Thesis Report

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Zero-Crossing Rates of Some Non-Gaussian Processes with Application to Detection and Estimation

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Abstract

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In this dissertation we present extensions of Rice's formula for the expected zero-crossing rate of a Gaussian process to some useful non-Gaussian cases. In particular, we extend Rice's formula to the class of stationary processes which are a monotone transformation of a Gaussian process, to countable mixtures of Gaussians, and to products of independent Gaussian processes. In all the above mentioned cases the expected zero-crossing rates are given for both continuous time and discrete time processes. We also investigate the application of parametric filtering, using zero-crossing count statistics, to the problem of frequency estimation in a mixed spectrum model and the application of mean-level-crossing counts of the envelope of a Gaussian process to a radar detection problem. For the radar problem we prove asymptotic normality of the level-crossings of the envelope of a Gaussian process and provide and expression for the asymptotic variance.

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by

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Dedication

To my wife Deborah Ann and our son Steven Thomas

for their love and patience.

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Chapter 1

Introduction

1.1 Rice's Formula

The origin of Rice's formula for the average level-crossing rate of a general class of random processes can be traced back to his 1936 notes on "Singing Transmission Lines," (see Rainal 1988). This celebrated formula and the basic mathematical techniques derived from Rice's analysis have been used in numerous other related problems such as first passage times, FM fading, and frequency estimation.

Oscillation as observed in time-series is ubiquitous. Simply considering a centered pure sinusoid, we see that there are two zero-crossings per cycle. This intimate connection between zero-crossings and frequency content will be the starting point of this thesis. We will see how zero-crossing counts and higher-order-crossing counts can be an efficient tool for performing discrete frequency estimation - competitive in accuracy and speed with the renowned Cooley-Tukey FFT algorithm.

After this illustrative example of a zero-crossings based frequency estimator we present extensions of Rice's formula to functions of Gaussian processes

(Chapter 2) and then to mixtures and products of Gaussian (Chapter 3) and answer a question regarding maximal level-crossing rates when the spectrum is specified. In the last chapter we present an application of a level-crossing based detector to a radar detection problem.

1.2 Discrete Frequency Estimation via Zero-Crossings

Several techniques have been investigated which use parametric families of linear filters for discrete frequency estimation. The proposed methods are similar in that they use iterative filtering procedures for estimating the frequencies of underlying periodic components embedded in noise. In this chapter we present a technique that combines parametric filtering with a contraction mapping principle to recursively estimate the frequencies of discrete spectral components. By incorporating the contraction mapping idea with parametric filtering a *fundamental property* is determined which when satisfied, guarantees the convergence of the iterative procedure. Several examples are provided which illustrate the method.

Frequency estimation is a classic problem in time series analysis. Aside from the purely mathematical interest of the problem, there are a number of engineering systems that require precise discrete frequency estimation. Communications systems, sonar receivers, and nuclear magnetic resonance spectroscopy devices are such examples.

For almost a hundred years the periodogram has been widely used for spectral estimation and analysis. The fast Fourier transform (FFT), which is an efficient algorithm for evaluating the periodogram at the Fourier frequencies, has helped to sustain the popularity of this important tool. However, over the last decade a number of authors have suggested iterative filtering techniques for discrete frequency estimation (see Dragošević and Stanković 1989, He and Kedem 1989, Kay 1984, Li and Kedem 1993, Mataušek et al. 1983, Troendle 1991 and Yakowitz 1991). Although there are similarities in the various methods, an important and notable aspect of the He-Kedem work is a so-called *fundamental property* required of the parametric filter family which guarantees convergence of the frequency estimates. As we will show, a number of parametric filter families can be defined which satisfy this property.

