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Composite Hypothesis Testing with Data Compression in a Distributed Environment

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COMPOSITE HYPOTHESIS TESTING WITH DATA COMPRESSION IN A DISTRIBUTED ENVIRONMENT

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ABSTRACT

We discuss the asymptotic performance of a multiterminal detection system comprising a central detector and two remote sensors that have access to discrete, spatially dependent, and temporally memoryless observations. We assume that prior to transmitting information to the central detector, each sensor compresses its observations at a rate which approaches zero as the sample size tends to infinity; and that on the basis of the compressed data from all sensors, the central detector seeks to determine whether the true distribution of the observations belongs to a null class Π or an alternative class Ξ. Under the criterion that stipulates minimization of the type II error rate subject to an upper bound ϵ on the type I error rate, we obtain error exponents for four different problems in the above framework, and contrast our results with the case of simple hypothesis testing ($|\Pi| = |\Xi| = 1$).

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1. Introduction

We consider the problem of testing a composite null hypothesis H_0 against a composite alternative H_1 on the basis of compressed data from a discrete-time, discrete-alphabet, memoryless multiple source. In its simplest form, our setup comprises two remote *sensors* S_X and S_Y which are linked to a *central detector*. The sensors S_X and S_Y observe the respective components of the random sequence $\{(X_i, Y_i)\}_{i=1}^n$, and encode their observations into a maximum of M_n and N_n messages, respectively. Upon receipt of the two codewords, the central detector accepts or rejects the null hypothesis in conformity with the classical criterion that stipulates minimization of the probability of falsely accepting H_0 (*type II error*) subject to a fixed upper bound ϵ on the probability of falsely rejecting H_0 (*type I error*). In other words, for disjoint classes Π and Ξ of bivariate distributions on $\mathcal{X} \times \mathcal{Y}$, we wish to test

$$H_0 \colon P_{XY} \in \Pi \qquad \text{against} \qquad H_1 \colon Q_{XY} \in \Xi \ .$$

We assume (asymptotically) *zero-rate* data compression:

$$R_X(n) \ = \ \frac{1}{n} \log M_n \ \to \ 0, \qquad R_Y(n) \ = \ \frac{1}{n} \log N_n \ \to \ 0.$$

In a previous work [1], we studied the corresponding simple hypothesis testing problem ($|\Pi| = |\Xi| = 1$). Let us briefly recapitulate the conclusions of this related work. Under a positivity assumption on the alternative distribution, we showed that the error exponent $\theta(\mathbf{M}, \mathbf{N}, \epsilon)$ of the minimum type II error exists and is independent of the sequences $\mathbf{M}$, $\mathbf{N}$ and the level ϵ. Furthermore, it is possible to specify a sequence of asymptotically optimal acceptance regions *solely* in terms of the null distribution P, and thus the alternative distribution enters the picture only in the computation of the error exponent $\theta(\mathbf{M}, \mathbf{N}, \epsilon)$.

In this paper we ascertain that the above conclusions are of limited validity in the case of composite hypothesis testing. That is, the error exponent for the above composite hypothesis test depends in general on the sequences $\mathbf{M}$, $\mathbf{N}$, and the level ϵ. Furthermore, the choice of optimal acceptance regions is influenced by *both* Π and Ξ.

More specifically, assuming a uniform positivity constraint on the distributions in Ξ, we show the following.

(a) If Π and Ξ are arbitrary classes, and the codebook sizes M_n, N_n are allowed to grow without bound subject to the above zero-rate constraints, then the error exponent has no further dependence on $\mathbf{M}$, $\mathbf{N}$, and ϵ, and is achieved by a sequence of acceptance regions specified solely in terms of Π.

Assuming that the null class Π is finite, and that the codebook sizes M_n and N_n are *fixed* at M and N, respectively, we also have:

(b) If no two distributions in Π share the same X or Y marginal, then the optimal acceptance regions and the resulting error exponents depend on Π, Ξ, M and N. There exist threshold values of M and N, above which we can specify optimal acceptance regions in terms of the null class Π alone.

(c) If two or more distributions in Π share the same X or Y marginal, then the solution of the problem depends explicitly on the level ϵ (in addition to Π, Ξ, M and N).

In (a) above, we consider the problem in its full generality and derive a compact expression for the error exponent. To illustrate (b), we produce a complete solution for the setup in which

$$|\Pi| = 2, \qquad |\Xi| = 1, \qquad M = 2,$$

and the S_Y encoder is nontrivial, i.e., $N \geq 2$. To illustrate (c), we consider the situation in which

$$|\Pi| < \infty, \qquad |\Xi| = 1, \qquad M = 2,$$

and N is greater than a certain threshold. The results in (b) and (c) admit extensions to larger codebooks and classes of distributions, albeit at some expense of compactness in the characterization of the error exponent. It seems to us that the general problem of determining error exponents for arbitrary Π, Ξ, $\mathbf{M}$ and $\mathbf{N}$ resists coherent treatment, and is thus placed outside the scope of this work.

The formulation of the general problem is given in Section 2, together with pertinent notation. The main results (a), (b), and (c) appear in Sections 3, 4, and 5, respectively.

2. Problem Statement and Preliminaries

(a) General notation. The observations of S_X and S_Y are denoted by the sequences $X^n = (X_1, \ldots, X_n) \in \mathcal{X}^n$ and $Y^n = (Y_1, \ldots, Y_n) \in \mathcal{Y}^n$, respectively, and the alphabets $\mathcal{X}$ and $\mathcal{Y}$ are assumed finite. Since the multiple source is memoryless, the sequence of pairs $((X_1, Y_1), \ldots, (X_n, Y_n)) \in (\mathcal{X} \times \mathcal{Y})^n$ is i.i.d. under both hypotheses. In what follows, it will be convenient to deal with the product space $\mathcal{X}^n \times \mathcal{Y}^n$ instead of $(\mathcal{X} \times \mathcal{Y})^n$, and thus the observations will be collectively represented by the pair $(X^n, Y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$.

By virtue of the aforementioned i.i.d. assumption, all distributions of interest can be specified through bivariate distributions on $\mathcal{X} \times \mathcal{Y}$. Under the null hypothesis, the distribution of any pair (X_i, Y_i) is usually denoted by P_{XY}, and its respective marginals by P_X and P_Y. The distributions of X^n, Y^n, and (X^n, Y^n) under the same hypothesis are denoted by P_X^n, P_Y^n and P_{XY}^n, respectively. The i.i.d. assumption then implies that for all (x^n, y^n) in $\mathcal{X}^n \times \mathcal{Y}^n$,

$$P_{XY}^n(x^n, y^n) = \prod_{i=1}^{n} P_{XY}(x_i, y_i) \ .$$

Analogous notation is employed for the alternative hypothesis, with Q replacing P. We will also have occasion to use distributions $\bar{P}_{XY}$, $\tilde{P}_{XY}$ and $\hat{P}_{XY}$ on $\mathcal{X} \times \mathcal{Y}$, which will yield marginals and higher-order distributions in the same manner as P_{XY} and Q_{XY}.

The spaces of all distributions on $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X} \times \mathcal{Y}$ will be denoted by $\mathcal{P}(\mathcal{X})$, $\mathcal{P}(\mathcal{Y})$ and $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, respectively.

The compression of X^n and Y^n is effected by encoders f_n, and g_n, respectively, where

$$f_n : \mathcal{X}^n \mapsto \{1, \ldots, M_n\} , \qquad \text{and} \qquad g_n : \mathcal{Y}^n \mapsto \{1, \ldots, N_n\} .$$

For one-sided zero-rate compression of X^n we assume that $N_n \geq |\mathcal{Y}|^n$ and

$$M_n \geq 2 , \qquad \lim_n \frac{1}{n} \log M_n = 0 , \tag{2.1}$$

and similarly for one-sided zero-rate compression of Y^n, we have $M_n \geq |\mathcal{X}|^n$ and

$$N_n \geq 2 , \qquad \lim_n \frac{1}{n} \log N_n = 0 . \tag{2.2}$$

For two-sided zero-rate compression, both (2.1) and (2.2) are assumed.

The central detector is represented by the function

$$\phi_n : \{1, \ldots, M_n\} \times \{1, \ldots, N_n\} \mapsto \{0, 1\} ,$$

where the output 0 signifies the acceptance of the null hypothesis H_0, and 1 its rejection. This induces a partition of the original (i.e., non-compressed) sample space $\mathcal{X}^n \times \mathcal{Y}^n$ into an *acceptance* region

$$\mathcal{A}_n \stackrel{\text{def}}{=} \{(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n : \phi_n(f_n(x^n), g_n(y^n)) = 0\} ,$$

and a *critical* (or rejection) region $\mathcal{A}_n^c$.

By nature of the encoding process, the acceptance region can be decomposed into M_n rectangles $C_i \times F_i$ in $\mathcal{X}^n \times \mathcal{Y}^n$ that possess disjoint projections C_i on $\mathcal{X}^n$. More precisely, if for every $1 \leq i \leq M_n$ we define

$$C_i = \{x^n \in \mathcal{X}^n : f_n(x^n) = i\} \qquad \text{and} \qquad F_i = \{y^n \in \mathcal{Y}^n : \phi_n(i, g(y^n)) = 0\} ,$$

then we can write

$$\mathcal{A}_n = \bigcup_{i=1}^{M_n} C_i \times F_i , \qquad \text{where} \qquad (\forall i \neq j) \quad C_i \cap C_j = \emptyset . \tag{2.3}$$

We can obtain an alternative representation for $\mathcal{A}_n$ by partitioning $\mathcal{Y}^n$ into N_n sets:

$$\mathcal{A}_n = \bigcup_{i=1}^{N_n} D_i \times G_i , \qquad \text{where} \qquad (\forall i \neq j) \quad D_i \cap D_j = \emptyset . \tag{2.4}$$

Note that conditions (2.3) and (2.4) *jointly* characterize all admissible acceptance regions under *two-sided* compression with codebook sizes M_n (for X^n) and N_n (for Y^n). Taken *separately*, the above conditions characterize the admissible acceptance regions under *one-sided* compression of X^n and Y^n, respectively.

