 4.2.4</u>. <u>For</u> $A \in Y^*$, $y \in Y$, <u>and</u> $u \in J(y)$ <u>we have</u> $A \subset \pi^{-1}(y)$ <u>and</u> $A \circ u = A \Longrightarrow A = \pi^{-1}(y) \circ u$.

<u>Proof</u>: If $A \subset \pi^{-1}(y)$ and $A \circ u = A$ then $A \in \gamma^{-1}(y) \circ u$. Also, $\pi^{-1}(y) \circ u \in \gamma^{-1}(y) \circ u$. Thus $A = \pi^{-1}(y) \circ u$, since γ is highly proximal. ||

Next we define X^* as a subset of $X \times Y^*$.

<u>Definition 4.2.5</u>. $X^* = \{(xu, \pi^{-1}(y) \circ u) \mid y \in Y, x \in \pi^{-1}(y), \text{ and } u \in J(y)\}$; $\delta : X^* \longrightarrow X$ and $\pi^* : X^* \longrightarrow Y^*$ are the projections.

For the rest of this section we assume both X and Y to be minimal.

<u>Proposition 4.2.6</u>. X^* <u>is a minimal set</u>.

<u>Proof</u>: Take a fixed $y_0 \in Y$, $x_0 \in \pi^{-1}(y_0)$, and $u_0 \in J(y_0)$. We'll show that $X^* = (x_0 u_0, \pi^{-1}(y_0) \circ u_0)M$.

Consider $p \in M$ and let $x^* = (x_0u_0, \pi^{-1}(y_0)\circ u_0)p$. Pick $u \in J$ so $pu = p$. Then $x^* = (x_0pu, \pi^{-1}(y_0)\circ(pu)) = (x_0pu, \pi^{-1}(y_0p)\circ u)$ and $\pi(x_0p) = y_0p$. Hence $x^* \in X^*$. Now consider an arbitrary $x' \in X^*$. Then $x' = (x_1u_1, \pi^{-1}(y_1)\circ u_1)$ for some $y_1 \in Y$, $u_1 \in J(y_1)$, and $x_1 \in \pi^{-1}(y_1)$. Pick $p \in M$ such that $x_1 = x_0p$. Then $x' = (x_1u_1, \pi^{-1}(y_1)\circ u_1) = (x_0pu_1, \pi^{-1}(y_0p)\circ u_1) = (x_0pu_1, \pi^{-1}(y_0)\circ(pu_1)) = (x_0u_0, \pi^{-1}(y_0)\circ u_0)pu_1$. Thus $x' \in (x_0u_0, \pi^{-1}(y_0)\circ u_0)M$. ||

It is clear from our construction that X^* is the unique minimal set in $\{(x,y^*) \mid x \in X,\ y^* \in Y^* \text{ and } \pi(x) = \gamma(y^*)\}$.

Proposition 4.2.7. *The homomorphism $\delta: X^* \longrightarrow X$ is highly proximal.*

Proof: Take $x \in X$ and $v \in J(x)$. Let $y = \pi(x)$. Then

$$\begin{aligned}\delta^{-1}(x) &= \{(xu, \pi^{-1}(y)\circ u) \mid u \in J(x)\} \\ &= \{x\} \times \{\pi^{-1}(y)\circ u \mid u \in J(x)\}.\end{aligned}$$

Therefore

$$\begin{aligned}\delta^{-1}(x)\circ v &= (\{x\} \times \{\pi^{-1}(y)\circ u \mid u \in J(x)\})v \\ &= \{x\} \times (\{\pi^{-1}(y)\circ u \mid u \in J(x)\}\circ v) \\ &\subset \{x\} \times (\{\pi^{-1}(y)\circ u \mid u \in J(y)\}\circ v) \\ &= \{x\} \times (\gamma^{-1}(y)\circ v).\end{aligned}$$

This last set is a singleton, since γ is highly proximal. Therefore δ is highly proximal. ||

Proposition 4.2.8. *The following statements are equivalent.*

(*i*) $\gamma: Y^* \longrightarrow Y$ *is an isomorphism.*

(*ii*) $\gamma: Y^* \longrightarrow Y$ *and* $\delta: X^* \longrightarrow X$ *are both isomorphisms.*

(*iii*) $\pi: X \longrightarrow Y$ *is open.*

Proof: The equivalence of (*i*) and (*ii*) is clear from the definitions.

(*i*) $\Longrightarrow$ (*iii*) Suppose γ is an isomorphism and pick $A \in Y^*$ and $y \in Y$ so $\gamma^{-1}(y) = \{A\}$. Consider $x \in \pi^{-1}(y)$ and $u \in J(x)$. Then $yu = y$ so we have $\pi^{-1}(y) \circ u \in \gamma^{-1}(y)$. Therefore $\pi^{-1}(y) \circ u = A$. Also $x \in \pi^{-1}(y) \circ u$, since $xu = x$. Since x was an arbitrary element of $\pi^{-1}(y)$, we have $\gamma^{-1}(y) = \{\pi^{-1}(y)\}$. Now for arbitrary $y \in Y$ and $p \in M$ we have $\pi^{-1}(y) \circ p \in \gamma^{-1}(yp)$. Therefore $\pi^{-1}(yp) = \pi^{-1}(y) \circ p$ and π is open by Lemma 4.1.1.

(*iii*) $\Longrightarrow$ (*i*) Suppose π is open and consider $y \in Y$. Then, by Lemma 4.1.1, $\gamma^{-1}(y) = \{\pi^{-1}(y) \circ u \mid u \in J(y)\} = \{\pi^{-1}(y)\}$. Thus γ is one-to-one and hence an isomorphism. ||

We now have enough machinery to construct a diagram as described in the beginning of the chapter by iteration. We make the following remarks:

(*i*) It is clear that an inverse system of diagrams of minimal point-transitive transformation groups of the type below has an inverse limit which is a diagram of the same type.

$$\begin{tikzcd}
(X_i,x_i) \arrow[r, "\pi_i"] \arrow[d, "\delta_i"] & (Y_i,y_i) \arrow[d, "\gamma_i"] \\
(X,x) \arrow[r, "\pi"'] & (Y,y)
\end{tikzcd}$$

(ii) We may produce an ordinal chain of such diagrams by starting with $\pi\colon X \longrightarrow Y$, applying the X^*, Y^* construction at successor ordinal stages, and taking pointed inverse limits at limit ordinal stages. It follows from Lemmas 4.1.8 and 4.1.12 that the homomorphisms δ_i and γ_i obtained at each stage are uniquely defined and highly proximal. Cardinality considerations imply that the procedure must terminate and, when it does, the last π_i obtained will be open by Proposition 4.2.8.

(iii) If we denote the last stage of this construction by $i = \infty$ and we define $\bar{\pi}_\infty = \pi \circ \delta_\infty = \gamma_\infty \circ \pi_\infty$, then $\bar{\pi}_\infty\colon X_\infty \longrightarrow Y$ is a point-transitive product of $\pi\colon X \longrightarrow Y$ and $\gamma_\infty\colon Y_\infty \longrightarrow Y$ for an appropriate choice of base points.

McMahon and Wu carried out a construction of this type in [13] without the notion of highly proximal.

Actually, such a construction is unnecessary as Glasner has recently shown that the homomorphism $\pi^*\colon X^* \longrightarrow Y^*$, obtained at the first stage, is always open. The argument which follows is essentially his, but we isolate a lemma concerning the structure of the fibers of π which may be of independent interest.

Given a homomorphism of minimal sets $\pi\colon X \longrightarrow Y$ and a point $y \in Y$ we let $J_\pi(y)$ denote the set of minimal idempotents $u \in J(y)$ such that $\pi^{-1}(y) \circ u$ is an inclusion minimal element of Y^*. It follows from Zorn's Lemma that $J_\pi(y)$ is always non-empty and that, in fact, given $A \in Y^*$ with $A \subset \pi^{-1}(y)$ there exists a $u \in J_\pi(y)$ such that $\pi^{-1}(y) \circ u \subset A$.

Lemma 4.2.9. *Suppose* $y \in Y$ *and* $u \in J_\pi(y)$. *Then* $\pi^{-1}(y) \circ u = \cup \{\pi^{-1}(y)v \mid v \in J(y) \text{ and } \pi^{-1}(y) \circ v = \pi^{-1}(y) \circ u\}$.

Proof: Let $K = \cup \{\pi^{-1}(y)v \mid v \in J(y) \text{ and } \pi^{-1}(y) \circ v = \pi^{-1}(y) \circ u\}$. Suppose $x \in K$. Then $x \in \pi^{-1}(y)v$ for some $v \in J(y)$ with $\pi^{-1}(y) \circ v = \pi^{-1}(y) \circ u$, and we have $\pi^{-1}(y)v \subset \pi^{-1}(y) \circ v = \pi^{-1}(y) \circ u$. Therefore $K \subset \pi^{-1}(y) \circ u$.