A useful mathematical model, as well as the one we will use for this example, is the following mixed spectrum stationary process,

$$Z_t = \sum_{j=1}^{p}(A_j \cos(\omega_j t) + B_j \sin(\omega_j t)) + \zeta_t \tag{1.1}$$

where, $t = 0, \pm 1, \pm 2, \cdots$, the A's and B's are all uncorrelated, $E(A_j) = E(B_j) = 0$, and $Var(A_j) = Var(B_j) = \sigma_j^2$. In general, one assumes $\{\zeta_t\}$ is colored stationary noise with mean 0 and variance σ_ζ^2, independent of the A's and B's. The noise is assumed to possess an absolutely continuous spectral distribution function $F_\zeta(\omega)$ with spectral density $f_\zeta(\omega)$, $\omega \in [-\pi, \pi]$. For our purposes we will assume $\{Z_t\}$ to be Gaussian. However, the Gaussianity assumption is not necessary for the parametric filtering method as Yakowitz shows in 1991.

Without loss of generality assume that the frequencies are ordered fixed constants,

$$0 < \omega_1 < \omega_2 < \cdots < \omega_p < \pi.$$

The general problem is to estimate the frequencies, $\omega_1, \omega_2, \cdots, \omega_p$, using a finite length observation from the time series, $Z_1, Z_2, \cdots, Z_N$.

In words, our basic strategy is to filter the observations $Z_1, Z_2, \cdots, Z_N$ with a filter from a given parametric family of linear filters, observe a zero-crossing statistic of the filtered output, then select another filter from the family based on this observed statistic. We will show, under some conditions, that this iterative procedure will converge and accurate frequency estimates may be obtained.

The remainder of the chapter is as follows. In the next section we present the formulas for the average zero-crossing rate of Gaussian processes. These are used in subsequent sections and chapters. After that, the basic iterative scheme for the case of a single sinusoid in Gaussian white noise is presented. This scheme is known as the HK algorithm (He and Kedem 1989) and uses an autoregressive order 1, $AR(1)$ filter family. Later we provide two other examples of the parametric filters applicable to this method. They are, a moving average order 1, $MA(1)$ family and an autoregressive order 2, $AR(2)$ family. The $AR(2)$ family is a particularly important example which illustrates the idea of contracting the bandwidth of the filter during the iterative procedure. The idea of shifting the center frequency of the filter and simultaneously contracting the bandwidth was first presented in Yakowitz 1991.

1.3 Expected zero-crossing rate of a Gaussian process

We present formulas for the expected zero-crossing rate of a Gaussian process. Both the continuous time and discrete time cases are given. We start with the well-known result of Rice for the expected zero-crossing rate of a continuous time Gaussian process.

If a zero-mean, stationary Gaussian process $\{Z(t)\}$, for $-\infty < t < \infty$, with normalized autocorrelation function $\rho(t)$ has sufficiently smooth sample paths, the average number of zero-crossings per unit time is given by Rice's formula (Rice 1944)

$$E[D] = \frac{1}{\pi}\sqrt{-\rho''(0)} \tag{1.2}$$

where D is the number of zero-crossings of $\{Z(t)\}$ for t in the unit interval $[0,1]$, and $\rho''(0)$ is the second derivative of the normalized autocorrelation of $\{Z(t)\}$ at 0. Ylvisaker 1965 proved Rice's formula (1.2) rigorously under mild conditions and proved that the expected number of zero-crossings is finite if and only if the autocorrelation function is twice differentiable at the origin.

The analogous formula for a discrete-time, zero-mean, stationary Gaussian sequence $\{Z_k\}$, $k = 0, \pm 1, \pm 2 \cdots$ has been obtained by many authors (see Kedem 1986, Ylvisaker 1965) and is given by

$$\rho_1 = \cos\frac{\pi E[D_1]}{N-1} \tag{1.3}$$

or, equivalently, by the inverse form

$$\frac{E[D_1]}{N-1} = \frac{1}{\pi}\cos^{-1}\rho_1$$

where D_1 is the number of sign-changes or zero-crossings in $Z_1, \cdots, Z_N$, $\rho_k = E[Z_{k+j}Z_j]/E[Z_j^2]$ is the correlation sequence of $\{Z_k\}$, and $E[D_1]/(N-1)$ is the expected zero-crossing rate in discrete time. We refer to (1.3) as the "cosine formula". Observe that, because of stationarity, the expected zero-crossing rate $E[D_1]/(N-1)$ is independent of N. In general $\cos\frac{\pi E[D_1]}{N-1}$ need not be a correlation, see Kedem (1991). Since a linearly filtered Gaussian process results in a

