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By

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Abstract

We propose a method to compute the spectral norm distance from a given matrix A to the set of matrices having at least an eigenvalue on the imaginary axis. It is shown that the distance is one of the roots of a suitably constructed polynomial in one variable. Our method can be easily generalized to compute the distance from A to the set of matrices having at least an eigenvalue on any straight line or circle. Thus, it can be applied to compute the distance from a stable matrix to the set of unstable matrices in either continuous or discrete sense.

1. Introduction

Let $\mathcal{W} \subset \mathbb{C}^{n\times n}$ be the set of matrices with at least one eigenvalue on the imaginary axis. For any $A \in \mathbb{C}^{n\times n}$, the distance from A to $\mathcal{W}$ is defined by [1]

$$\beta(A) = \min_{\Delta A \in \mathbb{C}^{n\times n}} \{\|\Delta A\| : A + \Delta A \in \mathcal{W}\} , \tag{1}$$

where $\|\cdot\|$ denotes the spectral norm. If A is stable in the sense that all its eigenvalues have negative real part, then $\beta(A)$ is the distance to the set of unstable matrices [1]. $\beta(A)$ is a measure of how "nearly unstable" the stable matrix A is. It can be shown that

$$\begin{aligned}\beta(A) &= \min_{\omega\in\mathbb{R}} \underline{\sigma}(A - j\omega I) \\ &= \frac{1}{\max\limits_{\omega\in\mathbb{R}} \overline{\sigma}\left((j\omega I - A)^{-1}\right)} ,\end{aligned} \tag{2}$$

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where $\underline{\sigma}(\cdot)$ and $\overline{\sigma}(\cdot)$ are the smallest and largest singular values, respectively, j the square root of -1, and I the $n \times n$ identity matrix. Therefore, $\beta(A)$ is also the reciprocal of the L_∞-norm (see, e.g., [2]) of the linear dynamic system

$$\dot{x} = Ax + u$$

$$y = x \ .$$

Applying direct local optimization to the function $a(\omega) = \underline{\sigma}(A - j\omega I)$ is not suitable for computing $\beta(A)$, since $a(\omega)$ may have local minima which are not global. In [1], Van Loan proposed a very simple algorithm to compute $\beta(A)$ which relies on a conjecture about the location of the eigenvalues of A. Demmel [3] later invalidated it by providing a counterexample to the conjecture. The most reliable method currently available for computing $\beta(A)$ may be the bisection method proposed by Byers [4] and Boyd *et. al* [5]. The bisection method iteratively gives upper and lower bounds on $\beta(A)$ and it is not affected by the number of local solutions $a(\omega)$ may have, nor does it require any starting value. However, it converges too slowly if high accuracy for the solution is desired.

In this paper, we propose a new method for computing $\beta(A)$. The method is based on the result (Theorem 1) that a certain necessary condition satisfied by the global minimizer of the optimization problem in (2) has only a *finite* number of solutions. Therefore, the set $\mathbb{R}$ in (2) could be replaced by a finite set in $\mathbb{R}$ without affecting the solution; and thus the computation of $\beta(A)$ becomes trivial. We will further show that this set needs not even be identified explicitly. The result in this paper borrows some ideas from [6].

The organization of the paper is as follows. In Section 2, we present the main results and summarize the section by giving an implementable procedure for computing $\beta(A)$ with a few remarks. The generalization of the method for applying to stability defined in different sense is discussed in Section 3. Finally, several numerical examples are provided in Section 4.

