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ABSTRACT

The area of any region on a surface

$$x^r = x^r(u^\alpha) \quad , \qquad (r = 1, 2, \cdots, n) \; ; \; (\alpha = 1, 2)$$

is assumed given by

$$\Sigma = \iint_R L \, du^1 \, du^2$$

L is a given function of x^r and $\frac{\partial x^r}{\partial \mu^\alpha}$. The purpose of the thesis is to investigate the differential properties of a space in which Σ is invariant.

Previous work in this subject has been restricted to the case where a g_{rs} defining the length of a vector exists. In this paper no such assumption is made.

It is shown that the given function depends only on the coordinates of a point, x^r, and the components of the Jacobians x^{rs}.

The calculus of variations is employed to obtain a normal vector to any surface which vanishes if the surface is minimal. It is shown that a ds^2 is determined on any surface, but that ds^2 is not necessarily the same for two surfaces tangent along a curve. Conditions are found in the form of the vanishing of a tensor in order that the space possess a g_{rs} with appropriate properties.

It is shown that the minimal equations define a displacement of a surface element passing through a curve, along itself

but away from the curve. These equations are extended to define a displacement of a surface element passing through a curve in any direction away from the curve.

In the last section of the thesis an intrinsic differential is determined which defines the parallel displacement of any bivector, $Z^{rs}(x^k, x^{pq})$ passing through a curve. When the curve is varied in a congruence of a type such that the area of the bivector formed by the tangent vector to the curve and the displacement vector does not change along the curve, two covariant derivative are obtained.

THE DIFFERENTIAL GEOMETRY ASSOCIATED WITH A GIVEN AREA METRIC

By

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INTRODUCTION

Since the introduction by Riemann of a geometry based upon a metric defined by[1]

$$S = \int \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\, dt , \qquad \frac{\partial g_{ij}}{\partial \dot{x}^k} = 0$$

there have been many generalizations of these ideas. One generalization, suggested by Riemann[2] himself, arose by removing the restriction that the function

$$g_{ij}\dot{x}^i\dot{x}^j$$

be a polynomial in the $\dot{x}^i$ of degree two; distance along a curve is then given by

$$S = \int L(x^i, \dot{x}^k)\, dt ,$$

where L is an arbitrary function, only subject to the condition that it be homogeneous of degree one in the $\dot{x}^i$. Such spaces are known as Finsler spaces.[3]

Apparently E. Cartan first conceived the idea of replacing a given distance metric with a given area metric[4]

$$\Sigma = \iint L(x^r, x^r_\alpha)\, du^1 du^2 ,$$

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[1] The notation of the absolute differential calculus will be employed and the summation convention adopted.

[2] B. Riemann. Ueber die Hypothesen welche der Geometrie zu grunde liegen. Habilitationsschrift, 1854. Goett. Abh., 13, 1868.

[3] P. Finsler. Ueber Kurven und Flaechen in allgemeinen Raumen. Dissert. Goettingen, 1918.

[4] E. Cartan. Les espaces metriques fondes sur la notion d'aire. Actualites scientifiques et industrielles, 72, 1933.

and this is the generalization with which this paper is concerned. In 1932 Cartan published an excellent and quite exhaustive treatise[6] to cover the case where the imbedding space is that of three dimensions, and in 1940 Kawaguchi and Hokari discussed the case of odd spaces.[5] In this paper no restrictions shall be made on the dimensionality of the imbedding space.

Cartan is able to take advantage of the fact, peculiar to three dimensions, that an elementary plane

$$dx^r \delta x^s - dx^s \delta x^r$$

determines a unique normal, l_r. Then

$$L(x^r, x^r_\alpha) = L(x^r, l_k).$$

He demonstrates that the function L generates a g_{rs}. A connection is determined of the form

$$\underline{d}\,\lambda^i = d\lambda^i + \Gamma^i_{mn}\lambda^m dx^n + C^i_{mn}\lambda^m \omega^r;$$

$$\omega^i = dl^i + \Gamma^i_{mn} l^m dx^n,$$

in which the coefficients Γ^i_{mn}, and C^i_{mn} are expressed in terms of g_{rs} and its derivatives.

Kawaguchi and Hokari state that

$$L(x^r, x^r_\alpha) = F^{1/2}(x^r, x^{ij}); \quad x^{ij} = x^i_\alpha x^j_\beta e^{\alpha\beta}.$$

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[5] A. Kawaguchi and S. Hokari. Die Grundlegung der Geometrie der fuenf-dimensionalen metrischen Raume auf Grund des Begriffs des zwei-dimensionalen Flaecheninhalts. Proc. Imp. Acad. Tokyo, Vol. XVI, No. 8. 1940.

[6] E. Cartan, Op. cit..

Without mention of the fact that the x^{rs} are not all independent a "tensor" g_{rsmn} is defined by means of

$$g_{rsmn} = \frac{\partial^2 F}{\partial x^{rs}\,\partial x^{mn}} .$$

A g_{ij} is then obtained by summing an appropriate number of $e^{i_1 i_2 \dots i_\lambda}$'s and g_{rsmn}'s. The authors claim that a somewhat complicated calculation reveals that

$$g_{rsmn} = \frac{1}{2}\left(g_{rm}\,g_{sn} - g_{rn}\,g_{sm}\right).$$

The relations which Kawaguchi and Hokari suppose the g_{rsmn} to satisfy permit it to have

$$\frac{n^2(n^2-1)}{12}$$

independent components, yet they find that the g_{rsmn} may be expressed in terms of $\frac{1}{2}n(n+1)$ quantities! The solution presented is valid, however, when $n = 3$. In this case the x^{rs} are all independent and one may put

$$g^2 = \left| e^{imn} e^{jpq}\, g_{mnpq} \right| , \qquad \varepsilon^{ijk} = g^{-1} e^{ijk} ,$$

$$g^{ij} = \varepsilon^{imn} \varepsilon^{jpq}\, g_{mnpq} , \qquad g^{ik} g_{kj} = \delta^i_j .$$