Gaussian process, the cosine formula holds for the filtered process where the correlation coefficient and zero-crossing count of the filtered process are used in the cosine formula (1.3). To be precise, let $\mathcal{L}_\alpha(Z)_t$ be the output at time t of a linear time invariant filter $\mathcal{L}_\alpha$ applied to $\{Z_t\}$. Using the cosine formula (1.3) and the spectral representation for stationary processes, the first-order correlation coefficient, $\rho_1(\alpha)$, of the filtered process $\{\mathcal{L}_\alpha(Z)_t\}$ is given by,

$$\rho_1(\alpha) = \cos \frac{\pi E[D_\alpha]}{N-1} = \frac{\int_{-\pi}^{\pi} \cos(\omega) |H(\omega;\alpha)|^2 dF_Z(\omega)}{\int_{-\pi}^{\pi} |H(\omega;\alpha)|^2 dF_Z(\omega)} \tag{1.4}$$

where, D_α is the zero-crossing count in $\{\mathcal{L}_\alpha(Z)_1, \cdots, \mathcal{L}_\alpha(Z)_N\}$, $F_Z(\omega)$ the spectral distribution function of the process $\{Z_t\}$, and $|H(\omega;\alpha)|^2$ the squared gain of the filter $\mathcal{L}_\alpha$. The zero-crossings, D_α, of filtered time series are referred to as Higher-Order-Crossings or HOC (see Kedem 1986).

For a given zero-mean time series $\{Z_k\}$ and parametric filter family with parameter space Θ, $\{\mathcal{L}_\alpha(\cdot), \alpha \in \Theta\}$, the corresponding HOC family is denoted by $\{D_\alpha, \alpha \in \Theta\}$.

1.4 The HK Algorithm

The iterative scheme described below illustrates a method for detecting a single frequency in Gaussian noise. Our model is (1.1) with $p = 1$ and $\{\zeta_t\}$ white Gaussian noise. As we will see the algorithm presented next guarantees convergence of a HOC sequence to the frequency ω_1 in our model. The filter family used is the exponential smoothing filter or autoregressive order 1, $AR(1)$ filter.

The $AR(1)$ filter known as the (α-filter) is defined by the operation,

$$Z_t(\alpha) = \mathcal{L}_\alpha(Z)_t = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots \tag{1.5}$$

or equivalently in its recursive form by,

$$Z_t(\alpha) = \alpha Z_{t-1}(\alpha) + Z_t$$

where the squared gain of the filter $|H(\omega;\alpha)|^2$ is given by

$$|H(\omega;\alpha)|^2 = \frac{1}{1 - 2\alpha\cos(\omega) + \alpha^2}, \quad \alpha \in (-1,1), \ \omega \in [0,\pi]. \tag{1.6}$$

Similarly define the output noise at time t by,

$$\zeta_t(\alpha) = \mathcal{L}_\alpha(\zeta)_t$$

and the contraction factor $C(\alpha)$ by,

$$C(\alpha) = \frac{Var(\zeta_t(\alpha))}{Var(Z_t(\alpha))}. \tag{1.7}$$

Then for $\alpha \in (-1,1)$,

$$0 < C(\alpha) < 1.$$

Clearly $C(\alpha)$ also depends on ω_1, but this is not included to keep the notation simple.

The following theorem from He and Kedem (1989) provides the theoretical basis for the parametric filtering and contraction mapping method and reveals the *fundamental property* in its proof.

