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Constructing Queueing Network Approximations for Mass Dispensing and Vaccination Clinics

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Abstract

This paper presents approximations for open queueing networks. The approximations include the wait-in-batch-time, the departure variability at self-service stations, and the arrival variability for process batches.

1. Introduction

This work in this paper is motivated by the study of mass dispensing and vaccination clinics. While the study of queueing networks has resulted in numerous results, the need to model queueing networks with batch processes performed by multiple parallel servers and self-service stations led us to develop new approximations. This paper presents results that justify the approximations in Pilehvar et al. (2006).

The fundamental problem is to evaluate the capacity and queueing of a given clinic design, given information about the arrival of residents to the clinic, the flow of residents through the clinic, and the processing at each workstation in the clinic.

Due to the nature of these facilities, we model a clinic as an open queueing network. In the clinic queueing model, county residents are the customers, and the servers are the clinic staff, who are the critical resource. Residents arrive according to an external (not necessarily Poisson) arrival process. When buses are used to transport residents to a clinic, arrivals will be batches of residents. The queueing network operates in the following manner. When a resident arrives, he goes to the first workstation. Based on that resident's personal information (including current state of health and medical allergies), the resident moves from one workstation to another in the clinic. Most residents will receive treatment (medication or vaccination) and then leave. However, some residents will leave without receiving treatment, and others will be transported to a hospital.

We decompose the queueing network by estimating the performance of each workstation using a combination of exact and approximate models. A key contribution of this paper is to introduce approximations for workstations with batch arrivals and multiple parallel servers, for workstations with batch processes and multiple parallel servers, and for self-service workstations.

Most of the workstations in a clinic have multiple, parallel servers that treat one resident at a time. For example, a vaccination workstation may have a dozen nurses, and each nurse vaccinates one resident at a time. However, some workstations in a clinic have batch processes that serve multiple residents simultaneously as a group. Moreover, there may be multiple servers so that multiple batches can be processed in parallel. For instance, at the education station, residents sit in classrooms in which they watch an informational video about the smallpox vaccine (under the direction of a staff member). Because there are multiple classrooms, different groups begin and end the process at different times.

There are also self-service stations where residents complete paperwork (typically, medical history questionnaires) on their own. Staff may be present to answer questions, but they are not the critical resource, and modeling the process by which residents ask for and receive assistance is not essential to estimate clinic performance. One could also model the time that residents spend walking from one station to another as a self-service station.

Motivated by the setup of typical clinics, we assume that there is no re-entrant flow. The overall model is described by Pilehvar et al. (2006). This paper provides supporting derivations and results of computational experiments.

2. Wait-in-batch-time derivation

This section considers the case with a general arrival process. Residents arrive to the workstation in batches and individually. The arrival batches may come from different batch processing workstations, and the batch sizes from each workstation may vary due to the routing probabilities. There are also individual arrivals from individual processing workstations. The workstation has multiple, parallel servers that serve residents individually. To analyze this case we model all of the arrivals as batches. Each batch must wait to get to the head of the queue, at which point it "opens" and at least one of the residents in the batch begins service. The other residents must wait in the batch for a server.

A key quantity is the estimate of the wait-in-batch-time, the average time that a resident spends in the batch from the time that the batch "opens" until the resident begins service.

We will use the following notation:

m_i = Number of staff at station i

t_i = Mean process time at station i (minutes)

λ_{Ai} = Batch arrival rate at station i (batches per minute)

$\bar{K}_{Ai}$ = Average batch size of all batches that come to station i

u_i = Utilization at station i

$p_n(i)$ = Steady-state probability of having n residents in station i.

U_i = Steady-state probability of all of the servers at station i being busy

X_i = Average number of residents that wait in the batch at station i.

$WIBT_i$ =Average wait in batch time at station i (minutes)

If we replace the workstation by the equivalent single-server workstation with a mean processing time equal to t_i / m_i, we would estimate the wait-in-batch-time as follows:

$$WIBT_i = \frac{(\overline{K}_{Ai} - 1)t_i}{2m_i}$$

As we will see, this estimate (which we call Formula 1) is not a good approximation, so we will derive a new formula for the wait-in-batch-time. To do so, we start by calculating the following terms:

$$u_i = \frac{\lambda_{Ai}\overline{K}_{Ai}t_i}{m_i}$$

$$U_i = \sum_{n=m_i}^{\infty} p_n(i) = 1 - \sum_{n=0}^{m_i-1} p_n(i)$$

It will be useful to note the following:

$$\sum_{n=0}^{m_i-1} np_n(i) = m_i u_i - m_i \sum_{n=m_i}^{\infty} p_n(i) = m_i\left(u_i - U_i\right)$$

If, when the batch arrives, the number of residents is greater or equal to the number of servers, all of the servers are busy, so the batch waits in the queue. Eventually, the batch is at the head of the queue and one of the servers completes a resident. Then the batch opens, one resident begins service without waiting in batch, and all of the others wait in the batch.

If, when the batch arrives, the number of residents is less than the number of servers, one or more servers are idle, so the batch opens and one or more residents begin service immediately.

From this we estimate X_i as follows:

$$\begin{aligned} X_i &= \sum_{n=0}^{m_i-1} p_n(i)\left(\overline{K}_{Ai} - m_i + n\right) + \sum_{n=m_i}^{\infty} p_n(i)\left(\overline{K}_{Ai} - 1\right) \\ &= \overline{K}_{Ai} - m_i\left(1 - U_i\right) + m_i\left(u_i - U_i\right) - U_i \\ &= \overline{K}_{Ai} - m_i + m_i u_i - U_i \end{aligned}$$

Thus, $\overline{K}_{Ai} - X_i$ residents go to servers immediately. For them the wait-in-batch-time is zero.

Assuming that the servers, when busy, complete a resident every $\frac{t_i}{m_i}$ minutes, the first resident of those remaining must wait-in-batch for $\frac{t_i}{m_i}$ minutes. The second waits $\frac{2t_i}{m_i}$ minutes, and so forth. The last resident in the batch waits for $\frac{X_i t_i}{m_i}$ minutes. We then estimate the average wait-in-batch-time as follows (we call this estimate Formula 2):

$$WIBT_i = \frac{1}{\bar{K}_{Ai}} \sum_{n=1}^{X_i} \frac{nt_i}{m_i} = \frac{X_i(X_i+1)}{2\bar{K}_{Ai}} \cdot \frac{t_i}{m_i}$$
$$= \frac{(\bar{K}_{Ai} - m_i + m_i u_i - U_i)(\bar{K}_{Ai} - m_i + m_i u_i - U_i + 1)}{2\bar{K}_{Ai}} \cdot \frac{t_i}{m_i}$$

The only remaining task is to estimate U_i. Following Shore (1988) and dropping the station subscript for the moment, we let $E(N_c)$ be the mean number of customers in the system and $E(N_1)$ be the mean number in of customers in the corresponding GI/G/1 queue having the same traffic intensity.

$$E(N_c) = mu + \left[\frac{u^{\sqrt{2m+2}}}{1-u}\right]\left[\frac{c^2{}_a + c^2{}_e}{2}\right]$$

$$E(N_1) = u + \left[\frac{u^2}{1-u}\right]\left[\frac{c^2{}_a + c^2{}_e}{2}\right]$$

Shore shows that

$$U = u(E(N_c) - mu)/(E(N_1) - u)$$

From this, we can determine that $U = u^{\sqrt{2m+2}-1}$. Since this is not affected by the arrival variability, we will use this result for our batch arrival case. Going back to the original notation, we have

$$U_i = u_i^{\sqrt{2m_i+2}-1}$$

3. Wait-in-batch-time experiments

To evaluate Formula 1 and Formula 2, we conducted a set of computational experiments using a discrete-event simulation model of the station. In each scenario, the arrival batches had a fixed size (either 5 or 20), and the interarrival times were exponentially distributed. The mean interarrival time varied from 0.1684 minutes to 0.3333 minutes. The distribution of the processing times was an exponential distribution or a gamma distribution. For the exponential distributions, the mean was either 0.0333 minutes or 0.10 minutes. For the gamma distributions, the alpha was always 0.5, while the beta was set to 0.0167, 0.050, 0.0667, and 0.20. The number of servers was either 1 or 3. Tables 1, 2, and 3 describes the scenarios. The name of each scenario refers to the processing time distribution, the batch size, the number of servers, and the expected utilization.

Table 1. Scenarios with exponentially distributed processing times.

Scenario	Batch size	Mean Interarrival Time (mins)	Mean Processing Time (mins)	Number of servers
E-5-1-99	5	0.1684	0.0333	1
E-5-1-95	5	0.1754	0.0333	1
E-5-1-90	5	0.1852	0.0333	1
E-5-1-80	5	0.2083	0.0333	1
E-5-1-50	5	0.3333	0.0333	1
E-5-3-99	5	0.1684	0.1000	3
E-5-3-95	5	0.1754	0.1000	3
E-5-3-90	5	0.1852	0.1000	3
E-5-3-80	5	0.2083	0.1000	3
E-5-3-50	5	0.3333	0.1000	3

Table 2. Scenarios with processing times that have a gamma distribution and one server.

Scenario	Batch size	Mean Interarrival Time (mins)	Mean Processing Time (mins)	Number of servers
G-5-1-99	5	0.1684	0.0333	1
G-5-1-95	5	0.1754	0.0333	1
G-5-1-90	5	0.1852	0.0333	1
G-5-1-80	5	0.2083	0.0333	1
G-5-1-50	5	0.3333	0.0333	1
G-20-1-99	20	0.1684	0.0083	1
G-20-1-95	20	0.1754	0.0083	1
G-20-1-90	20	0.1852	0.0083	1
G-20-1-80	20	0.2083	0.0083	1
G-20-1-50	20	0.3333	0.0083	1

Table 3. Scenarios with processing times that have a gamma distribution and three servers.

