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Global Stability of TCP with AQM Router

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Abstract

In this study we extend the global stability results of the previous report [1]. We focus on the scenario of a single TCP flow with proportional AQM controller at the router. By using Lyapunov functional method, we are able to prove the global stability of such nonlinear system with feedback delay.

I. Introduction

Recent studies on the behaviors of TCP and the design of AQM (Active Queueing Management) algorithms at bottleneck routers rely heavily on stability conditions for the TCP system. Based on local linearized analysis, performance of RED (Random Early Detection) has been studied ([2], [3]), and various AQM algorithms have been proposed ([4], [5], etc) to achieve certain performance metrics.

Since near equilibrium analysis only gives necessary stability conditions, it is very important to know whether the rules obtained from those local analyses can be applied to the global situation. Efforts have been made in this direction in the past few years. Due to the complexity of the TCP dynamics coupled with network feedback, results are often dealt with some simplified descriptions of TCP (no delay and/or single flow and single bottleneck router) or stability condition on a very restricted region around the equilibrium state ([6], [7]). A typical method is applying Razumikhin thoerem with quadratic Lyapunov functions. Since the conditions in Razumikhin theorem when using a simple quadratic Lyapunov function often result in a very conservative estimated stability region, it is desirable to expand the estimated stability region by using a more sophisticated Lyapunov functional. This report explores this approach on a single flow single bottleneck case.

There are also numerous studies on the global nonlinear stability problem of network communication system other than the TCP system we consider here. Rate control network is especially well covered. Some powerful tools, such as passivity theory [8] and ISS Small-Gain theorem [9], are successfully used to obtain robust stability conditions for this type of network. Recent study [10] shows the relationship between the fluid flow model and the underlying discrete time system in determining the global stability condition. However, due to different dynamical structure and feedback information, these methods cannot be directly applied to the system we consider in this report.

This work continues part of the research in [1], which considers the heterogeneous TCP/AQM network without feedback delay. We try to establish global stability condition for a single flow single bottleneck router network with proportional controller. It turns out that a properly chosen Lyapunov functional shows the system is globally stable.

II. Global Stability of TCP systems

We only study a single-flow scenario with a proportional (linear) queue marking function at the bottleneck router. Consider a TCP/AQM dynamics described as below,

$$\begin{aligned} \dot{r} &= \frac{1-kq(t-\tau)}{\tau^2} - \frac{1}{2}r(t)^2kq(t-\tau), \\ \dot{q} &= (r-C)_q^+, \end{aligned}$$

r and q are TCP rate and bottleneck queue size as usual and k is the slope of the marking function. By re-scaling the time $t = \tau s$ to normalize the delay, we get

$$\begin{aligned} \dot{\tilde{r}} &= -\frac{\tilde{q}(s-1)}{\tau q^*} - \tau(\tilde{r}/2 + C)kq(s-1)\tilde{r}(s) \\ \dot{\tilde{q}} &= \tau\tilde{r}. \end{aligned} \tag{1}$$

Here we let $r = r^* + \tilde{r}$ and $q = q^* + \tilde{q}$, where $r^* = C$ and $q^* = k^{-1}\left(\left(1 + C^2\tau^2/2\right)^{-1}\right)$ are equilibrium rate and queue length respectively.

Firstly, let us restate the local stability condition for (1) [1]. The linearized version of (1) is

$$\begin{aligned} \dot{\tilde{r}} &= -\frac{f'(q^*)}{\tau f(q^*)}\tilde{q}(s-1) - \tau f(q^*)C\tilde{r}(s) \\ \dot{\tilde{q}} &= \tau\tilde{r}, \end{aligned} \tag{2}$$

and we have

Lemma 1: Suppose $f'(q^*) < 0.5006$ and let t_k, $k = \{0, 1\}$ be the solutions of the following equation,

$$\frac{\omega_k^2}{\cos\omega_k} = f'(q^*)(1 + t_k^2 C^2/2), \tag{3}$$

and ω_0, ω_1 be respectively the first and the second smallest positive $\omega \in [0, \pi/2]$ satisfying

$$\frac{\cos\omega}{\omega^2} f'(q^*) + \frac{\omega^4\tan^2\omega}{2f'(q^*)\cos\omega} = 1. \tag{4}$$

If $t_0 < \tau < t_1$, then all the characteristic roots $\lambda(\tau)$ of Equations (2) have negative real parts. There is an infinite series $\tau_k, k = 0, 1, \ldots$, such that there are exactly 2 pure imaginary roots when $\tau = \tau_k$. $\lambda(\tau)$ is differentiable at $\tau = t_1$, and $\mathrm{Re}\lambda'(t_1) > 0$. If $f'(q^*) > 0.5006$, the system (2) is unstable for all τ. For $\tau > t_1$, there are precisely two characteristic roots λ of Equations (2) in the region $\mathrm{Re}\lambda > 0$ and $-\pi < \mathrm{Im}\lambda < \pi$.