It is found then, that

$$g_{rsmn} = \frac{1}{4}\,\delta^{ij}_{rs}\, g_{im}\, g_{jn} .$$

THE ALGEBRAIC PROPERTIES OF THE GIVEN FUNCTION

If a surface be defined by

$$x^r = x^r(u^\alpha), \qquad (r = 1, 2, \ldots, n), \quad (\alpha = 1, 2)$$

the area of a closed region of the surface will be given by the invariant integral

$$\Sigma = \iint_R L\, du^1\, du^2$$

where L is a given function of the x^r and their partial derivatives, x^r_α; L is not completely arbitrary, however. The integral is to be invariant under a change of parameter. If such a transformation be given by

$$u^{\bar\alpha} = u^{\bar\alpha}(u^\alpha),$$

the invariance of the integral implies that

$$L(x^r, x^r_\alpha) = \bar L(x^r, x^r_{\bar\alpha})\,|\delta^{\bar\alpha}_\alpha|,$$

where the notation

$$\delta^{\bar\alpha}_\alpha = \frac{\partial u^{\bar\alpha}}{\partial u^\alpha}$$

has been employed. When this relation, which is an identity in the $\delta^{\bar\alpha}_\alpha$ is differentiated partially with respect to $\delta^{\bar\beta}_\beta$ and use is made of the relations

$$x^r_\alpha = \delta^{\bar\alpha}_\alpha x^r_{\bar\alpha}, \quad \frac{\partial}{\partial \delta^{\bar\beta}_\beta}|\delta^{\bar\alpha}_\alpha| = \delta^\beta_{\bar\beta}\,|\delta^{\bar\alpha}_\alpha|,$$

one finds

$$L_{x^r_\beta} x^r_{\bar\beta} = \bar L\, \delta^\beta_{\bar\beta}\,|\delta^{\bar\alpha}_\alpha|;$$

or

$$L_{x^r_\beta} x^r_{\bar\beta}\,\delta^{\bar\beta}_\gamma = L\,\delta^\beta_{\bar\beta}\,\delta^{\bar\beta}_\gamma.$$

Thus,

$$L_{x^r_\beta} x^r_\gamma = \delta^\beta_\gamma L$$

This is an important identity; by differentiating this relation repeatedly with respect to x^r_α a series of further identities are obtained;

(1.1) $X^r_\alpha L^\beta_r = \delta^\beta_\alpha L.$

(1.2) $X^r_\alpha L^{\beta\gamma}_{ri} = \delta^\beta_\alpha L^\gamma_i - \delta^\gamma_\alpha L^\beta_i.$

(1.3) $X^r_\alpha L^{\beta\gamma\delta}_{rij} = \delta^\beta_\alpha L^{\gamma\delta}_{ij} - \delta^\gamma_\alpha L^{\beta\delta}_{ij} - \delta^\delta_\alpha L^{\gamma\beta}_{ij}.$

In the above and in subsequent material L^α_i, $L^{\alpha\beta}_{ij}$, etc. denote $\frac{\partial L}{\partial x^i_\alpha}$, $\frac{\partial^2 L}{\partial x^i_\alpha x^j_\beta}$, etc.

The relation (1.1) is a necessary and sufficient condition that the integral be invariant under a change of parameters. It shall be shown now that this implies that L must be a function of the x^r and of the Jacobians x^{rs}, where

$$X^{rs} = X^r_\alpha X^s_\beta e^{\alpha\beta}$$

Summing $e_{\beta\gamma} e^{\alpha\rho}$ on (1.2) reveals that

$$L^\rho_i = \tfrac{1}{2} L^{\beta\gamma}_{ri} e_{\beta\gamma} X^r_\alpha e^{\alpha\rho}.$$

Now

$$dL = L_{x^r}\, dx^r + L^\alpha_r\, dx^r_\alpha \quad ,$$

or

$$dL = L_{x^r}\, dx^r + \tfrac{1}{4} L^{\alpha\beta}_{rs}\, e_{\alpha\beta}\, dx^{rs}.$$

Thus

$$L = L(x^r, x^{rs}).$$

x^{rs} is a <u>bivector</u>, that is;

$$X^{rs} = \lambda^r \mu^s - \lambda^s \mu^r \quad ,$$

or in other words the x^{rs} satisfy

$$\delta^{ijkl}_{rspq}\, X^{rs} X^{pq} = 0$$

identically.

Since the x^{rs} are not independent

$$L_{rs} = \frac{\partial L}{\partial x^{rs}}$$

is not unique. Consider any function, $\varphi(x^{rs})$; there is a unique

$$\varphi_{rs} = \frac{\partial \varphi}{\partial x^{rs}}$$

such that φ_{rs} is a bivector. But

$$\varphi^{\alpha}_{r} = 2\varphi_{rs}\, x^{s}_{\rho}\, e^{\alpha\rho}. \tag{1.4}$$

Then

$$\varphi^{\alpha}_{r}\varphi^{\beta}_{s} e_{\alpha\beta} = 4\varphi_{ri}\,\varphi_{sj}\, x^{ij} = 2(\varphi_{ri}\varphi_{sj} - \varphi_{rj}\varphi_{si})x^{ij};$$

since φ_{rs} is a bivector

$$\varphi_{ri}\varphi_{sj} - \varphi_{rj}\varphi_{si} = \varphi_{rs}\varphi_{ij}.$$

It follows that

$$\varphi^{\alpha}_{r}\varphi^{\beta}_{s} e_{\alpha\beta} = 2\varphi_{rs}(\varphi_{ij}x^{ij}). \tag{1.5}$$

This result enables a direct calculation of φ_{rs} provided $\varphi_{ij}x^{ij} \neq 0$.

To obtain an alternate form for φ_{rs} (1.4) is differentiated with respect to x^{s}_{β} ; one finds

$$\varphi^{\alpha\beta}_{rs} e_{\alpha\beta} = 4\varphi_{rs} + 2\varphi_{rm}{}^{\rho}_{s}\, x^{m}_{\rho}.$$

It is always possible to arrange φ_{rsmn} so that

$$\varphi_{ijk\ell}\,\delta^{ijk\ell}_{rspq} = 0 .$$

Then

$$2\varphi_{rm}{}^{\rho}_{s}\, x^{m}_{\rho} = \varphi_{rs}{}^{\rho}_{m}\, x^{m}_{\rho} = 2\varphi_{rsmn}x^{mn},$$

and

$$\varphi^{\alpha\beta}_{rs} e_{\alpha\beta} = 4\varphi_{rs} + 2\varphi_{rsmn}x^{mn}. \tag{1.6}$$

Suppose

$$\varphi_{ij}x^{ij} = a\varphi,$$

then

$$\varphi_{rsmn}x^{mn} = (a-1)\varphi_{rs},$$

and (1.5) and (1.6) may be written

$$\varphi_r^\alpha \varphi_s^\beta e_{\alpha\beta} = a\varphi\varphi_{rs}.$$
$$\varphi_{rs}^{\alpha\beta} e_{\alpha\beta} = 2(a+1)\varphi_{rs},$$

or

(1.7) $$\varphi_{rs} = (a\varphi)^{-1}\varphi_r^\alpha \varphi_s^\beta e_{\alpha\beta} = \tfrac{1}{2}(a+1)^{-1}\varphi_{rs}^{\alpha\beta} e_{\alpha\beta}.$$

This may readily be iterated.