Scenario	Batch size	Mean Interarrival Time (mins)	Mean Processing Time (mins)	Number of servers
G-5-3-99	5	0.1684	0.1000	3
G-5-3-95	5	0.1754	0.1000	3
G-5-3-90	5	0.1852	0.1000	3
G-5-3-80	5	0.2083	0.1000	3
G-5-3-50	5	0.3333	0.1000	3
G-20-3-99	20	0.1684	0.0250	3
G-20-3-95	20	0.1754	0.0250	3
G-20-3-90	20	0.1852	0.0250	3
G-20-3-80	20	0.2083	0.0250	3
G-20-3-50	20	0.3333	0.0250	3

For each scenario, we ran a simulation model with 100 replications, each 30,000 minutes long with a warm-up period of 25,000 minutes. From the simulation model we could calculate the average wait-in-batch-time of residents. We also used Formula 1 and Formula 2 to estimate the average wait-in-batch-time. Tables 4, 5, and 6 shows the results. For each scenario, the table lists the average wait-in-batch-time from the simulation model, the estimate from Formula 1, and the estimate from Formula 2. Also listed are the relative errors for the estimates. We see that Formula 2 provides a much better estimate than Formula 1.

Table 4. Results for scenarios with exponentially distributed processing times.

Scenario	WIBT from simulation (mins)	WIBT from Formula 1 (mins)	Relative error, Formula 1	WIBT from Formula 2 (mins)	Relative error, Formula 2
E-5-1-99	0.0667	0.0667	0.050%	0.0667	0.050%
E-5-1-95	0.0667	0.0667	0.050%	0.0667	0.050%
E-5-1-90	0.0667	0.0667	0.050%	0.0667	0.050%
E-5-1-80	0.0667	0.0667	0.050%	0.0667	0.050%
E-5-1-50	0.0667	0.0667	0.050%	0.0667	0.050%
E-5-3-99	0.0660	0.0667	1.010%	0.0663	0.475%
E-5-3-95	0.0660	0.0667	1.010%	0.0649	1.720%
E-5-3-90	0.0600	0.0667	11.111%	0.0630	4.959%
E-5-3-80	0.0600	0.0667	11.111%	0.0590	1.748%
E-5-3-50	0.0480	0.0667	38.889%	0.0453	5.717%

Table 5. Results for scenarios with Gamma distributed processing times with 1 server.

Scenario	WIBT from simulation (mins)	WIBT from Formula 1 (mins)	Relative error, Formula 1	WIBT from Formula 2 (mins)	Relative error, Formula 2
G-5-1-99	0.0665	0.0667	0.251%	0.0667	0.251%
G-5-1-95	0.0665	0.0667	0.251%	0.0667	0.251%
G-5-1-90	0.0665	0.0667	0.251%	0.0667	0.251%
G-5-1-80	0.0665	0.0667	0.251%	0.0667	0.251%
G-5-1-50	0.0665	0.0667	0.251%	0.0667	0.251%
G-20-1-99	0.0790	0.0792	0.211%	0.0792	0.211%
G-20-1-95	0.0790	0.0792	0.211%	0.0792	0.211%
G-20-1-90	0.0790	0.0792	0.211%	0.0792	0.211%
G-20-1-80	0.0790	0.0792	0.211%	0.0792	0.211%
G-20-1-50	0.0790	0.0792	0.211%	0.0792	0.211%

Table 6. Results for scenarios with Gamma distributed processing times with 3 servers.

Scenario	WIBT from simulation (mins)	WIBT from Formula 1 (mins)	Relative error, Formula 1	WIBT from Formula 2 (mins)	Relative error, Formula 2
G-5-3-99	0.0660	0.0667	1.010%	0.0663	0.475%
G-5-3-95	0.0643	0.0667	3.681%	0.0649	0.878%
G-5-3-90	0.0621	0.0667	7.354%	0.0630	1.409%
G-5-3-80	0.0576	0.0667	15.741%	0.0590	2.346%
G-5-3-50	0.0429	0.0667	55.400%	0.0453	5.491%
G-20-3-99	0.0787	0.0792	0.593%	0.0729	7.373%
G-20-3-95	0.0779	0.0792	1.626%	0.0729	6.422%
G-20-3-90	0.0770	0.0792	2.814%	0.0729	5.328%
G-20-3-80	0.0750	0.0792	5.556%	0.0729	2.803%
G-20-3-50	0.0688	0.0792	15.068%	0.0729	5.956%

4. Self-service

In this case, residents arrive individually to the workstation. The residents perform the process themselves without any external resources. In this domain, an example would be a workstation where each resident must complete a form. Thus, the workstation can be modeled as a G/G/∞ queueing system.

We will use the following notation:

r_i = Arrival rate at station i (residents per minute)

c_{ai}^2 = Aggregate batch arrival SCV at station i

t_i = Mean process time at station i (minutes)

c_{ei}^2 = Processing time SCV at station i

$\rho_i = r_i t_i$ = Load

c_{di}^2 = Departure SCV at station i

To estimate the departure variability we first take into account the following facts. For a G/D/∞ system, the departure variability equals the arrival variability because the departure process is simply the arrival process shifted by a constant equal to the processing time. For a M/G/∞ system, the departure process is a Poisson process; thus the departure variability equals 1 (Burke, 1958, and Mirasol, 1963). For a G/G/∞ system, Whitt (1983) suggests that the departure variability approaches 1 as the load (the arrival rate divided by the service rate) goes to infinity. On the other hand, if the load is near 0, the service rate is relatively fast, implying that customers spend very little time in the system. Thus, we would expect the departure variability to equal the arrival variability. These imply that, in the general case (a G/G/∞ system with moderate load), the departure variability will be somewhere between the arrival variability and one. Exactly where will depend upon the load.

Therefore, we conducted experiments to characterize this relationship and to examine various weights for interpolating between the arrival variability and one as a function of the load $\rho_i = r_i t_i$. The weight should be between 0 and 1.

The general form of the interpolation can be expressed as follows:

$$c_{di}^2 = (1-\omega)c_{ai}^2 + \omega$$

Note that, if the arrival variability equals 1, then (for any weight), the departure variability equals 1. The purpose of the experiments was to evaluate various functions that could be used to determine the weight for this interpolation.

Three candidates were tried:

$$\omega_a = \frac{\rho_i^2 c_{ei}^2}{\left(1+\rho_i\sqrt{c_{ei}^2}\right)^2}$$

$$\omega_b = \frac{\rho_i^2 c_{ei}^2}{1+\rho_i^2 c_{ei}^2}$$

$$\omega_c = \frac{\rho_i c_{ei}^2}{1+\rho_i c_{ei}^2}$$

All of these are between 0 and 1 and have the following desirable properties:

- As the processing time variability goes to 0, the weight goes to 0, and the departure variability approaches the arrival variability.
- As the load goes to 0, the weight goes to 0, and the departure variability approaches the arrival variability.
- As the load goes to infinity, the weight goes to 1, and the departure variability approaches 1.

Based on the results (discussed in the next section), we decided to use ω_a, which yields the following approximation:

$$c_{di}^2 = c_{ai}^2\left(1-\frac{\rho_i^2 c_{ei}^2}{\left(1+\rho_i\sqrt{c_{ei}^2}\right)^2}\right)+\frac{\rho_i^2 c_{ei}^2}{\left(1+\rho_i\sqrt{c_{ei}^2}\right)^2}$$

5. Self-service experiments

To evaluate these weights, we conducted sets of computational experiments using a discrete-event simulation model of the station. In all cases, we ran five replications and measured the interdeparture times of the residents. We then calculated the departure SCV for each replication and calculated 95% confidence intervals. The run lengths and warmup periods were proportional to the mean interarrival time as indicated below.

In the first set (which we denote as Set DE), the interarrival times were constant, and the processing times were exponentially distributed. The mean interarrival time went from 0.0006 minutes to 100 minutes. The mean processing time was 3 minutes in all scenarios. Thus, the load varied from 0.03 to 5000. The run length was set equal to 260,000 times the mean interarrival time, and the warmup period was set equal to 200,000 times the mean interarrival time.

In the second set (Set GE), the interarrival times had a gamma distribution, and the processing times were exponentially distributed. The mean interarrival time went from 0.04 minutes to 40 minutes. The alpha parameter was always equal to 0.2, so the interarrival variability was always 5. The mean processing time was 3 minutes in all scenarios. Thus, the load varied from 0.075 to

750. The run length was set equal to 110,000 times the mean interarrival time, and the warmup period was set equal to 50,000 times the mean interarrival time.

In the third set (Set EG), the interarrival times were exponentially distributed, and the processing times had a gamma distribution. The mean interarrival time was always 4 minutes. The mean processing time varied from 0.05 to 2000 minutes. The alpha parameter was always equal to 0.5, so the processing time variability was always 2. Thus, the load varied from 0.0125 to 500. The run length was set equal to 865,000 times the mean interarrival time, and the warmup period was set equal to 800,000 times the mean interarrival time.

In the fourth set (Set GG), the interarrival times had a gamma distribution, and the processing times had a gamma distribution. The mean interarrival time was always 4 minutes. The alpha parameter was always 0.1, so the arrival variability was always 10. The mean processing time varied from 0.25 to 2000 minutes. The alpha parameter was always equal to 0.5, so the processing time variability was always 2. Thus, the load varied from 0.0625 to 500. The run length was set equal to 1,315,000 times the mean interarrival time, and the warmup period was set equal to 1,250,000 times the mean interarrival time.