Now suppose $x \in \pi^{-1}(y) \circ u$. Then there are nets $\langle x_i \rangle$ in $\pi^{-1}(y)$ and $\langle t_i \rangle$ in T such that $x_i t_i \longrightarrow x$ and $t_i \longrightarrow u$. Pick $p_i \in M$ such that $x_i = xp_i$ for each i. Taking a subnet if necessary, we may assume that $\langle p_i t_i \rangle$ converges to some $q \in M$. Then $xp_i t_i \longrightarrow x$, so $xq = x$. For each i, $\pi^{-1}(y) \circ (p_i t_i) = (\pi^{-1}(y) \circ p_i) t_i \subset (\pi^{-1}(yp_i)) t_i = \pi^{-1}(y) t_i$. Taking limits, $\pi^{-1}(y) \circ q \subset \pi^{-1}(y) \circ u$. Since $u \in J_\pi(y)$ we then have $\pi^{-1}(y) \circ q = \pi^{-1}(y) \circ u$ by minimality. Now $yq = y$, so we can find $v \in J(y)$ with $qv = q$. By Lemma 1.2.3 we have $\pi^{-1}(y) \circ q = \pi^{-1}(yq) \circ v = \pi^{-1}(y) \circ v = \pi^{-1}(y) \circ u$. Finally, $x = xq = xqv = xv$ so $x \in \pi^{-1}(y)v$. Therefore $K \supset \pi^{-1}(y) \circ u$. ||

Proposition 4.2.10. *The homomorphism* $\pi^*: X^* \longrightarrow Y^*$ *is open*.

<u>Proof</u>: By Lemma 4.1.2, it will suffice to find an element $y^* \in Y^*$ such that for any $p \in M$, $\pi^{*-1}(y^*)\circ p \supset \pi^{*-1}(y^*p)$. To this end we pick an arbitrary $y \in Y$ and an idempotent $u \in J_\pi(y)$ and let $y^* = \pi^{-1}(y)\circ u$. Letting $K = \cup\,\{\pi^{-1}(y)v \mid v \in J(y)$ and $\pi^{-1}(y)\circ v = \pi^{-1}(y)\circ u\}$ we have $\pi^{*-1}(y^*) = K \times \{y^*\}$ (by Definition 4.2.5) $=$ $(\pi^{-1}(y)\circ u) \times \{y^*\}$ (by Lemma 4.2.9). Now consider an arbitrary $p \in M$ and pick $v \in J$ such that $pv = p$. We have $y^*p = \pi^{-1}(y)\circ p = \pi^{-1}(yp)\circ v$ by Lemma 1.2.3. Thus

$$
\begin{aligned}
&\pi^{*-1}(y^*p)\\
&= \cup\,\{\pi^{-1}(yp)w \mid w \in J(yp) \text{ and } \pi^{-1}(yp)\circ w = \pi^{-1}(yp)\circ v\} \times \{y^*p\}\\
&\qquad\qquad \subset (\pi^{-1}(yp)\circ v) \times \{y^*p\}\\
&= (\pi^{-1}(y)\circ p) \times \{y^*p\}\\
&= (\pi^{-1}(y)\circ u\circ p) \times \{y^*p\}\\
&= ((\pi^{-1}(y)\circ u) \times \{y^*\})\circ p\\
&= \pi^{*-1}(y^*)\circ p
\end{aligned}
$$

completing the proof. ||

3. <u>An Abstract Characterization</u>.

Given a homomorphism of minimal sets $\pi: X \longrightarrow Y$ we have constructed a diagram of minimal sets

$$
\begin{tikzcd}
X^* \arrow[r, "\pi^*"] \arrow[d, "\delta"'] & Y^* \arrow[d, "\gamma"] \\
X \arrow[r, "\pi"'] & Y
\end{tikzcd}
$$

where γ and δ are highly proximal and π^* is open. Our construction is unique in that no undetermined choices are involved. In this section our goal is to obtain an abstract characterization of this diagram, up to isomorphism of diagrams in the obvious sense. By this we mean a description which uses only the "layout" of the diagram and the abstract properties of the arrows, e.g., δ and γ being highly proximal and π^* being open.

A first guess might be that given π, the diagram we've constructed is the only such diagram with δ and γ highly proximal and π^* open. However a trivial example shows that this won't work. Take $X = Y$ and $X^* = Y^*$ with Y^* being any highly proximal extension of Y and with π and π^* being the identity maps on Y and Y^* respectively. Then by varying which highly proximal extension of Y we take Y^* to be, we get nonisomorphic diagrams.

We will show that the diagram we constructed is, up to isomorphism, the unique, universally attracting object in the category of all such diagrams with γ and δ highly proximal and π^* open.

<u>Definition 4.3.1</u>. A diagram of minimal sets

$$\begin{array}{ccc} X_1 & \xrightarrow{\pi_1} & Y_1 \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ X_2 & \xrightarrow{\pi_2} & Y_2 \end{array}$$

is said to be <u>proper</u> if, for some $y_1 \in Y_1$, the pair $(\pi_2^{-1}(\beta(y_1)), \alpha(\pi_1^{-1}(y_1)))$ is proximal in 2^{X_2}. We note that this condition is equivalent to requiring that there exist some minimal right ideal I in βT such that for any $p \in I$

$$(\pi_2^{-1}(\beta(y_1)))\circ p = (\alpha(\pi_1^{-1}(y_1)))\circ p.$$

Our next lemma says that given a proper diagram, the pair (α,β) connecting the homomorphisms π_1 and π_2 may be lifted across the X^*, Y^* construction to connect π_1^* and π_2^*. Our definition of "proper" was chosen to provide a sufficient condition for the next lemma.

<u>Lemma 4.3.2</u>. <u>Suppose we have a proper diagram consisting of homomorphisms</u> $\pi_1\colon X_1 \longrightarrow Y_1$, $\pi_2\colon X_2 \longrightarrow Y_2$, $\alpha\colon X_1 \longrightarrow X_2$ <u>and</u> $\beta\colon Y_1 \longrightarrow Y_2$. <u>The</u> X^*, Y^* <u>construction is applied to obtain additional homomorphisms</u> $\pi_i^*\colon X_i^* \longrightarrow Y_i^*$, $\delta_i\colon X_i^* \longrightarrow X_i$, $\gamma_i\colon Y_i^* \longrightarrow Y_i$, <u>for</u> $i = 1,2$. <u>Then there exist homomorphisms</u> $\alpha^*\colon X_1^* \longrightarrow X_2^*$ <u>and</u> $\beta^*\colon Y_1^* \longrightarrow Y_2^*$ <u>which make the following diagram commute</u>:

$$\begin{tikzcd}
X_1^* \arrow[rr, "\pi_1^*"] \arrow[dr, "\alpha^*"'] \arrow[dd, "\delta_1"'] & & Y_1^* \arrow[dr, "\beta^*"] \arrow[dd, "\gamma_1"' near end] & \\
& X_2^* \arrow[rr, "\pi_2^*" near start] \arrow[dd, "\delta_2" near end] & & Y_2^* \arrow[dd, "\gamma_2"] \\
X_1 \arrow[rr, "\pi_1" near start] \arrow[dr, "\alpha"'] & & Y_1 \arrow[dr, "\beta"'] & \\
& X_2 \arrow[rr, "\pi_2"'] & & Y_2
\end{tikzcd}$$

<u>Proof</u>: Pick $y_1 \in Y_1$ and a minimal right ideal I in βT such that $(\pi_2^{-1}(\beta(y_1)))\circ p = (\alpha(\pi_1^{-1}(y_1)))\circ p$ for all $p \in I$. Pick $x_1 \in \pi_1^{-1}(y_1)$ and pick a minimal idempotent $u \in I$ such that $x_1 u = x_1$. Let $x_2 = \alpha(x_1)$, $y_1 = \pi_1(x_1)$, and $y_2 = \beta(y_1) = \pi_2(x_2)$. Also, let $y_1^* = \pi_1^{-1}(y_1)\circ u$,

$y_2^* = \pi_2^{-1}(y_2)\circ u$, $x_1^* = (x_1, y_1^*)$ and $x_2^* = (x_2, y_2^*)$. We'll show that there exist homomorphisms $\beta^*: Y_1^* \to Y_2^*$ and $\alpha^*: X_1^* \longrightarrow X_2^*$ such that $\beta^*(y_1^*) = y_2^*$ and $\alpha^*(x_1^*) = x_2^*$.

Checking β^* first, we need only show that for $p, q \in \beta T$,

$$y_1^* p = y_1^* q \Longrightarrow y_2^* p = y_2^* q.$$

Assume $y_1^* p = y_1^* q$. Then

$$\begin{aligned}
y_2^* p &= (\pi_2^{-1}(y_2)\circ u)\circ p = (\pi_2^{-1}(\beta(y_1)))\circ u\circ p \\
&= \text{(since the diagram is proper) } (\alpha(\pi_1^{-1}(y_1)))\circ u\circ p \\
&= \text{(by Lemma 1.2.3) } \alpha(\pi_1^{-1}(y_1)\circ u\circ p) \\
&= \text{(by hypothesis) } \alpha(\pi_1^{-1}(y_1)\circ u\circ q) \\
&= \text{(retracing our steps) } (\pi_2^{-1}(y_2)\circ u)\circ q \\
&= y_2^* q \text{ as required.}
\end{aligned}$$

Next we check α^*. Again, we need only show that for $p, q \in \beta T$,

$$x_1^* p = x_1^* q \Longrightarrow x_2^* p = x_2^* q.$$

Assuming $x_1^* p = x_1^* q$ we have $x_1 p = x_1 q$ and $y_1^* p = y_1^* q$. Thus

$$\begin{aligned}
x_2^* p &= (x_2 p, y_2^* p) = (\alpha(x_1)p, \beta^*(y_1^*)p) = (\alpha(x_1 p), \beta^*(x_1^* p)) \\
&= (\alpha(x_1 q), \beta^*(y_1^* q)) \\
&= \text{(retracing our steps) } (x_2 q, y_2^* q) \\
&= x_2^* q.
\end{aligned}$$

Clearly everything commutes. $||$

The next lemma shows that the notion of a proper diagram is not too strong for our purposes.

