ABSTRACT

Title of Dissertation: LOGARITHMIC CONNECTIONS ON ARITHMETIC SURFACES AND COHOMOLOGY COMPUTATION

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De Rham cohomology is important across a broad range of mathematical fields. The good properties of de Rham cohomology on smooth and complex manifolds are also shared by those schemes which most closely resemble complex manifolds, namely schemes that are (1) smooth, (2) proper, and (3) defined over the complex numbers or other another field of characteristic zero. In the absence of one or more of those three properties, one observes more pathological behavior. One of the most concerning is that for affine morphisms X/S, which are arguably analogous to contractible manifolds, the groups $\mathrm{H}^i(X/S)$ may be infinitely generated. In this case, when $S = \mathrm{Spec}(k), \mathrm{char}(k) > 0$, the *Cartier isomorphism* allows one to view the groups as finite dimensional over a different base: $\mathcal{O}_{X^{(p)}}$. However when S is a Dedekind ring of mixed characteristic, there is no good substitute for the Cartier isomorphism.

In this work we explore a method of calculating the de Rham cohomology of some affine schemes which occur as the complement of certain divisors on arithmetic surfaces over a Dedekind

scheme of mixed characteristic. The main tool will be (Koszul) connections on vector bundles, whose primary role is to generalize the exterior derivative $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_{X/S}$ to a map $\mathcal{F} \xrightarrow{\nabla} \Omega^1_{X/S} \otimes \mathcal{F}$ defined on more general quasi-coherent modules $\mathcal{F}$.

Given an suitable arithmetic surface X and divisor D with complement $U = X \setminus D$, the de Rham cohomology $\mathrm{H}^1(U/S)$ is infinitely generated. We use a natural filtration $\mathrm{Fil}^\bullet \mathcal{O}_U$ to construct a filtration $\mathrm{Fil}^\bullet \mathrm{H}^1(U/S)$. We show that associated graded of this filtration is the direct sum of finitely generated modules, and we give a formula to calculate them in terms of the structure sheaf $\mathcal{O}_D$ of the divisor as well as the different ideal $\mathcal{D}_D \subset \mathcal{O}_D$ of the finite, flat extension D/S.

LOGARITHMIC CONNECTIONS ON ARITHMETIC SURFACES AND COHOMOLOGY COMPUTATION

by

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Notation

All rings and algebras will be commutative and unital. Dedekind domains/schemes are one-dimensional (fields are not considered Dedekind domains).

For a scheme $\pi : X \longrightarrow S$ and a sheaf $\mathcal{F}$ on X, we denote the i^{th} sheaf cohomology of $\mathcal{F}$ on X by $\mathrm{H}^i(X, \mathcal{F})$ and the i^{th} relative de Rham cohomology of X/S by $\mathrm{H}^i_{\mathrm{dR}}(X/S)$ or simply $\mathrm{H}^i(X/S)$.

We will introduce a type of de Rham cohomology with coefficients, depending on a sheaf $\mathcal{F}$ and an algebraic connection ∇, and we will denote this by $\mathrm{H}^i(X/S; \mathcal{F}, \nabla)$.

Remark 0.0.1 (Note on page citations). The reader may notice some cited page numbers given as [p. X/Y]. What this means is that X is the usual ”analog” page number, whereas Y is the digital page number one may use if one happens to have a digital copy of the same book or article.

Chapter 1: Introduction

1.1 Intro

De Rham cohomology is a cohomology theory applicable across a wide range of geometric objects. Like many cohomology theories, a common method of computation is to take a "good cover" of a space, on which cohomology becomes trivial, and then reduce the computation to something more combinatorial. Over smooth manifolds, good covers are typically contractible open subsets (with contractible intersections). Over these, sheaf cohomology and de Rham cohomology are both trivial. Over schemes, good covers are typically affine open subschemes. Over affines, sheaf cohomology is trivial, but de Rham cohomology can be highly nontrivial, even infinitely generated.

In this work we explore one method of computing the de Rham cohomology of some affine open subschemes of arithmetic surfaces. When $X \xrightarrow{\pi} S$ is a nice arithmetic surface with a suitable divisor D, the affine open complement $U = X \setminus D$ usually has an infinitely generated module $\mathrm{H}^1_{\mathrm{dR}}(U/S)$. However there is a filtration $\mathrm{Fil}^\bullet \mathcal{O}_U$ associated to the divisor D, which induces in turn a filtration $\mathrm{Fil}^\bullet \mathrm{H}^1_{\mathrm{dR}}(U/S)$. The associated graded $\mathrm{Gr}^\bullet \mathrm{H}^1_{\mathrm{dR}}(U/S)$ is composed of finite dimensional modules, which can be expressed in terms of the structrure sheaf $\mathcal{O}_D$ of the divisor as well as a different ideal $\mathcal{D}_D \subset \mathcal{O}_D$ of the finite, flat extension D/S.

The computation proceeds by identifying the de Rham cohomology of U in terms of similar cohomologies defined on X: $\mathrm{H}^1_{\mathrm{dR}}(U/S) \cong \mathrm{H}^1(X/S; \mathcal{O}_U)$. Our main theorem is

Theorem 1.1.1. *The associated graded of* $\mathrm{Fil}^\bullet \mathrm{H}^1(X/S; \mathcal{O}_U)$ *satisfies*

$$\mathrm{Gr}\, \mathrm{H}^1(X/S; \mathcal{O}_U) \cong \mathrm{H}^1(X/S) \oplus \pi_* \Omega_{X/S}(D) \oplus \bigoplus_{n \geq 1} \pi_* \left(\mathcal{O}_D / (n \mathcal{D}_D) \right)$$

The cohomology modules $\mathrm{H}^i(X/S; \mathcal{L}, \nabla)$ on X are defined in terms of logarithmic connections $\mathcal{L} \xrightarrow{\nabla} \Omega_{X/S}(D) \otimes \mathcal{L}$.

The issue is that for an affine open subscheme $U \subset X$, the coordinate ring $\Gamma(U, \mathcal{O}_X)$ allows poles of arbitrarily large order on $X \setminus U = D$.

For example when working over the affine line $\mathbb{A}^1_{\mathbb{Z}}$ and computing with the ring $\mathbb{Z}[x]$, essentially each power of x contributes a dimension to the de Rham cohomology module $\mathrm{H}^1(\mathbb{A}^1_{\mathbb{Z}}/\mathbb{Z})$.

Furthermore, there are no small subrings of $\mathbb{Z}[x]$ that one can use to decompose the computation into smaller pieces. However, one can filter the ring $\mathbb{Z}[x]$ by the *modules* $x^n \mathbb{Z}[x]$, and each graded piece of this filtration is just a rank 1 module over $\mathbb{Z}$.

The guiding idea in this work is to use this strategy to filter the coordinate rings $\Gamma(U, \mathcal{O}_X)$ of certain affine subschemes $U \subset X$ by a sequence of sheaves of $\mathcal{O}_X$-modules that interpolate between the rings $\Gamma(X, \mathcal{O}_X)$ and $\Gamma(U, \mathcal{O}_X)$. Then we can analyze algebraic connections on these modules, as a way of studying the derivative on $\Gamma(U, \mathcal{O}_X)$, one small piece at a time.

Unfortunately we must trudge through no small number of technical details as we implement this strategy.

1.2 Motivating examples

1.2.1 de Rham cohomology of $\mathbb{A}^1_{\mathbb{Z}}, \mathbb{P}^1_{\mathbb{Z}}$

In algebraic geometry, de Rham cohomology often behaves very similarly to in topology, but there are some pathologies. We explore one of them here.

Let $S = \operatorname{Spec}(R)$ be an affine scheme, $Y := \mathbb{A}^1_S \subset \mathbb{P}^1_S =: X$.

From topology one expects

$$\mathrm{H}^i_{\mathrm{dR}}(\mathbb{P}^1_S/S) \cong \begin{cases} R & i = 0 \\ 0 & i = 1 \\ R & i = 2 \end{cases}$$

$$\mathrm{H}^i_{\mathrm{dR}}(\mathbb{A}^1_S/S) \cong \begin{cases} R & i = 0 \\ 0 & i = 1 \\ 0 & i > 1 \end{cases}$$

and indeed this is the result in the geometric case $S = \operatorname{Spec}(\mathbb{C})$ and furthermore over any field of characteristic zero.

Over other rings the result for the proper $\mathbb{P}^1_S$ continues to hold, but the result for the affine $\mathbb{A}^1_S$ can be less nice:

$$\mathrm{H}^i_{\mathrm{dR}}(\mathbb{A}^1/\mathbb{F}_p) \cong \begin{cases} \bigoplus_{n\geq 0} \mathbb{F}_p & i=0 \\ \\ \bigoplus_{n\geq 0} \mathbb{F}_p & i=1 \end{cases}$$

$$\mathrm{H}^i_{\mathrm{dR}}(\mathbb{A}^1/\mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i=0 \\ \\ \bigoplus_{n\geq 1} \mathbb{Z}/n\mathbb{Z} & i=1 \end{cases} \tag{1.2.1}$$

The positive characteristic case has been well studied, and there is the so-called Cartier isomorphism, which explains how to view the cohomology as finite-dimensional, in a sense. The integral/mixed characteristic case is somewhat less understood, and we explore some methods of analyzing it in this work.

The de Rham cohomology is computed as the hypercohomology of the de Rham complex (of $\mathcal{O}_S$-modules)

$$\Omega^\bullet_{X/S} = \left[\mathcal{O}_X \to \Omega^1_{X/S} \to \Omega^2_{X/S} \to ...\right]$$

which for affine $X = \mathrm{Spec}(A)$ equals the ordinary cohomology of the sequence

$$\Gamma(\mathcal{O}_X) \to \Gamma(\Omega^1_{X/S}) \to \Gamma(\Omega^2_{X/S}) \to ...$$

$$A \to \Omega^1_{A/R} \to \Omega^2_{A/R} \to ...$$

and if $X/S = \mathrm{Spec}(A)/\mathrm{Spec}(R)$ is smooth of relative dimension one, then the sheaves of higher differentials vanish, and we are left with only the R-linear map

$$A \to \Omega^1_{A/R}.$$

So the case $Y = \mathbb{A}^1_R$ comes down to

$$R[x] \xrightarrow{\mathrm{d}} R[x] \cdot \mathrm{d}x$$

$$f(x) \mapsto \mathrm{d}f = f' \,\mathrm{d}x$$

$$x^n \mapsto nx^{n-1} \,\mathrm{d}x$$

an explicit calculation of the cohomology of the map d reveals

$$\mathrm{H}^i_{\mathrm{dR}}(\mathbb{A}^1/R) \cong \begin{cases} \bigoplus_{n \geq 0} R_{n-\mathrm{tors}} & i = 0 \\ \bigoplus_{n \geq 1} R/nR & i = 1 \end{cases}$$

where the n-torsion and n-cotorsion of R comprise the cohomology of $R[x]/R$.

In a later section we will spell out the computations for these results more explicitly.

Example 1.2.1. To see how the noninvertibility of elements of $\mathbb{Z}$ affects the de Rham cohomology $\mathrm{H}^\bullet(\mathbb{A}^1_{\mathbb{Z}}/\mathbb{Z})$, we calculate the cohomology of the complex of $\mathbb{Z}$-modules

$$\mathbb{Z}[x] \xrightarrow{\mathrm{d}} \Omega_{\mathbb{Z}[x]/\mathbb{Z}} = \mathbb{Z}[x] \cdot \mathrm{d}x$$

which decomposes as $\mathbb{Z}$-modules

$$\bigoplus_{n \geq 0} \mathbb{Z} \cdot x^n \to \bigoplus_{n \geq 0} \mathbb{Z} \cdot x^n \,\mathrm{d}x$$

$$x^n \mapsto nx^{n-1} \,\mathrm{d}x$$

The map and its cohomology can be visualized

$$\begin{array}{ccccccccc}
\mathbb{Z} & & 0 & & 0 & & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \ldots \\
\downarrow \times 0 & & \downarrow \times 1 & & \downarrow \times 2 & & \downarrow \times 3 & & \\
0 & & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \mathbb{Z} & \oplus & \ldots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & \mathbb{Z}/(2) & & \mathbb{Z}/(3) & &
\end{array}$$

where the top row represents $\mathrm{H}^0_{\mathrm{dR}}$, and the bottom row represents $\mathrm{H}^1_{\mathrm{dR}}$.

Remark 1.2.2. Based on the above example, and the remark in the introduction that we would like to filter the ring $\mathbb{Z}[x]$ by submodules $x^n\mathbb{Z}[x]$, one might get the impression that the idea is to work with modules allowing zeros of increasingly large order. In fact what we will be doing will be allowing *poles* of increasingly large order.

In the above example, the elements x^n should be thought of as containing increasingly large poles in the complementary subring $\mathbb{Z}[x^{-1}]$.

Remark 1.2.3. In the descriptions of $\mathrm{H}^1(\mathbb{A}^1_R/R)$ above, we obtain a direct sum decomposition $\bigoplus_{n\geq 1} R/nR$. Unfortunately, in the more general case we explore in the body of the work, we are only able to obtain a formula for the associated graded module of the de Rham cohomology.

When we apply this technique in 1.7 to the example above, the associated graded is isomorphic to the direct sum we have seen already.

1.3 Algebra background

1.3.1 Abelian categories

Proposition 1.3.1 (Snake lemma)**.** *Let $\mathcal{A}$ be an abelian category. Let*

$$\begin{array}{ccccccc} & & x & \xrightarrow{f} & y & \xrightarrow{g} & z & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & u & \xrightarrow{k} & v & \xrightarrow{l} & w & & \end{array}$$

be a commutative diagram with exact rows.

(1) There exists a unique morphism $\delta : \mathrm{Ker}(\gamma) \to \mathrm{Coker}(\alpha)$ such that the diagram

$$\begin{array}{ccccc} y & \xleftarrow[\pi']{} & y \times_z \mathrm{Ker}(\gamma) & \xrightarrow{\pi} & \mathrm{Ker}(\gamma) \\ \downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\delta} \\ v & \xrightarrow{\iota'} & \mathrm{Coker}(\alpha) \coprod_u v & \xleftarrow[\iota]{} & \mathrm{Coker}(\alpha) \end{array}$$

commutes, where π and π' are the canonical projections and ι and ι' are the canonical coprojections.

(2) The induced sequence

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma)$$

is exact. If f is injective, then so is f', and if l is surjective, then so is l'.

Proof. [14, Tag 07JV], [14, Tag 010H] □

Definition 1.3.2 (Kernel-cokernel sequences)**.** The six-term exact sequence

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma) \xrightarrow{\delta} \mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma)$$

in 1.3.1 is called the *kernel-cokernel sequence*.

For convenience, we also use the following terminology in the context of a morphism of short exact sequences. The *kernel sequence* is

$$\mathrm{Ker}(\alpha) \xrightarrow{f'} \mathrm{Ker}(\beta) \xrightarrow{g'} \mathrm{Ker}(\gamma).$$

The *cokernel sequence* is

$$\mathrm{Coker}(\alpha) \xrightarrow{k'} \mathrm{Coker}(\beta) \xrightarrow{l'} \mathrm{Coker}(\gamma).$$

We say the kernel-cokernel sequence *splits* if one of the three equivalent conditions is satisfied:

1. The connecting homomorphism δ is the zero map.

2. The kernel sequence is right exact.

3. The cokernel sequence is left exact.

In particular, if both rows of the diagram in the snake lemma are short exact, then for the kernel-cokernel sequence to split means that the kernel sequence and the cokernel sequence each separately form a short exact sequence.

Remark 1.3.3. The following presents a uniform way of discussing when a square in an abelian category is commutative, Cartesian, or co-Cartesian. It can be taken as a definition or a proposition.

One can find the equivalence with other definitions in the following references: [14, Tag 00ZX], [17].

Definition 1.3.4. Let $\mathcal{A}$ be an abelian category, and let

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ X & \xrightarrow{k} & Z \end{array}$$

be a diagram. Consider the associated sequence

$$0 \longrightarrow W \xrightarrow{(g,f)} X \oplus Y \xrightarrow{(k,-h)} Z \longrightarrow 0$$

Then the diagram is

0. *commutative* iff the sequence is a complex,

1. *Cartesian* iff the sequence is a left exact complex,

2. *co-Cartesian* iff the sequence is right exact complex,

3. *semi-Cartesian* iff the sequence is a middle exact complex.

Corollary 1.3.5. Left exact additive functors preserve Cartesian squares.

Proof. Let $F : \mathcal{A} \to \mathcal{B}$ be an additive functor which is left exact. The square from 1.3.4 is Cartesian after applying F iff the following sequence is exact:

$$0 \longrightarrow FW \xrightarrow{F(g,f)} FX \oplus FY \xrightarrow{F(k,-h)} FZ.$$

Note that additive functors preserve direct sums ([14, Tag 010M]). By definition a left exact functor preserves the exactness of the sequence. Therefore the result follows. □

Remark 1.3.6. Suppose that our abelian category $\mathcal{A}$ is the category of modules over some commutative ring (or more generally that $\mathcal{A}$ is concrete, i.e. admits a faithful functor to $\mathbf{Set}$). Then we can regard subobjects (e.g. images) as subsets, and we can analyze containments and intersections of subobjects in a straightforward set-theoretic way.

In such an abelian category, given a diagram as in 1.3.4, it is easy to see the containments

$$\mathrm{im}(k \circ g) = \mathrm{im}(h \circ f) \subset \mathrm{im}(k) \cap \mathrm{im}(h) \subset z,$$

where the first equality results from the commutativity of the diagram.

If the square is in addition Cartesian, then it is also easy to see that the containment becomes an equality:

$$\mathrm{im}(k \circ g) = \mathrm{im}(h \circ f) = \mathrm{im}(k) \cap \mathrm{im}(h).$$

This is a consequence of the construction of the fiber product of sets.

Corollary 1.3.7. Let $\mathcal{A}$ be a concrete abelian category, and consider a commutative diagram

$$\begin{array}{ccc} W & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\ X & \xrightarrow{k} & Z \\ \Big\downarrow{\scriptstyle \overline{g}} & & \Big\downarrow{\scriptstyle \overline{h}} \\ \overline{X} & \xrightarrow{\overline{k}} & \overline{Z} \end{array}$$

where $\overline{X} = \operatorname{coker}(g)$, $\overline{Z} = \operatorname{coker}(h)$, and $\overline{k}$ is the morphism induced from k. Suppose that the upper square is Cartesian and that k is injective.

Then $\overline{k}$ is injective.

Proof. Let $\overline{x} \in \ker(\overline{k})$, i.e. $\overline{X} \ni x \mapsto 0 \in \overline{Z}$. Then $\overline{x}$ lifts to $x \in X$ such that $k(x) \in \operatorname{im}(h) \subset Z$. So $k(x) \in \operatorname{im}(k) \cap \operatorname{im}(h)$. By 1.3.6, $k(x) = (k \circ g)(w)$, for some $w \in W$.

Then $k(x) = k(g(w))$, so $0 = k(x) - k(g(w)) = k(x - g(w))$. But k is injective, so $x = g(w)$. Then $\overline{x} = \overline{g}(x) = \overline{g}(g(w)) = 0$, since $\overline{g} \circ g = 0$.

Therefore $\overline{k}$ is injective. □

We have a partial converse to 1.3.6:

Proposition 1.3.8. *Let $\mathcal{A}$ be a concrete abelian category, and consider a diagram*

$$\begin{array}{ccc} \ker(g) & \xrightarrow{f'} & \ker(h) \\ \Big\downarrow & & \Big\downarrow \\ W & \xrightarrow{f} & Y \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle h} \\ X & \xrightarrow{k} & Z \end{array}$$

where the bottom square satisfies the image containment property in 1.3.6:

$$\mathrm{im}(k \circ g) = \mathrm{im}(h \circ f) = \mathrm{im}(k) \cap \mathrm{im}(h).$$

1. *If the induced map f' on kernels is an isomorphism, then the bottom square is semi-Cartesian.*

2. *If in addition, either f is injective or g is injective, then the bottom square is Cartesian.*

Proof. We must show that the complex

$$0 \longrightarrow W \xrightarrow{\alpha=(g,f)} X \oplus Y \xrightarrow{\beta=(k,-h)} Z$$

is exact.

Let $(x, y) \in X \oplus Y$ such that $\beta(x, y) = k(x) - h(y) = 0 \in Z$. Then the element $k(x) = h(y) \in Z$ is in the image of both h and k. By the image containment property of 1.3.6, there is a $w \in W$ such that $(h \circ f)(w) = (k \circ g)(w) = k(x) = h(y)$.

Then $h(y) = h(f(w)) \Rightarrow h(y) - h(f(w)) = h(y - f(w)) = 0 \Rightarrow y - f(w) \in \ker(h)$. By the assumption on the kernels, $y - f(w)$ lifts to a unique element $w_0 \in \ker(g) \subset W$. So $f(w_0) = y - f(w)$, and $g(w_0) = 0$. Set $w_1 = w + w_0$.

Then

$$f(w_1) = f(w) + f(w_0) = f(w) + (y - f(w)) = y,$$

$$g(w_1) = g(w) + g(w_0) = g(w) = x.$$

So $\alpha(w_1) = (x, y)$, and the complex is exact in the middle, and so the bottom square is semi-

Cartesian.

If either f or g is injective, then the sum (g, f) is also injective, and the complex is exact on the left. So the bottom square is Cartesian. □

1.4 Algebraic geometry background

1.4.1 Sheaves and line bundles

Proposition 1.4.1. *Let A be a graded ring and $X = \mathrm{Proj}(A)$. The associated sheaf functor $M \mapsto \tilde{M}$ from the category of graded A-modules to the category of quasi-coherent $\mathcal{O}_X$-modules is covariant, exact, and commutes with direct sums, filtered inductive limits, and tensor products.*

Proof. We begin by defining for any homogeneous element $f \in A_+$ a functor $M \mapsto \left(M_{(f)}\right)\tilde{}$ from A-modules to quasi-coherent modules over the distinguished affine $D_+(f)$. Then we show that this functor has the properties desired of the functor in the statement of the proposition.

See [9, p. 373/379] for more details □

Definition 1.4.2. (Twisting sheaves on a projective scheme) Let X be a projective scheme with graded coordinate ring $A = \bigoplus_{n \geq 0} A_d$. Associated to the grading on A, from any graded A-module M we define new graded A-modules $M(n)$ by

$$M(n)_d := M_{n+d}, \ \forall d \in \mathbb{Z}.$$

The *twisting sheaves* of the scheme X are

$$\mathcal{O}_X(n) := (A(n))\tilde{}.$$

Lemma 1.4.3. *Let X be a projective scheme with graded coordinate ring $A = \bigoplus_{n\geq 0} A_d$, and M a graded A-module. Let $d > 0$ be an integer, and $f \in A_d$ a homogeneous element. For any multiple $n = dk$ of d, multiplication by f^{-k} yields an isomorphism of $\mathcal{O}_X$-modules*

$$\mathcal{O}_X(n)|_{D_+(f)} \xrightarrow{\sim} \mathcal{O}_X|_{D_+(f)}.$$

Proof. [9, p. 374/380] [11, p. 164/181] □

The following content on Cartier divisors is taken from [11, p. 256/273ff].

Definition 1.4.4 (Cartier divisors, linear equivalence)**.** Let X be a scheme. Denote the group $\mathrm{H}^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$ by $\mathrm{Div}(X)$. The elements of $\mathrm{Div}(X)$ are called *Cartier divisors*. Let $f \in \mathrm{H}^0(X, \mathcal{K}_X^\times)$; its image in $\mathrm{Div}(X)$ is called a principal Cartier divisor and denoted by $\mathrm{div}(f)$. The group law on $\mathrm{Div}(X)$ is noted additively. We say that two Cartier divisors D_1, D_2 are *linearly equivalent* if $D_1 - D_2$ is principal, and we write $D_1 \sim D_2$. A Cartier divisor is called *effective* if it is in the image of the canonical map $\mathrm{H}^0(X, \mathcal{O}_X \cap \mathcal{K}_X^\times) \to \mathrm{H}^0(X, \mathcal{K}_X^\times/\mathcal{O}_X^\times)$, and we write $D \geq 0$. The set of effective Cartier divisors is denoted $\mathrm{Div}_+(X)$.

Remark 1.4.5. By definition, we can represent a Cartier divisor by a system $\{(U_i, f_i)_i\}$, where the U_i are open subsets forming a cover of X, f_i is the quotient of two regular elements of $\mathcal{O}_X(U_i)$, and $f_i|_{U_i\cap U_j} \in f_j|_{U_i\cap U_j}\mathcal{O}_X(U_i \cap U_J)^\times$ for every i, j.

We denote by $\mathrm{CaCl}(X)$ the group of isomorphism classes of Cartier divisors modulo the linear equivalence relation. We will relate this group to the Picard group $\mathrm{Pic}(X)$ in the following proposition. To start, if D is a Cartier divisor represented by $\{(U_i, f_i)\}$, we can associate a subsheaf $\mathcal{O}_X(D) \subset \mathcal{K}_X$ defined by $\mathcal{O}_X(D)|_{U_i} = f_i^{-1}\mathcal{O}_X|_U$. It is independent of the choice of a representing system $\{(U_i, f_i)\}$, and it is also an invertible sheaf on X.

Definition 1.4.6 (Twist by a Cartier divisor)**.** Given any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and a Cartier divisor D, we form the *twist of* $\mathcal{F}$ *by* D as

$$\mathcal{F}(D) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D).$$

Proposition 1.4.7. *Let* X *be a scheme. The following properties are true.*

a The map $\rho : D \mapsto \mathcal{O}_X(D)$ *is additive, i.e.*

$$\rho(D_1 + D_2) = \mathcal{O}_X(D_1) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D_2).$$

b The map ρ *induces an injective homomorphism*

$$\mathrm{CaCl}(X) \to \mathrm{Pic}(X).$$

c The image of ρ *corresponds to the invertible sheaves contained in* $\mathcal{K}_X$*.*

d If X *is a Noetherian scheme without embedded point (e.g. reduced), then the canonical homomorphism* $\mathrm{CaCl}(X) \to \mathrm{Pic}(X)$ *is an isomorphism.*

Proof. [11, p. 257/274]. □

Proposition 1.4.8 (Exactness on affines)**.** *Let* X *be a scheme.*

1. A sequence

$$\mathcal{F} \to \mathcal{G} \to \mathcal{H}$$

of quasi-coherent sheaves on X is exact if and only if it is exact on every open set in a given affine open cover of X.