2. Main results

Let $A \in \mathbb{C}^{n\times n}$. In view of (2), $\beta^2(A)$ is the smallest possible eigenvalue the positive semidefinite matrix $(A - j\omega I)^H(A - j\omega I)$ may have, while ω varies in $\mathbb{R}$. Here, the superscript "H" denotes the complex conjugate and transpose. Therefore, $\beta^2(A)$ is also the smallest possible real value of α that satisfies $\det((A-j\omega I)^H(A-j\omega I)-\alpha I)=0$ for some $\omega \in \mathbb{R}$. As will be shown in Theorem 1, this optimality of $\beta(A)$ allows us to establish some simple necessary conditions on $\beta(A)$. Notice that $\det((A-j\omega I)^H(A-j\omega I)-\alpha I)$ is an element in $\mathbb{R}[\alpha,\omega]$, the ring of all polynomials in α and ω with real coefficients. If every element of A has real and imaginary parts in the set $\mathbb{Q}$ of all rational numbers, then $\det((A-j\omega I)^H(A-j\omega I)-\alpha I)$ is also an element of the ring $\mathbb{Q}[\alpha,\omega]$ of polynomials in α and ω with coefficients in $\mathbb{Q}$. For simplicity, let $\mathcal{D}$ denote $\mathbb{R}$ or $\mathbb{Q}$. A nonconstant element f in $\mathcal{D}[\alpha,\omega]$ is said to be *irreducible over $\mathcal{D}$* if $f = gh$ for some $g, h \in \mathcal{D}[\alpha,\omega]$ implies either g or h is a constant (see, e.g., [7]). Any $f(\alpha,\omega) \in \mathcal{D}[\alpha,\omega]$ can be written as

$$f(\alpha,\omega) = a_0\omega^m + a_1\omega^{m-1} + \cdots + a_m$$

for some nonnegative integer m, where a_i's are polynomials in α with coefficients in $\mathcal{D}$. The partial derivative of $f(\alpha,\omega)$ with respect to ω is of a similar form:

$$\frac{\partial f(\alpha,\omega)}{\partial\omega} = ma_0\omega^{m-1} + (m-1)a_1\omega^{m-2} + \cdots + a_{m-1} \ .$$

If $m \geq 1$, the Sylvester resultant (see, e.g., [7]) of $f(\alpha,\omega)$ and $\frac{\partial f(\alpha,\omega)}{\partial\omega}$, denoted here by $\Gamma_f(\alpha)$, is the determinant of the $(2m-1)\times(2m-1)$ Sylvester matrix, namely, (shown for $m=3$)

$$\Gamma_f(\alpha) = \det\left(\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & 0 \\ 0 & a_0 & a_1 & a_2 & a_3 \\ 3a_0 & 2a_1 & a_2 & 0 & 0 \\ 0 & 3a_0 & 2a_1 & a_2 & 0 \\ 0 & 0 & 3a_0 & 2a_1 & a_2 \end{bmatrix}\right) .$$

When $m = 1$, we use the convention that $\Gamma_f(\alpha) = a_0$. Thus $\Gamma_f(\alpha)$ is a polynomial in α with coefficients in $\mathcal{D}$. We are now ready to present the main theorem of this paper.

Theorem 1. Let $A \in \mathbb{C}^{n\times n}$ and define

$$f(\alpha,\omega) = \det\left((A - j\omega I)^H(A - j\omega I) - \alpha I\right) .$$

Suppose that $f(\alpha,\omega) \in \mathcal{D}[\alpha,\omega]$ ($\mathcal{D} = \mathbb{R}$ or $\mathbb{Q}$) is factored as

$$f(\alpha,\omega) = c\prod_{i=1}^{k} g_i^{n_i}(\alpha,\omega) , \tag{3}$$

where $c \in \mathcal{D}$, g_i's are nonconstant and irreducible over $\mathcal{D}$, and k and n_i's positive integers. Then the following statements hold.

(a) For every $i = 1,\dots,k$,

 (i) $g_i(\alpha,\omega)$ has degree at least one in ω,

 (ii) the system of polynomial equations $g_i(\alpha,\omega) = 0$ and $\frac{\partial g_i(\alpha,\omega)}{\partial\omega} = 0$ has only a finite number of (possibly complex) solutions, and

 (iii) the Sylvester resultant $\Gamma_{g_i}(\alpha)$ is not identically zero.

(b) There exist $l \in \{1,\dots,k\}$ and $\omega^* \in \mathbb{R}$ such that $(\beta^2(A),\omega^*)$ solves the system of polynomial equations $g_l(\alpha,\omega) = 0$ and $\frac{\partial g_l(\alpha,\omega)}{\partial\omega} = 0$. For any such w^*,

$$\beta(A) = \underline{\sigma}(A - j\omega^* I) .$$

Furthermore, let

$$A - j\omega^* I = U\Sigma V^H$$

be a singular value decomposition of $A - j\omega^* I$. Then the matrix

$$\Delta A = -\beta(A)U\begin{bmatrix} A_0 & 0 \\ 0 & 1 \end{bmatrix} V^H \tag{4}$$

satisfies

$$\|\Delta A\| = \beta(A) \tag{5}$$

and

$$\det(A + \Delta A - j\omega^* I) = 0 \tag{6}$$

for any $A_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ with $\|A_0\| \leq 1$.