Lemma 4.3.3. *Suppose that* β *is highly proximal in the diagram of minimal sets.*

$$\begin{tikzcd} X_1 \arrow[r, "\pi_1"] \arrow[d, "\alpha"'] & Y_1 \arrow[d, "\beta"] \\ X_2 \arrow[r, "\pi_2"'] & Y_2 \end{tikzcd}$$

Then the diagram is proper.

Proof: Pick $y_1 \in Y_1$ and a minimal idempotent $u \in M$ with $y_1 u = y_1$. It will suffice to show that

$$(\pi_2^{-1}(\beta(y_1)))\circ u \quad = \quad (\alpha(\pi_1^{-1}(y_1)))\circ u$$

Suppose $x_2 \in \alpha(\pi_1^{-1}(y_1))$. Then there exists an $x_1 \in X_1$ with $\alpha(x_1) = x_2$ and $\pi_1(x_1) = y_1$. Thus $\pi_2(x_2) = \pi_2(\alpha(x_1)) = \beta(\pi_1(x_1)) = \beta(y_1)$ and $x_2 \in \pi_2^{-1}(\beta(y_1))$. Therefore $\alpha(\pi_1^{-1}(y_1)) \subset \pi_2^{-1}(\beta(y_1)$ and consequently $(\alpha(\pi_1^{-1}(y_1)))\circ u \subset (\pi_2^{-1}(\beta(y_1)))\circ u$.

Obviously $\alpha^{-1}(\pi_2^{-1}(y)) = \pi_1^{-1}(\beta^{-1}(y))$ for all $y \in Y_2$. Also β is highly proximal and $y_1 u = y_1$ so

$$(\alpha^{-1}(\pi_2^{-1}(\beta(y_1)))\circ u$$

$$= \quad (\pi_1^{-1}(\beta^{-1}(\beta(y_1))))\circ u \subset \pi_1^{-1}(\beta^{-1}(\beta(y_1))\circ u) \quad = \quad \pi_1^{-1}(y_1).$$

Thus

$$(\pi_2^{-1}(\beta(y_1)))\circ u = (\alpha(\alpha^{-1}(\pi_2^{-1}(\beta(y_1)))))\circ u$$
$$= \text{(by Lemma 1.2.3) } \alpha(\alpha^{-1}(\pi_2^{-1}(\beta(y_1)))\circ u)$$
$$\subset \alpha(\pi_1^{-1}(y_1))$$

Finally we get from this

$$(\pi_2^{-1}(\beta(y_1)))\circ u = (((\pi_2^{-1}(\beta(y_1)))\circ u)\circ u \subset (\alpha(\pi_1^{-1}(y_1)))\circ u$$

completing the proof. ||

Given two diagrams

$$\Delta_1 = \begin{tikzcd} X_1 \arrow[r, "\pi_1"] \arrow[d, "\delta_1"'] & Y_1 \arrow[d, "\beta_1"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd} \quad \text{and} \quad \Delta_2 = \begin{tikzcd} X_2 \arrow[r, "\pi_2"] \arrow[d, "\delta_2"'] & Y_2 \arrow[d, "\beta_2"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

with the same lower arrow we say that Δ_1 attracts Δ_2 if there exist homomorphisms $\lambda\colon X_2 \longrightarrow X_1$ and $\varphi\colon Y_2 \longrightarrow Y_1$ such that everything commutes.

Theorem 4.3.4. The diagram of minimal sets

$$\Delta = \begin{tikzcd} X^* \arrow[r, "\pi^*"] \arrow[d, "\delta"'] & Y^* \arrow[d, "\gamma"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

may be characterized, up to isomorphism, as the unique, universally attracting, proper diagram with $\pi\colon X \longrightarrow Y$ as the lower arrow and with the upper arrow open.

Proof: We've already seen that δ and γ are highly proximal and π^* is open. Δ is proper by Lemma 4.3.3. Suppose we have another proper diagram

$$\Delta' = \begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "\alpha"'] & Y' \arrow[d, "\beta"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

with π' open. We apply Lemma 4.3.2 to construct the diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
X'^{*} \arrow[rr, "\pi'^{*}"] \arrow[dr, "\alpha^{*}"] \arrow[dd, "\delta'"'] & & Y'^{*} \arrow[dr, "\beta^{*}"] \arrow[dd, "\gamma'" near start] & \\
& X^{*} \arrow[rr, "\pi^{*}" near start] \arrow[dd, "\delta" near start] & & Y^{*} \arrow[dd, "\gamma"] \\
X' \arrow[rr, "\pi'" near start] \arrow[dr, "\alpha"'] & & Y' \arrow[dr, "\beta"] & \\
& X \arrow[rr, "\pi"'] & & Y
\end{tikzcd}$$

However, since π' is open, δ' and γ' are isomorphisms, by Proposition 4.2.8. Thus, defining $\lambda\colon X' \longrightarrow X^{*}$ and $\varphi\colon Y' \longrightarrow Y^{*}$ by $\lambda = \alpha^{*} \circ \delta'^{-1}$ and $\varphi = \beta^{*} \circ \gamma'^{-1}$ we see that Δ attracts Δ'.

Now suppose we have another universally attracting proper diagram

$$\Delta_1 = \begin{tikzcd} X_1 \arrow[r, "\pi_1"] \arrow[d, "\alpha_1"'] & Y_1 \arrow[d, "\beta_1"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

Then we have homomorphisms $\lambda\colon X_1 \rightarrow X^{*}$, $\varphi\colon Y_1 \longrightarrow Y^{*}$, $\lambda_1\colon X^{*} \longrightarrow X_1$, and $\varphi_1\colon Y^{*} \longrightarrow Y_1$ such that $\alpha_1 \circ \lambda_1 = \delta$, $\beta_1 \circ \varphi_1 = \gamma$, $\delta \circ \lambda = \alpha_1$, and $\gamma \circ \varphi = \beta_1$. δ and γ are coalescent (highly proximal $\Longrightarrow$ proximal $\Longrightarrow$ regular $\Longrightarrow$ coalescent), $\delta \circ (\lambda \circ \lambda_1) = \delta$, and $\gamma \circ (\varphi \circ \varphi_1) = \gamma$. Thus $\lambda \circ \lambda_1$ and $\varphi \circ \varphi_1$ are automorphisms of X^{*} and Y^{*} respectively which implies that λ_1 and φ_1 are isomorphisms. This proves the uniqueness. ||

Corollary 4.3.5. _The diagram_

$$\begin{array}{ccc} X^* & \xrightarrow{\pi^*} & Y^* \\ \delta\big\downarrow & & \big\downarrow\gamma \\ X & \xrightarrow{\pi} & Y \end{array}$$

is the unique (up to isomorphism), universally attracting object in the category of all such diagrams with δ _and_ γ _highly proximal and_ π^* _open_.

Proof: All such diagrams are proper, by Lemma 4.3.3. ||

This is the result that was promised at the beginning of the section.

4. Non-Metric Veech Structure Theorem.

A homomorphism of minimal flows $\pi: X \longrightarrow Y$ is said to be _point-distal_ with distal point x if $x \in X$ and x is proximal to no other point in its fiber. The minimal flow X is said to be point-distal with distal point $x \in X$ if the trivial homomorphism $X \longrightarrow 1$ has that property. Veech [16] showed that if X is metric and has a residual set of distal points it has an almost one-to-one extension which can be built up from the trivial flow by isometric (almost periodic) and almost one-to-one extensions. Ellis [8] extended this result to homomorphisms and showed that it is sufficient to assume a single distal point rather than a residual set. He also showed that the metrizability assumption could be replaced by the weaker condition of quasi-separability if proximal extensions were used instead of almost one-to-one extensions. Here we strengthen the second Ellis result by replacing his proximal extensions by highly proximal extensions. Since highly proximal and almost one-to-one extensions are the same in the metric case, this result includes the earlier ones.

We build a tower of highly proximal and almost periodic extensions in essentially the same way as was done by Veech and Ellis using Lemma 4.1.12, Proposition 4.2.10, and, most importantly, Lemma 7.4 of [8].

Given T-subalgebras $\mathcal{R}$ and $\mathcal{S}$ with $\mathcal{R} \subset \mathcal{S}$, Ellis defines $\mathcal{S}$ to be a quasi-separable extension of $\mathcal{R}$ if there exists a subset $\mathcal{L} \subset \mathcal{S}$ such that $\mathcal{S} = [\mathcal{L} \cup \mathcal{R}]$ and such that $[f]$ is separable for each $f \in \mathcal{L}$ (the brackets denote "T-subalgebra generated by"). If the point-transitive flows (X,x) and (Y,y) are associated with $\mathcal{S}$ and $\mathcal{R}$, respectively, this is equivalent to the existence of a family of metrizable point-transitive flows (W_i,w_i) such that (X,x) is isomorphic to the orbit closure of the point $(y,\langle w_i\rangle)$ in the product $Y \times \prod_i \{W_i\}$. This follows from Lemma 1.1.6 and the fact that metrizable transformation groups correspond to separable T-subalgebras. Thus the definition which follows is equivalent to that of Ellis.

Definition 4.4.1. A homomorphism of minimal sets $\pi: X \longrightarrow Y$ is quasi-separable if there exists $x \in X$, $y \in Y$, and a family (W_i,w_i) of metrizable point-transitive flows such that $\pi(x) = y$ and X is isomorphic to $\overline{(y,\langle w_i\rangle)T}$ in $Y \times \prod_i \{W_i\}$.

Clearly, since X is minimal in the above definition, the base point $x \in X$ can be chosen at will.

