ABSTRACT

Title of Dissertation: Topological T-duality: KK-monopoles, Gerbes and Automorphisms

Ashwin S. Pande, Doctor of Philosophy, April 2007

Dissertation directed by: Prof. J. Rosenberg
Department of Mathematics

We show that Topological T-duality proposed by Mathai and Rosenberg may be used to define a T-dual for a semi-free S^1-space. In particular, we argue that it gives the physical T-dual for a system of n Kaluza-Klein (KK) monopoles.

We show that the 'dyonic coordinates' well known in the physics literature may be incorporated within this formalism of Topological T-duality.

We study some formal properties of topological T-duality: We note that Topological T-duality naturally defines a T-dual of any semi-free S^1-space X. If $B \simeq X/S^1$, X is naturally associated to a Hitchin $2-$gerbe on B^+. We also note that T-duals of such spaces may be naturally associated to Hitchin $3-$gerbes on $B^+ \times S^1$. We demonstrate that Topological T-duality gives a natural mapping between these two gerbes.

We use the Equivariant Brauer Group to model a space with a $B-$field or a $H-$flux. We note that each step of the natural filtration on this group corresponds to one of the gauge fields of the $H-$flux. We note that given a T-dual pair of principal S^1-bundles $E, E^{\#}$ over B, T-duality gives a natural map $T : H^2(E, \mathbb{Z}) \rightarrow H^2(E^{\#}, \mathbb{Z})$. We define a classifying space for pairs over B consisting of a principal S^-1bundle $p : X \rightarrow B$ and a class $b \in H^2(X, \mathbb{Z})$. We characterize this space up to homotopy. We make a conjecture on the T-dual of an automorphism with nonzero $H-$flux.

Topological T-duality: KK-monopoles,

Gerbes and Automorphisms

by

Ashwin S. Pande

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Advisory Committee:

Prof. J. Rosenberg, Chair,
Prof. J. Millson,
Prof. J. Schafer,
Prof. K. Grove,
Prof. S. J. Gates, Jr. (Dean's Representative).

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Ashwin S. Pande

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KK-monopoles and T-duality

1.1 Introduction

String theory is widely regarded as a candidate for a 'Theory of Everything'. It replaces point particles by extended objects called strings propagating in a background spacetime called the 'Target Space'[1]. Torus duality or T-duality in string theory is an important symmetry of string theories: IIA string theory with a target spacetime X is identical to IIB string theory on the T-dual spacetime $X^{\#}$ [2]; i.e., the two quantum theories are identical up to a canonical transformation.

In the case that spacetime is a principal T^n bundle over a base, the dual spacetime $X^{\#}$ is roughly one in which each torus orbit in the original spacetime has been replaced by its dual torus (Following Ref. [3],if Λ is a lattice in $\mathbb{R}^n$ and Λ^* is the dual lattice in the dual vector space $(\mathbb{R}^n)^*$, then the torus dual to $\mathbb{R}^n/\Lambda$ is $(\mathbb{R}^n)^*/\Lambda^*$.) In the original string theory calculation (using gauged sigma models) the T-dual is obtained as a manifold together with various extra data (metric, B-field, dilaton and R-R charges.) [1].

[1] It was the study of the mapping of the R-R charges of string theory under such a transformation that gave the initial impetus to study T-duality purely topologically [4].

Mirror symmetry is conjectured [5] to be an example of T-duality. It also applies to spaces which are not T^n-spaces. However, it is only defined for Calabi-Yau manifolds [6] and the Kähler structure is essential for the theory to work.

Surprisingly, if we restrict ourselves to spacetimes which are principal T^n $(n = 1, 2)$ bundles over some base, it is possible to develop a theory of T-duality using only topological information [2]. That is, it is possible to specify the topological structure of the T-dual of a given spacetime using only the H-flux and topological structure of the original spacetime [3]. This is surprising since string theory usually requires a smooth, semi-Riemannian manifold (usually a Kähler manifold) as its target. The resulting theory has been the subject of the papers [4, 7, 3] and may be viewed as a "topological approximation" to T-duality in string theory. That is, it should be possible to take a spacetime X which posseses a T-dual in the sense of Ref. [3] and give it additional structure (metric, spin etc) so as to construct on X a type IIA string theory and to construct on $X^\#$ a type IIB string theory which form a dual pair (however, see Ref. [7].).

By the Gleason Slice Theorem, principal T^n-bundles X over Y are the same as spaces with a free T^n-action with $Y = X/T^n$. It is natural to ask whether the above formalism can be generalized to spaces with a non-free T^n-action. This is the aim of this thesis.

Suppose X is homeomorphic to a smooth closed, compact, connected Riemannian manifold. Suppose further that X possesses a smooth, semi-free, action of $T^n, (n = 1, 2)$ (by a semi-free action we mean an action with exactly two types

[2]The metric structure and finer structures (like the Kahler structure, needed for supersymmetry) are not needed to determine the topological type of the T-dual.

[3]provided certain Mackey obstructions vanish.This is automatic for $n = 1$.

of orbits: Free orbits and fixed points). Then we are in the basic setup of Ref. [3] if a class $\delta \in H^3(X, \mathbb{Z})$ is specified. Here X is to be viewed as a spacetime and the map $X \to X/T^n$ is a degenerate fibration. If the semi-free action has no fixed points, it is free and so by the Gleason slice theorem, X is a principal T^n-bundle over X/T^n. This case has already been extensively studied in Refs. [8, 3]. There, the authors associate to a space X with $H-$flux δ the continuous trace algebra $CT(X, \delta)$. If X is a principal T^n-bundle $(n = 1, 2)$, they demonstrate the following:

- If $n = 1$, there is a unique lifting α of the $\mathbb{R}-$action on X to $CT(X, \delta)$. The T-dual spacetime to X is given by the spectrum of $CT(X, \delta) \rtimes_{\alpha} \mathbb{R}$. In this case, the crossed product is always continuous trace.

- If $n = 2$ and if a certain condition is satisfied, the T-dual is unique and is given by the spectrum of the crossed product as above. If this condition is violated, however, there is no unique lifting of the $\mathbb{R}^n-$action on X to $CT(X, \delta)$. Also, the crossed product is not Type I. However, as explained in Ref. [3], the T-dual may be viewed as a noncommutative space.

In either case, the natural action of $\hat{\mathbb{R}}^n$ on the spectrum of the crossed product makes it into a principal T^n-bundle.

We will study the case of a T^n-action which has a fixed point. In Chap. (1), we show that the formalism of Ref. [3] extends to spaces containing KK-monopoles. In Chap. (2), we show that the result obtained in the previous chapter may be used to define an analogue of the dyonic coordinate of a system of KK-monopoles within the formalism of Topological T-Duality. In Chap. (3), we study the result of Chap. (2) using gerbes. In Chap. (4), we study a problem inspired by the work of Chap. (2).

1.2 KK-monopoles and T-duality

In this chapter, we study semi-free S^1-spaces with non-empty fixed point sets. Now, it is well known [9] that a smooth action of a compact Lie group G on a smooth manifold M in a neighbourhood of a fixed point is equivariantly-G-homeomorphic to an orthogonal action of G on a finite dimensional vector space. It would be helpful to consider this case first.

Thus, now we let X be $\mathbb{R}^k$ with a faithful orthogonal action of T^n. We view this as defining a fibration of $\mathbb{R}^k$ over the quotient space. We attempt to define a T-dual for such a fibration. Note that there can be no H-flux in this setting, since $\mathbb{R}^k$ is contractible (i.e., $\delta = 0$). As in Ref. [3], we let $\mathcal{A} = C_0(X, \mathcal{K})$. We lift[4] the S^1- action on X to a $\mathbb{R}$ action α on $\mathcal{A}$. As in Ref. [3], we may attempt to define the T-dual of X as the spectrum of the crossed product $\mathcal{A} \underset{\alpha}{\rtimes} \mathbb{R}$.

As a test example let $X = \mathbb{C}^2$ with the S^1 action

$$e^{2\pi i\theta}.(z, w) = (e^{2\pi i\theta} z, e^{2\pi i\theta} w).$$

This may be lifted to the obvious action of $\mathbb{R}$ namely,

$$\alpha_t(z, w) = (e^{2\pi it} z, e^{2\pi it} w).$$

Note that the S^1 action leaves each three-sphere

$$S^3_r = \{(z_1, z_2) \mid ||z_1|^2 + |z_2|^2 = r^2\}$$

invariant. The S^1 action on each S^3_r has S^2 as quotient. The quotient map is the Hopf fibration. The origin $(0, 0)$ is a fixed point for the $\mathbb{R}-$action.

We may also view this as a fibration of C^0S^3 over the base C^0S^2 (Here, $C^0S^3 \simeq (S^3 \times \mathbb{R}^+)/(S^3 \times 0))$ the open cone on S^3). The map sending C^0S^3 to

[4] For $n > 1$, if $H^3(X, \mathbb{Z}) \neq 0$, Mackey obstructions may arise which prevent such a lift.

C^0S^2 is the Hopf fibration when restricted to $S^2 \times \{t\}, t \neq 0$ and sends the vertex of C^0S^3 to the vertex of C^0S^2. (Note $S^3 \times (0, \infty)$ is a principal S^1 bundle over $S^2 \times (0, \infty)$.)

1.3 Physical T-duals

As we mentioned in the introduction, T-duality was first discovered in the theory of closed strings. This formalism only allows us to calculate the T-dual of space with a free T^n-action. It was soon realised that string theory contains a theory of extended objects called 'branes'. These are (roughly) submanifolds of spacetime on which strings can end. Due to the strings ending on such a submanifold[5], there is a quantum field theory defined on it. There are two types of branes: The Dp-branes are submanifolds which are sources of the RR-fields of string theory[6]; the NS5-branes, on the other hand are submanifolds which are sources of the Neveu-Schwartz B-field[7]. After the introduction of Dp-branes, T-duality was studied by putting Dp-brane probes into a geometry. The T-dual is the moduli space of the worldvolume theory on the Dp-brane. This approach is extremely flexible and enables the calculation of several T-duals unobtainable by previous methods.

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[5]The submanifold associated with a brane is termed its 'worldvolume.'

[6]Dp-branes possess $(p+1)$-dimensional worldvolumes. In type IIA string theory, p can only be even dimensional. In type IIB string theory, p can only be odd dimensional. Their worldvolume theory at low energies is a Super-Yang-Mills gauge theory.

[7]These have $6-$dimensional worldvolumes. Their worldvolume theory is a string theory (a "little string theory"). At low string coupling, they are extremely massive compared to Dp-branes.

It is useful to study the T-duals in the physics literature. This will constrain the mathematical models we create[8].

1. The T-dual of a NS5-brane is a Kaluza-Klein (KK) monopole (See Refs. [10, 11, 12]). Geometrically, an NS5-brane is a $6-$dimensional submanifold of X which is a source of $H-$flux. We model X topologically as a fibration $\mathbb{R}^6 \times C^0S^2 \times S^1 \xrightarrow{\pi} \mathbb{R}^6 \times C^0S^2$ where π is the projection map. The worldvolume of the NS5-brane intersects the S^1-fiber $\pi^{-1}(0)$ at a single point while its six worldvolume directions occupy $\mathbb{R}^6$. Since $\mathbb{R}^6$ is contractible, it does not affect the topological type of the T-dual. In the following we will model this by studying $C^0S^2 \times S^1 \xrightarrow{\pi} C^0S^2$. We will say that the NS5-brane is sitting at some location on the S^1-fiber over 0. This brane emits 1 unit of $H-$flux which we model as the cohomology class $[1] \in H^3(CS^2 \times S^1 - \{0\} \times S^1)$ where [1] is the generator of

$$H^3(S^2 \times S^1) \simeq H^3(S^2 \times (0,\infty) \times S^1) \simeq H^3(C^0S^2 \times S^1 - \pi^{-1}(0)).$$

A Kaluza-Klein monopole is a semi-riemannian manifold [9], which solves Einstein's equations. Topologically, this manifold is $\mathbb{R}^5$ with metric $g_{KK} = -dt^2 + g_{TN}$. Here g_{TN} is a certain Riemannian metric on $\mathbb{R}^4$ called the Taub-NUT metric[10]. The space $(\mathbb{R}^4, g_{TN})$ (termed 'Taub-NUT' space) possesses an isometric action of S^1 with one fixed point, and is S^1-equivariantly homeomorphic to $\mathbb{C}^2$ with the S^1 action $\lambda \cdot (z_1, z_2) = (\lambda \cdot z_1, \lambda \cdot z_2), \lambda \in S^1 \subseteq$

[8]To start with, we <u>assume</u> that topological information alone is enough to determine the topological type of the T-dual.

[9]see Refs. [12, 13, 14]

[10]This is an example of an ALF gravitational instanton metric.

$\mathbb{C}, z_i \in \mathbb{C}$. In the physics literature, the time direction is often ignored and $(\mathbb{R}^4, g_{TN})$ is also called the 'Kaluza-Klein monopole'. For our purposes, a Kaluza-Klein monopole is an S^1 fibration over $\mathbb{R}^3 \simeq C^0 S^2$. Over each sphere $\{t\} \times S^2$ in the base, the fibration is the Hopf fibration. Over 0 in the base, the fibration degenerates to a point. It may be viewed as a fibration $C^0 S^3 \xrightarrow{\pi^\#} C^0 S^2$.

Thus, we say that the string-theoretic T-dual of $C^0 S^3 \xrightarrow{\pi^\#} C^0 S^2$ is $C^0 S^2 \times S^1 \xrightarrow{\pi} C^0 S^2$ together with the $H-$flux emitted from a point source sitting at some point on the fiber $\pi^{-1}(0)$ (also termed a $H-$monopole). This is the test case discussed in Sec. (1.2). It is an important example, because (as will be seen below) most physical examples are built up from this one.

This T-dual may be calculated using Buscher's rules. We use polar coordinates (r, θ, ϕ) to parametrize the base and a periodic coordinate κ to parametrize the S^1 fiber. The Riemannian metric on Taub-NUT space may be written as

$$g_{TN} = H(r) d\vec{r} \cdot d\vec{r} + H(r)^{-1} (d\kappa + \frac{1}{2} \vec{\omega} \cdot d\vec{r})^2 \tag{1.1}$$

where $d\vec{r} = (dr, r \sin(\phi) d\theta, r d\phi)$, $H(r) = g^{-2} + (2r)^{-1}$ and $\omega_r = \omega_\theta = 0, \omega_\phi = (1 - \cos(\theta))$. We may write

$$\begin{aligned} g_{TN} &= H dr^2 + H r^2 \sin^2(\phi) (d\theta)^2 + H r^2 (d\phi)^2 + H^{-1} (d\kappa)^2 \\ &+ H^{-1} (1 - \cos(\theta)) d\phi d\kappa + H^{-1} (1 - \cos(\theta)) (d\phi)^2 \end{aligned} \tag{1.2}$$

where κ is the coordinate on the S^1 fiber. Let $x^0 = \kappa, x^1 = r, x^2 = \theta, x^3 = \phi$. Buscher's rules specify the T-dual metric $(\tilde{g})$ and $B-$field $(\tilde{b})$ in terms of the original metric (g) and $B-$field (b). By Ref. [2], we have

$$\tilde{g}_{00} = 1/g_{00}, \tilde{g}_{0\alpha} = b_{0\alpha}/g_{00},$$
$$\tilde{g}_{\alpha\beta} = g_{\alpha\beta} - (g_{0\alpha}g_{0\beta} - b_{0\alpha}b_{0\beta})/g_{00}. \tag{1.3}$$

Now, we have $b_{\alpha\beta} = 0, g_{00} = H, g_{0\alpha} = 0$ by Eq. (1.2) above. Therefore $\tilde{g}_{00} = H, \tilde{g}_{0\alpha} = 0 \forall\alpha$. Also by Eq. (1.3), $\tilde{g}_{\alpha\beta} = g_{\alpha\beta}$ if $\alpha \neq 3$ or $\beta \neq 3$. We also have

$$\tilde{g}_{33} = g_{33} - (g_{03})^2/(H^{-1}), =$$
$$= Hr^2 + H^{-1}(1-\cos(\theta))^2 - H^{-1}(1-\cos(\theta))^2 = Hr^2.$$

Hence,

$$g_H = H(d\kappa)^2 + H(dr)^2 + Hr^2\sin^2(\phi)(d\theta)^2 + Hr^2(d\phi)^2$$
$$= H((d\kappa)^2 + d\vec{r}\cdot d\vec{r}). \tag{1.4}$$

It is clear that $\tilde{g}$ is conformally equivalent to a product metric on $\mathbb{R}^3 \times S^1$. As $r \to 0, H \to \infty$ thus implying that the S^1-fiber over $0 \in \mathbb{R}^3$ is infinitely far away from the rest of the space. This is termed as a *smeared* $H-$monopole solution. Quantum effects [13] are supposed to modify H so that $\lim_{r\to 0} H(r,\theta)$ is finite except at the value of θ corresponding to the location of the $H-$monopole.

2. The T-dual of a set of p distinct, non-intersecting NS5-branes is a $p-$center KK monopole (See Ref. [10]). This is obtained from the previous example in the obvious fashion: We introduce p sources of $H-$flux in spacetime for the p NS5-branes. In the T-dual, we allow the S^1 fiber to degenerate to a point over p points in the base.

Let (X_p, g_p) be the spacetime containing p KK-monopoles [11]. This space posseses an isometric action of S^1 which is free except for p fixed points. The quotient of X_p by the S^1 action is $\mathbb{R}^3$ with the Euclidean metric. Let $\pi : X_p \to \mathbb{R}^3$ be the quotient map. If we use polar coordinates (r, θ, ϕ) on the base $\mathbb{R}^3$ then we may write the metric g_p in terms of these three coordinates and an additional coordinate κ on the S^1 fiber. Let

$$H(\mathbf{r}) = \frac{1}{g^2} + \sum_{i=1}^{p} \frac{1}{2|\mathbf{r} - \mathbf{r}_i|},$$

then,

$$g_p = H(\mathbf{r}) d\vec{r} \cdot d\vec{r} + H(\mathbf{r})^{-1} (d\kappa + \frac{1}{2} \vec{\omega} \cdot d\vec{r})^2 \tag{1.5}$$

where $d\vec{r} = (dr, r\sin(\phi)d\theta, rd\phi)$ and $\nabla H = \nabla \times \omega$. The above expression for g_p agrees with the expression for g_{TN} above (see Eq. (1.1)) except that the expression for H is different in the two cases. Since the form of H does not appear in the application of Buscher's rules above, the form of the metric on the T-dual is the same in both cases. Therefore, the metric on the T-dual of g_p is

$$\tilde{g} = H((d\kappa)^2 + d\vec{r} \cdot d\vec{r}).$$

This is conformally equivalent to a product metric on $\mathbb{R}^3 \times S^1$. The fibers over $r_i, i = 1, \ldots, p$ are infinitely far away from the rest of the spacetime, similar to the T-dual of one KK-monopole.