In the fifth set (Set UG), the interarrival times had a uniform distribution, and the processing times had a gamma distribution. The interarrival time distribution was always between 3 and 5 minutes. Thus, the arrival variability was always 0.02. The mean processing time varied from 0.25 to 2000 minutes. The alpha parameter was always equal to 0.5, so the processing time variability was always 2. Thus, the load varied from 0.0625 to 500. The run length was set equal to 140,000 times the mean interarrival time, and the warmup period was set equal to 75,000 times the mean interarrival time.

Tables 7 to 11 present the results for sets DE, GE, EG, GG and UG. For each scenario, the table lists the load, the departure SCV from the simulation, the lower and upper bound on the confidence interval. In addition, it provides the three departure SCV estimates (one using each weight) and the relative error for each. Figures 1, 2, 3, 4, and 5 compare the estimates using ω_a to the simulation results.

Table 7. Departure variability results for Set DE.

Load	SCV (sim.)	Lower bound	Upper bound	SCV a	Relative error	SCV b	Relative error	SCV c	Relative error
5000	1.0055	0.027	1.0325	0.9996	0.590%	1.0000	0.550%	0.9998	0.570%
3000	1.0081	0.0258	1.0339	0.9993	0.873%	1.0000	0.807%	0.9997	0.840%
1000	1.0014	0.03	1.0314	0.9980	0.343%	1.0000	0.143%	0.9990	0.243%
500	0.9956	0.023	1.0185	0.9960	0.043%	1.0000	0.443%	0.9980	0.243%
400	1.0027	0.0268	1.0295	0.9950	0.771%	1.0000	0.275%	0.9975	0.523%
300	1.0077	0.0271	1.0348	0.9934	1.420%	1.0000	0.763%	0.9967	1.092%
200	0.9882	0.0245	1.0127	0.9901	0.195%	1.0000	1.197%	0.9950	0.696%
100	1.0012	0.0303	1.0316	0.9803	2.092%	0.9999	0.134%	0.9901	1.113%
60	0.9776	0.019	0.9966	0.9675	1.032%	0.9997	2.266%	0.9836	0.617%
30	0.9545	0.0158	0.9703	0.9365	1.879%	0.9989	4.656%	0.9677	1.392%
6	0.8499	0.0199	0.8697	0.7347	13.553%	0.9730	14.484%	0.8571	0.855%
3	0.7564	0.0215	0.7778	0.5625	25.630%	0.9000	18.991%	0.7500	0.841%
2	0.6796	0.0176	0.6972	0.4444	34.599%	0.8000	17.722%	0.6667	1.898%
1.5	0.5933	0.0143	0.6076	0.3600	39.325%	0.6923	16.683%	0.6000	1.126%
0.75	0.3957	0.0099	0.4056	0.1837	53.578%	0.3600	9.012%	0.4286	8.319%
0.5	0.2753	0.0045	0.2798	0.1111	59.642%	0.2000	27.355%	0.3333	21.07%
0.3	0.1446	0.0013	0.1458	0.0533	63.160%	0.0826	42.881%	0.2308	59.64%
0.2	0.0741	0.0008	0.0749	0.0278	62.513%	0.0385	48.095%	0.1667	124.9%
0.15	0.0434	0.0007	0.0441	0.0170	60.754%	0.0220	49.239%	0.1304	200.8%
0.12	0.0280	0.0006	0.0286	0.0115	59.058%	0.0142	49.372%	0.1071	282.1%
0.1	0.0195	0.0004	0.0199	0.0083	57.639%	0.0099	49.251%	0.0909	365.9%
0.03	0.0018	3.3E-05	0.0018	0.0008	51.524%	0.0009	48.618%	0.0291	1564.%

Table 8. Departure variability results for Set GE.

Load	SCV (sim.)	Lower bound	Upper bound	SCV a	Relative error	SCV b	Relative error	SCV c	Relative error
750	0.9995	0.9755	1.0234	1.0106	1.115%	1.0000	0.051%	1.0053	0.583%
250	1.0074	0.9657	1.0491	1.0318	2.423%	1.0001	0.728%	1.0159	0.847%
125	1.0299	1.0144	1.0453	1.0632	3.237%	1.0003	2.878%	1.0317	0.179%
100	1.0075	0.9505	1.0645	1.0788	7.078%	1.0004	0.705%	1.0396	3.186%
75	1.0689	1.0372	1.1007	1.1046	3.337%	1.0007	6.379%	1.0526	1.522%
50	1.0607	1.042	1.0795	1.1553	8.921%	1.0016	5.572%	1.0784	1.672%
25	1.1834	1.1294	1.2374	1.3018	10.003%	1.0064	14.958%	1.1538	2.497%
15	1.3269	1.2303	1.4235	1.4844	11.868%	1.0177	23.303%	1.2500	5.795%
7.5	1.8037	1.5534	2.0539	1.8858	4.552%	1.0699	40.685%	1.4706	18.468%
1.5	3.7072	3.2701	4.1444	3.5600	3.971%	2.2308	39.826%	2.6000	29.866%
0.75	4.328	3.8798	4.7763	4.2653	1.449%	3.5600	17.745%	3.2857	24.082%
0.5	4.5708	4.1191	5.0226	4.5556	0.334%	4.2000	8.112%	3.6667	19.781%
0.375	4.6949	4.2422	5.1475	4.7025	0.161%	4.5068	4.005%	3.9091	16.738%
0.1875	4.8676	4.3642	5.371	4.9003	0.671%	4.8642	0.071%	4.3684	10.255%
0.125	4.9324	4.4765	5.3882	4.9506	0.369%	4.9385	0.123%	4.5556	7.640%
0.075	4.9721	4.5159	5.4284	4.9805	0.170%	4.9776	0.111%	4.7209	5.052%

Table 9. Departure variability results for Set EG.

Load	SCV (sim.)	Lower bound	Upper bound	SCV a	Relative error	SCV b	Relative error	SCV c	Relative error
500	0.9892	0.9729	1.0056	1.00	1.089%	1.00	1.089%	1.00	1.089%
250	0.9924	0.9737	1.0112	1.00	0.761%	1.00	0.761%	1.00	0.761%
125	0.9993	0.9714	1.0273	1.00	0.063%	1.00	0.063%	1.00	0.063%
87.5	1.0132	0.9892	1.0373	1.00	1.308%	1.00	1.308%	1.00	1.308%
62.5	0.9946	0.9686	1.0206	1.00	0.543%	1.00	0.543%	1.00	0.543%
50	0.9898	0.9769	1.0027	1.00	1.029%	1.00	1.029%	1.00	1.029%
37.5	0.9933	0.9717	1.015	1.00	0.670%	1.00	0.670%	1.00	0.670%
25	0.9972	0.962	1.0324	1.00	0.278%	1.00	0.278%	1.00	0.278%
12.5	0.9897	0.9704	1.0091	1.00	1.037%	1.00	1.037%	1.00	1.037%
6.25	0.9947	0.9679	1.0215	1.00	0.531%	1.00	0.531%	1.00	0.531%
2.5	0.9947	0.9831	1.0064	1.00	0.527%	1.00	0.527%	1.00	0.527%
1.25	0.9893	0.9764	1.0022	1.00	1.080%	1.00	1.080%	1.00	1.080%
0.63	0.9904	0.9694	1.0115	1.00	0.960%	1.00	0.960%	1.00	0.960%
0.13	0.9995	0.9802	1.0188	1.00	0.046%	1.00	0.046%	1.00	0.046%
0.06	1.0010	0.9822	1.02	1.00	0.107%	1.00	0.107%	1.00	0.107%
0.01	1.0021	0.984	1.0203	1.00	0.214%	1.00	0.214%	1.00	0.214%

Table 10. Departure variability results for Set GG.

Load	SCV (sim.)	Lower bound	Upper bound	SCV a	Relative error	SCV b	Relative error	SCV c	Relative error
500	1.0172	0.9984	1.0359	1.0254	0.806%	1.0000	1.689%	1.0090	0.807%
250	1.0557	1.0254	1.086	1.0507	0.474%	1.0001	5.269%	1.0180	3.574%
125	1.0935	1.0588	1.1281	1.1010	0.683%	1.0003	8.524%	1.0359	5.271%
87.5	1.158	1.1211	1.1949	1.1437	1.233%	1.0006	13.593%	1.0511	9.228%
62.5	1.2288	1.1585	1.2991	1.2002	2.324%	1.0012	18.526%	1.0714	12.807%
50	1.2752	1.1668	1.3837	1.2493	2.034%	1.0018	21.440%	1.0891	14.593%
37.5	1.4536	1.2319	1.6752	1.3300	8.500%	1.0032	30.985%	1.1184	23.059%
25	1.7081	1.3121	2.1041	1.4883	12.868%	1.0072	41.034%	1.1765	31.124%
12.5	2.723	1.667	3.779	1.9379	28.831%	1.0287	62.222%	1.3462	50.564%
6.25	4.2706	2.6963	5.845	2.7365	35.922%	1.1137	73.921%	1.6667	60.973%
2.5	6.9288	4.9615	8.8961	4.5312	34.604%	1.6667	75.946%	2.5000	63.919%
1.25	8.5373	6.5603	10.5144	6.3286	25.871%	3.1818	62.730%	3.5714	58.167%
0.625	9.7565	7.7646	11.7483	8.0188	17.811%	6.0526	37.963%	5.0000	48.752%
0.125	10.7301	8.7557	12.7045	9.7969	8.697%	9.7273	9.346%	8.2000	23.579%
0.062	10.8393	8.868	12.8107	9.9415	8.283%	9.9313	8.377%	9.0071	16.903%

Table 11. Departure variability results for Set UG.