2. *If*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

is a short exact sequence of quasi-coherent sheaves on X, then

$$0 \to \mathcal{F}(U) \to \mathcal{G}(U) \to \mathcal{H}(U)$$

is exact for any open $U \subset X$. Furthermore if U is affine, the sequence is also right exact. In other words, the global section functor is left exact.

[8, p. 382].

Corollary 1.4.9. Suppose we have a scheme X and a commutative square of quasi-coherent $\mathcal{O}_X$-modules

$$\begin{array}{ccc} \mathcal{E} & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ \mathcal{G} & \longrightarrow & \mathcal{H} \end{array}$$

Then the square is Cartesian if and only if it is Cartesian on every open set in a given affine cover of X.

Proof. Follows from 1.3.4 and 1.4.8. □

Similarly, we have

Corollary 1.4.10. Given a morphism $\varphi : X \to Y$ of schemes, the direct image functor φ_* takes Cartesian squares of $\mathcal{O}_X$-modules to Cartesian squares of $\mathcal{O}_Y$-modules.

Proof. Follows from 1.3.4 and the fact that the direct image functor is left exact. □

Proposition 1.4.11 (Pullback of locally free is locally free)**.** *Let $\varphi : X \to Y$ be a morphism of schemes, and $\mathcal{G}$ a locally free sheaf of rank r on Y.*

Then $\varphi^*\mathcal{G}$ is a locally free sheaf of rank r on X.

Proof. [8, p. 426] □

1.4.1.1 Sheaf cohomology

Proposition 1.4.12. *Let $\pi : X \to S$ be a morphism of schemes and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module. If π is affine, then π_* is an exact functor, i.e.*

- $\mathbf{R}^i\pi_*\mathcal{F} = 0 \ \ i > 0$

- $\mathrm{H}^i(X, \mathcal{F}) = \mathrm{H}^i(S, \pi_*\mathcal{F}) \ \ i \geq 0$

In particular the result holds if π is finite (e.g. closed immersions).

Proof. [14, Tag 01XC], [14, Tag 089W], □

Proposition 1.4.13. *Let $i : Z \to X$ be a closed immersion with ideal sheaf $\mathcal{I} \subset \mathcal{O}_X$. Then the pushforward functor*

$$i_* : \mathrm{QCoh}(\mathcal{O}_Z) \to \mathrm{Qcoh}(\mathcal{O}_X)$$

is exact, fully faithful, and with essential image those sheaves $\mathcal{G}$ such that $\mathcal{I}\mathcal{G} = 0$.

Furthermore, there is a right adjoint $i^!$, which is given by the pullback of $\mathcal{I}$-torsion.

Proof. The statement about the functor i_* is [14, Tag 01QY]. The statement about the adjoint $i^!$ is [14, Tag 01R0]. □

Proposition 1.4.14 (Support of coherent is closed)**.** *Let X be a locally Noetherian scheme and $\mathcal{F}$ a coherent $\mathcal{O}_X$-module.*

Then the support $\mathrm{Supp}(\mathcal{F})$ is closed, and $\mathcal{F}$ comes from a coherent sheaf on the scheme-theoretic support of $\mathcal{F}$.

Proof. [14, Tag 01Y5] □

Proposition 1.4.15 (Composition of right derived functors)**.** *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories, and $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functions. Assume $\mathcal{A}, \mathcal{B}$ have enough injectives. The following are equivalent:*

1. *$F(I)$ is right acyclic for G for each injective object I of $\mathcal{A}$,*
2. *The canonical map*

$$t : \mathbf{R}(G \circ F) \to \mathbf{R}G \circ \mathbf{R}F$$

   is an isomorphism of functors from $D^+(\mathcal{A})$ to $D^+(\mathcal{C})$

Proof. [14, Tag 015L] □

Proposition 1.4.16 (Composition of higher direct images)**.** *Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms of schemes. The canonical map of right derived functors*

$$t : \mathbf{R}(g \circ f)_* \to \mathbf{R}g_* \circ \mathbf{R}f_*$$

from $D_{\mathrm{qcoh}}(X)$ to $D_{\mathrm{qcoh}}(Z)$ is an isomorphism.

If f is an affine morphism and $\mathcal{F}$ an $\mathcal{O}_X$-module, then we also have

$$\mathbf{R}^i(g \circ f)_*\mathcal{F} \to \mathbf{R}^i g_*(f_*\mathcal{F}).$$

Proof. By [14, Tag 06WW] pushfoward preserves injectives, so by 1.4.15, the map t of functors is an isomorphism. The result for affine morphisms follows because if f is affine, then f_* is an exact functor.

□

Proposition 1.4.17. *$j : U \hookrightarrow X$ is an affine morphism, so j_* is an exact functor, and $\mathbf{R}^\bullet(\pi \circ j)_*\mathcal{O}_U = \mathbf{R}^\bullet\pi_*(j_*\mathcal{O}_U)$.*

Proposition 1.4.18 (Projection formula)**.** *Let $\varphi : X \to Y$ be a morphism of schemes, $\mathcal{F}$ a quasi-coherent sheaf on X, and $\mathcal{G}$ a quasi-coherent sheaf on Y. For any $i \geq 0$ there is a canonical homomorphism*

$$(\mathbf{R}^i\varphi_*\mathcal{F}) \otimes_{\mathcal{O}_Y} \mathcal{G} \to \mathbf{R}^i\varphi_* \left(\mathcal{F} \otimes_{\mathcal{O}_X} \varphi^*\mathcal{G}\right)$$

that is an isomorphism if $\mathcal{G}$ is flat over Y (e.g. locally free).

It is also an isomorphism if φ is affine.

Proof. See [11, p. 190/207]. The idea is to work affine-locally on Y, take a Čech complex for $\mathcal{F}$, and tensor with $\mathcal{G}$.

See [8, p. 428] for the affine case. □

1.4.1.2 Higher direct images

Proposition 1.4.19 (Grothendieck vanishing)**.** *Let $\varphi : X \to Y$ be a projective morphism to a locally Noetherian scheme (or more generally such that $\mathcal{O}_Y$ is coherent over itself).*

If d is the supremum of the dimensions of the fibers of φ, then $\mathbf{R}^i\pi_*\mathcal{F} = 0$ for any quasi-coherent $\mathcal{O}_X$-module and any $i > d$.

Proof. [8, p. 500] □

Definition 1.4.20 (Relatively flasque)**.** Let $\pi : X \to S$ be a map of topological spaces, and $\mathcal{F}$ a sheaf on X. Then $\mathcal{F}$ is *relatively flasque* (with respect to π) if for any opens $U \subset V \subset X$, any $x \in U$, and any $\sigma \in \Gamma(U, \mathcal{F})$, there exists any open W satisfying $\pi(x) \in W \subset S$ such that the restriction $\sigma|_{U \cap \pi^{-1}(W)}$ lies in the image of

$$\Gamma\left(V \cap \pi^{-1}(W), \mathcal{F}\right) \to \Gamma\left(U \cap \pi^{-1}(W), \mathcal{F}\right).$$

In particular a flasque sheaf on X is relatively flasque.

Proposition 1.4.21. *Suppose $\pi : X \to S$ is a map of topological spaces with X is quasi-compact, and let $\mathcal{F}$ be a relatively flasque sheaf of abelian groups on X over S.*

Then $\mathbf{R}^i\pi_*\mathcal{F} = 0$ for $i > 0$.

Proof. See [15, p. 23] or a detailed proof in [18]. □

Lemma 1.4.22. *Let $\pi : X \to S$ be a map of topological spaces, and let $\mathcal{G}$ be a sheaf on S. Then the inverse image sheaf $\pi^{-1}\mathcal{G}$ on X is relatively flasque over S.*

If furthermore $\mathcal{G}$ is a sheaf of abelian groups, and X is quasi-compact, then

$$\mathbf{R}^i\pi_*\left(\pi^{-1}\mathcal{G}\right) = 0, \ \ i > 0.$$

Proof. Let $x \in U \subset V \subset X$, and $\sigma \in \pi^{-1}\mathcal{G}(U)$, as in 1.4.20.

$$\pi^{-1}\mathcal{G}(U) := \varinjlim_{W \supset \pi(U)} \mathcal{G}(W)$$

so take $W_0 \supset \pi(U)$ and $\sigma_0 \in \mathcal{G}(W_0)$ representing σ. Then we have $\sigma_0 \in \pi^{-1}\mathcal{G}(\pi^{-1}(W_0))$, and by the restriction map $\pi^{-1}\mathcal{G}\big(\pi^{-1}(W_0)\big) \to \pi^{-1}\mathcal{G}\big(V \cap \pi^{-1}(W_0)\big)$ we obtain $\tau := \sigma_0|_{V\cap\pi^{-1}(W_0)}$ such that $\tau|_{U\cap\pi^{-1}(W_0)} = \sigma|_{U\cap\pi^{-1}(W_0)} = \sigma|_U = \sigma$.

The last claim follows from 1.4.21. □

1.4.2 Flatness

Proposition 1.4.23. *Let $f : X \to Y$ be an affine morphism, and $\mathcal{F}$ a quasi-coherent $\mathcal{O}_X$-module.*

Then $f_*\mathcal{F}$ is flat over Y iff $\mathcal{F}$ is flat over Y, and f is a flat morphism iff $\mathcal{O}_X$ is flat over Y.

Proof. [14, Tag 01U2], □

Proposition 1.4.24. *The pushforward of the structure sheaf along a proper flat map is locally free.*

Proof. [14, Tag 08EU], □

1.4.2.1 Cohomological flatness

This proposition applies in our situation due to the following characterization of cohomological flatness (in dimension zero)

Definition 1.4.25 (Cohomological flatness)**.** Let $\varphi : X \to Y$ be a flat, projective morphism, where Y is locally Noetherian. Then φ is *cohomologically flat* (in dimension zero) if the construction of $\varphi_*\mathcal{O}_X$ commutes with arbitrary base change.

See [11, p. 209/226]

Lemma 1.4.26. *Let S be the spectrum of a DVR, and suppose $\pi : X \to S$ is a flat, finite type, separated S-scheme of relative dimension 1, which is normal and with smooth generic fiber.*

Then π is cohomologically flat if and only if $\mathrm{H}^1(X, \mathcal{O}_X)$ is a torsion free $\mathcal{O}_S$-module.

Proof. [1, p. 4] □

Proposition 1.4.27. *(Raynaud's condition for cohomological flatness) Let X be a connected proper relative curve over S, where S is the spectrum of a DVR $(R, \mathfrak{m}, k)$. If $\mathrm{char}(k) = 0$, then X is cohomologically flat. If $\mathrm{char}(k) = p > 0$, then X is cohomologically flat if the GCD of the geometric multiplicities of the components of the closed fibers is prime to p, or a fortiori if $X \to S$ admits and etale quasi-section (e.g. a section).*

Proof. [1, p. 6] □

Proposition 1.4.28 (Cohomological flatness is a local property)**.** *Let Y be locally Noetherian, and $\varphi : X \to Y$ be a flat projective morphism. Then φ is cohomologically flat iff for every $y \in Y$, the base change $X \times_Y \mathrm{Spec}\mathcal{O}_{Y,y} \to \mathrm{Spec}\mathcal{O}_{Y,y}$ is cohomologically flat.*

Proof. [11, p. 209/226] □

Example 1.4.29. The relative projective line $\mathbb{P}^1_S$ is cohomologically flat, by 1.4.26 and 1.4.28.

Proposition 1.4.30. *If $\pi : X \to S$ is a proper, flat morphism with geometric fibers reduced and connected, then π is cohomologically flat.*

[16, p. 25]

1.4.3 Arithmetic surfaces

We use the terminology from Qing Liu's book [11] to define arithmetic surfaces. However whereas Liu allows a Dedekind domain/scheme to be zero or one-dimensional, we will follow the convention that a Dedekind domain/scheme must be one-dimensional.

Definition 1.4.31. A *surface* is a Noetherian, integral scheme of dimension 2, endowed with a projective, flat morphism to a base scheme S.

[11, p. 317/334]

Definition 1.4.32. Let S be a Dedekind scheme. A *fibered surface* (over S) is a 2-dimensional integral, projective, flat S-scheme $\pi : X \to S$.

An *arithmetic surface* is a fibered surface $X \to S$ which is in addition a regular scheme (hence normal).

[11, p. 347/364,353/370]

Remark 1.4.33. A common way one thinks about a fibered surface X/S over a Dedekind domain is as a compatible family of curves X_η, X_s over the generic point $\eta \in S$ and all the closed points $s \in S$. According to the next proposition, each fiber is a projective curve. When S is of mixed characteristic, the generic fiber X_η is a curve over a field of characteristic zero, and the closed fibers X_s are curves over finite fields $\mathbb{F}_q$. The ability to synthesize these different types of curves is what makes such fibered surfaces both interesting and difficult.

Proposition 1.4.34. *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme S, and let $\eta \in S$ be the generic point and $s \in S$ a closed point. Then*

- *The fiber X_s is a projective curve over $k(s)$, and we have an equality of arithmetic genera $p_a(X_s) = p_a(X_\eta)$.*

- *If X_η is geometrically connected (e.g. $\mathcal{O}_S \simeq \pi_*\mathcal{O}_X$), then the same holds for X_s.*

- *If X is geometrically integral, then the canonical homomorphism $\mathcal{O}_S \to \pi_*\mathcal{O}_X$ is an isomorphism.*

- *Suppose that X is a regular scheme. Then the morphisms $X \to S$ and $X_s \to \mathrm{Spec} k(s)$ are local complete intersections, and we have the relation $\omega_{X_s/k(s)} = \omega_{X/S}|_{X_s}$ between dualizing sheaves.*

Proof. [11, p. 350/367] □

Example 1.4.35. Here are some basic examples of fibered surfaces:

- $\mathbb{A}^1_{\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x])$, the affine line over $\mathbb{Z}$, is not proper and therefore not a fibered surface according to our restrictive definition, but it is the most important example to understand first. If the reader would like to gain some intuition, we point to Mumford's famous illustration:

- $\mathbb{P}^1_{\mathbb{Z}}$, the projective line over $\mathbb{Z}$, has a projective line $\mathbb{P}^1_k$ over each point in $\mathrm{Spec}(\mathbb{Z})$. $\mathbb{P}^1_{\mathbb{Z}}$ is the set-theoretic union of $\mathbb{A}^1_{\mathbb{Z}}$ with one copy of $\mathrm{Spec}(\mathbb{Z})$ at infinity.

- A relative elliptic curve E/R over a number ring R, where the generic fiber E_η is an elliptic curve over the fraction field $\mathrm{Frac}(R)$. Each closed fiber E_p will be a curve with arithmetic genus 1, but not necessarily smooth. The singular fibers are termed the fibers of bad reduction, and these have important consequences for the arithmetic of the scheme.

By a result of Fontaine, there are no abelian varieties over $\mathbb{Z}$, and so given a fibered surface $E/\mathbb{Z}$ with generic fiber E_η an elliptic curve, there will necessarily be singular fibers.

Remark 1.4.36 (Regularity vs smoothness). The distinction between regularity and smoothness is a potential point of confusion for those unfamiliar with these types of schemes. These conditions coincide for varieties over a perfect field, but for schemes over a Dedekind scheme we have only that smoothness implies regularity. Regularity requires that the Zariski cotangent spaces $\mathfrak{m}/\mathfrak{m}^2$ at each point have the correct dimension. This is an absolute condition, placed on the scheme X. Smoothness requires that the sheaf $\Omega^1_{X/S}$ of relative differentials is locally free. This is a relative condition, placed on the morphism $X \to S$.

The blowup $\tilde{X}$ of a smooth (hence regular) arithmetic surface X/S at a closed point will remain regular, but it will no longer be smooth. If E/S is an arithmetic surface with generic fiber E_η an elliptic curve, then E will be a regular scheme, but E/S will only be smooth if there are no fibers of bad reduction.

Proposition 1.4.37. *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme S, and suppose π admits an etale quasi-section (e.g. a section). Then π is cohomologically flat.*

Proof. By 1.4.28, we need only check the base change of X to the local rings $\mathcal{O}_{S,s}$. Since π admits an etale quasi-section, each such base change is cohomologically flat by 1.4.27. ☐

Proposition 1.4.38. *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme.*

Then for any quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and integer $i > 1$

$$\mathbf{R}^i\pi_*\mathcal{F} = 0.$$

Proof. Since π is projective with 1-dimensional fibers, the result follows from dimensional vanishing: 1.4.19 □

1.4.3.1 Local structure of an arithmetic surface

Proposition 1.4.39. *Let $\pi : X \to S$ be an arithmetic surface with at most normal crossing singularities (e.g. π smooth). Let $s \in S, x \in X_s$ be closed points, and fix a uniformizing parameter $i \in \mathcal{O}_{S,s}$. The following properties are true.*

(a) There exists a flat scheme of finite type Z over S of relative dimension 2, a closed point $z \in Z_s$ such that Z_s us regular at z, and a surjective homomorphism $\mathcal{O}_{Z,z} \to \mathcal{O}_{X,x}$, whose kernel is generated by an element $F \in \mathcal{O}_{Z,z}$. The form of F can be specified as follows:

(b) If a single irreducible component Γ_1 of X_s, of multiplicity d_1, passes through x, then $F = u^{d_1} - ta$, where u is part of a system of parameters of $\mathcal{O}_{Z,z}$, and where $a \in \mathcal{O}_{Z,z}^{\times}$. Moreover, if Γ_1 is smooth at x, then Z is smooth at z.

(c) If two irreducible components Γ_1, Γ_2 of X_s, of respective multiplicities d_1, d_2, pass through x, then $F = u^{d_1}v^{d_2} - ta$, where $\{t, u, v\}$ is a system of parameters of $\mathcal{O}_{Z,z}$, and $a \in \mathcal{O}_{Z,z}^{\times}$.

Proof. [11, p. 406/423] □

Proposition 1.4.40. *Let $\pi : X \to S$ be a morphism of schemes, and $x \in X$. The following conditions are equivalent:*

(a) π is smooth at x of relative dimension n,

(b) There exists an open neighborhood $U \in x$ and a commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{g} & \mathbb{A}^n_S \\ & \searrow^{f|_U} & \downarrow p \\ & & S \end{array}$$

where g is etale and p is the canonical projection.

Furthermore there are local sections $g_1, \dots, g_n$ of $\mathcal{O}_X$ such that $\mathrm{d}g_1, \dots, \mathrm{d}g_n$ generate $\Omega^1_{X/S}$ in a neighborhood of x.

Proof. [10, p. 42, Prop. 11] □

Corollary 1.4.41. Let $\pi : X \to S$ be a smooth arithmetic surface, and $x \in X$. Then there exists an open neighborhood $U \ni x$ and a section $z \in \mathcal{O}_X(U)$ such that $\mathrm{d}z$ generates $\Omega^1_{X/S}(U)$.

Proof. This follows from 1.4.40, and the fact that π is of relative dimension 1. □

1.4.3.2 Divisors on a fibered surface

Remark 1.4.42. In the body of this work we will deal exclusively with regular surfaces (hence normal), on which there is an equivalence between Cartier and Weil divisors.

The following distinction is important to the study of divisors on a fibered surface:

Definition 1.4.43. Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme, and $D \subset X$ an irreducible Weil divisor.

D is *horizontal* if $D \to S$ is surjective (hence finite). D is *vertical* if $\pi(D)$ is reduced to a point.

An arbitrary Weil divisor is said to be horizontal or vertical if all of its components are horizontal or vertical, respectively.

[11, p. 349/366]

Proposition 1.4.44. *Let $\pi : X \to S$ be a fibered surface over a Dedekind scheme. Then*

a Let x be a closed point of the generic fiber X_η. Then the closure $\overline{\{x\}}$ is an irreducible closed subset of X, finite and surjective to S.

b Let D be an irreducible closed subset of X. If $\dim D = 1$, then either D is an irreducible component of a closed fiber, or $D = \overline{\{x\}}$ for a closed point $x \in X_\eta$ of the generic fiber.

c Let x_0 be a closed point of X. Then $\dim \mathcal{O}_{X,x_0} = 2$.

In particular, if $D \subset X$ is a horizontal divisor, then D/S is a finite morphism, hence an affine morphism.

Proof. [11, p. 349/366] □

In a more general relative setting one has the following notion of relative divisor:

Definition 1.4.45. Let $\pi : X \to S$ be a morphism of schemes, which is flat and locally of finite presentation. A *relative effective Cartier divisor* on X/S is an effective Cartier divisor $D \subset X$ such that $D \to S$ is a flat morphism. A *relative Cartier divisor* is the sum or difference of relative effective Cartier divisors.

See [14, Tag 056P] or [10, p. 213].

A good reason for the definition of a relative effective Cartier divisor is:

Proposition 1.4.46 (Relative divisors pull back)**.** *Let $\pi : X \to S$ be a morphism of schemes (not necessarily flat or of finite presentation). Let $D \subset X$ be an effective Cartier divisor such that the induced $D \to S$ is flat.*

Then for every morphism of schemes $\psi : S' \to S$ and $\pi' : X' = X \times_S S' \to S'$ the pullback $(\pi')^{-1} D$ of D to X' is an effective Cartier divisor.

Proof. [14, Tag 056P], □

A further characterization of relative effective divisors is found in:

Proposition 1.4.47. *Let $\pi : X \to S$ be locally of finite presentation. Let $\mathcal{I} \subset \mathcal{O}_X$ be a quasi-coherent sheaf of ideals, which is locally of finite presentation, and let $D \subset X$ be the closed subscheme defined by $\mathcal{I}$. Let $x \in D$ and set $s := \pi(x)$. Then the following are equivalent:*

1. *$\mathcal{I}$ is invertible at x (i.e. $\mathcal{I}_x$ is generated by a regular element), and D is flat over S at x.*
2. *X and D are flat over S at x, and the restriction D_s of D onto the fiber X_s over s is an effective Cartier divisor on X_s at x.*
3. *X is flat over S at x, and $\mathcal{I}_x$ is generated by an element f_x which induces a regular element on X_s at x.*

Proof. [10, p. 213] □

We have a partial converse to 1.4.45:

Proposition 1.4.48 (Relative Cartier divisor is flat)**.** *Suppose $\pi : X \to Y$ is a flat morphism of Noetherian schemes, and $D \subset X$ a locally principal subscheme that is an effective divisor on all fibers of π.*

Then $D \to Y$ is also flat.

Proof. [8, p. 668] $\square$

Proposition 1.4.49 (Cartier divisor restricts to Cartier divisor)**.** *Let X be a regular, Noetherian, connected scheme of dimension 2. Let $x \in X$ be a closed point. If $D \xrightarrow{i} X, E \xrightarrow{j} X$ are effective Cartier divisors with no common component, then $D|_E := j^*D$ is an effective Cartier divisor on D, and*

$$\mathcal{O}_E(D)|_E \simeq \mathcal{O}_X(D)|_E.$$

Proof. [11, p. 377/394, Lemma 1.4(b)] $\square$

Corollary 1.4.50 (Horizontal divisor is flat)**.** Let $\pi : X \to S$ be an arithmetic surface, and $D \subset X$ a horizontal effective divisor.

Then D is flat over S.

Proof. Since X is regular, X is also normal, and there is an equivalence of Weil and Cartier divisors. For any closed point $s \in S$, the fiber X_s is a vertical divisor on X and does not share a common component with D. So by 1.4.49, the restriction $D|_{X_s}$ is an effective Cartier divisor. Then by 1.4.48, D is flat over S. $\square$

Proposition 1.4.51. *Let $\pi : X \to S$ be an arithmetic surface, η the generic point of S, and $s \in S$ a closed point. Then for any point $P \in X_\eta$, its closure $\overline{\{P\}}$ is a horizontal divisor on X, and its restriction P_s to X_s satisfies*

$$\deg_{k(s)} P_s = [K(P) : K(S)]$$

Proof. [11, p. 388/405, Lemma 1.29] $\square$

Proposition 1.4.52. *Let $\pi : X \to S$ be an arithmetic surface, and $D \subset X$ a sum of horizontal Cartier divisors.*

The restriction D_s to any (not necessarily closed) fiber is a Cartier divisor on a projective curve over $k(s)$, and furthermore the degree $\deg_{k(s)}(D_s)$ is independent of s.

Proof. By 1.4.34, each fiber X_s is a projective curve over the residue field $k(s)$ of s By 1.4.46, the restriction D_s to any (not necessarily closed) fiber is again a Cartier divisor.

The independence of degree follows from 1.4.51 [11, p. 388/405] □

Corollary 1.4.53. Let $D \subset X$ be a horizontal divisor, and $\psi : D \to S$ the induced morphism.

Then $\mathbf{R}\pi_*\mathcal{O}_X = \mathbf{R}\psi_*\mathcal{O}_D$ is a locally free $\mathcal{O}_S$-module.

Proof. Follows from 1.4.24, since D/S is finite (hence proper) and flat. This result is also stated in [11, p. 249/266]. □

1.4.3.3 Line bundles on arithmetic surfaces

Now we observe some important consequences of previous cohomological results applied to $\mathcal{O}_X$-modules whose support is contained in a horizontal divisor (more pithily: *horizontally supported* sheaves).

Proposition 1.4.54 (Cohomological vanishing for horizontal divisors)**.** *Let $X \xrightarrow{\pi} S$ be an arithmetic surface, and $D \xrightarrow{i} X$ a horizontal divisor, and $\psi : D \to S$ the induced morphism. Let $\mathcal{F}$ be a quasi-coherent sheaf on D.*

Then $\mathcal{F}$ is acyclic for ψ_*, i.e.

$$\mathbf{R}^i\psi_*\mathcal{F} = 0$$

for $i > 0$.

In particular the result holds when $\mathcal{F}$ is $\mathcal{O}_D(nD)$ or the restriction $\mathcal{G}|_D = i^*\mathcal{G}$ for a quasi-coherent sheaf $\mathcal{G}$ on X.

Proof. $\psi : D \to S$ is affine by 1.4.44, so by 1.4.12, ψ_* is an exact functor, so

$$\mathbf{R}^i\psi_*\mathcal{F} = 0$$

for $i > 0$.

□

Corollary 1.4.55 (Horizontally supported is acyclic)**.** Let $X \xrightarrow{\pi} S$ be an arithmetic surface and $D \xrightarrow{i} X$ a horizontal divisor. Let $\mathcal{G}$ be a quasi-coherent sheaf on X which is supported on D.

Then $\mathcal{G} \simeq i_*\mathcal{F}$ for some quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_D$-modules, and furthermore $\mathcal{G}$ is acyclic for π_*, i.e.

$$\mathbf{R}^i\pi_*\mathcal{G} = 0$$

for $i > 0$.