Theorem 1.1 (He and Kedem 1989)

Suppose

$$Z_t = A_1\cos(\omega_1 t) + B_1\sin(\omega_1 t) + \zeta_t, \ t = 0, \pm 1, \cdots$$

where $\omega_1 \in (0,\pi)$, A_1, B_1 are uncorrelated, normal, zero-mean, variance σ_1^2, (i.e. $N(0,\sigma_1^2)$) random variables, and $\{\zeta_t\}$ is Gaussian white noise with mean 0 and variance σ_ζ^2, independent of A_1, B_1. Let $\{D_\alpha\}$ be the HOC from the $AR(1)$ filter (1.5). Fix $\alpha_1 \in (-1, 1)$, and define

$$\alpha_{k+1} = \cos\left(\frac{\pi E[D_{\alpha_k}]}{N-1}\right), \ k = 1, 2, \cdots \tag{1.8}$$

Then, as $k \to \infty$,

$$\alpha_k \to \cos(\omega_1)$$

and

$$\frac{\pi E[D_{\alpha_k}]}{N-1} \to \omega_1 \tag{1.9}$$

Proof: Note that the special form (1.6) gives

$$\int_0^\pi |H(\omega; \alpha)|^2 d\omega = \frac{\pi}{1-\alpha^2}$$

and

$$\int_0^\pi \cos(\omega)|H(\omega; \alpha)|^2 d\omega = \frac{\pi}{1-\alpha^2} \times \alpha$$

Therefore, by symmetry, we obtain the *factorization*,

$$\int_{-\pi}^\pi \cos(\omega)|H(\omega; \alpha)|^2 d\omega = \alpha \times \int_{-\pi}^\pi |H(\omega; \alpha)|^2 d\omega \tag{1.10}$$

and so, from the zero-crossing spectral representation (1.4), and the cosine formula (1.3),

$$\rho_1(\alpha) = \cos\left(\frac{\pi E[D_\alpha]}{N-1}\right)$$

we have, with $dF_\zeta(\omega) = \frac{1}{2\pi}\sigma_\zeta^2 d\omega$,

$$\rho_1(\alpha) = \frac{\sigma_1^2|H(\omega_1; \alpha)|^2 \times \cos(\omega_1) + \int_{-\pi}^\pi |H(\omega; \alpha)|^2 dF_\zeta(\omega) \times \alpha}{\sigma_1^2|H(\omega_1; \alpha)|^2 + \int_{-\pi}^\pi |H(\omega; \alpha)|^2 dF_\zeta(\omega)} \tag{1.11}$$

or, from the definition of $C(\alpha)$ in (1.7),

$$\rho_1(\alpha) = [1 - C(\alpha)] \times \cos(\omega_1) + C(\alpha) \times \alpha \tag{1.12}$$

We can see that $\rho_1(\alpha)$ is a convex combination of $\cos(\omega_1)$ and α and that it also can be rewritten as a contraction mapping of the form,

$$\rho_1(\alpha) = \alpha^* + C(\alpha)(\alpha - \alpha^*) \tag{1.13}$$

where $\alpha^* = \cos(\omega_1)$. Invoke the cosine formula, and write the recursion (1.8) as,

$$\alpha_{k+1} = \rho_1(\alpha_k) \tag{1.14}$$

Starting with $k = 1$, substitute this in (1.13), iteratively, to obtain,

$$\rho_1(\alpha_k) = \alpha^* + [\prod_{j=1}^{k} C(\alpha_j)](\alpha_1 - \alpha^*).$$