Load	SCV (sim.)	Lower bound	Upper bound	SCV a	Relative error	SCV b	Relative error	SCV c	Relative error
500	0.9884	0.9591	1.0178	0.9972	0.894%	1.0000	1.173%	0.9990	1.075%
250	0.9938	0.9752	1.0125	0.9945	0.069%	1.0000	0.623%	0.9980	0.427%
125	0.9809	0.9551	1.0067	0.9890	0.827%	1.0000	1.944%	0.9961	1.549%
87.5	0.979	0.9636	0.9943	0.9844	0.548%	0.9999	2.139%	0.9944	1.577%
62.5	0.9727	0.9474	0.998	0.9782	0.567%	0.9999	2.794%	0.9922	2.008%
50	0.9703	0.9604	0.9802	0.9729	0.266%	0.9998	3.041%	0.9903	2.062%
37.5	0.9564	0.9403	0.9724	0.9641	0.804%	0.9997	4.522%	0.9871	3.212%
25	0.9402	0.9211	0.9594	0.9469	0.710%	0.9992	6.277%	0.9808	4.318%
12.5	0.8928	0.8797	0.9059	0.8980	0.578%	0.9969	11.657%	0.9623	7.789%
6.25	0.8288	0.8157	0.842	0.8111	2.139%	0.9876	19.163%	0.9275	11.905%
2.5	0.6933	0.6852	0.7013	0.6158	11.175%	0.9275	33.776%	0.8368	20.699%
1.25	0.5455	0.5358	0.5552	0.4203	22.957%	0.7626	39.803%	0.7202	32.032%
0.625	0.3736	0.3661	0.3811	0.2364	36.730%	0.4503	20.527%	0.5648	51.181%
0.125	0.0758	0.0738	0.0777	0.0429	43.369%	0.0505	33.375%	0.2167	185.83%
0.062	0.0366	0.0358	0.0374	0.0273	25.443%	0.0284	22.348%	0.1296	254.17%

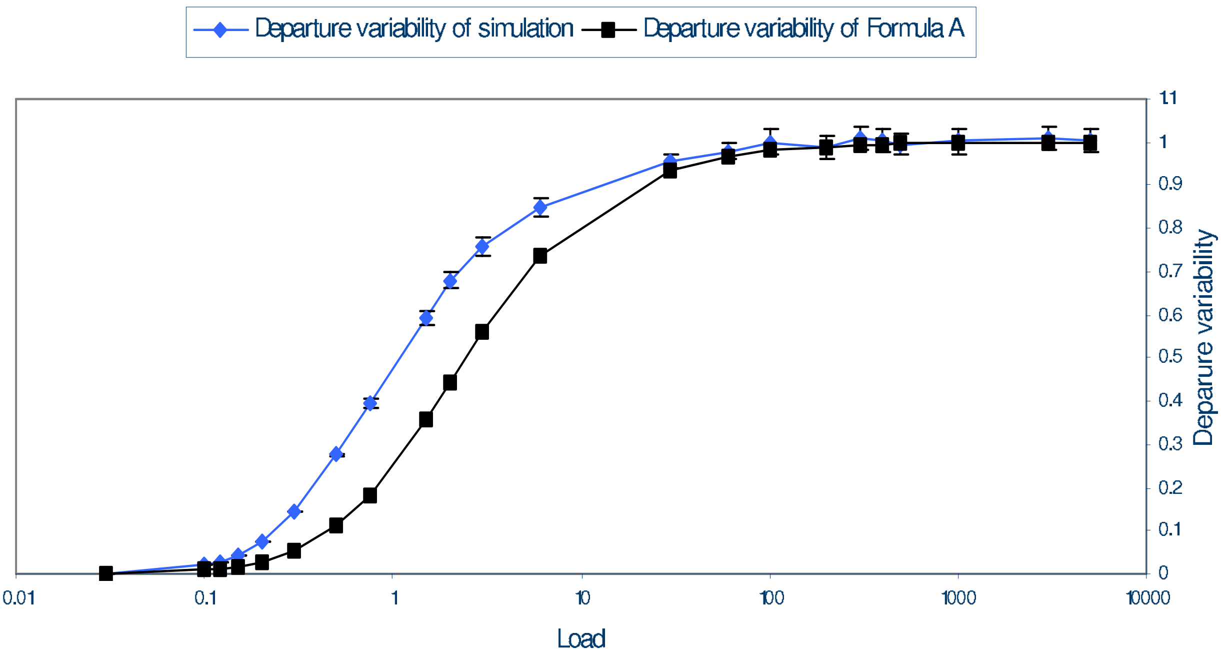

Figure 1. Departure variability results for Set DE.

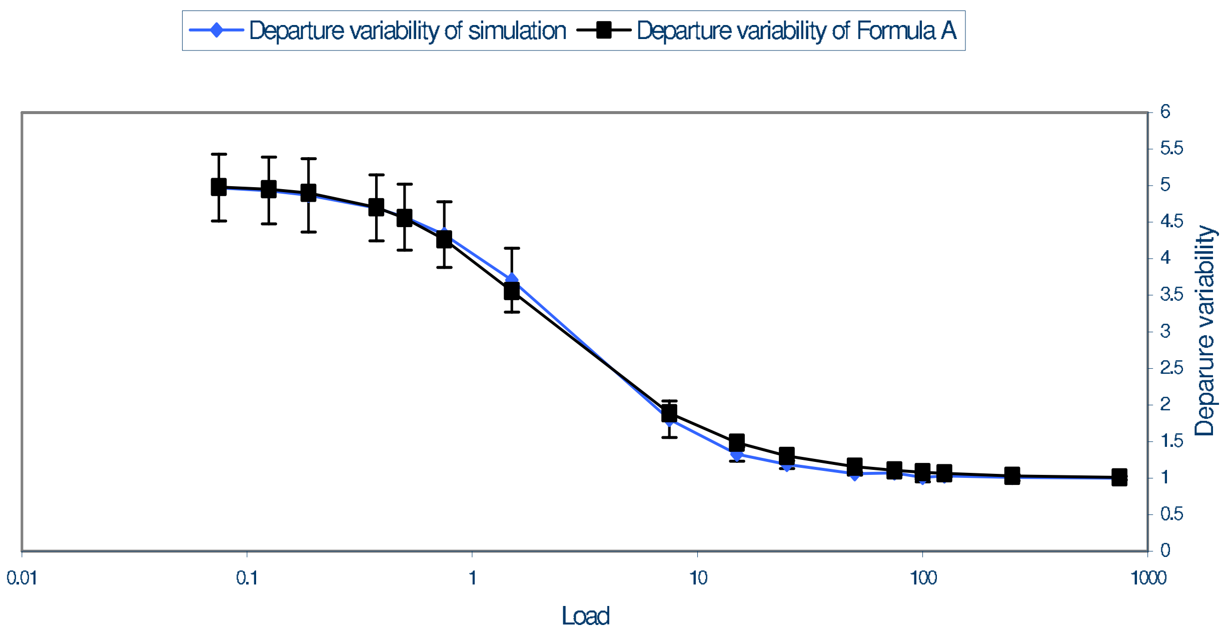

Figure 2. Departure variability results for Set GE.

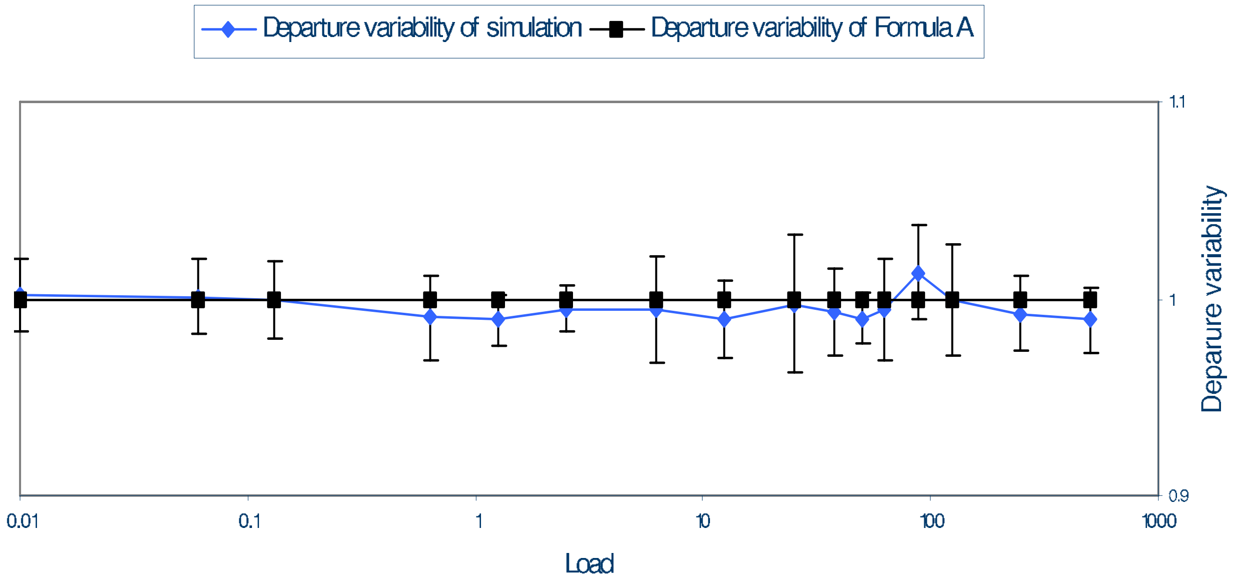

Figure 3. Departure variability results for Set EG.