(c) Suppose $(\alpha_1, \omega_1) \in \mathbb{R}^2$ solves the system of polynomial equations $g_i(\alpha, \omega) = 0$ and $\frac{\partial g_i(\alpha,\omega)}{\partial \omega} = 0$ for some $i \in \{1, \ldots, k\}$. Then $\alpha_1 \geq \beta^2(A)$. $\square$

We will employ the following facts in proving Theorem 1.

Fact 1. [7] Any nonconstant element $f(\alpha, \omega) \in \mathcal{D}[\alpha, \omega]$ can be factored as

$$f(\alpha, \omega) = c \prod_{i=1}^{k} g_i^{n_i}(\alpha, \omega) ,$$

where $c \in \mathcal{D}$, g_i's are nonconstant and irreducible over $\mathcal{D}$, and k and n_i's positive integers. The constant term, c, can be dropped when $\mathcal{D} = \mathbb{R}$. $\square$

Fact 2. [7] Let $g(\alpha, \omega) \in \mathcal{D}[\alpha, \omega]$ have degree at least one in ω. Then the following statements hold.

(a) For any $\alpha_1 \in \mathbb{C}$, $\Gamma_g(\alpha_1) = 0$ if, and only if, (α_1, ω_1) solves the system of polynomial equations $g(\alpha, \omega) = 0$ and $\frac{\partial g(\alpha,\omega)}{\partial \omega} = 0$ for some $\omega_1 \in \mathbb{C}$.

(b) If, in addition, $g(\alpha, \omega)$ is irreducible over $\mathcal{D}$, then the Sylvester resultant $\Gamma_g(\alpha)$ is not identically zero, and the system of polynomial equations $g(\alpha, \omega) = 0$ and $\frac{\partial g(\alpha,\omega)}{\partial \omega} = 0$ has only a finite number of (possibly complex) solutions. $\square$

Implicit Function Theorem. (see, e.g., [8]) Let $S \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open set, and $f : S \to \mathbb{R}^n$ a differentiable function on S. Suppose $(x_0, y_0) \in S$ is such that $f(x_0, y_0) = 0$ and

$$\det\left(\frac{\partial f(x_0, y_0)}{\partial y}\right) \neq 0 ,$$

where $\frac{\partial f}{\partial y}$ denotes the Jacobian of f with respect to the variable y. Then there exist an open neighborhood Ω of x_0 in $\mathbb{R}^m$ and a continuous function $\phi : \Omega \to \mathbb{R}^n$ which satisfy $\phi(x_0) = y_0$ and $f(x, \phi(x)) = 0$ for all $x \in \Omega$. □

Proof of Theorem 1. (a) Notice that the factorization of $f(\alpha, \omega)$ in (3) is guaranteed by Fact 1. Let $i \in \{1, \dots, k\}$. Suppose g_i has degree zero in ω. Then, since g_i is nonconstant, there is some (possibly complex) α_i which satisfies $g_i(\alpha_i, \omega) = 0$ for all $\omega \in \mathbb{R}$. Hence $f(\alpha_i, \omega) = 0$ for all $\omega \in \mathbb{R}$. This implies that the matrix $(A - j\omega I)^H(A - j\omega I)$ has the eigenvalue α_i for all $\omega \in \mathbb{R}$, which is clearly impossible. Therefore (i) holds. Then (ii) and (iii) follow from Fact 2.

(b) Let ω^* satisfy $\beta(A) = \underline{\sigma}(A - j\omega^* I)$. Then $f(\beta^2(A), \omega^*) = 0$. Therefore $g_i(\beta^2(A), \omega^*) = 0$ for some $i \in \{1, \dots, k\}$. Suppose $\frac{\partial g_i(\beta^2(A), \omega^*)}{\partial \omega} \neq 0$. Then, by the Implicit Function Theorem, there is a neighborhood Ω of $\beta^2(A)$ in $\mathbb{R}$ and a function $\phi : \Omega \to \mathbb{R}$ such that $g_i(x, \phi(x)) = 0$ for all $x \in \Omega$. This implies $f(x, \phi(x)) = 0$ for all $x \in \Omega$. In particular, there exists $x \in \mathbb{R}$ such that $x < \beta^2(A)$ and

$$\det((A - j\omega I)^H(A - j\omega I) - xI) = 0$$

for some real ω. This contradicts to the fact that $\beta^2(A)$ is the smallest possible value of such x's. Hence $\frac{\partial g_i(\beta^2(A), \omega^*)}{\partial \omega} = 0$. Then it is straightforward to check that ΔA defined in (4) satisfies (5) and (6) for any $A_0 \in \mathbb{C}^{(n-1)\times(n-1)}$ with $\|A_0\| \leq 1$.

