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ROBUSTNESS UNDER UNCERTAINTY WITH PHASE INFORMATION

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Abstract. The framework of Doyle's structured singular value is extended to take advantage of possibly available phase information on the dynamic uncertainty. A computable upper bound is obtained for this phase-sensitive structured singular value.

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1. Introduction and preliminaries

Let RH_∞ denote the set of real rational stable proper scalar transfer functions and let $RH_\infty^{n\times n}$ be the set of $n\times n$ matrices with entries in RH_∞. Given two nonnegative integers r and c and a list of integers (*block-structure*) $\mathcal{K} = (k_1, \cdots, k_r; k_{r+1}, \cdots, k_{r+c})$, with $\sum_{i=1}^{r+c} k_i = n$, consider the subspace $\mathcal{X}_\mathcal{K}$ of $RH_\infty^{n\times n}$ given by

$$\mathcal{X}_\mathcal{K} = \{\text{block diag}(\delta_1 I_{k_1}, \cdots, \delta_{r+c} I_{k_{r+c}}) : \\ \delta_i \in \mathbb{R},\ i = 1, \cdots, r;\ \delta_{r+i} \in RH_\infty,\ i = 1, \cdots, c\}.$$

Many robust stability analysis issues can be addressed via the following question for the feedback system S of Figure 1 below [1–3]: Given $P \in RH_\infty^{n\times n}$ and $\delta > 0$, is S well-formed and internally stable for any $\Delta \in \mathcal{X}_\mathcal{K}$ satisfying $\|\Delta\|_\infty \le \delta$, where $\|\cdot\|_\infty$ denotes the H_∞ norm? Here δ_i, $i = 1, \cdots, r$, correspond to real parametric uncertainties and δ_{r+i}, $i = 1, \cdots, c$, to unmodeled dynamics. In [1,2] Doyle *et al.* showed that this question can be answered by means of the *structured singular value* (*SSV*) $\mu_\mathcal{K}$, defined for any complex $n \times n$ matrix M by $\mu_\mathcal{K}(M) = 0$ if $\det(I - \Delta M) \ne 0$ for all $\Delta \in X_\mathcal{K}$, and

$$\mu_\mathcal{K}(M) = \left(\min_{\Delta \in X_\mathcal{K}} \{\bar\sigma(\Delta) : \det(I - \Delta M) = 0\} \right)^{-1}$$

otherwise, where the subspace $X_\mathcal{K}$ of $\mathbb{C}^{n\times n}$ is given by[2]

$$X_\mathcal{K} = \{\text{block diag}(\delta_1 I_{k_1}, \cdots, \delta_{r+c} I_{k_{r+c}}) : \\ \delta_i \in \mathbb{R},\ i = 1, \cdots, r;\ \delta_{r+i} \in \mathbb{C},\ i = 1, \cdots, c\}$$

and where $\bar\sigma(\cdot)$ denotes the largest singular value. Specifically, the "Small μ Theorem" [2] asserts that S is well-formed and internally stable for any $\Delta \in \mathcal{X}_\mathcal{K}$, $\|\Delta\|_\infty \le \delta$ if, and only if,

$$\sup_\omega \mu_\mathcal{K}(P(j\omega)) < 1/\delta \ .$$

While exact numerical evaluation of the SSV appears to be generally computationally prohibitive when $r \ne 0$, a reasonably good upper bound (yielding a *sufficient* stability condition) can be obtained at moderate cost [4,5].