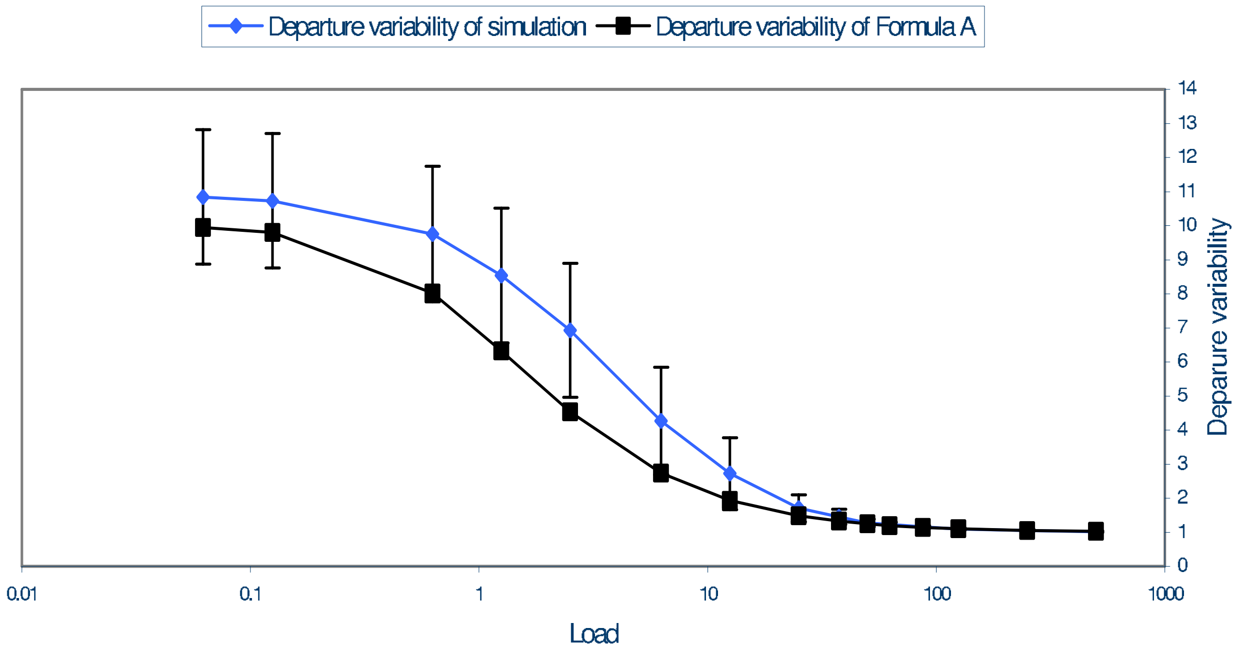

Figure 4. Departure variability results for Set GG.

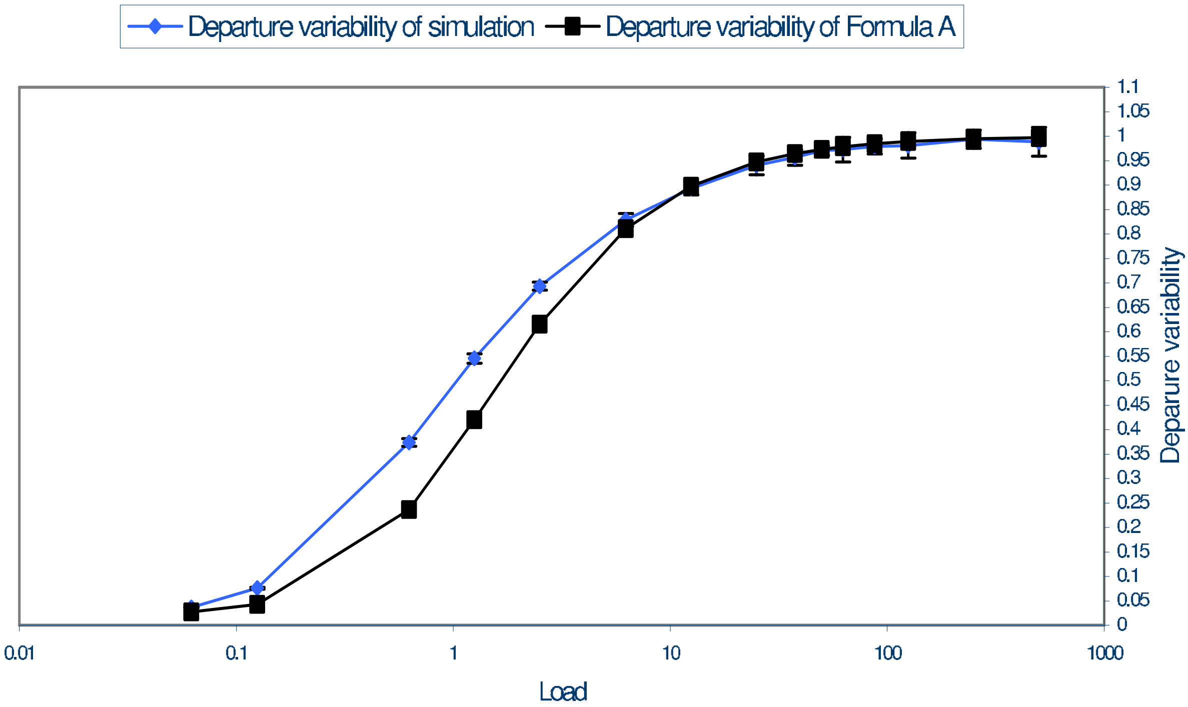

Figure 5. Departure variability results for Set UG.

6. Batch formation

In this case, residents arrive in batches (*arrival batches*) to the workstation. The arriving batches may come from multiple workstations and may be of different sizes. Arriving residents are grouped into *process batches* of a given size to perform the process. There may be multiple servers that can process different batches in parallel. We assume that the process batches are larger than the arrival batches. In this domain, an example would be a workstation where residents must view an educational video. The process batch is the group of residents watching the video at the same time.

Arriving residents enter a batch formation queue. A process batch is formed whenever there are k_i residents waiting in this queue. These residents then leave this queue, and the newly formed process batch enters a process queue, where it waits for a server to process it.

We will use the following notation:

c_{ai}^2 = Aggregate batch arrival SCV at station i

c_{bi}^2 = Arrival SCV for process batches at station i (after being formed)

k_i = Processing batch size at station i

$\overline{K}_{Ai}$ = Average batch size of all batches that come to station i

$\overline{C}_{Ai}^2$ = SCV of the batch size of all batches that come to station *i*

$\overline{K}_{Bji}$ = Average batch size of batches that come to station *i* from station *j*

λ_{Bji} = Batch flow rate from station *j* to station *i* (batches per minute)

c_{Bji}^2 = SCV of the inter-arrival times for batches that come to station *i* from station *j*

A key quantity for estimating the performance of such a workstation is the variability associated with the formation of process batches. The time between two consecutive process batches forming is a random variable with a SCV of c_{bi}^2, which we call the *batch formation variability*. There is no established estimate for this term. Thus, we developed and tested four different estimates X_i^h, for $h = 1, 2, 3$, and 4.

Next, we consider two special cases. First, if all of the residents arrive individually, then it is easy to see that the variability is pooled:

$$c_{bi}^2 = \frac{c_{ai}^2}{k_i}$$

Second, if all of the arrival batches have exactly $\overline{K}_{Ai}$ residents, then each process batch has exactly $k_i / \overline{K}_{Ai}$ arrival batches:

$$c_{bi}^2 = X_i^1 = \frac{\overline{K}_{Ai} c_{ai}^2}{k_i}$$

In general, however, the size of the arrival batches varies and has SCV of $\overline{C}_{Ai}^2$. Thus, in the general case, the above equation is only an approximation.

Intuitively it is clear that the arrival batch size variability $\overline{C}_{Ai}^2$ affects the batch formation variability. Therefore, we decided to create and test a second estimate:

$$X_i^2 = \frac{\overline{K}_{Ai}\left(c_{ai}^2 + \overline{C}_{Ai}^2\right)}{k_i}$$

The next section will discuss the results of these tests.

7. Initial batch formation experiments

To evaluate these first two estimates, we conducted sets of computational experiments using a discrete-event simulation model of the station. Each simulation replication was 150,000 to 600,000 minutes long, with a warm-up period of 100,000 to 400,000 minutes. Ten replications were conducted for each scenario. Simulation results are shown as 95% confidence intervals.

Initially, seven scenarios were tested. In all of the scenarios, three workstations (1, 2, and 3) sent batches to a fourth workstation, which is the workstation of interest. This forms three arrival streams, one from each workstation. In this set of scenarios, the batch size for each arrival stream is a constant (that is, all of the batches in each arrival stream has the same number of residents), and the interarrival times are exponentially distributed. The batch sizes and mean interarrival times for each stream were changed. The process batch size k_i varied as well. Table 12 describes the seven scenarios, and Table 13 describes the results for the scenarios.

Table 12. Description of Scenarios 1 to 7.

Scenario	k_i	Mean interrarrival times (mins)			Arrival batch size		
		1	2	3	1	2	3
1	10	6	7	10	1	2	2
2	10	6	7	10	1	3	5
3	10	6	7	10	2	4	6
4	10	6	7	10	8	1	5
5	15	6	4	10	8	7	6
6	30	6	4	10	6	2	12
7	30	6	4	10	3	11	7

Table 13. Results for Scenarios 1 to 7.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
1	0.159	0.174	0.173	0.165	0.182	8.128%	0.609%
2	0.267	0.361	0.37	0.361	0.379	27.718%	2.468%
3	0.367	0.435	0.452	0.443	0.461	18.689%	3.641%
4	0.483	0.673	0.722	0.713	0.732	33.183%	6.746%
5	0.467	0.474	0.504	0.496	0.512	7.377%	6.002%
6	0.236	0.313	0.325	0.318	0.332	27.277%	3.753%
7	0.221	0.272	0.283	0.275	0.292	21.907%	4.088%

Next, we tested scenarios in which the interarrival time distributions of each arrival stream in Scenario 4 were changed in order to vary the variability in each arrival stream. The mean interarrival times and other parameters remained as specified for Scenario 4, and the other two arrival streams kept exponentially distributed interarrival times. Scenarios 4.1.1 to 4.1.8 changed the first arrival stream as shown in Table 14. (Note Scenario 4.1.4 is the same as the original Scenario 4.)