(b) Composite hypothesis testing. Let Π and Ξ be disjoint subsets of $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$. For testing $H_0 : P_{XY} \in \Pi$ versus $H_1 : Q_{XY} \in \Xi$ at a given level ϵ, we employ the *uniformly most powerful* (UMP) test. Thus for a given level $\epsilon \in (0,1)$, we seek to minimize the quantity

$$\sup_{Q \in \Xi} Q^n_{XY}(\mathcal{A}_n)$$

over all acceptance regions $\mathcal{A}_n$ that meet the constraints

(C1) $P^n_{XY}(\mathcal{A}^c_n) \leq \epsilon$ for *all* P_{XY} in Π;

and

(C2) satisfy the appropriate compression constraints; namely

- (2.1) and (2.3) for one-sided compression of X^n;
- (2.2) and (2.4) for one-sided compression of Y^n;
- (2.1), (2.2), (2.3) and (2.4) for two-sided compression.

We use the notation

$$\beta_n(M_n, N_n, \epsilon) \stackrel{\text{def}}{=} \min_{\mathcal{A}_n} \sup_{Q \in \Xi} Q^n_{XY}(\mathcal{A}_n) ,$$

and define the associated error exponent as

$$\theta(\mathbf{M}, \mathbf{N}, \epsilon) \stackrel{\text{def}}{=} -\lim_n \frac{1}{n} \log \beta_n(M_n, N_n, \epsilon) ,$$

provided the limit on the right-hand side exists.

(c) Typical sequences. Our proofs rely on the concept of a typical sequence, as developed in [5]. We cite some basic definitions and facts on typical sequences.

The *type* of a sequence $x^n \in \mathcal{X}^n$ is the distribution λ_x on $\mathcal{X}$ defined by the relationship

$$(\forall a \in \mathcal{X}) \qquad \lambda_x(a) \stackrel{\text{def}}{=} \frac{1}{n} N(a|x^n),$$

where $N(a|x^n)$ is the number of terms in x^n equal to a. The set of all types of sequences in $\mathcal{X}^n$, namely $\{\lambda_x : x^n \in \mathcal{X}^n\}$, will be denoted by $\mathcal{P}_n(\mathcal{X})$.

Given a type $\hat{P}_X \in \mathcal{P}_n(\mathcal{X})$, we will denote by $\hat{T}^n_X$ the set of sequences $x^n \in \mathcal{X}^n$ of type $\hat{P}_X$:

$$\hat{T}^n_X \stackrel{\text{def}}{=} \{x^n \in \mathcal{X}^n : \lambda_x = \hat{P}_X\} .$$

Also, for an arbitrary distribution $\tilde{P}_X$ on $\mathcal{X}$ and a constant $\eta > 0$, we will denote by $\tilde{T}^n_{X,\eta}$ the set of $(\tilde{P}_X, \eta)$-*typical* sequences in $\mathcal{X}^n$. A sequence x^n is $(\tilde{P}_X, \eta)$-typical if $|\lambda_x(a) - \tilde{P}_X(a)| \leq$

η for every letter $a \in \mathcal{X}$ and, in addition, $\lambda_x(a) = 0$ for every a such that $\tilde{P}_X(a) = 0$. Thus, if $||\cdot||$ denotes the sup norm and $\ll$ denotes absolute continuity, we have

$$\tilde{T}^n_{X,\eta} \stackrel{\text{def}}{=} \{x^n \in \mathcal{X}^n\colon\ ||\lambda_x - \tilde{P}_X|| \le \eta,\ \lambda_x \ll \tilde{P}_X\}\ .$$

In the same manner, we will denote by $T^n_{X,\eta}$ and $\bar{T}^n_{X,\eta}$ the sets of (P_X, η)- and $(\bar{P}_X, \eta)$- (respectively) typical sequences in $\mathcal{X}^n$. We will have no need to consider sequences with exact or approximate type Q_X.

The proofs of the following lemmas appear in [5]. As usual, $|\mathcal{A}|$ denotes the size of $\mathcal{A}$.

LEMMA 2.1. *The size of $\mathcal{P}_n(\mathcal{X})$ is at most $(n+1)^{|\mathcal{X}|}$. For any $\hat{P}_X$ in $\mathcal{P}_n(\mathcal{X})$ and Q_X in $\mathcal{P}(\mathcal{X})$,*

$$(n+1)^{-|\mathcal{X}|}\exp[nH(\hat{P}_X)]\ \le\ |\hat{T}^n_X|\ \le\ \exp[nH(\hat{P}_X)]\ ,$$

and

$$(n+1)^{-|\mathcal{X}|}\ \exp[-nD(\hat{P}_X||Q_X)]\ \le\ Q^n_X(\hat{T}^n_X)\ \le\ \exp[-nD(\hat{P}_X||Q_X)]\ .$$

LEMMA 2.2. *For any distribution P_X on $\mathcal{X}$ and $\eta > 0$,*

$$P^n_X(T^n_{X,\eta})\ \ge\ 1\ -\ \frac{|\mathcal{X}|}{4n\eta^2}\ .$$

One can easily modify the above exposition to accommodate pairs $(x^n, y^n) \in \mathcal{X}^n \times \mathcal{Y}^n$ by reverting to their representation in $(\mathcal{X}\times\mathcal{Y})^n$. Thus the type of (x^n, y^n) is the distribution λ_{xy} on $\mathcal{X} \times \mathcal{Y}$ such that

$$\lambda_{xy}(a,b)\ =\ \frac{1}{n}\Big|\{i\ :\ (x_i, y_i) = (a,b)\}\Big|\ ,$$

and the class $\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})$, as well as the sets $\hat{T}^n_{XY} \subset \mathcal{X}^n \times \mathcal{Y}^n$ and $\tilde{T}^n_{XY,\eta} \subset \mathcal{X}^n \times \mathcal{Y}^n$, are defined accordingly.

In this and the following sections, we will omit the superscript n from T^n, as n will be essentially constant.

The following theorem can be obtained by an argument parallel to the proof of Theorem 3.1 in [1].

THEOREM 2.3. *Fix $\rho > 0$ and $\epsilon \in (0,1)$, and let M_n be a sequence of integers satisfying (2.1). Then there exists a sequence*

$$\nu_n\ =\ \nu_n(\rho, \epsilon, M_n, |\mathcal{X}|, |\mathcal{Y}|)\ \to\ 0$$

such that for every $\tilde{Q}_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ that satisfies $\tilde{Q}_{XY} \ge \rho$, and every $\tilde{P}_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$, $C \in \mathcal{X}^n$, $F \in \mathcal{Y}^n$ that satisfy either

$$(\exists P_{XY}:\ P_X = \tilde{P}_X,\ P_Y = \tilde{P}_Y) \qquad P_{XY}(C \times F)\ \ge\ \frac{1-\epsilon}{M_n}$$

or, more generally,

$$\tilde{P}_X(C) \geq \frac{1-\epsilon}{M_n}, \qquad \tilde{P}_Y(F) \geq \frac{1-\epsilon}{M_n},$$

the following is true:

$$\tilde{Q}^n_{XY}(C \times F) \geq \exp[-n(D(\tilde{P}_{XY}||\tilde{Q}_{XY}) + \nu_n] . \qquad \triangle$$

The following notation will be used in the remaining sections.

(i) For a class of distributions on $\mathcal{X} \times \mathcal{Y}$, the corresponding classes of marginals are denoted by

$$\Pi_X = \{P_X \in \mathcal{P}(\mathcal{X}) \colon \exists P_{XY} \in \Pi\}, \quad \text{and} \quad \Pi_Y = \{P_Y \in \mathcal{P}(\mathcal{Y}) \colon \exists P_{XY} \in \Pi\}.$$

(ii) In the space $\mathcal{P}(\mathcal{X})$, we define a ball of radius η centered at P_X by

$$\mathcal{B}_\eta(P_X) \stackrel{\text{def}}{=} \{\tilde{P}_X \in \mathcal{P}(\mathcal{X}) \colon ||\tilde{P}_X - P_X|| \leq \eta, \ \tilde{P}_X \ll P_X\},$$

and we extend the domain of definition to subsets of $\mathcal{P}(\mathcal{X})$ in the obvious way.

(iii) If P_X, P_Y, Q_{XY} are distributions on $\mathcal{X}$, $\mathcal{Y}$, and $\mathcal{X} \times \mathcal{Y}$, respectively, we let

$$d(P_X, P_Y||Q) \stackrel{\text{def}}{=} \min_{\substack{\tilde{P}_{XY}: \\ \tilde{P}_X = P_X, \ \tilde{P}_Y = P_Y}} D(\tilde{P}_{XY}||Q_{XY}).$$

More generally, if Δ, Λ and Ξ are classes of distributions on the same spaces (respectively) as above, then

$$d(\Delta, \Lambda||\Xi) \stackrel{\text{def}}{=} \inf_{\substack{Q_{XY} \in \Xi, \\ \tilde{P}_{XY}: \ \tilde{P}_X \in \Delta, \ \tilde{P}_Y \in \Lambda}} D(\tilde{P}_{XY}||Q_{XY}).$$

(iv) Finally, if Φ is a subset of $\mathcal{P}(\mathcal{X})$, we will write

$$\bigcup_{P_X \in \Phi} T_X \quad \text{and} \quad \bigcup_{P_X \in \Phi} T_{X,\eta}$$

for

$$\bigcup_{\hat{P}_X \in \Phi \cap \mathcal{P}_n(\mathcal{X})} \hat{T}_X \quad \text{and} \quad \bigcup_{\hat{P}_X \in \mathcal{B}_\eta(\Phi) \cap \mathcal{P}_n(\mathcal{X})} \hat{T}_X ,$$

respectively.