As $k \to \infty$, we have that

$\prod_{j=1}^{k} C(\alpha_j) \to 0$, and this implies $\alpha_k \to \alpha^*$, and that α^* is a fixed point of $\rho_1(\cdot)$,

$$\alpha^* = \rho_1(\alpha^*)$$

or

$$\cos(\omega_1) = \cos\left(\frac{\pi E[D_{\alpha^*}]}{N-1}\right)$$

By the monotonicity of $\cos(x)$, $x \in [0, \pi]$,

$$\omega_1 = \frac{\pi E[D_{\alpha^*}]}{N-1}.$$

The most important single fact in the preceding proof is the factorization equation (1.10) in which the parameter α is factorized outside the integral. This factorization is the basis for extending Theorem 1 as was done in Li and Kedem 1993 and Yakowitz 1991. The fact that the parameter is "kicked out" in (1.10) is somewhat more apparent if we rewrite (1.10) as

$$\alpha = \rho_{1,\zeta}(\alpha) = \frac{\int_{-\pi}^{\pi} \cos(\omega)|H(\omega;\alpha)|^2 d\omega}{\int_{-\pi}^{\pi} |H(\omega;\alpha)|^2 d\omega} \tag{1.15}$$

where $\rho_{1,\zeta}(\alpha)$ is the first-order autocorrelation of the filtered noise. The property (1.15) is what we call the *fundamental property* relative to a given family of filters. Thus, the $AR(1)$ parametric filter possesses the fundamental property relative to white noise. This together with the correlation representation (1.12) lead to the contraction mapping (1.13), and eventually to the convergent HOC sequences α_k, and $\frac{\pi E[D_{\alpha_k}]}{N-1}$. Fortunately, as we shall see in the next section, factorizations of the form (1.10) are readily available. In actual practice, the observed or empirical zero-crossing rate is used in place of $E[D_{\alpha_k}]$ at each stage in the iteration and the noise process need not be white - it simply needs to possess a sufficiently continuous spectrum. Computer simulation results using the α-filter for a single sinusoid in white Gaussian noise are given in Table 1.1

1.5 Examples of Parametric Families and Contraction Mappings

In this section we give two examples of parametric filter families which satisfy the *fundamental property* (1.15). They are, an $MA(1)$ family, which is similar to the α-filter family, and an $AR(2)$ filter family. With the $AR(2)$ filters it is possible to simultaneously shift the center frequency of the filter and contract the bandwidth. This allows for a faster rate of convergence of the HOC sequence and greater accuracy in comparison with the α-filter family. General conditions relating filter bandwidth contraction rate and the convergence rate of the HOC sequences may be found in Li and Kedem 1993 and Troendle 1991.

1.5.1 An example using a MA(1) filter

Again let our model be as in Theorem 1.1 with $\{Z_t\}$, a zero-mean stationary Gaussian time series defined by,

$$Z_t = A_1 \cos(\omega_1 t) + B_1 \sin(\omega_1 t) + \zeta_t, \ t = 0, \pm 1, \cdots \tag{1.16}$$

where we restrict $\omega_1 \in (\frac{\pi}{3}, \frac{2\pi}{3})$ for convenience.

Consider the family $\{\mathcal{L}_r\}$ of moving average order one, $MA(1)$, filters indexed by parameter r, $r \in (-1, 1)$ and defined by,

$$Z_t(r) = \mathcal{L}_r(Z)_t = Z_t + rZ_{t-1}, \tag{1.17}$$

and whose squared gain $|H(\omega; r)|^2$ is,

$$|H(\omega; r)|^2 = 1 + 2r\cos(\omega) + r^2, \quad r \in (-1, 1), \quad \omega \in [0, \pi]. \tag{1.18}$$

This family consists of simple finite impulse response filters which exhibit lowpass characteristics for values of the parameter r which are positive and exhibits highpass characteristics for values which are negative.