<u>The First Polarity</u>. There is a polarity established by means of the given function L. We put

$$L_r^\alpha = p_r^\alpha .$$

These equations may be solved for the x_α^r in terms of the p_r^α provided the $2n^{th}$ order determinant

$$\begin{vmatrix} L_{rs}^{11} & L_{rs}^{12} \\ L_{rs}^{21} & L_{rs}^{22} \end{vmatrix}$$

does not vanish.[7]

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[7]The non-vanishing of this determinant is an analogous condition to the one usually imposed in Finsler space;

$$\left| L^2_{\dot{x}^r \dot{x}^s} \right| \neq 0 .$$

It guarantees that the equations

$$L^2_{\dot{x}^r} = p_r$$

may be reversed. Cartan, <u>op</u>. <u>cit</u>., does not impose this restriction. In his example of a singular space

$$\Sigma = \iint (p^2+q^2)\,dx\,dy = \iint \frac{u^2+v^2}{\omega}\,d\xi_1 d\xi_2 ,$$

where

$$u_i = \tfrac{1}{2} e_{ijk} x_\alpha^j x_\beta^k e^{\alpha\beta}, \quad x^{jk} = e^{rjk} u_r ,$$

then $L_i^\alpha = L_{u_r} e_{rik} x_\beta^\alpha e^{\alpha\beta}$; noting that $u_r L_{u_r} = L$, one finds

$$2LL_{u_r} = e^{rst} L_s^\alpha L_t^\beta e_{\alpha\beta}, \quad L_s^\alpha L_t^\beta e_{\alpha\beta} = e_{rst} LL_{u_r}$$

so that expressing L uniquely in terms of the p_i^α depends on expressing L uniquely in terms of the L_{u_r}. This, however, When the relation

$$x^r_\alpha p^\alpha_r = 2L$$

is differentiated with respect to p^α_γ, it is found that

$$x^r_\alpha + p^\beta_s \frac{\partial x^s_\beta}{\partial p^\alpha_r} = 2L^r_\alpha .$$

Here and in the future

L^r_α $L^{rs}_{\alpha\beta}$, etc. denote $L_{p^\alpha_r}$, $L_{p^\alpha_r p^\beta_s}$, etc.

On the other hand

$$L^r_\alpha = L^\beta_s \frac{\partial x^s_\beta}{\partial p^\alpha_r} = p^\beta_s \frac{\partial x^s_\beta}{\partial p^\alpha_r} ,$$

so that

$$x^r_\alpha = L^r_\alpha .$$

Computation of $\frac{\partial x^r_\alpha}{\partial x^s_\beta}$ results in the inverse matrix relation

(1.8) $$L^{\alpha\gamma}_{r\rho} L^{\rho s}_{\gamma\beta} = \delta^\alpha_\beta \delta^s_r$$

<u>The Second Polarity</u>. Another polarity, closely related to the first may be demonstrated. Reference to (1.7) reveals that

$$\overset{2}{L}_{rs} \equiv L^\alpha_r L^\beta_s e_{\alpha\beta} \equiv p^\alpha_r p^\beta_s e_{\alpha\beta} .$$

We put

(1.9) $$\overset{2}{L}_{rs} = p_{rs} .$$

p_{rs} is a bivector; it will be shown that it is possible to

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is clearly not possible for,

$$L_{u_r} = \left(\frac{2u}{\omega}, \frac{2v}{\omega}, -\frac{u^2+v^2}{\omega^2} \right) .$$

$$|L_{u_r u_s}| = \begin{vmatrix} \frac{2}{\omega} & 0 & -\frac{2u}{\omega^2} \\ 0 & \frac{2}{\omega} & -\frac{2v}{\omega^2} \\ -\frac{2u}{\omega^2} & -\frac{2v}{\omega^2} & \frac{2(u^2+v^2)}{\omega^3} \end{vmatrix} \equiv 0$$

reverse (1.9) and express the bivector x^{rs} in terms of p_{rs}.

It has been demonstrated in the preceding section that L may be expressed as a function of x^r, and p^α_s. The equation (1.1) may be written in the form

$$p^\alpha_r L^r_\beta = \delta^\alpha_\beta L,$$

which implies that

$$L = L(x^r, p_{rs}).$$

The notation

$$L^{2\,rs} = \frac{\partial L^2}{\partial p_{rs}}, \quad L^{2\,rsmn} = \frac{\partial^2 L^2}{\partial p_{rs}\,\partial p_{mn}}, \text{ etc.}$$

will be used. Reference to (1.7) shows that

$$L^{2\,rs} = L^r_\alpha L^s_\beta e^{\alpha\beta};$$

that is,

$$x^{rs} = L^{2\,rs}.$$

Since

$$\frac{\partial x^{ij}}{\partial x^{rs}} = \delta^{ij}_{rs};$$

$$L^2_{rsmn} L^{2\,mnij} = \delta^{ij}_{rs}.$$

APPLICATIONS OF THE CALCULUS OF VARIATIONS

The Principal Normal Vector To A Surface. Consider the variation of the integral

$$\Sigma = \iint_R L\,du^1 du^2 .$$

We have

$$\delta\Sigma = \iint_R (L_{x^r}\delta x^r + L^\alpha_r \delta x^r_\alpha)\, du^1 du^2,$$

or

$$\delta\Sigma = \iint_R (L_{x^r} - \partial_\alpha L^\alpha_r)\delta x^r du^1 du^2 + \int_C L^\alpha_r \delta x^r e_{\alpha\beta} du^\beta .$$

In this manner one obtains the spacial invariant of weight zero, and the surface invariant of weight one

$$(\partial_\alpha L^\alpha_r - L_{x^r})\delta x^r .$$

Thus

$$L_r \equiv L^{-1}(\partial_\alpha L^\alpha_r - L_{x^r})$$

is a vector in space and an invariant on the surface.