We have

Lemma 4.4.2. *If* $\pi\colon X \longrightarrow Y$ *is a quasi-separable, point-distal, open homomorphism of minimal sets then* π *has a non-trivial almost periodic factor, i.e., there exists a minimal set* Z *and homomorphisms* $\psi\colon X \longrightarrow Z$ *and* $\varphi\colon Z \longrightarrow Y$ *such that* $\pi = \varphi \circ \psi$ *and* φ *is non-trivial and almost periodic.*

$$\begin{tikzcd} X \arrow[dr, "\psi"] \arrow[dd, "\pi"'] & \\ & Z \arrow[dl, "\varphi"] \\ Y & \end{tikzcd}$$

Proof: This is Lemma 7.4 of [8], modulo the correspondence between point-transitive flows and T-subalgebras. ||

We will need the lemma which follows several times.

Lemma 4.4.3. *Suppose we have a diagram of point-transitive flows*

$$\begin{tikzcd} (X',x') \arrow[r, "\pi'"] \arrow[d, "\delta"'] & (Y',y') \arrow[d, "\gamma"] \\ (X,x) \arrow[r, "\pi"'] & (Y,y) \end{tikzcd}$$

with π *quasi-separable and* (X',x') *isomorphic to* $\overline{(x,y')T}$ *in* $X \times Y'$. *Then* π' *is quasi-separable.*

Proof: By hypothesis, there exists a family of metrizable point-transitive flows (W_i,w_i) such that

(a) $(X,x) \simeq (\overline{(y,\langle w_i\rangle)T}, (y,\langle w_i\rangle)) \subset Y \times \prod_i W_i$.

Since we have a homomorphism $\gamma\colon (Y',y') \longrightarrow (Y,y)$,

(b) $(Y',y') \simeq (\overline{(y',y)T}, (y',y)) \subset Y' \times Y$,

and by hypothesis

(c) $(X',x') \simeq (\overline{(x,y')T}, (x,y')) \subset X \times Y'$.

Therefore, by (c) and (a)

$$(X',x') \simeq (\overline{(y,\langle w_i\rangle,y')T}, (y,\langle w_i\rangle,y')) \subset Y \times \prod_i W_i \times Y'$$

and by (b)

$$(X',x') \simeq (\overline{(y',\langle w_i\rangle)T}, (y',\langle w_i\rangle)) \subset Y' \times \prod_i W_i$$

Thus π' is quasi-separable. ||

Lemma 4.4.4. Suppose we have a diagram of minimal sets

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "\delta"'] & Y' \arrow[d, "\gamma"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd}$$

where π *is point distal and* X', δ, *and* π' *may be obtained by taking* X' *a minimal subset of the product* $\{(x,y') \mid \pi(x) = \gamma(y')\} \subset X \times Y'$ *and* δ *and* π' *the projections. Then* π' *is point distal.*

Proof: Suppose $x_0 \in X$ is a distal point for π. Pick $y_0' \in Y$ so $(x_0,y_0') \in X'$ and let $x_0' = (x_0,y_0')$. Then $\delta(x_0') = x_0$ and we'll show that x_0' is a distal point for π'. Suppose $x_1' \in X'$, x_1' and x_0' proximal, and $\pi'(x_1') = \pi'(x_0')$. Then $x_1' = (x_1,y_0')$ for some $x_1 \in X$. x_1 and x_0

are proximal, since $x_1 = \delta(x_1')$, and $x_0 = \delta(x_0')$. Also, $\pi(x_1) = \pi\circ\delta(x_1') = \gamma\circ\pi'(x_1') = \gamma(y_0') = \pi(x_0)$. Therefore $x_1 = x_0$, since x_0 is a distal point for π and we have x_0' a distal point for π' as needed.

Theorem 4.4.5. Suppose $\pi: X \longrightarrow Y$ is a point-distal, quasi-separable homomorphism. Then there exists an ordinal sequence of minimal sets $\{Y_\alpha \mid \alpha \le \nu\}$ such that

(*i*) $Y_0 = Y$.

(*ii*) Y_ν is a highly proximal extension of X.

(*iii*) $Y_{\alpha+1}$ is either an almost periodic or a highly proximal extension of Y_α, for each successor ordinal $\alpha+1 \le \nu$.

(*iv*) Y_λ is a uniquely determined pointed inverse limit of the system $\{Y_\alpha \mid \alpha < \lambda\}$ for each limit ordinal $\lambda \le \nu$.

(*v*) If $\eta_\nu: Y_\nu \longrightarrow X$, and $\mu_{\alpha,\beta}: Y_\alpha \longrightarrow Y_\beta$ for ordinals $\alpha > \beta$ are the homomorphisms implicitly defined by (*i*) - (*iv*), then $\pi\circ\eta_\nu = \mu_{\nu,0}$.

Proof: The proof is by transfinite induction. At each ordinal stage $\beta \le \nu$ we'll get a diagram

$$\begin{tikzcd}
X_\beta \arrow[r, "\pi_\beta"] \arrow[dddd, "\eta_\beta"'] & Y_\beta \arrow[d] \\
& \vdots \arrow[d] \\
& Y_1 \arrow[d] \\
& \\
X_0 = X \arrow[r, "\pi=\pi_0"'] & Y = Y_0
\end{tikzcd}$$

such that Y_β is as promised, η_β is highly proximal, and π_β is point-distal and quasi-separable. The procedure stops when π_β is almost periodic.

We'll call an ordinal <u>odd</u> if it's of the form $\lambda + n$ where λ is a limit ordinal or 0, and n is an odd natural number. We proceed as follows.

(a) Take $X_0 = X$, $Y_0 = Y$, and $\pi_0 = \pi$.

(b) Suppose we are at stage $\alpha+1$, with $\alpha+1$ an odd successor ordinal, having completed all stages through stage α. If π_α is almost periodic we stop, taking $\nu = \alpha+1$ and $Y_\nu = X_\alpha$. Otherwise, we construct the diagram

$$\begin{tikzcd}
X^*_\alpha \arrow[rr, "\pi^*_\alpha"] \arrow[dr, "\psi"'] \arrow[dd, "\delta"'] & & Y^*_\alpha \arrow[dd, "\gamma"] \\
& Z \arrow[ur, "\varphi"'] & \\
X \arrow[rr, "\pi_\alpha"'] & & Y
\end{tikzcd}$$

as follows. First we obtain π^*_α as usual. Then π^*_α is open (Proposition 4.2.10), γ and δ are highly proximal, and π^*_α is point-distal and quasi-separable (Lemmas 4.4.3 and 4.4.4). By Lemma 4.4.2, π^*_α has a non-trivial almost periodic factor, which we take to be $\varphi\colon Z \longrightarrow Y^*_\alpha$. Clearly since π^*_α is point-distal, so is $\psi\colon X^*_\alpha \longrightarrow Z$. An argument like that for Lemma 4.4.3 shows that ψ is quasi-separable, since π^*_α is. We now take $Y_{\alpha+1} = Y^*_\alpha$, $Y_{\alpha+2} = Z$, $X_{\alpha+1} = X_{\alpha+2} = X^*_\alpha$, $\pi_{\alpha+1} = \pi^*_\alpha$, and $\pi_{\alpha+2} = \psi$. We now have

$$\begin{tikzcd}
X_{\alpha+2} \arrow[r, "\pi_{\alpha+2}"] \arrow[d, "\text{identity}"'] & Y_{\alpha+2} \arrow[d, "\varphi \text{ almost periodic}"] \\
X_{\alpha+1} \arrow[r, "\pi_{\alpha+1}"] \arrow[d, "\delta"', "\text{highly proximal}"] & Y_{\alpha+1} \arrow[d, "\gamma \text{ highly proximal}"] \\
X_{\alpha} \arrow[r, "\pi_{\alpha}"] \arrow[ddd, "\eta_{\alpha}"'] & Y_{\alpha} \arrow[d] \\
 & \vdots \arrow[d] \\
 & Y_1 \arrow[d] \\
X_0 \arrow[r, "\pi_0"'] & Y_0
\end{tikzcd}$$

and everything is as needed, taking $\eta_{\alpha+1} = \eta_{\alpha+2} = \eta_{\alpha} \circ \delta$. This gets us up to the next odd successor ordinal.

(c) Suppose we're at stage λ, where λ is a limit ordinal. We take "inverse limits" in the following way:

Let X_{λ} be the unique (by Lemma 4.1.12) pointed inverse limit of the system $\{X_{\alpha} \mid \alpha < \lambda\}$ and let $P_{\alpha}\colon X_{\lambda} \longrightarrow X_{\alpha}$ be the projections which are highly proximal by Lemma 4.1.12. Define a homomorphism

$$Q\colon \prod \{X_{\alpha} \mid \alpha < \lambda\} \longrightarrow \prod \{Y_{\alpha} \mid \alpha < \lambda\}$$

by $Q(\langle x_{\alpha} \rangle) = \langle \pi_{\alpha} \circ P_{\alpha}(x_{\alpha}) \rangle$. Let $Y_{\lambda} = Q(X_{\lambda})$, let $\pi_{\lambda} = Q|X_{\lambda}$, and let $\eta_{\lambda} = P_0|X_{\lambda}$. π_{λ} is point-distal and quasi-separable by Lemmas 4.4.3 and 4.4.4 and everything is as needed.

$$\begin{tikzcd}
X_{\lambda} \arrow[r, "\pi_{\lambda}"] \arrow[ddd, bend right=60, "\eta_{\lambda}"'] & Y_{\lambda} \\
\vdots \arrow[d] & \vdots \arrow[d] \\
X_1 \arrow[r, "\pi_1"] \arrow[d] & Y_1 \arrow[d] \\
X_0 \arrow[r, "\pi_0"'] & Y_0
\end{tikzcd}$$

In the foregoing, the definition of the maps $\mu_{\alpha,\beta}\colon Y_\alpha \longrightarrow Y_\beta$ has been left to context.