(c) If $g_l(\alpha_1, \omega_1) = 0$ for some $\alpha_1, \omega_1 \in \mathbb{R}$ and $l \in \{1, \dots, k\}$, then $f(\alpha_1, \omega_1) = 0$. Again, by the fact that $\beta^2(A)$ is the smallest real value α that makes $f(\alpha, \omega) = 0$ for some $\omega \in \mathbb{R}$, we have the result. □

The following Procedure for computing $\beta(A)$ is based on Theorem 1.

Procedure 1. (Computation of $\beta(A)$)

Step 0. Set $\alpha^* = \infty$ and $f(\alpha, \omega) = \det\big((A - j\omega I)^H(A - j\omega I) - \alpha I\big)$.

Step 1. Factor $f(\alpha,\omega)$ as

$$f(\alpha,\omega) = \prod_{i=1}^{k} g_i^{n_i}(\alpha,\omega) ,$$

where g_i's are nonconstant and irreducible in $\mathbb{R}[\alpha,\omega]$, k and n_i's positive integers.

Step 2. For each successive $i = 1,\ldots,k$, set α_i to be the smallest nonnegative real root of $\Gamma_{g_i}(\alpha) = 0$ for which $g_i(\alpha_i,\omega)$ and $\frac{\partial g_i(\alpha_i,\omega)}{\partial\omega}$ (as functions of ω) have at least one common *real* root, if any. If no such root exists, set $\alpha_i = \infty$. Then, set $\alpha^* = \min(\alpha^*,\alpha_i)$ and go back to the loop.

Step 3. $\beta(A) = \sqrt{\alpha^*}$. □

Remark 1. It can be easily checked that each nonconstant factor, $g_i(\alpha,\omega)$, of $f(\alpha,\omega)$ must have degree at least one in α as well. □

Remark 2. Let

$$g(\alpha,\omega) = \prod_{i=1}^{k} g_i(\alpha,\omega),$$

where g_i's are the irreducible factors of $f(\alpha,\omega)$. Since $g(\alpha,\omega)$ does not contain a square of any nonconstant irreducible factor, $\Gamma_g(\alpha)$ is not identically zero and the system of polynomial equations $g(\alpha,\omega) = \frac{\partial g(\alpha,\omega)}{\partial\omega} = 0$ has only a finite number of solutions [7]. Hence, instead of considering each $g_i(\alpha,\omega)$ separately, one may apply Step 2 of Procedure 1 directly on $g(\alpha,\omega)$ to obtain α^*. However, this approach may require more computations because the system of polynomial equations $g(\alpha,\omega) = \frac{\partial g(\alpha,\omega)}{\partial\omega} = 0$ may have more solutions than those obtained from each of the systems of equations $g_i(\alpha,\omega) = \frac{\partial g_i(\alpha,\omega)}{\partial\omega} = 0$ for $i = 1,\ldots,k$. □

Remark 3. The factorization of $f(\alpha,\omega)$ in Step 1 is not necessary in most cases, since $f(\alpha,\omega)$ is often irreducible. One can simply assume so and go ahead to perform Step 2. Factorization of $f(\alpha,\omega)$ is necessary only if $\Gamma_f(\alpha)$ is found to be identically zero. □

Remark 4. The factorization of $f(\alpha, \omega)$ may be difficult if any coefficient of $f(\alpha, \omega)$ is not rational. We may approximate A by $\hat{A}$ for which all its elements have rational real and imaginary parts, thus this would imply that the polynomial

$$\hat{f}(\alpha, \omega) = \det\left((\hat{A} - j\omega I)^H (\hat{A} - j\omega I) - \alpha I\right)$$

has only real rational coefficients. Effective software for factorizing $\hat{f}(\alpha, \omega)$ into irreducible factors over rational numbers can be found in, e.g., MACSYMA [9]. Theorem 1 then asserts that any irreducible factor over Q and its partial derivative with respect to ω may only have a finite number of common (possibly complex) roots. Hence, Steps 2 and 3 can be followed and yield $\beta(\hat{A})$. Finally, an upper bound of the error $|\beta(A) - \beta(\hat{A})|$ can be obtained by using the following result due to Van Loan [1].