In particular $i_*i^*\mathcal{G}$ is acyclic for π_*, for any quasi-coherent $\mathcal{O}_X$-modules $\mathcal{G}$.

Proof. For $\mathcal{G}$ to be supported on D is equivalent to $\mathcal{G}$ being annihilated by the ideal sheaf $\mathcal{I}_D = \mathcal{O}_X(-D)$ of D. So by 1.4.13, $\mathcal{G} \simeq i_*\mathcal{F}$ for some quasi-coherent sheaf $\mathcal{F}$ on D.

Then by 1.4.16 and 1.4.54,

$$\mathbf{R}^i\pi_*\mathcal{G} \simeq \mathbf{R}^i\pi_*(i_*\mathcal{F}) = \mathbf{R}^i(\pi_* \circ i_*)\mathcal{F} = \mathbf{R}^i\psi_*\mathcal{F} = 0$$

for $i > 0$.

□

Proposition 1.4.56 (Injection with horizontal cokernel). *Let $\pi : X \to S$ be an arithmetic surface over a Dedekind scheme, and $\mathcal{F}' \xrightarrow{\lambda} \mathcal{F}$ an injection of $\mathcal{O}_X$-modules with cokernel $\mathcal{T}$ which is supported on a horizontal divisor.*

Then the induced maps

$$\mathbf{R}^0\pi_*\mathcal{F}' \to \mathbf{R}^0\pi_*\mathcal{F}$$

$$\mathbf{R}^1\pi_*\mathcal{F}' \to \mathbf{R}^1\pi_*\mathcal{F}$$

are injective and surjective, respectively.

Proof. We have a short exact sequence of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F}' \xrightarrow{\lambda} \mathcal{F} \to \mathcal{T} \to 0,$$

where $\mathcal{T}$ is supported on a horizontal divisor.

By 1.4.55, $\mathbf{R}^1\pi_*\mathcal{T} = 0$, so the associated long exact sequence reads

$$0 \longrightarrow \mathbf{R}^0\pi_*\mathcal{F}' \longrightarrow \mathbf{R}^0\pi_*\mathcal{F} \longrightarrow \mathbf{R}^0\pi_*\mathcal{T} \longrightarrow$$
$$\longrightarrow \mathbf{R}^1\pi_*\mathcal{F}' \longrightarrow \mathbf{R}^1\pi_*\mathcal{F} \longrightarrow 0$$

and the result follows. □

Remark 1.4.57. We will primarily be applying 1.4.56 to the cases

$$\mathcal{O}_X\big((n-1)D\big) \to \mathcal{O}_X(nD),$$

$$\Omega^1_{X/S}\big((n-1)D\big) \to \Omega^1_{X/S}(nD),$$

where $D \subset X$ is a horizontal Cartier divisor, and $\Omega^1_{X/S}$ is the sheaf of differentials 1.4.64.

So we are considering the sequences.

$$0 \longrightarrow \mathcal{O}_X\big((n-1)D\big) \longrightarrow \mathcal{O}_X(nD) \longrightarrow \mathcal{O}_D(nD) \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big) \longrightarrow \Omega^1_{X/S} \otimes \mathcal{O}_X(nD) \longrightarrow \Omega^1_{X/S}\mathcal{O}_D(nD) \longrightarrow 0$$

Note that the support of a sheaf is not changed upon tensoring with a locally free sheaf, e.g. in this case $\mathcal{O}_D(nD)$ and $\Omega^1_{X/S} \otimes \mathcal{O}_D(nD)$ are both supported on D. One way to see that the support is not changed is to combine 1.4.18 and 1.4.13.

1.4.4 Sheaves of differentials

Definition 1.4.58 (Derivation)**.** If A is an R algebra and M an A-module, an *R-derivation of A with values in M* is an R-linear map

$$\partial : A \to M$$

satisfying the Leibnitz rule

$$\partial(ab) = a\partial(b) + b\partial(a)$$

The collection of such derivations is denoted $\mathrm{Der}_R(A, M)$

The collection of first-order differential operators is the special case of derivations from A to itself: $\mathrm{Der}_R(A, A)$

Definition 1.4.59 (Differential, universal derivation)**.** If A is an R-algebra, the *module of relative* *differentials of A over R* is an A-module $\Omega_{A/R}$ together with an R-derivation

$$\mathrm{d} : A \to \Omega_{A/R}$$

satisfying the following universal property:

For any A-module M and R-derivation $\partial : A \to M$, there is a unique A-module homomorphism $\phi : \Omega_{A/R} \to M$ such that $\partial = \phi \circ \mathrm{d}$.

d is referred to as the *universal derivation* of A over R.

Proposition 1.4.60. *The module $\Omega_{A/R}$ exists and is unique up to unique isomorphism.*

Proof. [11, p. 211/228] The uniqueness is proved as usual with universal properties.

For existence, take the free module F generated by the symbols $\mathrm{d}a, a \in A$, and $E \subset F$ the submodule generated by

$$\mathrm{d}r$$

$$\mathrm{d}(a_1 + a_2) - \mathrm{d}a_1 - \mathrm{d}a_2$$

$$\mathrm{d}(a_1 a_2) - a_1 \,\mathrm{d}a_2 - a_2 \,\mathrm{d}a_1$$

for $r \in R$, $a_i \in A$.

Define

$$\Omega_{A/R} := F/E$$

$$\mathrm{d}(a) := \mathrm{d}a \pmod{E}$$

It is straightforward to show the universal property is satisfied. ∎

Remark 1.4.61. For an A-module M, there is a map

$$\operatorname{Hom}_A(\Omega_{A/R}, M) \to \operatorname{Der}_R(A, M)$$

$$\phi \mapsto \phi \circ \mathrm{d}$$

The universal property of $\Omega_{A/R}$ is equivalent to this map being an isomorphism.

Definition 1.4.62 (Module of higher differentials)**.** There is a complex $\Omega^{\bullet}_{A/R}$ containing differentials of all orders (See [14, Tag 0FKF]), where we use the wedge product ([14, Tag 00DM]) to define

$$\Omega^p_{A/R} := \bigwedge^p \Omega_{A/R}$$

$$\Omega^{\bullet}_{A/R} = \left[A \xrightarrow{\mathrm{d}} \Omega_{A/R} \xrightarrow{\mathrm{d}} \Omega^2_{A/R} \to \ldots \right]$$

For this reason $\Omega_{A/R}$ is sometimes written as $\Omega^1_{A/R}$.

Remark 1.4.63. The derivation d is defined by sending $a \mapsto \mathrm{d}a$, using the notation of the construction in 1.4.60.

For each $k \geq 1$, the derivation $\Omega^k_{A/R} \xrightarrow{\mathrm{d}} \Omega^{k+1}_{A/R}$ is the unique R-linear map such that

$$\mathrm{d}\left(a_0 \,\mathrm{d}a_1 \wedge \mathrm{d}a_2 \wedge \ldots \wedge \mathrm{d}a_k\right) = \mathrm{d}a_0 \wedge \mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_k \,.$$

Since $\Omega^1_{A/R}$ is generated over A by the elements $\mathrm{d}a$, the A-module $\Omega^k_{A/R}$ is generated over A by the elements $a_0 \,\mathrm{d}a_1 \wedge \ldots \wedge \mathrm{d}a_k$.

The composition $\mathrm{d}^2 : \Omega^0_{A/R} \to \Omega^1_{A/R} \to \Omega^2_{A/R}$ is zero, as

$$\mathrm{d}(\mathrm{d}(a)) = \mathrm{d}(1\,\mathrm{d}a) = \mathrm{d}(1) \wedge \mathrm{d}(a) = 0 \wedge \mathrm{d}a = 0.$$

The composition $\mathrm{d}^2 : \Omega^k_{A/R} \to \Omega^{k+1}_{A/R} \to \Omega^{k+2}_{A/R}$ is zero by a similar computation.

See [14, Tag 0FKF] for more details.

We use the notation $M\tilde{}$ to denote the sheaf on $\mathrm{Spec}(A)$ associated to the A-module M.

Definition 1.4.64 (Sheaf of relative differentials)**.** Let $\varphi : X \to Y$ be a morphism of schemes. The *sheaf of relative differentials* of φ is the unique quasi-coherent sheaf $\Omega_{X/Y}$ on X such that for any affine open $V \subset Y$, and any affine open $U \subset \varphi^{-1}(V)$, we have

$$\Omega^1_{X/Y}|_U \cong (\Omega^1_{\mathcal{O}_X(U)/\mathcal{O}_Y(V)})\tilde{}$$

The universal derivations on the affine opens glue to form a derivation

$$\mathrm{d} = \underset{X/Y}{\mathrm{d}} : \mathcal{O}_X \to \Omega_{X/S}.$$

The map d is referred to as the *universal derivation* or *exterior differentiation* of $X \to S$. We may sometimes write Ω_X in place of $\Omega_{X/S}$, if S is understood.

The definition can be found in [11, p. 216/233], along with more details.

Remark 1.4.65. The sheaf of relative differentials satisfies a universal property analogous to 1.4.61:

Let $\varphi : X \to Y$ be a morphism of schemes. There is a natural map

$$\mathrm{Hom}_{\mathcal{O}_X}\left(\Omega_{X/Y}, \mathcal{F}\right) \to \mathrm{Der}_Y\left(\mathcal{O}_X, \mathcal{F}\right)$$

$$\alpha \mapsto \alpha \circ \underset{X/Y}{\mathrm{d}}$$

This map is an isomorphism of functors, by the universal property of $\Omega_{X/Y}$ ([14, Tag 01UM]).

1.4.4.1 The different ideal

Recall from 1.4.44 that a horizontal divisor D of X/S is finite over S.

Definition 1.4.66 (Codifferent, different ideal)**.** Let $A \to B$ be an injection of Noetherian integral domains, such that B is a finite A-module and the associated map of fraction fields $K \to L$ is separable. Then there is a trace map on L/K which restricts to B/A:

$$\mathrm{Tr}_{L/K} : L \to K$$

If B/A is flat, the trace restricts to

$$\mathrm{Tr}_{L/K}|_B = \mathrm{Tr}_{B/A} : B \to A$$

Set

$$W_{B/A} := \left\{\beta \in L | \,\mathrm{Tr}_{L/K}(\beta B) \subset A\right\}$$

When A, B are Dedekind domains, $W_{B/A}$ is the *codifferent* of the extension B/A. $W_{B/A}$ is a fractional ideal, and its inverse $\mathcal{D}_{B/A}$ is an ideal of B called the *(Dedekind) different* ideal.

[11, p. 250/267] [14, Tag 0BW0,0BW0],

Remark 1.4.67. Assume as in 1.4.66 that $A \to B$ is an injection of Noetherian integral domains, such that B is a finite A-module and the associated map of fraction fields $K \to L$ is separable.

There is an isomorphism

$$W_{B/A} \to \mathrm{Hom}_A(B, A)$$

$$\beta \mapsto \left[b \mapsto \mathrm{Tr}_{L/K}(\beta b)\right]$$

showing that $W_{B/A}$ is a relative dualizing module for the extension B/A.

The inclusion $B \subset W_{B/A}$, provides a canonical map $B \mapsto \mathrm{Hom}_A(B, A)$.

[11, p. 250/267] [14, Tag 0BW2],

Definition 1.4.68. If $\pi : X \to S$ is an arithmetic surface and $D \subset X$ a horizontal divisor, then the induced map $D \to \mathrm{Spec}(R)$ is a finite, affine morphism (1.4.44). So $\pi_* \mathcal{O}_D / R$ is a finite extension of rings, whose different ideal we denote by $\mathcal{D}_{\pi_* \mathcal{O}_D / R} \subset \mathcal{O}_D$. We denote the corresponding sheaf of ideals $\mathcal{D}_D \subset \mathcal{O}_D$.

$\mathcal{D}_D$ may also be viewed as an $\mathcal{O}_X$-module supported on D.

Proposition 1.4.69. *The different ideal of a divisor D is the annihilator of its R-differentials:*

$$\mathcal{D}_D = \mathrm{Ann}_{\mathcal{O}_D}(\Omega_{D/R})$$

[14, Tag 0BW5],

Proposition 1.4.70. *The support of the different ideal $\mathcal{D}_D$ is precisely the ramification locus of D/S.*

Proof. [14, Tag 0BW9], ☐

Remark 1.4.71. Although the module $\Omega_{A/R}$ can be viewed as a stand-in for $\mathrm{Der}(A, A)$, it actually provides more information. For instance, if $\mathcal{D}_{L/K}$ is the different ideal of a finite extension $\mathcal{O}_L/\mathcal{O}_K$ of number rings, then $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ is torsion, but $\mathcal{O}_L$ is torsion free. So $\mathrm{Der}(\mathcal{O}_L, \mathcal{O}_L) = 0$, but $\Omega_{\mathcal{O}_L/\mathcal{O}_K}$ can be nonzero (in the presence of ramification).

Remark 1.4.72. The definition of derivation (or R-derivation) requires the multiplicative structure of a ring (or an R-algebra) in order to state the Leibniz property

$$\mathrm{d} : A \to \Omega_{A/R}$$

$$\mathrm{d}(fg) = f\,\mathrm{d}g + g\,\mathrm{d}f$$

If instead of a ring A we have only an R-module M, the appropriate analog is a *connection*

$$\nabla : M \to \Omega_{A/R} \otimes M$$

$$\nabla(f\alpha) = f\nabla(\alpha) + df \otimes \alpha$$

The point is that we want to generalize the Leibniz formula from ring multiplication to scalar multiplication.

Definition 1.4.73 (Connection)**.** Given a morphism of schemes $\pi : X \to S$, a *connection* on a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules (with respect to π) is a $\pi^{-1}\mathcal{O}_S$-linear homomorphism

$$\mathcal{F} \xrightarrow{\nabla} \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{F}$$

which locally satisfies the Leibniz formula for scalar multiplication:

$$\nabla(f\alpha) = f\nabla(\alpha) + \mathrm{d}f \otimes \alpha.$$

To emphasize the $\mathcal{O}_S$-linearity we may call ∇ an *S-connection*. Whenever we refer to a connection on X, it will be assumed to be an S-connection unless otherwise specified.

Remark 1.4.74. Another type of connection one uses is a *logarithmic connection* (with respect to an effective Cartier divisor D), defined by a $\pi^{-1}\mathcal{O}_S$-linear map

$$\mathcal{F} \xrightarrow{\nabla} \Omega_{X/S}(D) \otimes_{\mathcal{O}_X} \mathcal{F}.$$

Note that often the term logarithmic differentials refers to a slightly different sheaf $\Omega_{X/S}(\log D)$ (See [14, Tag 0FMV]). However in this work we will only use the term to refer to the sheaf $\Omega_{X/S}(D) = \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{O}_X(D)$, so there should not be any confusion.

Recall that in 1.4.6 we defined the twist $\mathcal{F}(D)$ of a quasi-coherent sheaf $\mathcal{F}$ by a Cartier divisor D.

In order to unify the notions of a connection and a logarithmic connection, we introduce the following notion.

Definition 1.4.75 (Ω-valued connection, coefficient sheaf, value sheaf)**.** Suppose we have a morphism of schemes $\pi : X \to S$ and a quasicoherent sheaf Ω of $\mathcal{O}_X$-modules together with an $\mathcal{O}_X$-module homomorphism

$$\Omega_{X/S} \xrightarrow{\iota} \Omega.$$

An Ω-*valued connection* on a quasi-coherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules is a $\pi^{-1}\mathcal{O}_S$-linear homomorphism

$$\nabla : \mathcal{F} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{F},$$

which locally satisfies the Leibniz formula for scalar multiplication:

$$\nabla(f\alpha) = f\nabla(\alpha) + \iota(\mathrm{d}f) \otimes \alpha.$$

We will refer to $\mathcal{F}$ as the *coefficient sheaf* of ∇ and Ω as the *value sheaf* of ∇.

Remark 1.4.76 (General morphism of connections). Suppose we have two coefficient sheaves $\mathcal{F}, \mathcal{F}'$ and two value sheaves Ω, Ω', with maps

$$\Omega_{X/S} \xrightarrow{\iota} \Omega$$

$$\Omega_{X/S} \xrightarrow{\iota'} \Omega'$$

In order to define a morphism between connections

$$\nabla' : \mathcal{F}' \to \Omega' \otimes_{\mathcal{O}_X} \mathcal{F}'$$

$$\nabla : \mathcal{F} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{F}$$

we need maps

$$\mathcal{F}' \xrightarrow{\lambda} \mathcal{F}$$

$$\Omega' \xrightarrow{\mu} \Omega$$

such that $\iota = \mu \circ \iota'$.

This will induce diagrams

$$\begin{array}{ccc} & \Omega_{X/S} & \\ {}^{\iota'}\swarrow & & \searrow^{\iota} \\ \Omega' & \xrightarrow{\quad\mu\quad} & \Omega \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{F}' & \xrightarrow{\nabla'} & \Omega \otimes \mathcal{F}' \\ \downarrow{\scriptstyle\lambda} & & \downarrow{\scriptstyle\mu\otimes\lambda} \\ \mathcal{F} & \xrightarrow{\nabla} & \Omega' \otimes \mathcal{F} \end{array}$$

We require the left diagram to commute, but the right diagram need not commute. The case when the right diagram does commute is called a horizontal morphism (See 1.4.80).

Although we have described general morphisms of connections here, in all the cases we deal with only one of the sheaves will change at a time (e.g. 1.5.4). So instead of defining a morphism of connections by tuples (λ, μ), we will speak of morphisms of connections

1. induced by a morphism $\mathcal{F} \xrightarrow{\lambda} \mathcal{F}'$, meaning the tuple is $(\lambda, \mathrm{id}_\Omega)$.

2. induced by a morphism $\Omega \xrightarrow{\mu} \Omega'$, meaning the tuple is $(\mathrm{id}_\mathcal{F}, \mu)$.

Furthermore, the value sheaves we deal with will be $\Omega_{X/S}$ or $\Omega_{X/S}(D)$, and so we leave the natural map $\Omega_{X/S} \xrightarrow{\iota} \Omega$ implicit.

Definition 1.4.77 (Constants of a connection)**.** The constants of an Ω-valued connection $(\mathcal{F}, \nabla)$, or the ∇-*constants*, are denoted and defined as the $\pi^{-1}\mathcal{O}_S$-submodule

$$\mathcal{F}^{\nabla} := \ker \left[\mathcal{F} \xrightarrow{\nabla} \Omega \otimes \mathcal{F} \right] \subset \mathcal{F}$$

Remark 1.4.78. The ∇ constants of $\mathcal{F}$ are equivalent to the 0^{th} cohomology sheaf of the complex $\Omega^{\bullet}_{\nabla}$, which we will define in 1.5.1.

Operations on vector bundles or quasi-coherent modules often have analogs for bundles with connection. For example:

Definition 1.4.79 (Tensor, Hom, and horizontal sections/morphisms)**.** Let $\mathcal{F}, \mathcal{G}$ be quasi-coherent $\mathcal{O}_X$-modules, and let

$$\nabla_{\mathcal{F}} : \mathcal{F} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{F}$$

$$\nabla_{\mathcal{G}} : \mathcal{G} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{G}$$

be Ω-valued connections, for some quasi-coherent sheaf Ω.

Then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$ have Ω-valued connections $\nabla_{\mathcal{F}\otimes\mathcal{G}}$ and $\nabla_{\mathcal{F},\mathcal{G}}$ defined by:

$$\nabla_{\mathcal{F}\otimes\mathcal{G}}(f \otimes g) = \nabla_{\mathcal{F}}(f) \otimes g + f \otimes \nabla_{\mathcal{G}}(g)$$

$$\nabla_{\mathcal{F},\mathcal{G}}(h)(f) = \nabla_{\mathcal{G}}(h(f)) - h\left(\nabla_{\mathcal{F}}(f)\right)$$

for $h \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

A section $f \in \mathcal{F}$ is called *horizontal* if

$$\nabla_{\mathcal{F}}(f) = 0,$$

and a homomorphism $h : \mathcal{F} \to \mathcal{G}$ is called *horizontal* if

$$\nabla_{\mathcal{F},\mathcal{G}}(h) = 0.$$

See [2, p. 180/7] or [3, p. 394/3].

Definition 1.4.80 (Horizontal morphism of connections)**.** A morphism of connections

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\nabla} & \Omega \otimes \mathcal{F}' \\ \big\downarrow{\scriptstyle \lambda} & & \big\downarrow{\scriptstyle \mu\otimes\lambda} \\ \mathcal{F} & \xrightarrow{\nabla'} & \Omega' \otimes \mathcal{F} \end{array}$$

is called *horizontal* if the square commutes.

See [2, p. 180/7].

Proposition 1.4.81. *Let $\pi : X \to S$ be a morphism of schemes, with quasi-coherent $\mathcal{O}_X$-modules $\mathcal{L}, \mathcal{L}'$ and Ω-valued S-connections*

$$\nabla : \mathcal{L} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L} =: \Omega^1_\nabla$$

$$\nabla' : \mathcal{L}' \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L}' =: \Omega^1_{\nabla'}$$

Let $\lambda : \nabla' \to \nabla$ be a horizontal morphism of connections, where $\mathcal{L}' \overset{\lambda}{\to} \mathcal{L}$ is injective and induces the exact sequence

$$0 \to \mathcal{L}' \overset{\lambda}{\to} \mathcal{L} \to \overline{\mathcal{L}} \to 0$$

Then the connection ∇ descends to an Ω-valued connection on the quotient:

$$\overline{\nabla} : \overline{\mathcal{L}} \to \Omega \otimes \overline{\mathcal{L}}$$

Proof. The fact that ∇ descends to a connection on the quotient in a well-defined way follows directly from the horizontality of λ. □

1.4.4.2 Connections on Cartier divisors

Now we turn to the particular bundles and connections which will be the primary study in this work. By restricting the standard derivation on rational functions

$$\mathcal{K}_X \xrightarrow{\mathrm{d}} \Omega_X \otimes \mathcal{K}_X$$

we obtain a map on any quasicoherent $\mathcal{O}_X$-submodule $\mathcal{F} \subset \mathcal{K}_X$ (e.g. fractional ideal sheaves)

$$\mathrm{d}\,|_{\mathcal{F}} : \mathcal{F} \to \Omega_X \otimes \mathcal{K}_X$$

This morphism is R-linear and satisfies the Leibniz rule, however the restriction $\mathrm{d}\,|_{\mathcal{F}}$ is not necessarily a connection, because it does not necessarily factor through

$$\mathcal{F} \to \Omega_X \otimes \mathcal{F} \hookrightarrow \Omega_X \otimes \mathcal{K}_X.$$

This means that not all subsheaves $\mathcal{F}$ have a natural connection (an Ω_X-valued connection), but we will see that our sheaves of interest have a logarithmic connection (an $\Omega_X(D)$-valued connection).

Remark 1.4.82. The standard derivation $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X$ extends naturally to a logarithmic derivation $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X(D)$ via the composition

$$\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X \hookrightarrow \Omega_X(D)$$

After we define de Rham cohomology of a connection, one may observe that passing from the standard connection to a logarithmic connection will preserve $\mathrm{H}^0_{\mathrm{dR}}$ and enlarge $\mathrm{H}^1_{\mathrm{dR}}$ (1.5.34).

We will now introduce the connection ∇_D associated to an effective Cartier divisor D. This will be an $\Omega_{X/S}(D)$-valued connection defined on certain $\mathcal{O}_X$-submodule of the rational functions $\mathcal{K}_X$, namely those with poles supported on D. Then it will be the case that $\mathrm{H}^\bullet_{\mathrm{dR}}(X; \mathcal{O}_X, \mathrm{d})$ computes the ordinary de Rham cohomology, and $\mathrm{H}^\bullet_{\mathrm{dR}}(X; \mathcal{O}_X, \nabla_D)$ computes the logarithmic de Rham cohomology with respect to D.

Remark 1.4.83. The reason for considering Cartier divisors in particular is that they may be realized as subsheaves of the rational functions $\mathcal{K}_X$ and thereby inherit a derivation.

Indeed we will consider each $\mathcal{O}_X(nD)$ to be a subsheaf of the sheaf $\mathcal{K}_X$ of rational functions, and so we have natural inclusions

$$\mathcal{O}_X(nD) \subset \mathcal{O}_X\big((n+1)D\big) \subset j_*(\mathcal{O}_U) \subset \mathcal{K}_X$$

Note that the sheaves $\mathcal{O}_X, \mathcal{O}_X(nD), j_*\mathcal{O}_U$ all trivialize over $U = X \setminus D$ and when restricted to U are equal as subsheaves of $\mathcal{K}_X|_U$.

Because X is smooth, Ω_X is locally free, so it preserves inclusions upon tensoring:

$$\Omega_X(nD) \subset \Omega_X\big((n+1)D\big) \subset \Omega_U \subset \Omega_X \otimes \mathcal{K}_X.$$

The sheaf $\mathcal{K}_X$ carries a natural derivation

$$\mathcal{K}_X \xrightarrow{D} \Omega_{X/S} \otimes \mathcal{K}_X,$$

which restricts to derivations on quasi-coherent sheaves of $\mathcal{O}_X$-subalgebras of $\mathcal{K}_X$, such as $\mathcal{O}_X$ and $j_*\mathcal{O}_U$.

Proposition 1.4.84. *The restriction of the standard derivation from $j_*\mathcal{O}_U$ to $\mathcal{O}_X(nD)$*

$$\mathrm{d} : \mathcal{O}_X(nD) \to \Omega_X \otimes j_*(\mathcal{O}_U)$$

factors through $\Omega_X(D) \otimes \mathcal{O}_X(nD)$:

$$\mathrm{d} : \mathcal{O}_X(nD) \to \Omega_X(D) \otimes \mathcal{O}_X(nD) \to \Omega_X \otimes j_*(\mathcal{O}_U)$$

We denote this map by

$$\nabla_{D,n} : \mathcal{O}_X(nD) \to \Omega_X(D) \otimes \mathcal{O}_X(nD)$$

Then $\nabla_{D,n}$ is a logarithmic connection. Since all $\nabla_{D,n}$ agree on their shared domains, we will refer to $\nabla_{D,n}$ simply as ∇_D, or ∇ if D is understood.

Proof. We proceed by induction on n. The case $n = 0$ is clear. Assume the result holds for n.

For notational simplicity, for a locally free sheaf $\mathcal{F}$ we will write $f \in \mathcal{F}$ to mean $f \in \mathcal{F}(V)$

for some trivializing affine open $V \subset X$. We may work locally in this way, as each $x \in X$ has such a neighborhood.