The *fundamental property* would require,

$$\int_{-\pi}^{\pi} \cos(\omega)|H(\omega; r)|^2 d\omega = r \times \int_{-\pi}^{\pi} |H(\omega; r)|^2 d\omega, \tag{1.19}$$

since we assume the noise to be white. However, evaluating the particular integrals yields,

$$\frac{\int_{-\pi}^{\pi} \cos(\omega)|H(\omega; r)|^2 d\omega}{\int_{-\pi}^{\pi} |H(\omega; r)|^2 d\omega} = \frac{r}{1 + r^2}. \tag{1.20}$$

Thus, we need to reparameterize. To obtain a reparameterization which will satisfy the *fundamental property*, set

$$\beta = \frac{r}{1 + r^2},$$

and solve for r in terms of β. This gives,

$$r = \frac{1 - \sqrt{1 - 4\beta^2}}{2\beta}.$$

Thus, the fundamental property is satisfied by the family reparameterized by β. Note that $\beta \in (-\frac{1}{2}, \frac{1}{2})$, hence the reason for restricting $\omega_1 \in (\frac{\pi}{3}, \frac{2\pi}{3})$.

1.5.2 An AR(2) filter family

The next example illustrates how to enhance the HK algorithm by selecting a parametric family that allows for adjustable narrow bandwidth filters.

Our model again will be $\{Z_t\}$ as in (1.16) with $\omega_1 \in (0, \pi)$. Consider the family $\{\mathcal{L}_{(\beta,\gamma)}\}$ of autoregressive order 2, $AR(2)$ filters indexed by the 2-vector parameter (β, γ) and defined by,

$$Z_t(\beta, \gamma) = \mathcal{L}_{(\beta,\gamma)}(Z)_t = \beta Z_{t-1}(\beta, \gamma) + \gamma Z_{t-2}(\beta, \gamma) + Z_t. \tag{1.21}$$

The squared gain of the $AR(2)$ filter, $|H(\omega; (\beta, \gamma))|^2$, is given by

$$|H(\omega; (\beta, \gamma))|^2 = \frac{1}{1 + \beta^2 + \gamma^2 + 2\beta(\gamma - 1)\cos(\omega) - 2\gamma\cos(2\omega)}, \quad \omega \in [0, \pi] \tag{1.22}$$

Evaluating the integrals yields,

$$\frac{\int_{-\pi}^{\pi} \cos(\omega)|H(\omega; (\beta, \gamma))|^2 d\omega}{\int_{-\pi}^{\pi} |H(\omega; (\beta, \gamma))|^2 d\omega} = \frac{\beta}{1 - \gamma} \tag{1.23}$$

It is seen from (1.23) that we need to reparameterize the filter family in order to satisfy (15).

Before reparameterizing the filter family, note that these filters have poles at

$$\frac{\beta \pm \sqrt{\beta^2 + 4\gamma}}{2},$$

which are inside the unit circle for values of the parameters given by,

$$-1 < \gamma < 0 \quad \text{and} \quad |\beta| < \frac{4\gamma}{\gamma - 1} \tag{1.24}$$

and approach the unit circle as $\gamma \to -1$. That is the poles go to $\exp(\pm i\theta)$ as $\gamma \to -1$ and $\theta \to \cos^{-1}(\beta/2)$. Thus, we will restrict our parameter space according to (1.24) to guarantee stable filters.

If γ is fixed to some $\gamma_0 \in (-1, 0)$, then $\delta = \frac{\beta}{1-\gamma_0}$ gives a parameterization (in terms of $\frac{\beta}{1-\gamma_0}$, with only β free) satisfying the *fundamental property.* Furthermore, by allowing γ to vary in a prescribed way with each iteration (i. e. let γ approach -1 during the iteration process for narrower bandwidths), the filters can be made to simultaneously shift there center frequency and contract the bandwidth. Thus, for suitably chosen sequences $\{\gamma_k\}$, with $\gamma_k \to -1$, we can also satisfy *fundamental property* with the added bonus of accelerated convergence of the HOC sequence and greater accuracy of the estimates (see Li 1992, Troendle 1991 and Yakowitz 1991). Simulation results for a single sinusoid in white Gaussian noise using the $AR(2)$ filter with $\gamma = -.9$ are given in Table 1.2

1.6 Summary

In this opening chapter a parametric filtering technique was presented for application to the problem of discrete frequency estimation. By incorporating a contraction mapping principle with parametric filtering a theoretical basis for this new method was established (Theorem 1). The theorem also provides a *fundamental property* (1.15), which places a condition on the parametric filter family guaranteeing convergence of the iterative filtering method for frequency estimation. Two contrasting examples were given which illustrate the utility of the method.