Now consider the case $p = 2$: Let Y be a line segment joining the image of the 2 centers in $\mathbb{R}^3$. We have $\mathbb{R}^3 - Y \simeq S^2 \times (0, \infty)$. Let $W = \pi^{-1}(Y)$ and consider $X_2 - W$. Now, $\forall t, \pi^{-1}(S^2 \times \{t\})$ is homeomorphic to $S^3/\mathbb{Z}_2$ since

[11] This is termed a multi-Taub-NUT space.

each S^2 encloses Y in $\mathbb{R}^3$. Therefore[12], $X_2 - W \simeq (S^3/\mathbb{Z}_2) \times (0, \infty)$ (This may also be seen by examining the expression for g_2.) Suppose we collapse Y to a point in X_2 and $\pi(Y)$ to a point in $\mathbb{R}^3$ We would obtain an equivariant fibration $\tilde{\pi} : C(S^3/\mathbb{Z}_2) \to C(S^2)$. Note that both $C(S^3/\mathbb{Z}_2) \simeq X_2/W$ and $C(S^2) \simeq \mathbb{R}^3/W$ are contractible to their vertices. This implies that X_2 is homotopy equivalent to $W \simeq S^2 \simeq \pi^{-1}(Y)$ (in fact equivariantly so).

If $p > 2$, we may always change the total space by a homeomorphism so that the image of the p centers in $\mathbb{R}^3$ under π lie on a straight line W. The inverse image of W under π is a collection of $(p-1)$ spheres joined to each other at one point and is homeomorphic to a wedge of $(p-1)$ spheres. By an exactly similar argument to the above, X_p is homotopy equivalent to this wedge of $(p-1)$ spheres.

3. The T-dual of a H-monopole of charge p is a KK-monopole of charge p. [13] A $H-$monopole of charge p is a fibration of the form $C^0S^2 \times S^1 \xrightarrow{\pi} C^0S^2$. The $H-$monopole sits at some point in $\pi^{-1}(0)$ and emits p units of $H-$flux on $C^0S^2 - \pi^{-1}(0)$. We represent the $H-$flux as the cohomology class $[p] \in H^3(CS^2 - \{0\} \times S^1) \simeq H^3(S^2 \times S^1)$. The KK-monopole of charge p is similar to the KK-monopole configuration above, except that the fiber over each $S^2 \times \{t\}$ in the base is a S^1 bundle of Chern class p, i.e., we have a fibration like $CL(1,p) \to CS^2$ where $L(1,p) \to S^2$ is the lens space viewed

[12] i.e., the map π restricted to any $\pi^{-1}(S^2) \subset (X_2 - W)$ will be the projection map of the S^1-bundle over S^2 of Chern class 2.

[13] A space with a KK-monopole of charge $p > 1$ is not a smooth manifold, it possesses a conical singularity at the location of the monopole.

as a principal S^1-bundle[14].

4. The T-dual of a spacetime of the form $\mathbb{C}^2/\Gamma$, with Γ a discrete subgroup of $SU(2)$ (in its fundamental representation) is a collection of interesecting NS5-branes (See Ref. [15]). (Here Γ can only be isomorphic to a cyclic group of order k, a dihedral group of order k or one of the groups of symmetries of a regular polyhedron in $\mathbb{R}^3$.) Note that we only get a principal S^1-bundle for $\Gamma = \mathbb{Z}_k$. The other cases are examples of mirror symmetry. However, since the T-dual in the other cases is a collection of NS5-branes, once we understand the test example above, we might be able to extend the C^*-formalism of T-duality to these cases as well.

5. The T-dual of a spacetime of the form $\mathbb{C}^3/\Gamma$, with Γ a discrete abelian subgroup of $SU(3)$ is a "brane box" which is a principal T^2 fibration with a pair of NS5-branes winding around a particular fiber[15]. There is also an appropriate $H-$flux in the bulk which is emitted by these branes. Here $\Gamma \simeq \mathbb{Z}_k \times \mathbb{Z}_l$ only and this turns $\mathbb{C}^3/\Gamma$ into a singular T^2-fibration. Its T-dual may be viewed as the fiber product of the example above with itself.

1.4 Test Example

To the test example we will associate $\mathcal{A} = CT(C^0S^3, 0)$ as there is no $H-$flux on $\mathbb{C}^2$. By Ref. [8], the $\mathbb{R}-$action on $X \simeq C^0S^3$ lifts to a unique $\mathbb{R}-$action α on $\mathcal{A}$. We work with the example of Sec. (1.2). Suppose we consider $CT(X, 0)$, then, as shown in [8], the S^1 action on X lifts uniquely to a $\mathbb{R}$ action on X. By Thm. 4.8 of Ref. [8], the spectrum of the crossed product is homeomorphic to $(X \times \hat{\mathbb{R}})/\sim$

[14]Note that $CL(1,p) \simeq \mathbb{C}^2/\mathbb{Z}_p$ where $\mathbb{Z}_p \subseteq SU(2)$ in its fundamental representation.

where $\sim$ is the equivalence relation given by

$$(x,\gamma) \sim (y,\chi) \Leftrightarrow \overline{\mathbb{R}.x} = \overline{\mathbb{R}.y}$$

and $\gamma\bar{\chi} \in (Stab_x)^{\perp}$. For $\mathbb{R}$, all irreps are one-dimensional, and of the form $\pi_k : x \mapsto e^{ikx}$. Distinct values of k correspond to non-unitarily-equivalent irreps. (As a topological space $\hat{\mathbb{R}}$ is homeomorphic to $\mathbb{R}$.) If $\gamma = k_1$ and $\chi = k_2$ then $\gamma\bar{\chi}$ corresponds to $x \mapsto e^{i(k_1-k_2)x}$. We have $\gamma\bar{\chi} \in (Stab_x)^{\perp}$ iff $\gamma\bar{\chi}(l) = 1, \forall l \in (Stab_x)$. If $x \neq 0$, the stabilizer is $\mathbb{Z}$, so, we have $e^{i(k_1-k_2)n} = 1, \forall n \in \mathbb{Z}$. This implies that $(k_1 - k_2) = 2l\pi, \forall l \in \mathbb{Z}$. Thus, points in $\hat{\mathbb{R}}$ are periodically identified. Further, points in the same torus orbit are identified. So, for $x \neq 0$, we have the dual principal bundle $S^2 \times S^1$ as described in Ref. [3]. If $x = 0$, the stabilizer is $\mathbb{R}$, so, we have $e^{i(k_1-k_2)x} = 1, \forall x \in \mathbb{R}$. This implies that $k_1 - k_2 = 0$. Thus, at the fixed point there is no quotienting.

Pick a S^1-invariant neighbourhood U of $0 \in \mathbb{C}^2$, and an $\epsilon-$neighbourhood V_i^{ϵ} of k_i. Then, $W_i^{\epsilon} = U \times V_i^{\epsilon}$ is a neighbourhood of $\{0\} \times k_i$ in $X \times \hat{\mathbb{R}}$. The W_i^{ϵ} form a neighbourhood base k_i in $X \times \hat{\mathbb{R}}$. Note that the quotient map associated to $\sim$ is open. The saturation of W_i^{ϵ} with respect to $\sim$ is

$$\tilde{W_i^{\epsilon}} = U \times (\coprod_j V_i + 2\pi j)$$

Thus, if $k_i - k_j \neq 2\pi l, l \in \mathbb{Z}$, $\tilde{W_i}$ can be chosen to be disjoint from $\tilde{W_j}$ by taking V_i small enough. Conversely, if $k_i - k_j = 2\pi l$, it is impossible to choose disjoint neighbourhoods for them in $X \times \hat{\mathbb{R}}$ (since the $\tilde{W_i}$ form a neighbourhood basis at k_i.)

So, we see that the crossed product has a very non-Hausdorff spectrum. In particular, its spectrum is $S^2 \times S^1 \times (0,\infty)$ with the line $\hat{\mathbb{R}}$ glued on at 0. The gluing is such that if a sequence $\{x_i\} \in S^2 \times S^1 \times (0,\infty)$ converges to $x_\infty \in (0 \times \hat{\mathbb{R}})$, then it converges to $x_\infty + 2\pi l, \forall l \in \hat{\mathbb{R}}$. Note that if we remove the fixed point of the $\mathbb{R}$ action on X, the crossed product is a nontrivial continuous-trace algebra $CT(S^2 \times S^1 \times (0, \infty), \delta^{\#})$. This example can be viewed as calculating the spectrum of the group C^*-algebra of the motion group $\mathbb{C}^2 \rtimes \mathbb{R}$.

If we view the crossed product as a C^*-bundle over the maximal Hausdorff quotient of its spectrum, we obtain a C^*-bundle over $CS^2 \times S^1$. Note that $CS^2 \times S^1$ is the physical T-dual. Thus, we define the physical spacetime to be the maximal Hausdorff regularisation (this is canonical [16]) of the spectrum of the crossed product.

Most physical examples of T-duality are built up from the T-duality of a NS5-brane with a KK-monopole. Following Ref. [3], a topological space X with $H-$flux δ may be naturally associated to the continuous trace algebra $CT(X, \delta)$. This $H-$flux is <u>sourceless</u>, i.e., we can pick a three-form which represents this $H-$flux in a neighbourhood of every point of X. However, if the space possesses a <u>source</u> of $H-$flux, we cannot pick such a three-form in any neighbourhood of the source. (It might be helpful to keep in mind the description of the Dirac monopole in electromagnetism: Recall, that the flux is not defined at the location of the monopole.) Here, the flux is only defined on $X - Y$ and so we only have a cohomology class $\delta \in H^3(X - Y)$. If we could find a natural definition of a C^*-algebra $\mathcal{A}$ which encodes the structure of a space with a source of $H-$flux, then we hope that the spectrum of its crossed product with $\mathbb{R}^n$ would still give the T-dual. Note that in all the examples above, we are dualizing spaces with a non-free T^n action and no $H-$flux, to spaces containing a source of $H-$flux.

Assume that we are in the set-up of Sec. (1.1), with a space X with a source of $H-$flux represented by a cohomology class $\delta^{\#}$ in $H^3((B - F) \times S^1, \mathbb{Z})$. We assume that the source is located somewhere in the S^1-fiber over F (where $F \times S^1 \subset B \times S^1$). We will replace continuous-trace algebra $CT(B \times S^1, \delta)$ by another C^*-algebra $\mathcal{B}$ so that the maximal Hausdorff regularisation of $\hat{\mathcal{B}}$ is $B \times S^1$ and we can naturally obtain a $H-$flux in $(B-F) \times S^1$. We suppose that $\mathcal{B}$ is an extension of the form

$$0 \to CT((B-F) \times S^1, \delta^{\#}) \to \mathcal{B} \to \mathcal{C}(F) \to 0, \tag{1.6}$$

where $\mathcal{C}(F) \simeq C_0(\mathbb{R}) \otimes C_0(F) \otimes \mathcal{K}$ ($\mathcal{K}$ is the C^*-algebra of compact operators on a separable infinite dimensional Hilbert space.).

For most spaces there are many extensions $\mathcal{B}$ with regularisation of $\hat{\mathcal{B}}$ $B \times S^1$ [17]. However, in this case, we can see that $\hat{\mathcal{B}}$ is uniquely determined.

Lemma 1.4.1. *Suppose we are given the following data*

- *topological spaces B, F such that $F \subseteq B$,*
- *a cohomology class $\delta^{\#} \in H^3((B-F) \times S^1, \mathbb{Z})$.*

Then, topological T-duality for principal S^1-bundles with $H-$flux uniquely determines the algebra $\mathcal{B}$ in Eq. (7).

Proof. Topological T-duality applied to the trivial principal S^1-bundle $(B-F) \times S^1 \to (B-F)$ with $H-$flux $\delta^{\#}$ gives a principal S^1-bundle E over $(B-F)$. The characteristic class of this bundle may be obtained by integrating $\delta^{\#}$ over the S^1-fiber.

This completely determines the T-dual space X up to equivariant homomorphism as follows: Any orbit of a semi-free S^1-action on a space X can only have two stabilizers namely, the identity and S^1. As a result, the spaces X can only have two orbit types: Fixed points and free orbits. Hence, if F is the subset of

B such that $\pi^{-1}(F)$ is the fixed point subset of X, by the classification theorem for spaces with two orbit types (see Ref. [9], Chap. V Sec. (5)), the space X is completely specified up to equivariant homomorphism by $F \subset B$ and the class of the principal S^1-bundle over $B - F$.

Let $\mathcal{A} = C_0(X, \mathcal{K})$. The S^1-action on X may always be lifted to $\mathcal{A}$ uniquely, and so determines a C^*-dynamical system $(\mathcal{A}, \alpha)$ up to exterior equivalence. Now, $\mathcal{B}$ may be defined as $\mathcal{A} \underset{\alpha}{\rtimes} \mathbb{R}$. The result is unique up to C^*-algebra isomorphism. $\square$

Note that topological T-duality is a geometric operation on CW complexes by Ref. [3, 18] and hence may be freely used in computations. It is clear that $\mathcal{B}$ has a $\hat{\mathbb{R}}-$action β and the crossed product $\hat{\mathcal{B}} \underset{\beta}{\rtimes} \hat{\mathbb{R}}$ is isomorphic to $\mathcal{A}$.

In general, the spectrum of $\mathcal{B}$ may not be a CW-complex[15]. We emphasize that this does not imply that the physical spacetime is non-Hausdorff, only that for calculational purposes, it is convenient to take a non-Hausdorff space whose regularisation is the physical spacetime.

Hence, we make the following dictionary

- Spacetime a principal S^1-bundle $X \to B$ with a sourceless $H-$flux $\delta \equiv (CT(X, \delta), \alpha)$ where α is the lift of the S^1-action on X to $CT(X, \delta)$, as in Ref. [3].

- Spacetime X with a NS5-brane of charge $\delta^{\#}$ wrapped on $F \times S^1 \subseteq B \times S^1$, $\equiv$ The unique extension like Eq. (1.6) above.

If the NS5-brane is not wrapped around a S^1-orbit, for consistency, we should associate to the space an extension like Eq. (1.6) above. However, it is not clear

[15]We saw this for the test example above.

how unique such an extension is. We do not address this question here, since we are T-dualizing semi-free S^1-spaces without $H-$flux and the NS5-branes we encounter will always be wrapped around an S^1-orbit.

Chapter 2

Dyonic Coordinates and KK-monopoles

2.1 Physical Background

In Sec. (1.2) of Chap. (1) we noted that the T-dual of a Kaluza-Klein monopole is a source of $H-$flux[1]. If we apply Buscher's rules [13] to a KK-monopole we obtain a $H-$monopole *smeared* over the S^1 fiber over $\{0\}$. Quantum corrections are expected to localize the $H-$monopole to a particular point in the fiber. Recall that we identified a Kaluza-Klein monopole with the space $(\mathbb{R}^4, g_{TN})$ (termed 'Taub-NUT' space). This space possesses an isometric action of S^1 with one fixed point and is S^1-equivariantly homeomorphic to $\mathbb{C}^2 \simeq \mathbb{R}^4$ with the S^1 action $\lambda \cdot (z_1, z_2) = (\lambda \cdot z_1, \lambda \cdot z_2), \lambda \in S^1 \subseteq \mathbb{C}, z_i \in \mathbb{C}$. We have $\mathbb{R}^4/S^1 \simeq \mathbb{R}^3$. It might be objected that the result obtained in the previous section is accidental. To give further evidence that it is nontrivial, we reproduce the dyonic coordinate of Refs. [19, 13, 14, 20] within the current formalism.

Suppose we have a KK-monopole located somewhere in $\mathbb{R}^4$. This space is a semi-free S^1-space. In the previous Chapter, in Sec. (3.4) we saw that such a space is specified up to equivariant homeomorphism by two data: The class of the

[1] Also termed a $H-$monopole.

principal S^1-bundle $\mathbb{R}^4 - \mathbf{x} \to \mathbb{R}^3 - \mathbf{x}$, and the fixed point set $\mathbf{x} \subseteq \mathbb{R}^3$. For the case of a single KK-monopole, the isomorphism class of the principal S^1-bundle is fixed. Thus, we only need to know $\mathbf{x}$ to fix the KK-monopole space. Therefore, only three numbers are needed to specify the KK-monopole[2], namely, the three coordinates of the location of the image of the center in $\mathbb{R}^3$ under the quotient map. In the T-dual picture, we have a source of $H-$flux somewhere in the fiber over $\{0\} \times S^1$. This is specified by the position of the source in $\mathbb{R}^3$ and the location of the source in $\{0\} \times S^1$. Thus four parameters are needed to specify the T-dual. Since T-dual spaces are physically equivalent, we should need the same number of parameters on both sides. It is interesting, therefore, to ask which datum of the KK-monopole changes when we change the location of the source of $H-$flux in the S^1 fiber of the T-dual. According to a result of A. Sen, [19], this may be obtained as follows: On the total space of the KK-monopole, we have a zero $H-$flux. This implies that the gauge field B of the $H-$flux is a closed two-form (as $H = dB$)[3]. It is this $B-$field that corresponds to the position of the $H-$monopole in the T-dual. It is termed a 'dyonic coordinate' in Refs. [13, 14] by analogy with the case of monopoles in gauge theories [20].

Taub-NUT space posseses [4] a nontrivial L^2-normalizable harmonic two-form Ω on its total space and every such harmonic form is a multiple[5] of Ω. If the $B-$field changes in time according to $B(t) = \beta(t)\Omega$ then, (see below) on the T-dual side, this corresponds to changing the angular coordinate of the S^1 factor

[2]following Ref. [13], pp. $2-3$

[3]Since $\mathbb{R}^4$ is contractible, B is also exact.

[4]Note that the total space is noncompact, so the usual Hodge theorem does not apply here.