When the relation (1.1)

$$X^r_\gamma L^\alpha_r - \delta^\alpha_\gamma L = 0$$

is differentiated with respect to u^α one finds

$$X^r_{\gamma\alpha} L^\alpha_r + X^r_\gamma \partial_\alpha L^\alpha_r - L^\alpha_r X^r_{\alpha\gamma} - L_{x^r} X^r_\gamma = 0 ,$$

or

(2.1) $\quad X^r_\gamma L_r = 0 .$

In view of this identity it will be said that L_r is normal to the surface.

The ds^2 And The Mean Curvature Induced On A Surface. If one is able to measure distance in space, one may determine an absolute invariant on any curve immersed in this space, namely the principal curvature. In an analogous manner if one is able to measure area in space, then an invariant and a symmetric double tensor may be determined on the surface. In the case of area geometry we put

$$H^2 g_{\alpha\beta} = L L_r L_s L^{rs}_{\alpha\beta}$$

then on identifying $|g_{\alpha\beta}|$ with L^2, we obtain the invariant of weight zero

$$H^4 = |L^{rs}_{\alpha\beta} L_r L_s|, \tag{2.2}$$

and the tensor of weight zero

$$g_{\alpha\beta} = H^{-2} L L_r L_s L^{rs}_{\alpha\beta} \quad . \tag{2.3}$$

Suppose that a symmetric double tensor, g^*_{rs}, exists with the following properties:

(2.4)

A. $g^*_{rs} = g^*_{rs}(x^\kappa, x^\kappa_\alpha)$.

B. $g^*_{rs\kappa}{}^{\alpha} x^\kappa_\lambda = 0$.

C. $g^*_{rs\kappa}{}^{\alpha} x^s_\lambda = 0$.

D. If $g^*_{\alpha\beta}$ denotes $g^*_{rs} X^r_\alpha X^s_\beta$, $L^2 = |g^*_{\alpha\beta}|$.

Differentiation of the equation

$$L^2 = |g^*_{\alpha\beta}|$$

with respect to x^r_α yields

$$L^{-1} L^\alpha_r = g^*_{rs} X^s_\beta e^{\alpha\beta}$$

Whence

$$X^r_\alpha L = L L^r_\alpha = g^{*rl} L^\beta_l g^*_{\alpha\beta} \quad .$$

The latter equation is differentiated with respect to L^β_s and there results the identity

$$L^r_\alpha L^s_\beta + L L^{rs}_{\alpha\beta} = g^{*rs} g^*_{\alpha\beta} + g^{*ri} L^\rho_i g^*_{mn} \left(L^n_\rho L^{ms}_{\alpha\beta} + L^{ns}_{\rho\beta} L^m_\alpha \right). \tag{2.5}$$

When (2.5) is solved for g^{*rs} one finds

(2.6) $$g^{*rs} g^{*}_{\alpha\beta} = \frac{1}{2} L\left(L^{rs}_{\alpha\beta} + L^{sr}_{\alpha\beta}\right) + X^r_\rho X^s_\sigma\, g^{*\rho\sigma} g^{*}_{\alpha\beta}.$$

$L_r L_s$ is summed on (2.6); when note is taken of (2.1) we obtain

$$L_r L_s L L^{rs}_{\alpha\beta} = g^{*rs} L_r L_s\, g^{*}_{\alpha\beta}.$$

In other words; if a g^*_{rs} exists satisfying the conditions listed in (2.4), H defined in (2.2) has the value given by

$$H^2 = g^{*rs} L_r L_s ,$$

and $g_{\alpha\beta}$ defined in (2.3) is identical with $g^*_{\alpha\beta}$.

The properties that g^*_{rs} enjoys are possessed by the g_{rs} employed by Cartan. If such a g_{rs} exist we shall refer to the geometry as that of a Cartan space. These properties imply that:

A. g_{rs} depends on the coordinates x^k and on the direction of the surface element x^{ij}.

B. If P^r is vector perpendicular to dx^s, that is;

$$P^r g_{rs}\, dx^s = 0 ,$$

and if the contravariant components of P^r are unaltered, then this relation is unaltered when x^{ij} is turned about dx^s.

C. The area measured by means of g_{rs} is the same as that measured by the given function L.

__Conditions For The Existence of A Cartan Space.__ The surface tensor $g_{\alpha\beta}$ is a function of the x^r, the x^r_α, and the $x^r_{\alpha\beta}$. If two surfaces are tangent along a curve, the distance measured along the curve on each of the surfaces by means of

$$ds^2 = g_{\alpha\beta}\, du^\alpha du^\beta$$

will not, in general, have the same value. These values will be equal if and only if the tensor

$$g_{\alpha\beta r}^{\ \ \ \gamma\delta} = \frac{\partial}{\partial x^r_{\gamma\delta}} g_{\alpha\beta}$$

vanishes.

Differentiation of the relation

$$H^2 g_{\alpha\beta} = L L_r L_s L^{rs}_{\alpha\beta}$$

partially with respect to $x^\kappa_{\gamma\delta}$ yields

$$2 H H_\kappa^{\gamma\delta} g_{\alpha\beta} + H^2 g_{\alpha\beta\kappa}^{\ \ \ \gamma\delta} = 2 L^{\gamma\delta}_{r\kappa} L^{rs}_{\alpha\beta} L_s ,$$

while (2.2) reveals that

$$2 H H_\kappa^{\gamma\delta} = L_s L^{\gamma\delta}_{r\kappa} L^{rs}_{\rho\sigma} g^{\rho\sigma}.$$

Then

$$H^2 g_{\alpha\beta r}^{\ \ \ \gamma\delta} = L_s L^{\gamma\delta}_{r\kappa} \left(2 L^{rs}_{\alpha\beta} - L^{rs}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta}\right).$$