3. Unboundedly growing codebook sizes.

We consider the composite hypothesis testing problem in which the null class Π is an arbitrary subset of $\mathcal{P}(\mathcal{X} \times \mathcal{Y})$, and the alternative class Ξ satisfies the uniform positivity constraint

$$\rho_{\inf} \stackrel{\text{def}}{=} \inf_{Q \in \Xi} \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} Q_{XY}(x,y) > 0 . \tag{3.1}$$

The above condition ensures that the convex function $D(\cdot||\cdot)$ is bounded on $\mathcal{P}(\mathcal{X}\times\mathcal{Y})\times\Xi$ and is thus uniformly continuous. For the codebook sizes, we assume

$$\lim_n \frac{1}{n}\log M_n = \lim_n \frac{1}{n}\log N_n = 0, \qquad \text{and} \qquad \lim_n M_n = \lim_n N_n = \infty .$$

The above size constraint allows each of the two encoders to specify the type of the observed sequence with arbitrary accuracy. Indeed, if we let

$$a_n = \lfloor M_n^{1/|\mathcal{X}|}\rfloor,$$

then by an elementary geometrical construction we can partition $\mathcal{P}(\mathcal{X})$ into at most $a_n^{|\mathcal{X}|}\le M_n$ cells $\mathcal{C}_i^n$ of maximum dimension (measured by sup norm) not exceeding a_n^{-1}; clearly $a_n^{-1}\to 0$ since $M_n\to\infty$. The same is true for $\mathcal{P}(\mathcal{Y})$ with b_n replacing a_n:

$$b_n = \lfloor N_n^{1/|\mathcal{Y}|}\rfloor .$$

We denote the $\mathcal{P}(\mathcal{Y})$-counterpart of $\mathcal{C}_i^n$ by $\mathcal{F}_j^n$, and we write

$$C_i^n = \bigcup_{\hat{P}_X\in\mathcal{C}_i^n} \hat{T}_X, \qquad F_j^n = \bigcup_{\hat{P}_Y\in\mathcal{F}_j^n} \hat{T}_Y .$$

Based on the above partition, we devise a compression/decision scheme as follows. First, we require that each encoder transmit the cell index corresponding to the observed type, i.e.,

$$\begin{aligned} f_n(x^n) &= i \quad \text{iff} \quad x^n\in C_i^n;\\ g_n(y^n) &= j \quad \text{iff} \quad y^n\in F_j^n. \end{aligned}$$

Next, we seek an acceptance region $\mathcal{A}_n\subset\mathcal{X}^n\times\mathcal{Y}^n$ such that

$$\mathcal{A}_n \supset \bigcup_{P_{XY}\in\Pi} T^n_{XY,\eta} \tag{3.2}$$

for some *fixed* $\eta>0$. This is because the above set has P^n_{XY}-probability that *uniformly* approaches unity for all $P_{XY}\in\Pi$ (by Lemma 2.2), and this automatically ensures that the type I error bound is met for every $\epsilon\in(0,1)$. We define $\mathcal{A}_n$ as the smallest union of rectangles $C_i^n\times F_j^n$ that contains

$$\bigcup_{P_{XY}\in\Pi} T_{X,\xi}\times T_{Y,\xi} ,$$

where ξ is a multiple of η chosen so as to ensure that (3.2) holds.

Since ξ is fixed and the dimension of each C_i^n and F_j^n shrinks to zero as n approaches infinity, it is also true that for n sufficiently large,

$$\mathcal{A}_n \subset \bigcup_{P_{XY} \in \Pi} T_{X,2\xi} \times T_{Y,2\xi} .$$

By a standard argument based on the definition of typicality, we also have

$$T_{X,2\xi} \times T_{Y,2\xi} \subset \bigcup_{\substack{\tilde{P}_{XY}: \\ \tilde{P}_X = P_X,\ \tilde{P}_Y = P_Y}} \tilde{T}_{XY,\zeta} ,$$

where ζ is a fixed multiple of ξ and η. We conclude that

$$\mathcal{A}_n \subset \bigcup_{\substack{\tilde{P}_{XY}: \\ (\exists P_{XY} \in \Pi)\ \tilde{P}_X = P_X,\ \tilde{P}_Y = P_Y}} \tilde{T}_{XY,\zeta} .$$

A union bound on $Q^n(\mathcal{A}_n)$ for $Q \in \Xi$ can now be established using Lemma 2.1 and the fact that $D(\cdot||\cdot)$ is uniformly continuous on $\mathcal{P}(\mathcal{X} \times \mathcal{Y}) \times \Xi$:

$$\begin{aligned} Q(\mathcal{A}_n) &\leq |\mathcal{P}_n(\mathcal{X} \times \mathcal{Y})| \exp[-n \inf_{\substack{\tilde{P}_{XY}: \\ (\exists P_{XY} \in \Pi)\ \tilde{P}_X = P_X, \tilde{P}_Y = P_Y}} (D(\tilde{P}_{XY}||Q_{XY}) - \mu'(\zeta))] \\ &\leq \exp[-n \inf_{P_{XY} \in \Pi} (d(P_X, P_Y||Q_{XY}) - \mu(\zeta))] , \end{aligned}$$

where $\mu(\zeta)$ goes to zero together with ζ (and hence also η). We therefore have

$$\beta_n(M_n, N_n, \epsilon) \leq \exp[-n \inf_{P_{XY} \in \Pi, Q_{XY} \in \Xi} (d(P_X, P_Y||Q_{XY}) - \mu(\zeta))] .$$

Since $\mu(\zeta)$ can be made arbitrarily small by choice of η, we conclude that

$$\theta(\mathbf{M}, \mathbf{N}, \epsilon) \geq \inf_{P_{XY} \in \Pi, Q_{XY} \in \Xi} d(P_X, P_Y||Q_{XY}) .$$

To show the reverse inequality, we consider an admissible acceptance region $\mathcal{A}_n$. By (2.3), for every distribution P_{XY} in Π, we can find a rectangle $C \times F \subset \mathcal{X}^n \times \mathcal{Y}^n$ such that

$$P_{XY}^n(C \times F) \geq (1 - \epsilon)/M_n .$$

Applying Theorem 2.3 with $\rho = \rho_{\inf}$, we obtain a universal sequence $\nu_n \to 0$ with the property that for every $Q_{XY} \in \Xi$, $P_{XY} \in \Pi$ and $\tilde{P}_{XY} \in \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ such that $\tilde{P}_X = P_X$, $\tilde{P}_Y = P_Y$, the following is true:

$$Q_{XY}^n(\mathcal{A}_n) \geq \exp[-n(D(\tilde{P}_{XY}||Q_{XY}) + \nu_n)] .$$

We conclude that

$$\beta_n(M_n, N_n, \epsilon) \;\geq\; \exp[-n \inf_{\substack{\tilde{P}_{XY}:\ (\exists P_{XY} \in \Pi)\ \tilde{P}_X = P_X,\ \tilde{P}_Y = P_Y, \\ Q_{XY} \in \Xi}} (D(\tilde{P}_{XY} || Q_{XY}) + \nu_n)]$$

and hence

$$\theta(\mathbf{M}, \mathbf{N}, \epsilon) \;\leq\; \inf_{P_{XY} \in \Pi,\ Q_{XY} \in \Xi} d(P_X, P_Y || Q_{XY}) \ .$$

We thus have proved

THEOREM 3.1. *If $\Pi \subset \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is arbitrary, $\Xi \subset \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ is such that*

$$\inf_{Q \in \Xi} \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} Q_{XY}(x, y) \;>\; 0 \ ,$$

and the sequences $\mathbf{M}$, $\mathbf{N}$ *satisfy*

$$\lim_n \frac{1}{n} \log M_n \;=\; \lim_n \frac{1}{n} \log N_n \;=\; 0, \qquad \text{and} \qquad \lim_n M_n \;=\; \lim_n N_n \;=\; \infty \ ,$$

then

$$\theta(\mathbf{M}, \mathbf{N}, \epsilon) \;=\; \inf_{P_{XY} \in \Pi,\ Q_{XY} \in \Xi} d(P_X, P_Y || Q_{XY}) \ . \qquad \triangle$$

4. Fixed codebook sizes

In this and the following section we assume that the codebook sizes are fixed:

$$(\forall n) \qquad M_n = M, \qquad N_n = N \ .$$

Under the above constraint, it is no longer possible to encode the type of the observed sequences with arbitrary accuracy, and the conclusion of Theorem 3.1 does not hold in general. As we shall see, the optimal system design depends on the distribution classes Π and Ξ, the actual codebook sizes M and N, and (somewhat surprisingly) the value of the level ϵ.

Throughout the remainder of this work, we will assume for simplicity that the class Π is finite. As we pointed out earlier, some of our proofs admit cumbersome but straightforward generalizations to situations in which Π is infinite. However, since our aim is to highlight salient differences from the simple hypothesis testing problem, we choose to restrict our attention to the simplest possible setups.

Our first observation is that given Π finite and Ξ satisfying the uniform positivity constraint (3.1), there exist threshold values of M and N, above which the error exponent of Theorem 3.1 obtains. Indeed, if

$$M \;\geq\; |\Pi_X| + 1, \qquad N \;\geq\; |\Pi_Y| + 1 \ ,$$

then the S_X encoder can specify which one (if any) of the distributions $P_X \in \Pi_X$ lies within distance η from the type of the observed sequence x^n; similarly for S_Y. This allows us to employ an acceptance region

$$\mathcal{A}_n = \bigcup_{P_{XY} \in \Pi} T_{X,\eta} \times T_{Y,\eta} .$$

As in the proof of the positive part of Theorem 3.1, we obtain

$$\theta(\mathbf{M}, \mathbf{N}, \epsilon) \geq \inf_{P_{XY} \in \Pi,\ Q_{XY} \in \Xi} d(P_X, P_Y || Q_{XY}) .$$

The converse part of Theorem 3.1 clearly suffices for this problem. We thus have

THEOREM 4.1. *If* $\Pi \subset \mathcal{P}(\mathcal{X} \times \mathcal{Y})$ *is finite,* $\Xi \subset \mathcal{P}(\mathcal{Y})$ *is such that*

$$\inf_{Q \in \Xi} \min_{(x,y) \in \mathcal{X} \times \mathcal{Y}} Q_{XY}(x, y) > 0 ,$$

and

$$M \geq |\Pi_X| + 1, \qquad N \geq |\Pi_Y| + 1 ,$$

then

$$\theta(M, N, \epsilon) = \inf_{P_{XY} \in \Pi,\ Q_{XY} \in \Xi} d(P_X, P_Y || Q_{XY}) . \qquad \triangle$$

We now consider the situation in which either one or both codebook sizes M, N are smaller than the threshold values given in the hypothesis of Theorem 4.1. For simplicity, we will assume that Π consists of two distributions P_{XY}, $\bar{P}_{XY}$ with distinct X, Y marginals, and that the alternative hypothesis is simple, i.e., $\Xi = \{Q_{XY}\}$. The threshold values are then both equal to 3, and it clearly suffices to consider two cases: (i) $(M, N) = (2, 3)$ and (ii) $(M, N) = (2, 2)$.

We consider case (i) first.