[5]See Ref. [21] Sec. (7).

of $\mathbb{R}^3 \times S^1$ via an isometry

$$\theta(t) = \theta(0) - \beta(t). \tag{2.1}$$

We may explicitly calculate the above effect using Eq. (1.3). Recall, from Chap. 1, that the Taub-NUT metric was

$$g_{TN}(r, \theta, \phi, \kappa) = Hd\vec{r} \cdot d\vec{r} + H^{-1}(d\kappa^2 + (1 - \cos(\theta))d\phi d\kappa + (1 - \cos(\theta))^2 d\phi^2).$$

Its T-dual in the absence of a $B-$field was the H-monopole metric

$$g_H(r, \theta, \phi, \kappa) = H(d\kappa)^2 + H(dr)^2 + Hr^2 \sin^2(\phi)(d\theta)^2 + Hr^2(d\phi)^2$$

Note that for the Taub-NUT metric, $g_{00} = H^{-1}$; the harmonic form B discussed above is given by

$$\begin{aligned} B = \beta\Omega = \beta d\Lambda &= \beta d(\frac{1}{g^2 H}(d\kappa + \frac{(1 - \cos(\theta))}{2} d\phi)) \\ &= -\frac{\beta H'}{g^2 H^2} dr(d\kappa + \frac{(1 - \cos(\theta))}{2} d\phi) - \frac{\beta}{g^2 H} \sin\theta d\theta d\phi \end{aligned} \tag{2.2}$$

Hence we have $b_{01} = -(\beta H')/(g^2 H^2)$ and so

$$\tilde{g}_{00} = H, \tilde{g}_{01} = -(\beta H H')/(g^2 H^2) = -(\beta H')/(g^2 H).$$

Thus, the T-dual is given by

$$\begin{aligned} \tilde{g} &= H(r)(d\kappa^2 - (\beta H')/(g^2 H) d\kappa dr) + \{\text{terms of} g_{\alpha\beta}, \beta \neq 0, \alpha \neq 0\} \\ &= H(r)(d\kappa^2 - (\beta H')(g^2 H^2) d\kappa dr + (\beta^2 H'^2)/(4 g^4 H^4) dr^2) + ... \\ &= H(r)((d\kappa - (\beta H')/(g^2 H^2) dr)^2) + ... \\ &= H(r)(d(\kappa + \beta/(g^2 H)))^2 + ... \end{aligned}$$

Therefore, if we take a diffeomorphism Γ of the T-dual, $\Gamma : \mathbb{R}^3 \times S^1 \to \mathbb{R}^3 \times S^1$ given by

$$r = r, \theta = \theta, \phi = \phi, K = \kappa + \beta/(g^2 H(r)),$$

we see that $\tilde{g}(r,\theta,\phi,\kappa) = \Gamma^*(g_H(r,\theta,\phi,K))$. Also, Γ is an isometric diffeomorphism[6] between the distinct Riemannian manifolds $(\mathbb{R}^3 \times S^1, g_H)$ and $(\mathbb{R}^3 \times S^1, \tilde{g}.)$ Note that as $r \to \infty$, Γ approaches the isometry $\kappa \to \kappa + \beta$. In general, it is preferable if such transformations approach the identity at infinity. Thus, we consider instead the diffeomorphism Λ of the T-dual, $\Lambda : \mathbb{R}^3 \times S^1 \to \mathbb{R}^3 \times S^1$ given by

$$r = r, \theta = \theta, \phi = \phi, \tilde{K} = \kappa + \beta/(g^2 H(r)) - \beta,$$

which approaches the identity as $r \to \infty$: We obtain that

$$\tilde{g}(r,\theta,\phi,\kappa) = \Lambda^*(g_H(r,\theta,\phi,\tilde{K} + \beta)).$$

We expect that it should be possible to model the dyonic coordinate discussed above within the formalism of Refs. [3, 4]. We expect this because the T-dual of the KK-monopole is also obtained from Buscher's rules. If we could mimic this effect within the topological formalism, this would give added evidence that the T-dual in Ch. (1) is the 'correct' one.

2.2 Mathematical Formalism

We first simplify the problem by passing to a suitable compactification: We view $\mathbb{C}^2$ as an open subset of $\mathbb{CP}^2$: i.e., we have compactified $\mathbb{C}^2$ by adding an S^2 at ∞ and collapsing each S^1-orbit to a point. As $H^2(\mathbb{CP}^2, \mathbb{Z}) \simeq \mathbb{Z}$, and $\mathbb{CP}^2$ is compact, there is a unique harmonic form on $\mathbb{CP}^2$ corresponding to the usual generator of $H^2(\mathbb{CP}^2, \mathbb{Z})$ (viewed as sitting in $H^2(\mathbb{CP}^2, \mathbb{R}) \cong H^2_{deRham}(\mathbb{CP}^2, \mathbb{R})$). It is shown in Ref. [21] that the restriction of this form to $\mathbb{C}^2 \subseteq \mathbb{CP}^2$ is exactly Ω.

[6] Therefore, by the principle of general covariance, these two are indistinguishable at the level of general relativity, as expected.

Also $\beta(t)\Omega$ is a closed two-form on $\mathbb{CP}^2$ and we may identify it with an element of $H^2(\mathbb{CP}^2, \mathbb{R})$. It is not clear which topological object may be associated with a real cohomology class. However, if we restrict ourselves to integral $B-$fields (i.e., elements of the form $m\Omega, m \in \mathbb{Z}$) we may reformulate the above as follows: If we change the $B-$ field by adding an element of $H^2(\mathbb{CP}^2, \mathbb{Z})$ to it, on the T-dual side, this should correspond to rotating the S^1 fiber via Eq. (2.1).

We use homogenous coordinates $[x_1 : x_2 : x_3]$ on $\mathbb{CP}^2$ (with $x_i \in \mathbb{C}$ and $(x_1 : x_2 : x_3) \sim (\lambda x_1 : \lambda x_2 : \lambda x_3), \forall \lambda \in \mathbb{C})$. Then $\mathbb{C}^2$ corresponds to the subset

$$U = \{[x_1 : x_2 : x_3] | x_3 \neq 0\}$$

and the sphere at infinity to the subset

$$W = \{[x_1 : x_2 : x_3] | x_3 = 0\} .$$

We consider the action $\lambda.[x_1 : x_2 : x_3] = [\lambda x_1 : \lambda x_2 : x_3]$, $\lambda \in S^1$ on $\mathbb{CP}^2$. This is obviously well-defined. Note that on U it turns into the action of Sec. (1.2) because the following commutes

$$\begin{array}{ccc} [x_1 : x_2 : x_3] & \longrightarrow & [\lambda x_1 : \lambda x_2 : x_3] \\ \big\downarrow & & \big\downarrow \\ (x_1/x_3, x_2/x_3) & \longrightarrow & (\lambda x_1/x_3, \lambda x_2/x_3). \end{array} \tag{2.3}$$

Hence we have an S^1 action on $\mathbb{CP}^2$ which we may lift to a $\mathbb{R}$ action α_t on $C(\mathbb{CP}^2, \mathcal{K})$. We recall that spectrum fixing automorphisms of C^*-algebras are classified upto inner automorphisms by their Phillips-Raeburn invariant which is a homomorphism $\zeta : \mathrm{Aut}_{C_0(X)}(\mathcal{A}) \to H^2(X, \mathbb{Z})$ (see Ref. [22] for more details). The following theorem shows that this automorphism may always be 'dualized':

Theorem 2.2.1. *Let X be a finite CW-complex with a semi-free S^1-action. Let α_t be a lift of the S^1-action on X to $C(X, \mathcal{K})$.*

1. *Let $[\lambda] \in H^2(X,\mathbb{Z})$. Then, there is a $\mathbb{R}-$action β on $C(X,\mathcal{K})$ exterior equivalent to α and a spectrum-fixing $\mathbb{Z}$ action λ on $C(X,\mathcal{K})$ which has Phillips-Raeburn obstruction $[\lambda]$ such that β and λ commute.*

2. *With the notation above, the action λ induces a $\mathbb{Z}$ action $\tilde{\lambda}$ on $C(X,\mathcal{K}) \underset{\alpha}{\rtimes} \mathbb{R}$. The induced action on the crossed product is locally unitary on the spectrum of the crossed product and is thus spectrum fixing.*

Proof: Let $\mathcal{A} = C(X,\mathcal{K})$ then $X = \hat{\mathcal{A}}$.

1. We have a short exact sequence

$$0 \to Inn(\mathcal{A}) \to Aut_{C_0(X)}(\mathcal{A}) \xrightarrow{\zeta} H^2(X,\mathbb{Z}) \to 0.$$

   Pick any $\mathbb{Z}-$action $\tilde{\lambda}$ with $\zeta(\lambda) = [\lambda]$. Note that $\zeta(\alpha_{-t}\tilde{\lambda}\alpha_t) = \alpha_t^*(\zeta(\lambda))$ (By Lemma 4.4 of Ref. [23]) Since $\alpha_t^* = id, \forall t \in \mathbb{R}$ (as $\alpha_1^* = id$), we have $\zeta(\alpha_{-t}\tilde{\lambda}\alpha_t) = [\lambda], \forall t \in \mathbb{R}$. We see that $\alpha_{-t}\lambda\alpha_t$ is equal to λ up to inner automorphisms. If it were exactly equal to λ we would obtain a map $\phi : \mathbb{R} \times \mathbb{Z} \to \mathrm{Aut}(\mathcal{A})$. As it is, we obtain a map $\tilde{\phi} : \mathbb{R} \times \mathbb{Z} \to \mathrm{Out}(\mathcal{A})$. To lift this to $Aut(\mathcal{A})$, by Ref. [23] Lemma 4.6, an obstruction class in $H^3_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T}))$ must vanish. To calculate this cohomology group, we use the fact that $H^3_M(G,A) \simeq \underline{H}^3_M(G,A)$ if G is second countable and locally compact and A is a polish abelian G-module. (See [24] Thm. 7.4). Now, by Ref. [25], Thm. 9, there is a spectral sequence[7] converging to $\underline{H}^*(\mathbb{R} \times \mathbb{Z}, C(X,\mathbb{T}))$ whose E^2 term is $E^2_{p,q} = \underline{H}^p(\mathbb{Z}, \underline{H}^q(\mathbb{R}, C(X,\mathbb{T}))$. By Ref. [8], Thm. 4.1, we have that $H^q(\mathbb{R}, C(X,\mathbb{T})) = 0, q > 1$. Note that the $\mathbb{Z}-$module $H^*_M(\mathbb{R}, C(X,\mathbb{T}))$ has a trivial $\mathbb{Z}-$action, since $C(X,\mathbb{T})$ has a trivial $\mathbb{Z}$ action. Since $\mathbb{Z}$ has

[7] See also Ref. [24], pg. 190.

the discrete topology, the Borel cochains with values in a Polish abelian group A with a trivial $\mathbb{Z}-$action are simply all the cochains. Therefore, $H^*_M(\mathbb{Z}, A) \simeq H^*(\mathbb{Z}, A)$, where the last cohomology group is calculated by the usual group cohomology theory. Since $H^k(\mathbb{Z}, A) = 0$ for $k > 1$, we see that $E_2^{p,q} = 0$ for $p > 1$ as well as for $q > 1$. Thus $E_2^{p,q} = 0$ for all $p+q = 3$ and H^3 vanishes. As a result, the action $\tilde{\phi}$ lifts to a twisted action ϕ'. By Raeburn's Stabilization trick, this is exterior equivalent to an ordinary action ϕ.

Note that the restriction of ϕ to the $\mathbb{R}-$factor gives an $\mathbb{R}-$action β exterior equivalent to α (since the lift of the S^1-action is unique up to exterior equivalence [8]).

2. Since λ is a locally unitary action we may pick a sufficiently small open set $U \subset X$ such that λ is unitary on the localization $\mathcal{A}_U$ of $\mathcal{A}$ to U [24]. Then there exists $u_\alpha \in UM(\mathcal{A}_U)$ such that $\lambda(x) = \mathrm{Ad}_{u_\alpha}(x)$ for every $x \in \mathcal{A}_U$. This defines an element $f \in C_b(\mathbb{R}, UM(\mathcal{A}_U))$ by $f(t) = u_\alpha, \forall t$.

   Note that U may be taken to be invariant under the S^1-action on X. Since there are only two orbit types and we're assuming everything is homotopy finite, we can choose U to be equivariantly homeomorphic to either $S^1 \times V$ with V contractible (if we are away from the fixed points) or to a cone times V, V a contractible open subset of the fixed set. For both these, $H^2(U, \mathbb{Z}) \simeq 0$, and so λ localized to these sets is unitary.

   Since U is a union of S^1-orbits, we have that $C_b(\mathbb{R}, UM(\mathcal{A}_U)) \subseteq UM(\mathcal{A}_U \underset{\alpha}{\rtimes} \mathbb{R})$. The induced automorphism on $\mathcal{A}_U \underset{\alpha}{\rtimes} \mathbb{R}$ is given on $C_0(\mathbb{R}, \mathcal{A}_U)$ (which is a dense subspace of $\mathcal{A}_U \underset{\alpha}{\rtimes} \mathbb{R}$) by $\tilde{\lambda}(g)(s) = f(s)^* g(s) f(s), \forall s \in \mathbb{R}$. This extends to a unitary automorphism of $\mathcal{A}_U \underset{\alpha}{\rtimes} \mathbb{R}$. Hence $\tilde{\lambda}$ is locally unitary on the spectrum of the crossed product.

$\square$

Thus we see that under T-duality a class in $H^2(\mathbb{CP}^2, \mathbb{Z})$ gives rise to a spectrum fixing automorphism of the crossed product algebra. We identify this with a rotation of the form Eq. (2.1) with $\beta = 2m\pi, m \in \mathbb{Z}$.

Note that in our example, the spectrum $X^\#$ of the crossed product is not Hausdorff. Hence, the crossed product algebra is not continuous-trace. Thus, this spectrum fixing automorphism does not define a cohomology class in $H^2(Y, \mathbb{Z})$ where Y is the Hausdorff regularization of $X^\#$. Physically, this is reasonable, since there is an H-flux in the T-dual so we would not expect a $B-$field there.

2.3 Multiple KK-monopoles

Consider the Multi-Taub-NUT space (X_k) defined in Chap. (1). Using the coordinates in Eq. (1.5), the metric on X_k is given by

$$g_{kTN} = H(\vec{\mathbf{r}})d\vec{\mathbf{r}}.d\vec{\mathbf{r}} + H(\vec{\mathbf{r}})^{-1}(d\kappa + \frac{1}{2}\omega.d\vec{\mathbf{r}})^2$$

$$\text{where } H(\vec{\mathbf{r}}) = 1 + \sum_{i=1}^{k} \frac{1}{|\vec{\mathbf{r}} - \vec{\mathbf{r}}_i|}.$$

The T-dual is given by

$$g_{H_k}(r, \theta, \phi, \kappa) = H((d\kappa)^2 + d\vec{r}.d\vec{r}) \quad (2.4)$$

The harmonic forms B_k on X_k are given by

$$
\begin{aligned}
B_i &= \beta\Omega_i = \beta d\xi_i, \\
\xi_i &= \alpha_i - \frac{H_i}{H}(d\kappa + \alpha), \\
H_i &= \frac{1}{|r - r_i|}, \\
B_i &= -\beta\frac{\partial}{\partial r}\left(\frac{H_i}{H}\right) d\kappa dr + \{\text{ terms not containing } d\kappa\}, \\
B_{01i} &= -\beta\frac{\partial}{\partial r}\left(\frac{H_i}{H}\right)
\end{aligned}
\tag{2.5}
$$

Hence, for the T-dual metric,

$$
g_{00} = H, g_{01} = -\beta\frac{\partial}{\partial r}\left(\frac{H_i}{H}\right)
$$

As in the previous section, the T-dual is given by

$$
g_{H_k} = H(r)(d(\kappa + \beta\frac{\partial}{\partial r}\left(\frac{H_i}{H}\right)) + \{\text{ terms of } g_{\alpha\beta}, \alpha \neq 0, \beta \neq 0\}.
$$

We prefer to take the following as a basis for the set of harmonic forms

$$
\tilde{B}_i = \sum_{j, j \neq i} B_i = -\beta\frac{\partial}{\partial r}\left(1 - \frac{H_i}{H}\right) d\kappa dr + \dots
$$

Then, the T-dual is given by

$$
g_{H_k} = H(r)(d(\kappa + \beta\frac{\partial}{\partial r}\left(1 - \frac{H_i}{H}\right)) + \{\text{ terms of } g_{\alpha\beta}, \alpha \neq 0, \beta \neq 0\}.
$$

We take the diffeomorphism Γ of $\mathbb{R}^3 \times S^1$ given by

$$
\begin{aligned}
r &= r \\
\theta &= \theta \\
\phi &= \phi \\
\kappa &= \kappa + \beta\frac{\partial}{\partial r}\left(1 - \frac{H_i}{H}\right) - \beta
\end{aligned}
$$

Hence, exactly in the previous section, $g_{TN_k}(r, \theta, \phi, \kappa) = \Lambda^*(g_{H_k}(r, \theta, \phi, \kappa + \beta))$.

The calculation in the previous section trivially extends to Multi-Taub-NUT spaces. As in Sec. (3.1.2) we need a suitable compactification $\tilde{X}_k$ of X_k such that the harmonic forms on X_k are related to the cohomology of $\tilde{X}_k$. Since g_{TN_k} is a metric of fibered boundary type [21], we use the compactification given in that paper. Thus, $\tilde{X}_k$ is obtained by collapsing the S^1-fibres of $S^3/\mathbb{Z}_k$ (which is the boundary of X_k at ∞) to points to obtain an S^2. Now, X_k has an S^1 action which extends to $\tilde{X}_k$ by fixing the S^2 at ∞. In Thm. (2.2.1), we did not use the fact that $X = \mathbb{CP}^2$ anywhere. Therefore, the theorem is actually true for any finite CW complex X which possesses an S^1 action. This would be the analogue of the dyonic coordinate for X_k. In general, we can repeat the above construction for any Riemannian manifold of fibered boundary type using the compactification in Ref. [21].

Chapter 3

Gerbes and T-duality

In this Chapter, we show that we may naturally associate a $2-$gerbe (see below) to a semi-free S^1-space. Similarly, we show that we may associate a $3-$gerbe to a space with a source of $H-$flux. We show that topological T-duality induces a natural map between these two gerbes.