When the above is multiplied by $L^{\kappa i}_{\delta\mu}$ one finds

$$H^2 g_{\alpha\beta\kappa}^{\ \ \ \gamma\delta} L^{\kappa i}_{\delta\mu} = \delta^\gamma_\mu L_s \left(2 L^{is}_{\alpha\beta} - L^{is}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta}\right).$$

Thus, the vanishing of $Hg_{\alpha\beta\kappa}^{\ \ \ \gamma\delta}$ implies that and is implied by

$$L_s \left[2 L^{is}_{\alpha\beta} - L^{is}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta} \right] = 0 .$$

This relation is differentiated once more with respect to $x^\kappa_{\gamma\delta}$ to obtain the tensor relation

$$\left(L^{\gamma\delta}_{s\kappa} + L^{\gamma\delta}_{\kappa s}\right)\left(2 L^{is}_{\alpha\beta} - L^{is}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta}\right) = 0 .$$

Reference to (1.7) and (1.2) shows that the above is symmetric in α, β and it may be written in the form

$$2 L^{\gamma\delta}_{s\kappa} \left(L^{is}_{\alpha\beta} + L^{si}_{\alpha\beta} - L^{is}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta} \right) = 0 .$$

Since

$$L^{\gamma\delta}_{s\kappa} L^{\kappa r}_{\delta\alpha} = \delta^\gamma_\alpha \delta^r_s ,$$

the vanishing of $Hg_{\alpha\beta\kappa}^{\ \ \ \gamma\delta}$ implies

(2.7) $$A^{ij}_{\alpha\beta} \equiv L^{ij}_{\alpha\beta} + L^{ji}_{\alpha\beta} - L^{ij}_{\rho\sigma} g^{\rho\sigma} g_{\alpha\beta} = 0 .$$

But

$$H^2 g_{\alpha\beta\kappa}^{\ \ \ \gamma\delta} = L^{\gamma\delta}_{\kappa i} L_s A^{is}_{\alpha\beta}$$

A necessary and sufficient condition that $Hg_{\alpha\beta\kappa}^{\ \ \ \gamma\delta}$ vanish is that $A^{ij}_{\alpha\beta}$ vanish.

In view of the fact that $A^{rs}_{\alpha\beta}$ satisfies the 6n conditions

$$A^{ij}_{\alpha\beta} L^{\gamma}_{j} = 0,$$

and the $\frac{1}{2}n(n+1) - 2n$ further conditions

$$A^{ij}_{\alpha\beta}\, g^{\alpha\beta} = 0 .$$

$A^{ij}_{\alpha\beta}$ has altogether $n(n-3)$ independent components.

If the space is a Cartan space it is necessary that g^{rs} be given by (2.6) and that $A^{rs}_{\alpha\beta}$ vanish. In this case

$$g^{rs} = \tfrac{1}{2} L L^{rs}_{\rho\sigma}\, g^{\rho\sigma} + x^{r}_{\rho} x^{s}_{\sigma}\, g^{\rho\sigma} . \tag{2.8}$$

It will be found, however, that these are not sufficient conditions for the existence of a Cartan space.

When (2.7) is differentiated with respect to x^{r}_{γ} one finds

$$S^{ij} g_{\alpha\beta}{}^{\gamma}_{r} = L^{ij\,\gamma}_{\alpha\beta r} + L^{ji\,\gamma}_{\alpha\beta r} - S^{ij\gamma}_{\;\;\;r}\, g_{\alpha\beta}, \tag{2.9}$$

where $S^{ij} = L^{ij}_{\beta\sigma} g^{\rho\sigma}$

$$S^{ij\gamma}_{\;\;\;r} = L^{ij\,\gamma}_{\rho\sigma r} g^{\rho\sigma} - L^{-1} L^{\gamma}_{r} S^{ij}.$$

Differentiation of (2.8) yields

$$g^{rs\gamma}_{\;\;\;\kappa} = \tfrac{1}{2} L^{\gamma}_{\kappa} S^{rs} + \tfrac{1}{2} L S^{rs\gamma}_{\;\;\;\kappa} + (\delta^{r}_{\kappa} x^{s}_{\sigma} + \delta^{s}_{\kappa} x^{r}_{\sigma}) g^{\gamma\sigma} + x^{r}_{\rho} x^{s}_{\sigma} g^{\rho\sigma\gamma}_{\;\;\;\;\kappa}.$$

Reference to (1.3), (2.7), and (2.9) demonstrates that

$$g^{rs\gamma}_{\;\;\;\kappa} x^{\kappa}_{\lambda} = 0, \quad \text{i.e.;} \quad g_{rs\kappa}{}^{\gamma} x^{\kappa}_{\lambda} = 0 .$$

But one finds that the satisfaction of the equations

$$g_{rs\kappa}{}^{\gamma} x^{s}_{\lambda} = 0$$

implies and is implied by

$$g_{\alpha\beta r}{}^{\gamma} - L^{-1} L^{\rho}_{r} (g_{\rho\alpha} \delta^{\gamma}_{\beta} + g_{\rho\beta} \delta^{\gamma}_{\alpha}) = 0 \tag{2.10}$$

Application of (2.9) permits (2.10) to be written in the form

$$A^{ij\,n}_{\alpha\beta r} \equiv L^{-1} S^{ij} (2 L^{n}_{\delta} g_{\alpha\beta} - L^{n}_{\alpha} g_{\beta\delta} - L^{n}_{\beta} g_{\alpha\delta}) - L^{ij\,n}_{\alpha\beta\delta} - L^{ji\,n}_{\alpha\beta\delta} + S^{ij\,n}_{\;\;\;\delta}\, g_{\alpha\beta} = 0 .$$

On the other hand

$$A^{i n j}_{\alpha \beta \gamma} L^{\delta}_{n} = \delta^{\delta}_{\gamma} A^{ij}_{\alpha\beta}$$

Thus, the vanishing of $A^{ijn}_{\alpha\beta\gamma}$ implies that $A^{ij}_{\alpha\beta} = 0$. Therefore, the necessary and sufficient condition that the space be a Cartan space is that

$$A^{ijk}_{\alpha\beta\gamma} = 0 .$$

The Fundamental Displacement Of A Surface Strip. Suppose that a curve together with surface elements passing through the curve be given. It will be shown that a unique minimal surface contains the given strip. This surface may be regarded as a "parallel" displacement of the surface elements along themselves. Furthermore, the differential equations for this displacement will enable us to define a displacement of the strip in any direction.