For the induction step, we would like an $h \in \mathcal{O}_X$ such that $h \cdot \mathcal{O}_X\big((n+1)D\big) \subset \mathcal{O}_X(nD)$. In other words, $h \in \mathcal{O}_X(nD) \otimes \mathcal{O}_X\big((n+1)D\big)^{-1} \cong \mathcal{O}_X(-D) = \mathcal{I}_D$. Such an h exists because D is a Cartier divisor: $\mathcal{O}_X(-D)$ is locally principal, so we may choose h generating $\mathcal{O}_X(-D)$.

Now take $\alpha \in \mathcal{O}_X\big((n+1)D\big)$. We want to show that $\nabla_{D,n}(\alpha) \in \Omega_X(D) \otimes \mathcal{O}_X\big((n+1)D\big)$.

The Leibniz property gives

$$\nabla_{D,n}(h\alpha) = h\nabla_{D,n}(\alpha) + \mathrm{d}h \otimes \alpha$$

and by induction $h\alpha \in \mathcal{O}_X(nD)$ implies $\nabla_{D,n}(h\alpha) \in \Omega_X(D) \otimes \mathcal{O}_X(nD)$.

$$\nabla_{D,n}(\alpha) = \frac{\nabla_{D,n}(h\alpha) - \mathrm{d}h \otimes \alpha}{h} = \frac{\nabla_{D,n}(h\alpha)}{h} + \frac{\mathrm{d}h}{h} \otimes \alpha$$

Both $\frac{\nabla_{D,n}(h\alpha)}{h}, \frac{\mathrm{d}h}{h} \otimes \alpha \in \Omega_X(D) \otimes \mathcal{O}_X\big((n+1)D\big)$, so the result holds. □

Remark 1.4.85. Since the $\nabla_{D,n}$ are all restrictions of a common connection, the inclusion $\mathcal{O}_X\big((n-1)D\big) \to \mathcal{O}_X(nD)$ induces a horizontal morphism of connections, i.e.

$$\begin{array}{ccc} \mathcal{O}_X\big((n-1)D\big) & \longrightarrow & \Omega_{X/S}(D) \otimes \mathcal{O}_X\big((n-1)D\big) \\ \big\downarrow & & \big\downarrow \\ \mathcal{O}_X(nD) & \longrightarrow & \Omega_{X/S}(D) \otimes \mathcal{O}_X(nD) \end{array}$$

commutes.

Remark 1.4.86. The connection ∇_D on $\mathcal{O}_X(nD)$ can also be obtained from the connection ∇_D

on $\mathcal{O}_X(D)$ via the tensor product of connections defined above.

Remark 1.4.87. The reason why $\Omega_X(D)$-valued connections are needed for ∇_D, and Ω_X-valued connections are insufficient, is exemplified by the calculation

$$\mathrm{d}\left(\frac{1}{x}\right) = \frac{-\,\mathrm{d}x}{x^2} = \frac{-1}{x} \cdot \frac{\mathrm{d}x}{x}$$

If we allow a single pole along x, we must allow an additional pole to accommodate the derivatives.

The way to view this in terms of the bundle $\mathcal{O}_X(D)$ is as follows. The distinguished map $\mathcal{O}_X \to \mathcal{O}_X(D)$ is equivalent to a choice of global section $1_D \in \Gamma(X, \mathcal{O}_X(D))$. If ∇ extends $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_X$, then $\nabla(1_D) = 0$.

If $e \in \mathcal{O}_X(D)$ is a local basis, then there is a unique $h \in \mathcal{O}_X$ such that $he = 1_D$, so we can regard e as $\frac{1}{h}$, and

$$0 = \nabla(1_D) = \nabla(he) = h\nabla(e) + \mathrm{d}h \otimes e$$

$$\Rightarrow \nabla(e) = \frac{\mathrm{d}h}{h} \otimes -e$$

If $s \in \mathcal{O}_X(D)$ is an arbitrary section, then there is a unique $f \in \mathcal{O}_X$ such that $fe = s$, and

$$\nabla(s) = \nabla(fe) = f\nabla(e) + \mathrm{d}f \otimes e$$

$$= \left(\frac{h\,\mathrm{d}f - f\,\mathrm{d}h}{h}\right) \otimes e \in \Omega_X(D) \otimes \mathcal{O}_X(D)$$

$$= \big((h\,\mathrm{d}f - f\,\mathrm{d}h) \otimes e\big) \otimes e \in \big(\Omega_X \otimes \mathcal{O}_X(D)\big) \otimes \mathcal{O}_X(D)$$

After investigating some generalities on connections and cohomology, we will return to the connection ∇_D in 1.5.7. There we will see another explanation of the presence of logarithmic differentials.

1.4.4.3 Strictly horizontal morphisms of connections

Definition 1.4.88. A morphism of connections $\nabla' \xrightarrow{\lambda} \nabla$ is *strictly horizontal* if the diagram commutes

$$\begin{array}{ccc} \mathcal{L}' & \xrightarrow{\lambda} & \mathcal{L} \\ \downarrow{\scriptstyle \nabla'} & & \downarrow{\scriptstyle \nabla} \\ \Omega' \otimes \mathcal{L}' & \xrightarrow{\nu} & \Omega \otimes \mathcal{L} \end{array}$$

and is furthermore a Cartesian square.

Remark 1.4.89. The definition of strict horizontality is intended to guarantee a compatibility between the filtrations

$$\begin{array}{ccccccccc} \ldots & \subset & \mathcal{O}_X\big((n-1)D\big) & \subset & \mathcal{O}_X(nD) & \subset & \mathcal{O}_X\big((n+1)D\big) & \subset & \ldots \\ \ldots & \subset & \Omega \otimes \mathcal{O}_X\big((n-1)D\big) & \subset & \Omega \otimes \mathcal{O}_X(nD) & \subset & \Omega \otimes \mathcal{O}_X\big((n+1)D\big) & \subset & \ldots \end{array}$$

Proposition 1.4.90. *Suppose $\nabla' \xrightarrow{\lambda} \nabla$ is strictly horizontal. Then the square obtained by applying the direct image functor π_*:*

$$\begin{array}{ccc} \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} \\ \downarrow & & \downarrow \\ \mathbf{R}^0\pi_*(\Omega' \otimes \mathcal{L}') & \longrightarrow & \mathbf{R}^0\pi_*(\Omega \otimes \mathcal{L}) \end{array}$$

is also Cartesian.

Proof. This follows from 1.4.10, which says that the direct image functor preserves Cartesian squares. □

Remark 1.4.91. If $\nabla' \xrightarrow{\lambda} \nabla$ is a strictly horizontal morphism of connections, and S is affine, then on the diagram in 1.4.90 we also have the image containment property of 1.3.6, which says that the containment

$$\mathrm{im}(\nabla \circ \lambda) = \mathrm{im}(\nu \circ \nabla') \subset \mathrm{im}(\nabla) \cap \mathrm{im}(\nu)$$

induced by the commutativity of the diagram is in fact an equality:

$$\mathrm{im}(\nabla \circ \lambda) = \mathrm{im}(\nu \circ \nabla') = \mathrm{im}(\nabla) \cap \mathrm{im}(\nu)$$

The reason for asking S to be affine is that the direct images $\pi_*\mathcal{L}, \pi_*\Omega \otimes \mathcal{L}$ correspond to modules instead of sheaves, which makes it easier to talk of the intersection of images.

1.4.5 The de Rham complex

Definition 1.4.92 (de Rham complex)**.** The *de Rham complex* of X over S is the sequence of sheaves

$$\Omega^\bullet_{X/S} := \left[\mathcal{O}_X \xrightarrow{\mathrm{d}=\mathrm{d}_0} \Omega^1_{X/S} \xrightarrow{\mathrm{d}_1} \Omega^2_{X/S} \xrightarrow{\mathrm{d}_2} ...\right],$$

where $\Omega^p_{X/S} := \bigwedge^p \Omega_{X/S}$ is the p^{th} exterior power ([14, Tag 00DM]). This is a complex by [14, Tag 00DM].

Remark 1.4.93. The dual of the map

$$\Omega^1 \xrightarrow{\mathrm{d}_1} \Omega^2$$

is the Lie bracket on the tangent bundle. The relation $\mathrm{d}^2 = 0$ translates to the Jacobi identity on the Lie bracket. See [13, p. 469/487]

Definition 1.4.94 (de Rham cohomology)**.** The q^{th} *de Rham cohomology sheaf* of X over S is the q^{th} cohomology sheaf of the de Rham complex:

$$\mathcal{H}^q(X/S) := \mathrm{H}^q(\Omega^\bullet_{X/S}) = \ker(\mathrm{d}_q)/\operatorname{im}(\mathrm{d}_{q-1})$$

and the q^{th} *de Rham cohomology* of X over S is the q^{th} hypercohomology of the de Rham complex:

$$\mathrm{H}^q(X/S) := \mathbb{H}^q(\Omega^\bullet_{X/S})$$

Remark 1.4.95. Each term of the complex is an $\mathcal{O}_X$-module, but the differential d is only a $\pi^{-1}\mathcal{O}_S$-module homomorphism. So $\Omega^\bullet_{X/S}$ is a complex of $\pi^{-1}\mathcal{O}_S$-modules, and the de Rham cohomology sheaves $\mathcal{H}^q(X/S)$ are also sheaves of $\pi^{-1}\mathcal{O}_S$-modules.

If $\pi : X \to S$ is an affine morphism, then the de Rham cohomology is just the global sections of the de Rham cohomology sheaves.

In the relative setting hypercohomology refers to the derived pushforward.

Lemma 1.4.96. *Let X/S be a smooth relative curve, and suppose*

$$\mathrm{H}^1(X, \mathcal{O}_X) = 0,$$

$$\mathrm{H}^0(X, \Omega^1_{X/S}) = 0.$$

Then

$$\mathrm{H}^1_{\mathrm{dR}}(X/S) = 0.$$

Proof. Let $\Omega^\bullet_{X/S}$ be the de Rham complex of $X/S, \mathcal{U}$ a $\check{C}$ech cover of X, and $C^{i,j}$ the associated double complex, where i is the Čech index and j the de Rham index.

Let $C^k = \bigoplus_{i+j=k} C^{i,j}$ be the total complex of $C^{i,j}$. and Z^k, B^k the cochains and coboundaries of C^k.

The double complex looks like

$$\begin{array}{ccc} C^{0,0} & \stackrel{\mathrm{d}}{\to} & C^{1,0} \\ \downarrow \partial & & \downarrow \partial \\ C^{0,1} & \stackrel{\mathrm{d}}{\to} & C^{1,1} \end{array}$$

and the total complex is

$$C^0 \stackrel{\mathbf{d}}{\to} C^1 \stackrel{\mathbf{d}}{\to} C^2$$

$$C^{0,0} \stackrel{\mathbf{d}}{\to} C^{0,1} \oplus C^{1,0} \stackrel{\mathbf{d}}{\to} C^{1,1}$$

where we denote horizontal, vertical, and total differentials by $\mathrm{d}, \partial,$ **d**, respectively, and we use the convention that the vertical arrows in the j^{th} column have an additional sign of $(-1)^j$ when viewed as part of the total differential **d**. The squares in the double complex commute, but when viewed as the total complex the additional sign causes it to anti-commute.

Now let $\gamma = (\alpha, \beta) \in Z^1 \subset C^1 = C^{0,1} \oplus C^{1,0}$. Then

$$0 = \mathbf{d}(\gamma) = (\mathrm{d}(\,\alpha) - \partial(\beta))$$

$$\mathrm{d}(\alpha) = \partial(\beta)$$

α is a Čech 1-cochain with values in $\mathcal{O}_X$. By assumption $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, so α lifts to $\hat{\alpha} \in C^{0,0}$.

Since the square commutes, we have

$$\partial(\beta) = \mathrm{d}(\alpha) = \mathrm{d}(\partial(\hat{\alpha})) = \partial(\mathrm{d}(\hat{\alpha}))$$

and $\beta, \mathrm{d}(\alpha) \in C^{1,0}$ have the same image in $C^{1,1}$. Therefore the difference

$$\mathrm{d}(\alpha) - \beta =: \sigma \in \mathrm{H}^0(X, \Omega_{X/S})$$

defines an global section.

But we assumed $\mathrm{H}^0(X, \Omega_{X/S}) = 0$, so $\beta = \mathrm{d}(\alpha)$, and

$$\mathbf{d}(\hat{\alpha}) = (\partial(\hat{\alpha}), \mathrm{d}(\hat{\alpha})) = (\alpha, \beta) = \gamma$$

Therefore the chosen cochain γ is trivial and thus

$$\mathrm{H}^1(X/S) = 0.$$

□

Remark 1.4.97. It is not hard to show also that if $\mathrm{H}^1(X, \mathcal{O}_X) = 0$, then

$$\mathrm{H}^1_{\mathrm{dR}}(X/S) \simeq \mathrm{H}^0(X, \Omega^1_{X/S})$$

This follows from the observation in the previous proof that

$$\mathrm{d}(\alpha) - \beta \in \mathrm{H}^0(X, \Omega_{X/S}).$$

More generally we have

Proposition 1.4.98. *Suppose that $\pi : X \to S$ is a proper and cohomologically flat relative curve (1.4.25), with S affine. Then there are canonical isomorphisms of free $\mathcal{O}_S$-modules of rank 1*

$$\mathrm{H}^0(X/S) \simeq \mathrm{H}^0(X, \mathcal{O}_X)$$

$$\mathrm{H}^2(X/S) \simeq \mathrm{H}^1(X, \omega_{X/S})$$

and a short exact sequence of finite free $\mathcal{O}_S$-modules

$$0 \to \mathrm{H}^0(X, \omega_{X/S}) \to \mathrm{H}^1(X/S) \to \mathrm{H}^1(X, \mathcal{O}_X) \to 0$$

Proof. [1, p. 18] □

Remark 1.4.99. Note that in the complex $\Omega^\bullet_\nabla$, though the differentials are not $\mathcal{O}_X$-linear but $\pi^{-1}\mathcal{O}_S$ linear, global sections still map to global sections. This makes some of the hypercohomology computations easier. This can be seem from the (anti-)commutativity of the square in the double complex, in particular zeroness is preserved.

1.5 Cohomology of a connection

Definition 1.5.1 (de Rham complex and integrability of a connection)**.** A connection $\nabla : \mathcal{E} \to \Omega \otimes \mathcal{E}$ can be iterated to define a sequence

$$\Omega^\bullet_\nabla := \left[\mathcal{E} \xrightarrow{\nabla = \nabla^0} \Omega \otimes \mathcal{E} \xrightarrow{\nabla^1} \Omega^2 \otimes \mathcal{E} \xrightarrow{\nabla^2} ...\right]$$

resembling the de Rham complex, where

$$\Omega^i := \bigwedge^i \Omega$$

$$\nabla^i : \Omega^i \otimes \mathcal{E} \to \Omega^{i+1} \otimes \mathcal{E}$$

$$\nabla^i(\omega \otimes s) = \mathrm{d}\omega \otimes s + (-1)^i \omega \wedge \nabla(s)$$

The sequence $\Omega^\bullet_\nabla$ is in fact a complex if and only if ∇ is *integrable*, which is the condition that the curvature $\mathcal{R}_\nabla$ vanishes:

$$\mathcal{R}_\nabla := \nabla^1 \circ \nabla = 0$$

When the connection is integrable, we can use the complex to define an analog of de Rham cohomology, and this will serve as de Rham cohomology *with coefficients* in $\mathcal{E}$.

Remark 1.5.2. The notation $\Omega^\bullet_\nabla$ sacrifices clarity for concision, in that

$$\Omega^q_\nabla = \Omega^q \otimes \mathcal{E} \neq (\Omega^1_\nabla)^q$$

In particular, $\Omega^0_\nabla = \mathcal{E} \neq \mathcal{O}_X$.

Remark 1.5.3. When X/S is smooth of relative dimension 1, the higher differentials $\Omega^i_{X/S},\ i > 1$ vanish, and so the connections are automatically integrable.

Remark 1.5.4. Though ∇, ∇_1 are only R-linear, their composition, the curvature $\mathcal{R}_\nabla$, is actually $\mathcal{O}_X$-linear. See [4].

Definition 1.5.5 (De Rham cohomology of a connection)**.** Suppose that $\mathcal{E}$ is a quasi-coherent sheaf on X and $\nabla : \mathcal{E} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{E}$ an integrable Ω-valued connection. Define

$$\mathcal{H}^\bullet(X/S; \mathcal{E}, \nabla) := \mathrm{H}^\bullet\left(\Omega^\bullet_\nabla\right)$$

to be the cohomology sheaves of the complex $\Omega^\bullet_\nabla$.

Define

$$\mathrm{H}^\bullet(X/S; \mathcal{E}, \nabla) := \mathbb{H}^\bullet\left[\Omega^\bullet_\nabla\right]$$

to be the hypercohomology of the complex $\Omega^\bullet_\nabla$.

We may omit X and/or $\mathcal{E}$ and/or ∇ from the notation, when understood.

Remark 1.5.6. Ordinary de Rham cohomology and de Rham cohomology of a connection both arise from the hyperderived functor of the pushforward applied to their respective complexes:

$$\mathrm{H}^q(X/S) = \mathrm{R}^q\,\pi_*\left(\Omega^\bullet_{X/S}\right)$$

$$\mathrm{H}^q(X/S; \mathcal{E}, \nabla) = \mathrm{R}^q\,\pi_*\left(\Omega^\bullet_\nabla\right)$$

See [2, p. 361/8] or [5, p. 7]

1.5.1 Hodge-to-de Rham Spectral sequence

Proposition 1.5.7. *Given an $\mathcal{O}_X$-module with S-connection ∇, the hypercohomology of the complex $\Omega^\bullet_\nabla$ can be computed via a spectral sequence*

$$E_1^{pq} = \mathrm{H}^q(X, \Omega^p_\nabla)$$

Proof. This is just the usual spectral sequence for a double complex, using the fact that the category of $\pi^{-1}\mathcal{O}_S$-modules has enough injectives in order to form a Cartan-Eilenberg resolution for $\Omega^\bullet_\nabla$. See [14, Tag 0FM6] for details in the case of the complex $\Omega^\bullet_{X/S}$. □

Remark 1.5.8. As in [1, p. 18], when X/S is cohomologically flat, the degeneration of the Hodge-to-de Rham spectral sequence for the generic fiber X_K implies the degeneration for the spectral sequence of X.

This result is stated in 1.5.9 below.

Proposition 1.5.9. *Suppose that $\pi : X \to S$ is a smooth arithmetic surface, and X is cohomologically flat. Then the Hodge-to-de Rham spectral sequence*

$$E_1^{ij} = \mathrm{H}^j(X, \Omega^1_{X/S}) \Rightarrow \mathrm{H}^{i+j}(X/S)$$

degenerates at the E_1-stage.

Proof. This is stated in [1, p. 18] for the dualizing complex $\omega^\bullet_{X/S}$ in place of the de Rham complex $\Omega^\bullet_{X/S}$. But since our morphism π is smooth, the two complexes coincide (See e.g. [11, p. 247/264]). □

Remark 1.5.10. Let us see what the Hodge-de Rham spectral sequence looks like using a $\check{C}$ech resolution.

Let

$$\mathcal{F} \xrightarrow{\nabla} \Omega \otimes_{\mathcal{O}_X} \mathcal{F} =: \Omega^1_\nabla$$

be an S-connection on $\mathcal{F}$.

We will use a Čech resolution, but the following can also be done with an injective or other acyclic resolution. The Čech differential will be denoted ∂.

Since X/S is relative dimension 1, the complex will have only two rows, and since X/S is also smooth, the complex will have only two columns.

Choose a Čech cover $\mathcal{U}$ of X. Denote

$$C^{i,j} := \check{C}^j(\mathcal{U}, \Omega^1_\nabla)$$

Consider the double complex $C^{\bullet,\bullet}$:

$$\begin{array}{ccc} C^{0,0} & \xrightarrow{\nabla} & C^{1,0} \\ \downarrow{\scriptstyle\partial} & & \downarrow{\scriptstyle\partial} \\ C^{0,1} & \xrightarrow{\nabla} & C^{1,1} \end{array}$$

and the total complex $C^\bullet$:

$$C^k := \bigoplus_{i+j=k} C^{i,j}$$

The Hodge-to-de Rham spectral sequence arises by taking the spectral sequence for the double complex $C^{i,j}$ with vertical orientation.

The first two pages have the form:

E_0:

$$\begin{array}{ccc} \check{C}^0(\mathcal{U}, \mathcal{F}) & & \check{C}^0(\mathcal{U}, \Omega^1_\nabla) \\ \downarrow & & \downarrow \\ \check{C}^1(\mathcal{U}, \mathcal{F}) & & \check{C}^1(\mathcal{U}, \Omega^1_\nabla) \end{array}$$

E_1:

$$\begin{array}{ccc} \mathrm{H}^0(X, \mathcal{F}) & \longrightarrow & \mathrm{H}^0(X, \Omega^1_\nabla) \\ & & \\ \mathrm{H}^1(X, \mathcal{F}) & \longrightarrow & \mathrm{H}^1(X, \Omega^1_\nabla) \end{array}$$

Remark 1.5.11. By looking at the page E_1 above, one can see that the degeneration at E_1 is equivalent to the vanishing of all maps

$$\mathrm{H}^j(X, \mathcal{F}) \xrightarrow{\mathrm{H}^j(\nabla)} \mathrm{H}^j(X, \Omega^1_\nabla)$$

in E_1. See [1, p. 18] for more details.

This is the primary consequence of the degeneration that we will utilize.

Remark 1.5.12. [14, Tags 0FW5, 0FW6] shows that, for X/k smooth, proper, and equidimensional over a field, the maps in the previous remark vanish for $j = 0, d$. This implies the degeneration of the spectral sequence for curves.

Remark 1.5.13. When X/S is a smooth arithmetic surface which is cohomologically flat, then the Hodge-to-de Rham spectral sequence degenerates at E_1 (1.5.9). When we deal with more general connections on such a surface, there is still a spectral sequence, but it will not necessarily degenerate at stage E_1. Because the complex $\Omega^\bullet_\nabla$ has length two, the spectral sequence will have only two nonzero columns, and so at least we will have degeneration by stage E_2.

1.5.2 Analysis of constants

Here we analyze the constants of a connection defined in 1.4.77

Remark 1.5.14. In this section we deal with two notions called "constant":

- A (locally) constant sheaf,
- Elements of a sheaf which map to zero under a derivation or connection.

If there is a chance of confusion, we will disambiguate by referring to them respectively as *topologically constant* and *differentially constant*.

For instance, the sheaf $\mathcal{K}_X$ of rational functions is topologically constant, but only the subsheaf $\ker(\mathrm{d})$ is differentially constant.

We will demonstrate below that $\ker(\mathrm{d})$ is in fact topologically constant as well.

Proposition 1.5.15. *Suppose that $\pi : X \to S$ is a smooth morphism of schemes of relative dimension 1, where X is geometrically integral, and the fraction field $K(S)$ is a number field. Let $\mathcal{K}_X^{\mathrm{d}}$ be the sheaf of (differentially) constant rational functions, i.e.*

$$\mathcal{K}_X^{\mathrm{d}} = \ker\left[\mathcal{K}_X \xrightarrow{\mathrm{d}} \Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_X\right].$$

Then $\mathcal{K}_X^{\mathrm{d}}$ is a (topologically) constant sheaf, which has values in $K(S) \subset K(X)$.

Hence $\mathcal{K}_X^{\mathrm{d}}$ is flasque and acyclic.

Proof. By [14, Tag 01X5], $\mathcal{K}_X$ and $\Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{K}_X$ are both constant sheaves, which respectively have the values $\mathcal{O}_{X,\xi}$ and $\Omega_{X/S,\xi}$ of the stalks at the generic point ξ of X. We also have that $\mathcal{O}_{X,\xi} = \mathcal{K}_{X,\xi} = K(X)$ is the function field of X.

By [14, Tag 093P], the subcategory of $\mathrm{Sh}(X)$ consisting of locally constant sheaves contains the kernel and cokernel of every $\mathrm{Sh}(X)$-morphism between locally constant sheaves (i.e. is a weak Serre subcategory). Hence the kernel $\mathcal{K}_X^{\mathrm{d}}$ is also a constant sheaf on X, which takes values in some subset of $K(X)$. By taking global sections we have the constants $\Gamma(X, \mathcal{K}_X^{\mathrm{d}}) = \Gamma(X, \mathcal{K}_X)^{\mathrm{d}} = K(X)^{\mathrm{d}}$.

An elementary calculation shows that $K(X)^{\mathrm{d}}$ is closed under addition and multiplication, and is a subfield.

A slightly less elementary calculation shows that $K(X)^{\mathrm{d}}$ is separably closed in $K(X)$: Suppose $\alpha \in K(X)$ satisfies a separable polynomial $P(x) \in K(X)^{\mathrm{d}}[x]$. As the coefficients of P are (differentially) constants, this implies that the derivation d is $K(X)^{\mathrm{d}}$-linear, so we can take the derivative P' in the usual way, and we obtain $\mathrm{d}(P(\alpha)) = P'(\alpha)\,\mathrm{d}\alpha = 0$. But as P is separable, its root α cannot also be a root of P'. And since X/S smooth implies $\Omega_{K/S,\xi}$ is a torsion-free $K(X)$-module, $P'(\alpha) \neq 0 = \mathrm{d}\alpha$. Thus $\alpha \in K(X)^{\mathrm{d}}$, and $K(X)^{\mathrm{d}}$ is separably closed.

Since $K(X)/K(S)$ has transcendence degree 1, and assuming $\mathrm{char}(K(X)) = 0$ and X geometrically integral, the only subfields of $K(X)$ which are separably closed in $K(X)$ are $K(S)$ and $K(X)$ (in positive characteristic we would also have $K(X)^p$). But an element of $K(X)$ transcendental over K is not constant with respect to d, and so the constants must be $K(X)^{\mathrm{d}} = K(S)$.

A Zariski-locally constant sheaf on an irreducible scheme is flasque, and flasque sheaves are acyclic. □

Remark 1.5.16. Consider the morphism $\mathbb{P}^1_{\mathbb{Z}} \to \mathrm{Spec}\mathbb{Z}$ and the sheaf $\tilde{\mathbb{Z}} = \pi^{-1}\mathcal{O}_{\mathrm{Spec}\mathbb{Z}} \subset \mathcal{O}_{\mathbb{P}^1_{\mathbb{Z}}}$. We would certainly expect $\tilde{\mathbb{Z}}$ to be differentially constant with respect to any derivation on $\mathbb{P}^1_{\mathbb{Z}}$, but it is not flasque or topologically constant for the same reason that $\mathcal{O}_{\mathrm{Spec}\mathbb{Z}}$ is not flasque or topologically constant.