Table 14. Description of Scenarios 4.1.1 to 4.1.8.

Scenario	Interrarrival time distribution	Arrival variability (SCV)
4.1.1	Constant	0
4.1.2	Gamma(2, 3.5)	0.5
4.1.3	Gamma(4/3, 21/4)	0.75
4.1.4	Exponential	1
4.1.5	Gamma(2/3, 21/2)	1.5
4.1.6	Gamma(1/2, 14)	2
4.1.7	Gamma(1/4, 28)	4
4.1.8	Gamma(1/5, 35)	5

Scenarios 4.2.1 through 4.2.8 modified the interarrival time distributions of the second arrival stream to increase the arrival variability in the same way. The mean interarrival time remained 7 minutes for these eight scenarios. Likewise, Scenarios 4.3.1 through 4.3.8 modified the interarrival time distributions of the third arrival stream to increase the arrival variability in the same way. The mean interarrival time remained 10 minutes for these eight scenarios.

Table 15. Results for Scenarios 4.1.1 to 4.1.8.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
4.1.1	0.29	0.48	0.21	0.201	0.219	36.5%	127.6%
4.1.2	0.38	0.58	0.50	0.495	0.513	23.7%	14.2%
4.1.3	0.43	0.62	0.63	0.620	0.638	31.1%	0.7%
4.1.4	0.48	0.67	0.72	0.713	0.731	33.2%	6.7%
4.1.5	0.58	0.77	0.90	0.893	0.911	35.6%	14.4%
4.1.6	0.68	0.87	1.04	1.027	1.045	34.5%	16.0%
4.1.7	1.07	1.26	1.36	1.355	1.373	21.4%	7.4%
4.1.8	1.27	1.46	1.48	1.472	1.490	14.4%	1.5%

Table 16. Results for Scenarios 4.2.1 to 4.2.8.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
4.2.1	0.31	0.51	0.717	0.708	0.726	56.2%	29.5%
4.2.2	0.40	0.59	0.707	0.698	0.716	43.7%	16.7%
4.2.3	0.44	0.63	0.727	0.718	0.736	39.4%	13.1%
4.2.4	0.48	0.67	0.722	0.713	0.731	33.2%	6.7%
4.2.5	0.57	0.76	0.739	0.730	0.748	23.3%	2.5%
4.2.6	0.65	0.84	0.737	0.728	0.745	11.6%	14.3%
4.2.7	0.99	1.18	0.754	0.745	0.763	31.0%	56.4%
4.2.8	1.16	1.35	0.743	0.714	0.771	55.6%	81.4%

Table 17. Results for Scenarios 4.3.1 to 4.3.8.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
4.3.1	0.36	0.56	0.62	0.612	0.636	41.5%	10.9%
4.3.2	0.42	0.61	0.68	0.660	0.701	37.7%	9.6%
4.3.3	0.45	0.64	0.70	0.683	0.715	35.2%	7.9%
4.3.4	0.48	0.67	0.73	0.716	0.742	33.8%	7.6%
4.3.5	0.54	0.73	0.77	0.756	0.785	29.7%	5.0%
4.3.6	0.60	0.79	0.78	0.734	0.831	23.3%	1.1%
4.3.7	0.84	1.03	0.86	0.817	0.901	2.7%	19.5%
4.3.8	0.95	1.14	0.90	0.861	0.935	6.2%	27.5%

Scenarios 7.1.1 through 7.1.8 modified the interarrival time distributions of the first arrival stream in Scenario 7 to increase the arrival variability, but the mean interarrival time remained 6 minutes. Scenarios 7.2.1 through 7.2.8 modified the interarrival time distributions of the second arrival stream to increase the arrival variability, but the mean interarrival time remained 4 minutes.

Table 18. Results for Scenarios 7.1.1 to 7.1.8.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
7.1.1	0.18	0.23	0.27	0.257	0.275	33.5%	14.5%
7.1.2	0.20	0.25	0.27	0.266	0.280	27.1%	8.6%
7.1.3	0.21	0.26	0.28	0.273	0.297	26.2%	8.5%
7.1.4	0.22	0.27	0.27	0.268	0.282	19.5%	1.2%
7.1.5	0.24	0.29	0.29	0.257	0.325	16.3%	1.0%
7.1.6	0.27	0.32	0.29	0.276	0.308	9.2%	8.0%
7.1.7	0.35	0.40	0.31	0.290	0.332	14.0%	30.2%
7.1.8	0.40	0.45	0.31	0.294	0.330	27.6%	43.7%

Table 19. Results for Scenarios 7.2.1 to 7.2.8.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
7.2.1	0.19	0.24	0.126	0.118	0.134	49.3%	89.3%
7.2.2	0.20	0.26	0.213	0.204	0.222	4.0%	19.7%
7.2.3	0.21	0.26	0.251	0.242	0.260	15.1%	5.0%
7.2.4	0.22	0.27	0.283	0.271	0.295	21.8%	3.9%
7.2.5	0.24	0.29	0.337	0.328	0.347	29.5%	14.5%
7.2.6	0.25	0.30	0.377	0.354	0.400	32.5%	19.1%
7.2.7	0.32	0.37	0.474	0.466	0.481	32.3%	21.6%
7.2.8	0.35	0.40	0.504	0.502	0.506	29.8%	19.8%

The next step was to look at the impact of varying the arrival rates. To do this, we created Scenarios 4.4.1 through 4.4.5 and Scenarios 4.5.1 through 4.5.5 from the original Scenario 4. The interarrival time distributions of the first and second arrival streams remained as exponential distributions. For Scenarios 4.4.1 through 4.4.5, the interarrival times for the third arrival stream were constant. For Scenarios 4.5.1 through 4.5.5, the interarrival times for the third arrival stream had a gamma distribution with alpha equal to 0.5. Thus, the arrival SCV equals 2. The mean interarrival times were varied as shown in Table 20.

Table 20. Description of Scenarios 4.4.1 to 4.4.5 and Scenarios 4.5.1 through 4.5.5.

	Interrarrival time means (mins)		
Scenario	Arrival stream 1	Arrival stream 2	Arrival stream 3
4.4.1 (4.5.1)	6	10	10
4.4.2 (4.5.2)	7	5	10
4.4.3 (4.5.3)	6	7	10
4.4.4 (4.5.4)	15	10	6
4.4.5 (4.5.5)	10	15	5

In addition, we created Scenarios 7.3.1 through 7.3.5 and Scenarios 7.4.1 through 7.4.5 from the original Scenario 7. In Scenarios 7.3.1 to 7.3.5, the interarrival time distributions of the second and third arrival streams remained as exponential distributions, but the interarrival times for the first arrival stream had a gamma distribution with alpha equal to 2. In Scenarios 7.4.1 to 7.4.5, the interarrival time distributions of the first and third arrival streams remained as exponential distributions, but the interarrival times for the second arrival stream had a gamma distribution with alpha equal to 2/3. The mean interarrival times were varied as shown in Table 21.

Table 21. Description of Scenarios 7.3.1 to 7.3.5 and Scenarios 7.4.1 to 7.4.5.

	Interrarrival time means (mins)		
Scenario	Arrival stream 1	Arrival stream 2	Arrival stream 3
7.3.1 (7.4.1)	3	12	3
7.3.2 (7.4.2)	6	6	12
7.3.3 (7.4.3)	12	10	20/3
7.3.4 (7.4.4)	4	20	15
7.3.5 (7.4.5)	12	4	15

Table 22. Results for Scenarios 4.4.1 to 4.4.5.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
4.4.1	0.38	0.54	0.63	0.623	0.641	39.141%	13.993%
4.4.2	0.33	0.55	0.57	0.558	0.576	42.506%	2.572%
4.4.3	0.60	0.79	0.62	0.611	0.636	3.732%	26.886%
4.4.4	0.29	0.43	0.31	0.295	0.319	5.564%	40.660%
4.4.5	0.24	0.34	0.35	0.331	0.375	32.410%	2.595%

Table 23. Results for Scenarios 4.5.1 to 4.5.5.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
4.5.1	0.67	0.83	0.81	0.800	0.829	17.796%	1.707%
4.5.2	0.51	0.73	0.74	0.730	0.758	31.971%	1.526%
4.5.3	0.60	0.79	0.77	0.752	0.796	22.463%	2.201%
4.5.4	0.59	0.73	0.81	0.801	0.819	27.133%	9.625%
4.5.5	0.78	0.88	0.88	0.869	0.884	11.043%	0.958%

Table 24. Results for Scenarios 7.3.1 to 7.3.5.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
7.3.1	0.20	0.25	0.27	0.249	0.293	26.611%	8.006%
7.3.2	0.22	0.28	0.30	0.283	0.318	25.971%	5.678%
7.3.3	0.23	0.27	0.29	0.278	0.302	19.323%	5.334%
7.3.4	0.14	0.20	0.19	0.184	0.202	25.015%	4.874%
7.3.5	0.28	0.32	0.35	0.338	0.353	19.547%	7.798%

Table 25. Results for Scenarios 7.4.1 to 7.4.5.