THEOREM 4.2. *Let* $\Pi = \{P_{XY}, \bar{P}_{XY}\}$, *where* $P_X \neq \bar{P}_X$ *and* $P_Y \neq \bar{P}_Y$. *If* $Q_{XY} > 0$, *then for* $0 < \epsilon < 1$,

$$\theta(2, 3, \epsilon) = \theta^{(1)} \vee \theta^{(2)} ,$$

where

$$\theta^{(1)} \stackrel{\text{def}}{=} d(\Pi_X, \Pi_Y || Q)$$

and

$$\theta^{(2)} \stackrel{\text{def}}{=} d(P_X, P_Y || Q) \wedge d(\bar{P}_X, \bar{P}_Y || Q) \wedge \min_{\tilde{P}_X \in \mathcal{P}(\mathcal{X})} \{ d(\tilde{P}_X, P_Y || Q) \vee d(\tilde{P}_X, \bar{P}_Y || Q) \}.$$

PROOF. *Positive part.* As before, we restrict our attention to encoders that group sequences of the same type together. Since $N = 3$, a sensible choice for the S_Y encoder is one that specifies whether the sequence y^n lies in $T_{Y,\eta}$, $\bar{T}_{Y,\eta}$ or $(T_{Y,\eta} \cup \bar{T}_{Y,\eta})^c$.

The choice of the S_X encoder is less straightforward. At first sight it would seem that since $M = 2$, the S_X encoder should specify whether or not the type of the observed sequence x^n is close to *either one* or *none* of the distributions P_X, $\bar{P}_X$, i.e.,

$$C_1 = T_{X,\eta} \cup \bar{T}_{X,\eta} , \qquad C_2 = (T_{X,\eta} \cup \bar{T}_{X,\eta})^c .$$

With this choice of encoders, the smallest acceptance region that satisfies the type I error constraint under both P_{XY} and $\bar{P}_{XY}$ is

$$\mathcal{A}_n^{(1)} = (T_{X,\eta} \cup \bar{T}_{X,\eta}) \times (T_{Y,\eta} \cup \bar{T}_{Y,\eta}) .$$

The Q^n-probability of the above set can be upper-bounded in the standard fashion (viz. the proof of Theorem (3.1):

$$Q_{XY}^n(\mathcal{A}_n) \leq \exp[-n(\min_{\tilde{P}_X \in \{P_X, \bar{P}_X\}, \tilde{P}_Y \in \{P_Y, \bar{P}_Y\}} D(\tilde{P}_{XY} || Q_{XY}) - \mu(\eta))]$$

where $\mu(\eta) \to 0$ as $\eta \to 0$. This yields, since η is arbitrary small,

$$\theta(2, 3, \epsilon) \geq \theta^{(1)} = d(\Pi_X, \Pi_Y | Q) . \tag{4.1}$$

Another (somewhat less prominent) candidate for the S_X encoder is one that *separates* sequences of approximate type P_X from ones of approximate type $\bar{P}_X$. Since only two codewords are available, this separation entails grouping some types in $\mathcal{P}(\mathcal{X})$ together with P_X, and the remaining types with $\bar{P}_X$. More formally, if

$$\Phi \subset \mathcal{P}(\mathcal{X}) - \mathcal{B}_\eta(P_X) - \mathcal{B}_\eta(\bar{P}_X) \qquad \text{and} \qquad \bar{\Phi} = \mathcal{P}(\mathcal{X}) - \mathcal{B}_\eta(P_X) - \mathcal{B}_\eta(\bar{P}_X) - \Phi ,$$

then this encoder partitions $\mathcal{X}^n$ into

$$C_1' = T_{X,\eta} \cup \bigcup_{\tilde{P}_X \in \Phi} \tilde{T}_X \qquad \text{and} \qquad C_2' = (C_1')^c = \bar{T}_{X,\eta} \cup \bigcup_{\tilde{P}_X \in \bar{\Phi}} \tilde{T}_X . \tag{4.2}$$

With this choice of S_X encoder (together with the S_Y encoder introduced in the beginning of the proof), the smallest acceptance region that satisfies the type I error constraint is

$$\mathcal{A}_n^{(2)} = (C_1' \times T_{Y,\eta}) \cup (C_2' \times \bar{T}_{Y,\eta})$$

Note that unlike $\mathcal{A}_n^{(1)}$, $\mathcal{A}_n^{(2)}$ does not contain $T_{X,\eta} \times \bar{T}_{Y,\eta}$ or $\bar{T}_{X,\eta} \times T_{Y,\eta}$. It does, however, contain pairs (x^n, y^n) whose marginal type λ_x is close to neither P_X nor $\bar{P}_X$.

To estimate $Q^n(\mathcal{A}_n^{(2)})$, we decompose each of C_1' and C_2' into two sets as in definition (4.2). We then treat $\mathcal{A}_n^{(2)}$ as a union of four disjoint sets, and upper-bound their Q^n-probabilities in the usual way:

$$\begin{aligned}
Q_{XY}^n(T_{X,\eta} \times T_{Y,\eta}) &\le \exp[-n(d(P_X, P_Y||Q) - \mu(\eta))] , \\
Q_{XY}^n(\bar{T}_{X,\eta} \times \bar{T}_{Y,\eta}) &\le \exp[-n(d(\bar{P}_X, \bar{P}_Y||Q) - \mu(\eta))] , \\
Q_{XY}^n\Big(\bigcup_{\tilde{P}_X \in \Phi} \tilde{T}_X \times T_{Y,\eta} \Big) &\le \exp[-n(\inf_{\tilde{P}_X \in \Phi} d(\tilde{P}_X, P_Y||Q) - \mu(\eta))] , \\
Q_{XY}^n\Big(\bigcup_{\tilde{P}_X \in \bar{\Phi}} \tilde{T}_X \times \bar{T}_{Y,\eta} \Big) &\le \exp[-n(\inf_{\tilde{P}_X \in \bar{\Phi}} d(\tilde{P}_X, \bar{P}_Y||Q) - \mu(\eta))] ,
\end{aligned}$$

where $\mu(\eta) \to 0$ as $\eta \to 0$.

Thus the error exponent associated with this choice of acceptance region is greater than or equal to the minimum of the four exponents appearing in the above bounds, namely the quantity

$$d(P_X, P_Y||Q) \ \wedge \ d(\bar{P}_X, \bar{P}_Y||Q) \ \wedge \ \inf_{\tilde{P}_X \in \Phi} d(\tilde{P}_X, P_Y||Q) \ \wedge \ \inf_{\tilde{P}_X \in \bar{\Phi}} d(\tilde{P}_X, \bar{P}_Y||Q) \ .$$

At this point we should note that by letting η shrink to zero, we have expanded the classes Φ and $\bar{\Phi}$ in the vicinity P_X and $\bar{P}_X$ so that $\Phi \cup \bar{\Phi} = \mathcal{P}(\mathcal{X}) - \{P_X\} - \{\bar{P}_X\}$. This is justified by continuity of $d(\cdot, \cdot||Q)$, which further allows us to treat Φ and $\bar{\Phi}$ in the above expression as constituting a partition of $\mathcal{P}(\mathcal{X})$.

It remains to find that partition $\{\Phi, \bar{\Phi}\}$ of $\mathcal{P}(\mathcal{X})$ which maximizes

$$v(P_X) \ \wedge \ \bar{v}(\bar{P}_X) \ \wedge \ \inf_{\tilde{P}_X \in \Phi} v(\tilde{P}_X) \ \wedge \ \inf_{\tilde{P}_X \in \bar{\Phi}} \bar{v}(\tilde{P}_X) \ ,$$

where $v(\cdot) \stackrel{\text{def}}{=} d(\cdot, P_Y||Q)$ and $\bar{v}(\cdot) \stackrel{\text{def}}{=} d(\cdot, \bar{P}_Y||Q)$. This is easily accomplished by noting that

$$\begin{aligned}
\inf_{\Phi} v(\tilde{P}_X) \ \wedge \ \inf_{\bar{\Phi}} \bar{v}(\tilde{P}_X) &\le \inf_{\Phi}[v(\tilde{P}_X) \ \vee \ \bar{v}(\tilde{P}_X)] \ \wedge \ \inf_{\bar{\Phi}}[v(\tilde{P}_X) \ \vee \ \bar{v}(\tilde{P}_X)] \\
&= \inf_{\mathcal{P}(\mathcal{X})} [v(\tilde{P}_X) \ \vee \ \bar{v}(\tilde{P}_X)] \\
&= \inf_{\tilde{P}_X : v(\tilde{P}_X) \ge \bar{v}(\tilde{P}_X)} v(\tilde{P}_X) \ \wedge \ \inf_{\tilde{P}_X : v(\tilde{P}_X) < \bar{v}(\tilde{P}_X)} \bar{v}(\tilde{P}_X) \ .
\end{aligned}$$

Thus an optimal partition consists of the sets

$$\Phi = \{\tilde{P}_X : v(\tilde{P}_X) \ge \bar{v}(\tilde{P}_X)\}, \quad \text{and} \quad \bar{\Phi} = \{\tilde{P}_X : v(\tilde{P}_X) < \bar{v}(\tilde{P}_X)\} ,$$

and the error exponent associated with the corresponding $\mathcal{A}^{(2)}$ is given by

$$\begin{aligned}
\theta^{(2)} = \ & d(P_X, P_Y||Q) \ \wedge \ d(\bar{P}_X, \bar{P}_Y||Q) \\
& \wedge \min_{\tilde{P}_X \in \mathcal{P}(\mathcal{X})} \{d(\tilde{P}_X, P_Y||Q) \ \vee \ d(\tilde{P}_X, \bar{P}_Y||Q)\} \ .
\end{aligned}$$

We conclude that $\theta(2,3,\epsilon) \;\geq\; \theta^{(2)}$, and in light of (4.1),

$$\theta(2,3,\epsilon) \;\geq\; \theta^{(1)} \vee \theta^{(2)} \ .$$

Converse part. For fixed n, consider an admissible acceptance region $\mathcal{A}_n$. By nature of the encoding, $\mathcal{A}_n$ can be written as

$$\mathcal{A}_n \;=\; (C_1 \times F_1) \;\cup\; (C_2 \times F_2) \ ,$$

where C_1 and C_2 form a partition of $\mathcal{X}^n$, and at most one of F_1, F_2 may be empty. From the type I error constraint

$$P_{XY}(\mathcal{A}_n) \geq 1-\epsilon \qquad \text{and} \qquad \bar{P}_{XY}(\mathcal{A}_n) \geq 1-\epsilon \ ,$$

it follows that two cases may arise.