Fact 3. Suppose $A_1, A_2 \in \mathbb{C}^{n \times n}$. Then

$$|\beta(A_1) - \beta(A_2)| \leq \|A_1 - A_2\| \, .$$

□

Remark 5. Roots of $\Gamma_{g_i}(\alpha) = 0$ in Step 2 can be equivalently determined by finding the eigenvalues of the companion matrix of $\Gamma_{g_i}(\alpha)$. Reliable software for calculating eigenvalues can be found in, e.g., Eispack [10]. □

Remark 6. In Step 2, for any nonnegative root $\hat{\alpha}$ of $\Gamma_{g_i}(\alpha)$, whether $g_i(\hat{\alpha}, \omega)$ and $\frac{\partial g_i(\hat{\alpha}, \omega)}{\partial \omega}$ (as functions of ω) have at least one common real root can be determined in a finite number of steps without actually computing their roots. First we apply the Euclidean Algorithm to find the greatest common divisor of $g_i(\hat{\alpha}, \omega)$ and $\frac{\partial g_i(\hat{\alpha}, \omega)}{\partial \omega}$, say, $h(\omega)$. Then a Sturm method can be used to test whether $h(\omega)$ has real roots (see, e.g., [11]). Define the polynomial sequence consisting of $h(\omega)$, it's derivative $\frac{dh(\omega)}{d\omega}$, and the successive remainder (with their sign changed) in the process of finding the greatest common divisor of $h(\omega)$ and $\frac{dh(\omega)}{d\omega}$, i.e., apply the Euclidean algorithm to $h(\omega)$ and $\frac{dh(\omega)}{d\omega}$: Let $p_1 = h$, $p_2 = \frac{dh}{d\omega}$ and recursively divide

$$\begin{aligned} p_1 &= q_1 p_2 - p_3 \\ p_2 &= q_2 p_3 - p_4 \\ . &= \quad . \\ . &= \quad . \\ . &= \quad . \end{aligned}$$

until, for some l, p_l is the greatest common divisor of h and $\frac{dh}{d\omega}$. Then the difference between the number of sign changes in the sequence $\{p_1, \ldots, p_l\}$ when ∞ is substituted for ω (i.e., the number of sign changes in the leading coefficients of the sequence $\{p_1, \ldots, p_l\}$) and the number when $-\infty$ is substituted for ω expresses exactly the number of distinct real roots of the equation $h(\omega) = 0$. $\square$

3. Generalization

Our method can be extended easily to compute the distance from A to the set of matrices having at least an eigenvalue on the unit circle, namely, to compute $\gamma(A)$ defined by

$$\gamma(A) = \min_{\Delta A \in \mathbb{C}^{n \times n}} \{\|\Delta A\| : A + \Delta A \in \mathcal{W}\} ,$$

where $\mathcal{W}$ here denotes the set of matrices having at least one eigenvalue on the unit circle. Thus $\gamma(A)$ is a measure of how "nearly unstable" A is, if A is stable in the sense that all its eigenvalues lie inside the unit circle. By applying the method to simple modifications of A, it should be obvious that the unit circle can be replaced by an arbitrary circle and the imaginary axis by an arbitrary straight line.

It can be easily shown that

$$\gamma(A) = \inf_{\omega \in \mathbb{R}} \underline{\sigma} \left(A - \frac{1 + j\omega}{1 - j\omega} I \right) .$$