We are trying to model a space containing a source of $H-$flux, a 3-form field. It is useful to begin by studying a simpler example, a Dirac monopole (this is a source of a $2-$form field, the electromagnetic flux). We begin by reviewing a construction of J.-L. Brylinski [26]

3.1 The Dirac Monopole

It is well known that a Dirac monopole situated at $\mathbf{x} \in \mathbb{R}^3$ is defined by a line bundle E on $\mathbb{R}^3 - \mathbf{x}$, the gauge bundle, together with a connection ∇_X on this line bundle. The connection is only specified upto a gauge transformation, i.e., we consider ∇_X equivalent to $U\nabla_X U^*$ where U is a section of the bundle $End(E)$.

The curvature of this connection is identified with the electromagnetic field strength F emitted by the monopole. It is a closed two-form on $\mathbb{R}^3 - \mathbf{x}$. Maxwell's equations for the field generated by the monopole are $dF = l$ and $d{*}F = h$ where

l, h are distribution-valued differential forms with support at the point $\mathbf{x}$ in $\mathbb{R}^3$; these model the monopole as a source of the electromagnetic field.

(From now on, we assume that the monopole is located at $\mathbf{x} = 0$. We will relax this restriction later.) From the equations governing F, it follows that F diverges as $F \sim 1/|\mathbf{y}|^\alpha$ for some constant α as $\mathbf{y} \to 0$ Following Refs. [27, 26] we view the field strength of the monopole F as a distribution-valued $2-$form on $\mathbb{R}^3$.

Classically, a line bundle with connection has a characteristic class which is the de Rham cohomology class of its curvature two-form. In Refs. [27, 26] the authors obtain a generalization of this characteristic class for the monopole as follows: Suppose we attempt to use the curvature of the line bundle associated to the monopole to obtain a cohomology class. This class would most naturally reside in the cohomology of the complex $\Omega^*_{\{0\}}(\mathbb{R}^3)$ which consists of distribution-valued differential forms on $\mathbb{R}^3$ which have singular support at $0 \in \mathbb{R}^3$. It can be shown that the cohomology of this complex is $H^*(\mathbb{R}^3, \mathbb{R}^3 - 0)$. Using the long exact sequence of the pair $(\mathbb{R}^3, \mathbb{R}^3 - 0)$, we obtain the exact sequence

$$H^2(\mathbb{R}^3) \to H^2(\mathbb{R}^3 - 0) \to H^3(\mathbb{R}^3, \mathbb{R}^3 - 0) \to H^3(\mathbb{R}^3) \to H^3(\mathbb{R}^3 - 0). \tag{3.1}$$

Since all the cohomology groups of $\mathbb{R}^3$ vanish, we obtain an isomorphism $H^2(\mathbb{R}^3 - 0) \to H^3(\mathbb{R}^3, \mathbb{R}^3 - 0)$ under which $[F|\mathbb{R}^3 - \{0\}] \to [dF = l]$. Now, as explained in Ref. [26] we would like to move the monopole about $\mathbb{R}^3$ without affecting the above class. This may be done by passing to S^3 (which is to be viewed as) the one point compactification of $\mathbb{R}^3$. We assume given an inclusion $i : \mathbb{R}^3 \to S^3$ which induces maps i^*, j^* as shown below.

$$\begin{tikzcd}
H^2(\mathbb{R}^3) \arrow[r] & H^2(\mathbb{R}^3 - 0) \arrow[r] & H^3(\mathbb{R}^3, \mathbb{R}^3 - 0) \arrow[r] & H^3(\mathbb{R}^3) & \\
H^2(S^3) \arrow[r] \arrow[u, "i^*"] & H^2(S^3 - 0) \arrow[r] \arrow[u, "j^*"] & H^3(S^3, S^3 - 0) \arrow[r] \arrow[u, "k^*"] & H^3(S^3) \arrow[r] \arrow[u, "i^*"] & 0
\end{tikzcd} \tag{3.2}$$

Here k^* is an isomorphism by excision. This gives the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\ \uparrow{\scriptstyle i^*} & & \uparrow{\scriptstyle j^*} & & \uparrow{\scriptstyle k^*} & & \uparrow{\scriptstyle i^*} \\ 0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Z} \end{array} \tag{3.3}$$

and so an isomorphism of $H^3(\mathbb{R}^3, \mathbb{R}^3 - 0)$ with $H^3(S^3)$.

Hence, given a monopole situated at 0 in $\mathbb{R}^3$, we view its curvature two-form as a distribution-valued two form defined on $\mathbb{R}^3$ with singular support at 0. Its cohomology class gives an element of $H^3(\mathbb{R}^3, \mathbb{R}^3 - 0)$ which, by the above isomorphism, gives a class in $H^3(S^3)$. It is also clear by the above argument that changing the location of the monopole from 0 to $\mathbf{x} \in \mathbb{R}^3$ will not change the above class in $H^3(S^3)$.

It is interesting to view the class obtained above in another way. Elements of $H^3(S^3)$ are in one-to-one correspondence with stable continuous-trace C^*-algebras with spectrum S^3. Given a monopole on $\mathbb{R}^3$, is it possible to uniquely obtain a continuous trace algebra on S^3? Also, does every continuous trace algebra on S^3 arise in this way?

Suppose we are given a monopole located at a point $\mathbf{x} \in \mathbb{R}^3$. This gives rise to a closed two-form (the gauge field strength) on $\mathbb{R}^3 - \mathbf{x}$. The de Rham cohomology class ω of this two-form defines an element in $H^2(\mathbb{R}^3 - \mathbf{x})$ and thus a map $(\mathbb{R}^3 - \mathbf{x}) \to K(\mathbb{Z}, 2)$. Since $K(\mathbb{Z}, 2)$ is homotopy equivalent to $\mathcal{PU}$ we obtain a map from $(\mathbb{R}^3 - \mathbf{x}) \to \mathcal{PU}$. (Note that $\mathbb{R}^3 - \mathbf{x} \simeq S^3 - \{\mathbf{x}, \infty\}$.) Using the gluing construction of continuous trace algebras on S^3 given in Ref. [17] we see that we obtain a unique stable continuous trace algebra on S^3 associated to the monopole [1]. Every class in $H^3(S^3)$ arises in this way because the Dixmier-Douady invariant

[1]Take trivial continuous-trace algebras over $S^3 - \mathbf{x}$ and $S^3 - \infty$ and glue on the overlap

of the continuous trace algebra, which classifies the algebra up to isomorphism, is the image of ω in $H^3(S^3)$ via the isomorphism $H^2(S^2) \to H^3(S^3)$.

Note that the same algebra on S^3 represents the gauge bundle of a monopole at any other[2] point $\mathbf{x}$ in $\mathbb{R}^3$. Pick as the open sets $U_{\mathbf{x}} = S^3 - \mathbf{x}$ and $U_\infty = S^3 - \infty$. Since these open sets are contractible, the algebra localized to these sets is $C_0(U_*, \mathcal{K})$. We obtain transition functions on $U_{\mathbf{x}} \cap U_\infty$ which define the same class in $H^2(S^2)$ as above, since the Dixmier-Douady invariant is the same. This shows that we have a line bundle defined on $U_{\mathbf{x}} \cap U_\infty$ which we identify with the gauge bundle of a monopole located at $\mathbf{x}$.

3.2 Sources of $H-$flux

For the first T-dual pair, we have a source of $H-$flux situated at $0 \times S^1$ on $C^0S^2 \times S^1$. We represent such a situation by a distribution-valued three-form on $C^0S^2 \times S^1$ with singular support at $0 \times S^1$. As above, we expect that its cohomology class in a suitably defined group should give a topological invariant of this situation. By an argument similar to the one given above, the cohomology class should lie in $H^4(C^0S^2 \times S^1, 0 \times S^1)$. We have an exact sequence,

$$\begin{tikzcd} H^3(C^0S^2 \times S^1) \arrow[r] & H^3((C^0S^2 - 0) \times S^1) \arrow[d, "\phi^*"'] & \\ & H^4(C^0S^2 \times S^1, (C^0S^2 - 0) \times S^1) \arrow[r] & H^4(C^0S^2 \times S^1) \end{tikzcd} \tag{3.4}$$

As $C^0S^2 \times S^1$ is homotopy equivalent to S^1, we get an isomorphism $\phi^* : H^3((C^0S^2 - 0) \times S^1) \to H^4(C^0S^2 \times S^1, (C^0S^2 - 0) \times S^1))$ Now consider the

using the above function, noting that $\mathcal{P}U \simeq Aut(\mathcal{K})$.

[2]Hence the associated continuous-trace algebra does not determine the monopole completely.

inclusion $C^0S^2 \times S^1 \to S^3 \times S^1$ where we view S^3 as the one point compactification of C^0S^2. We get a commutative diagram

$$\begin{array}{ccccc}
H^3(C^0S^2 \times S^1) & \longrightarrow & H^4(C^0S^2 \times S^1, (C^0S^2 - 0) \times S^1) & \longrightarrow \\
\Big\uparrow{\scriptstyle i^*} & & \Big\uparrow{\scriptstyle j^*} & \\
H^3((S^3 - 0) \times S^1) & \longrightarrow & H^4(S^3 \times S^1, (S^3 - 0) \times S^1) & \longrightarrow \\
H^4(C^0S^2 \times S^1) & \longrightarrow & H^4((C^0S^2 - 0) \times S^1) & \\
\Big\uparrow{\scriptstyle k^*} & & \Big\uparrow{\scriptstyle i^*} & \\
H^4(S^3 \times S^1) & \longrightarrow & H^4((S^3 - 0) \times S^1) &
\end{array} \tag{3.5}$$

and hence an isomorphism $H^4(C^0S^2 \times S^1, (C^0S^2 - 0) \times S^1) \simeq H^4(S^3 \times S^1)$.

We would like a geometric version of this isomorphism; i.e., given an extension like Eq.(1.6), we would like to naturally associate a class in $H^4(S^3 \times S^1)$. We use the above argument to associate a $2-$gerbe (see below) on B^+ to a semi-free S^1-space X with $X/S^1 = B$. We show in Thm. 3.4.1 below, that T-duality gives a natural mapping between $2-$gerbes on B and $3-$gerbes on $B^+ \times S^1$. The characteristic class of this $3-$gerbe is exactly the class obtained above.

3.3 Gerbes

Following Ref. [28], we define a $k-$gerbe on a space to be a geometric object naturally associated to a class in $H^k(X; \underline{\mathbb{C}^*}) \simeq H^{k+1}(X; \mathbb{Z})$. Here, $\underline{\mathbb{C}^*}$ is the sheaf of continuous $\mathbb{C}^*$-valued functions on X. Therefore[3], a $0-$gerbe is a homotopy class of functions in $C(X, \mathbb{C}^*)$. A $1-$gerbe is a complex line bundle or a principal S^1-bundle (since such objects are in $1-1$ correspondence with elements of $H^1(X; \underline{\mathbb{C}^*})$). A $2-$gerbe is an object which is naturally associated to a class in

[3]The word "gerbe" in the following always refers to gerbes in the sense of Ref. [28] i.e., strict gerbes in the sense of Brylinski

$H^2(X; \underline{\mathbb{C}^*}) \simeq H^3(X;\mathbb{Z})$ and may be identified with a continuous-trace algebra on X. No explicit realizations of gerbes above degree 2 are known, but they may be easily specified in terms of data similar to that given below for $2-$gerbes.

Def 3.3.1. *A abelian, locally-trivialised $2-$gerbe* [4] *on a space X is specified by the following data*

- *An open cover of X*

$$\{U_i : i \in I\} \quad \textit{with} \quad \bigcup_I U_i = X$$

  (and we write $U_{i,j} = U_i \cap U_j$ and so on for other tuples of indices.)

- *A complex line bundle Γ^i_j over $U_{i,j}$ for each ordered pair $(i,j), i \neq j$, such that Γ^j_i and Γ^i_j are dual to each other.*

- *For each ordered triple of distinct indices (i,j,k), a global nonzero section*

$$\theta_{i,j,k} \in \Gamma(U_{i,j,k}; \Gamma^j_i \otimes \Gamma^k_j \otimes \Gamma^i_k)$$

  such that the sections $\theta_{i,j,k}$ of reorderings of a triple (i,j,k) are related in the natural way.

- *On four-fold intersections we require that $\delta\theta = 1$ where δ is the Cech coboundary operator.*

We refer to Ref. [29] for the notion of a refinement of a $2-$gerbe and a proof of the fact that a $2-$gerbe naturally gives rise to a class in $H^3(X;\mathbb{Z})$. Note that by passing to a sufficiently fine cover, the Γ^i_j could be trivialized. Then, the above

[4] We follow Ref. [29] Def. 2.1.1, here. These are termed 'gerbs' in Ref. [29] and are shown to be identical with $2-$gerbes in the sense of Ref. [28] later on in that paper.

definition would reduce to that of a Cech $3-$cocyle. However, it is more useful to keep the above definition.

Note that the above definition of a $2-$gerbe may be used to construct a continuous-trace algebra on X. The bundles Γ^i_j give maps from $U_{i,j}$ to $\mathcal{P}U$. Since $\mathcal{P}U$ is isomorphic to $Aut(\mathcal{K})$, these maps may be used to glue $C_0(U_i, \mathcal{K})$ together along the $U_{i,j}$ to get a continuous trace algebra as in Ref. [17]. Conversely, given a continuous-trace algebra on X, we obtain a gerbe, since the $\mathcal{P}U$ cocyles defining the continuous-trace algebra will give the line bundles Γ^i_j. The remaining conditions are automatically satisfied, by definition. In particular, the image of the cohomology class of $\theta_{i,j,k}$ via the Bockstein map will be the Dixmier-Douady invariant.

Def 3.3.2. *A locally trivialized $3-$gerbe*[5] *on a space X consists of the following data:*

- *An open cover of X*

$$\{U_i : i \in I\} \quad \textit{with} \quad \bigcup_I U_i = X$$

(and we write $U_{ij} = U_i \cap U_j$ and so on for other tuples of indices.)

- *A $2-$gerbe, i.e., a continuous-trace algebra $\mathcal{A}^j_i$ over U_{ij} for each ordered pair $(i,j), i \neq j$, such that the classes of $\mathcal{A}^j_i$ and $\mathcal{A}^i_j$ in $H^3(U_{ij}, \mathbb{Z})$ are inverses of each other.*

- *A canonical trivialization Γ_{ijk} of the tensor product $\mathcal{A}^j_i|_{U_{ijk}} \otimes \mathcal{A}^k_j|_{U_{ijk}} \otimes \mathcal{A}^i_k|_{U_{ijk}}$ (This would be a line bundle.) The bundles Γ are related in the natural way under reorderings of (i,j,k).*

[5] We follow Ref. [29] Sec. (4.5) here

- *A trivialization of the coboundary of the Γ_{ijk} on four-fold intersections U_{ijkl}, i.e., a canonical nonzero section η_{ijkl} of*

$$\Gamma_{ijk}|_{U_{ijkl}} \otimes \Gamma_{ijl}^{-1}|_{U_{ijkl}} \otimes \Gamma_{ikl}|_{U_{ijkl}} \otimes \Gamma_{jkl}^{-1}|_{U_{ijkl}}$$

  and all the sections η are related in the natural way under reorderings of (i, j, k, l)

- *On five-fold intersections, we require that $\delta\eta = 1$ where δ is the Cech coboundary operator.*

The characteristic class of this 3−gerbe is the cohomology class of $\eta \in H^4(X;\mathbb{Z})$. (Thus a 3−gerbe would define a cohomology class in $H^4(X;\mathbb{Z})$ in exactly a similar manner as a 2−gerbe determines a cohomology class in $H^3(X;\mathbb{Z})$.) From the above definitions, we can see that a k−gerbe possesses $(k-1)$−gerbes as "local sections". For example, a non-trivial line bundle (a 1−gerbe) has continuous functions as local nonzero sections. Similarly, a continuous-trace algebra (a 2−gerbe) has local *objects* which are line bundles. This may be seen as follows: Continuous-trace algebras satisfy Fell's condition, which guarantees the existence of a local rank-one projection in some neighbourhood U_x of each $x \in X$. This is the same as a map $U_x \to \mathrm{Gr}(1, \mathcal{H}_x)$ for some Hilbert space $\mathcal{H}_x$. However, $\mathrm{Gr}(1, \mathcal{H}_x)$ is the classifying space for line bundles over U_x. The algebra is trivial if and only if there is a global rank-one projection. This would be the same as specifying a global line bundle on X which would be the analogue, for a 2−gerbe, of a global section of a line bundle. Similarly, a 3−gerbe on X (which would be classified by an element of $H^4(X;\mathbb{Z})$) would have stable continuous-trace algebras as local sections. In the case of a line bundle, it is impossible to pick a global nonzero section unless the bundle is trivial, similarly, for a gerbe it is impossible to pick a global object, unless the gerbe is trivial. Hence, $2-$gerbes may be used to study situations in which we have 'partially defined' line bundles. This is the case, for example, in the monopole of the previous section. It also explains why we could naturally associate to it a class in H^3. Similarly, we expect $3-$gerbes to be useful for describing situations where we have 'partially defined' continuous-trace algebras, that is, sources of $H-$flux.

3.4 Application to T-duality

We consider a special case of T-duality formulated in Ref. [3], Lemma 4.5: Continuous-trace algebras on $B \times S^1$ are T-dual to $U(1)-$bundles E on B. We may restate this by saying that T-duality gives a correspondence between $1-$gerbes on B and $2-$gerbes on $B \times S^1$. If the Euler class of the bundle E is $[E]$, the $H-$flux of the T-dual is given by $[E] \times z$ where z is the canonical generator of $H^1(S^1)$.

We would like to extend this correspondence to semi-free S^1-actions. Any orbit of a semi-free S^1-action on a space X can only have two stabilizers namely, the identity and S^1. As a result, the spaces X can only have two orbit types: Fixed points and free orbits. Hence, if F is the subset of $B \simeq X/S^1$ such that $\pi^{-1}(F)$ is the fixed point subset of X, by the classification theorem for spaces with two orbit types (see Ref. [9] Chap. V sec. 5), the space X is completely specified by the class of the principal S^1-bundle over $B - F$. Thus, it is specified by a class λ in $H^2(B - F)$.