Case 1. For i and j distinct, we have

$$P^n_{XY}(C_i \times F_i) \;\geq\; P^n_{XY}(C_j \times F_j) \qquad \text{and} \qquad \bar{P}^n_{XY}(C_i \times F_i) \;\geq\; \bar{P}^n_{XY}(C_j \times F_j).$$

This clearly implies that

$$\tilde{P}^n_X(C_i) \;\geq\; \frac{1-\epsilon}{2} \qquad \text{and} \qquad \tilde{P}^n_Y(F_i) \;\geq\; \frac{1-\epsilon}{2} \ ,$$

for any $\tilde{P}_X \in \Pi_X$, $\tilde{P}_Y \in \Pi_Y$. From Theorem 2.3, we obtain

$$-\frac{1}{n}\log Q^n_{XY}(C_i \times F_i) \;\leq\; d(\Pi_X, \Pi_Y||Q) + \nu_n \;=\; \theta^{(1)} + \nu_n \ ,$$

where $\nu_n \to 0$ as $n \to \infty$, and thus also

$$-\frac{1}{n}\log Q^n(\mathcal{A}_n) \;\leq\; \theta^{(1)} + \nu_n \ . \tag{4.3}$$

Case 2. For i and j distinct, we have

$$P^n_{XY}(C_i \times F_i) \;\geq\; P^n_{XY}(C_j \times F_j) \qquad \text{and} \qquad \bar{P}^n_{XY}(C_i \times F_i) \;<\; \bar{P}^n_{XY}(C_j \times F_j) \ . \tag{4.4}$$

Using Theorem 2.3 once again, we obtain respectively

$$-\frac{1}{n}\log Q^n_{XY}(C_i \times F_i) \;\leq\; d(P_X, P_Y||Q) + \nu_n$$

and

$$-\frac{1}{n}\log Q^n_{XY}(C_j \times F_j) \;\leq\; d(\bar{P}_X, \bar{P}_Y||Q) + \nu_n \ .$$

Hence

$$-\frac{1}{n}\log Q^n(\mathcal{A}_n) \;\le\; d(P_X, P_Y||Q) \;\wedge\; d(\bar{P}_X, \bar{P}_Y||Q) + \nu_n \;. \tag{4.5}$$

Relationship (4.4) also implies that

$$P_Y^n(F_i) \;\ge\; \frac{1-\epsilon}{2} \qquad \text{and} \qquad \bar{P}_Y^n(F_j) \;\ge\; \frac{1-\epsilon}{2}\;.$$

By virtue of Theorem 2.3, the above inequalities can lead to a further upper bound on $Q^n(\mathcal{A}_n)$ provided there exists a distribution $\tilde{P}_X \in \mathcal{P}(\mathcal{X})$ for which either $\tilde{P}_X^n(C_i)$ or $\tilde{P}_X^n(C_j)$ exceeds a fixed value independent of n. But the last disjunction is true for *every* $\tilde{P}_X$, since C_i and C_j are complementary events. We thus obtain the upper bound

$$-\frac{1}{n}\log Q_{XY}^n(\mathcal{A}_n) \;\le\; \min_{\tilde{P}_X \in \mathcal{P}(\mathcal{X})}\{d(\tilde{P}_X, P_Y||Q) \;\vee\; d(\tilde{P}_X, \bar{P}_Y||Q)\} + \nu_n \;,$$

which, together with (4.5), yields

$$-\frac{1}{n}\log Q_{XY}^n(\mathcal{A}_n) \;\le\; \theta^{(2)} + \nu_n \;.$$

Finally, by combining the bound for case 1 (eq. (4.3)) with the above bound for case 2, we obtain the converse statement

$$\theta(2,3,\epsilon) \;\le\; \theta^{(1)} \vee \theta^{(2)} \;. \qquad \triangle$$

For the system in which both encoders use two codewords, i.e., $M = N = 2$, we have the following result.

THEOREM 4.3. *Let* $\Pi = \{P_{XY}, \bar{P}_{XY}\}$, *where* $P_X \ne \bar{P}_X$ *and* $P_Y \ne \bar{P}_Y$. *If* $Q_{XY} > 0$, *then for* $0 < \epsilon < 1$,

$$\theta(2,2,\epsilon) \;=\; \theta^{(1)} \;\vee\; \theta^{(3)} \;,$$

where $\theta^{(1)}$ *is as defined in Theorem 4.2, and and* $\theta^{(3)}$ *is the supremum, over all partitions* $\{\Phi, \bar{\Phi}\}$ *of* $\mathcal{P}(\mathcal{X})$ *and* $\{\Psi, \bar{\Psi}\}$ *of* $\mathcal{P}(\mathcal{Y})$, *of the quantity*

$$d(\Phi \cup \{P_X\},\, \Psi \cup \{P_Y\}\,||Q) \;\wedge\; d(\bar{\Phi} \cup \{\bar{P}_X\},\, \bar{\Psi} \cup \{\bar{P}_Y\}\,||Q)\;. \tag{4.6}$$

PROOF. *Direct part.* Since $M = 2$ as in the previous problem, we consider the same two candidates for the S_X encoder:

$$f: \quad C_1 \;=\; T_{X,\eta} \cup \bar{T}_{X,\eta}\;, \qquad C_2 \;=\; (T_{X,\eta} \cup \bar{T}_{X,\eta})^c$$

and

$$f': \quad C_1' \;=\; T_{X,\eta} \cup \bigcup_{\tilde{P}_X \in \Phi} \tilde{T}_X\;, \qquad C_2' \;=\; \bar{T}_{X,\eta} \cup \bigcup_{\tilde{P}_X \in \bar{\Phi}} \tilde{T}_X\;,$$

where $(\Phi, \bar{\Phi})$ form a partition of $\mathcal{P}(\mathcal{X}) - \mathcal{B}_\eta(P_X) - \mathcal{B}_\eta(\bar{P}_X)$. Observe that in this case $N = 2$ also, and thus it is no longer possible for the S_Y encoder to specify whether y^n lies in $T_{Y,\eta}$, $\bar{T}_{Y,\eta}$ or $(T_{Y,\eta} \cup \bar{T}_{Y,\eta})^c$. Proceeding as for S_X, we propose the following two encoders for S_Y:

$$g: \quad F_1 = T_{Y,\eta} \cup \bar{T}_{Y,\eta} , \qquad F_2 = (T_{Y,\eta} \cup \bar{T}_{Y,\eta})^c$$

and

$$g': \quad F_1' = T_{Y,\eta} \cup \bigcup_{\tilde{P}_Y \in \Psi} \tilde{T}_Y , \qquad F_2' = \bar{T}_{Y,\eta} \cup \bigcup_{\tilde{P}_Y \in \bar{\Psi}} \tilde{T}_Y ,$$

where $(\Psi, \bar{\Psi})$ are defined in a similar manner.

Given the above possibilities for encoding S_X and S_Y, there are only two reasonable choices for the acceptance region $\mathcal{A}_n$:

$$\mathcal{A}_n^{(1)} = C_1 \times F_1 \qquad \text{and} \qquad \mathcal{A}_n^{(3)} = (C_1' \times F_1') \cup (C_2' \times F_2') .$$

Note that the region $\mathcal{A}_n^{(1)}$ is identical to the one used in the proof of the previous theorem, whence we obtain

$$\theta(2,2,\epsilon) \geq \theta^{(1)} = d(\Pi_X, \Pi_Y || Q) .$$

To evaluate the error exponent associated with $\mathcal{A}_n^{(3)}$, we follow the corresponding procedure for $\mathcal{A}_n^{(2)}$ in the proof of Theorem 4.2. Since

$$\mathcal{A}_n^{(3)} = \Big(\bigcup_{\Phi \cup \mathcal{B}_\eta(P_X)} \tilde{T}_X \times \bigcup_{\Psi \cup \mathcal{B}_\eta(P_Y)} \tilde{T}_Y \Big) \cup \Big(\bigcup_{\bar{\Phi} \cup \mathcal{B}_\eta(\bar{P}_X)} \tilde{T}_X \times \bigcup_{\bar{\Psi} \cup \mathcal{B}_\eta(\bar{P}_Y)} \tilde{T}_Y \Big) ,$$

we obtain

$$-\lim_n \frac{1}{n} \log Q^n(\mathcal{A}_n^{(3)}) = d(\Phi \cup \{P_X\}, \Psi \cup \{P_Y\} || Q) \wedge d(\bar{\Phi} \cup \{\bar{P}_X\}, \bar{\Psi} \cup \{\bar{P}_Y\} || Q) .$$

Once again, it is legitimate to assume that in the above equation, $\{\Phi, \bar{\Phi}\}$, $\{\Psi, \bar{\Psi}\}$ constitute partitions of the entire spaces $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$, respectively. The best error exponent attainable by a sequence of acceptance regions of the form $\mathcal{A}_n^{(3)}$ is therefore

$$\theta^{(3)} = \sup_{\Phi, \Psi} \{ d(\Phi \cup \{P_X\}, \Psi \cup \{P_Y\} || Q) \wedge d(\bar{\Phi} \cup \{\bar{P}_X\}, \bar{\Psi} \cup \{\bar{P}_Y\} || Q) \} .$$

We conclude that

$$\theta(2,2,\epsilon) \geq \theta^{(1)} \vee \theta^{(3)} .$$

Converse part. In this case every admissible acceptance region $\mathcal{A}_n$ can be written as

$$\mathcal{A}_n = (C_1 \times F_1) \cup (C_2 \times F_2),$$

where C_1, C_2 are complementary, while F_1, F_2 are constrained by $F_2 \in \{\emptyset, \mathcal{Y}^n, F_1^c\}$. As in the proof of Theorem 4.2, two cases may arise.

Case 1. For i and j distinct, we have

$$P^n_{XY}(C_i \times F_i) \;\geq\; P^n_{XY}(C_j \times F_j) \qquad \text{and} \qquad \bar{P}^n_{XY}(C_i \times F_i) \;\geq\; \bar{P}^n_{XY}(C_j \times F_j) \;.$$

This is same as Case 1 in the proof of Theorem 4.2, whence we obtain

$$-\frac{1}{n}\log Q^n(\mathcal{A}_n) \;\leq\; \theta^{(1)} + \nu_n \;.$$

Note that this case subsumes the situation in which F_2 is empty.