By cardinality considerations and the coalescence of the universal minimal set, the procedure must terminate at some stage of (b). This proves the theorem. $||$

The original Veech theorem has a valid converse, i.e., any minimal flow which can be built up in the way specified in the conclusion of the theorem must necessarily be point-distal. This does not appear to be the case with our theorem, unless point-distal is replaced by some weaker condition. Such a condition should be sufficient for our theorem, equivalent to point-distal in the metric case, and valid for any quasi-separable extension which can be built from almost periodic and highly proximal extensions in the manner described. The determination of such a condition appears to be a difficult problem.

CHAPTER V

GENERALIZED ALMOST FINITE HOMOMORPHISMS

The regular homomorphisms discussed in Chapter II are objects of a very general nature. The general properties obtained are very similar to those which arise in the absolute case, i.e., the properties of regular minimal sets. It is therefore natural to look for restrictive conditions which make further classification possible. Restricting ourselves to situations in which the fibers are in some sense "small" has the additional advantange of eliminating the absolute case, since regular minimal sets are generally quite big. Thus we are led to considering finiteness conditions on the fibers.

The notion of generalized almost finiteness which we study here is so named because, in the metric case, it coincides with the property of having at least one fiber finite. When combined with regularity, the consequences of this condition are quite powerful.

1. Definition and Basic Properties.

Once again, we'll work with a fixed homomorphism $\pi: X \longrightarrow Y$, with Y assumed minimal. Recall that the minimal set Y^* consists of the elements of 2^π of the form $\pi^{-1}(y)\circ u$, where u is a minimal idempotent with $yu = y$.

Proposition 5.1.1. The following conditions are equivalent:

(*i*) $\lim \langle \pi^{-1}(y)t_n \rangle$ is finite, for some $y \in Y$ and some net $\langle t_n \rangle$ in T.

(*ii*) There exists a least positive integer N such that for any $y \in Y$ there exists $p \in M$ with $\mathrm{card}(\pi^{-1}(y) \circ p) = N$.

(*iii*) There exists a least positive integer N such that $\mathrm{card}(A) \leq N$ whenever A is an almost periodic element in 2^{π}.

(*iv*) There exists a finite integer N such that each element of Y^* has cardinality N.

Moreover, the numbers N in (*ii*), (*iii*), and (*iv*) are the same.

Proof: (*i*) $\Longrightarrow$ (*ii*) Suppose $\pi^{-1}(y)t_n \longrightarrow \{x_1, \cdots, x_k\}$. Consider $y' \in Y$. Taking a subnet if necessary, we can find $q \in \beta T$ such that $t_n \longrightarrow q$. Pick $r \in M$ so $y'r = y$. Let $p = rq$. Then $p \in M$ and $\pi^{-1}(y') \circ p = \pi^{-1}(y') \circ (rq) \subset \pi^{-1}(y'r) \circ q = \pi^{-1}(y) \circ q = \{x_1, \cdots, x_k\}$. Thus $\mathrm{card}(\pi^{-1}(y' \circ p) \leq k$. We take N to be the least positive integer such that $\mathrm{card}(\lim \pi^{-1}(y)t_n) = N$ for some $y \in Y$ and some net $\langle t_n \rangle$ in T.

(*ii*) $\Longrightarrow$ (*iii*) Suppose A is an almost periodic element in 2^{π}. We can pick $y \in Y$ and $u \in J$ such that $A \subset \pi^{-1}(y)$, $A \circ u = A$, and $yu = y$. By (*ii*), there exists a $p \in M$ and points $x_1, \cdots, x_N$ in X such that $\pi^{-1}(y) \circ p = \{x_1, \cdots, x_N\}$.

We pick $q \in M$ such that $pq = u$. Then $A = A \circ u \subset \pi^{-1}(y) \circ u = \pi^{-1}(y) \circ (pq) = \{x_1, \cdots, x_n\} \circ q = \{x_1 q, \cdots, x_N q\}$. Thus $\operatorname{card} A \leq N$.

(iii) $\Longrightarrow$ *(iv)* *(iii)* clearly implies that some inclusion maximal almost periodic element of 2^π has cardinality N, and such elements always belong to Y^*. Thus we can find $y \in Y$ and $u \in J(y)$ such that $\operatorname{card}(\pi^{-1}(y) \circ u) = N$. Consider some other element of Y^*, say $\pi^{-1}(y') \circ u'$ where $y' \in Y$ and $u' \in J(y')$. We can find $p \in M$ such that $pu' = p$ and $yp = y'$. Since $\pi^{-1}(y) \circ u$ is finite, we have $(\pi^{-1}(y) \circ u)u = (\pi^{-1}(y) \circ u) \circ u = \pi^{-1}(y) \circ u$. We can pick $q \in M$ so $pq = u$. Thus $\pi^{-1}(y) \circ u = (\pi^{-1}(y) \circ u)pq$. Therefore $\operatorname{card}(\pi^{-1}(y) \circ u)p \geq \operatorname{card}(\pi^{-1}(y) \circ u) = N$. Now $(\pi^{-1}(y) \circ u)p = (\pi^{-1}(y) \circ u) \circ p = \pi^{-1}(y) \circ p = (\pi^{-1}(y) \circ p) \circ u' = \pi^{-1}(yp) \circ u' = \pi^{-1}(y') \circ u'$. Thus $\operatorname{card}(\pi^{-1}(y') \circ u') = \operatorname{card}(\pi^{-1}(y) \circ u)p \geq N$. By *(iii)*, $\operatorname{card}(\pi^{-1}(y') \circ u') \leq N$. Therefore $\operatorname{card}(\pi^{-1}(y') \circ u') = N$.

(iv) $\Longrightarrow$ *(i)* Obvious.

It is also obvious that the number N which satisfies *(iv)* will also satisfy *(ii)* and *(iii)*. ||

Definition 5.1.2. We say that a homomorphism with minimal range is generalized almost finite, or generalized almost N to one, if it satisfies conditions *(i)* - *(iv)* of Proposition 5.1.1.

Lemma 5.1.3. *If* $\pi: X \longrightarrow Y$ *is generalized almost finite,* $y \in Y$, *and* $u \in J(y)$, *then* $\pi^{-1}(y) \circ u = \pi^{-1}(y)u$.

Proof: We have $\pi^{-1}(y)u \subset \pi^{-1}(y)\circ u$ always. Since π is generalized almost finite, $\pi^{-1}(y)\circ u$ is finite and we have

$$\pi^{-1}(y)\circ u = (\pi^{-1}(y)\circ u)\circ u = (\pi^{-1}(y)\circ u)u \subset \pi^{-1}(y)u. \quad ||$$

Proposition 5.1.4. *Suppose the homomorphism* $\pi: X \longrightarrow Y$ *is generalized almost* N *to one. Then, for each point* $y \in Y$, *the following statements are equivalent:*

(*i*) π *is open at all points of* $\pi^{-1}(y)$.

(*ii*) $\mathrm{card}(\pi^{-1}(y)) = N$.

(*iii*) $\pi^{-1}(y)$ *is an almost periodic set.*

(*iv*) $\pi^{-1}(y) \in Y^*$.

Proof: (*i*) $\Longrightarrow$ (*iv*). Pick $u \in J(y)$. By Lemma 4.1.1, we have $\pi^{-1}(y) = \pi^{-1}(y)\circ u$ so $\pi^{-1}(y) \in Y^*$.

(*iv*) $\Longrightarrow$ (*i*) Suppose $y' \in Y$ and $p \in M$ with $y'p = y$. By Lemma 4.1.1, we need only show that $\pi^{-1}(y')\circ p = \pi^{-1}(y)$. Certainly $\pi^{-1}(y')\circ p \subset \pi^{-1}(y)$ and $\pi^{-1}(y')\circ p \in Y^*$. By (*iv*), $\pi^{-1}(y) \in Y^*$ also. Since $\pi^{-1}(y')\circ p$ and $\pi^{-1}(y)$ have the same finite cardinality, they are equal.

(*iv*) $\Longrightarrow$ (*ii*) Obvious.

(*ii*) $\Longrightarrow$ (*iv*) We have $\pi^{-1}(y) \supset \pi^{-1}(y)\circ u$, for $u \in J(y)$. By (*ii*) both sets are of the same finite cardinality and therefore equal.

(*iv*) $\Longrightarrow$ (*iii*) If (*iv*) holds, we have $\pi^{-1}(y) = \pi^{-1}(y)\circ u$ for some $u \in J(y)$ and $\pi^{-1}(y)u = \pi^{-1}(y)\circ u$ by Lemma 5.1.3. Thus $\pi^{-1}(y)$ is an almost periodic set.

(*iii*) $\Longrightarrow$ (*iv*) If (*iii*) holds, we have $\pi^{-1}(y) = \pi^{-1}(y)u$ for some $u \in J(y)$. We have $\pi^{-1}(y) = \pi^{-1}(y)u = \pi^{-1}(y)\circ u$ by Lemma 5.1.3. Thus $\pi^{-1}(y) \in Y^*$. ||

Next we see that, in the metric case, the notion of a generalized almost finite homomorphism reduces to something simpler.

Proposition 5.1.5. *Suppose we have a homomorphism* $\pi: X \longrightarrow Y$ *with both* X *and* Y *metric and* Y *minimal. Then the following conditions are equivalent.*

(*i*) π *is generalized almost* N *to one* (N *finite*).

(*ii*) π *has at least one fiber of finite cardinality* N, *and no fibers of cardinality less than* N.