This class may be used to construct a $2-$gerbe on B^+ by taking the image of

λ in $H^3(B^+)$ by the following sequence[6] (see the previous section):

$$\to H^2(B-F) \to H^3(B, B-F) \to H^3(B^+, B^+ - F) \to H^3(B^+) \to \tag{3.6}$$

The T-dual of such a space is, by the argument presented in Chap. 1, the space $B \times S^1$ with a source of $H-$flux located at $F \times S^1$. Such a source emits a $H-$flux which defines a class $\lambda \times z$ in $H^3((B-F) \times S^1)$.

This class may be used to construct a 3−gerbe on $B^+ \times S^1$ by taking the image of $\lambda \times z$ in $H^3(B^+)$ via the following sequence[7] (see the previous section):

$$\begin{aligned} \to H^3((B-F)\times S^1) \to H^4(B\times S^1, (B-F)\times S^1) \to H^4(B^+ \times S^1, (B^+ - F)\times S^1) \\ \to H^4(B^+ \times S^1) \to \end{aligned} \tag{3.7}$$

Thus, there seems to be a map between 2−gerbes on B^+ and 3−gerbes on $B^+ \times S^1$ induced by T-duality. This may be understood as follows: If we fix a generator z of $H^1(S^1)$, then taking the cross product of a cohomology class $\lambda \in H^k(X)$ with z gives a homomorphism $\times : H^k(X) \to H^{k+1}(X \times S^1)$. If $k = 2$, this homomorphism is exactly the one which is induced by sending a principal S^1 bundle to the T-dual trivial bundle with $H-$flux. It is interesting therefore, that the map given by the cross product in degree 3 is also induced by T-duality:

Theorem 3.4.1. *Let X be a 2−gerbe on B^+ with characteristic class $\eta \in H^3(B^+)$. T-duality defines a map which sends X to a 3−gerbe Y on $B^+ \times S^1$ with characteristic class $(\eta \times z) \in H^4(B^+ \times S^1)$.*

Proof: Pick an open cover $\{U_i\}$ for B^+. Then, the 2−gerbe on B^+ induces principal S^1-bundles $p_{ij} : L_i^j \to U_{ij}$. Let $[p_{ij}]$ denote the characteristic class of

[6]This sequence is not exact at $H^3(B, B-F)$ otherwise this class would always be zero!

[7]This sequence too is not exact at $H^3(B, B-F)$ otherwise this class would always be zero!

L^i_j. By the definition of a 2-gerbe (see Def. 3.3.1), on U_{ijk} the bundle

$$L^j_i|_{U_{ijk}} \otimes L^k_j|_{U_{ijk}} \otimes L^i_k|_{U_{ijk}} \tag{3.8}$$

is trivial with a canonical section θ_{ijk}. The definition requires $\delta\theta = 1$. We take the cohomology class of θ to be η. T-dualizing each of the bundles L^i_j gives continuous trace algebras A^i_j on $U_{ij} \times S^1$ with characteristic class $[p_{ij}] \times z$. Note that the characteristic class of A^i_j and A^j_i are inverses of each other in $H^3(U_{ij})$ since

$$([p_{ij}] \times z) + ([p_{ji}] \times z) = ([p_{ij}] + [p_{ji}]) \times z = 0.$$

Let $w_\alpha : U_{ijk} \to U_\alpha, \alpha = ij, jk, ki$ denote the inclusion map. Then, since the tensor product in Eq. (3.8) is trivial, we see that

$$w^*_{ij}([p_{ij}]) + w^*_{jk}([p_{jk}]) + w^*_{ki}([p_{ki}]) = 0 \tag{3.9}$$

Let us try to compute the characteristic class of the tensor product

$$\mathcal{A}^j_i|_{U_{ijk}} \otimes \mathcal{A}^k_j|_{U_{ijk}} \otimes \mathcal{A}^i_k|_{U_{ijk}} \tag{3.10}$$

This would be given by

$$(w_{ij} \times 1)^*([p_{ij}] \times z) + (w_{jk} \times 1)^*([p_{jk}] \times z) + (w_{ki} \times 1)^*([p_{ki}] \times z)$$

where

$$w_\alpha \times 1 : (U_{ijk} \times S^1) \to (U_\alpha \times S^1), \alpha = \{ij, jk, ki\}$$

are the induced inclusion maps on the T-dual side. This may be simplified as follows

$$\begin{aligned}
&(w_{ij} \times 1)^*([p_{ij}] \times z) + (w_{jk} \times 1)^*([p_{jk}] \times z) + (w_{ki} \times 1)^*([p_{ki}] \times z) \\
&= w^*_{ij}([p_{ij}]) \times z + w^*_{jk}([p_{jk}]) \times z + w^*_{ki}([p_{ki}]) \times z \\
&= (w^*_{ij}([p_{ij}]) + w^*_{jk}([p_{jk}]) + w^*_{ki}([p_{ki}])) \times z = 0
\end{aligned} \tag{3.11}$$

Thus, the continuous-trace algebra defined in Eq. (3.10) is trivial. Thus, it must possess a section [8] which would be a line bundle Γ_{ijk} over $U_{ijk} \times S^1$. To obtain this section, we note that θ_{ijk} defines an element $[\theta_{ijk}] \in H^1(U_{ijk};\mathbb{Z})$ and so we obtain an element $([\theta_{ijk}] \times z) \in H^2(U_{ijk} \times S^1;\mathbb{Z})$ which defines Γ_{ijk}.

Now, by Def. (3.3.2), restricting these Γ to U_{ijkl} and calculating the tensor product

$$\Gamma_{ijk}|_{U_{ijkl}} \otimes \Gamma^{-1}_{ijl}|_{U_{ijkl}} \otimes \Gamma_{ikl}|_{U_{ijkl}} \otimes \Gamma^{-1}_{jkl}|_{U_{ijkl}} \tag{3.12}$$

should give us a trivial bundle and a canonical section η_{ijkl} which is a Cech cocycle. To show that the tensor product Eq. (3.12) is trivial, we once again calculate the characteristic class of this tensor product line bundle. If $w_\alpha : U_{ijkl} \to U_\alpha, \alpha = \{ijk, ijl, ikl, jkl\}$ is the inclusion map, the class we want to calculate is

$$(w_{ijk}\times 1)^*([\theta_{ijk}]\times z)-(w_{ijl}\times 1)^*([\theta_{ijl}]\times z)+(w_{ikl}\times 1)^*([\theta_{ikl}]\times z)-(w_{jkl}\times 1)^*([\theta_{jkl}]\times z)$$

This may be simplified as follows

$$\begin{aligned} &(w_{ijk} \times 1)^*([\theta_{ijk}] \times z) - (w_{ijl} \times 1)^*([\theta_{ijl}] \times z) \\ &+(w_{ikl} \times 1)^*([\theta_{ikl}] \times z) - (w_{jkl} \times 1)^*([\theta_{jkl}] \times z) \\ &= (w^*_{ijk}([\theta_{ijk}]) - w^*_{ijl}([\theta_{ijl}]) + w^*_{ikl}([\theta_{ikl}]) - w^*_{jkl}([\theta_{jkl}])) \times z \end{aligned} \tag{3.13}$$

The term in parenthesis in the last equation is the class in $H^1(U_{ijkl})$ induced by $\delta\theta$. Since $\delta\theta = 1$, the expression vanishes. Note that if we change θ by a coboundary, the Γ will change, but the tensor product will still remain trivial as its characteristic class will shift by the class in $H^1(U_{ijkl})$ of the coboundary of a coboundary.

We now need a trivialization of this tensor product on five-fold intersections. This is given by any representative of the cross product cocycle $\theta \times z$ which gives

[8] See Def. 3.3.2

a cocycle on five-fold intersections and so a $\mathbb{C}^*-$valued function on this space. Changing the cocycle within its cohomology class will not change the gerbe as the characteristic class of the gerbe will remain the same.

Changing the original cover $\{U_i\}$ will not affect the answer, as following the above construction through on the new cover will show. It is also clear that the characteristic class of the 3$-$gerbe so constructed will be $\eta \times z$.

$\square$

Now, if we are given a space with a semi-free S^1-action with fixed point set whose image is $F \subset B$, then, as argued above, we get a class in $H^2(B - F)$. This gives rise to a gerbe on B^+. If we pick an open cover of B^+ containing $B - F$ and $B - \{+\}$, we will obtain, by the above theorem, a 3$-$gerbe on $B^+ \times S^1$ whose restriction to $B - F$ is exactly the continuous-trace algebra which is the T-dual of the line bundle we had over $B - F$.

We saw above that a semi-free S^1-space X with quotient space a (compact, closed, connected) manifold B is classified up to equivariant homeomorphism by the fixed point set $F \subset B$ and the characteristic class of the principal S^1-bundle $(X - \pi^{-1}(F)) \xrightarrow{\pi} (B - F)$. We now assume that F is a smooth embedded submanifold of B. We associated to X a cohomology class in $H^3(B, B - F)$ which gave us a class in $H^3(B^+, B^+ - F)$ by the excision isomorphism and finally gave us a class in $H^3(B^+)$ (using the long exact sequence of the pair $(B^+, B^+ - F)$). However, we could have obtained a class in $H^3(\tilde{B})$ for any compactification $\tilde{B}$ of B. (B^+ is not always a manifold even if B is, so in applications we might need to use another compactification $\tilde{B}$.)

Lemma 3.4.1. *There is a space* Y, *a map* $H^3(B, B - F) \to H^3(Y)$ *together with a natural map* $\phi : H^3(Y) \to H^3(\tilde{B})$ *such that every map* $H^3(B, B - F) \to H^3(\tilde{B})$

factors through ϕ.

Proof. This space may be constructed as follows: If $N(F)$ is a tubular neighbourhood of F in B, then, by the tubular neighbourhood theorem, $N(F)$ is diffeomorphic to the normal bundle of F in B. Let $D(F)$ be the closure of $N(F)$ in B, then, $D(F)$ is homeomorphic to a disc bundle over F. By excision, and homotopy,

$$H^k(B, B-F) \simeq H^k(D(F), D(F)-F) \simeq H^k(D(F), S(F))$$

where $S(F)$ is the sphere bundle which is the boundary of the disc bundle $D(F)$. Now, $H^k(D(F), S(F)) \simeq H^k(D(F)/S(F)) \simeq H^k(TD(F))$ where $TD(F)$ is the Thom space of the disc bundle $D(F)$ (See Ref. [30], pp. 441 for details).

Now, for any space $\tilde{B}$ containing F, there is a collapse map $\lambda : \tilde{B} \to TD(F)$ obtained by collapsing everything outside $N(F) \subset B \subset \tilde{B}$ to a point. The following diagram commutes:

$$\begin{array}{ccccc} H^3(D(F), S(F)) & \xrightarrow{\simeq} & H^3(TD(F), *) & \longrightarrow & H^3(TD(F)) \\ \simeq\Big\downarrow & & & & \Big\downarrow\lambda^* \\ H^3(B, B-F) & \xrightarrow{\simeq} & H^3(\tilde{B}, \tilde{B}-F) & \longrightarrow & H^3(\tilde{B}) \end{array} \tag{3.14}$$

From this it follows that the image of any class in $H^3(\tilde{B})$ which is the image of a class γ in $H^3(B, B-F)$ is actually pulled back from the image of γ in $H^3(TD(F))$ via λ^*. Thus, the image of γ in $H^3(TD(F))$ is a *universal* invariant.

□

It follows from this construction that the $2-$gerbe we constructed in the previous section is actually pulled back from the $2-$gerbe on Y via the collapse map. It also follows from this construction that the invariant is zero once the codimension k of F is more than 3. For, by a property of the Thom space (See

Ref. [30], pp. 441) $H^i(TD(F)) = 0$ if $i < k$. Further, we have the Thom isomorphism $\Phi : H^i(F) \to H^{i+k}(TD(F))$ This should enable us to calculate the invariant explicitly.

On the T-dual side, we have a trivial S^1-bundle $B \times S^1$ with the NS5-brane sitting somewhere in $F \times S^1$, transverse to the S^1-fiber. This would have a total charge given by a cohomology class in $H^4((B \times S^1), (B - F) \times S^1)$. By an argument similar to the above, we would have a commutative diagram

$$\begin{array}{ccccc} H^3(D(F \times S^1), S(F \times S^1)) & \xrightarrow{\simeq} & H^3(TD(F \times S^1), *) & \to & H^3(TD(F \times S^1)) \\ \simeq\downarrow & & & & \lambda^*\downarrow \\ H^3(B \times S^1, (B - F) \times S^1) & \xrightarrow{\simeq} & H^3(\tilde{B} \times S^1, (\tilde{B} - F) \times S^1) & \to & H^3(\tilde{B} \times S^1) \end{array} \tag{3.15}$$

We also have a Thom isomorphism $\tilde{\Phi} : H^i(F \times S^1) \to H^{i+k}(TD(F \times S^1))$. Recall that T-duality gave a map $\times : H^3(B - F) \to H^3((B - F) \times S^1)$ given by the cross product. We saw in the previous section that this induced a map $\times : H^3(B^+) \to H^3(B^+ \times S^1)$. An argument similar to the one given in that section would give also give a map $\times : H^3(\tilde{B}) \to H^3(\tilde{B} \times S^1)$. Then we have a commutative diagram

$$\begin{array}{ccccc} H^{4-k}(F \times S^1) & \xrightarrow{\Phi^{-1}} & H^4(TD(F \times S^1)) & \xrightarrow{\lambda^*} & H^4(\tilde{B} \times S^1) \\ \times\uparrow & & & & \times\uparrow \\ H^{3-k}(F) & \xrightarrow{\Phi^{-1}} & H^3(TD(F)) & \xrightarrow{\lambda^*} & H^3(\tilde{B}) \end{array} \tag{3.16}$$

This may be used to calculate the invariant in H^4 for an NS5$-$brane configuration from the one in H^3.

3.5 Open Questions

Let X be a finite CW complex with a semi-free S^1 action β; let F be the fixed point set of this action. Let $\mathcal{A} = C(X, \mathcal{K})$ and α the lifting of the S^1 action β to a $\mathbb{R}-$action on $\mathcal{A}$, and let $B = X/S^1$.

1. We have noted in Ch. (1) that $\mathcal{E} = \mathcal{A} \rtimes_{\lambda} \mathbb{R}$ is the extension of algebras

   $$0 \to CT((B-F) \times S^1, \delta^{\#}) \to \mathcal{E} \to C_0(F, \mathcal{K}) \otimes C_0(\mathbb{R}) \to 0 \tag{3.17}$$

   Is there an explicit characterisation of $\mathcal{E}$?

   Suppose we did not know X but were only given

   - topological spaces B, F such that $F \subseteq B$,
   - a cohomology class $\delta^{\#} \in H^3((B-F) \times S^1, \mathbb{Z})$.

   It is then possible to **uniquely** determine the C$^*-$algebra $\mathcal{E}$ from T-duality: Integrate $\delta^{\#}$ over the S^1-fiber to obtain a cohomology class $\eta \in H^2(B-F, \mathbb{Z})$. This may be viewed as the characteristic class of a principal S^1-bundle on $B-F$. Since we know the fixed point set $F \subseteq B$, then, as mentioned in Ch. 2, we may recover the semi-free S^1-space X upto equivariant homeomorphism. Hence, we know $\mathcal{A}$ and α. Now, $\mathcal{E}$ may be defined as $\mathcal{A} \rtimes_{\alpha} \mathbb{R}$. The result is unique upto C$^*-$algebra isomorphism.

   However, we are looking for an explicit characterisation of $\mathcal{E}$ as a C^*-algebra without reference to T-duality.

2. In Ch. 2, before Eq. (2.6), we obtained an invariant $\eta \in H^3(B^+, \mathbb{Z})$ which specified the local structure of X near F. Is this related to any physical

quantity (for example the NUT charge of the singularity)? (An analogous statement for the Dirac Monopole is true, see Ref. [26] .)

3. In Ch. 2, we noted that a continuous trace algebra on X was equivalent to a 2-gerbe in the sense of Hitchin et al. In Ref. [29], a gerbe connection on a 2-gerbe is defined, which generalizes a connection on a vector bundle. Is it possible to define a C^*-algebraic construction which, when applied to a given continuous-trace algebra gives a gerbe connection on the associated 2-gerbe? Since the $H-$flux is the curvature of this gerbe connection, this question is interesting physically. It would also be interesting to study the moduli space of instantons of the higher gauge theory defined using this connection and to compare the result with Ref. [31].

4. In Thm. (2.2.1) we showed that under T-duality, a class in $H^2(X,\mathbb{Z})$ gives rise to a locally unitary automorphism of $\mathcal{E}$. If $F \neq \phi$ the spectrum of $\mathcal{E}$ is non-Hausdorff and we cannot associate this to a class in $H^2(\hat{\mathcal{E}},\mathbb{Z})$. Assume, for this item only, that F is empty, i.e., X is a principal S^1-bundle over B so that the spectrum of $\mathcal{E}$ is $B \times S^1$. Then under T-duality a class in $H^2(X,\mathbb{Z})$ does give rise to a class in $H^2(B \times S^1,\mathbb{Z})$. Is there an explicit expression for this T-dual class?

5. Can we calculate the T-dual of a semi-free S^1-space with a non-zero $H-$flux? (The method used in Chs. 1-3 will not work.)

Chapter 4

T-duality and Automorphisms

In this chapter we attempt to answer problem 4 of the previous chapter. The work in this chapter is still somewhat preliminary. In Sec. (1) I define the problem to be studied more formally. In Sec. (2) I define a classifying space for the problem. In Sec. (3) I close with some speculations on the case when $H \neq 0$.

4.1 Introduction

Let $p : X \to W$ be a semi-free S^1-space and $\mathcal{A} = C_0(X, \mathcal{K})$. In Thm. (2.2.1) we showed that given any class $[\lambda] \in H^2(X; \mathbb{Z})$, there exists an action α of $\mathbb{R}$ on $\mathcal{A}$ inducing the given action of $S^1 = \mathbb{R}/\mathbb{Z}$ on X and a commuting action λ of $\mathbb{Z}$ on $\mathcal{A}$ with Phillips-Raeburn obstruction $[\lambda]$. Then λ passes to a locally unitary action on $\mathcal{E} \simeq \mathcal{A} \underset{\alpha}{\rtimes} \mathbb{R}$. Now the actions α and λ are (individually) unique up to exterior equivalence, but unfortunately the pair (α, λ), as an action of $\mathbb{R} \times \mathbb{Z}$, is NOT necessarily unique, so this construction is not entirely canonical. We therefore introduce here another point of view, based on the equivariant Brauer group, which measures precisely this lack of canonicity.