Case 2. For i and j distinct, we have

$$P^n_{XY}(C_i \times F_i) \;\geq\; P^n_{XY}(C_j \times F_j) \qquad \text{and} \qquad \bar{P}^n_{XY}(C_i \times F_i) \;<\; \bar{P}^n_{XY}(C_j \times F_j) \;.$$

We easily deduce that

$$P^n_X(C_i) \;\geq\; \frac{1-\epsilon}{2}\;, \qquad\qquad P^n_Y(F_i) \;\geq\; \frac{1-\epsilon}{2}\;,$$

and

$$\bar{P}^n_X(C_j) \;\geq\; \frac{1-\epsilon}{2}\;, \qquad\qquad \bar{P}^n_Y(F_j) \;\geq\; \frac{1-\epsilon}{2}\;.$$

Let us define the classes

$$\Phi_n \;=\; \{\tilde{P}_X :\; \tilde{P}^n_X(C_i) \;\geq\; \frac{1}{2}\}\;, \qquad \Psi_n \;=\; \{\tilde{P}_Y :\; \tilde{P}^n_Y(F_i) \;\geq\; \frac{1}{2}\}\;,$$

and

$$\Phi^*_n \;=\; \{\tilde{P}_X :\; \tilde{P}^n_X(C_j) \;>\; \frac{1}{2}\}\;, \qquad \Psi^*_n \;=\; \{\tilde{P}_Y :\; \tilde{P}^n_Y(F_j) \;>\; \frac{1}{2}\}\;.$$

Since C_1 and C_2 are complementary, $\Phi^*_n = \Phi^c_n$. For F_1 and F_2, we have either $F_2 = F_1^c$ or $F_2 = \mathcal{Y}^n$. In the former case we have again $\Psi^*_n = \Psi^c_n$, while in the latter, either Ψ_n or Ψ^*_n is equal to $\mathcal{P}(\mathcal{Y})$.

By the foregoing discussion, all marginal distributions $\tilde{P}_X \in \Phi_n \cup \{P_X\}$, $\tilde{P}_Y \in \Psi_n \cup \{P_Y\}$, satisfy

$$\tilde{P}^n_X(C_i) \;\geq\; \frac{1-\epsilon}{2} \qquad \text{and} \qquad \tilde{P}^n_Y(F_i) \;\geq\; \frac{1-\epsilon}{2}\;.$$

Applying Theorem 2.3, we obtain

$$-\frac{1}{n}\log Q^n_{XY}(C_i \times F_i) \;\leq\; d(\Phi_n \cup \{P_X\},\, \Psi_n \cup \{P_Y\}\,||Q) + \nu_n \;. \tag{4.7}$$

Similarly for $C_j \times F_j$ we have

$$-\frac{1}{n}\log Q^n_{XY}(C_j \times F_j) \;\leq\; d(\Phi^*_n \cup \{\bar{P}_X\},\, \Psi^*_n \cup \{\bar{P}_Y\}\,||Q) + \nu_n \;. \tag{4.8}$$

We must show that the smaller of the two bounds appearing in equations (4.7) and (4.8) is less than or equal to $\theta^{(3)}$ as defined in the statement of the theorem. This is certainly true if $\Psi_n^* = \Psi_n^c$, since we can then take

$$\{\Phi, \bar{\Phi}\} = \{\Phi_n, \Phi_n^*\} \qquad \text{and} \qquad \{\Psi, \bar{\Psi}\} = \{\Psi_n, \Psi_n^*\}$$

in the definition of $\theta^{(3)}$. Otherwise, if w.l.o.g. $\Psi_n^* = \mathcal{P}(\mathcal{Y})$, the same conclusion can be reached by taking

$$\{\Phi, \bar{\Phi}\} = \{\Phi_n, \Phi_n^*\} \qquad \text{and} \qquad \{\Psi, \bar{\Psi}\} = \{\Psi_n, \Psi_n^c\} .$$

Thus we have obtained

$$-\frac{1}{n} \log Q_{XY}^n(\mathcal{A}_n) \ \leq \ \theta^{(3)} + \nu_n .$$

This, together with our result for Case 1, yields the converse statement

$$\theta(2, 2, \epsilon) \ \leq \ \theta^{(1)} \vee \theta^{(3)} . \qquad \triangle$$

REMARKS. (a) It is shown in the Appendix that $\theta^{(3)}$ can also expressed in the simpler form

$$\theta^{(3)} \ = \ \{D(P_X||Q_X) \ \wedge \ D(\bar{P}_Y||Q_Y)\} \ \vee \ \{D(\bar{P}_X||Q_X) \ \wedge \ D(P_Y||Q_Y)\} . \qquad (4.9)$$

This characterization also simplifies the determination of the maximizing classes Φ and Ψ in the original definition of $\theta^{(3)}$.

(b) The definition of the asymptotically optimal acceptance regions $\mathcal{A}_n^{(2)}$ and $\mathcal{A}_n^{(3)}$ in the proofs of Theorems 4.2 and 4.3 depends implicitly on the alternative distribution Q_{XY} through the choice of the optimal classes Φ and Ψ; this is not the case with $\mathcal{A}_n^{(1)}$.

(c) Which of the alternative exponents is the dominant one depends on Π and Q_{XY}. To show this, in what follows we let Q_{XY} be a product distribution on $\mathcal{X} \times \mathcal{Y}$, i.e., $Q_{XY} = Q_X \times Q_Y$, where $Q_X > 0$, $Q_Y > 0$. Then it is quite straightforward to show that

$$\begin{aligned}
\theta^{(1)} &= \{D(P_X||Q_X) \ \wedge D(\bar{P}_X||Q_X)\} \ + \ \{D(P_Y||Q_Y) \ \wedge D(\bar{P}_Y||Q_Y)\} ; \\
\theta^{(2)} &= \{(D(P_X||Q_X) + D(P_Y||Q_Y)) \ \wedge \ D(\bar{P}_Y||Q_Y)\} \\
&\qquad \vee \ \{D(P_Y||Q_Y) \ \wedge \ (D(\bar{P}_X||Q_X) + D(\bar{P}_Y||Q_Y))\} ; \\
\theta^{(3)} &= \{D(P_X||Q_X) \ \wedge \ D(\bar{P}_Y||Q_Y)\} \ \vee \ \{D(\bar{P}_X||Q_X) \ \wedge \ D(P_Y||Q_Y)\} .
\end{aligned}$$

Consider first the situation in which $\bar{P}_X = Q_X$ and $D(\bar{P}_Y||Q_Y) > D(P_X||Q_X) + D(P_Y||Q_Y)$. From the above we obtain

$$\theta^{(1)} \ = \ D(P_Y||Q_Y) , \qquad \theta^{(2)} \ = \ D(P_X||Q_X) + D(P_Y||Q_Y) , \qquad \theta^{(3)} \ = \ D(P_X||Q_X) .$$

Thus provided all above divergences are positive and distinct, we obtain either $\theta^{(2)} > \theta^{(3)} > \theta^{(1)}$ or $\theta^{(2)} > \theta^{(1)} > \theta^{(3)}$.

As another example, consider the case in which all distributions are distinct, and

$$D(P_X||Q_X) \;=\; D(\bar{P}_X||Q_X)\,, \qquad D(P_Y||Q_Y) \;=\; D(\bar{P}_Y||Q_Y)\,.$$

Then

$$\theta^{(1)} \;=\; D(P_X||Q_X) \;+\; D(P_Y||Q_Y)\,, \qquad \theta^{(2)} \;=\; D(P_Y||Q_Y)\,,$$
$$\theta^{(3)} \;=\; D(P_X||Q_X) \;\wedge\; D(P_Y||Q_Y)\,.$$

We thus obtain either $\theta^{(1)} > \theta^{(2)} > \theta^{(3)}$ or $\theta^{(1)} > \theta^{(2)} = \theta^{(3)}$.

5. Dependence of the error exponent on ϵ.

Theorems 4.2 and 4.3 were derived under the assumption that the distributions P_{XY} and $\bar{P}_{XY}$ have distinct X and Y marginals. As it turns out, the conclusions of these theorems are true even if this assumption is not. Indeed, it is easy to show that if $P_X = \bar{P}_X$ or $P_Y = \bar{P}_Y$, then $\mathcal{A}_n^{(1)}$ is optimal, and $\theta^{(1)}$ dominates both $\theta^{(2)}$ and $\theta^{(3)}$.

If $|\Pi| > 2$, and the codebook sizes M and N are fixed at levels below the thresholds given in Theorem 4.1, then it is still possible to derive versions of Theorems 4.2 and 4.3 in which the acceptance regions $\mathcal{A}_n^{(2)}$ and $\mathcal{A}_n^{(3)}$ are constructed by first grouping distributions in Π_X and Π_Y together, and then partitioning $\mathcal{P}(\mathcal{X})$ and $\mathcal{P}(\mathcal{Y})$ appropriately. Our final result illustrates this procedure, and more importantly, it reveals a hitherto unseen aspect of this problem: specifically, if the marginals of some distributions in Π coincide, the error exponent may depend on the level ϵ. This is certainly a surprising discovery, considering the chain of strong converse theorems which have been derived in [1-4], and in this work.

NOTATION. $1_{\mathcal{X}}$ denotes the set of degenerate distributions on $\mathcal{P}(\mathcal{X})$.

THEOREM 5.1. *Let* $\Pi < \infty$, $M = 2$, *and* $N \geq |\Pi_Y| + 1$. *Also, let* $\{\Delta, \bar{\Delta}\}$ *denote a partition of* Π. *If* $Q_{XY} > 0$, *then for* $\epsilon \in (0, 1/2) \cup (1/2, 1)$, *the following is true:*

$$\theta(2, N, \epsilon) \;=\; \theta^{(1)} \;\vee\; \theta^{(4)}(\epsilon),$$

where

$$\theta^{(1)} \;=\; d(\Pi_X, \Pi_Y||Q)\,,$$

$$\theta^{(4)}(\epsilon) \;=\; \begin{cases} \max_{\Delta,\bar{\Delta}:\ \Delta_X \cap \bar{\Delta}_X = \emptyset}\ \tau(\Delta,\bar{\Delta}), & \text{if } 0 < \epsilon < \frac{1}{2}; \\ \max\limits_{\substack{\Delta,\bar{\Delta}: \\ \Delta_X \cap \bar{\Delta}_X \cap 1_{\mathcal{X}} = \emptyset}} \tau(\Delta,\bar{\Delta}), & \text{if } \frac{1}{2} < \epsilon < 1, \end{cases}$$

and

$$\tau(\Delta,\bar{\Delta}) = d(\Delta_X, \Delta_Y||Q) \;\wedge\; d(\bar{\Delta}_X, \bar{\Delta}_Y||Q)$$
$$\wedge \inf_{\tilde{P}_X}\{d(\tilde{P}_X, \Delta_Y||Q) \;\vee\; d(\tilde{P}_X, \bar{\Delta}_Y||Q)\}\,.$$

REMARK. We have been unable to evaluate $\theta(2, N, 1/2)$.