However, it turns out that such a sheaf is *relatively flasque* (1.4.20), and this is sufficient to be ayclic for the direct image functor π_*.

Lemma 1.5.17. *Suppose that $\pi : X \to S$ is a smooth morphism of schemes of relative dimension 1, where X is geometrically integral, and the fraction field $K(S)$ is a number field.*

Let $D \subset X$ be an effective horizontal Cartier divisor D (1.4.43). and let

$$\mathcal{O}_X(nD) \xrightarrow{\nabla_D} \Omega_{X/S}(D) \otimes \mathcal{O}_X(nD)$$

be the connection defined in 1.4.84.

Then the constants $\mathcal{O}_X(nD)^{\nabla_D}$ are equal to the $\pi^{-1}\mathcal{O}_S$-submodule of $\mathcal{O}_X(nD)$ generated by the image of 1 under the natural map $\mathcal{O}_X \to \mathcal{O}_X(nD)$:

$$\mathcal{O}_X(nD)^{\nabla_D} = \pi^{-1}\mathcal{O}_S \cdot 1 \subset \mathcal{O}_X \subset \mathcal{O}_X(nD).$$

Proof. Let $U = X \setminus D$. By definition, ∇_D is the restriction to $\mathcal{O}_X(nD) \subset \mathcal{K}_X$ of the derivation $\mathcal{K}_X \to \Omega_{X/S} \otimes \mathcal{K}_X$ So we can compute the constants as

$$\mathcal{O}_X(nD)^{\nabla_D}(V) = \mathcal{O}_X(nD)(V) \cap \mathcal{K}_X^{\mathrm{d}}(V)$$

$$= \pi^{-1}\mathcal{O}_S(V).$$

The rational constants $\mathcal{K}_X^{\mathrm{d}} = K(S)$ consist of those rational functions in $K(X)$ whose poles may only lie along vertical divisors (1.4.43), and the elements of $\mathcal{O}_X(nD)(V)$ which have only poles along vertical divisors consist of $\mathcal{O}_S(V) \cdot 1$.

□

Proposition 1.5.18. *The sheaf of constants $\mathcal{O}_X(nD)^{\nabla}$ is acyclic with respect to the direct image functor π_*.*

Proof. By 1.5.17, we have an isomorphism of $\pi^{-1}\mathcal{O}_S$-modules $\mathcal{O}_X(nD)^{\nabla_D} \cong \pi^{-1}\mathcal{O}_S$. By 1.4.22, an inverse image sheaf is relatively flasque, and by 1.4.21 relatively flasque sheaves are acyclic with respect to the direct image functor. □

1.5.3 Seven term exact sequence

Proposition 1.5.19. *Let $\pi : X \to S$ be a morphism of schemes. There is a natural exact sequence computing the relative de Rham cohomology $H^\bullet(X/S)$:*

$$0 \to \mathrm{H}^0(X/S) \to \mathbf{R}^0\pi_*\mathcal{O}_X \to \mathbf{R}^0\pi_*\Omega_{X/S} \to \mathrm{H}^1(X/S) \to \mathbf{R}^1\pi_*\mathcal{O}_X \to \mathbf{R}^1\pi_*\Omega_{X/S} \to \mathrm{H}^2(X/S) \to 0$$

If S is affine, the sequence becomes:

$$0 \to \mathrm{H}^0(X/S) \to \mathrm{H}^0(X, \mathcal{O}_X) \to \mathrm{H}^0(X, \Omega_{X/S}) \to \mathrm{H}^1(X/S) \to \mathrm{H}^1(X, \mathcal{O}_X) \to \mathrm{H}^1(X, \Omega_{X/S}) \to \mathrm{H}^2(X/S) \to 0$$

Proof. This is a consequence of the following more general statement. □

Lemma 1.5.20 (Seven term exact sequence)**.** *Let $\pi : X \to S$ be a smooth arithmetic surface (1.4.32), with a quasi-coherent $\mathcal{O}_X$-module $\mathcal{F}$ and an Ω-valued S-connection*

$$\nabla : \mathcal{F} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{F} =: \Omega^1_\nabla$$

There is a natural exact sequence computing $H^\bullet(X/S;\nabla)$, which we will denote by $\mathcal{S}^i_\nabla$, for $i = 0, ..., 6$.

$$0 \to \mathrm{H}^0(X/S;\nabla) \to \mathbf{R}^0\pi_*\mathcal{F} \to \mathbf{R}^0\pi_*\Omega^1_\nabla \to \mathrm{H}^1(X/S;\nabla) \to \mathbf{R}^1\pi_*\mathcal{F} \to \mathbf{R}^1\pi_*\Omega^1_\nabla \to \mathrm{H}^2(X/S;\nabla) \to 0$$

If S is affine, the sequence becomes:

$$0 \to \mathrm{H}^0(X/S;\nabla) \to \mathrm{H}^0(X,\mathcal{F}) \to \mathrm{H}^0(X,\Omega^1_\nabla) \to \mathrm{H}^1(X/S;\nabla) \to \mathrm{H}^1(X,\mathcal{F}) \to \mathrm{H}^1(X,\Omega^1_\nabla) \to \mathrm{H}^2(X/S;\nabla) \to$$

The sequence is functorial in ∇.

Proof. The cohomology $\mathrm{H}^\bullet(X/S;\nabla)$, $\mathbf{R}^\bullet\pi_*\mathcal{F}$ are sheaves of $\mathcal{O}_S$-modules on S. Because the formation of higher direct images and all the differentials commute with flat base change on S, we can compute the cohomology locally on S. Therefore we assume S is affine and work with sheaf cohomology on X in place of higher direct images. [12, p. 249/268, Cor. 8.2]

Choose an affine cover $\mathcal{U}$ of X. We compute the hypercohomology using a Čech resolution with respect to $\mathcal{U}$. We denote by

$$\mathrm{H}^\bullet(X/S;\nabla),\ C^\bullet(\mathcal{U};\nabla),\ Z^\bullet(\mathcal{U};\nabla),\ B^\bullet(\mathcal{U};\nabla)$$

the cohomology, cochains, cocycles, and coboundaries of the total complex of the connection ∇, respectively, and we denote by

$$\mathrm{H}^\bullet(X,\mathcal{F}),\ C^\bullet(\mathcal{U},\mathcal{F}),\ Z^\bullet(\mathcal{U},\mathcal{F}),\ B^\bullet(\mathcal{U},\mathcal{F})$$

the cohomology, cochains, cocycles, and coboundaries of the sheaf $\mathcal{F}$.

We will exhibit a sequence

$$\begin{aligned} 0 \longrightarrow C^0(\mathcal{U};\nabla) &\xrightarrow{\check{\sigma}_0} C^0(\mathcal{U},\mathcal{F}) \xrightarrow{\check{\sigma}_1} C^0(\mathcal{U},\Omega^1_\nabla) \xrightarrow{\check{\sigma}_2} \\ \longrightarrow C^1(\mathcal{U};\nabla) &\xrightarrow{\check{\sigma}_3} C^1(\mathcal{U},\mathcal{F}) \xrightarrow{\check{\sigma}_4} C^1(\mathcal{U},\Omega^1_\nabla) \xrightarrow{\check{\sigma}_5} \\ \longrightarrow C^2(\mathcal{U};\nabla) &\longrightarrow 0 \end{aligned} \tag{1.5.1}$$

which when restricted to cocycles descends to the desired sequence on cohomology

$$\begin{aligned} 0 \longrightarrow \mathrm{H}^0(X/S;\nabla) &\xrightarrow{\sigma_0} \mathrm{H}^0(X,\mathcal{F}) \xrightarrow{\sigma_1} \mathrm{H}^0(X,\Omega^1_\nabla) \xrightarrow{\sigma_2} \\ \longrightarrow \mathrm{H}^1(X/S;\nabla) &\xrightarrow{\sigma_3} \mathrm{H}^1(X,\mathcal{F}) \xrightarrow{\sigma_4} \mathrm{H}^1(X,\Omega^1_\nabla) \xrightarrow{\sigma_5} \\ \longrightarrow \mathrm{H}^2(X/S;\nabla) &\longrightarrow 0 \end{aligned} \tag{1.5.2}$$

The proof is straightforward but tedious. We present a couple of diagrams which are helpful in visualizing all the maps.

In terms of the components $C^{i,j}$ of the double complex, the sequence 1.5.1 is equivalent to

$$0 \longrightarrow C^{0,0} \xrightarrow{\check{\sigma}_0} C^{0,0} \xrightarrow{\check{\sigma}_1} C^{1,0} \xrightarrow{\check{\sigma}_2} C^{1,0} \oplus C^{0,1} \xrightarrow{\check{\sigma}_3} C^{0,1} \xrightarrow{\check{\sigma}_4} C^{1,1} \xrightarrow{\check{\sigma}_5} C^{1,1} \longrightarrow 0$$

and the arrows are all the natural ones: $\check{\sigma}_0, \check{\sigma}_5$ are equality, $\check{\sigma}_1, \check{\sigma}_4$ are induced by the connection ∇, $\check{\sigma}_2$ is inclusion into the first factor, and $\check{\sigma}_3$ is projection onto the second factor.

We can view the $\check{\sigma}_i$ along with the total differential $\mathbf{d}$, the Čech differential ∂, and the connection ∇ in either of the following equivalent diagrams

$$\begin{array}{ccccccc}
0 \longrightarrow & C^{0,0} & \xrightarrow{\check{\sigma}_0} & C^{0,0} & \xrightarrow[\nabla]{\check{\sigma}_1} & C^{1,0} & \xrightarrow{\check{\sigma}_2} \\
 & \big\downarrow \mathbf{d} & & \big\downarrow \partial & & \big\downarrow \partial & \\
\longrightarrow & C^{1,0} \oplus C^{0,1} & \xrightarrow{\check{\sigma}_3} & C^{0,1} & \xrightarrow[\nabla]{\check{\sigma}_4} & C^{1,1} & \xrightarrow{\check{\sigma}_5} \\
 & \big\downarrow \mathbf{d} & & & & & \\
\longrightarrow & C^{1,1} & \longrightarrow & 0 & & &
\end{array}$$

$$\begin{array}{ccccccc}
 & & & 0 \longrightarrow & C^{0,0} & \xrightarrow{\check{\sigma}_0} & \\
 & & & & \big\downarrow \mathbf{d} & & \\
\longrightarrow C^{0,0} & \xrightarrow[\nabla]{\check{\sigma}_1} & C^{1,0} & \xrightarrow{\check{\sigma}_2} & C^{1,0} \oplus C^{0,1} & \xrightarrow{\check{\sigma}_3} & \\
\big\downarrow \partial & & \big\downarrow \partial & & \big\downarrow \mathbf{d} & & \\
\longrightarrow C^{0,1} & \xrightarrow[\nabla]{\check{\sigma}_4} & C^{1,1} & \xrightarrow{\check{\sigma}_5} & C^{1,1} & \longrightarrow & 0
\end{array}$$

We establish exactness of the maps σ_i at each of the seven terms of the sequence, starting at the 0^{th} term and working toward the 6^{th}. Keep in mind that the $\check{\sigma}_i$ denote maps on cochains, and σ_i denote the corresponding maps on cohomology.

[0^{th} term]: The injectivity of σ_0 follows immediately from the injectivity of $\check{\sigma}_0$ and the surjectivity of $\check{\sigma}_3$.

[1st term]: Let $\gamma_1 \in Z^0(\mathcal{U}, \mathcal{L}) \subset C^{0,0}$. Then $0 = \sigma_1(\gamma_1) = \nabla(\gamma_1) \Longleftrightarrow \mathbf{d}(\gamma_1) = \big(\nabla(\gamma_1), \partial(\gamma_1)\big) = 0 \Longleftrightarrow \gamma_1 \in \check{\sigma}_0\left(Z^0(\mathcal{U}; \nabla)\right)$

[2nd term]: Let $\gamma_2 \in C^{1,0}$. Then $\sigma_2(\gamma_2) = 0 \Longleftrightarrow$ There exists $\gamma_0 \in C^{0,0}$ such that $\mathbf{d}(\gamma_0) = \big(\nabla(\gamma_0), \partial(\gamma_0)\big) = \check{\sigma}_2(\gamma_2) = (\gamma_2, 0) \Longleftrightarrow \gamma_2 \in \check{\sigma}_1\left(Z^0(\mathcal{U}, \mathcal{L})\right)$

[3rd term]: Let $\gamma_3 = (\alpha, \beta) \in C^{1,0} \oplus C^{0,1}$. Then $\sigma_3(\gamma_3) = 0 \Longleftrightarrow \check{\sigma}_3(\gamma_3) = \beta = \partial(\hat{\beta})$ for some $\hat{\beta} \in C^{0,0}$. Then $\gamma_3 - \mathbf{d}(\hat{\beta}) = (\alpha, \beta) - \big(\nabla(\hat{\beta}), \partial(\hat{\beta})\big) = \big(\alpha - \nabla(\hat{\beta}), 0\big) = \check{\sigma}_2\big(\alpha - \nabla(\hat{\beta})\big)$ $\Longleftrightarrow \gamma_3$ is cohomologous to an element in the image of $\check{\sigma}_2$.

[4th term]: Let $\gamma_4 \in C^{0,1}$. $\sigma_4(\gamma_4) = \nabla(\gamma_4) = 0 \Longleftrightarrow$ There exists $\alpha \in C^{1,0}$ such that $\nabla(\gamma_4) = \partial(\alpha) \Longleftrightarrow \mathbf{d}\big((\alpha, \gamma_4)\big) = \nabla(\gamma_4) - \partial(\alpha) = 0 \Longleftrightarrow \gamma_4 \in \mathrm{im}(\sigma_3)$.

[5th term]: Let $\gamma_5 \in C^{1,1}$. Then $\sigma_5(\gamma_5) = 0 \Longleftrightarrow$ There is $(\alpha, \beta) \in C^{1,0} \oplus C^{0,1}$ such that $\check{\sigma}_5(\gamma_5) = \mathbf{d}\big((\alpha, \beta)\big) = \nabla(\alpha) - \partial(\beta) \in C^{1,1} \Longleftrightarrow \sigma_5(\gamma_5) = \nabla(\alpha) \in \mathrm{H}^1(X, \Omega^1_\nabla)$ for some α in $\mathrm{H}^1(X, \mathcal{L})$.

[6th term]: $\check{\sigma}_5$ is surjective, so σ_5 is surjective.

Thus the sequence of cohomology groups is exact. Functoriality follows because each group of cochains is functorial, and the differentials are compatible.

□

Remark 1.5.21. Due to 1.5.11, the degeneration of the Hodge-to-de Rham spectral sequence for ∇ is equivalent to the degeneration of the seven term exact sequence, in the sense that the maps σ_1, σ_4 vanish, and so $\mathcal{S}_\nabla$ splits into the three sequences

$$0 \to \mathrm{H}^0(X/S; \nabla) \to \mathbf{R}^0\pi_*\mathcal{F} \to 0$$

$$0 \to \mathbf{R}^0\pi_*\Omega^1_\nabla \to \mathrm{H}^1(X/S; \nabla) \to \mathbf{R}^1\pi_*\mathcal{F} \to 0$$

$$0 \to \mathbf{R}^1\pi_*\Omega^1_{\nabla} \to \mathrm{H}^2(X/S;\nabla) \to 0$$

1.5.4 Functoriality of seven term sequence

We now analyze the behavior of the seven term sequence $\mathcal{S}^{\bullet}_{\nabla}$ (1.5.20) under morphisms of connections. Our aim is to compare the cohomology of a connection ∇ with the cohomology of d, i.e. ordinary de Rham cohomology. This will chiefly be to determine the cokernel of the map

$$\mathrm{H}^{\bullet}(X/S;\mathcal{O}_X,\mathrm{d}) \to \mathrm{H}^{\bullet}(X/S;\mathcal{O}_X(nD),\nabla_D)$$

where D is an effective Cartier divisor. We will achieve this by analyzing the following two types of morphisms of connections

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\nabla'} & \Omega\otimes\mathcal{L} \\ \downarrow & & \downarrow \\ \mathcal{L}(D) & \xrightarrow{\nabla} & \Omega\otimes\mathcal{L}(D) \end{array} \qquad\qquad \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\nabla} & \Omega_{X/S} \\ \downarrow & & \downarrow \\ \mathcal{O}_X & \xrightarrow{\widehat{\nabla}} & \Omega_{X/S}(D) \end{array}$$

which we refer to respectively as *extension of the coefficient sheaf* $\mathcal{L}$ and *extension of the value sheaf* Ω. In both of these cases each (non-identity) map of line bundles is an inclusion with horizontal cokernel, and we will make use of the earlier result 1.4.56 about cohomological vanishing of such maps.

Remark 1.5.22. When the Hodge-to-de Rham spectral sequence degenerates for one of the connections ∇', ∇, then the functoriality of the seven term exact sequence becomes easier to analyze. The Hodge-to-de Rham spectral sequence does not degenerate at E_1 for most connections, but it does degenerate at E_1 for the standard derivation d on a cohomologically flat morphism (1.5.9).

We will use this fact to bootstrap our analysis, in the extension of value sheaves 1.5.6.

Let $\pi : X \to S$ be a morphism of schemes, with quasi-coherent $\mathcal{O}_X$-modules $\mathcal{L}, \mathcal{L}'$ and S-connections

$$\nabla : \mathcal{L} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L} =: \Omega^1_\nabla$$

$$\nabla' : \mathcal{L}' \to \Omega' \otimes_{\mathcal{O}_X} \mathcal{L}' =: \Omega^1_{\nabla'}$$

Let $\chi : \nabla' \to \nabla$ be a horizontal morphism of connections.

We display the functoriality of the seven term exact sequences $\mathcal{S}^\bullet_{\nabla'} \to \mathcal{S}^\bullet_\nabla$ in the diagram

$$\begin{array}{ccccccccccccc}
\mathrm{H}^0(X/S;\nabla') & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla'} & \longrightarrow & \mathrm{H}^1(X/S;\nabla') & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^1\pi_*\Omega^1_{\nabla'} & \longrightarrow & \mathrm{H}^2(X/S;\nabla') \\
\downarrow{\scriptstyle\chi_0} & & \downarrow{\scriptstyle\chi_1} & & \downarrow{\scriptstyle\chi_2} & & \downarrow{\scriptstyle\chi_3} & & \downarrow{\scriptstyle\chi_4} & & \downarrow{\scriptstyle\chi_5} & & \downarrow{\scriptstyle\chi_6} \\
\mathrm{H}^0(X/S;\nabla) & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla} & \longrightarrow & \mathrm{H}^1(X/S;\nabla) & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^1\pi_*\Omega^1_{\nabla} & \longrightarrow & \mathrm{H}^2(X/S;\nabla)
\end{array}$$

where columns $0, 3, 6$ are of primary interest. Our aim will be to analyze columns $0, 3, 6$ by examining the kernels and cokernels in the exact sequences

$$0 \longrightarrow \mathcal{P}^i \longrightarrow \mathrm{H}^i(X/S;\nabla') \longrightarrow \mathrm{H}^i(X/S;\nabla) \longrightarrow \mathcal{Q}^i \longrightarrow 0.$$

We extend the map of seven term sequences to include the sequences of kernels and cokernels:

$$
\begin{array}{ccccccccccccc}
\mathcal{P}^0 & \longrightarrow & E_1^{1,0} & \longrightarrow & E_1^{2,0} & \longrightarrow & \mathcal{P}^1 & \longrightarrow & E_1^{4,0} & \longrightarrow & E_1^{5,0} & \longrightarrow & \mathcal{P}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^0(X/S;\nabla') & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla'} & \longrightarrow & \mathrm{H}^1(X/S;\nabla') & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^1\pi_*\Omega^1_{\nabla'} & \longrightarrow & \mathrm{H}^2(X/S;\nabla') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^0(X/S;\nabla) & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla} & \longrightarrow & \mathrm{H}^1(X/S;\nabla) & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^1\pi_*\Omega^1_{\nabla} & \longrightarrow & \mathrm{H}^2(X/S;\nabla) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Q}^0 & \longrightarrow & E_1^{1,1} & \longrightarrow & E_1^{2,1} & \longrightarrow & \mathcal{Q}^1 & \longrightarrow & E_1^{4,1} & \longrightarrow & E_1^{5,1} & \longrightarrow & \mathcal{Q}^2
\end{array}
$$

The terms $\mathcal{P}^i$, $\mathcal{Q}^i$ are the ones we want to determine, and the terms $E_k^{i,j}$ are terms of the spectral sequence we will describe shortly.

Unfortunately the kernels and cokernels of a map of exact sequences *are not necessarily exact* themselves, and this is what necessitates the use of a spectral sequence.

Remark 1.5.23. WARNING: The vertical orientation of our spectral sequences is reversed, in order to match the map of seven term sequences.

Remark 1.5.24. The fact that the kernel and cokernel sequences of a morphism of exact complexes are not necessarily exact is related to the fact that the derived category is not an abelian category. It is however a triangulated category.

Proposition 1.5.25. *Given a morphism $\nabla' \xrightarrow{\chi} \nabla$ of connections and the induced morphism $\mathcal{S}^\bullet_{\nabla'} \to \mathcal{S}^\bullet_{\nabla}$ of seven term exact sequences, there is associated a two-row spectral sequence $E_k^{i,j}$ with the following properties:*

E_0 contains the seven term sequences:

$$E_0^{\bullet,0} = \mathcal{S}^{\bullet}_{\nabla'},$$

$$E_0^{\bullet,1} = \mathcal{S}^{\bullet}_{\nabla}.$$

E_1 contains the kernels and cokernels:

$$E_1^{\bullet,0} = \ker\left[\mathcal{S}^{\bullet}_{\nabla'} \xrightarrow{\chi} \mathcal{S}^{\bullet}_{\nabla}\right],$$

$$E_1^{\bullet,1} = \operatorname{coker}\left[\mathcal{S}^{\bullet}_{\nabla'} \xrightarrow{\chi} \mathcal{S}^{\bullet}_{\nabla}\right].$$

$E_3 = E_\infty = 0$, and all morphisms on page E_2 are isomorphisms.

Proof. This spectral sequence is just the spectral sequence for a double complex, in the special case with only two nonzero rows. We regard the map χ of seven term sequences as a double complex and use the machinery of spectral sequences of a double complex as bookkeeping for the failure of the kernel and cokernel sequences to be exact. We mimic the analysis in [8, p. 62, 1.7.5].

Since the seven term sequences are exact, the spectral sequence with horizontal orientation converges immediately to zero on page E_1. Therefore we will take the spectral sequence with vertical orientation, and know that it will also converge to zero eventually. Because the only nonzero terms on E_0 are two consecutive rows, the spectral sequence must converge no later than page E_3. This implies are morphisms on page E_2 are isomorphisms.

E_0:

$$\begin{array}{ccccccc}
\mathrm{H}^0(X/S;\nabla') & \mathbf{R}^0\pi_*\mathcal{L}' & \mathbf{R}^0\pi_*\Omega^1_{\nabla'} & \mathrm{H}^1(X/S;\nabla') & \mathbf{R}^1\pi_*\mathcal{L}' & \mathbf{R}^1\pi_*\Omega^1_{\nabla'} & \mathrm{H}^2(X/S;\nabla') \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathrm{H}^0(X/S;\nabla) & \mathbf{R}^0\pi_*\mathcal{L} & \mathbf{R}^0\pi_*\Omega^1_{\nabla} & \mathrm{H}^1(X/S;\nabla) & \mathbf{R}^1\pi_*\mathcal{L} & \mathbf{R}^1\pi_*\Omega^1_{\nabla} & \mathrm{H}^2(X/S;\nabla)
\end{array}$$

E_1:

$$\mathcal{P}^0 \longrightarrow E_1^{1,0} \longrightarrow E_1^{2,0} \longrightarrow \mathcal{P}^1 \longrightarrow E_1^{4,0} \longrightarrow E_1^{5,0} \longrightarrow \mathcal{P}^2$$

$$\mathcal{Q}^0 \longrightarrow E_1^{1,1} \longrightarrow E_1^{2,1} \longrightarrow \mathcal{Q}^1 \longrightarrow E_1^{4,1} \longrightarrow E_1^{5,1} \longrightarrow \mathcal{Q}^2$$

E_2:

$$\begin{array}{ccccccccccc}
0 & 0 & E_2^{0,0} & E_2^{1,0} & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & E_2^{5,0} & E_2^{6,0} & 0 & 0 \\
 & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow \\
0 & 0 & E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & E_2^{5,1} & E_2^{6,1} & 0 & 0
\end{array}$$

Accounting for the zeros in columns $i < 0$ and $i > 6$, the isomorphisms in E_2 allow us to simplify to

E_2:

$$\begin{array}{ccccccc}
0 & 0 & E_2^{2,0} & E_2^{3,0} & E_2^{4,0} & E_2^{5,0} & E_2^{6,0} \\
 & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \nearrow\sim & \\
E_2^{0,1} & E_2^{1,1} & E_2^{2,1} & E_2^{3,1} & E_2^{4,1} & 0 & 0
\end{array}$$

□

Proposition 1.5.26. *Let $\pi : X \to S$ be a morphism of schemes, with quasi-coherent $\mathcal{O}_X$-modules $\mathcal{L}, \mathcal{L}'$ and Ω-valued S-connections*

$$\nabla : \mathcal{L} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L} =: \Omega^1_{\nabla}$$

$$\nabla' : \mathcal{L}' \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L}' =: \Omega^1_{\nabla'}$$

Let $\lambda : \nabla' \to \nabla$ be a horizontal morphism of connections, where $\mathcal{L}' \xrightarrow{\lambda} \mathcal{L}$ is injective and induces the exact sequence

$$0 \to \mathcal{L}' \xrightarrow{\lambda} \mathcal{L} \to \overline{\mathcal{L}} \to 0$$

The cohomology $\mathrm{H}^{\bullet}(X/S; \overline{\nabla})$ of the quotient connection $\overline{\nabla}$ from 1.4.81 fits into the seven term exact sequence

$$\mathrm{H}^0(X/S;\overline{\nabla}) \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathrm{H}^1(X/S;\overline{\nabla}) \longrightarrow \mathbf{R}^1\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^1\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathrm{H}^2(X/S;\overline{\nabla})$$

Proof. This is just a restatement of the existence of the seven term exact sequence, which is valid for any quasi-coherent coefficient sheaf Ω. □

1.5.5 Extension of coefficient sheaves

We will find it useful to have the following positivity condition on a line bundle, which guarantees the vanishing of certain terms in the spectral sequence 1.5.25.