Scenario	Batch formation variability estimates		Confidence interval on batch formation variability from simulation results			Relative errors	
	X_i^1	X_i^2	Midpoint	Lower limit	Upper limit	X_i^1	X_i^2
7.4.1	0.24	0.29	0.26	0.193	0.317	6.749%	13.046%
7.4.2	0.24	0.31	0.41	0.383	0.427	39.675%	24.625%
7.4.3	0.25	0.29	0.34	0.332	0.356	28.100%	16.311%
7.4.4	0.16	0.22	0.25	0.217	0.275	33.411%	9.983%
7.4.5	0.32	0.36	0.49	0.472	0.500	33.833%	25.479%

8. Discussion of initial batch formation experiments

Based on these results, we see that X_i^1, the first estimate for batch formation variability, is generally much worse than X_i^2, the second estimate for batch formation variability. The latter estimate is, however, only acceptable when all of the arrival streams have interarrival time distributions with moderate variability, which occurs in Scenarios 1 to 7, Scenarios 4.1.3 to 4.1.5, Scenarios 4.2.3 to 4.2.5, Scenarios 4.3.3 to 4.3.5, Scenarios 7.1.3 to 7.1.5, and Scenarios 7.2.3 to 7.2.5.

Clearly, changes to arrival variability affect the batch formation variability. Moreover, changes in the arrival variability of smaller batches have less impact than changes in the arrival variability of larger batches. For example, the batch formation variability changes much more across Scenarios 4.1.1 to 4.1.8 (which modifies the arrival stream with the largest batch size) than it does across Scenarios 4.2.1 to 4.2.8, which modifies the arrival stream with the smallest batch size.

Similarly, the batch formation variability changes much more across Scenarios 7.2.1 to 7.2.8 (which modifies the arrival stream with the largest batch size) than it does across Scenarios 7.1.1 to 7.1.8, which modifies the arrival stream with the smallest batch size.

However, X_i^2, the second estimate for batch formation variability, does not include information about the batch sizes. Based on these observations, we developed two more estimates that replace the aggregate batch arrival variability term with terms that explicitly incorporate batch size information:

$$X_i^3 = \frac{\overline{K}_{Ai}}{k_i}\left(\frac{\sum_{j \in S_i} \overline{K}_{Bji}\lambda_{Bji}c_{Bji}^2}{\sum_{j \in S_i} \overline{K}_{Bji}\lambda_{Bji}} + \overline{C}_{Ai}^2\right)$$

$$X_i^4 = \frac{\overline{K}_{Ai}}{k_i}\left(\frac{\sum_{j \in S_i} \overline{K}_{Bji}^2\lambda_{Bji}c_{Bji}^2}{\sum_{j \in S_i} \overline{K}_{Bji}^2\lambda_{Bji}} + \overline{C}_{Ai}^2\right)$$

9. Evaluation of additional estimates

To evaluate these two new estimates, we calculated them for the scenarios discussed in Section 2. Tables 26 to 34 show the results, along with the X_i^2 estimates previously calculated. These results show the fourth estimate X_i^4 is more accurate than the others. It is especially good with the arrival variability is moderate (between 0.5 and 1.5). Figures 6 to 8 show the results graphically. (In these graphs, "Estimate 2" refers to X_i^2, "Estimate 3" refers to X_i^3, and "Estimate 4" refers to X_i^4.)

Table 26. Results for Scenarios 4.1.1 to 4.1.8.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
4.1.1	0.48	0.348	0.287	127.6%	65.8%	36.6%
4.1.2	0.58	0.511	0.480	14.2%	2.2%	3.9%
4.1.3	0.62	0.592	0.577	0.7%	6.0%	8.4%
4.1.4	0.67	0.674	0.674	6.7%	6.4%	6.4%
4.1.5	0.77	0.837	0.867	14.4%	7.1%	3.7%
4.1.6	0.87	0.999	1.060	16.0%	3.9%	2.0%
4.1.7	1.26	1.651	1.834	7.4%	21.4%	34.9%
4.1.8	1.46	1.976	2.221	1.5%	33.5%	50.1%

Table 27. Results for Scenarios 4.2.1 to 4.2.8.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
4.2.1	0.51	0.639	0.669	29.5%	10.9%	6.7%
4.2.2	0.59	0.656	0.671	16.7%	7.2%	5.1%
4.2.3	0.63	0.665	0.672	13.1%	8.5%	7.5%
4.2.4	0.67	0.674	0.674	6.7%	6.7%	6.7%
4.2.5	0.76	0.691	0.676	2.5%	6.5%	8.5%
4.2.6	0.84	0.709	0.679	14.3%	3.9%	7.9%
4.2.7	1.18	0.778	0.689	56.4%	3.2%	8.6%
4.2.8	1.35	0.813	0.694	81.4%	9.5%	6.5%

Table 28. Results for Scenarios 4.3.1 to 4.3.8.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
4.3.1	0.56	0.552	0.583	10.9%	11.0%	6.0%
4.3.2	0.61	0.613	0.628	9.6%	9.9%	7.6%
4.3.3	0.64	0.643	0.651	7.9%	8.1%	7.0%
4.3.4	0.67	0.674	0.674	7.6%	7.7%	7.7%
4.3.5	0.73	0.735	0.719	5.0%	4.6%	6.6%
4.3.6	0.79	0.796	0.764	1.1%	2.0%	2.0%
4.3.7	1.03	1.040	0.946	19.5%	20.9%	10.0%
4.3.8	1.14	1.162	1.036	27.5%	29.1%	15.1%

Table 29. Results for Scenarios 7.1.1 to 7.1.8.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
7.1.1	0.23	0.24	0.26	14.5%	12.9%	4.4%
7.1.2	0.25	0.25	0.26	8.6%	6.1%	1.9%
7.1.3	0.26	0.26	0.27	8.5%	6.2%	4.2%
7.1.4	0.27	0.27	0.27	1.2%	0.6%	0.6%
7.1.5	0.29	0.29	0.28	1.0%	0.1%	4.0%
7.1.6	0.32	0.31	0.29	8.0%	6.2%	1.7%
7.1.7	0.40	0.38	0.31	30.2%	22.8%	0.6%
7.1.8	0.45	0.42	0.33	43.7%	34.5%	4.9%

Table 30. Results for Scenarios 7.2.1 to 7.2.8.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
7.2.1	0.24	0.17	0.14	89.3%	36.2%	8.5%
7.2.2	0.26	0.22	0.20	19.7%	4.1%	4.2%
7.2.3	0.26	0.25	0.24	5.0%	1.7%	5.2%
7.2.4	0.27	0.27	0.27	3.9%	4.0%	4.0%
7.2.5	0.29	0.32	0.34	14.5%	4.6%	0.6%
7.2.6	0.30	0.37	0.41	19.1%	1.4%	7.8%
7.2.7	0.37	0.57	0.68	21.6%	20.6%	42.7%
7.2.8	0.40	0.67	0.81	19.8%	33.3%	61.0%

Table 31. Results for Scenarios 4.4.1 to 4.4.5.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
4.4.1	0.54	0.55	0.586	13.9%	12.9%	7.0%
4.4.2	0.55	0.53	0.553	2.6%	7.4%	3.0%
4.4.3	0.79	0.55	0.585	26.9%	10.6%	5.6%
4.4.4	0.43	0.33	0.366	40.7%	6.7%	18.1%
4.4.5	0.34	0.34	0.394	2.6%	1.9%	12.6%

Table 32. Results for Scenarios 4.5.1 to 4.5.5.

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
4.5.1	0.83	0.82	0.784	1.7%	1.4%	3.2%
4.5.2	0.73	0.75	0.728	1.5%	1.8%	1.6%
4.5.3	0.79	0.80	0.767	2.2%	3.7%	0.4%
4.5.4	0.73	0.83	0.796	9.6%	2.6%	1.8%
4.5.5	0.88	0.89	0.838	0.9%	1.0%	4.8%

Table 33. Results for Scenarios 7.3.1 to 7.3.5

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
7.3.1	0.25	0.25	0.26	8.0%	8.8%	3.4%
7.3.2	0.28	0.27	0.29	5.7%	8.7%	4.3%
7.3.3	0.27	0.27	0.28	5.4%	7.5%	4.7%
7.3.4	0.20	0.18	0.20	4.9%	2.9%	6.8%
7.3.5	0.32	0.32	0.33	7.8%	8.8%	6.8%

Table 34. Results for Scenarios 7.4.1 to 7.4.5

Scenario	Batch formation variability estimates			Relative errors		
	X_i^2	X_i^3	X_i^4	X_i^2	X_i^3	X_i^4
7.4.1	0.29	0.30	0.31	13%	13.9%	19.3%
7.4.2	0.31	0.37	0.39	24.7%	10.4%	6.0%
7.4.3	0.29	0.34	0.35	16.3%	1.2%	3.7%
7.4.4	0.22	0.24	0.26	9.9%	2.6%	4.2%
7.4.5	0.36	0.44	0.46	25.5%	9.4%	6.7%