We shall suppose the strip to be given by the equations

(3.1) $$x^r = x^r(u), \quad x^{rs} = x^{rs}(u).$$

Since the surface elements pass through the curve and since the x^{rs} are bivectors, the functions in (3.1) are subject to the further conditions

$$\delta^{ijk}_{rsp} x^{rs} \dot{x}^p = 0,$$
$$\delta^{ijk}_{rsp} x^{rs} x^{pq} = 0 .$$

Without loss of generality coordinates on any surface containing the strip may be taken so that on the curve

$$x^r_1 = \dot{x}^r(u) .$$

Then the equations of the strip may be taken in the form

$$x^r = x^r(u), \quad x^r_2 = x^r_2(u) .$$

Consider the vector

$$4F_r \equiv \partial_\alpha F^\alpha_r - F_{x^r} + \tfrac{1}{2} F^{-1} F'_r \dot{F} = 2(L^2 L_r - L_\alpha L^\alpha_r + L'_r \dot{L})$$

where

$$F = L^2$$

Since $L_r x^r_\alpha = 0$, F_r vanishes if and only if L_r and L_2 vanish. For the sake of convenience we shall put

$$\tfrac{1}{2} F_{rsmn} = g_{rsmn}$$
$$4 h_{rs} = F_{rs}^{22} = 4 g_{risj} \dot{x}^i \dot{x}^j$$

F_r may then be written in the form

$$F_r = h_{rs} x_{22}^s + 2\gamma_r(\dot{x}_\alpha^\kappa, x_\alpha^\kappa, x^\kappa).$$

The maximum rank of h_{rs} is plainly no greater than $n - 1$, for

$$h_{rs} \dot{x}^s = 0 .$$

But since g_{rsmn} is assumed to have the highest possible rank,

$$h_{rs} \lambda^s = 0$$

if and only if

$$\dot{x}^r \lambda^s - \dot{x}^s \lambda^r = 0 .$$

In other words the rank of h_{rs} is exactly $n - 1$. It is apparent that the equations

$$h_{rs} \lambda^s = \lambda_r , \quad \textit{where} \quad \lambda_r \dot{x}^r = 0$$

are not reversible, but one may express

$$\dot{x}^r \lambda^s - \dot{x}^s \lambda^r$$

in terms of λ_r

A synthetic demonstration of this result suggests the algebraic solution. At a point, the tensor g_{rsmn} determines a quadratic line complex in an n-1 space. $h_{rs} y^r y^s = 0$ may be regarded as a point cone of this complex and $\dot{x}^r$ is the vertex. To each [n-2], λ_r, through the vertex of the cone corresponds a unique line, $\lambda^s \dot{x}^r - \lambda^r \dot{x}^s$, through the vertex.

The details of the algebraic solution are now undertaken. If H^{rs} denotes the cofactor of h_{rs}, then

$$H^{rs} = k \dot{x}^r \dot{x}^s$$

If $\varphi_r \dot{x}^r = 1$,

$$K \equiv (H^{rs}\varphi_r\varphi_s)^{-1}$$

Let H^{rsij} denote the cofactor of

$$\begin{vmatrix} h_{ri} & h_{si} \\ h_{rj} & h_{sj} \end{vmatrix}$$

It is seen that

$$K^{-1} H^{rsij} = h^{rsij}$$

is a tensor. We put

$$h^{rsij}\varphi_s = h^{rij},$$

then

$$h^{rij} h_{rp} \lambda^p = \dot{x}^p \lambda^q \delta^{ij}_{pq}.$$

This is the desired result. Any φ_r such that $\varphi_r \dot{x}^r = 1$ will suffice; one may take

$$\varphi_r = \frac{1}{2} F^{-1} F'_r .$$

One has

$$F_r \dot{x}^r = 0 .$$

Thus F_r uniquely determines the bivector

$$n^{ij} = h^{kij} F_k$$

which may be written in the form

$$n^{rs} = \dot{x}^p (x^q_{\lambda\lambda} + \lambda \gamma^q)\delta^{rs}_{pq}$$

It may be noted that n^{rs} passes thru $\dot{x}^k$. When n^{rs} vanishes, $x^{rs}_{,\lambda}$ is expressed in terms of $\dot{x}^i_\alpha$, x^i_α, and x^i.

The anulling of n^{rs} defines a propagation of a given surface strip away from the initial curve in the directions of the strip. A more general displacement may be found which will enable the displacement of the strip in any direction away from the curve.

Consider the transformation of $\dfrac{\partial n^{rs}}{\partial x^k_\lambda} \equiv n^{rs\lambda}_k$.

$$n^{rs\,2}_{\;\;\kappa}\,\delta^{\bar r}_{r}\,\delta^{\bar s}_{s} = n^{\bar r\bar s\,2}_{\;\;\bar\kappa}\,\delta^{\bar\kappa}_{\kappa} + n^{\bar r\bar s\,12}_{\;\;\bar\kappa}\,\delta^{\bar\kappa}_{\kappa n}\,\dot x^{n} + 2n^{\bar r\bar s\,22}_{\;\;\bar\kappa}\,\delta^{\bar\kappa}_{\kappa n}\,x^{n}_{2}$$

It is seen that

$$\omega^{rs} = n^{rs\,22}_{\;\;\kappa}\,dx^{\kappa}_{2} + \frac{1}{2}n^{rs\,2}_{\;\;\kappa}\,dx^{\kappa} + \frac{1}{2}n^{rs\,12}_{\;\;\kappa}\,d\dot x^{\kappa}$$

is a tensor. ω^{rs} takes the form

$$\omega^{rs} = \dot x^{p}\left(dx^{q} + \gamma^{q\,2}_{\;\kappa}\,dx^{\kappa} + \gamma^{q\,12}_{\;\kappa}\,d\dot x^{\kappa}\right)\delta^{rs}_{pq}$$

It is a bivector passing through $\dot x^{\kappa}$. If ω^{rs} is annulled dx^{rs} is expressed in terms of x^{κ}, x^{κ}_{α}, $\dot x^{\kappa}_{\alpha}$, and dx^{κ}, $d\dot x^{\kappa}$.