We have the following result for the global stability condition for (1).

Theorem 1: Suppose $\tau C \gg 1$, we can choose k to make the system (1) globally stable.

Proof: Local linear stability result in Lemma 1 tells us when $\tau C \gg 1$, we can choose $k = 4C^{-3}\tau^{-3}$ to obtain local stability. For simplicity, let us denote $\psi(t) = [\tilde{r}(t), \tilde{q}(t)]^T$. Also denote η as a 2×2 matrix-valued function with bounded variation on $[-1, 0]$:

$$\eta(s) = \begin{cases} [0 \quad 0; 0 \quad 0], & s = -1 \\ [0 - \tau^{-1}/q^*; 0 \quad 0], & -1 < s < 0 \\ [-\tau C k q^* - \tau^{-1}/q^*; \tau \quad 0], & s = 0 \end{cases}$$

It is easy to see that the linearized version of (1) can be written as

$$\dot{\psi}(t) = \int_{-1}^{0} d\eta(\theta)\psi_t(\theta).$$

From [11], we can choose the following Lyapunov functional for (1),

$$\begin{aligned} V(\psi) &= \psi^T(0)Y(0)\psi(0) \\ &+ 2\psi^T(0)\int_{-1}^{0}\int_{u}^{0} Y(-u+\theta)d\eta(u)\psi(\theta)d\theta \\ &+ \int_{-1}^{0}\int_{-h}^{0} ds\psi^T(s)d\eta^T(h) \\ &\times \int_{-1}^{0}\int_{-1}^{0} Y(-s+h-u+\theta)d\eta(u)\psi(\theta)d\theta \end{aligned} \tag{5}$$

It is known that if (1) is locally stable, we can find $Y \in C([-1, 1], \mathrm{R}^{2\times 2})$ satisfies $\dot{Y}(0) + \dot{Y}^T(0) = -W$ where W is a positive definite matrix. Here $\dot{Y}(0)$ is defined as $d^+Y(0)/dt$. Then $V(\psi)$ is positive definite if Y satisfies additionally:

(i) $Y(t)$ is continuously differentiable for $t \neq 0$.
(ii) $Y(0)$ is symmetric and $Y(t) = Y^T(-t)$.
(iii) $Y(t) = \int_{-1}^{0} d\eta^T(s)Y(s+t)$.

Take the derivative of $V(\psi)$ we can show

$$\begin{aligned} \dot{V}\psi &= -\psi^T(0)W\psi(0) - 2k\tau(\tilde{r}(0)q(-1)/2 + \tilde{q}(-1)C) \\ &\times (\tilde{r}(0)\tilde{q}(0)Y_{12}(0) + \tilde{r}(0)^2 Y_{11}(0)) \end{aligned}$$

Denote $Y(t) = [y_1 \quad y_2; y_3 \quad y_4]$. From the conditions at which $Y(t)$ has to satisfy we have

$$\begin{bmatrix} \dot{y}_1(t) & \dot{y}_2(t) \\ \dot{y}_3(t) & \dot{y}_4(t) \end{bmatrix} = \begin{bmatrix} -kCq^*\tau & \tau \\ 0 & 0 \end{bmatrix} \begin{bmatrix} y_1(t) & y_2(t) \\ y_3(t) & y_4(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ -\frac{1}{\tau q^*} & 0 \end{bmatrix} \begin{bmatrix} y_1(1-t) & y_2(1-t) \\ y_3(1-t) & y_4(1-t) \end{bmatrix}.$$

We can solve the above equation explicitly. By some calculation, $y_1(t)$ is the solution of the following fourth-degree differential equation,

$$y_1^{(4)} - k^2C^2\tau^2q^{*2}y_1^{(2)} - \frac{1}{q^{*2}}y_1 = 0.$$

Given initial condition $\dot{y}_1(0) = -1/2$, a solution of the above equation is

$$y_1(t) = \frac{1}{2u}\frac{(u^2 + kC\tau q^* u)e^{-ut} - q^{*-1}e^{u(t-1)}}{u^2 + kC\tau q^* u + q^{*-1}e^{-u}}$$

where

$$u = \sqrt{\frac{k^2C^2\tau^2q^{*2}}{2} + \sqrt{\frac{k^4C^4\tau^4q^{*4}}{4} + \frac{1}{q^{*2}}}}.$$

It is easy to see that when $\tau C \gg 1$, $u = \sqrt{2/(\tau C)}$ and $y_1(t) = (1-t)/2$. We can also deduct the following relations,

$$y_3(1) + \frac{1}{\tau q^*} = y_3(0),$$

and for $\dot{Y}(0) + \dot{Y}^T(0) = -W = -[W_1 \quad W_2; W_2 \quad W_4]$,

$$\begin{cases} 2\dot{y}(0) = -1, \\ -kC\tau q^* y_2(0) + \tau y_4(0) - (\tau q^*)^{-1}y_1(1) = W_2, \\ -2(\tau q^*)^{-1}y_3(1) = W_4. \end{cases}$$