Therefore, $\gamma^2(A)$ is the smallest possible real value of α that satisfies

$$\det\left(\left(A-\frac{1+j\omega}{1-j\omega}I\right)^H\left(A-\frac{1+j\omega}{1-j\omega}I\right)-\alpha I\right)=0$$

for some $\omega \in \mathbb{R}\cup\{\infty\}$. Then we claim that one can perform Procedure 1 with $f(\alpha,\omega)$ replaced by

$$\tilde{f}(\alpha,\omega)=(1+\omega^2)^n\det\left(\left(A-\frac{1+j\omega}{1-j\omega}I\right)^H\left(A-\frac{1+j\omega}{1-j\omega}I\right)-\alpha I\right)$$

$$=\det\left(((1-j\omega)A-(1+j\omega)I)^H((1-j\omega)A-(1+j\omega)I)-(1+\omega^2)\alpha I\right)$$

and yield

$$\gamma(A)=\min\left\{\underline{\sigma}\left(A-\frac{1+j\infty}{1-j\infty}I\right),\sqrt{\alpha^*}\right\}$$

$$=\min\left\{\underline{\sigma}(A+I),\sqrt{\alpha^*}\right\},$$

where α^* is obtained at the end of Step 2 in Procedure 1. However, unlike the case of $f(\alpha,\omega)$, $\tilde{f}(\alpha,\omega)$ may have some nonconstant irreducible factor, say $g_i(\alpha,\omega)$, which has degree zero in α or ω (e.g., if $A=0$, then $\tilde{f}(\alpha,\omega)=(1+w^2)^n(1-\alpha)^n$). Recall that in Step 2 of Procedure 1, α_i is set to be the smallest nonnegative real value of α that satisfies $g_i(\alpha,\omega)=\frac{\partial g_i(\alpha,\omega)}{\partial\omega}=0$ for some $\omega\in\mathbb{R}$, if there exists such α. Otherwise, we set $\alpha_i=\infty$. Hence, if $g_i(\alpha,\omega)=g_i(\alpha)$ does not contain any ω terms, then α_i in Step 2 of Procedure 1 is set to be the smallest nonnegative real root of $g_i(\alpha)=0$, if it exists; otherwise we set $\alpha_i=\infty$. On the other hand, if $g_i(\alpha,\omega)=g_i(\omega)$ does not contain any α terms, then, since $g_i(\alpha,\omega)$ is irreducible, the Sylvester resultant $\Gamma_{g_i}(\alpha)$ must be a nonzero constant. Therefore, no α will satisfy $\Gamma_{g_i}(\alpha)=0$, and the system of polynomial equations $g_i(\alpha,\omega)=0$ and $\frac{\partial g_i(\alpha,\omega)}{\partial\omega}=0$ has no solutions. Hence, in such case, α_i is always set to be ∞. Finally, Step 2 is followed without modification when $g_i(\alpha,\omega)$ contains both α and ω terms.

4. Numerical examples

We present four examples to illustrate the use of our method for computing $\beta(A)$ or $\gamma(A)$.

Example 1. Let

$$A = \begin{bmatrix} -2 & 2+j \\ 3-j & -4 \end{bmatrix} .$$

Then

$$f(\omega,\alpha) = \det\left((A - j\omega I)^H (A - j\omega I) - \alpha I\right)$$
$$= \omega^4 - 2\alpha\omega^2 + 34\omega^2 - 12\omega + \alpha^2 - 35\alpha + 2$$

and

$$\Gamma_f(\alpha) = 256\alpha^4 + 101120\alpha^3 + 17865472\alpha^2 - 758496000\alpha + 20384000 .$$

$\Gamma_f(\alpha) = 0$ has real roots at $\alpha = 0.0269,\ 34.9$. The greatest common divisor of $f(0.0269,\omega)$ and $\frac{\partial f(0.0269,\omega)}{\partial \omega}$ is $h(\omega) = (\omega - 0.176)$. Thus $h(\omega)$ has a real root. This implies $\beta(A) = \sqrt{0.0269} = 0.164$, and any nearest unstable matrix from A must have $j0.176$ as an eigenvalue. Let $A - j0.176I = U\Sigma V^H$ be a singular value decomposition of $A - j0.176I$, where

$$U = \begin{bmatrix} -0.5050 - j0.0466 & 0.8618 + j0.0110 \\ 0.8172 - j0.2739 & 0.4666 - j0.1988 \end{bmatrix} ,$$

$$\Sigma = \begin{bmatrix} 5.9190 & 0 \\ 0 & 0.1640 \end{bmatrix} ,$$

$$V = \begin{bmatrix} 0.6325 & -0.7746 \\ -0.7226 + j0.2789 & -0.5900 + j0.2278 \end{bmatrix} .$$

Then the matrix

$$A + \Delta A = A - \beta(A) U V^H = \begin{bmatrix} -1.8381 + j0.0062 & 2.0253 + j1.0046 \\ 2.9745 - j0.9968 & -3.8381 + j0.0031 \end{bmatrix}$$

is a nearest unstable matrix from A.