Let $\mathcal{A}$ be a stable continuous-trace algebra with spectrum X which is a principal S^1-bundle with $W \simeq X/S^1$. In addition to the $H-$flux considered in Ref.

[3], we would like to study the $B-$field using C^*-algebraic techniques. In Chap. (2) we noted that, for the purposes of this thesis, we would only model integral $B-$fields which we identified with classes in $H^2(X,\mathbb{Z})$. We had also noted that for $\mathcal{A} = C_0(X,\mathcal{K})$, a class in $H^2(X,\mathbb{Z})$ can be lifted to a spectrum preserving automorphism of $\mathcal{A}$ which commutes with some lift of the S^1-action on X to an $\mathbb{R}-$action on $\mathcal{A}$. Such an automorphism gives an action of $\mathbb{R}\times\mathbb{Z}$ on $\mathcal{A}$.

It will be useful to recall the notion of the Equivariant Brauer Group[1]: Let X be a second countable, locally compact, Hausdorff topological space and let G be a second countable, locally compact, Hausdorff topological group acting on X. Let $\mathfrak{Br}_G(X)$ denote the class of C^*-dynamical systems consisting of continuous-trace algebras $\mathcal{A}$ on X together with a lift α of the G action on X to $\mathcal{A}$. We say that the dynamical sytem $(\mathcal{A},\alpha)$ is equivalent to $(\mathcal{B},\beta)$ if there is a Morita equivalence bimodule ${}_{\mathcal{A}}X_{\mathcal{B}}$ together with a strongly continuous $G-$action by linear transformations ϕ_s on X such that for every $s\in G$, $\alpha_s(\langle x,y\rangle_{\mathcal{A}}) = \langle\phi_s(x),\phi_s(y)\rangle_{\mathcal{A}}$, and $\beta_s(\langle x,y\rangle_{\mathcal{B}}) = \langle\phi_s(x),\phi_s(y)\rangle_{\mathcal{B}}$. (Recall, the image of the inner product $\langle,\rangle_{\mathcal{A}} : X\times X\to\mathcal{A}$ is dense in $\mathcal{A}$ and similarly for $\langle,\rangle_{\mathcal{B}}$.) It is shown in Ref. [23] that this is an equivalence relation and that the quotient is a *group*. This group is termed the Equivariant Brauer Group $\mathrm{Br}_G(X)$ of X. The group operation is the $C_0(X)-$tensor product of continuous-trace algebras and group actions.

Note that the $\mathbb{R}\times\mathbb{Z}$ action on $\mathcal{A}$ above gives rise to an element of $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$. Here, the $\mathbb{R}$ action on $\mathcal{A}$ factors through the S^1-action on X, while the $\mathbb{Z}$ action on X is trivial. Also, the $\mathbb{R}\times\mathbb{Z}$ action is not unique, as we really only care about the restriction of the action to the $\mathbb{R}$ factor and to the $\mathbb{Z}$ factor up to exterior

[1] We use Ref. [23] here.

equivalence. Hence it is really only the class of the action in $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ that is important. We conjecture that elements of this group are a good model for a space with a $B-$field. Let $F_1 : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}_{\mathbb{Z}}(X)$ and $F : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}_{\mathbb{R}}(X)$ be the forgetful maps. We modify the Basic Setup as follows:

Def 4.1.1. *Let X be a locally compact, finite dimensional CW-complex homotopy equivalent to a finite CW-complex. Let X also be a free S^1-space with $W = X/S^1$, so that we have a principal S^1-bundle $p : X \to W$. An element*[2] *$y = [\mathcal{A}, \alpha \times \phi]$ of $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ is now defined to be a model for a space with $H-$flux and a $B-$field. The $H-$flux H is $H = F(y) = [\mathcal{A}]$. Given the value of H, the $B-$field is the unique class*[3] *in $H^2(X, \mathbb{Z})$ which determines $F_1(y)$. This class is equal to the Phillips-Raeburn class of ϕ.*

By Ref. [23], there is a natural filtration of $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ given by $0 < B_1 < \ker(F) < \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$, where B_1 is a quotient of $H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$. We argue below that each step in this filtration corresponds to one of the gauge fields in the problem.

We need the following

Theorem 4.1.1. *Let $p : X \to W$ be as above.*

1. *We have a split short exact sequence*

$$0 \to \ker(F) \to \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}} \xrightarrow{F} \mathrm{Br}(X) \to 0$$

   where $F : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}(X)$ is the forgetful map.

---

[2] Here α is a lift of the S^1-action on X to a $\mathbb{R}-$action on $\mathcal{A}$, while ϕ is a commuting spectrum-fixing $\mathbb{Z}-$action on $\mathcal{A}$.

[3] Note that $\mathrm{Br}_{\mathbb{Z}}(X) \simeq H^3(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z})$.

2. *We have a surjective map* $\eta : \ker(F) \to H^2(X, \mathbb{Z})$.

3. *We have a natural isomorphism* $H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \simeq C(W, \mathbb{R})$.

4. *The group* $H^2_M(\mathbb{R}\times\mathbb{Z}, C(X, \mathbb{T}))$ *is connected and there is a natural surjective map* $q : H^2_M(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T})) \to C(W, \mathbb{T})_0$.

Proof. 1. We have a forgetful homomorphism $F_1 : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}_{\mathbb{R}}(X)$, where $F_1 : [\mathcal{A}, \alpha \times \phi] \to [\mathcal{A}, \alpha]$. This map is obviously surjective, since we have a section $s : [\mathcal{A}, \alpha] \to [\mathcal{A}, \alpha \times \mathrm{id}]$.

Since $\mathrm{Br}_{\mathbb{R}}(X) = H^3(X, \mathbb{Z})$ (by Sec. (6.1) of Ref. [23]), the kernel of F_1 consists of Morita equivalence classes of dynamical systems $[\mathcal{A}, \alpha \times \phi]$ such that $\delta(\mathcal{A}) = 0$. Thus, it actually consists of the group $\ker(F)$, where $F : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}(X)$ is the map forgetting the group action.

2. By Thm. (5.1) of Ref. [23], we have a homomorphism $\eta : \ker(F) \to H^1_M(\mathbb{R}\times \mathbb{Z}, H^2(X, \mathbb{Z}))$. Now, by Thm. (4.2) of Ref. [3], we have that $H^1_M(\mathbb{R}\times\mathbb{Z}, M) \simeq H^1(B(\mathbb{R} \times \mathbb{Z}), M)$ for any discrete $\mathbb{R} \times \mathbb{Z}$ module M. Also, $B(\mathbb{R} \times \mathbb{Z}) \simeq S^1$, so $H^1_M(\mathbb{R} \times \mathbb{Z}, H^2(X, \mathbb{Z})) \simeq H^2(X, \mathbb{Z})$.

By Thm. (5.1) item (2) of [23], the image of η has range which is all of $H^2(X, \mathbb{Z})$ since $H^3_M(\mathbb{R} \times \mathbb{Z}, C(X, S^1)) = 0$ by Thm. (2.2.1). Hence we have a surjective homomorphism $\eta : \ker(F) \to H^2(X, \mathbb{Z})$.

3. We have the following short exact sequence of $\mathbb{R}-$modules

$$0 \to H^0(X, \mathbb{Z}) \to C(X, \mathbb{R}) \to C(X, \mathbb{T})_0 \to 0$$

From the associated long exact sequence for H^*_M, we find that $H^i_M(\mathbb{R}, H^0(X, \mathbb{Z})) \simeq 0, i = 1, 2$ by Cor. (4.3) of Ref. [3]; hence $H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \simeq H^1_M(\mathbb{R}, C(X, \mathbb{R}))$.

By Thms. 4.5,4.6,4.7 of Ref. [3], $H^1_M(\mathbb{R}, C(X,\mathbb{R})) \simeq H^1_{\mathrm{Lie}}(\mathbb{R}, C(X,\mathbb{R})_\infty)$. Here $C(X,\mathbb{R})_\infty$ are the $C^\infty-$vectors for the $\mathbb{R}-$action on $C(X,\mathbb{R})$ and so are the functions which are smooth along the S^1-orbits.

The complex computing the Lie algebra cohomology of $\mathbb{R}$ shows that this group is exactly the functions in $C(X,\mathbb{R})_\infty$ modulo derivatives of functions in $C(X,\mathbb{R})_\infty$ by the generator of the $\mathbb{R}-$action.

This group is isomorphic to $C(W,\mathbb{R})$ via the 'averaging' map $f \to \int_{S^1} \phi_t \circ f \, dt$ where $\phi_t \circ f$ is f shifted by the S^1-action on X.

4. We use the spectral sequence calculation of Chap. (2) to note that this group is isomorphic to $H^1_M(\mathbb{R}, H^1_M(\mathbb{Z}, C(X,\mathbb{T})))$. Since $\mathbb{Z}$ is discrete and acts trivially on $C(X,\mathbb{T})$, we have $H^1_M(\mathbb{Z}, C(X,\mathbb{T})) \simeq H^1(\mathbb{Z}, C(X,\mathbb{T})) \simeq C(X,\mathbb{T})$. Hence we need to calculate $H^1_M(\mathbb{R}, C(X,\mathbb{T}))$.

   We have the following short exact sequence of $\mathbb{R}-$modules

$$0 \to C(X,\mathbb{T})_0 \to C(X,\mathbb{T}) \to H^1(X,\mathbb{Z}) \to 0$$

   where $C(X,\mathbb{T})_0$ is the connected component of $C(X,\mathbb{T})$ containing the constant maps.

   This gives us a long exact sequence

$$\begin{aligned} H^0_M(\mathbb{R}, C(X,\mathbb{T})) \to H^0_M(\mathbb{R}, H^1(X,\mathbb{Z})) \to H^1_M(\mathbb{R}, C(X,\mathbb{T})_0) \to \\ H^1_M(\mathbb{R}, C(X,\mathbb{T})) \to H^1_M(\mathbb{R}, H^1(X,\mathbb{Z})) \to \dots \end{aligned} \tag{4.1}$$

   Also, by Cor. (4.3) of Ref. [3], we have that $H^1_M(\mathbb{R}, H^1(X,\mathbb{Z})) \simeq 0$. Again, by Cor. (4.3) of Ref. [3], we find that $H^0_M(\mathbb{R}, H^1(X,\mathbb{Z})) \simeq H^1(X,\mathbb{Z})$ (since

$B\mathbb{R}$ is contractible). Also $H^0_M(\mathbb{R}, C(X, \mathbb{T}))$ consists of the $\mathbb{R}-$invariant functions in $C(X, \mathbb{T})$ and hence is naturally isomorphic to $C(W, \mathbb{T})$.

Hence we find an exact sequence

$$C(W, \mathbb{T}) \to H^1(X, \mathbb{Z}) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \to 0. \tag{4.2}$$

The map $C(W, \mathbb{T}) \to H^1(X, \mathbb{Z})$ is the composite $C(W, \mathbb{T}) \to H^1(W, \mathbb{Z}) \stackrel{p^*}{\to} H^1(X, \mathbb{Z})$. Its cokernel is $H^1(X, \mathbb{Z})/p^*(H^1(W, \mathbb{Z}))$ which is the image of $p_! : H^1(X, \mathbb{Z}) \to H^0(W, \mathbb{Z})$ by the Gysin sequence. The image can only be 0 or $\mathbb{Z}$ if X is connected. Using the isomorphism mentioned in the previous item of this lemma, we see that we need to find the connecting map $\mathrm{im}(p_!) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})_0) \simeq C(W, \mathbb{R})$. This map sends any class in $H^0(W, \mathbb{Z})$ to a constant $\mathbb{Z}-$valued function on W.

The above exact sequence now becomes

$$0 \to \mathrm{im}(p_!) \to C(W, \mathbb{R}) \to H^1_M(\mathbb{R}, C(X, \mathbb{T})) \to 0. \tag{4.3}$$

So $H^1_M(\mathbb{R}, C(X, \mathbb{T}))$ is isomorphic to the quotient of $C(W, \mathbb{R})$ by $\mathrm{im}(p_!)$. It surjects onto the quotient of $C(W, \mathbb{R})$ by all of $H^0(W, \mathbb{Z})$ which is isomorphic to $C(W, \mathbb{T})_0$.

$\square$

Lemma 4.1.1. *We have a commutative diagram with exact rows and columns*

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & H^2_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T})) \arrow[r, "q"] \arrow[d] & C(W,\mathbb{T})_0 \arrow[r] & 0 & \\
0 \arrow[r] & \ker(F) \arrow[r] \arrow[d, "\eta"'] & \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \arrow[r, "F"] & \mathrm{Br}_{\mathbb{R}}(X) \arrow[r] & 0 \\
 & H^2(X,\mathbb{Z}) \arrow[d] & & & \\
 & 0 & & &
\end{tikzcd}$$

Proof. The vertical and horizontal short exact sequences above are of the form $0 \to B_i \to B_{i+1} \to B_{i+1}/B_i \to 0$ where the B_i are the groups in the filtration of $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ described in the unnumbered Theorem on page (153) of Ref. [23]. All we need to check is that $B_1 = H^2_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T}))$. This will follow from the fact that $B_1 = H^2_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T}))/\operatorname{im}(d_2')$ and the fact that $\operatorname{im}(d_2') \subseteq H^3_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T}))$. The last group vanishes by the result in Chap. (2). The maps F, η, q were defined in the previous lemma. $\square$

We now make the following dictionary

- $y \in \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$, y not in $\ker(F) \Leftrightarrow$ Space X with $H \neq 0$. Here, $H = F(y)$.
- Element $y \in \ker(F) \subseteq \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}$, y not in $H^2_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T})) \Leftrightarrow$ Space X with $H = 0, B \neq 0$. Here, $B = \eta(y)$.
- $y \in H^2_M(\mathbb{R}\times\mathbb{Z}, C(X,\mathbb{T})) \subseteq \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \Leftrightarrow$ Space X with $H = 0, B = 0, A \neq 0$. Unfortunately, we do not obtain a class in $H^1(X,\mathbb{Z})$ as would be expected instead we obtain an element of $C(W,\mathbb{T})_0$.

- $C_0(X, \mathcal{K})$ with the lift of the $\mathbb{R}-$action and the trivial $\mathbb{Z}-$action $\Leftrightarrow$ Space X with $H = 0, B = 0, A = 0$.

Note that the last item above is exactly the C^*-dynamical system assigned to a space X with zero $H-$flux in Ref. [3]. If $y \in \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$, we have $H = F(y)$. Then $F^{-1}(y)$ is the coset $s(y) \circ \ker(F)$ and changing y by an element x of $\ker(F)$ corresponds to making a gauge transformation of the $B-$field $(H = dB) \to (H = d(B + B'))$. Hence, $dB' = 0$ and $B' \in H^2(X, \mathbb{R})$. We are restricted to integer $B-$fields in this formalism and so we only allow shifts by $B' \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{R})$. We see that $B' = \eta(x)$ here.

Similarly, if $y \in \ker(F)$, then $H = 0, B = \eta(y)$ and changing y by an element z of $H^2(\mathbb{R} \times \mathbb{Z}, C(X, \mathbb{T}))$ doesn't change B but corresponds to making a change in the $A-$field, the gauge field of the $B-$field. Note that an element of $C(W, \mathbb{T})$ naturally gives an element of $H^1(W, \mathbb{Z})$: In our case, we find an element of $C(W, \mathbb{T})_0$ which always gives rise to the 0 element of $H^1(W, \mathbb{Z})$.

We suspect that $A = 0$ in our formalism for the following reason: If we inspect the derivation of Buscher's rules [2], we see that they are obtained from a nonlinear sigma model of the form $\int_\Sigma (g_{ij} + b_{ij}) \partial X^i \bar{\partial} X^j$. In this model, if we set $B = dA$ globally, we see that A will only couple to fields on the boundary of the worldsheet. Hence, we suspect that it would only be of interest in a theory containing $Dp-$branes. However, Topological T-duality, as defined in Ref. [3], is obtained from Buscher's rules for theory of *closed* strings. Thus, we suspect that the theory does not allow for a topologically nontrivial A field.

Further, it may be argued that integer changes of B or A are physically equivalent and hence uninteresting. (Since the string action only depends on $\int_\Sigma e^{i\phi^* B}$.) However, such changes correspond, for example, to rotating a $H-$monopole around a S^1-fiber as we saw in Chap. (2). Hence, we consider them here. We will also need them later in this section in order to study T-duality mathematically.

By Ref. [3], we know that T-duality maps elements of $\mathfrak{Br}_{\mathbb{R}}(X)$ as $(\mathcal{A}, \alpha) \to (\mathcal{A} \rtimes_{\alpha} \mathbb{R}, \alpha^{\#})$. It follows from [3], that $(\mathcal{A} \rtimes_{\alpha} \mathbb{R}, \alpha^{\#})$ is a continuous trace algebra with $\mathbb{R}-$action $\alpha^{\#}$. Thus we might suspect that T-duality induces a well-defined map $T : \mathrm{Br}_{\mathbb{R}}(X) \to \mathrm{Br}_{\hat{\mathbb{R}}}(X^{\#})$. It is well known that[4] if the dynamical system $(\mathcal{A}, \alpha)$ is Morita equivalent to $(\mathcal{B}, \beta)$ then $(\mathcal{A} \rtimes_{\alpha} \mathbb{R}, \alpha^{\#})$ is Morita equivalent to $(\mathcal{B} \rtimes_{\beta} \mathbb{R}, \beta^{\#})$. Hence, we get a map $T : \mathrm{Br}_{\mathbb{R}}(X) \to \mathrm{Br}_{\mathbb{R}}(X^{\#})$ induced by T-duality. The map T may be calculated for any CW-complex X using the proof of Thm. (4.1.2) of Ref. [8].