So we can set $y_2(0) = y_3(0) = \tau^{-1}q^{*-1}$, and choose $y_4(0)$ properly so that

$$W = \begin{bmatrix} -1 & 0 \\ 0 & -\frac{2}{\tau^2 q^{*2}} \end{bmatrix} < 0.$$

Now $\dot{V}(\psi)$ becomes

$$\begin{aligned} \dot{V}(\psi) &= -\tilde{r}(0)^2 - 2\tau^{-2}/q^*\tilde{q}(0)^2 - 2k\tau(\tilde{r}(0)q(-1)/2 + \tilde{q}(-1)C) \\ &\times (\frac{2}{\tau q^*}\tilde{r}(0)\tilde{q}(0) + \frac{1}{2}\tilde{r}(0)^2). \end{aligned}$$

Suppose q is confined in the region $[0, mC\tau]$, for any positive integer m. We can upperbound the above equation by

$$\dot{V}(\psi) \le \begin{cases} -\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 + \frac{4\tilde{r}(0)}{C\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}), & \tilde{r}(0)(\tilde{r}(0)/2 + 2\tilde{q}(0)/(\tau q^*)) \ge 0 \\ -\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 - (2mC + \tilde{r}(0)(m+1/2))\frac{4\tilde{r}(0)}{C^2\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}), & \tilde{r}(0)(\tilde{r}(0)/2 + 2\tilde{q}(0)/(\tau q^*)) \le 0. \end{cases} \tag{6}$$

We can observe from Figure 1 in Region I and Region III the upper condition of (6) is satisfied and $\dot{V}(\psi)$ becomes

$$\begin{aligned} \dot{V}(\psi) &\le -\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 + \frac{4\tilde{r}(0)}{C\tau}(\frac{\tilde{r}(0)}{2} + \frac{2\tilde{q}(0)}{\tau q^*}) \\ &= -(1 - 2(C\tau)^{-1})\tilde{r}(0)^2 - \frac{2}{\tau^2q^{*2}}\tilde{q}(0)^2 + \frac{4}{\tau q^{*2}}\tilde{r}(0)\tilde{q}(0) \end{aligned}$$

The last part of the above equation is strictly below zero when ψ is other than 0 if the condition

$$4\left(1 - \frac{2}{C\tau}\right)\frac{2}{q^{*2}} > \frac{16}{q^{*4}}$$

holds. This is true from our assumption $C\tau \gg 1$.

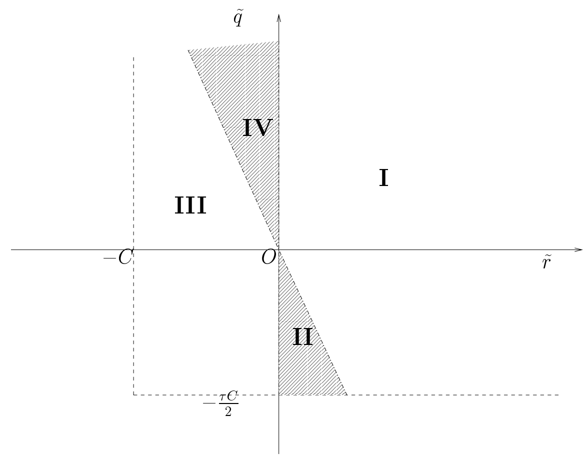

Fig. 1. Region of $\dot{V}(\psi)$ in (6)

In Region II and Region IV of Figure 1, the maximum value of $\dot{V}(\psi)$ should be reached for $\tilde{r} \geq 0$ (Region II). This is because for every $\tilde{r}$ and $\tilde{q}$ in Region IV, we can choose $-\tilde{r}$ and $-\tilde{q}$ in Region II so that the right handside of (6) in the second situation is larger. This argument can be immediately verified from the existence of the $(2mC + \tilde{r}(0)(m + 1/2))$ term. So now let us focus on Region II. The part which involves $\tilde{q}$ is

$$-\frac{2}{\tau^2 q^{*2}}\tilde{q}(0)^2 - (2mC + \tilde{r}(0)(m + 1/2)))\frac{8\tilde{r}(0)}{C^2\tau^2 q^*)}\tilde{q}(0).$$

This reaches maximum in Region II at $\tilde{q}(0) = 0$. So we showed in Region II and IV we also have

$$\dot{V}(\psi) \leq 0.$$

Therefore we conclude that in all regions of the state space, the derivative of the Lyapunov functional (5) is strictly below 0 except when the trajectory is at the equilibrium state. Consequently global stability holds. ■

III. Conclusion

We have shown in this report by applying the right Lyapunov functional, the global stability condition for the TCP/AQM network can be obtained. From the analysis of a single flow single bottleneck scenario, we may conjecture that for general heterogeneous TCP/AQM networks, local stability might imply global stability. It is rather computationally demanding to apply this method to multi-flow multi-bottleneck system. We are currently looking for more efficient ways to analyse this system.

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