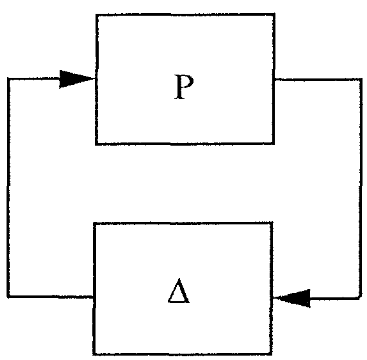

Figure 1

[2] Note that functions in $\mathcal{X}_\mathcal{K}$ take values in $X_\mathcal{K}$

The purpose of this paper is to extend the framework of the SSV to allow for the case when, besides structure– and magnitude information, *phase* information is available for the uncertainty (see [6,7] for other work on robustness with phase information). Specifically, we consider the situation in which given some function $\Theta : \mathbb{R} \rightarrow [0, \pi]^c$, Δ is known to lie in the set $\mathcal{X}_{\mathcal{K}}^{\Theta} \subset RH_{\infty}^{n \times n}$ defined by [3] [4]

$$\mathcal{X}_{\mathcal{K}}^{\Theta} = \{\Delta \in \mathcal{X}_{\mathcal{K}} : |\angle \delta_{r+i}(j\omega)| \leq \Theta_i(\omega),\ i = 1, \cdots, c\} \cup \{0\},$$

where given any $z \in \mathbb{C} \backslash \{0\}$, $\angle z$ denotes its phase in $(-\pi, \pi]$. We are naturally lead to define a "phase-sensitive" SSV.

In the sequel, for $\theta \in [0, \pi]^c$, we make use of the set $X_{\mathcal{K}}^{\theta} \subset \mathbb{C}^{n \times n}$, defined by

$$X_{\mathcal{K}}^{\theta} = \{\Delta \in X_{\mathcal{K}} : |\angle \delta_{r+i}| \leq \theta_i,\ i = 1, \cdots, c\} \cup \{0\}.$$

Definition 1. Given a block-structure $\mathcal{K}$ and a vector $\theta \in [0, \pi]^c$, *the phase-sensitive structured singular value* $\mu_{\mathcal{K}}^{\theta}(M)$ of M with respect to block-structure $\mathcal{K}$ and phase θ is given by $\mu_{\mathcal{K}}^{\theta}(M) = 0$ if there is no $\Delta \in X_{\mathcal{K}}^{\theta}$ such that $\det(I - \Delta M) = 0$, and

$$\mu_{\mathcal{K}}^{\theta}(M) = \left(\min_{\Delta \in X_{\mathcal{K}}^{\theta}} \{\bar{\sigma}(\Delta) : \det(I - \Delta M) = 0\} \right)^{-1} \tag{1}$$

otherwise. $\square$

Below, it is first shown that a natural extension of the Small μ Theorem holds in this case. A more tractable formula is then proposed for $\mu_{\mathcal{K}}^{\theta}(M)$, a direct extension of one obtained in [4] for $\mu_{\mathcal{K}}(M)$. Finally, an efficiently computable upper bound (yielding a *sufficient* condition of stability) is given, again directly related to that for $\mu_{\mathcal{K}}(M)$ derived in [4].

2. Extended Small μ Theorem

For any $\delta > 0$, let

$$\mathcal{X}_{\mathcal{K}}^{\Theta}(\delta) = \{\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta} : \ \|\Delta\|_{\infty} \leq \delta\}; \qquad X_{\mathcal{K}}^{\theta}(\delta) = \{\Delta \in X_{\mathcal{K}}^{\theta} : \ \bar{\sigma}(\Delta) \leq \delta\}\ .$$

Theorem 1.[5] Given $P \in RH_{\infty}^{n \times n}$, $\delta > 0$, and $\Theta : \mathbb{R} \rightarrow [0, \pi]^c$ continuous, the following two statements are equivalent: (i) the feedback system S of Figure 1 is well-formed and internally stable for all $\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$; (ii)

$$\sup_{\omega} \mu_{\mathcal{K}}^{\Theta(\omega)}(P(j\omega)) < 1/\delta\ .$$

[3] The more natural situation in which $\Phi_i(\omega) \leq \angle \delta_{r+i}(j\omega) \leq \Psi_i(\omega)$, for some Φ, Ψ : $\mathbb{R} \rightarrow (-\pi, \pi]^c$ can be easily reduced to the case considered here.

[4] $\Theta_i(\omega) = \pi$ for all ω accounts for blocks with no phase information.