Table 1.1: **Illustration of the HK algorithm using the $AR(1)$ filter family, convergence based on the observed zero-crossing count, $(\alpha_{k+1}) = \pi\hat{D}_{\alpha_k}/(N-1)$, $k \to \infty$, towards $\omega_1 = 0.8$ as a function of SNR.** $N = 10,000$.

	$1dB$	$0dB$	$-1.94dB$	$-6.02dB$
k	$\alpha_1 = -.1$	$\alpha_1 = .9$	$\alpha_1 = .2$	$\alpha_1 = .5$
1	0.8848	0.5194	0.9127	0.9291
2	0.8006	0.5904	0.8222	0.8713
3	0.7987	0.6563	0.8015	0.8411
4	0.7987	0.7142	0.7965	0.8191
5	0.7987	0.7600	0.7952	0.8053
6	0.7987	0.7864	0.7952	0.8015
7	0.7987	0.8002	0.7952	0.7990
8	0.7987	0.8065	0.7952	0.7984
9	0.7987	0.8065	0.7952	0.7977
10	0.7987	0.8065	0.7952	0.7971
11	0.7987	0.8065	0.7952	0.7971
12	0.7987	0.8065	0.7952	0.7971
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Table 1.2: **Illustration of the HK algorithm using the AR(2) filter family with $\gamma = -.9$, convergence based on the observed zero-crossing count, $(\alpha_{k+1}) = \pi \hat{D}_{\alpha_k}/(N-1)$, $k \to \infty$, towards $\omega_1 = 0.8$ as a function of SNR. $N = 2,000$.**

	$0dB$	$0dB$	$-6.02dB$	$-6.02dB$
k	$\alpha_1 = .9$	$\alpha_1 = -.5$	$\alpha_1 = .9$	$\alpha_1 = .2$
1	0.5613	2.0377	0.4685	1.3349
2	0.6839	2.0000	0.4890	1.3050
3	0.7987	1.9717	0.5141	1.2830
4	0.8003	1.9198	0.5424	1.2547
5	0.8003	1.8679	0.5833	1.2390
6	0.8003	1.8050	0.6336	1.1918
7	0.8003	1.7689	0.6965	1.1516
8	0.8003	1.7233	0.7704	1.1242
9	0.8003	1.6651	0.7987	1.0676
10	0.8003	1.6274	0.8034	0.9119
11	0.8003	1.5645	0.8034	0.8302
12	0.8003	1.5016	0.8034	0.8019
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
20	0.8003	0.8003	0.8019	0.8019

Chapter 2

Zero-Crossing Rates of Functions of Gaussian Processes

2.1 Introduction

Formulas for the expected zero-crossing rates of random processes that are monotone transformations of Gaussian processes can be obtained using two different techniques. The first technique involves the derivation of the expected zero-crossing rate for discrete-time processes and then extends the result to the continuous-time case by using an appropriate limiting argument. The second is a direct method that makes use, successively, of Price's theorem, the chain rule for derivatives, and Rice's formula for the expected zero-crossing rate of a Gaussian process. A constant, which depends on the variance of the transformed process and a second-moment of its derivative, is derived. Multiplying Rice's original expression by this constant yields the zero-crossing formula for the transformed process. The two methods can be used for the general level-crossing problem of random processes that are monotone functions of a Gaussian process.