This map is not a homomorphism in general due to the following argument: Let $p : X \to W$ be a principal S^1-bundle and let $q : X^{\#} \to W$ be the T-dual principal bundle. Suppose H_1, H_2 were $H-$fluxes on X T-dual to the $H-$fluxes $H_1^{\#}, H_2^{\#}$ on $X^{\#}$. Since $p_! : H^3(X, \mathbb{Z}) \to H^2(X, \mathbb{Z})$ is a homomorphism, $p_!(H_1) + p_!(H_2) = p_!(H_1 + H_2)$. If $(H_1 + H_2)$ was the T-dual of $(H_1^{\#} + H_2^{\#})$, then we would have a contradiction since $p_!(H) = [q]$ always by Ref. [3]. Thus the map T above need not be a homomorpism.

We also have a similar map $T : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X^{\#})$ given by the following

Lemma 4.1.2. *There is a well-defined map $T : \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \to \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X^{\#})$ induced by the crossed product.*

Proof. We need the following well-known fact. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras with $G-$action α, β respectively. Let $C_c(G, \mathcal{A})$ the $\alpha-$twisted convolution algebra of $\mathcal{A}-$valued functions on G which are of compact support on G. Similarly, let

[4] See Ref. [23], Sec. (6.2).

$C_c(G, \mathcal{B})$ be the $\beta-$twisted convolution algebra of $\mathcal{B}-$valued functions on G which are of compact support on G. Give $C_c(G, \mathcal{A})$ and $C_c(G, \mathcal{B})$ the inductive limit topology[5].

Theorem 4.1.2. *Suppose that $(\mathcal{A}, G, \alpha)$ and $(\mathcal{B}, G, \beta)$ are dynamical systems and $\phi : \mathcal{A} \to \mathcal{B}$ is an equivariant homomorphism. Then, there is a homomorphism $\phi \rtimes \mathrm{id} : \mathcal{A} \underset{\alpha}{\rtimes} G \to \mathcal{B} \underset{\beta}{\rtimes} G$ mapping $C_c(G, \mathcal{A})$ into $C_c(G, \mathcal{B})$ such that $\phi \rtimes \mathrm{id}(f)(s) = \phi(f(s))$.*

From the proof of this theorem, it is clear that the extension $\phi \rtimes \mathrm{id}$ is unique.

Here, $\mathcal{A} = \mathcal{B}$ and $G = \mathbb{R}$. Suppose $y \in \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$, and we pick a representative $(\mathcal{A}, \alpha \times \phi)$ of y. We may define $T(y)$ to be the dynamical system $(\mathcal{A} \underset{\alpha}{\rtimes} \mathbb{R}, \alpha^\# \times \phi^\#)$ where $\phi^\#$ is the map induced on the crossed product by $\phi^\#(f)(t) = \phi \rtimes \mathrm{id}(f)(t)$. It is clear that it is unique and commutes with the $\mathbb{R}-$action.

Changing the representative to a Morita equivalent one $(\mathcal{A}', \alpha' \times \phi')$ will not change the Morita equivalence class of the answer because, by the theorem cited in Ref. [23], Sec. (6.2), $\phi^{\#'}$ has the same Phillips-Raeburn obstruction as $\phi^\#$ and hence $\phi^{\#'}$ is exterior equivalent to $\phi^\#$. $\square$

By construction, we know that if we 'forget' the $\mathbb{Z}-$action on $\mathcal{A}$, the T-dual doesn't change in either $H-$flux or topology. Thus, the following diagram commutes

$$\begin{array}{ccc} \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) & \xrightarrow{T} & \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X^\#) \\ {\scriptstyle F}\big\downarrow & & {\scriptstyle F}\big\downarrow \\ \mathrm{Br}_{\mathbb{R}}(X) & \xrightarrow{T} & \mathrm{Br}_{\mathbb{R}}(X^\#). \end{array}$$

[5]See Ref. [32], Corollary 2.48.

This implies that, for any X, $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ is partitioned into sets $S_H = F^{-1}(H)$, $H \in H^3(X,\mathbb{Z}) \simeq \mathrm{Br}_{\mathbb{R}}(X)$. The T-duality map gives a well defined family of maps parametrized by $H \in H^3(X,\mathbb{Z})$, denoted $T_H : S_H \to S_{H^\#}$. Thus, it is enough to determine the T_H. Now, given an element $H \in \mathrm{Br}_{\mathbb{R}}(X)$, we obtain a unique element $s(H) \in \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ by adding a trivial $\mathbb{Z}-$action. (s was the section map of Thm. 4.1.1.) This enables us to identify S_H with $s(H)\cdot\ker(F)$ (we had shown $\mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X) \simeq \mathrm{Br}_{\mathbb{R}}(X) \oplus \ker(F)$). We may assign a B-field $B = \eta(s(H)^{-1}\circ x)$ to every $x \in S_H$. Thus, elements of S_H give *triples* consisting of a principal circle bundle $p : X \to W$, a class $b \in H^2(X,\mathbb{Z})$, and a class $H \in H^3(X,\mathbb{Z})$. All elements of S_H which differ by an element of B_1 give rise to the same triple. The T-duality map T gives rise to a family of maps parametrized by H, $T_H : b \to b^\#([p], b, H)$.

Physically, we have fixed H and B as $H = dB$ and view any other B' with $H = B'$ as giving rise to a cohomology class $(B - B') \in H^2_{\text{de Rham}}(X)$. Such a gauge-fixing is unphysical, unless $H = 0$, but it is useful so that we may determine the T-duality map. We are essentially claiming that it is possible to determine $(B - B')^\#$ as a function of $[p], H$ and $(B - B')$. In the next section we attempt to determine a classifying space for 'pairs' **(principal bundle, b-field).** Such a pair will not specify an element in $S_0 \subset \mathrm{Br}_{\mathbb{R}\times\mathbb{Z}}(X)$ uniquely, but only up to elements of B_1. Note that in the paper of Bunke et al. [18], a classifying space was constructed for pairs of the form **(principal bundle, H-flux)**. This data does specify a unique element of $\mathrm{Br}_{\mathbb{R}}(X)$ because $\mathrm{Br}_{\mathbb{R}}(X) \simeq H^3(X,\mathbb{Z})$.

4.2 The Classifying Space of $k-$pairs

In this section, we use the method of Ref. [18] to determine the map T of Sec. (1). In this section and the next we try to T-dualize automorphisms of $C_0(X,\mathcal{K})$.

Let $\mathcal{SET}$ be the category of sets with functions as morphisms. Let $\mathcal{C}$ be the category of unbased CW complexes with unbased homotopy classes of continuous maps as morphisms. Let $\mathcal{C}_0$ be the category of CW complexes which are finite subcomplexes of some fixed countably infinite dimensional standard simplicial complex. $(\mathcal{C}, \mathcal{C}_0)$ is a homotopy category in the sense of Ref. [34] (see Thm. 2.5 in [34]).

Let X be a *fixed* CW complex. We define a k-pair over X to consist of a principal S^1-bundle $p : E \to X$ together with a cohomology class $b \in H^k(X, \mathbb{Z})$. We denote a k-pair as $([p], b)$. (Here the space X is understood from the context as is the value of k.) Note that a 'pair' in the sense of Ref. [18] would be a termed a 3-pair here.

Def 4.2.1. *We declare two k-pairs (**same** k) $([p], b)$ and $([q], b')$ over X equivalent if*

- *We are given two principal S^1-bundles $p : E \to X$ and $q : E' \to X$ such that*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E' \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{id} & X \end{array}$$

  commutes.

- *We also require that $b' = \phi^*(b)$.*

It is clear that the collection of equivalence classes of k-pairs over a fixed space X (denoted $P_k(X)$) is a *set*. For all X, we have a distinguished pair consisting of the trivial S^1-bundle over X with the zero class in $H^k(X, \mathbb{Z})$. Thus, $P_k(X)$ is actually a pointed set.

Def 4.2.2. *Let X and Y be two CW-complexes and let $f : X \to Y$ be a continuous map. Let $([p], b) \in P_k(Y)$ be represented by a principal S^1-bundle $p : E \to Y$ and a class $b \in H^k(Y, \mathbb{Z})$. We define the pullback of $([p], b)$ via f, denoted $f^*([p], b)$, to be the following data*

- *The unique principal S^1-bundle $f^*p : f^*E \to X$ such that the following diagram commutes*
$$\begin{array}{ccc} f^*E & \xrightarrow{\tilde{\phi_f}} & E \\ {\scriptstyle f^*p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$
- *The cohomology class $\phi_f^*(b)$ in $H^k(f^*E, \mathbb{Z})$.*

That is,we define $f^*([p], b) = (f^*[p], \phi_f^*(b))$.

Lemma 4.2.1. *Let $f_0, f_1 : X \to Y$ be freely homotopic. For any pair $([p], b) \in P_2(Y)$, $f_0^*([p], b)$ is equivalent to $f_1^*([p], b)$.*

Proof. Let $p : E \to Y$ be a principal S^1-bundle. We have pullback squares for $i = 0, 1$
$$\begin{array}{ccc} f_i^*E & \xrightarrow{\tilde{\phi_i}} & E \\ {\scriptstyle f_i^*p}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X & \xrightarrow{f_i} & Y. \end{array}$$
Then, by Ref. [33], Cor. (1.8), the pullback bundles $f_0^*p : f_0^*E \to X$ and $f_1^* : f_1^*E \to X$ are isomorphic. Further, by the same lemma, this isomorphism is implemented by a map $\psi : f_0^*E \to f_1^*E$. This map induces isomorphisms on the cohomology groups such that $\psi \circ f_0^* = f_1^*$. As a result, by the above definition, $f_0^*([p], b) = f_1^*([p], b)$. $\square$

Hence, $P_k(X)$ is a pointed set depending only on the homotopy type of X. Given a map $f : X \to Y$, define $P_k(f) : P_k(Y) \to P_k(X)$ to be the map induced by pullback of pairs. It is clear that $P_k(1) = \mathrm{Id}$. (This is just the condition that two pairs be equivalent). Hence, P_k extends to a functor (also denoted P_k) $P_k : \mathcal{C} \to \mathcal{SET}$.

Theorem 4.2.1. *For every k, the functor P_k above satisfies the conditions of the Brown Representability Theorem. Hence, for every k, there exists a classifying space R_k for P_k.*

Proof. There are two conditions we need to prove.

1. Consider an arbitrary family $\{X_\mu\}, \mu \in I$ of of objects in $\mathcal{C}$. Let $Y = \bigsqcup_{\mu \in I} X_\mu$. Let $h_\mu : X_\mu \to \bigsqcup_{\mu \in I} X_\mu$ be the inclusion maps.

   We have a pullback square (for every $\mu \in I$)

$$\begin{array}{ccc} h_\mu^* E & \xrightarrow{\tilde{h}_\mu} & E \\ {\scriptstyle p_\mu}\big\downarrow & & {\scriptstyle p}\big\downarrow \\ X_\mu & \xrightarrow{h_\mu} & Y. \end{array}$$

   Here $p_\mu = h_\mu^* p$. Since $H^2(X, \mathbb{Z}) \simeq \prod_{\mu \in I} H^2(X_\mu, \mathbb{Z})$, we have that $[p] = ([p_\mu]), \mu \in I$.

   Let $h_\mu^* E = E_\mu$, then, we also have that $E = \bigsqcup_{\mu \in I} E_\mu$ and $H^k(E, \mathbb{Z}) \simeq \prod_{\mu \in I} H^k(E_\mu, \mathbb{Z})$. Hence, every class $b \in H^k(E, \mathbb{Z})$ may be written as $(b_\mu), \mu \in I$ with $b_\mu = \tilde{h}_\mu^*(b)$. Hence, we have an isomorphism

$$\Pi_\mu P(h_\mu) : P(\bigsqcup_\mu X_\mu) \approx \Pi_\mu P(X_\mu).$$

2. Suppose we are given CW complexes A, X_1, X_2 and continuous maps $f_i : A \to X_i, g_i : X_i \to Z, i = 1, 2$ such that

$$\begin{array}{ccc} A & \xrightarrow{f_1} & X_1 \\ {\scriptstyle f_2}\big\downarrow & & {\scriptstyle g_1}\big\downarrow \\ X_2 & \xrightarrow{g_2} & Z \end{array}$$

commutes up to homotopy and is a pushout square in $\mathcal{C}$. We may take f_i to be inclusions into the X_i and Z the result of gluing X_1 to X_2 along A. Suppose $u_i \in P(X_i)$ satisfy $P(f_1)u_1 = P(f_2)u_2$. For $i = 1, 2$, let u_i correspond to the pair $([p_i], b_i)$ over X_i, where $p_i : E_i \to X_i$ are principal S^1-bundles. Then, since $P(f_1)u_1 = P(f_2)u_2$, $f_1^*E_i \simeq f_2^*E_2$. This implies that the restrictions of $f_i^*E_i$ to A are the same. Hence, these two bundles may be glued into a unique bundle $p : E \to Z$. Note that $G_i = f_i^*E_i \subset E, i = 1, 2$ and $G_1 \cup G_2 = E$. We have a pullback square

$$\begin{array}{ccc} G_i & \xrightarrow{\tilde{g}_i} & E \\ {\scriptstyle p_i}\downarrow & & \downarrow{\scriptstyle p} \\ X_i & \xrightarrow{g_i} & Z \end{array}$$

By the Mayer-Vietoris theorem, we have

$$H^k(E, \mathbb{Z}) \to H^k(G_1, \mathbb{Z}) \oplus H^k(G_2, \mathbb{Z}) \to H^k(G_1 \cap G_2, \mathbb{Z})$$

Now $f_1^*(b_1) = f_2^*(b_2)$ and so the image of (b_1, b_2) via the second map above is zero. Hence, by exactness, there is an element in $c \in H^k(E, \mathbb{Z})$ such that $\tilde{g}_i(c) = b_i, i = 1, 2$. Thus, we define an element $v \in P(Z)$ by $v = ([p], c)$. It is clear that $P(g_i)v = u_i, i = 1, 2$.

As a result, for every k, there is a CW complex R_k such that isomorphism classes of k-pairs over a space X correspond to *unbased* homotopy classes of maps from $X \to R_k$. □

Bunke et al. [18] have considered the case $k = 3$. We denote their classifying space R_3 here. For the remainder of this section and the next we work with $k = 2$. We abbreviate 2-pair to 'pair'.

A priori, R_2 is an unbased CW complex. We now arbitrarily pick a basepoint r_0 in R_2.

Lemma 4.2.2. *Let X be any CW complex. Pick a basepoint $x_0 \in X$. Any unbased map $f : X \to R_2$ may be freely homotoped to a based map $g : (X, x_0) \to (R_2, r_0)$.*

Proof. Suppose X was *any* CW complex, and $f : X \to R_2$ an unbased map. By the Lemma that follows, we know that R_2 is a fibration of a connected space over a connected base space. Hence R_2 is connected. Pick a basepoint $x_0 \in X$. Pick a path $q : I \to R_2$ connecting $f(x_0)$ to r_0. Extend the data f, q to a free homotopy $H : X \times I \to R_2$. Then, $g = H(1, .) : X \to R_2$ is map such that $g(x_0) = r_0$. The map g classifies the same pair that f does, since R_2 is an *unbased classifying space.* □

Lemma 4.2.3. *There is a fibration $K(\mathbb{Z}, 2) \to R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$.*

Proof. Given a pair $([p], b)$ over X, we obtain two natural cohomology classes $[p] \in H^2(X, \mathbb{Z})$ and $p_!(b) \in H^1(X, \mathbb{Z})$. As a result, there is a natural map $\phi \times \psi : R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. Given a pair $([p], b)$ over any space X, classified by $f : X \to R_2$, the map $f \mapsto \phi \circ f$ corresponds to the map $([p], b) \mapsto [p]$. Similarly, $f \mapsto \psi \circ f$ corresponds to the map $([p], b) \mapsto p_!(b)$. We pick a basepoint in $K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $\phi \times \psi$ is a based map. Suppose $f : X \to R_2$ classified a pair $([p], b)$ over X. Pick a basepoint $x_0 \in X$. By Lemma 4.2.2 above, f may be freely homotoped to a based map $g : (X, x_0) \to (R_2, r_0)$. Suppose g was in the homotopy fiber of $\phi \times \psi$. Then, we would obtain a pair $([p], b)$ over X such that $p_!(b) = 0, [p] = 0$. This would correspond to the trivial bundle $X \times S^1 \to X$ equipped with the cohomology class $1 \times a, a \in H^2(X, \mathbb{Z})$. Hence we would get a natural based map $X \to K(\mathbb{Z}, 2)$. Conversely, given a class a in $H^2(X, \mathbb{Z})$, we could obtain a pair $(0, 1 \times a)$ over X which would have $[p] = 0$ and $p_!(1 \times a) = 0$. By Lemma 4.2.2, this pair would be classified by a based map $g : (X, x_0) \to (R_2, r_0)$. Obviously, $(\phi \times \psi) \circ g$ would be nullhomotopic.

Thus, the homotopy fiber of $\phi \times \psi$ is $K(\mathbb{Z}, 2)$. $\square$

Lemma 4.2.4. *The homotopy groups of R_2 are as follows*

- $\pi_1(R_2) = \mathbb{Z}$,
- $\pi_2(R_2) \simeq \mathbb{Z}^2$,
- $\pi_i(R_2) = 0, i > 2$.

Proof. We had picked a basepoint for R_2. Hence, we may calculate $\pi_i(R_2)$ from the long exact sequence of the fibration in Lemma 4.2.3. We find that the nozero part of the sequence is

$$0 \to \mathbb{Z} \to \pi_2(R_2) \to \mathbb{Z} \to 0 \to \pi_1(R_2) \to \mathbb{Z} \to 0.$$

Thus, $\pi_1(R_2) = \mathbb{Z}, \pi_2(R_2) \simeq \mathbb{Z}^2$, and $\pi_i(R_2) = 0, i > 2$. $\square$

We may characterize R_2 as follows

Lemma 4.2.5. *Let $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 3)$ be the based map which induces the cup product. Then R_2 is the homotopy fiber of c.*

Proof. Let $f : X \to R_2$ be a map inducing the pair $([p], b)$ over X. Fixing a basepoint $x_0 \in X$, we may replace f by a based map $g : (X, x_0) \to (R_2, r_0)$ by Lemma 4.2.2. It is clear that we may take $\phi \times \psi$ to be based. Then we have a principal S^1-bundle $p : E \to X$. By the Gysin sequence of this bundle we have that $[p] \cup p_!(b) = 0$. This implies that $c \circ (\phi \times \psi) \circ f$ is nullhomotopic via a based homotopy, since c is exactly the based map which gives the cup product.