PROOF. *Direct part.* Once again it is feasible to construct $\mathcal{A}_n^{(1)}$ as defined in the proof of Theorem 4.2, whence we obtain $\theta(2, N, \epsilon) \geq \theta^{(1)}$.

To construct $\mathcal{A}_n^{(2)}$ by analogy to Theorem 4.2, we partition the space Π_X into Λ, $\bar{\Lambda}$, and the space $\mathcal{P}(\mathcal{X}) - \mathcal{B}_\eta(\Pi_X)$ into Φ, $\bar{\Phi}$. We then have

$$\mathcal{A}_n^{(2)} = \Big(\bigcup_{\tilde{P}_X \in \Phi \cup \mathcal{B}_\eta(\Lambda)} \tilde{T}_X \times \bigcup_{\tilde{P}_{XY} \in \Pi:\ \tilde{P}_X \in \Lambda} \tilde{T}_{Y,\eta} \Big) \cup \Big(\bigcup_{\tilde{P}_X \in \bar{\Phi} \cup \mathcal{B}_\eta(\bar{\Lambda})} \tilde{T}_X \times \bigcup_{\tilde{P}_{XY} \in \Pi:\ \tilde{P}_X \in \bar{\Lambda}} \tilde{T}_{Y,\eta} \Big)$$

which is readily seen to satisfy the type I error constraint for every ϵ and every distribution in Π.

Note that instead of partitioning Π_X into Λ and $\bar{\Lambda}$, one can begin by partitioning Π itself into Δ and $\bar{\Delta}$ such that $\Delta_X \cap \bar{\Delta}_X = \emptyset$. Then one can write equivalently

$$\mathcal{A}_n^{(2)} = \Big(\bigcup_{\tilde{P}_X \in \Phi \cup \mathcal{B}_\eta(\Delta_X)} \tilde{T}_X \times \bigcup_{\tilde{P}_Y \in \Delta_Y} \tilde{T}_{Y,\eta} \Big) \cup \Big(\bigcup_{\tilde{P}_X \in \bar{\Phi} \cup \mathcal{B}_\eta(\bar{\Delta}_X)} \tilde{T}_X \times \bigcup_{\tilde{P}_Y \in \bar{\Delta}_Y} \tilde{T}_{Y,\eta} \Big) ,$$

and by the argument given in the proof of Theorem 4.2,

$$\begin{aligned} \theta(2, N, \epsilon) \ \geq\ \tau(\Delta, \bar{\Delta}) = d(\Delta_X, \Delta_Y \| Q) \ \wedge\ d(\bar{\Delta}_X, \bar{\Delta}_Y \| Q) \\ \wedge \inf_{\tilde{P}_X} \{ d(\tilde{P}_X, \Delta_Y \| Q) \ \vee\ d(\tilde{P}_X, \bar{\Delta}_Y \| Q) \} \ . \end{aligned}$$

Taking the maximum over all partitions $\{\Delta, \bar{\Delta}\}$ of Π satisfying $\Delta_X \cap \bar{\Delta}_X = \emptyset$, we obtain for all $\epsilon \in (0, 1)$,

$$\theta(2, N, \epsilon) \ \geq \max_{\Delta, \bar{\Delta}:\ \Delta_X \cap \bar{\Delta}_X = \emptyset} \tau(\Delta, \bar{\Delta}) \ .$$

The constraint $\Delta_X \cap \bar{\Delta}_X = \emptyset$ is essential in the above construction of $\mathcal{A}_n^{(2)}$; its removal would allow

$$C_1' \ = \bigcup_{\tilde{P}_X \in \Phi \cup \mathcal{B}_\eta(\Delta_X)} \tilde{T}_X \qquad \text{and} \qquad C_2' \ = \bigcup_{\tilde{P}_X \in \bar{\Phi} \cup \mathcal{B}_\eta(\bar{\Delta}_X)} \tilde{T}_X$$

to have nonempty intersection and hence be inadmissible under the given compression scheme. If, however, $1/2 < \epsilon < 1$, then it is possible to relax the said constraint to

$$\Delta_X \ \cap\ \bar{\Delta}_X \ \cap\ 1_X \ =\ \emptyset$$

in the following manner. For every $\tilde{P}_X$ that lies in $\mathcal{B}_\eta(\Delta_X \cap \bar{\Delta}_X)$ (and hence not in 1_X if η is properly chosen), we can partition $\tilde{T}_X$ into two sets $\tilde{T}_X^+$ and $\tilde{T}_X^-$ of sizes that differ by at most 1, and redefine C_1' and C_2' by

$$C_1' \ = \bigcup_{\tilde{P}_X \in \Phi\, \cup \mathcal{B}_\eta(\Delta_X - \bar{\Delta}_X)} \tilde{T}_X \cup \bigcup_{\tilde{P}_X \in \mathcal{B}_\eta(\Delta_X \cap \bar{\Delta}_X)} \tilde{T}_X^+$$

and

$$C_2' = \bigcup_{\tilde{P}_X \in \bar{\Phi} \cup \mathcal{B}_\eta(\bar{\Delta}_X - \Delta_X)} \tilde{T}_X \cup \bigcup_{\tilde{P}_X \in \mathcal{B}_\eta(\Delta_X \cap \bar{\Delta}_X)} \tilde{T}_X^- \ .$$

We can then complete the construction of $\mathcal{A}_n^{(2)}$ in the usual manner.

It is easily seen that for every $P_{XY} \in \Pi$ such that $P_X \notin \Delta_X \cap \bar{\Delta}_X$, and every $\epsilon \in (0,1)$,

$$P_{XY}^n(\mathcal{A}_n^{(2)}) \ \geq \ 1 - \epsilon$$

for n sufficiently large. The same is true for for every $P_{XY} \in \Pi$ such that $P_X \in \Delta_X \cap \bar{\Delta}_X$, if $\epsilon \in (1/2, 1)$. To see this, let w.l.o.g. $P_{XY} \in \Delta$. Then

$$\begin{aligned} P_{XY}^n(\mathcal{A}_n^{(2)}) &\geq P_{XY}^n\Big(\bigcup_{\tilde{P}_X \in \mathcal{B}_\eta(P_X)} \tilde{T}_X^+ \times T_{Y,\eta} \Big) \\ &\geq P_X^n\Big(\bigcup_{\tilde{P}_X \in \mathcal{B}_\eta(P_X)} \tilde{T}_X^+ \Big) + P_Y^n(T_{Y,\eta}) - 1 \\ &\geq \frac{1}{2} - \lambda_n + 1 - \frac{|\mathcal{Y}|}{4n\eta^2} - 1 \end{aligned}$$

where $\lambda_n \to 0$ since $\mathcal{B}_\eta(P_X)$ contains no degenerate distributions. We conclude that for n sufficiently large,

$$P_{XY}^n(\mathcal{A}_n^{(2)}) \ \geq \ 1 - \epsilon \ .$$

By computing the error exponent as before, we obtain for $1/2 < \epsilon < 1$,

$$\theta(2, N, \epsilon) \ \geq \max_{\substack{\Delta, \bar{\Delta}: \\ \Delta_X \cap \bar{\Delta}_X \cap 1_X = \emptyset}} \tau(\Delta, \bar{\Delta}) \ .$$

This concludes the proof of the positive part.

Converse part. As in the proof of the converse part of Theorem 4.2, we express $\mathcal{A}_n$ as

$$\mathcal{A}_n \ = \ (C_1 \times F_1) \ \cup \ (C_2 \times F_2),$$

where C_1 and C_2 form a partition of $\mathcal{X}^n$, and at most one of F_1, F_2 may be empty. Once again, two cases may arise.

Case 1. For i and j distinct, we have

$$(\forall P_{XY} \in \Pi) \qquad P_{XY}^n(C_i \times F_i) \ \geq \ P_{XY}^n(C_j \times F_j) \ .$$

This implies that

$$-\frac{1}{n} \log Q^n(\mathcal{A}_n) \ \leq \ \theta^{(1)} + \nu_n \ .$$

Case 2. The sets Δ and $\bar{\Delta}$ defined below form a nontrivial partition of Π

$$\Delta = \{P_{XY} \in \Pi\colon\ P^n_{XY}(C_1 \times F_1) \geq P^n_{XY}(C_2 \times F_2)\}\ ,$$

$$\bar{\Delta} = \{P_{XY} \in \Pi\colon\ P^n_{XY}(C_1 \times F_1) < P^n_{XY}(C_2 \times F_2)\}\ .$$

We claim further that $\Delta_X \cap \bar{\Delta}_X \cap 1_X = \emptyset$. Indeed, if there exist $P_{XY} \in \Delta$ and $\tilde{P}_{XY} \in \bar{\Delta}$ such that $P_X = \tilde{P}_X$, then

$$P^n_X(C_1) \ \geq\ \frac{1-\epsilon}{2}, \qquad\qquad \tilde{P}^n_X(C_2) \ =\ P^n_X(C_2) \ >\ \frac{1-\epsilon}{2}\ .$$

Since C_1 and C_2 are complementary and have positive probability under P^n_X, P_X cannot be degenerate.

As in Case 2 in the proof of the converse of Theorem 4.2, we obtain for all $\epsilon \in (0,1)$,

$$-\frac{1}{n}\log Q^n(\mathcal{A}_n) \ \leq\ \tau(\Delta, \bar{\Delta}) + \nu_n\ .$$

It remains to show that if $\epsilon \in (0, 1/2)$, the above bound is also valid for a partition $\{\Omega, \bar{\Omega}\}$ of Π such that $\Omega_X \cap \bar{\Omega}_X = \emptyset$. To construct such a partition, we argue as follows.