<u>The Fundamental Connection</u>. In the preceding paragraphs only the fundamental tensors have been considered, those derivable from the function L itself. We now consider other tensors. The most important of these will be bivectors and tensors of higher order, which when contracted on bivectors produce bivectors. We shall suppose that the components of these tensors depend on the coordinates of a point, x^{κ}, and on the components of a surface element x^{rs} at the point. The combination (x^{κ}, x^{rs}) will be referred to as the element of support of the tensor.

Consider a bivector, Z^{rs}, passing through a given curve and suppose that the element of support also passes through the given curve. It will be shown that an intrinsic differential may be determined, which may be regarded as defining the geometric change in Z^{rs} as its element of support is displaced away from the curve.

The bivector and its element of support pass through a common curve; thus for all values of the element of support

$$\delta^{rspq}_{ijkl}\,x^{ij}\,Z^{kl}$$

must be identically zero. If the equations of the curve are

$$x^{r} = x^{r}(t)$$

Then

$$\delta^{ijk}_{rsp} X^{rs} \dot{x}^{p} = 0, \quad \delta^{ijk}_{rsp} Z^{rs} \dot{x}^{p} = 0,$$

and X^{rs} and Z^{rs} may be put in the form

$$X^{rs} = \dot{x}^{i} x'^{j} \delta^{rs}_{ij}$$
$$Z^{rs} = \dot{x}^{i} \lambda^{j} \delta^{rs}_{ij}$$

We shall suppose the intrinsic differential to have the form $\underline{d} Z^{ij} = \dot{x}^{r}(d\lambda^{s} + \lambda^{m}[\Gamma^{s}_{mk}\, dx^{k} + C^{s}_{mk}\, dx'^{k}])\delta^{ij}_{rs}$.

Then dZ^{rs} and $Z^{rs} + dZ^{rs}$ are bivectors passing thru the given curve.

The coefficients

$$\Gamma^{s}_{mk}, \quad C^{s}_{mk}$$

will be determined from the following conditions.

1. The intrinsic derivative in the direction of the curve

$$\underline{\dot{Z}}^{ij}$$

and the differential

$$\underline{d}(\dot{x}^{r}\dot{x}^{s} - \dot{x}^{s}\dot{x}^{r})$$

shall vanish.

2. The area of Z^{rs} measured by

$$Z^{2} = g_{rsmn} Z^{rs} Z^{mn}$$

shall remain unaltered when Z^{rs} is displaced by means of $\underline{d}Z^{rs} = 0$.

3. If Z^{rs} and Y^{rs} are two bivectors with a common element of support and passing through a common curve, then when their contravariant components remain fixed while their element of support turns thru the curve, the invariant

$$g_{rspq}(Y^{rs}\, \underline{d}Z^{pq} - Z^{rs}\, \underline{d}Y^{pq})$$

shall vanish.

4. When $dx'^s = 0$, the relation

$$\underline{d}(\dot{x}^r \delta x^s - \dot{x}^s \delta x^r) = \underline{\delta}(\dot{x}^r dx^s - \dot{x}^s dx^r)$$

shall hold. That is, the coefficient of $\delta x^m dx^n$ in this case, Γ^s_{*mn} shall be symmetric in the indices m and n.

5. Along the curve the displacement vector dx^r, shall satisfy a differential equation of the form $d\dot{x}^r = \Delta^r_k dx^k$.

Application of the second condition reveals that

$$dh_{rs} = (\Gamma_{rsk} + \Gamma_{srk})dx^k + (C_{rsk} + C_{srk})dx'^k \tag{3.2}$$

where $\Gamma_{rsk} = h_{ri}\Gamma^i_{sk}$, $C_{rsk} = h_{ri}C^i_{sk}$.

One finds at once that

$$C_{rsk} + C_{srk} = h_{rsk} \tag{3.3}$$

where $h_{rsk} = \frac{\partial}{\partial x'^k} h_{rs} = 2g_{risjkn}\dot{x}^i\dot{x}^j\dot{x}^n$

The implication of the third condition is that

$$C_{rsk} = C_{srk}$$

Thus (3.3) yields

$$C_{rsk} = \frac{1}{2}h_{rsk}$$

C_{rsk} is completely symmetric in r,s,k and enjoys the property

$$C_{rsk}\dot{x}^s = 0, \quad C_{rsk}x'^s = 0$$

Employing these results, one finds

$$\underline{d}\,x^{rs} = \dot{x}^i(dx^{ij} + \Gamma^j_{mk}dx^k x'^m)\delta^{rs}_{ij}$$

and $\underline{d}Z^{rs}$ may be written in the form

$$\underline{d}\,Z^{rs} = \dot{x}^p\left[d\lambda^q + \lambda^m(\Gamma^q_{*mk}\,dx^k + C^q_{mk}\,\underline{d}x'^k)\right]\delta^{rs}_{pq}. \tag{3.4}$$

where $\Gamma^s_{*mk} = \Gamma^s_{mk} - C^s_{mp}\Gamma^p_{nk}x'^n$

and $\underline{d}x'^k = dx'^k + \Gamma^k_{np}x'^n dx^p$

In the future summation on x'^k will be denoted by a "o" in the place of the index summed, and summation on $\dot{x}^k$ will be denoted by a "1" in the place of the in dex summed.

Note is made of the relation

$$\Gamma^{s}_{*\,0k} = \Gamma^{s}_{0k}$$

so that

(3.5) $$\Gamma^{s}_{mk} = \Gamma^{s}_{*\,mk} + C^{s}_{mp}\Gamma^{p}_{*\,0k}$$

$\Gamma^{r}_{*\,sk}$ is the $\Gamma^{r}_{*\,sk}$ referred to in the fourth condition and accordingly it is symmetric in the last two indices. When the substitution given in the fifth condition and (3.7) are introduced into (3.2) one obtains

$$\partial_k h_{rs} + \dot{\partial}_p h_{rs}\Delta^p_k = \Gamma_{*\,rsk} + \Gamma_{*\,srk} + h_{rsp}\Gamma^{p}_{*\,0k}$$