[5] More generally the theorem holds with $\mathcal{X}_{\mathcal{K}}^{\Theta}$ replaced by any subset $\mathcal{X}$ of $RH_{\infty}^{n \times n}$ containing the origin such that, for any $\delta > 0$, $\{\Delta \in \mathcal{X} : \|\Delta\|_{\infty} \leq \delta\}$ is pathwise connected, and $\mu(M)$ defined accordingly.

Proof. We first show that (i) is equivalent to

$$\det(I - \Delta(j\omega)P(j\omega)) \neq 0 \quad \forall \Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta),\ \omega \in [-\infty, \infty]. \tag{2}$$

Since $P \in RH_\infty^{n\times n}$ and $\mathcal{X}_{\mathcal{K}}^{\Theta}(\delta) \subset RH_\infty^{n\times n}$, (i) holds if and only if, for all $\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$, $(I - \Delta(s)P(s))^{-1}$ is well defined and has no poles in $\mathbb{C}_+ \cup \{\infty\}$, i.e., $\det(I - \Delta(s)P(s)) \neq 0\ \forall s \in \mathbb{C}_+ \cup \{\infty\}$. Since P and Δ are analytic and bounded in $\mathbb{C}_+$, in view of the Principle of the Argument, it follows that (i) holds if and only if for all $\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$ the closed curve

$$\Gamma := \{\det(I - \Delta(j\omega)P(j\omega)) : \omega \in [-\infty, \infty]\}$$

does not encircle or pass through the origin. Thus, in particular, (i) implies (2). Conversely suppose that (2) holds. Let $\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$ and let $\varphi : [0,1] \to [-\infty, \infty]$ be continuous and onto. In view of the definition of $\mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$, $\alpha\Delta \in \mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$ for any $\alpha \in [0, 1]$. In view of (2), it follows that the continuous function $h : [0,1] \times [0,1] \to \mathbb{C}$ given by

$$h(\alpha, t) = \det(I - \alpha\Delta(j\varphi(t))P(j\varphi(t)))$$

defines a homotopy in the punctured plane $\mathbb{C}\backslash\{0\}$ between $h(0,\cdot) = 1$ and $h(1,\cdot) = \det(I - \Delta(j\varphi(\cdot))P(j\varphi(\cdot)))$. Thus the family of closed curves generated by $h(\alpha,\cdot)$, as α ranges over $[0,1]$, corresponds to a single element in the fundamental group of equivalence classes of closed curves in the punctured plane, i.e., all these curves encircle the origin the same number of times (see, e.g., [8, Section 8-5] for details). Thus the curve Γ $(= \{h(1,t) : t \in [0,1]\})$ does not encircle the origin, so that S is well-formed and internally stable for the given Δ. Thus (i) is equivalent to (2). To complete the proof, we now show that (2) is equivalent to (ii). First, note that, for given ω, when Δ ranges over $\mathcal{X}_{\mathcal{K}}^{\Theta}(\delta)$, its value $\Delta(j\omega)$ ranges over $X_{\mathcal{K}}^{\Theta(\omega)}(\delta)$. Thus (2) holds if, and only if,

$$\det(I - \Delta P(j\omega)) \neq 0 \quad \forall \Delta \in X_{\mathcal{K}}^{\Theta(\omega)}(\delta), \quad \forall \omega \in [-\infty, \infty]$$

or equivalently, since Θ is continuous,

$$\inf_{\omega}\{\min_{\Delta \in X_{\mathcal{K}}^{\Theta(\omega)}} \{\bar{\sigma}(\Delta) : \det(I - \Delta P(j\omega)) = 0\}\} > \delta \ .$$

In view of Definition 1, the last statement is equivalent to (ii). $\square$

Note that the standard Small μ Theorem [2] is obtained as a particular case of Theorem 1, corresponding to $\Theta_i(\omega) = \pi,\ i = 1, \cdots, c$, for all ω.