Definition 1.5.27. Let $\pi : X \to S$ be an arithmetic surface. A line bundle $\mathcal{L}$ on X is *adequate* (with respect to π) if

- $\mathbf{R}^1\pi_*\mathcal{L} = 0$
- $\mathbf{R}^1\pi_*(\Omega_{X/S} \otimes_{\mathcal{O}_X} \mathcal{L}) = 0$

If $D \subset X$ is a divisor and $\mathcal{O}_X(D)$ is adequate, then we call D adequate as well.

Remark 1.5.28. Note that on projective curves X/k, by Serre duality, the second vanishing condition is entailed in $\deg D > 0$. Furthermore, by Riemann-Roch all divisors D with $\deg D > 2g-2$ are adequate.

In particular, for $g = 0$, ample divisors, adequate divisors, and divisors of positive degree all coincide.

When X/S is an arithmetic surface that is also smooth, we have the same vanishing results whenever we replace the degree of a line bundle with the degree of its restriction to each vertical fiber. See [6, p. 17, Prop.4.3].

The primary utility of adequacy lies in:

Proposition 1.5.29. *Let $\mathcal{L}' \to \mathcal{L}$ be an injection of line bundles on an arithmetic surface X.*

We have the short exact sequences

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L} \longrightarrow \overline{\mathcal{L}} \longrightarrow 0$$

$$0 \longrightarrow \Omega_{X/S} \otimes \mathcal{L}' \longrightarrow \Omega_{X/S} \otimes \mathcal{L} \longrightarrow \Omega_{X/S} \otimes \overline{\mathcal{L}} \longrightarrow 0$$

If $\mathcal{L}'$ is adequate, then the associated long exact sequences of higher direct images induces *the short exact sequences of low degree terms:*

$$0 \longrightarrow \mathbf{R}^0\pi_*\mathcal{L}' \longrightarrow \mathbf{R}^0\pi_*\mathcal{L} \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L}' \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L} \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \overline{\mathcal{L}} \longrightarrow 0.$$

Proof. This follows directly from looking at the long exact sequences of higher direct images and taking into account that $\mathcal{L}'$ and $\Omega_{X/S} \otimes \mathcal{L}'$ are acyclic by definition of adequacy:

$$0 \longrightarrow \mathbf{R}^0\pi_*\mathcal{L}' \longrightarrow \mathbf{R}^0\pi_*\mathcal{L} \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow$$

$$\longrightarrow 0 \longrightarrow \mathbf{R}^1\pi_*\mathcal{L} \longrightarrow \mathbf{R}^1\pi_*\overline{\mathcal{L}} \longrightarrow 0$$

$$0 \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L}' \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L} \longrightarrow \mathbf{R}^0\pi_*\Omega \otimes \overline{\mathcal{L}} \longrightarrow$$

$$\longrightarrow 0 \longrightarrow \mathbf{R}^1\pi_*\Omega \otimes \mathcal{L} \longrightarrow \mathbf{R}^1\pi_*\Omega \otimes \overline{\mathcal{L}} \longrightarrow 0.$$

□

Corollary 1.5.30. Let $\mathcal{L}' \to \mathcal{L}$ be an injection of line bundles on an arithmetic surface X. If $\mathcal{L}'$ is adequate, and the cokernel $\overline{\mathcal{L}}$ is supported on a horizontal divisor, then $\mathcal{L}$ is also adequate.

In particular, considering the injection $\mathcal{L} \to \mathcal{L}(D)$ for an effective divisor D,

$$\mathbf{R}^1\pi_*\left(\Omega_{X/S} \otimes \mathcal{L}(D)\right) = \mathbf{R}^1\pi_*\left(\Omega_{X/S}(D) \otimes \mathcal{L}\right) = 0.$$

This says that if $\mathcal{L}$ is adequate, then not only do the higher direct images of the differentials

$\Omega_{X/S} \otimes \mathcal{L}$ vanish, but also the higher direct images of the logarithmic differentials $\Omega_{X/S}(D) \otimes \mathcal{L}$ vanish.

Proof. This follows from looking at the long exact sequence, just as in the previous proposition.

If the cokernel $\overline{\mathcal{L}}$ is supported on a horizontal divisor, then the terms $\mathbf{R}^1\pi_*\overline{\mathcal{L}}$ and $\mathbf{R}^1\pi_*\Omega\otimes\overline{\mathcal{L}}$ vanish by 1.4.55, so by exactness, $\mathbf{R}^1\pi_*\mathcal{L},\ \mathbf{R}^1\pi_*\Omega\otimes\mathcal{L}$ vanish as well. So the long exact sequences simplify entirely to the given short exact sequences in the statement of the corollary. □

Proposition 1.5.31 (Extension of adequate bundles)**.** *Let $\pi : X \to S$ be a smooth arithmetic surface. Let $\mathcal{L}', \mathcal{L}$ be line bundles with Ω-valued connections ∇', ∇. Let $\nabla' \xrightarrow{\lambda} \nabla$ be a horizontal morphism induced by an injection $\mathcal{L}' \xrightarrow{\lambda} \mathcal{L}$.*

Suppose that $\operatorname{coker}(\lambda)$ *is supported on a horizontal divisor, and that either one of the following conditions is satisfied*

- *$\mathcal{L}, \mathcal{L}'$ are adequate,*
- *$\mathcal{L}$ is adequate, and $\mathbf{R}^1\pi_*\mathcal{L}' = \mathbf{R}^1\pi_*\Omega \otimes \mathcal{L}' = 0$*

Then the induced map

$$\mathrm{H}^2(X/S; \nabla') \to \mathrm{H}^2(X/S; \nabla)$$

is an isomorphism, and there is an exact sequence relating the cohomology of the connections $\nabla', \nabla, \overline{\nabla}$:

$$0 \to \mathrm{H}^0(X/S;\nabla') \to \mathrm{H}^0(X/S;\nabla) \to \mathrm{H}^0(X/S;\overline{\nabla}) \to$$

$$\hookrightarrow \mathrm{H}^1(X/S;\nabla') \to \mathrm{H}^1(X/S;\nabla) \to \mathrm{H}^1(X/S;\overline{\nabla}) \to 0$$

Proof. The spectral sequence $E_k^{i,j}$ (1.5.25) associated to the morphism $\nabla' \to \nabla$ has pages

E_0:

$$\begin{array}{ccccccc}
\mathrm{H}^0(X/S;\nabla') & \mathbf{R}^0\pi_*\mathcal{L}' & \mathbf{R}^0\pi_*\Omega^1_{\nabla'} & \mathrm{H}^1(X/S;\nabla') & \mathbf{R}^1\pi_*\mathcal{L}' & \mathbf{R}^1\pi_*\Omega^1_{\nabla'} & \mathrm{H}^2(X/S;\nabla') \\
\downarrow{\lambda_0} & \downarrow{\lambda_1} & \downarrow{\lambda_2} & \downarrow{\lambda_3} & \downarrow{\lambda_4} & \downarrow{\lambda_5} & \downarrow{\lambda_6} \\
\mathrm{H}^0(X/S;\nabla) & \mathbf{R}^0\pi_*\mathcal{L} & \mathbf{R}^0\pi_*\Omega^1_{\nabla} & \mathrm{H}^1(X/S;\nabla) & \mathbf{R}^1\pi_*\mathcal{L} & \mathbf{R}^1\pi_*\Omega^1_{\nabla} & \mathrm{H}^2(X/S;\nabla)
\end{array}$$

E_1:

$$\mathcal{P}^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{P}^1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{P}^2$$

$$\mathcal{Q}^0 \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathcal{Q}^1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{Q}^2$$

1.5.25

The vanishing terms in columns $1, 2$ are due to λ being an injection. The vanishing terms in columns $4, 5$ are implied by 1.5.30 together with either one of the conditions 1.5.31 in the statement of the proposition. The $\mathcal{P}^i$, $\mathcal{Q}^i$ are defined as the corresponding terms in E_1, i.e. $\mathcal{P}^i = \ker(\lambda_{3i})$, $\mathcal{Q}^i = \mathrm{coker}(\lambda_{3i})$.

E_2:

$$\begin{tikzcd}
E_2^{0,0} & 0 & 0 & E_2^{3,0} & 0 & 0 & E_2^{6,0} \\
E_2^{0,1} \arrow[urr] & E_2^{1,1} \arrow[urr] & E_2^{2,1} \arrow[urr] & E_2^{3,1} \arrow[urr] & 0 \arrow[urr] & 0 & E_2^{6,1}
\end{tikzcd}$$

where all arrows are isomorphisms. So this simplifies to

$$\begin{array}{ccccccc}
0 & 0 & 0 & E_2^{3,0} & 0 & 0 & 0 \\
0 & E_2^{1,1} & 0 & 0 & 0 & 0 & 0
\end{array}$$

The vanishing of E_2 at indices $(0,0), (0,1), (6,0), (6,1)$ implies the exactness of E_1 at the same indices, which implies that $\mathcal{P}^0, \mathcal{P}^2, \mathcal{Q}^2$ vanish. Then E_0 is also exact at indices $(0,0), (6,0), (6,1)$. This implies the injection on H^0 and the isomorphism on H^2.

Now we restrict to the first half of the seven term sequences ($\mathcal{S}^i, 0 \leq i \leq 4$), together with the kernels and cokernels:

$$
\begin{array}{ccccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{P}^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(X/S;\nabla') & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla'} & \longrightarrow & \mathrm{H}^1(X/S;\nabla') & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{H}^0(X/S;\nabla) & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\nabla} & \longrightarrow & \mathrm{H}^1(X/S;\nabla) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{Q}^0 & \longrightarrow & \mathbf{R}^0\pi_*\overline{\mathcal{L}} & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}} & \longrightarrow & \mathcal{Q}^1 & \longrightarrow & 0
\end{array}
\tag{1.5.3}
$$

So 1.5.3 is a diagram with the middle two rows exact and all columns exact.

Observe that columns $1, 2$ are the short exact sequences from 1.5.29 that arise from the initial segment of the long exact sequence for the higher direct images of adequate bundles. We isolate those two columns in the map of short exact sequences:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\overline{\mathcal{L}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \nabla'} & & \downarrow{\scriptstyle \nabla} & & \downarrow{\scriptstyle \overline{\nabla}} & & \\
0 & \longrightarrow & \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\Omega \otimes \mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\overline{\Omega} & \longrightarrow & 0
\end{array}
$$

We produce the exact sequence in the statement of the proposition from the kernels and cokernels by using the snake lemma 1.3.1:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{H}^0(X/S;\nabla') & \longrightarrow & \mathrm{H}^0(X/S;\nabla) & \longrightarrow & \mathrm{H}^0(X/S;\overline{\nabla}) & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\overline{\mathcal{L}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbf{R}^0\pi_*\Omega\otimes\mathcal{L}' & \longrightarrow & \mathbf{R}^0\pi_*\Omega\otimes\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\overline{\Omega} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & \longrightarrow & \mathrm{H}^1(X/S;\nabla') & \longrightarrow & \mathrm{H}^1(X/S;\nabla) & \longrightarrow & \mathrm{H}^1(X/S;\overline{\nabla}) & \longrightarrow & 0
\end{array}
\tag{1.5.4}
$$

□

Remark 1.5.32. Note that because only the middle two rows of 1.5.3 are exact, we cannot conclude $\mathcal{Q}^0 = \mathrm{H}^0(X/S;\overline{\nabla})$, unless the connecting homomorphism of the kernel-cokernel sequence is zero. This is because $\mathcal{Q}^0 := \operatorname{coker}\left[\mathrm{H}^0(X/S;\nabla') \to \mathrm{H}^0(X/S;\nabla)\right]$. Equivalently, $E_2^{1,1}$ is not necessarily zero, and so the row $E_1^{\bullet,1}$ is not necessarily exact.

We will remedy this issue by upgrading the condition on our morphisms of connections from horizontal to strictly horizontal (1.4.88).

Proposition 1.5.33 (Strictly horizontal morphism of adequate bundles)**.** *Use the same notation as the previous proposition 1.5.31, and continue to assume that the cokernel $\overline{\mathcal{L}}$ of λ is supported on a horizontal divisor.*

Assume now that λ is strictly horizontal (1.4.88).

Then the connecting homomorphism of 1.5.4 vanishes, and the kernel-cokernel sequence *splits into two short exact sequences:*

$$0 \to \mathrm{H}^0(X/S;\nabla') \to \mathrm{H}^0(X/S;\nabla) \to \mathrm{H}^0(X/S;\overline{\nabla}) \to 0$$

$$0 \to \mathrm{H}^1(X/S;\nabla') \to \mathrm{H}^1(X/S;\nabla) \to \mathrm{H}^1(X/S;\overline{\nabla}) \to 0$$

In particular, the maps on H^0, H^1 *are injective.*

Proof. We use the Cartesian property to extended our analysis of the diagram 1.5.4.

The vanishing of the connecting homomorphism is equivalent to the right-exactness of the kernel sequence, i.e. surjectivity of the map $\mathrm{H}^0(X/S;\nabla) \to \mathrm{H}^0(X/S;\overline{\nabla})$, and it is also equivalent to the left-exactness of the cokernel sequence, i.e. injectivity of the map $\mathrm{H}^1(X/S;\nabla') \to \mathrm{H}^1(X/S;\nabla)$. We will demonstrate the injectivity of the latter map.

Consider the lower left portion of the diagram 1.5.4:

$$\begin{array}{ccccc} 0 & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L}' & \xrightarrow{\lambda} & \mathbf{R}^0\pi_*\mathcal{L} \\ & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbf{R}^0\pi_*\Omega\otimes\mathcal{L}' & \xrightarrow{\mu} & \mathbf{R}^0\pi_*\Omega\otimes\mathcal{L} \\ & & \downarrow & & \downarrow \\ & & \mathrm{H}^1(X/S;\nabla') & \xrightarrow{\overline{\mu}} & \mathrm{H}^1(X/S;\nabla) \end{array}$$

The map in question is $\overline{\mu}$ on the bottom row. Since S is affine, we are working in the category of modules over the ring $\Gamma(\mathcal{O}_S)$, and so we can apply 1.3.7. This says precisely that since the upper square is Cartesian (by 1.4.90), and μ is injective, then $\overline{\mu}$ is also injective. $\square$

1.5.6 Extension of value sheaves

Proposition 1.5.34 (Extension of value sheaves)**.** *Let $\pi : X \to S$ be a smooth arithmetic surface.*

Let $\mathcal{L}$ be a line bundle on X with two S-connections

$$\nabla : \mathcal{L} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L} =: \Omega^1_{\nabla}$$

$$\widehat{\nabla} : \mathcal{L} \to \widehat{\Omega} \otimes_{\mathcal{O}_X} \mathcal{L} =: \widehat{\Omega}^1_{\widehat{\nabla}}$$

and an $\mathcal{O}_X$-module homomorphism $\eta : \Omega \to \widehat{\Omega}$ which is compatible with the connections, i.e.

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\nabla} & \Omega \otimes_{\mathcal{O}_X} \mathcal{L} \\ \downarrow \mathrm{id} & & \downarrow \eta\otimes\mathrm{id} \\ \mathcal{L} & \xrightarrow{\widehat{\nabla}} & \widehat{\Omega} \otimes_{\mathcal{O}_X} \mathcal{L} \end{array}$$

is commutative. So η induces a horizontal morphism of connections.

We assume that η is injective and the cokernel of η is supported on a horizontal divisor.

Then the induced maps on cohomology satisfy:

$$\mathrm{H}^0(X/S; \nabla') \xrightarrow{\simeq} \mathrm{H}^0(X/S; \nabla) \text{ is an isomorphism},$$

$$\mathrm{H}^1(X/S; \nabla') \to \mathrm{H}^1(X/S; \nabla) \text{ is injective},$$

$$\mathrm{H}^2(X/S; \nabla') \to \mathrm{H}^2(X/S; \nabla) \text{ is surjective}.$$

If furthermore, the Hodge-to-de Rham spectral sequence 1.5.7 for ∇ degenerates at E_1, *then*

- *The Hodge-to-de Rham spectral sequence for* $\widehat{\nabla}$ *also degenerates at* E_1*,*

- *The spectral sequence 1.5.25 associated to* η *converges at* E_2*.*

Proof. As in proposition 1.5.31 we have a map $\mathcal{S}^\bullet_\nabla \to \mathcal{S}^\bullet_{\widehat{\nabla}}$ between the seven term sequence of connections, and the associated spectral sequence 1.5.25 has pages

E_0:

$$\begin{array}{ccccccc}
\mathrm{H}^0(X/S;\nabla) & \mathbf{R}^0\pi_*\mathcal{L} & \mathbf{R}^0\pi_*\Omega^1_\nabla & \mathrm{H}^1(X/S;\nabla) & \mathbf{R}^1\pi_*\mathcal{L} & \mathbf{R}^1\pi_*\Omega^1_\nabla & \mathrm{H}^2(X/S;\nabla) \\
\downarrow \eta_0 & = \downarrow \eta_1 & \downarrow \eta_2 & \downarrow \eta_3 & = \downarrow \eta_4 & \downarrow \eta_5 & \downarrow \eta_6 \\
\mathrm{H}^0(X/S;\widehat{\nabla}) & \mathbf{R}^0\pi_*\mathcal{L} & \mathbf{R}^0\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} & \mathrm{H}^1(X/S;\widehat{\nabla}) & \mathbf{R}^1\pi_*\mathcal{L} & \mathbf{R}^1\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} & \mathrm{H}^2(X/S;\widehat{\nabla})
\end{array}$$

E_1:

$$\mathcal{P}^0 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{P}^1 \longrightarrow 0 \longrightarrow E_1^{5,0} \longrightarrow \mathcal{P}^2$$

$$\mathcal{Q}^0 \longrightarrow 0 \longrightarrow E_1^{2,1} \longrightarrow \mathcal{Q}^1 \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathcal{Q}^2$$

where the zeros in columns $1, 4$ are the kernels and cokernels of the isomorphisms η_1, η_4 and the zeros in columns $2, 5$ come from and the injection of value sheaves with horizontally supported cokernel (using 1.4.56).

E_2:

$$
\begin{array}{ccccccc}
E_2^{0,0} & 0 & 0 & E_2^{3,0} & 0 & E_2^{5,0} & E_2^{6,0} \\
 & & \nearrow & \nearrow & \nearrow & \nearrow & \nearrow \\
E_2^{0,1} & 0 & E_2^{2,1} & E_2^{3,1} & 0 & 0 & E_2^{6,1}
\end{array}
$$

On E_2 all arrows are isomorphisms, so E_2 simplifies to

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & E_2^{5,0} & 0 \\
 & & & & & \overset{\sim}{\nearrow} & \\
0 & 0 & 0 & E_2^{3,1} & 0 & 0 & 0
\end{array}
$$

Now we view the map of seven term sequences, along with the kernel and cokernel sequences. To reiterate, in the following diagram the second and third rows are the sequences $\mathcal{S}^{\bullet}_{\nabla}$ and $\mathcal{S}^{\bullet}_{\widehat{\nabla}}$, which constitute the nonzero part of the page E_0, and the first and fourth rows are the nonzero rows of the page E_1.

$$
\begin{array}{ccccccccccccc}
\mathcal{P}^0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{P}^1 & \longrightarrow & 0 & \longrightarrow & E_1^{5,0} & \longrightarrow & \mathcal{P}^2 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^0(X/S;\nabla) & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\Omega^1_\nabla & \longrightarrow & \mathrm{H}^1(X/S;\nabla) & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^1\pi_*\Omega^1_\nabla & \longrightarrow & \mathrm{H}^2(X/S;\nabla) \\
\downarrow{\scriptstyle \eta_0} & & \downarrow{\scriptstyle \eta_1} & & \downarrow{\scriptstyle \eta_2} & & \downarrow{\scriptstyle \eta_3} & & \downarrow{\scriptstyle \eta_4} & & \downarrow{\scriptstyle \eta_5} & & \downarrow{\scriptstyle \eta_6} \\
\mathrm{H}^0(X/S;\widehat{\nabla}) & \longrightarrow & \mathbf{R}^0\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^0\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} & \longrightarrow & \mathrm{H}^1(X/S;\widehat{\nabla}) & \longrightarrow & \mathbf{R}^1\pi_*\mathcal{L} & \longrightarrow & \mathbf{R}^1\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} & \longrightarrow & \mathrm{H}^2(X/S;\widehat{\nabla}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{Q}^0 & \longrightarrow & 0 & \longrightarrow & E_1^{2,1} & \longrightarrow & \mathcal{Q}^1 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{Q}^2
\end{array}
\tag{1.5.5}
$$

Because the only possibly nonzero terms on E_2 are $E_2^{3,1}$ and $E_2^{5,0}$, the corresponding terms $E_1^{2,1}, E_1^{5,0}$ are the only places where E_1 is possibly not exact. In particular, E_1 is exact at $\mathcal{P}^0, \mathcal{P}^1, \mathcal{P}^2, \mathcal{Q}^0, \mathcal{Q}^2$. From this we can conclude $\mathrm{H}^0(X/S)$ is an isomorphism, $\mathrm{H}^1(X/S)$ is injective, and $\mathrm{H}^2(X/S)$ is surjective.

Finally, suppose that the Hodge-to-de Rham spectral sequence 1.5.7 degenerates for ∇. As mentioned in 1.5.11 and 1.5.21, this is equivalent to the vanishing of the maps σ_1, σ_4 in $\mathcal{S}^\bullet_\nabla$:

$$\mathcal{S}^1_\nabla = \mathbf{R}^0\pi_*\mathcal{L} \xrightarrow{\sigma_1} \mathbf{R}^0\pi_*\Omega^1_\nabla = \mathcal{S}^2_\nabla$$

$$\mathcal{S}^4_\nabla = \mathbf{R}^1\pi_*\mathcal{L} \xrightarrow{\sigma_4} \mathbf{R}^1\pi_*\Omega^1_\nabla = \mathcal{S}^5_\nabla.$$

But because η_1, η_4 are the identity map, the maps $\widehat{\sigma}_1, \widehat{\sigma}_4$ factor through the zero map:

$$\mathcal{S}^1_{\widehat{\nabla}} = \mathbf{R}^0\pi_*\mathcal{L} \xrightarrow{\sigma_1=0} \mathbf{R}^0\pi_*\Omega^1_\nabla \xrightarrow{\eta_2} \mathbf{R}^0\pi_*\Omega^1_{\widehat{\nabla}} = \mathcal{S}^2_{\widehat{\nabla}}$$

$$\mathcal{S}^4_{\widehat{\nabla}} = \mathbf{R}^1\pi_*\mathcal{L} \xrightarrow{\sigma_4=0} \mathbf{R}^1\pi_*\Omega^1_{\nabla} \xrightarrow{\eta_2} \mathbf{R}^1\pi_*\Omega^1_{\widehat{\nabla}} = \mathcal{S}^5_{\widehat{\nabla}}$$

Hence $\widehat{\sigma}_1, \widehat{\sigma}_4$ vanish, and the Hodge-to-de Rham spectral sequence 1.5.7 for $\widehat{\nabla}$ degenerates on E_1 as well.

Finally we show that the morphism of connection spectral sequence 1.5.25 associated to η degenerates on E_2, or equivalently that the rows in E_1 are exact. Since $E_2^{3,1}, E_2^{5,0}$ are the only terms in E_2 which are possibly nonzero, and they are isomorphic, the exactness of E_1 can be achieved either by demonstrating the surjectivity of $E_1^{2,1} \to E_1^{3,1} =: \mathcal{Q}^1$ or the injectivity of $E_1^{5,0} \to E_1^{6,0} =: \mathcal{P}^2$. We will pursue the latter.

Because the Hodge-to-de Rham spectral sequence 1.5.7 for ∇ degenerates at E_1, the map $\mathbf{R}^1\pi_*\Omega^1_{\nabla} \to \mathrm{H}^2(X/S;\nabla)$ is an isomorphism (1.5.21), and this directly implies the injectivity of the kernels $\ker(\eta_5) = E_1^{5,0} \to E_1^{6,0} = \ker(\eta_6)$.

Therefore all terms on page E_2 are zero, and so the spectral sequence associated to η converges at E_2.

□

Remark 1.5.35. Set $\overline{\Omega}^1_{\widehat{\nabla}} := \mathrm{coker}(\eta)$ to be the cokernel of the morphism of values sheaves from 1.5.34. Since we assume $\Omega^1_{\nabla} \xrightarrow{\eta} \Omega^1_{\widehat{\nabla}}$ is an injection of line bundles on X with horizontally supported cokernel, the cokernel is acyclic (1.4.55).

The exact sequence

$$0 \longrightarrow \Omega^1_{\nabla} \xrightarrow{\eta} \widehat{\Omega}^1_{\widehat{\nabla}} \longrightarrow \overline{\Omega}^1_{\widehat{\nabla}} \longrightarrow 0$$

produces a long exact sequence via $R\pi_*$, which has five nonzero terms (as the quotient sheaf $\overline{\Omega}^1_{\overline{\nabla}}$ is acyclic):

$$0 \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\nabla} \longrightarrow \mathbf{R}^0\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} \longrightarrow \mathbf{R}^0\pi_*\overline{\Omega}^1_{\overline{\nabla}} \longrightarrow \mathbf{R}^1\pi_*\Omega^1_{\nabla} \longrightarrow \mathbf{R}^1\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} \longrightarrow 0$$

Using the usual method of decomposing a long exact sequence into short exact sequences, we can split it into into the following short exact sequences

$$\begin{gathered} 0 \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\nabla} \longrightarrow \mathbf{R}^0\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} \longrightarrow Q_\eta \longrightarrow 0 \\ 0 \longrightarrow P_\eta \longrightarrow \mathbf{R}^1\pi_*\Omega^1_{\nabla} \longrightarrow \mathbf{R}^1\pi_*\widehat{\Omega}^1_{\widehat{\nabla}} \longrightarrow 0. \end{gathered} \tag{1.5.6}$$

We can also see that $P_\eta = E_1^{5,0}, Q_\eta = E_1^{2,1}$ in the proof of 1.5.34 above, since $E_1^{5,0}$, $E_1^{2,1}$ are defined respectively as the kernel of η_5 and cokernel of η_2, which are the same maps as in 1.5.6.

Corollary 1.5.36. As in the previous proposition 1.5.34, let $\mathcal{L}$ be a line bundle on X with two S-connections

$$\nabla : \mathcal{L} \to \Omega \otimes_{\mathcal{O}_X} \mathcal{L} =: \Omega^1_{\nabla}$$

$$\widehat{\nabla} : \mathcal{L} \to \widehat{\Omega} \otimes_{\mathcal{O}_X} \mathcal{L} =: \widehat{\Omega}^1_{\widehat{\nabla}}$$

and an $\mathcal{O}_X$-module homomorphism $\eta : \Omega \to \hat{\Omega}$ which is compatible with the connections. Assume that η is injective with cokernel supported on a horizontal divisor. Suppose that the Hodge-to-de Rham spectral sequence 1.5.7 for ∇ degenerates at E_1.