(*iii*) $\text{card}(\pi^{-1}(y)) = N$ *for a dense,* G_δ *set of points* $y \in Y$ *and no fiber has cardinality less than* N (N *finite*).

Proof: Corollary 4.1.4, Proposition 5.1.1, and Proposition 5.1.4. ||

The relationship between the general and metric cases with generalized almost finite homomorphisms is the same as with highly proximal homomorphisms. The condition of generalized almost finiteness is homogeneous, in that it is defined without reference to any distinguished points in Y. However, distinguished points appear in the metric case just as they do with highly proximal homomorphisms. It is clear that a homomorphism is highly proximal if and only if it is generalized almost N to one, with $N = 1$, and that, in the metric case, the notions of an almost one-to-one and generalized almost one-to-one homomorphisms coincide.

2. Regular Generalized Almost Finite Homomorphisms.

In this section we see that if a homomorphism is both regular and generalized almost finite, its structure is very simple. In fact such a homomorphism may always be uniquely represented as the composition of a highly proximal extension and a finite group extension.

Proposition 5.2.1. *Suppose $\pi: X \longrightarrow Y$ is a regular generalized almost finite homomorphism of minimal sets. Then there exists a homomorphism $\psi: X \longrightarrow Y^*$ such that ψ is a group extension and $\pi = \gamma \circ \psi$, where γ is the canonical homomorphism from Y^* to Y.*

Proof: We have $\pi^{-1}(y) \circ u = \pi^{-1}(y)u$ for all $y \in Y$ and $u \in J(y)$, by Lemma 5.1.3.

The regularity of π implies that the sets of the form $\pi^{-1}(y)u$, for idempotents $u \in J(y)$, partition $\pi^{-1}(y)$, for each $y \in Y$ (Corollary 2.2.9). Thus the elements of Y^* actually partition X and we can define a function $\psi: X \longrightarrow Y^*$ by $\psi(x) = A \Longleftrightarrow x \in A$. Clearly ψ is equivariant and $\pi = \gamma \circ \psi$.

To verify that ψ is a homomorphism we check continuity. Suppose $\langle x_i \rangle$ is a net in X and $x_i \longrightarrow x_0$ for some $x_0 \in X$.

We must show that $\psi(x_i) \longrightarrow \psi(x_0)$. Taking a subnet if necessary, assume that $\psi(x_i) \longrightarrow A$, for some $A \in 2^X$. Certainly $A \in Y^*$, since Y^* is a closed subset of 2^X. We have $x_0 \in A$ since $\psi(x_i) \longrightarrow A$, $x_i \in \psi(x_i)$ for each i, and $x_i \longrightarrow x_0$. Thus $\psi(x_0) = A$ as required.

Finally we show that ψ is a group extension. Suppose $\psi(x) = \psi(x')$ for $x, x' \in X$. Then $\pi(x) = \pi(x') = y$. Also we have $\psi(x) = \psi(x') = \pi^{-1}(y)u$ for $u \in J(y)$. Therefore $(x,x')u = (x,x')$ and (x,x') is an almost periodic pair. Thus, by Proposition 2.2.8, there is an automorphism $\theta\colon X \longrightarrow X$ such that $\theta(x) = x'$. Therefore ψ is a group extension. ||

Proposition 5.2.2. The group of automorphisms associated with the group extension ψ in the preceding proposition is of cardinality N, assuming that π is almost N to one.

Proof: This follows immediately from the fact that this group acts freely and transitively on the fibers of ψ. These fibers are elements of Y^* and hence of cardinality N. ||

The next lemma shows that the representation of a regular generalized almost finite homomorphism as the composition of a finite group extension and a highly proximal homomorphism is unique.

Lemma 5.2.3. Suppose we have the diagram of minimal sets

$$\begin{tikzcd}
 & X \arrow[dl, "\psi_1"'] \arrow[dr, "\psi_2"] & \\
Z \arrow[dr, "\lambda_1"'] & & Z \arrow[dl, "\lambda_2"] \\
 & Y &
\end{tikzcd}$$

where ψ_1 *and* ψ_2 *are group extensions and* λ_1 *and* λ_2 *are proximal. Then there exists an isomorphism* $\theta\colon Z_2 \longrightarrow Z_1$ *such that* $\lambda_1 \circ \theta = \lambda_2$.

Proof: Let $G_i = \{\sigma \mid \sigma$ an automorphism of X and $\psi_i \circ \sigma = \psi_i\}$ for $i = 1,2$. Since ψ_1 and ψ_2 are group extensions, it suffices to show that $G_1 = G_2$. Consider $\sigma \in G_1$. Pick $x \in X$ and $u \in J(x)$. Let $y = \lambda_1 \circ \psi_1(x)$. Then also $y = \lambda_2 \circ \psi_2(x) = \lambda_1 \circ \psi_1 \circ \sigma(x) = \lambda_2 \circ \psi_2 \circ \sigma(x)$. Since $xu = x$ we have also $\psi_2(x) = \psi_2(x)u$ and $\psi_2 \circ \sigma(x) = (\psi_2 \circ \sigma(x))u$. Therefore $\psi_2(x)$ and $\psi_2 \circ \sigma(x)$ are both in $\lambda_2^{-1}(y)u$ and, since λ_2 is proximal, $\psi_2(x) = \psi_2 \circ \sigma(x)$. Thus $\sigma \in G_2$ and we've shown $G_1 \subset G_2$. The same argument shows $G_2 \subset G_1$. ||

Theorem 5.2.4. *If* $\pi\colon X \longrightarrow Y$ *is a homomorphism of minimal sets, the following conditions are equivalent:*

(*i*) π *is regular and generalized almost finite.*

(*ii*) π *can be represented as a composition* $\gamma \circ \psi$, *where* γ *is highly proximal and* ψ *is a finite group extension.*

Moreover the representation in (*ii*) *is unique.*

Proof: We've already proved everything except (*ii*) $\Longrightarrow$

(i). Suppose we have a finite group extension $\psi: X \longrightarrow W$ and a highly proximal extension $\gamma: W \longrightarrow Y$ such that $\gamma \circ \psi = \pi$. Suppose (x,x') is an almost periodic pair in X with $\pi(x) = \pi(x') = y$ and $u \in J(y)$ such that $(x,x')u = (x,x')$. Then $\psi(x), \psi(x') \in \lambda^{-1}(y) \circ u$ and $\psi(x) = \psi(x')$, since λ is highly proximal. Since ψ is regular, there exists an automorphism taking x into x'. Thus π is regular. Also

$$\begin{aligned}\pi^{-1}(y) \circ u &= (\psi^{-1}(\gamma^{-1}(y))) \circ u \subset \psi^{-1}(\gamma^{-1}(y) \circ u) \\ &= \psi^{-1}(w) \quad \text{for some } w \text{ (since } \gamma \text{ is} \\ &\qquad\qquad \text{is highly proximal)} \\ &= \text{a finite set.}\end{aligned}$$

Thus π is generalized almost finite. ||

The Ellis two-circle minimal set (Example 5.29 of [7]) with irrational rotation 1 is an example of a homomorphism which is highly proximal but not almost one-to-one. Here Y is the unit circle, Z is $Y \times \{1,2\}$ with a special topology, T is the integers, and t rotates each circle by t radians counterclockwise. $\gamma: Z \longrightarrow Y$ is defined by $\gamma(y,i) = y$ If I is a minimal right ideal in $E(Z)$, then I has exactly two idempotents u_1 and u_2 and $(y,j)u_i = y_i$ for each i,j. A finite group extension $\psi: X \longrightarrow Z$ is most easily constructed by taking X to have the same phase space as Z but with an irrational rotation of $1/N$, N finite, and defining $\psi((y,i)) = (Ny,i)$. Defining $\pi: X \longrightarrow Y$ by $\pi = \gamma \circ \psi$ we have π generalized almost finite and regular. We observe that π could also have been constructed by first taking a finite group extension of Y and then taking a highly proximal extension. We shall see that this is possible under fairly general circumstances.

Recall that if $\pi_1: X_1 \longrightarrow Y$ and $\pi_2: X_2 \longrightarrow Y$ are homomorphisms, with Y minimal, we have a naturally defined product $\pi: X \longrightarrow Y$, where $X = \{(x_1, x_2) \mid \pi_1(x_1) = \pi_2(x_2)\}$. Thus, given some other transformation group Z and homomorphisms $\psi_1: Z \longrightarrow X_2$ and $\psi_2: Z \longrightarrow X_2$ with $\pi_1 \circ \psi_1 = \pi_2 \circ \psi_2$ we have a canonical diagram

$$\begin{tikzcd}
& Z \arrow[dl, "\psi_1"'] \arrow[d, "\psi"] \arrow[dr, "\psi_2"] & \\
X_1 \arrow[dr, "\pi_1"'] & X \arrow[l, "P_1"] \arrow[r, "P_2"'] \arrow[d, "\pi"] & X_2 \arrow[dl, "\pi_2"] \\
& Y &
\end{tikzcd}$$

where P_1 and P_2 are the projections and ψ is uniquely determined. If ψ is always onto for any choice of Z, ψ_1, and ψ_2, we say that π_1 and π_2 are *disjoint* [see 10 and 15]. It is easy to see that, when X_1 and X_2 are also minimal, π_1 and π_2 are disjoint if and only if X is minimal.

The following lemma is well-known.