Example 2. Let

$$A = \frac{1}{25}\begin{bmatrix} -1 & 1 & 9 & 12 \\ 0 & -1 & -12 & 16 \\ -16 & -12 & -1 & 0 \\ 12 & 9 & 0 & -1 \end{bmatrix} .$$

Then

$$\begin{aligned} f(\alpha,\omega) &= \det\left((A - j\omega I)^H (A - j\omega I) - \alpha I\right) \\ &= (\omega^4 - 2\alpha\omega^2 + 2\omega^2 + \alpha^2 - 3\alpha + 1)^2 \\ &= g^2(\alpha,\omega) \end{aligned}$$

and

$$\Gamma_g(\alpha) = 256\alpha^4 - 768\alpha^3 + 256\alpha^2 .$$

($\Gamma_f(\alpha)$ is identically zero). $\Gamma_g(\alpha) = 0$ has roots at $\alpha = 0$, $(3-\sqrt{5})/2$ and $(3+\sqrt{5})/2$. The greatest common divisor of $g(0,\omega)$ and $\frac{\partial g(0,\omega)}{\partial \omega}$ is $h_1(\omega) = (\omega^2+1)$, thus $h_1(\omega)$ has no real root. The greatest common divisor of $g((3-\sqrt{5})/2,\omega)$ and $\frac{\partial g((3-\sqrt{5})/2,\omega)}{\partial \omega}$ is $h_2(\omega) = \omega^2$. Thus $h_2(\omega)$ has real roots, and we conclude that $\beta(A) = \sqrt{(3-\sqrt{5})/2}$.

Example 3. Let

$$A = \begin{bmatrix} -\sqrt{2} & 1 & 1 \\ 0 & -\sqrt{2} & -1 \\ 0 & 0 & -\sqrt{3} \end{bmatrix}$$

and, for the purpose of computing $\beta(A)$, we approximate A by

$$\hat{A} = \begin{bmatrix} -1.4 & 1 & 1 \\ 0 & -1.4 & -1 \\ 0 & 0 & -1.7 \end{bmatrix} .$$

Following Procedure 1 we have $\beta(\hat{A}) = 0.9661$. In view of Fact 3, we have

$$|\beta(A) - \beta(\hat{A})| \le \|A - \hat{A}\| = 0.0321 ,$$

i.e.,

$$0.9340 \le \beta(A) \le 0.9982 .$$

Example 4. We compute $\gamma(A)$ for

$$A = \frac{1}{10}\begin{bmatrix} 1 & 3 & -2 \\ 0 & 5 & 0 \\ -2 & -6 & 4 \end{bmatrix} .$$

Then

$$\begin{aligned} f(\alpha,\omega) &= \det\Big(((1-j\omega)A-(1+j\omega)I)^H((1-j\omega)A-(1+j\omega)I)-(1+\omega^2)\alpha I\Big) \\ &= g_1(\alpha,\omega)g_2(\alpha,\omega)g_3(\alpha,\omega) , \end{aligned}$$

where

$$\begin{aligned} g_1(\alpha,\omega) &= 1-\alpha, \\ g_2(\alpha,\omega) &= \omega^2+1, \\ g_3(\alpha,\omega) &= \frac{1}{80}(80\alpha\omega-396\alpha\omega+405\omega+160\alpha^2\omega^2-472\alpha\omega^2+90\omega^2+80\alpha^2-76\alpha+5) \end{aligned}$$

are irreducible factors. Apply Procedure 1 and follow the discussion in Section 3, we have $\alpha_1 = 1$, $\alpha_2 = \infty$ and $\alpha_3 = 0.8789$. Thus $\alpha^* = \min\{\alpha_1,\alpha_2,\alpha_3\} = 0.8789$ and we have

$$\begin{aligned} \gamma(A) &= \min\left\{\underline{\sigma}(A+I),\ \sqrt{\alpha^*}\right\} \\ &= \min\{0.5024,\ 0.9375\} \\ &= 0.5024 . \end{aligned}$$

We also conclude that any nearest unstable matrix from A will have at least an eigenvalue at -1.

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