3. Computation of $\mu_{\mathcal{K}}^{\theta}(M)$

Computing $\mu_{\mathcal{K}}^{\theta}(M)$ by solving optimization problem (1) is impractical as this problem may have many local minimizers that are not global. Such local minima yield *lower bounds* to $\mu_{\mathcal{K}}^{\theta}(M)$, and thus sufficient conditions for *instability* may be tested. Short of computing $\mu_{\mathcal{K}}^{\theta}(M)$ exactly, of more interest would be an *upper bound* to it, allowing a sufficient condition for *stability* to be checked. As a first step toward this goal, we express $\mu_{\mathcal{K}}^{\theta}(M)$ as the optimal value of a smooth constrained maximization problem (note that problem (1) is nonsmooth). In this section, we sacrifice generality for clarity and restrict ourselves to the structure $\mathcal{K} = (\ ;1,1)$, i.e., to two scalar dynamic uncertainty blocks, and we assume that no phase information is available concerning the first block, i.e., $\theta_1 = \pi$. For notational simplicity, θ_2 is renamed θ and all $\mathcal{K}$ subscripts are dropped. Extension to more general structures presents no conceptual difficulties.

For any $\beta \in \mathbb{R}$, let $G_\beta \in \mathbb{C}^{2\times 2}$ be given by

$$G_\beta = \begin{bmatrix} 0 & 0 \\ 0 & 1+j\beta \end{bmatrix}.$$

Also let $P_1 = \mathrm{diag}(1,0)$, and $P_2 = \mathrm{diag}(0,1)$. Finally, we denote by ∂B the unit Euclidean ball in $\mathbb{C}^n$, i.e., $\partial B = \{x \in \mathbb{C}^n : \|x\|_2 \leq 1\}$ and supercript H indicates conjugate transpose. The phase sensitive SSV can be expressed as the optimal solution of a smooth constrained optimization problem as follows.

Theorem 2.

$$\mu^\theta(M) = \begin{cases} 0 & \text{if } \mathcal{S}^\theta(M) = \emptyset; \\ \max\limits_{\substack{x\in\mathcal{S}^\theta(M) \\ \gamma\geq 0}} \{\gamma : \|P_i M x\|_2 \geq \gamma \|P_i x\|_2, \quad i=1,2\} & \text{otherwise.} \end{cases}$$

where $\mathcal{S}^\theta(M)$ is defined as follows:

(i) $\mathcal{S}^0(M) = \{x \in \partial B : x^H(M^H P_2 - P_2 M)x = 0,$
$$x^H(M^H P_2 + P_2 M)x \geq 0\}$$

(ii) for $\theta \in (0, \frac{\pi}{2}]$,

$$\mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M^H G_\beta + G_\beta^H M)x \geq 0 \quad \forall \beta \in \{\pm \cot\theta\}\}$$

(iii) for $\theta \in (\frac{\pi}{2}, \pi)$

$$\begin{aligned} \mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M^H G_\beta + G_\beta^H M)x \geq 0,\ \beta = -\cot\theta, \\ j x^H(M^H P_2 - P_2 M)x \geq 0\} \\ \cup \{x \in \partial B : x^H(M^H G_\beta + G_\beta^H M)x \geq 0,\ \beta = \cot\theta, \\ j x^H(M^H P_2 - P_2 M)x \leq 0\}. \end{aligned}$$

□

Note that the second inequalities in each of the components of $\mathcal{S}^\theta(M)$ for the case $\theta \in (\frac{\pi}{2}, \pi)$ correspond to the limit of inequality

$$x^H(M^H G_\beta + G_\beta^H M)x \geq 0 \tag{3}$$

as β tends to $+\infty$ and $-\infty$, respectively. This leads to the next step, which is to observe that, as (3) is affine in β, for $\theta \in (0, \pi)$, the set $\mathcal{S}^\theta(M)$ can be equivalently written as

$$\mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M^H G + G^H M)x) \geq 0 \quad \forall G \in \mathcal{G}^\theta\}$$