Lemma 5.2.5. *Suppose $\pi_1: X_1 \longrightarrow Y$ and $\pi_2: X_2 \longrightarrow Y$ are homomorphisms of minimal sets with π_1 proximal and π_2 distal. Then π_1 and π_2 are disjoint.*

Proof: We must show that X is minimal, where $X = \{(x_1,x_2) \mid \pi_1(x_1) = \pi_2(x_2)\}$. Consider two points $x = (x_1,x_2)$ and $x' = (x_1',x_2')$. Pick an idempotent $u' \in M$ so $x_1'u' = x_1'$. Letting $y' = \pi(x')$, we have $y'u' = y'$ and hence $x_2'u' = x_2'$, since π_2 is distal. Pick $p \in M$ so $pu' = p$ and $x_2p = x_2'$. $\pi_1(x_1p) = \pi_2(x_2p) = y' = \pi_1(x_1')$ and $(x_1p,x_1')u' = (x_1p,x_1')$. Since π_1 is proximal, we have $x_1p = x_1'$. Therefore $xp = x'$ and $x' \in \overline{xT}$. Thus X is minimal. ||

Lemma 5.2.6. If $\pi_1\colon X_1 \longrightarrow Y$ and $\pi_2\colon X_2 \longrightarrow Y$ are homomorphisms of minimal sets with π_1 highly proximal and π_2 a finite group extension, then the product $\pi\colon X \longrightarrow Y$ is generalized almost finite and regular and may also be represented as the composition of a highly proximal extension followed by a finite group extension. Moreover, the cardinalities of both finite groups are the same.

Proof: Looking at the diagram

$$\begin{tikzcd} & X \arrow[dl, "P_1"'] \arrow[dr, "P_2"] \arrow[dd, "\pi"] & \\ X_1 \arrow[dr, "\pi_1"'] & & X_2 \arrow[dl, "\pi_2"] \\ & Y & \end{tikzcd}$$

where P_1 and P_2 are the projections, we show that P_1 is a finite group extension and P_2 is highly proximal. For each automorphism $\theta\colon X_2 \longrightarrow X_2$ with $\pi_2 \circ \theta = \pi_2$, we define

$\hat{\theta}: X \longrightarrow X$ by $\hat{\theta}(x_1,x_2) = (x_1,\theta(x_2))$. Then $\hat{\theta}$ is an automorphism and $P_1 \circ \hat{\theta} = P_1$. Clearly, since π_2 is a group extension, this defines a group of automorphisms of X, connecting all pairs of points in the same fiber of P_1, and this group has the same cardinality as $\{\theta: X_2 \longrightarrow X_2 \mid \pi_2 \circ \theta = \theta\}$ which is finite.

Consider $x_2 \in X_2$ and an idempotent $u \in M$ with $x_2 u = x_2$. Let $y = \pi_2(x_2)$. Since π_1 is highly proximal and $yu = y$, we have $\pi_1^{-1}(y) \circ u = \{x_1\}$ for some $x_1 \in X_1$. We'll show that $P_2^{-1}(x_2) \circ u = \{(x_1,x_2)\}$ from which we conclude that P_2 is highly proximal. Suppose $(x_1',x_2') \in P_2^{-1}(x_2) \circ u$. Then there exist nets $\langle x_{1n} \rangle$ in $\pi_1^{-1}(y)$, and $\langle t_n \rangle$ in T such that $(x_{1n}t_n,\ x_2 t_n) \longrightarrow (x_1',x_2')$ and $t_n \longrightarrow u$. Clearly $x_2' = x_2 u = x_2$ and $x_1' \in \pi_1^{-1}(y) \circ u$; hence $x_1' = x_1$.

Since X is minimal, by Lemma 5.2.5, this completes the proof. ||

<u>Theorem 5.2.7</u>. <u>Suppose $\pi: X \longrightarrow Y$ is a regular generalized almost finite homomorphism. Then the following conditions are equivalent:</u>

(*i*) <u>The relative proximal relation on X is closed.</u>

(*ii*) <u>π can be represented as a composition of a highly proximal extension followed by a finite group extension.</u>

(*iii*) <u>π may be represented as the product (pullback) of a highly proximal extension and a finite group extension.</u>

<u>Proof</u>: (*iii*) ==> (*ii*) Lemma 5.2.6.

$(ii) \Longrightarrow (i)$ Follows from Proposition 3.4.

$(i) \Longrightarrow (ii)$ Suppose the relative proximal relation on X is closed. Applying Proposition 5.2.1 and Proposition 3.4 we get a diagram of minimal sets

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr, "\alpha"] \arrow[dd, "\pi"] & \\ Y^{*} \arrow[dr, "\gamma"'] & & W \arrow[dl, "\beta"] \\ & Y & \end{tikzcd}$$

where γ is highly proximal, ψ is a finite group extension, α is proximal, and β is distal. We still must show that α is highly proximal and β is a finite group extension.

Consider $w \in W$ and an idempotent $u \in J(w)$. Let $y = \beta(w)$. We have $\alpha^{-1}(w) \circ u \subset \pi^{-1}(y) \circ u$ and $\pi^{-1}(y) \circ u$ is a finite set. Thus α is generalized almost finite and $\alpha^{-1}(w) \circ u = \alpha^{-1}(w)u$ which is a singleton, since α is proximal (Lemma 2.5.8). Thus α is highly proximal.

Consider $y \in Y$ and an idempotent $u \in J(y)$. Clearly $\alpha(\alpha^{-1}(A)u) = Au$ for any subset $A \subset W$. Also, since π is generalized almost finite we have $\pi^{-1}(y) \circ u = \pi^{-1}(y)u$. Since β is distal, we have $\beta^{-1}(y) = \beta^{-1}(y)u = \alpha(\alpha^{-1}(\beta^{-1}(y))u) = \alpha(\pi^{-1}(y)u)$ which is of the same finite cardinality as $\pi^{-1}(y)u$ since α is proximal and $\pi^{-1}(y)u$ is an almost periodic set. Thus for arbitrary points w_1 and w_2 in $\beta^{-1}(y)$ we need only show that there exists an endomorphism $\hat{\theta}$ with $\hat{\theta}(w_1) = w_2$. To establish this it will suffice to show that given $p, q \in \beta T$ with $w_1 p = w_1 q$ we also have $w_2 p = w_2 q$. Pick $x_1 \in \alpha^{-1}(w_1)$ and $x_2 \in \alpha^{-1}(w_2)$ such that

$(x_1,x_2)u = (x_1,x_2)$. Then $(\psi(x_1),\psi(x_2)) = (\psi(x_1),\psi(x_2))u$ and $\gamma\circ\psi(x_1) = \gamma\circ\psi(x_2) = y$. Since γ is proximal, we have $\psi(x_1) = \psi(x_2)$. Thus since ψ is a group extension, there exists an automorphism $\theta\colon X \longrightarrow X$ such that $\theta(x_1) = x_2$. Let $y' = yp$ and pick an idempotent $u' \in J(y')$. Then $yp = yq = y' = y'u'$. Since $\beta(w_2p) = \beta(w_2q) = y' = y'u'$ and β is distal we have $(w_2p,w_2q)u' = (w_2p,w_2q)$. Now (x_1pu',x_1qu') is an almost periodic pair, $\alpha(x_1pu') = \alpha(x_1qu')$, and α is proximal. Therefore $x_1pu' = x_1qu'$. Thus $w_2p = w_2pu' = \alpha(x_2pu') = \alpha\circ\theta(x_1pu') = \alpha\circ\theta(x_1qu') = \alpha(\theta(x_1)qu') = \alpha(x_2qu') = w_2qu' = w_2q$.

(ii) $\Longrightarrow$ *(iii)* If *(ii)* is satisfied, we have a diagram

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[dr, "\alpha"] \arrow[dd, "\pi"] & \\ Y^* \arrow[dr, "\gamma"'] & & W \arrow[dl, "\beta"] \\ & Y & \end{tikzcd}$$

where ψ and β are finite group extensions and α and γ are highly proximal. Letting $\overline{\pi}\colon \overline{X} \longrightarrow \overline{Y}$ denote the product of γ and β we get a diagram

$$\begin{tikzcd} & X \arrow[dl, "\psi"'] \arrow[d, "\delta"] \arrow[dr, "\alpha"] & \\ Y^* \arrow[dr, "\gamma"'] & \overline{X} \arrow[l, "P_1"'] \arrow[r, "P_2"] \arrow[d, "\overline{\pi}"] & W \arrow[dl, "\beta"] \\ & \overline{Y} & \end{tikzcd}$$

where P_1 and P_2 are the projections and δ is defined in the natural way. We also have that $\overline{X}$ is minimal

(Lemma 5.2.5), that P_1 is a finite group extension, P_2 is highly proximal, and that the fibers of β and P_1 have the same finite cardinality (Lemma 5.2.6). Thus, if we can show that the fibers of ψ and β have the same cardinality, this will prove that δ is one-to-one and hence an isomorphism.

Take $y \in Y$, $u \in J(y)$, and $y^* \in Y^*$ with $y^* = \pi^{-1}(y) \circ u$. Then $\psi^{-1}(y^*) = y^* = \pi^{-1}(y) \circ u = \pi^{-1}(y)u$ by construction. $\alpha | \pi^{-1}(y)u$ is one-to-one, since α is proximal, and $\beta^{-1}(y) = \beta^{-1}(y)u$, since β is distal. It's easy to see that $\beta^{-1}(y)u = \alpha(\pi^{-1}(y)u)$. Thus $\alpha | \pi^{-1}(y)u$ is a bijection from $\psi^{-1}(y^*)$ to $\beta^{-1}(y)$ and the proof is finished. $||$

Next we show how to construct examples of homomorphisms which are highly proximal but which have all fibers infinite. Suppose we have a homomorphism of minimal sets $\pi: X \longrightarrow Y$ which is almost one-to-one but which has an infinite fiber also, and suppose Y is distal regular. Let $\{\theta_i\}$ be the set of automorphisms of Y and let $\pi_i = \theta_i \circ \pi$. Let $\bar{\pi}: N \longrightarrow Y$ be the product $\prod_i \{\pi_i: X \to Y\}$, so that $N = \{\langle x_i \rangle \mid x_i \in X$ and $\pi_i(x_i)$ the same for all $i\}$, and let $\tilde{\pi}: \tilde{N} \longrightarrow Y$ be the restriction of $\bar{\pi}$ to a minimal subset. Then $\tilde{\pi}: N \longrightarrow Y$ is essentially independent of the choice made, by Proposition 2.5.1, and we have the following proposition.