Then we can give a more precise description of the map $\mathrm{H}^\bullet(X/S;\nabla) \to \mathrm{H}^\bullet(X/S;\widehat{\nabla})$

using 1.5.6 and the following exact sequences:

$$0 \longrightarrow \mathrm{H}^0(X/S;\nabla) \longrightarrow \mathrm{H}^0(X/S;\widehat{\nabla}) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{H}^1(X/S;\nabla) \longrightarrow \mathrm{H}^1(X/S;\widehat{\nabla}) \longrightarrow Q_\eta \longrightarrow 0$$

$$0 \longrightarrow P_\eta \longrightarrow \mathrm{H}^2(X/S;\nabla) \longrightarrow \mathrm{H}^2(X/S;\widehat{\nabla}) \longrightarrow 0.$$

Proof. The isomorphism on H^0 was already established in the previous proposition 1.5.34.

Two obtain the two sequences for $\mathrm{H}^1, \mathrm{H}^2$, we first consider the sequences

$$0 \longrightarrow \mathrm{H}^1(X/S;\nabla) \longrightarrow \mathrm{H}^1(X/S;\widehat{\nabla}) \longrightarrow Q_\eta \longrightarrow 0$$

$$0 \longrightarrow P_\eta \longrightarrow \mathrm{H}^2(X/S;\nabla) \longrightarrow \mathrm{H}^2(X/S;\widehat{\nabla}) \longrightarrow 0.$$

which are the columns 3 and 6 in 1.5.5,

and then we apply the isomorphisms

$$\mathcal{Q}_\eta \cong E_1^{2,1} \cong \mathcal{Q}^1,$$

$$\mathcal{P}_\eta \cong E_1^{5,0} \cong \mathcal{P}^2.$$

The isomorphisms $\mathcal{Q}_\eta \cong E_1^{2,1}$ and $\mathcal{P}_\eta \cong E_1^{5,0}$ were established in 1.5.35, and the isomorphisms $E_1^{2,1} \cong \mathcal{Q}^1$ and $\cong E_1^{5,0} \cong \mathcal{P}^2$ are a direct consequence of the convergence on page E_2 of the spectral sequence associated to η, because the convergence on page E_2 is equivalent to the exactness of the rows in E_1.

So the following subsequences of 1.5.5 are exact:

$$0 \to E_1^{5,0} \to \mathcal{P}^2 \to 0,$$

$$0 \to E_1^{2,1} \to \mathcal{Q}^1 \to 0.$$

and $\mathcal{P}^2 = \ker(\eta_6),\ \mathcal{Q}^1 = \operatorname{coker}(\eta_3)$.

□

Remark 1.5.37. In the previous corollary 1.5.36, in the special case $\Omega := \Omega_{X/S}$, $\widehat{\Omega} := \Omega_{X/S}(D)$, with D adequate, and the two connections are the natural maps

$$\mathrm{d} : \mathcal{O}_X \to \Omega_{X/S}$$

$$\nabla : \mathcal{O}_X \to \Omega_{X/S}(D)$$

(where ∇ factors through d).

Then the map $\Omega_{X/S} \xrightarrow{\eta} \Omega_{X/S}(D)$ induces the long exact sequence

$$\begin{aligned} 0 \longrightarrow \mathbf{R}^0\pi_*\Omega_{X/S} \longrightarrow \mathbf{R}^0\pi_*\Omega_{X/S}(D) \longrightarrow \mathbf{R}^0\pi_*\Omega_{X/S}(D)|_D \longrightarrow \\ \longrightarrow \mathbf{R}^1\pi_*\Omega_{X/S} \longrightarrow 0 \longrightarrow 0 \end{aligned}$$

which splits into the two short exact sequences

$$\begin{aligned} 0 \longrightarrow \mathbf{R}^0\pi_*\Omega_{X/S} \longrightarrow \mathbf{R}^0\pi_*\Omega_{X/S}(D) \longrightarrow \mathcal{Q}_\eta \longrightarrow 0 \\ 0 \longrightarrow \mathcal{P}_\eta \longrightarrow \mathbf{R}^1\pi_*\Omega_{X/S} \longrightarrow 0 \longrightarrow 0. \end{aligned}$$

By the analysis in 1.5.36,

$$\operatorname{coker}\left[\mathbf{R}^0\pi_*\Omega_{X/S} \to \mathbf{R}^0\pi_*\Omega_{X/S}(D)\right] = \mathcal{Q}_\eta \cong \operatorname{coker}\left[\mathrm{H}^1(X/S;\mathrm{d}) \to \mathrm{H}^1(X/S;\nabla)\right]$$

In 1.6.2 we define a filtration on $\mathrm{H}^1(X/S;\mathcal{O}_U,\nabla_D)$, such that by definition the following two sequences are equal:

$$0 \longrightarrow \mathrm{H}^1(X/S;\mathcal{O}_X,\mathrm{d}) \longrightarrow \mathrm{H}^1(X/S;\mathcal{O}_X,\nabla) \longrightarrow \mathcal{Q}_\eta \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Fil}^{-1}\,\mathrm{H}^1(X/S;\mathcal{O}_U) \longrightarrow \mathrm{Fil}^0\,\mathrm{H}^1(X/S;\mathcal{O}_U) \longrightarrow \mathrm{Gr}^0\,\mathrm{H}^1(X/S;\mathcal{O}_U) \longrightarrow 0$$

1.5.7 Analysis of the connection ∇_D

1.5.7.1 Local sections of the sheaves associated to the divisor D

Remark 1.5.38. In this section, unless otherwise specified, $\pi : X \to S$ will be a smooth arithmetic surface, and D an irreducible horizontal effective divisor on X.

Definition 1.5.39 (Canonical section of Cartier divisor)**.** For an effective Cartier divisor D on a scheme X, each $\mathcal{O}_X(nD)$, $n \geq 0$ has a canonical global section $1 = 1_{nD}$, defined by the inclusion

$$\mathcal{O}_X \hookrightarrow \mathcal{O}_X(nD)$$

$$1 \mapsto 1 = 1_{nD}$$

1_{nD} is the *canonical section* of $\mathcal{O}_X(nD)$.

Remark 1.5.40. The canonical section $1_{nD} \in \Gamma(X, \mathcal{O}_X(nD))$ defines the divisor nD.

Proposition 1.5.41. *Let X/S be an arithmetic surface, $V \subset X$ be a trivializing affine open, and $D \subset X$ an effective Cartier divisor. Given an $h \in \mathcal{O}_X(-D)(V)$ defining the divisor D, there is a compatible collection of local $\mathcal{O}_X$-module bases*

$$e_n \in \mathcal{O}_X(nD)(V)$$

such that

$$e_0 = 1 \in \mathcal{O}_X(V)$$

$$h^n e_n = 1_{nD} \in \mathcal{O}_X(nD)(V)$$

$$h^m e_n = e_{n-m} \in \mathcal{O}_X\big((n-m)D\big) \subset \mathcal{O}_X(nD)$$

So we can view e_n as the rational function $\frac{1}{h^n} \in \mathcal{K}_X$.

Furthermore the e_n map to local $\mathcal{O}_X$-module generators of $\mathcal{O}_D(nD) = \mathcal{O}_X(nD)|_D$ under the restriction map

$$\mathcal{O}_X(nD) \to \mathcal{O}_X(nD)|_D$$

$$e_n \mapsto \overline{e}_n$$

Proof. The result is closely related to 1.4.3.

Since D is a Cartier divisor, its ideal sheaf $\mathcal{O}_X(-D)$ is locally principal. For $h \in \mathcal{O}_X(-D)(V)$ to define the divisor D means that h generates the ideal sheaf $\mathcal{O}_X(-D)$, i.e. defines an isomorphism

$$\mathcal{O}_X(V) \xrightarrow[\sim]{\times h} \mathcal{O}_X(-D)(V)$$

Furthermore for $n \geq 0$, h^n generates the ideal $\mathcal{O}_X(-nD)(V)$, and h^{-n} generates the fractional ideal $\mathcal{O}_X(nD)(V)$.

We set $e_n := h^{-n}$, and the compatibility conditions described in the proposition are immediate.

□

Proposition 1.5.42 (Differentiation with respect to a local coordinate)**.** *Let $\pi : X \to S$ be a smooth arithmetic surface.*

Given a point $x \in X$, let $V \subset X$ be a trivializing affine open neighborhood for $\mathcal{O}_X$ and $\Omega^1_{X/S}$. There is a differential $\omega \in \Omega_{X/S}(V)$ which generates $\Omega_{X/S}(V)$ as a rank 1 free $\mathcal{O}_X(V)$-module.

Furthermore, possibly after shrinking V, we may assume that $\omega = \mathrm{d}z$ for some $z \in \mathcal{O}_X(V)$. Such a section z is called a local coordinate at $x \in X$.

We can differentiate any function f with respect to a local generating differential ω, in the sense that for each $f \in \mathcal{O}_X(V)$, there is a unique function $f' \in \mathcal{O}_X(V)$ with

$$\mathrm{d}f = f'\omega \in \Omega^1_{X/S}(V).$$

The assignment $f \mapsto f'$ satisfies the Leibniz rule:

$$(fg)' = f'g + fg'.$$

Proof. The existence of an ω and the function f' is a restatement of the fact that $\Omega_{X/S}$ is a locally free $\mathcal{O}_X$-module of rank 1, which is a result of the fact that X/S is smooth of relative dimension 1.

The fact that the local basis ω can be taken to be the differential of a function is the statement of 1.4.41.

The fact that the derivative satisfies the Leibniz rule follows directly from the Leibniz rule for the derivation $\mathcal{O}_X \to \Omega_{X/S}$:

$$(fg)'\omega = \mathrm{d}(fg) = g\,\mathrm{d}f + f\,\mathrm{d}g = f'g\omega + fg'\omega = (f'g + fg')\omega.$$

□

Remark 1.5.43. Let X/S be a smooth arithmetic surface, $V \subset X$ be a trivializing affine open for $\mathcal{O}_X(D)$ and $\Omega^1_{X/S}$. Given a local coordinate z as in 1.5.42 and a function h and bases e_n as in 1.5.41, we have the following presentations of the sheaf of differentials and twists:

$$\mathcal{O}_X(nD)(V) = \mathcal{O}_X(V) \cdot \langle e_n \rangle$$

$$\Omega^1_{X/S}(V) = \mathcal{O}_X(V) \cdot \langle \mathrm{d}z \rangle$$

$$\Omega^1_{X/S}(D)(V) = \mathcal{O}_X(V) \cdot \langle \frac{\mathrm{d}z}{h} \rangle$$

$$\Omega^1_{X/S}|_D(V) = \mathcal{O}_D(V) \cdot \langle \mathrm{d}z \rangle$$

$$\left(\Omega^1_{X/S}(D) \otimes \mathcal{O}_X(nD)\right)(V) = \mathcal{O}_X(V) \cdot \langle \frac{\mathrm{d}z}{h} \otimes e^n \rangle$$

$$\left(\Omega^1_{X/S}(D) \otimes \mathcal{O}_D(nD)\right)(V) = \mathcal{O}_D(V) \cdot \langle \frac{\mathrm{d}z}{h} \otimes \overline{e}_n \rangle$$

1.5.7.2 Local expression for the connection

Now we explicitly describe the connection

$$\nabla_D : \mathcal{O}_X(nD) \to \Omega_{X/S}(D) \otimes \mathcal{O}_X(nD)$$

defined in 1.4.84.

Remark 1.5.44. Whenever we use the derivative notation f' in this section, it will always be with respect to the local coordinate at hand, as described in 1.5.42.

Proposition 1.5.45. *Under the hypotheses of 1.5.38, the values of ∇_D on the canonical section 1_{nD} and the basis element $e_n \in \mathcal{O}_X(nD)(V)$ over an affine open $V \subset X$ are given by*

$$\nabla_D(1_{nD}) = 0$$

$$\nabla_D(e_n) = \frac{h'\,\mathrm{d}z}{h} \otimes (-ne_n)$$

Proof. Since the connection ∇_D is restricted from the subsheaf $\mathcal{O}_U \subset \mathcal{K}_X$, we have that $\nabla_D(1_{nD}) = \nabla_D(1) = \mathrm{d}(1) = 0$.

Furthermore

$$0 = \nabla(1_{nD}) = \nabla(h^n e_n) = h^n \nabla(e_n) + \mathrm{d}(h^n) \otimes e_n$$

$$= h^n \nabla(e_n) + (nh^{n-1}\,\mathrm{d}h) \otimes e_n$$

$$\implies \nabla(e_n) = \frac{h^{n-1}\,\mathrm{d}h}{h^n} \otimes (-ne_n) = \frac{\mathrm{d}h}{h} \otimes (-ne_n)$$

$$= \frac{h'\,\mathrm{d}z}{h} \otimes (-ne_n)$$

□

Proposition 1.5.46. *With the hypotheses of 1.5.38, for an affine open $V \subset X$, an arbitrary element in $\mathcal{O}_X(nD)(V)$ has the form fe_n for some $f \in \mathcal{O}_X(V)$, and*

$$\nabla_n(fe_n) = \frac{h'\,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f'\,\mathrm{d}z}{h} \otimes he_n.$$

An arbitrary element in $\mathcal{O}_D(nD)(V) = \mathcal{O}_X(nD)(V)|_D$ has the form $f\overline{e}_n$, and

$$\nabla_n(f\overline{e}_n) = \frac{h'\,\mathrm{d}z}{h} \otimes (-nf\overline{e}_n)$$

Proof. Since e_n is a local basis, over V an arbitrary element of $\mathcal{O}_X(nD)$ is of the form fe_n for some $f \in \mathcal{O}_X(V)$, and then we have the computation.

$$\nabla(fe_n) = f\nabla(e_n) + \mathrm{d}f \otimes e_n$$

$$= f\frac{h'\,\mathrm{d}z}{h} \otimes (-ne_n) + f'\,\mathrm{d}z \otimes e_n$$

$$= \frac{h'\,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f'\,\mathrm{d}z}{h} \otimes he_n$$

$$= \frac{h'\,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f'\,\mathrm{d}z}{h} \otimes e_{n-1}.$$

But $f'\frac{\mathrm{d}z}{h} \otimes e_{n-1}$ is in the image of $\Omega_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big)$

so this is equivalent mod $\Omega_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big)$ to

$$\frac{h' \, \mathrm{d}z}{h} \otimes (-nfe_n)$$

and then the statement follows for $\mathcal{O}_D(nD) = \mathcal{O}_X(nD)/\mathcal{O}_X\big((n-1)D\big)$. $\square$

The following is the first step in establishing that 1.5.7 is a Cartesian square, i.e. that the connection is strictly horizontal (1.4.88).

Proposition 1.5.47. *Let X/S be a smooth arithmetic surface, $D \subset X$ an irreducible horizontal effective Cartier divisor, and $n \in \mathbb{Z}$ an integer which is also a nonzerodivisor in $\mathcal{O}_S$. So either $\mathcal{O}_S$ is characteristic zero, or n coprime to the characteristic.*

Let $\mathcal{O}_X\big((n-1)D\big) \xrightarrow{\lambda} \mathcal{O}_X(nD)$ be the natural inclusion, which induces a horizontal morphism of the connections $\nabla_{n-1} \xrightarrow{\lambda} \nabla_n$:

$$\begin{array}{ccc} \mathcal{O}_X\big((n-1)D\big) & \xrightarrow{\nabla_{n-1}} & \Omega_{X/S}(D) \otimes \mathcal{O}_X\big((n-1)D\big) \\ \Big\downarrow{\scriptstyle\lambda} & & \Big\downarrow{\scriptstyle\mu} \\ \mathcal{O}_X(nD) & \xrightarrow{\nabla_n} & \Omega_{X/S}(D) \otimes \mathcal{O}_X(nD) \end{array} \tag{1.5.7}$$

Then on an affine open $V \subset X$ which trivializes $\mathcal{O}_X(D)$ and $\Omega^1_{X/S}$, the containment

$$\mathrm{im}(\nabla_n \circ \lambda) = \mathrm{im}(\mu \circ \nabla_{n-1}) \subset \mathrm{im}(\nabla_n) \cap \mathrm{im}(\mu)$$

is an equality.

Proof. The morphism $\nabla_{n-1} \xrightarrow{\lambda} \nabla_n$ is depicted by 1.5.7.

Let $V \subset X$ be affine open, and denote by $\theta = \frac{\mathrm{d}z}{h}$ the local basis of $\Omega_{X/S}(D)$ and $\theta \otimes e_n$ the local basis of $\Omega_{X/S}(D) \otimes \mathcal{O}_X(nD)$ mentioned in 1.5.43.

Let $\alpha \in \mathcal{O}_X(nD)(V)$. Then by 1.5.46

$$\alpha = f e_n$$

for some $f \in \mathcal{O}_X(V)$, and again by 1.5.46,

$$\nabla_n(\alpha) = \nabla_n(fe_n) = \frac{h' \,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f' \,\mathrm{d}z}{h} \otimes he_n$$

$$= \frac{h' \,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f' \,\mathrm{d}z}{h} \otimes e_{n-1}.$$

Since

$$\frac{f' \,\mathrm{d}z}{h} \otimes e_{n-1} \in \operatorname{im}(\mu),$$

We have the implications

$$\nabla_n(\alpha) \in \operatorname{im}(\mu) \iff$$

$$\frac{h' \,\mathrm{d}z}{h} \otimes (-nfe_n) \in \operatorname{im}(\mu) \iff$$

$$nfh'e_n \in \mathcal{O}_X\big((n-1)D\big) = h\mathcal{O}_X(nD) \iff$$

$$nfh' \in h\mathcal{O}_X(V) \iff$$

$$nfh' = 0 \in \mathcal{O}_X(V)/(h) \simeq (\mathcal{O}_X|_D)(V) = \mathcal{O}_D(V).$$

Since D is a horizontal divisor, $\mathcal{O}_D$ is a flat $\mathcal{O}_S$-algebra (1.4.50 and 1.4.23). In particular

$\mathcal{O}_D$ is a torsion-free as an $\mathcal{O}_S$-module. Since D irreducible, $\mathcal{O}_D$ contains no zero divisors. Then the localization $\mathcal{O}_D(V)$ is also torsion-free and contains no zero divisors. This implies the image of f is zero in $\mathcal{O}_D(V)$, so $f \in h\mathcal{O}_X(V)$, and so $\alpha \in \operatorname{im}(\lambda)$. □

In order to conclude that the connection is strictly horizontal, we first make the following observation:

Proposition 1.5.48. *In addition to the hypotheses of 1.5.38, assume also that X is geometrically connected, and that S is the spectrum of a number ring. If $n > 0$, then $\ker(\nabla_{n-1}) = \ker(\nabla_n)$, i.e. all ∇_n have the same constants, when n is positive.*

Proof. By 1.5.17, whenever $\mathcal{O}_X(nD)$ contains $\mathcal{O}_X$ (i.e. $n \geq 0$), the constants will be the $\mathcal{O}_S$-submodule generated by $1 \in \Gamma(X, \mathcal{O}_X)$.

The geometrically connected condition ensures that the constants consist only of $\mathcal{O}_S$ and not some algebraic extension of $\mathcal{O}_S$. □

Corollary 1.5.49. With the hypotheses of 1.5.38, assume also that X is geometrically connected and S is the spectrum of a number ring. When $n > 0$, the connection $\nabla_{n-1} \xrightarrow{\lambda} \nabla_n$ is strictly horizontal, i.e. the diagram 1.5.7 is Cartesian.

Proof. By 1.3.4 and 1.4.9, in order to show that the square 1.5.7 of $\mathcal{O}_X$-modules is Cartesian it suffices to check that the following complex is left exact over every affine in a given cover:

$$0 \longrightarrow \mathcal{O}_X\big((n-1)D\big) \xrightarrow{\alpha} \mathcal{O}_X(nD) \oplus \Big(\Omega_{X/S}(D) \otimes \mathcal{O}_X\big((n-1)D\big)\Big) \xrightarrow{\beta} \Omega_{X/S}(D) \otimes \mathcal{O}_X(nD)$$

where $\alpha := (\lambda, \nabla_{n-1}), \beta := (\nabla_n, -\mu)$.

We can obtain the result by using 1.3.8, which applies because 1.5.47 establishes the image containment property, 1.5.48 establishes the isomorphism of the kernels, and lastly the morphism $\mathcal{O}_X\big((n-1)D\big)(V) \to \mathcal{O}_X(nD)(V)$ is injective.

Therefore the diagram sequence is exact and 1.5.7 is Cartesian. $\square$

Remark 1.5.50. As stated in 1.4.91, the image containment property of 1.5.47 is the primary consequence of the Cartesian property that we will use.

1.5.7.3 Cohomology of the quotient connection $\overline{\nabla}_n$

Proposition 1.5.51. *With the hypotheses of 1.5.38, let $h \in \mathcal{O}_X(-D)(V)$ be a function locally defining the Cartier divisor D, and let $z \in \mathcal{O}_X(V)$ be the local parameter described in 1.5.42.*

Then $\mathrm{d}h = h' \,\mathrm{d}z$ for some $h' \in \mathcal{O}_X(V)$, and h' locally generates the different ideal $\mathcal{D}_D \subset \mathcal{O}_D$ defined in 1.4.66.

Proof. The different ideal $\mathcal{D}_D$ of $\mathcal{O}_D$ can be defined as the 0^{th} Fitting ideal of $\Omega_{D/X}$: [14, Tag 0BVW], [14, Tag 0BWG], As $D|_V$ is a Cartier divisor defined by h, the function h' generates the relations of $\Omega_{D/S}(V)$ and hence the 0^{th} Fitting ideal. $\square$

Corollary 1.5.52 (Image of quotient connection)**.** Over an affine open V, the image of

$$\overline{\nabla}_n : \mathcal{O}_D(nD) \to \Omega_{X/S}(D) \otimes \mathcal{O}_D(nD)$$

is given by

$$\mathrm{im}(\overline{\nabla}_n) = \begin{cases} 0 & n = 0, \\ n\mathcal{D}_D \cdot \left(\frac{\mathrm{d}z}{h} \otimes \overline{e}_n\right) & n > 0. \end{cases}$$

where $\frac{\mathrm{d}z}{h} \otimes \overline{e}_n$ is a generator of $\Omega_{X/S}(D) \otimes \mathcal{O}_D(nD)$ over V.

Proof. From 1.5.46 we have that

$$\nabla(fe_n) = \frac{h' \,\mathrm{d}z}{h} \otimes (-nfe_n) + \frac{f' \,\mathrm{d}z}{h} \otimes e_{n-1}$$

and $\frac{f' \,\mathrm{d}z}{h} \otimes e_{n-1}$ is a generator of $\Omega_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big)$ over V so this is equivalent modulo $\Omega_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big)$ to

$$\frac{h' \,\mathrm{d}z}{h} \otimes (-nfe_n) = nh'f\left(\frac{\mathrm{d}z}{h} \otimes (-e_n)\right)$$

But by 1.5.51, h' generates the different ideal $\mathcal{D}_D(V) \subset \mathcal{O}_D(V)$, so nh' generates $n\mathcal{D}_D(V)$.

Since f is an arbitrary element of $\mathcal{O}_D(V)$, the result follows.

□

Corollary 1.5.53 (Kernel of quotient connection)**.** Let X/S be a smooth arithmetic surface and $D \subset X$ an irreducible horizontal effective Cartier divisor. Assume X is geometrically connected and S is the spectrum of a number ring.

For $n > 0$, the kernel of

$$\overline{\nabla}_n : \mathcal{O}_D(nD) \to \Omega_{X/S}(D) \otimes \mathcal{O}_D(nD)$$

is given by

$$\ker(\overline{\nabla}_n) = \begin{cases} \mathcal{O}_D(nD) & n = 0, \\ 0 & n > 0. \end{cases}$$

So $\overline{\nabla}_n$ is zero when $n = 0$ and injective when $n > 0$.

Proof. From 1.5.46, an arbitrary element in $\mathcal{O}_D(nD)(V) = \mathcal{O}_X(nD)(V)|_D$ has the form $f\overline{e}_n$, and

$$\nabla_n(f\overline{e}_n) = \frac{h' \,\mathrm{d}z}{h} \otimes -nf\overline{e}_n$$

$$= -nfh' \left(\frac{\mathrm{d}z}{h} \otimes \overline{e}_n\right)$$

where $\frac{\mathrm{d}z}{h} \otimes \overline{e}_n$ is an $\mathcal{O}_D(V)$-generator of $\left(\Omega^1_{X/S}(D) \otimes \mathcal{O}_D(nD)\right)(V)$ (1.5.43).

Since X/S is smooth, $\Omega_{X/S}$ and $\Omega_{X/S}(nD)$ are torsion-free $\mathcal{O}_X$-modules. Since D is irreducible, $\mathcal{O}_D$ is an integral domain, so $\Omega_{X/S}(nD) \otimes \mathcal{O}_D$ is a torsion-free $\mathcal{O}_D$-module

Hence $-nfh' \left(\frac{\mathrm{d}z}{h} \otimes \overline{e}_n\right)$ can only be zero if n or f is zero in $\mathcal{O}_D(V)$, but we have assume $n > 0$ and S is characteristic zero, so f must be zero. Therefore $\overline{\nabla}_n$ is injective.

□

Corollary 1.5.54 (Cohomology sheaves of quotient connection)**.** Let X/S be a smooth arithmetic surface and $D \subset X$ an irreducible horizontal effective Cartier divisor. Assume X is geometrically connected and S is the spectrum of a number ring. Let $n > 0$ be an integer. Then

$$\mathcal{H}^i\left(X/S; \mathcal{O}_D(nD), \overline{\nabla}_n\right) \cong \begin{cases} 0 & i = 0 \\ \mathcal{O}_D/(n\mathcal{D}_D) & i = 1 \\ 0 & i = 2 \end{cases}$$

Proof.

$$\mathcal{H}^0\left(X/S; \mathcal{O}_D(nD), \overline{\nabla}_n\right) := \ker\left[\mathcal{O}_D(nD) \xrightarrow{\overline{\nabla}_n} \Omega_{X/S}(D) \otimes \mathcal{O}_D(nD)\right]$$

$$\mathcal{H}^1\left(X/S;\mathcal{O}_D(nD),\overline{\nabla}_n\right) := \operatorname{coker}\left[\mathcal{O}_D(nD) \xrightarrow{\overline{\nabla}_n} \Omega_{X/S}(D)\otimes\mathcal{O}_D(nD)\right]$$

So the case $i=0$ follows from the computation of the kernel in 1.5.53, and the case $i=1$ follows from the computation of the image in 1.5.52.