For $P_X \in \Pi_X$, we consider the set $\mathcal{H}(P_X)$ of distributions in Π that have P_X as X-marginal:

$$\mathcal{H}(P_X) \stackrel{\text{def}}{=} \{\tilde{P}_{XY} \in \Pi\colon \tilde{P}_X = P_X\}\ .$$

We let $\lambda > 0$ be independent of n, and we assume for the moment that for every $P_X \in \Pi_X$, we can find $i \in \{1, 2\}$ such that

$$(\forall \tilde{P}_{XY} \in \mathcal{H}(P_X)) \qquad \tilde{P}^n_{XY}(C_i \times F_i) \geq \lambda\ . \tag{5.1}$$

If so, then we can partition Π_X into Λ_1 and Λ_2 by placing each of the members P_X of Π_X in Λ_i iff i is the smallest index for which the above relationship holds. This in turn yields a partition $\Omega, \bar{\Omega}$ of Π through

$$\Omega \ =\ \bigcup_{P_X \in \Lambda_1} \mathcal{H}(P_X) \qquad \text{and} \qquad \bar{\Omega} \ =\ \bigcup_{P_X \in \Lambda_2} \mathcal{H}(P_X)\ .$$

Clearly $\Omega_X = \Lambda_1$, $\bar{\Omega}_X = \Lambda_2$, and from the definition of Λ_i and relationship (5.1), we obtain the desired bound

$$-\frac{1}{n}\log Q^n(\mathcal{A}_n) \ \leq\ \tau(\Omega, \bar{\Omega}) + \nu_n\ .$$

Thus the issue is to prove that for suitable $\lambda > 0$, every $P_X \in \Pi_X$ is such that (5.1) holds for $i = 1$ or $i = 2$. By definition of the classes Δ and $\bar{\Delta}$, this is true for $P_X \in \Delta_X - \bar{\Delta}_X$ and $P_X \in \bar{\Delta}_X - \Delta_X$. To show that it is also true for $P_X \in \Delta_X \cap \bar{\Delta}_X$, assume the contrary, namely that there exists $P_{XY} \in \Delta$ and $\tilde{P}_{XY} \in \bar{\Delta}$ with $\tilde{P}_X = P_X$ and

$$P^n_{XY}(C_1 \times F_1) < \lambda, \qquad \tilde{P}^n_{XY}(C_2 \times F_2) < \lambda\ .$$

This implies that

$$P_X^n(C_1) \;\geq\; \tilde{P}_{XY}^n(C_1 \times F_1) \;>\; 1-\epsilon-\lambda\;,$$
$$P_X^n(C_2) \;\geq\; P_{XY}^n(C_2 \times F_2) \;>\; 1-\epsilon-\lambda\;,$$

and hence

$$P_X^n(C_1) + P_X^n(C_2) \;>\; 2-2\epsilon-2\lambda\;.$$

Thus if $\epsilon < 1/2$, we can set $\lambda = (1-2\epsilon)/3 > 0$ to obtain the desired contradiction:

$$P_X^n(C_1) + P_X^n(C_2) \;>\; 1+\lambda\;. \qquad \triangle$$

As a final remark, the dependence of $\theta^{(4)}(\epsilon)$ on ϵ is nontrivial. If we consider the simple setup in which $\Pi = \{P_{XY}, \bar{P}_{XY}, \hat{P}_{XY}\}$ with $P_X = \hat{P}_X$ and $\bar{P}_Y = \hat{P}_Y$, then it is possible to choose the above distributions so that the error exponent for $0 < \epsilon < 1/2$ is strictly less than for $1/2 < \epsilon < 1$.

6. Concluding remarks

The positivity assumption on the alternative hypothesis was essential for the derivation of the converse results in this paper. As was mentioned in [1], without this assumption, we could not have applied the blowing-up lemma in the proof of the pivotal Theorem 2.3. We hope that this obstacle will eventually be removed.

APPENDIX

If we let, for all $\Phi \subset \mathcal{P}(\mathcal{X})$ and $\Psi \subset \mathcal{P}(\mathcal{Y})$,

$$\alpha(\Phi,\Psi) \;\stackrel{\text{def}}{=}\; \inf_{(\tilde{P}_X,\tilde{P}_Y)\in(\Phi\times\Psi)\cup(\Phi^c\times\Psi^c)} d(\tilde{P}_X,\tilde{P}_Y||Q)\;,$$

then the definition of $\theta^{(3)}$ becomes

$$\theta^{(3)} = \sup_{\substack{\Phi,\Psi:\\ (P_X,P_Y)\in\Phi\times\Psi,\\ (\bar{P}_X,\bar{P}_Y)\in\Phi^c\times\Psi^c}} \alpha(\Phi,\Psi)\;. \tag{A.1}$$

We must show that $\theta^{(3)}$ can be expressed as in (4.9), or equivalently, that $\theta^{(3)} = \theta'$, where

$$\theta' \;\stackrel{\text{def}}{=}\; \{D(P_X||Q_X) \;\wedge\; D(\bar{P}_Y||Q_Y)\} \;\vee\; \{D(\bar{P}_X||Q_X) \;\wedge\; D(P_Y||Q_Y)\}\;.$$

(i) To show that $\theta^{(3)} \geq \theta'$, let $\Phi = \{P_X\}$ and $\Psi = \{\bar{P}_Y\}^c$. Then

$$\begin{aligned}\theta^{(3)} \;\geq\; \alpha(\Phi,\Psi) &= d(P_X,\{\bar{P}_Y\}^c||Q) \wedge d(\{P_X\}^c,\bar{P}_Y||Q)\\ &= D(P_X||Q_X) \wedge D(\bar{P}_Y||Q_Y)\;,\end{aligned} \tag{A.2}$$

where the last equality follows by continuity of divergence. Similarly, if $\Phi = \{\bar{P}_X\}^c$ and $\Psi = \{P_Y\}$, we have

$$\theta^{(3)} \;\geq\; D(P_Y||Q_Y) \wedge D(\bar{P}_X||Q_X)\;. \tag{A.3}$$

Combining (A.2) with (A.3) we obtain $\theta^{(3)} \geq \theta'$.

(ii) To show the reverse inequality $\theta^{(3)} \leq \theta'$, let

$$A = \text{cl}\Phi, \quad \bar{A} = \text{cl}\Phi^c, \quad B = \text{cl}\Psi, \quad \bar{B} = \text{cl}\Psi^c ,$$

where cl denotes closure under sup norm. Then by continuity of divergence,

$$\alpha(\Phi, \Psi) = \min_{(\tilde{P}_X, \tilde{P}_Y) \in (A \times B) \cup (\bar{A} \times \bar{B})} d(\tilde{P}_X, \tilde{P}_Y || Q) .$$

We must show that $\alpha(\Phi, \Psi) \leq \theta'$ for every Φ and Ψ. This is trivially true if $(Q_X, Q_Y) \in (A \times B) \cup (\bar{A} \times \bar{B})$, in which case we have

$$\alpha(\Phi, \Psi) = d(Q_X, Q_Y || Q) = 0 .$$

Hence we may assume that

$$(Q_X, Q_Y) \notin (A \times B) \cup (\bar{A} \times \bar{B}) . \tag{A.4}$$

We provide an upper bound $\alpha(\Phi, \Psi)$ as follows. First we note that

$$(A \cap \bar{A}) \times \mathcal{P}(\mathcal{Y}) \subset (A \times B) \cup (\bar{A} \times \bar{B}) ,$$

so that

$$\begin{aligned} \alpha(\Phi, \Psi) &\leq \min_{(\tilde{P}_X, \tilde{P}_Y) \in (A \cap \bar{A}) \times \mathcal{P}(\mathcal{Y})} d(\tilde{P}_X, \tilde{P}_Y || Q) \\ &= \min_{(\tilde{P}_X, \tilde{P}_Y):\ \tilde{P}_X \in A \cap \bar{A}} D(\tilde{P}_{XY} || Q_{XY}) . \end{aligned}$$

Using the log-sum inequality, we can show that above minimum is equal to

$$\min_{\tilde{P}_X \in A \cap \bar{A}} D(\tilde{P}_X || Q_X) .$$

By symmetry we conclude that

$$\alpha(\Phi, \Psi) \leq \min_{\tilde{P}_X \in A \cap \bar{A}} D(\tilde{P}_X || Q_X) \ \wedge \ \min_{\tilde{P}_Y \in B \cap \bar{B}} D(\tilde{P}_Y || Q_Y) . \tag{A.5}$$

Two cases may arise, according to whether Q_X lies in A or $\bar{A}$ (note that it cannot lie in $A \cap \bar{A}$ by (A.4)).

Case 1. $Q_X \in A$: Since $\bar{P}_X \in \bar{A}$, there exists $\lambda \in (0, 1]$ such that

$$\hat{P}_X = \lambda \bar{P}_X + (1 - \lambda) Q_X \in A \cap \bar{A} .$$

This yields

$$\begin{aligned}\min_{\tilde{P}_X \in A \cap \bar{A}} D(\tilde{P}_X||Q_X) &\le D(\hat{P}_X||Q_X) \\ &\le \lambda D(\bar{P}_X||Q_X) + (1-\lambda) D(Q_X||Q_X) \\ &\le D(\bar{P}_X||Q_X) \ ,\end{aligned}$$

where the last inequality follows by convexity of divergence.

From (A.4), we also have that $Q_Y \in \bar{B}$. An analogous argument for $Q_Y \in \bar{B}$ and $P_Y \in B$ yields

$$\min_{\tilde{P}_Y \in B \cap \bar{B}} D(\tilde{P}_Y||Q_Y) \le D(P_Y||Q_Y) \ .$$

From (A.5), we conclude that

$$\alpha(\Phi, \Psi) \le D(\bar{P}_X||Q_X) \ \wedge \ D(P_Y||Q_Y) \ . \tag{A.6}$$

Case 2. $Q_X \in \bar{A}$: Again (A.4) implies that $Q_Y \in B$. As in Case 1 above, we obtain

$$\alpha(\Phi, \Psi) \le D(P_X||Q_X) \ \wedge \ D(\bar{P}_Y||Q_Y) \ . \tag{A.7}$$

From (A.6) and (A.7) we conclude that $\alpha(\Phi, \Psi) \le \theta'$, and hence also $\theta^{(3)} \le \theta'$. △

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