Proposition 5.2.8. $\tilde{\pi}: \tilde{N} \longrightarrow Y$, as described above, is highly proximal and has all fibers infinite.

Proof: $\tilde{\pi}$ is highly proximal since π is and since highly proximal is an admissible property. Pick $y_0 \in Y$ so $\pi^{-1}(y_0)$ is infinite. For an arbitrary $y_i \in Y$, there exists an automorphism θ_i such that $\theta_i(y_0) = y_i$, since Y is distal and regular. Take $\langle \bar{x}_i \rangle \in \tilde{N} \subset N$, so we have $\tilde{N} = \overline{\langle \bar{x}_i \rangle T}$. We'll show that $\tilde{\pi}^{-1}(y_j)$ is infinite for all $y_j \in Y$. We have

$$\begin{aligned}
\tilde{\pi}^{-1}(y_j) &= \{\langle \bar{x}_i \rangle q \mid q \in \beta T \text{ and } \pi_i(\bar{x}_i q) = y_j \text{ for all } i\} \\
&= \{\langle \bar{x}_i \rangle q \mid q \in \beta T \text{ and } \pi_j(\bar{x}_j q) = y_j\} \\
&= \{\langle \bar{x}_i \rangle q \mid q \in \beta T \text{ and } \theta_j \circ \pi(\bar{x}_j q) = y_j\} \\
&= \{\langle \bar{x}_i \rangle q \mid q \in \beta T \text{ and } \pi(\bar{x}_j q) = y_0\}
\end{aligned}$$

which is infinite, since X is minimal and $\pi^{-1}(y_0)$ is infinite. ||

The Floyd minimal set [see 1], taken as an extension of the triadic group provides an example of the type needed. The fibers are line segments and points and the triadic group is distal regular since it's a monothetic group.

In the case of an arbitrary generalized almost finite homomorphism $\pi: X \longrightarrow Y$ there is a connection between Y^* and the regularizer.

Proposition 5.2.9. Suppose $\pi: X \longrightarrow Y$ is a generalized almost finite homomorphism, $\bar{\pi}: N \longrightarrow Y$ is its regularizer, *and* $x^* \in X^*$. *Then we have a diagram*

$$\begin{tikzcd}
X^* \arrow[r, "\pi^*"'] \arrow[d, "\delta"'] & Y^* \arrow[d, "\gamma"] & N \arrow[l, "P"'] \arrow[ll, bend right, "\theta_{x^*}"'] \arrow[dl, "\overline{\pi}"] \\
X \arrow[r, "\pi"'] & Y &
\end{tikzcd}$$

where P *is a finite group extension and* θ_{x^*} *can be chosen so* $\theta_{x^*}(z) = x^*$ *whenever* $P(z) = \pi^*(x^*)$. *Furthermore* Y^* *is the same minimal set (up to isomorphism) as would be obtained from applying the* Y^* *construction to* $\overline{\pi}\colon N \longrightarrow Y$. *Finally,* P *is the regularizer of* π^*.

Proof: Pick $y \in Y$, $u \in J(y)$ and represent N as $\overline{z_y uT}$. We'll show that $\overline{\pi}$ is generalized almost finite.

Now $\overline{\pi}^{-1}(y)u = \{z_y upu \mid p \in \beta T \text{ and } yp = y\}$. Consider $p, q \in \beta T$ such that $yp = yq = y$. Then $z_y upu = z_y uqu \iff xpu = xqu$ for all $x \in \pi^{-1}(y)u$. Thus the cardinality of $\overline{\pi}^{-1}(y)u$ is not greater than the number of functions from $\pi^{-1}(y)u$ into itself, which is finite. Now for $z \in N$, we have

$z \in \overline{\pi}^{-1}(y)\circ u$

$\Longrightarrow$ there exists $\langle p_n \rangle$ in βT, $\langle t_n \rangle$ in T such that
$yp_n = y$ and

$z_y up_n t_n \longrightarrow z$

$\Longrightarrow$ $(z_y up_n t_n)(x) \longrightarrow z(x)$ for all $x \in \pi^{-1}(y)$

$\Longrightarrow$ (taking $x_n = (z_y up_n)(x)$) there exists $\langle x_n \rangle$ in $\pi^{-1}(y)$ such that $x_n t_n \longrightarrow z(x)$ for all $x \in \pi^{-1}(y)$

$\Longrightarrow z(x) \in \pi^{-1}(y)\circ u$ for all $x \in \pi^{-1}(y)$

$\Longrightarrow z(x) \in \pi^{-1}(y)u$ for all $x \in \pi^{-1}(y)$

$\Longrightarrow z \in \overline{\pi}^{-1}(y)u.$

Thus $\overline{\pi}^{-1}(y)\circ u \subset \overline{\pi}^{-1}(y)u$ which is finite, so we have $\overline{\pi}$ generalized almost finite.

We define $P\colon N \longrightarrow Y^*$ by $P(z) = \text{Range}(z)$. If $z = z_y up$, $\text{Range}(z) = \text{Range}(z_y)up = \pi^{-1}(y)up = (\pi^{-1}(y)\circ u)p = (\pi^{-1}(y)\circ u)\circ p \in Y^*$, so P is well-defined. P is easily seen to be a homomorphism. If $P(z) = P(z')$, then (z,z') is almost periodic and $\overline{\pi}(z) = \overline{\pi}(z')$ so we can find an automorphism taking z into z' by the regularity of $\overline{\pi}$. Hence P is a group extension.

We now have that $\overline{\pi}$ is the composition of a finite group extension followed by a highly proximal extension, and it follows from Proposition 5.2.3 that an isomorphic decomposition would be obtained from applying the Y^* construction to $\overline{\pi}$.

Suppose $z \in N$, $P(z) = \text{Range}(z) = \pi^*(x^*) = \pi^{-1}(y)u$, $x^* = (x,\pi^{-1}(y)u)$ and $x \in \pi^{-1}(y)u$. Then for $p,q \in \beta T$, $zp = zq \Longrightarrow x^*p = x^*q$ and we can construct θ_{x^*} as promised. It now follows that P is regular with respect to π^* according to Definition 2.2.1; the homomorphisms $\theta_{x^*}\colon N \longrightarrow X^*$ being exactly what's required. An argument like the proof of Theorem 2.2.7 shows that P attracts any other homomorphism which is regular with respect to π^* and it follows that P is the regularizer of π^*. $\qquad ||$

Starting with a generalized almost finite $\pi: X \longrightarrow Y$ we have shown that π and its regularizer yield the same Y^* and that the regularizers of π and π^* have the same domain. One might ask what, if any, portion of this remains true without the generalized almost finiteness assumption or with some weaker assumption. Conditions of the form $Au = A\circ u$ for certain sets $A \in 2^{\pi}$ and idempotents $u \in J$ appear crucial.

We have the following corollary:

Corollary 5.2.10. *If $\pi: X \longrightarrow Y$ is a generalized almost finite homomorphism of minimal sets, then π^* (constructed in the usual way) is almost periodic.*

Proof: Referring to the diagram in Proposition 5.2.9, we see that P is a finite group extension. Aut P is finite, so the discrete topology makes Aut P compact Hausdorff and makes its action on N jointly continuous. Hence P is almost periodic, by Proposition 2.2.12, and π^* is almost periodic, by Proposition 2.5.12, since P is its regularizer. $||$

SELECTED BIBLIOGRAPHY

1. J. Auslander, *Mean L-stable systems*, Illinois J. Math., 3(1959), 566-579.

2. J. Auslander, *Endomorphisms of minimal sets*, Duke Math. J., 30(1963), 605-614.

3. J. Auslander, *Regular minimal sets, I*, Trans. Amer. Math. Soc., 123(1966), 469-479.

4. J. Auslander, *Homomorphisms of minimal transformation groups*, Topology, 9(1970), 195-203.

5. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

6. R. Ellis, *Locally compact transformation groups*, Duke Math. J., 24(1957), 119-125.

7. R. Ellis, Lectures on Topological Dynamics, W.A. Benjamin, New York, 1969.

8. R. Ellis, *The Veech structure theorem*, Trans. Amer. Math. Soc., to appear.

9. R. Ellis, S. Glasner, and L. Shapiro, *PI flows*, Advances in Mathematics, to appear.

10. H. Furstenberg, *Disjointness in ergodic theory, minimal sets, and a problem in diophantine approximation*, Math. Systems Theory, 1(1967), 1 - 50.

11. S. Glasner, *Compressibility properties in topological dynamics*, Amer. J. Math., to appear.

12. W.H. Gottschalk and G.A. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloquium Publications, Vol.36, Providence, 1955.

13. D. McMahon and T.S. Wu, *Relative equicontinuity and its variations*, submitted for publication.

14. N. Markley, *Locally circular minimal sets*, Pacific J. Math., 47(1973), 177-197.

15. L. Shapiro, *Distal and proximal extensions of minimal flows*, Math. Systems Theory, 5(1971), 76-88.

16. W. Veech, *Point distal flows*, Amer. J. Math., 92(1970), 205-242.