for $\theta \in [0, \frac{\pi}{2}]$, and

$$\mathcal{S}^\theta(M) = \{x \in \partial B : x^H(M^H G + G^H M)x \geq 0 \quad \forall G \in \mathcal{G}_+^\theta\}$$
$$\cup \{x \in \partial B : x^H(M^H G + G^H M)x \geq 0 \quad \forall G \in \mathcal{G}_-^\theta\}$$

for $\theta \in (\frac{\pi}{2}, \pi)$, with

$$\mathcal{G}^0 = \{\alpha G_\beta : \alpha \geq 0,\ \beta \in \mathbb{R}\},$$

$$\mathcal{G}^\theta = \{\alpha G_\beta : \alpha \geq 0,\ |\beta| \leq \cot\theta\},\ \theta \in (0, \frac{\pi}{2}],$$

$$\mathcal{G}_+^\theta = \{\alpha G_\beta : \alpha \geq 0,\ \beta \geq -\cot\theta\},$$

$$\mathcal{G}_-^\theta = \{\alpha G_\beta : \alpha \geq 0,\ \beta \leq \cot\theta\}.$$

Using an argument similar to that employed in [4] one can then prove the following.

Theorem 3.

$$\mu^\theta(M) \leq \nu^\theta(M) \leq \bar{\sigma}(M) \qquad \forall \theta \in [0, \pi)$$

where

$$\nu^\theta(M)^2 = \begin{cases} \max\{0, \inf\limits_{G \in \mathcal{G}^\theta} \bar{\lambda}(M^H M + M^H G + G^H M)\} & \theta \in [0, \frac{\pi}{2}] \\ \max\{0, \inf\limits_{G \in \mathcal{G}_+^\theta} \bar{\lambda}(M^H M + M^H G + G^H M), & \\ \qquad \inf\limits_{G \in \mathcal{G}_-^\theta} \bar{\lambda}(M^H M + M^H G + G^H M)\} & \theta \in (\frac{\pi}{2}, \pi) \end{cases} \tag{4}$$

□

It is readily checked that, like $\mu(M)$, $\mu^\theta(M)$ satisfies $\mu^\theta(DMD^{-1}) = \mu^\theta(M)$ for any nonsingular diagonal matrix D (with our current assumption of scalar uncertainty blocks). It follows that

$$\mu^\theta(M) \leq \hat{\mu}^\theta(M) := \inf\{\nu^\theta(DMD^{-1}) : D = \mathrm{diag}(d_1, d_2), d_1 \neq 0 \neq d_2\} \tag{5}$$

It turns out that the value of the infimum is unchanged if d_1 and d_2 are constrained to be real positive. The algorithm proposed in [5] can be modified to compute $\hat{\mu}^\theta(M)$.

4. Discussion

As mentioned above, the results of Section 3 can be readily extended to more general structures. It should be noted however that if, say, k of the components of θ lie in $(\pi/2, \pi)$, the expression for $\nu^\theta(M)$ will involve 2^k optimization problems instead of 2. If this is computationally prohibitive, an alternate upper bound to $\mu_\mathcal{K}^\theta(M)$ can be obtained as follows. Rewrite the constraint $|\angle\delta_i| \leq \theta$, with $\theta \in [\pi/2, \pi)$, as $\delta_i = \delta_i^1 \delta_i^2$, with $|\angle\delta_i^1| \leq \pi/2$, $|\angle\delta_i^2| \leq \theta - \pi/2$. The magnitude constraint $|\delta_i| \leq \delta$ can be expressed, e.g., as $|\delta_i^1| \leq \delta$, $|\delta_i^2| \leq 1$ and, by elementary block diagram transformations, the system can be represented as in Figure 1 with δ_i^1 and δ_i^2 corresponding to distinct diagonal blocks in Δ. The results of Section 3 can be extended to such structure.

Preliminary numerical tests have been carried out based on (4) and (5), and on the algorithm proposed in [5]. The results are promising in that the computed upper bound is typically lower (yielding a less conservative sufficient stability test) than when the phase information is not taken into account.

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