Conversely, suppose we are given a based map $f : X \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$ such that $c \circ f$ is nullhomotopic. Then, this corresponds to a class $a \in H^1(X, \mathbb{Z})$ and a class $[p] \in H^2(X, \mathbb{Z})$ such that $[p] \cup a = 0$. Pick a principal S^1-bundle

$p : E \to X$ with characteristic class $[p]$. From the Gysin sequence of this bundle we see that $[p] \cup a = 0$ implies that $a = p_!(b)$ for some $b \in H^2(E, \mathbb{Z})$. Thus, we obtain a pair $([p], b)$ over X and hence an unbased map $g : X \to R_2$. By the above argument, we may replace it with a based map h classifying the *same* pair over X. Obviously, $(\phi \times \psi) \circ h = f$ as a based map.

Hence, R_2 is the homotopy fiber of the based map c in the category of based CW complexes with basepoint preserving homotopy classes of maps between them. There is a forgetful functor from this category to the category $\mathcal{C}$. We take the image of the homotopy fiber of c via this functor. This determines R_2 up to homotopy equivalence in $\mathcal{C}$. □

Since $\pi_1(R_2) \neq 0$, the choice of basepoints might be important. Indeed, we have the following

Lemma 4.2.6. *The space R_2 is not simple.*

Proof. Suppose R_2 was simple: Then, from Postnikov theory, we see that R_2 would be homotopy equivalent to the product $K(\mathbb{Z}, 1) \times K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 2)$ via a based homotopy. Given $f : X \to R_2$ by Lemma 4.2.2, we could obtain a based map $g : X \to R_2$ classifying the same pair over X as f. Hence we would obtain based maps $X \to K(\mathbb{Z}, 2)$, $X \to K(\mathbb{Z}, 2)$ and $X \to K(\mathbb{Z}, 1)$. The pair would then be *determined* by classes $[p], a \in H^2(X, \mathbb{Z})$ and $p_!(b) \in H^1(X, \mathbb{Z})$. Here $[p]$ would be the characteristic class of a principal S^1-bundle $p : E \to X$. This would imply in turn that b would be determined by $p_!(b)$ and a and hence that the Gysin sequence for $p : E \to X$ would split at degree two for *any* principal S^1-bundle E over X. Since X, E were arbitrary, this is obviously impossible. □

Lemma 4.2.7. *The cohomology of R_2 up to degree* 3 *is*

- $H^0(R_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^1(R_2, \mathbb{Z}) \simeq \mathbb{Z}$,
- $H^2(R_2, \mathbb{Z}) \simeq \mathbb{Z}$.
- $H^3(R_2, \mathbb{Z}) \simeq \mathbb{Z}$.

Proof. Consider the fibration $K(\mathbb{Z}, 2) \to R_2 \to K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1)$. We have that $H^*(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z}[a]$ where a is a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z}$. This ring has no automorphisms apart from $a \to -a$. Since the fibration is oriented, the generator of $\pi_1(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1))$ acts trivially on the cohomology of $K(\mathbb{Z}, 2)$. As a result, we may use the Serre spectral sequence using cohomology with untwisted coefficients to calculate $H^*(R_2, \mathbb{Z})$.

We note that the above fibration is pulled back from the path-loop fibration over $K(\mathbb{Z}, 3)$ via the map $c : K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1) \to K(\mathbb{Z}, 3)$ inducing the cup product. Suppose μ was a generator of $H^1(K(\mathbb{Z}, 1), \mathbb{Z})$ and that λ was a generator of $H^2(K(\mathbb{Z}, 2), \mathbb{Z})$. Let $\tilde{\mu} = (\phi \times \psi)^*(\mu)$, and $\tilde{\lambda} = (\phi \times \psi)^*(\lambda)$. Then we have that $\tilde{\mu} \cup \tilde{\lambda} = 0$. This shows that the transgression $E_2^{2,0} \to E_2^{0,3}$ must be a map $k : H^2(K(\mathbb{Z}, 2), \mathbb{Z}) \simeq \mathbb{Z} \to H^2(K(\mathbb{Z}, 2) \times K(\mathbb{Z}, 1), \mathbb{Z}) \simeq \mathbb{Z}$ sending $a \to a(\mu \cup \lambda)$.

From the spectral sequence table, we see that $H^0(R_2, \mathbb{Z}) \simeq \mathbb{Z}$, $H^1(R_2, \mathbb{Z}) \simeq \mathbb{Z}$, $H^2(R_2, \mathbb{Z}) \simeq \mathbb{Z}$ and $H^3(R_2, \mathbb{Z}) \simeq \mathbb{Z}$. □

We now determine the action of $\pi_1(R_2)$ on $\pi_2(R_2)$.

Theorem 4.2.2. *The action of the generator S of $\pi_1(R_2) \simeq \mathbb{Z}$ on $\pi_2(R_2)$ is given by*

$$S(a, b) = (a + b, b).$$

Proof. From the long exact sequence of homotopy groups of the fibration in Lemma 4.2.3, we see that we have a sequence $0 \to \mathbb{Z} \to \mathbb{Z}^2 \to \mathbb{Z} \to 0$. Now, $\pi_1(R_2)$ acts on each term of this sequence by $\mathbb{Z}-$module automorphisms with the trivial action on the first and last $\mathbb{Z}$ factors and by an action θ on the middle factor.

This implies that θ may be taken to be the homomorphism induced by the matrix

$$\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}.$$

We claim that

$$\theta \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

We have a fibration $K(\mathbb{Z},2) \to R_2 \to K(\mathbb{Z},2) \times K(\mathbb{Z},1)$. We could view this as a fibration over the $K(\mathbb{Z},1) \simeq S^1$ factor $K(\mathbb{Z}^2,2) \to R_2 \to S^1$. From the long exact sequence of a fibration, it is clear that $K(\mathbb{Z}^2,2)$ is the universal cover $\tilde{R}_2$ of R_2. Now $\pi_1(S^1)$ acts on $\tilde{R}_2$ by deck transformations. Hence, using the Serre spectral sequence with *twisted* coefficients (See Ref. [35] for details), we have a spectral sequence with $E_2^{p,q} = H^p(\mathbb{Z} = \pi_1(S^1), H^q(K(\mathbb{Z}^2,2),\mathbb{Z})) \Rightarrow H^*(R_2,\mathbb{Z})$ (here $H^*(\mathbb{Z},M)$ denotes the group cohomology of $\mathbb{Z}$ with coefficients in a module M). This sequence collapses at the E_2 term itself, since $E_2^{p,q} \simeq 0$ for $p \geq 2$. Thus $\mathbb{Z} \simeq H^2(R_2,\mathbb{Z}) \simeq H^0(\mathbb{Z}, H^2(K(\mathbb{Z}^2,2),\mathbb{Z})) \simeq H^0(\mathbb{Z},\mathbb{Z}^2)$, and so the fixed points of θ on $\mathbb{Z}^2$ are $\mathbb{Z} \neq \mathbb{Z}^2$. Hence the action θ is *not* trivial and R_2 is *not* simple.

Now, $\mathbb{Z} \simeq H^3(R_2,\mathbb{Z}) \simeq H^1(\mathbb{Z}, H^2(K(\mathbb{Z}^2,2),\mathbb{Z})) \simeq H^1(\mathbb{Z},\mathbb{Z}^2)$. If $\mathbb{Z}$ acts on $\mathbb{Z}^2$ with an action θ, $H^*(\mathbb{Z},\mathbb{Z}^2)$ is the cohomology of the complex $\mathbb{Z}^2 \stackrel{\theta-1}{\to} \mathbb{Z}^2$. Hence,

$\mathbb{Z}^2/(\theta-1)\mathbb{Z}^2 \simeq \mathbb{Z}$ here, and using the above form for θ,

$$\theta \simeq \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

□

Note that since R_2 was defined in the unbased category, for any CW-complex X, pairs over X are classified by unbased maps from X to R_2. Since the space of unbased maps from X to R_2 is the quotient of the space of based maps from X to R_2 by the action of $\pi_1(R_2)$, we see that the non-trivial action of $\pi_1(R_2)$ does not affect our results. We simply have to be careful to use unbased maps in all our constructions. We can see an example of this when we try to determine all the pairs over S^2.

Lemma 4.2.8. *$P(S^2)$ is not a group.*

Proof. For any CW complex X, we have a natural map $\phi : P(X) \to H^2(X,\mathbb{Z})$ which sends a pair $([p], b)$ over X to $[p]$. Now, for every $a \in H^2(S^2,\mathbb{Z}) \simeq \mathbb{Z}$, we claim that the set $\phi^{-1}(a)$ has cardinality $|a|$. To see this, it is enough to note that if $E_p \to S^2$ is a principal S^1-bundle of Chern class $[p]$, then $H^2(E_p,\mathbb{Z}) \simeq \mathbb{Z}_p$. This implies that $P(S^2) \to H^2(S^2,\mathbb{Z})$ is not a group homomorphism, and hence that $P(S^2)$ is not a group. □

In fact, $P(S^2)$ is the quotient of the group $\pi_2(R_2) \simeq \mathbb{Z}^2$ by the action of $\pi_1(R_2)$ calculated in Thm. 4.2.2 above.

We hope to study the map T of Section (1) using the classifying space R_2 studied above.

4.3 T-duality for automorphisms is not involutive

By the proof of Thm. 2.2.1, we know that the T-dual of an automorphism even with $H-$flux is always unique. However, T-duality for automorphisms is not involutive: If we perform two successive T-dualities we may not get the automorphism we started with. For example, if $X = S^2$ with 1 unit of $H-$flux on $S^2 \times S^1$, the T-dual is S^3 with no $H-$flux. Since $H^2(S^2 \times S^1, \mathbb{Z}) \simeq \mathbb{Z}$, but $H^2(S^3, \mathbb{Z}) \simeq 0$, every locally unitary (but not necessarily unitary) automorphism of $CT(S^2 \times S^1, 1)$ dualizes to a unitary automorphism of $C(S^3, \mathcal{K})$. Taking one more T-dual gives a unitary automorphism of $CT(S^2 \times S^1, 1)$.

While we cannot calculate the T-duality map when $H \neq 0$, we conjecture what it must be below, by studying a series of examples.

We begin with the following

Lemma 4.3.1. *Let X be connected and simply connected. Let $p : E \to X$ be a principal S^1-bundle with $H-$flux H and $b \in H^2(E, \mathbb{Z})$. Let $q : E^\# \to X$ be the T-dual principal S^1-bundle with $H-$flux $H^\#$ and $b^\# \in H^2(E^\#, \mathbb{Z})$ where $b^\# = T(b)$. Then, for all $b \in H^2(E, \mathbb{Z}), \forall l, m \in \mathbb{Z}$, the Gysin sequence induces a bijection between the cosets*

$$\{b + lp^* p_!(H)\} \text{ and } \{b^\# + mq^* q_!(H^\#)\}.$$

Proof: The Gysin sequence of $p : E \to X$ is $\mathbb{Z} \xrightarrow{[p]} H^2(X, \mathbb{Z}) \xrightarrow{p^*} H^2(E, \mathbb{Z}) \to \cdots$ The kernel of p^* is the subgroup[6] $\langle [p] \rangle$. Similarly, the kernel of q^* is the

[6] $\langle a_1, a_2, \cdots, a_n \rangle$ denotes the subgroup generated by $a_1, a_2, \cdots, a_n$. The ambient group is understood from context.

subgroup $\langle [q] \rangle$. Let $G = \langle [p], [q] \rangle = \langle p_!(H), q_!(H^\#) \rangle$. Since $H^1(X, \mathbb{Z}) = 0$, p^*, q^* are surjective by the Gysin sequence. Note that $p^*G \simeq \langle p^* p_!(H) \rangle$ and $q^*G \simeq \langle q^* q_!(H^\#) \rangle$. Thus, we have isomorphisms $H^2(X, \mathbb{Z})/G \simeq H^2(E, \mathbb{Z})/\langle p^* p_!(H) \rangle \simeq H^2(E^\#, \mathbb{Z})/\langle q^* q_!(H^\#) \rangle$.

□

Note that in the special case $E^\# = X \times S^1, H^\# = [p] \times z$, where $p : E \to X$ is a principal S^1-bundle, $H = 0$, the above theorem states that there is a natural map between the coset $\{a \times 1 + l[p] \times 1\}$ and $b^\#$. Here, b is always of the form $a \times 1, a \in H^2(X, \mathbb{Z})$, and $b^\# = p^*(a)$ by the Gysin sequence.

We conjecture that each coset is precisely the collection of b-fields with the same T-dual (even when $H \neq 0$). As support for this, note the following: Suppose $E^\# = X \times S^1$, with k units of $H-$flux. Let $q = \pi : X \times S^1 \to X$ be the projection map. Then, if $H^1(X, \mathbb{Z}) \neq 0$, the above theorem would not be expected to hold: For one thing, $\mathrm{im}(\pi^*)$ would not be all of $H^2(X \times S^1, \mathbb{Z})$. It is strange then, that $H^2(E, \mathbb{Z})/\langle p^* p_!(H) \rangle$ is isomorphic to $H^2(E^\#, \mathbb{Z})/\langle q^* q_!(H^\#) \rangle$ in all the following cases[7] (I use Ref. [4] for the examples):

1. <u>$X = T^2$:</u> For $E^\# = X \times S^1$, $H^0(E^\#, \mathbb{Z}) = \mathbb{Z}, H^1 = \mathbb{Z}^3, H^2 = \mathbb{Z}^3, H^3 = \mathbb{Z}$. The $H-$flux is a class $[p] \times z \in H^2(T^2) \otimes H^1(S^1) \simeq H^3(T^2 \times S^1) \simeq \mathbb{Z}$. The T-dual is the nilmanifold $p : N \to T^2$ whose cohomology is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}^2, H^2 = \mathbb{Z}^2 \oplus \mathbb{Z}_p, H^3 = \mathbb{Z}$. It is clear that $\mathbb{Z}^3/p\mathbb{Z} \simeq \mathbb{Z}^2 \oplus \mathbb{Z}_p$.

2. <u>$X = M$, an orientable surface of genus $g > 1$:</u> The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}^{2g+1}, H^2 = \mathbb{Z}^{2g+1}, H^3 = \mathbb{Z}$. The $H-$flux is a class

[7] Note that in most of these cases, $H^1(X, \mathbb{Z}) \neq 0$.

$j \times z \in H^3 \simeq H^2(M) \otimes H^1(S^1) \simeq \mathbb{Z}$. The T-dual is a bundle $j : E \to M$ with $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}^{2g}, H^2 = \mathbb{Z}^{2g} \oplus \mathbb{Z}_j, H^3 = \mathbb{Z}$. Here, $\mathbb{Z}^{2g+1}/j\mathbb{Z} \simeq \mathbb{Z}^{2g} \oplus \mathbb{Z}_j$.

3. $\underline{X = \mathbb{RP}^2 :}$ The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = \mathbb{Z}_2 \simeq H^2(X) \otimes H^0(S^1), H^3 \simeq H^2(X) \otimes H^1(S^1) \simeq \mathbb{Z}_2$. The $H-$flux is the class $1 \times z \in H^2(X) \otimes H^1(S^1)$. The T-dual is a bundle $k : E \to \mathbb{RP}^2$ with $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = 0, H^3 = \mathbb{Z}_2$. Once again, $\mathbb{Z}_2/1\mathbb{Z}_2 \simeq 0$.

4. $\underline{X = \mathbb{RP}^3 :}$ The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = \mathbb{Z}_2, H^3 = \mathbb{Z} \oplus \mathbb{Z}_2, H^4 = \mathbb{Z}$. Then, $H^3 \simeq H^2(X) \otimes H^1(S^1) \oplus H^3(X) \otimes H^0(S^1)$. The $H-$flux is $1 \times z + k \times 1$. Now, $\pi^* \pi_!(H) = 1 \times 1$. The T-dual is $q : S^1 \times S^3 \to \mathbb{RP}^3$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^3 = \mathbb{Z}, H^4 = \mathbb{Z}$. The T-dual has no $B-$field and $H-$flux $k \in \mathbb{Z}$.

5. $\underline{X = \mathbb{RP}^{2m}(m > 1)}$: The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \cdots, m-1, H^{2m} = \mathbb{Z}, H^{2m+1} = \mathbb{Z}_2$. The $H-$flux is the class $1 \times z \in H^2(\mathbb{RP}^{2m}) \otimes H^1(S^1)$. The T-dual is a bundle $q : E \to \mathbb{RP}^{2m}$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}_2$. Here, $\mathbb{Z}_2/1\mathbb{Z}_2 \simeq 0$.

6. $\underline{X = \mathbb{RP}^{2m+1}(m > 1)}$: The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^q = \mathbb{Z}_2, q = 2, \cdots, m-1, H^{2m+1} = \mathbb{Z} \oplus \mathbb{Z}_2, H^{2m+2} = \mathbb{Z}$. Note $H^3 = \mathbb{Z}_2$; we have that $H^3 \simeq H^2(\mathbb{RP}^{2m+1}) \otimes H^1(S^1)$. The $H-$flux is the class $1 \times z \in H^3 \simeq \mathbb{Z}_2$. The T-dual is a principal bundle $q : S^1 \times S^{2m+1} \to \mathbb{RP}^{2m+1}$ with cohomology $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^{2m+1} = \mathbb{Z}, H^{2m+2} = \mathbb{Z}$. The T-dual has no second cohomology as expected.

7. $\underline{X = \mathbb{CP}^2 :}$ The cohomology of $X \times S^1$ is $H^0 = \mathbb{Z}, H^1 = \mathbb{Z}, H^2 = \mathbb{Z}, H^3 = \mathbb{Z}, H^4 = \mathbb{Z}, H^5 = \mathbb{Z}$. We have $H^3 \simeq H^2(X) \oplus H^1(S^1) \simeq \mathbb{Z}$. The $H-$flux

is the class $j \times z \in H^3$. The T-dual is the Lens space $L(2, j) \rightarrow \mathbb{CP}^2$ if $j \neq 0$. It has cohomology $H^0 = \mathbb{Z}, H^2 = \mathbb{Z}_j, H^4 = \mathbb{Z}_j, H^5 = \mathbb{Z}$. Once again, $\mathbb{Z}/j\mathbb{Z} \simeq \mathbb{Z}_j$.

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