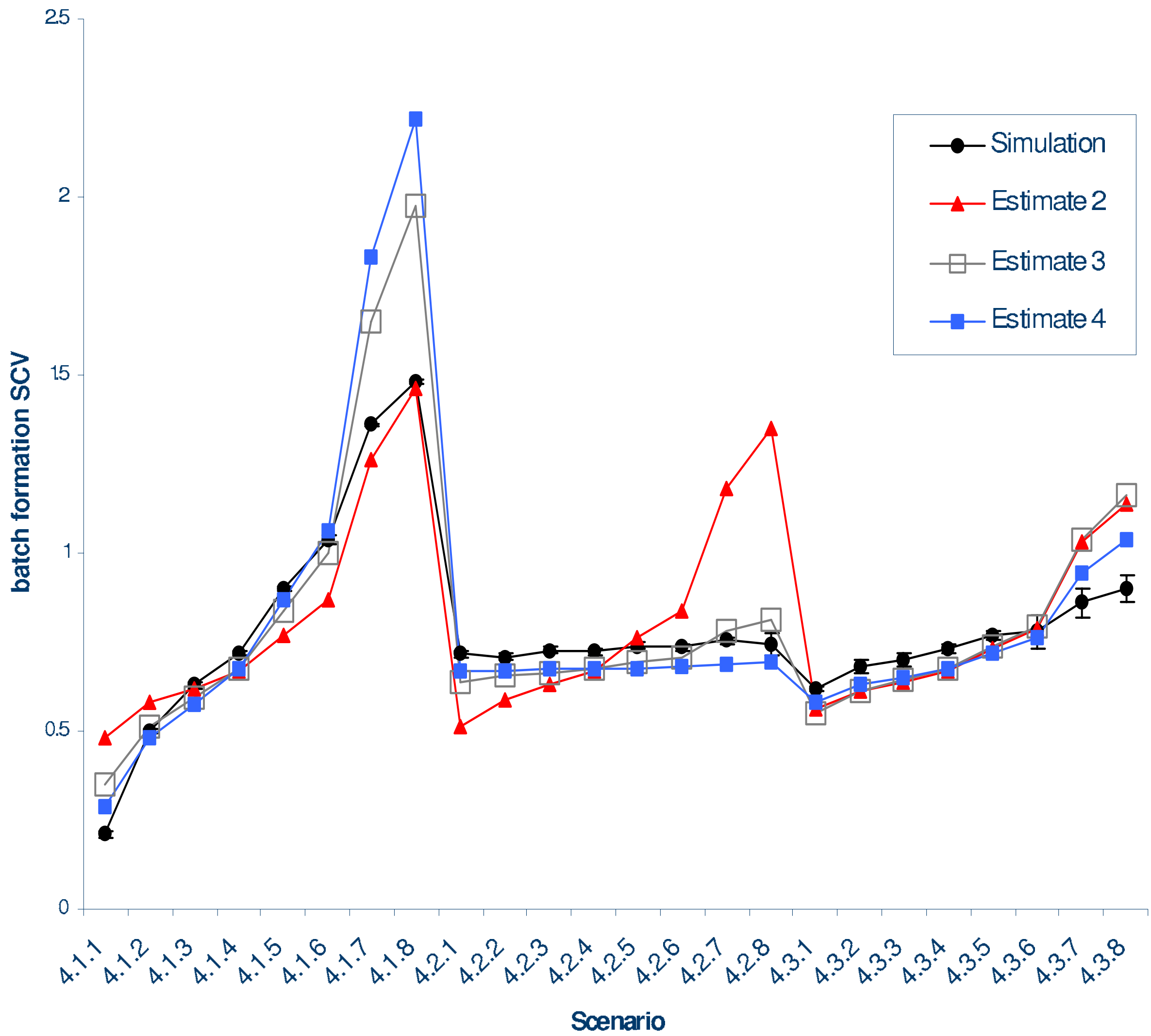

Figure 6. Results for Scenarios 4.1.1 to 4.3.8

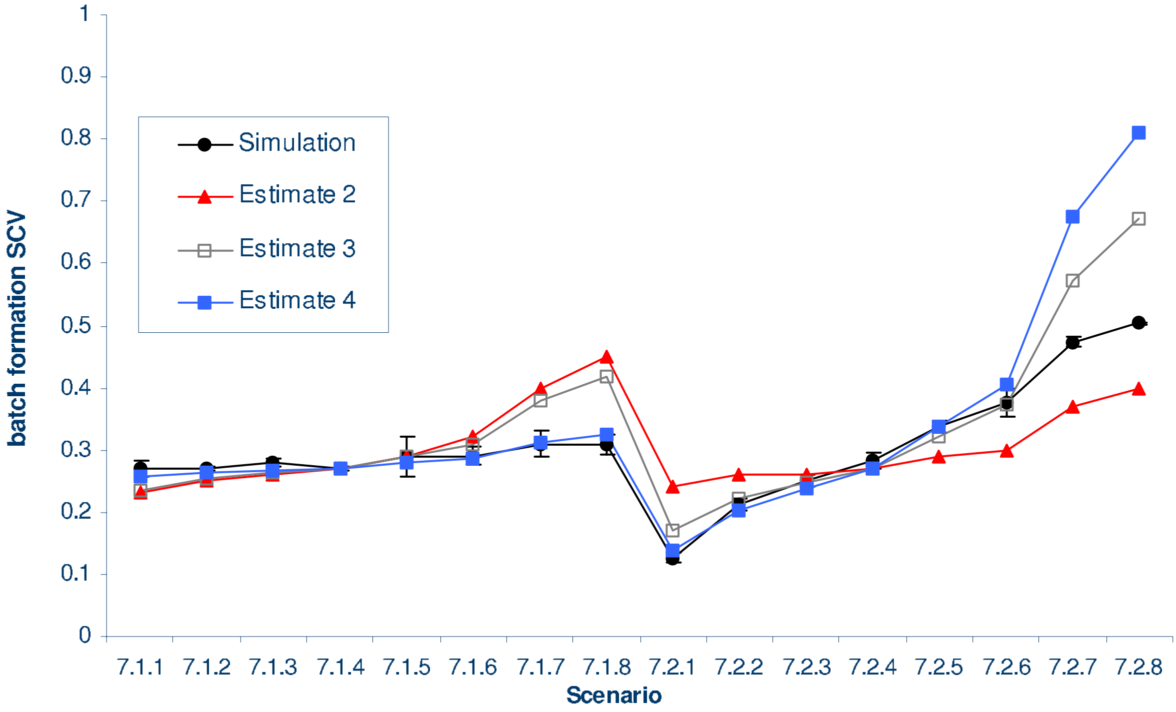

Figure 7. Results for Scenarios 7.1.1 to 7.2.8

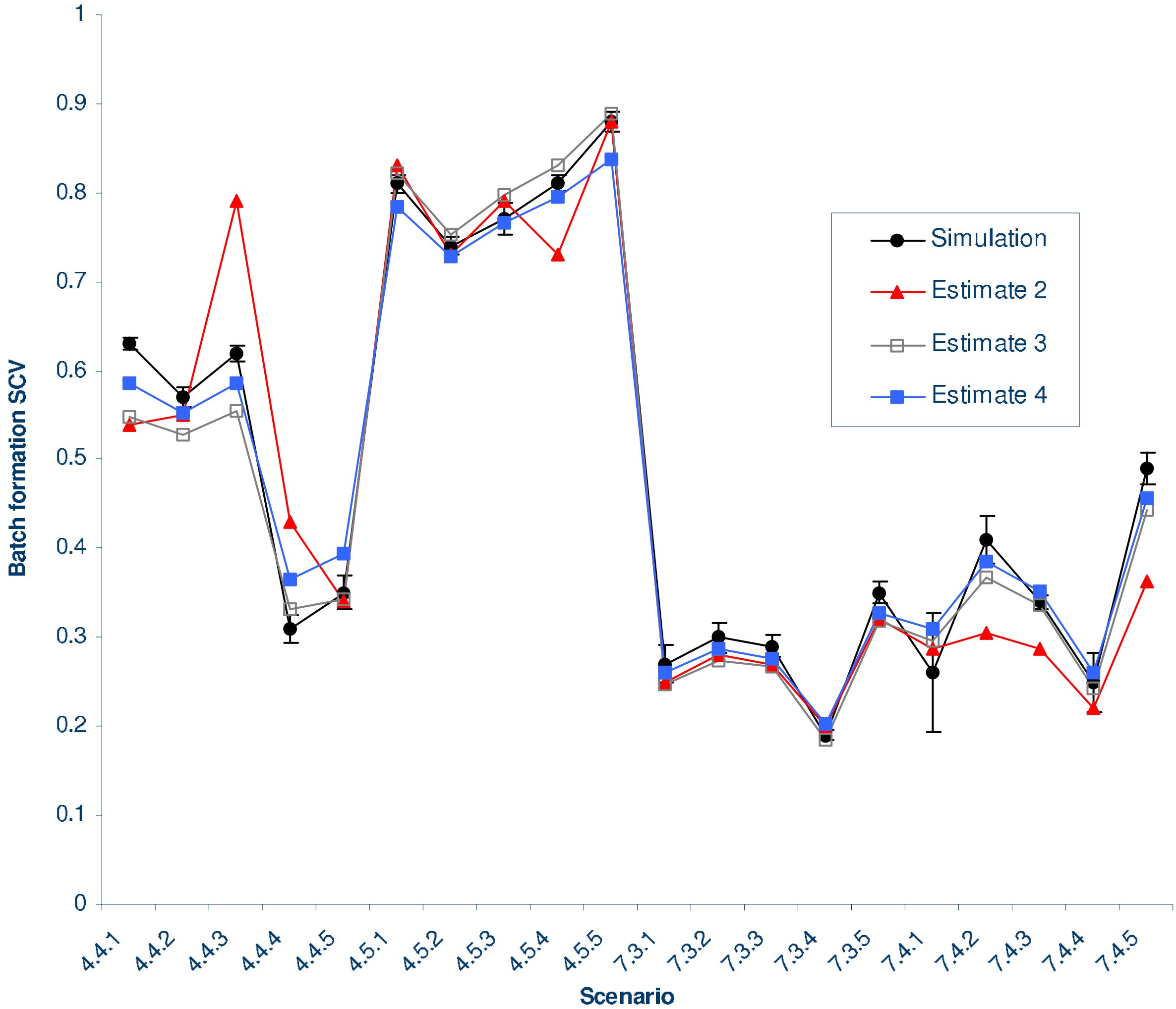

Figure 8. Results for Scenarios 4.4.1 to 4.5.5 and Scenarios 7.3.1 to 7.4.5

10. Summary and Conclusions

This paper has presented the results of computational experiments completed to evaluate different estimates for wait-in-batch-time, departure variabiility, and batch formation variability. While these results suggest that some approximations are better than others, we cannot guarantee the accuracy of any. Additional work would be useful to characterize their accuracy in other scenarios and to seek better approximations for those scenarios where they perform poorly.

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