Since the de Rham complex $\Omega^\bullet_{\overline{\nabla}_n}$ has only two terms, concentrated in degrees 0,1, the de Rham cohomology sheaves vanish in all other degrees.

□

Corollary 1.5.55 (Cohomology of quotient connection)**.** Let X/S be a smooth arithmetic surface and $D \subset X$ an irreducible horizontal effective Cartier divisor. Assume X is geometrically connected and S is the spectrum of a number ring. Let $n>0$ be an integer. Then

$$\mathrm{H}^i\left(X/S;\mathcal{O}_D(nD),\overline{\nabla}_n\right) \cong \begin{cases} 0 & i=0 \\ \pi_*\mathcal{O}_D/(n\mathcal{D}_D) & i=1 \\ 0 & i=2 \end{cases}$$

Proof. This follows from the computation of the cohomology sheaves in 1.5.54 and the fact that the sheaves $\mathcal{O}_D(nD)$ and $\Omega_{X/S}(D)\otimes\mathcal{O}_D(nD)$ are acyclic for $R\pi_*$, as follows:

Consider the seven term exact sequence $\mathcal{S}^\bullet_{\overline{\nabla}}$ from 1.5.26:

$$\mathrm{H}^0(X/S;\overline{\nabla}) \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathrm{H}^1(X/S;\overline{\nabla}) \longrightarrow \mathbf{R}^1\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^1\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathrm{H}^2(X/S;\overline{\nabla})$$

where

$$\overline{\mathcal{L}} = \mathcal{O}_D(nD),$$

$$\Omega^1_{\overline{\nabla}} = \Omega_{X/S}(D) \otimes \mathcal{O}_D(nD).$$

(We omit the zeros on the end for space.)

Since the sheaves $\mathcal{O}_D(nD)$ and $\Omega_{X/S}(D)\otimes\mathcal{O}_D(nD)$ are supported on the horizontal divisor D, by 1.4.55 they are acyclic for the direct image functor π_*.

So the seven term exact sequence simplifies to

$$0 \longrightarrow \mathrm{H}^0(X/S;\overline{\nabla}) \longrightarrow \mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}} \longrightarrow \mathrm{H}^1(X/S;\overline{\nabla}) \longrightarrow 0$$

Thus the cohomology modules $\mathrm{H}^i(X/S;\overline{\nabla})$ can be calculated simply as the kernel and cokernel of the map

$$\mathbf{R}^0\pi_*\overline{\mathcal{L}} \longrightarrow \mathbf{R}^0\pi_*\Omega^1_{\overline{\nabla}}.$$

□

1.5.8 Direct limits of connections

Proposition 1.5.56. *Let $\{\mathcal{F}_n\}_{n\geq 0}$ be a direct system of quasicoherent $\mathcal{O}_X$-modules with injective transition maps, and suppose that each $\mathcal{F}_n$ has an Ω-valued S-connection ∇_n, which is compatible with the maps in the direct system, i.e. the induced morphisms of connections are horizontal, or equivalently*

$$\begin{array}{ccc} \mathcal{F}_n & \xrightarrow{\nabla_n} & \Omega\otimes_{\mathcal{O}_X}\mathcal{F}_n \\ \downarrow & & \downarrow \\ \mathcal{F}_{n+1} & \xrightarrow{\nabla_{n+1}} & \Omega\otimes_{\mathcal{O}_X}\mathcal{F}_{n+1} \end{array}$$

commutes.

Let $\mathcal{F} = \lim\limits_{n\to} \mathcal{F}_n$ *be the direct limit. Then* $\mathcal{F}$ *has a unique* Ω*-valued connection* ∇ *compatible with the* $\mathcal{F}_n$*, and*

$$\mathrm{H}^\bullet(X/S; \mathcal{F}, \nabla) = \mathrm{H}^\bullet(X/S; \lim_{n\to} \mathcal{F}_n, \nabla_n) \cong \lim_{n\to} \mathrm{H}^\bullet(X/S; \mathcal{F}_n, \nabla_n).$$

Proof. Filtered colimits commute with tensor products ([14, Tag 00DD]), so

$$\lim_{n\to} (\Omega \otimes \mathcal{F}_n) \simeq \Omega \otimes \left(\lim_{n\to} \mathcal{F}_n\right) = \Omega \otimes \mathcal{F}.$$

Each $\mathcal{F}_n$ has a seven term exact sequence $\mathcal{S}^\bullet_{\nabla_n}$ (1.5.20). Let $\mathcal{S}^\bullet = \lim\limits_{n\to} \mathcal{S}^\bullet_{\nabla_n}$ be the limit sequence.

Filtered colimits preserve exactness ([14, Tag 00DB]), so the limit sequence $\mathcal{S}^\bullet$ is exact.

Filtered colimits commute with higher direct images ([14, Tag 07TA]), so the limit sequence $\mathcal{S}^\bullet$ is the same as the seven term exact sequence $\mathcal{S}^\bullet_\nabla$ for the limit $\mathcal{F}$.

Then $\mathcal{S}^\bullet = \mathcal{S}^\bullet_\nabla$ computes the cohomology of the connection ∇ on $\mathcal{F}$, and

$$\mathrm{H}^i(X/S; \mathcal{F}, \nabla) \simeq \lim_{n\to} \mathrm{H}^i(X/S; \mathcal{F}_n, \nabla_n)$$

for each i. □

1.6 Cohomology of affine complements

For this section, let S be the spectrum of a number ring, $\pi : X \to S$ be an arithmetic surface of genus 0 which is smooth, geometrically connected, and cohomologically flat, and $D \subset X$ an effective horizontal divisor (1.4.43). Let $U = X \setminus D \xrightarrow{j} X$ be the inclusion of the complement.

Assume also that D is irreducible and adequate (1.5.27).

The sheaf $j_*\mathcal{O}_U$ has a connection ∇_D which restricts to the connections ∇_D on the line bundles $\mathcal{O}_X(nD)$ (1.4.84).

By 1.4.44, 1.4.50 , D/S is a finite, flat extension, and so there is a different ideal $\mathcal{D}_D \subset \mathcal{O}_D$ (1.4.66), which is supported on the ramification locus of D/S (1.4.70).

Proposition 1.6.1. *The de Rham cohomology of the complement U/S can be computed on X via:*

$$\mathrm{H}^\bullet(U/S; \mathcal{O}_U, \mathrm{d}) \cong \mathrm{H}^\bullet(X/S; j_*\mathcal{O}_U, \mathrm{d}) \cong \mathrm{H}^\bullet(X/S; j_*\mathcal{O}_U, \nabla_D)$$

Proof. The first isomorphism holds because $j : U \hookrightarrow X$ is an affine morphism, so j_* is an exact functor, and $\mathbf{R}^\bullet(\pi \circ j)_*\mathcal{O}_U = \mathbf{R}^\bullet\pi_*(j_*\mathcal{O}_U)$. (1.4.16) Then the seven term exact sequences 1.5.20 for $\mathcal{O}_U$ and $j_*\mathcal{O}_U$ are isomorphic as complexes of $\mathcal{O}_S$-modules.

The second isomorphism holds because $j_*\mathcal{O}_U$ and $\mathcal{O}_X(D) \otimes j_*\mathcal{O}_U$ are canonically isomorphic, so

$$\Omega_{X/S}(D) \otimes j_*\mathcal{O}_U \simeq \Omega_{X/S} \otimes \mathcal{O}_X(D) \otimes j_*\mathcal{O}_U \simeq \Omega_{X/S} \otimes j_*\mathcal{O}_U,$$

and so the connections

$$\mathrm{d} : j_*\mathcal{O}_U \to \Omega_{X/S} \otimes j_*\mathcal{O}_U$$

$$\nabla_D : j_*\mathcal{O}_U \to \Omega_{X/S}(D) \otimes j_*\mathcal{O}_U$$

are isomorphic via the natural isomorphism $\Omega_{X/S} \otimes j_*\mathcal{O}_U \to \Omega_{X/S}(D) \otimes j_*\mathcal{O}_U$. □

Definition 1.6.2 (Polar filtration on H^1)**.** The ascending filtration on $\mathcal{O}_U$ given by

$$\mathrm{Fil}^n\mathcal{O}_U := \mathcal{O}_X(nD), \quad n \geq 0$$

induces an ascending filtration on $\mathrm{H}^1\left(X/S;\mathcal{O}_U\right)$ given by

$$\mathrm{Fil}^n\,\mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right) := \mathrm{im}\left[\mathrm{H}^1\left(X/S;\mathcal{O}_X(nD);\nabla_D\right) \to \mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right)\right].$$

Furthermore we extend the filtration to indices $-1,-2$ by setting

$$\mathrm{Fil}^{-1}\,\mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right) := \mathrm{im}\left[\mathrm{H}^1\left(X/S;\mathcal{O}_X,\mathrm{d}\right) \to \mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right)\right].$$

$$\mathrm{Fil}^{-2}\,\mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right) := 0.$$

Remark 1.6.3. We can depict the filtration with the following diagram, using the notation of 1.4.84:

$$\begin{array}{ccc}
\mathcal{O}_X & \xrightarrow{\mathrm{d}=\nabla_{-1}} & \Omega_{X/S}\otimes\mathcal{O}_X \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle i\otimes\mathrm{id}} \\
\mathcal{O}_X & \xrightarrow{\nabla_{D,0}} & \Omega_{X/S}(D)\otimes\mathcal{O}_X \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}\otimes i} \\
\mathcal{O}_X(D) & \xrightarrow{\nabla_{D,1}} & \Omega_{X/S}(D)\otimes\mathcal{O}_X(D) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}\otimes i} \\
\mathcal{O}_X(2D) & \xrightarrow{\nabla_{D,2}} & \Omega_{X/S}(D)\otimes\mathcal{O}_X(2D) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}\otimes i} \\
\cdots & & \cdots \\
\downarrow & & \downarrow \\
j_*\mathcal{O}_U & \xrightarrow{\nabla_D} & \Omega_{X/S}(D)\otimes j_*\mathcal{O}_U.
\end{array}$$

where in the vertical arrows, id is the identity and i is the natural inclusion. So each vertical arrow is an inclusion.

By 1.5.33 and 1.5.34, each step of this filtration induces an injection on first de Rham cohomology $\mathrm{H}^1(X/S;*)$.

Remark 1.6.4. Note how the connections in the -1^{st} and 0^{th} steps have the same coefficient sheaf but different value sheaves:

$$\mathrm{Fil}^{-1}\,\mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right) := \mathrm{im}\left[\mathrm{H}^1\left(X/S;\mathcal{O}_X,\mathrm{d}\right) \to \mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right)\right].$$

$$\mathrm{Fil}^{0}\,\mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right) := \mathrm{im}\left[\mathrm{H}^1\left(X/S;\mathcal{O}_X;\nabla_D\right) \to \mathrm{H}^1\left(X/S;\mathcal{O}_U,\nabla_D\right)\right].$$

Fil^{-1} is the usual de Rham cohomology of X, and Fil^0 is what is often referred to as the logarithmic de Rham cohomology of X (with respect to D).

If we were working over a field of characteristic zero, $S = \mathrm{Spec}(k)$, then the filtration would stabilize already at the 0^{th} step. But we are "fortunate" that things are more interesting over Dedekind domains.

Remark 1.6.5 (The conditions our connections must satisfy).

The polar filtration (1.6.2) consists of two types of connections (foreshadowed in 1.5.4):

$$\begin{array}{ccc} \mathcal{O}_X\big((n-1)D\big) & \xrightarrow{\nabla_{D,n-1}} & \Omega\otimes\mathcal{O}_X\big((n-1)D\big) \\ \downarrow\lambda & & \downarrow \\ \mathcal{O}_X(nD) & \xrightarrow{\nabla_{D,n}} & \Omega\otimes\mathcal{O}_X(nD) \end{array} \qquad \begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\mathrm{d}} & \Omega_{X/S} \\ \downarrow & & \downarrow\eta \\ \mathcal{O}_X & \xrightarrow{\nabla_D} & \Omega_{X/S}(D). \end{array}$$

We referred to the type on the left as an extension of coefficient sheaves (analyzed in 1.5.31) or more specifically a strictly horizontal extension of coefficient sheaves (analyzed in 1.5.33), and this is the type of morphism that makes up the positively indexed portion of the polar filtration:

$$\operatorname{Gr}^n \mathrm{H}^1(X/S; \mathcal{O}_U),\ n > 0.$$

We referred to the type on the right as an extension of value sheaves (analyzed in 1.5.34), and this is the type of morphism that makes up

$$\operatorname{Gr}^0 \mathrm{H}^1(X/S; \mathcal{O}_U).$$

In order to apply the result on strictly horizontal morphisms (1.5.33), we need:

1. The map of sheaves λ is injective with cokernel $\operatorname{coker}(\lambda)$ supported on a horizontal divisor,
2. The induced morphism of connections $\nabla_{D,n-1} \xrightarrow{\lambda} \nabla_{D,n}$ is strictly horizontal,
3. Either
   - $\mathcal{O}_X\big((n-1)D\big), \mathcal{O}_X(nD)$ both adequate, or
   - $\mathcal{O}_X(nD)$ adequate and

$$\mathbf{R}^1\pi_*\mathcal{O}_X\big((n-1)D\big) = \mathbf{R}^1\pi_*\Omega_{X/S}(D) \otimes \mathcal{O}_X\big((n-1)D\big) = 0.$$

Note that the condition for $\mathcal{O}_X\big((n-1)D\big)$ to be adequate is the similar looking:

$$\mathbf{R}^1\pi_*\mathcal{O}_X\big((n-1)D\big) = \mathbf{R}^1\pi_*\Omega_{X/S} \otimes \mathcal{O}_X\big((n-1)D\big) = 0.$$

For the result on extension of value sheaves (1.5.34), we need:

1. The map of sheaves $\Omega_{X/S} \xrightarrow{\eta} \Omega_{X/S}(D)$ is injective with horizontally supported cokernel,

2. The induced morphism of connections $\mathrm{d} \xrightarrow{\eta} \nabla_D$ is horizontal,

3. The Hodge-to-de Rham spectral sequence for d degenerates at the page E_1.

We will show these conditions are satisfied in the following propositions.

Proposition 1.6.6. *Assume the hypotheses stated at the beginning of the section (1.6), in particular X is a smooth arithmetic surface of genus 0.*

Suppose $n > 0$. The morphism of connections

$$\begin{array}{ccc} \mathcal{O}_X\big((n-1)D\big) & \xrightarrow{\nabla_{D,n-1}} & \Omega \otimes \mathcal{O}_X\big((n-1)D\big) \\ \downarrow \lambda & & \downarrow \\ \mathcal{O}_X(nD) & \xrightarrow{\nabla_{D,n}} & \Omega \otimes \mathcal{O}_X(nD) \end{array}$$

satisfies the conditions of 1.5.33, which were recounted in 1.6.5 above.

Proof. 1. The twisted ideal sequence

$$0 \to \mathcal{O}_X\big((n-1)D\big) \xrightarrow{\lambda} \mathcal{O}_X(nD) \to \mathcal{O}_D(nD) \to 0$$

demonstrates that λ is injective with cokernel supported on the horizontal divisor D.

2. The map $\nabla_{D,n-1} \xrightarrow{\lambda} \nabla_{D,n}$ is strictly horizontal by 1.5.49.

3. If $n > 1$, then both sheaves are adequate, by the assumption on the divisor D.

If $n = 1$, the sheaf $\mathcal{O}_X(nD) = \mathcal{O}_X(D)$ is again adequate by assumption on D, but $\mathcal{O}_X\big((n-1)D\big) = \mathcal{O}_X$ is not adequate. However when X is genus 0 and cohomologically flat, $\mathcal{O}_X$ only acyclic: $\mathbf{R}^1\pi_*\mathcal{O}_X = 0$. Additionally, in 1.5.30 it was shown that if $\mathcal{L}(D)$ is adequate, then $\Omega_{X/S}(D) \otimes \mathcal{L}$ is acyclic, so $\mathbf{R}^1\pi_*\Omega_{X/S}(D) \otimes \mathcal{L} = 0$.

□

Proposition 1.6.7. *Assume the hypotheses stated at the beginning of the section (1.6), in particular X/S is a smooth arithmetic surface, which is cohomologically flat.*

The morphism of connections

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{\ \mathrm{d}\ } & \Omega_{X/S} \\ \big\downarrow & & \big\downarrow \eta \\ \mathcal{O}_X & \xrightarrow{\nabla_D} & \Omega_{X/S}(D). \end{array}$$

satisfies the conditions of 1.5.34, which were recounted in 1.6.5 above.

Proof. 1. By tensoring the ideal sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

with the locally free sheaf $\Omega_{X/S}(D)$, we get that the map $\Omega_{X/S} \xrightarrow{\eta} \Omega_{X/S}(D)$ is injective with cokernel supported on the horizontal divisor D.

2. Since the map of connections $\mathrm{d} \xrightarrow{\eta} \nabla_D$ is induced by the natural inclusion $\Omega_{X/S} \xrightarrow{\eta} \Omega_{X/S}(D)$, the connection $\mathcal{O}_X \xrightarrow{\nabla_D} \Omega_{X/S}(D)$ factors through the connection $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/S}$. Therefore the map $\mathrm{d} \xrightarrow{\eta} \nabla_D$ is horizontal.

3. Since X/S is a smooth arithmetic surface with X cohomologically flat, 1.5.9 implies that the Hodge-to-de Rham spectral sequence for $\mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega_{X/S}$ degenerates at the page E_1.

□

Proposition 1.6.8. *The steps of the filtration are isomorphic images:*

$$\operatorname{Fil}^n \mathrm{H}^1\left(X/S; \mathcal{O}_U, \nabla_D\right) \cong \mathrm{H}^1\left(X/S; \mathcal{O}_X(nD), \nabla_D\right)$$

$$\operatorname{Fil}^{-1} \mathrm{H}^1\left(X/S; \mathcal{O}_U, \nabla_D\right) \cong \mathrm{H}^1\left(X/S; \mathcal{O}_X, \mathrm{d}\right)$$

Proof. By 1.6.6 and 1.6.7, we may apply 1.5.33 and 1.5.34, which imply that the morphisms defining the filtration 1.6.2 on H^1

$$\mathrm{H}^1\left(X/S; \mathcal{O}_X, \mathrm{d}\right) \to \mathrm{H}^1\left(X/S; \mathcal{O}_X, \nabla_D\right)$$

$$\mathrm{H}^1\left(X/S; \mathcal{O}_X\big((n-1)D\big), \nabla_{D,n-1}\right) \to \mathrm{H}^1\left(X/S; \mathcal{O}_X(nD), \nabla_{D,n}\right)$$

are all injective. Therefore they are isomorphic to their images in $\mathrm{H}^1(X/S; \mathcal{O}_U, \nabla_D$. □

Remark 1.6.9. If X has positive genus, we still get an ascending filtration, since each connection factors through the next, but the images are not necessarily isomorphic.

Proposition 1.6.10. *The limit of the polar filtration commutes with cohomology:*

$$\mathrm{H}^\bullet(X/S; \mathcal{O}_U, \nabla) \cong \lim_{n\to} \mathrm{H}^\bullet(X/S; \mathcal{O}_X(nD), \nabla).$$

Proof. This follows from $\mathcal{O}_U = \lim\limits_{n\to} \mathcal{O}_X(nD)$ and the fact that de Rham cohomology commutes with limits: 1.5.56. ☐

Proposition 1.6.11. *The graded pieces* $\mathrm{Gr}^n\,\mathrm{H}^1\left(X/S;\mathcal{O}_U\right) := \mathrm{Fil}^n/\mathrm{Fil}^{n-1}$ *have the following description via the connection 1.4.81 on* $\mathcal{O}_D(nD)$

$$\mathrm{Gr}^0\,\mathrm{H}^1(X/S;\mathcal{O}_U) \cong \mathrm{coker}\left[\mathbf{R}^0\pi_*\Omega_{X/S} \to \mathbf{R}^0\pi_*\Omega_{X/S}(D)\right]$$

$$\mathrm{Gr}^n\,\mathrm{H}^1(X/S;\mathcal{O}_U) \cong \mathrm{H}^1\left(X/S;\mathcal{O}_D(nD),\overline{\nabla}\right),\quad n>0$$

Proof. Using 1.5.33 with the morphism $\nabla_{D,n-1} \xrightarrow{\lambda} \nabla_{D,n}$ we obtain the sequence

$$0 \to \mathrm{H}^1(X/S;\mathcal{O}_X\big((n-1)D\big),\nabla_{D,n-1}) \to \mathrm{H}^1(X/S;\mathcal{O}_X(nD),\nabla_{D,n}) \to \mathrm{H}^1(X/S;\mathcal{O}_D(nD),\overline{\nabla}n) \to 0$$

which by definition is the same as

$$0 \longrightarrow \mathrm{Fil}^{n-1}\,\mathrm{H}^1(X/S;\mathcal{O}_U,\nabla_D) \longrightarrow \mathrm{Fil}^n\,\mathrm{H}^1(X/S;\mathcal{O}_U,\nabla_D) \longrightarrow \mathrm{Gr}^n\,\mathrm{H}^1(X/S;\mathcal{O}_U,\nabla_D) \longrightarrow 0$$

The degree 0 graded portion $\mathrm{Gr}^0\,\mathrm{H}^1(X/S;\mathcal{O}_U)$ follows from 1.5.36. The analysis for this special case is carried out in the subsequent remark 1.5.37. ☐

Proposition 1.6.12. *We have the more concrete description of the graded pieces:*

$$\mathrm{Gr}^0\,\mathrm{H}^1(X/S;\mathcal{O}_U) \cong \pi_*\Omega_{X/S}(D)$$

$$\mathrm{Gr}^n\,\mathrm{H}^1(X/S;\mathcal{O}_U) \cong \pi_*\left(\mathcal{O}_D/(n\mathcal{D}_D)\right),\quad n>0$$

Proof. When $n > 0$, we compose the isomorphism

$$\mathrm{Gr}^n \mathrm{H}^1(X/S; \mathcal{O}_U) \cong \mathrm{H}^1\left(X/S; \mathcal{O}_D(nD), \overline{\nabla}\right)$$

shown in 1.6.11 and the isomorphism

$$\mathrm{H}^1\left(X/S; \mathcal{O}_D(nD), \overline{\nabla}\right) \cong \pi_* \mathcal{O}_D/(n\mathcal{D}_D)$$

shown in 1.5.55.

For the $n = 0$ case, 1.6.11 shows that

$$\mathrm{Gr}^0 \mathrm{H}^1(X/S; \mathcal{O}_U) \cong \operatorname{coker}\left[\mathbf{R}^0\pi_*\Omega_{X/S} \to \mathbf{R}^0\pi_*\Omega_{X/S}(D)\right]$$

Since we have assumed the genus of X is zero, we have $\mathbf{R}^0\pi_*\Omega_{X/S} = 0$, and this implies the case $n = 0$:

$$\mathrm{Gr}^0 \mathrm{H}^1(X/S; \mathcal{O}_U) \cong \mathbf{R}^0\pi_*\Omega_{X/S}(D).$$

□

Theorem 1.6.13. *The associated graded of* $\mathrm{Fil}^\bullet \mathrm{H}^1(X/S; \mathcal{O}_U)$ *satisfies*

$$\mathrm{Gr}\, \mathrm{H}^1(X/S; \mathcal{O}_U) \cong \mathrm{H}^1(X/S) \oplus \pi_*\Omega_{X/S}(D) \oplus \bigoplus_{n \geq 1} \pi_*\left(\mathcal{O}_D/(n\mathcal{D}_D)\right)$$

Proof. This follows from the concrete description in 1.6.12 and the fact that by definition $\mathrm{Gr}^{-1} \mathrm{H}^1(X/S; \mathcal{O}_U) = \mathrm{H}^1(X/S)$, as discussed in 1.6.4. □

1.7 Example

Here we apply the results of the previous section to the example 1.2.1 from the introduction. There the first de Rham cohomology of the affine line over a ring R was given as

$$\mathrm{H}^1(\mathbb{A}^1/R) \cong \bigoplus_{n\geq 1} R/nR.$$

Let R be a number ring, $S = \mathrm{Spec}(R)$, and $X = \mathbb{P}^1_S \xrightarrow{\pi} S$. An integral point $S \to X$ corresponds to a horizontal effective Cartier divisor $D \subset X$ of degree 1, and $U := X \setminus D \cong \mathbb{A}^1_S$.

The induced morphism $D \to S$ is an isomorphism, in particular étale, so the different ideal is the unit ideal: $\mathcal{D}_D = (1) \subset \mathcal{O}_D$. Therefore we have

$$\pi_*\big(\mathcal{O}_D/(n\mathcal{D}_D)\big) = \pi_*\big(\mathcal{O}_D/n\mathcal{O}_D\big) = R/nR \tag{1.7.1}$$

The sheaf cohomology of the structure sheaf and sheaf of differentials satisfy

$$\mathrm{H}^1(X, \mathcal{O}_X) = 0$$

$$\mathrm{H}^0(X, \Omega^1_{X/S}) = 0$$

so by 1.4.96 we obtain the vanishing of the first de Rham cohomology:

$$\mathrm{H}^1(X/S) = 0. \tag{1.7.2}$$

Since

$$\deg\left(\Omega_{X/S}(D)\right) = \deg\left(\Omega_{X/S}\right) + \deg D = -2 + 1 = -1 < 0$$

we have

$$\pi_*\Omega_{X/S}(D) = 0. \tag{1.7.3}$$

Now we apply the theorem 1.6.13, using 1.7.2, 1.7.3, 1.7.1:

$$\begin{aligned}
\operatorname{Gr} \mathrm{H}^1(X/S; \mathcal{O}_U) &\cong \mathrm{H}^1(X/S) \oplus \pi_*\Omega_{X/S}(D) \oplus \bigoplus_{n\geq 1} \pi_*\left(\mathcal{O}_D/(n\mathcal{D}_D)\right) \\
&\cong 0 \oplus 0 \oplus \bigoplus_{n\geq 1} R/nR \\
&\cong \bigoplus_{n\geq 1} R/nR.
\end{aligned}$$

It turns out in the case when the divisor D is degree 1, the cohomology $\mathrm{H}^1(U/S)$ is actually isomorphic to its associated graded.

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