We let δ_k denote the operator

$$\frac{\partial}{\partial x^k} + \Delta^p_k \frac{\partial}{\partial \dot{x}^p}$$

then the usual permutation of indices reveals that

(3.6) $$\Gamma_{*\,krs} = \gamma_{krs} + C_{rsp}\Gamma^{p}_{*\,0k} - C_{ksp}\Gamma^{p}_{*\,0r} - C_{rkp}\Gamma^{p}_{*\,0s}$$

where

$$\gamma_{krs} = \frac{1}{2}\left(\delta_r h_{ks} + \delta_s h_{rk} - \delta_k h_{rs}\right).$$

Summation on $x'^r\, x'^s$ and x'^r respectively yields

$$\Gamma_{*\,k00} = \gamma_{k00}$$

$$\Gamma_{*\,k0s} = \gamma_{k0s} - C_{ksp}\Gamma^{p}_{*\,00} = \gamma_{k0s} - C_{ksp}\gamma^{p}_{\cdot\cdot}$$

where

$$\dot{x}^p \gamma^q_{mn}\delta^{rs}_{pq} = h^{prs}\gamma_{pmn}$$

Then one may put

$$\dot{x}^p\gamma^q_{00} - \dot{x}^q\gamma^p_{00} = 2G^{pq} = 2\left(\dot{x}^p G^q - \dot{x}^q G^p\right)$$

and it is found that

$$\dot{x}^q\Gamma^{p}_{*\,0s} - \dot{x}^p\Gamma^{q}_{*\,0s} = \frac{\partial}{\partial x'^s}G^{pq}$$

This result is substituted into (3.6) and one has

$$\Gamma_{*\,krs} = \gamma_{krs} + G_{rsp}\frac{\partial G^p}{\partial x'^k} - C_{ksp}\frac{\partial G^p}{\partial x'^r} - C_{rkp}\frac{\partial G^p}{\partial x'^s}.$$

The implication of the first condition is that

$$d\dot{x}^r + \Gamma^{\;r}_{*\;1K} = \rho\dot{x}^r$$

When the substitution

$$d\dot{x}^r = \Delta^r_K\, dx^K$$

is made, one finds

$$\Delta_{rK} + \Gamma_{*\,r1K} = 0 \tag{3.7}$$

Summation of $\dot{x}^r$ on (3.6) reveals that

$$\Gamma_{*K1S} = \gamma_{K1S} - C_{KS\rho}\Gamma^{\;\rho}_{*01}$$

The expansion of this result and use of (3.7) shows

$$\Delta_{SK} + \Delta_{KS} = -\left(\partial_\rho h_{KS}\,\dot{x}^\rho + \partial_{\dot{\rho}} h_{KS}\Delta^\rho_1 + h_{KS\rho}\Delta^\rho_0\right).$$

But

$$\Delta^\rho_1 = \dot{x}^\rho\Delta^\rho_0 = \dot{x}'^\rho$$

and one obtains

$$-\dot{h}_{KS} = \Delta_{KS} + \Delta_{SK}$$

This equation may be given a geometric interpretation. The given curve may be regarded as one member of a congruence with the property that for every adjacent curve the area of

$$\dot{x}^r dx^s - \dot{x}^s dx^r$$

is constant along the curve.

Consider such a congruence given by equations of the form

$$\dot{x}^r = \rho\,\Delta^r(x^K)$$

Then

$$d\dot{x}^r = \rho\Delta^r_K dx^K + d\rho\Delta^r.$$

By a proper choice of the parameter we may have $d\rho = 0$. If

$$\rho\left[\Delta_{rK} + \Delta_{Kr}\right] = -\dot{h}_{Kr}$$

the congruence is an admissible one and will have the property described above.

<u>The Covariant Derivatives</u>. When the substitutions in (3.4)

$$\lambda^m = Z^{nm}\varphi_n \,, \quad dx^r = \Delta^r_K \, dx^K$$

are made, one may write $\underline{d}Z^{rs}$ in the form

$$\underline{d}Z^{rs} = dZ^{rs} + Z^{mn}\left(\Gamma^{rs}{}_{mnK}\, dx^K + C^{rs}{}_{mnij}\, \underline{d}X^{ij}\right) \tag{3.8}$$

where
$$\Gamma^{rs}{}_{mnK} = \frac{1}{2}\left(h^{prs}\Gamma_{\alpha pjK} - \Delta^p_K \delta^{rs}_{pj}\right)\delta^{ij}_{mn}\varphi_i$$

$$C^{rs}{}_{mnij} = \frac{1}{2}h^{prs}\, g_{pomnij}$$

On the other hand

$$dZ^{rs} = \frac{\partial Z^{rs}}{\partial x^K}dx^K + \frac{\partial Z^{rs}}{\partial x^{ij}}\left(\underline{d}X^{ij} - \Gamma^{ij}{}_{mnK}X^{mn}dx^K\right)$$

then one finds

$$\underline{d}Z^{rs} = Z^{rs}{}_{,K}\, dx^K + Z^{rs}{}_{,ij}\, \underline{d}X^{ij}$$

where the first covariant derivative, $Z^{rs}{}_{,K}$, is given by

$$Z^{rs}{}_{,K} = \frac{dZ^{rs}}{dx^K} + Z^{mn}\Gamma^{rs}{}_{mnK} - Z^{rs}{}_{ij}\Gamma^{ij}{}_{pqK}X^{pq}$$

and the covariant derivative of the second kind is given by

$$Z^{rs}{}_{,ij} = Z^{rs}{}_{ij} + Z^{mn}C^{rs}{}_{mnij}$$

where $Z^{rs}{}_{ij} = \dfrac{\partial Z^{rs}}{\partial x^{ij}}$

CONCLUSION

The character of the area space discussed in the preceeding pages is seen to be much more general than that of a Cartan space. In a non-Cartan space almost all trace of the notion of the length of a vector disappears, and local parallelism at a point is replaced by local parallelism through a curve.

We have shown that a necessary and sufficient condition for the existence of a Cartan space is the vanishing of the tensor $A^{ijk}_{\alpha\beta\gamma}$. In this case the intrinsic differential given above should stratify into that given by Cartan.[8] This appears reasonable in view of the formal similarity of the coefficients in the respective connections. No difficulty should be encountered in obtaining curvature tensors.

By imposing a minimum number of conditions on the space we have obtained a geometry in which all metric notions are firmly based on the concept of area, and in which the bivector assumes the important role that it should have.

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[8]E. Cartan, Op. cit..

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