Recall that if a zero-mean, stationary Gaussian process $\{Z(t)\}$, for $-\infty < t < \infty$ with normalized autocorrelation function $\rho(t)$ has sufficiently smooth sample paths, the average number of zero-crossings per unit time is given by Rice's formula (Rice (1944))

$$E[D] = \frac{1}{\pi}\sqrt{-\rho''(0)} \tag{2.1}$$

where D is the number of zero-crossings of $\{Z(t)\}$ for t in the unit interval $[0,1]$, and $\rho''(0)$ is the second derivative of the normalized autocorrelation of $\{Z(t)\}$ at 0.

Recall as well the analogous formula for a discrete-time, zero-mean, stationary Gaussian process $\{Z_k\}$, $k = 0, \pm 1, \pm 2 \cdots$ has been obtained by many authors (see McFadden, Ylvisaker, Ruchkin, Kedem (1980a), Kedem (1986)) and is given by

$$\rho_1 = \cos\frac{\pi E[D_1]}{N-1} \tag{2.2}$$

or, equivalently, by the inverse form

$$\frac{E[D_1]}{N-1} = \frac{1}{\pi}\cos^{-1}\rho_1$$

where D_1 is the number of sign-changes or zero-crossings in $Z_1, \cdots, Z_N$, $\rho_k = E[Z_{k+j}Z_j]/E[Z_j^2]$ is the correlation sequence of $\{Z_k\}$, and $E[D_1]/(N-1)$ is the expected zero-crossing rate in discrete time.

The cosine formula only reaffirms the intuitive notion that in general, that is, regardless of gaussianity, the expected number of zero-crossings D_1 is *inversely* related to ρ_1 (i.e., as $E[D_1]$ increases, ρ_1 decreases, and vice versa). This inverse relationship is also exhibited by some new "cosine formulas" [eqs. (2.19), (2.22), (2.25)] that we derive in this chapter for some special cases.

Rice's formula (2.1) is closely related to the cosine formula (2.2) . In fact, Rice's formula can be derived from the cosine formula as is shown in McFadden (eq. (13)), Ylvisaker (1965), and Kedem (1980b, p. 23). That is, the continuous-time version of the expected zero-crossing rate can be obtained from the discrete-time analog, a fact that is exploited later in the chapter.

Extension of these formulas to non-Gaussian processes has generally been found problematical. The difficulty with generalizing Rice's approach is that it requires knowledge of the joint density of $Z(t)$ and its derivative $Z'(t)$, which in general is not tractable when $Z(t)$ is not Gaussian. Similarly, for the discrete-time case, an extension to non-Gaussian sequences requires knowledge of the orthant probability $\Pr(Z_k \geq 0, Z_{k-1} \geq 0)$ and its functional relationship to ρ_1. In general this functional relationship is not known when Z is not Gaussian.

For some cases, however, these difficulties can be overcome. To illustrate this, we include in the next section a result from He and Kedem (1989) that extends the cosine formula to the class of processes whose finite-dimensional distributions are elliptically symmetric (Theorem 2.1), along with a new result that under some stationarity conditions, extends the cosine formula to processes that are purely sinusoidal (Theorem 2.2).

A general way to extend (2.1) and (2.2) to a class of non-Gaussian processes is to transform a Gaussian process by a strictly monotone transformation. By using such non-linear transformations of Gaussian processes, one can obtain new zero-crossing formulas using (2.1) and (2.2), and second-moment properties of the transformed processes. This is explained with the help of some specific cases later in the chapter.

2.2 Extension to Ellipsoidal and Purely Sinusoidal Processes

A distribution in R^n defined by the n-dimensional density

$$f(\mathbf{x}) = |\Sigma|^{-1/2}\psi(\mathbf{x}'\Sigma^{-1}\mathbf{x}) \tag{2.3}$$

for some function $\psi(u)$ defined on $[0, \infty)$ and parametrized by a symmetric positive-definite matrix $\mathbf{\Sigma}$ is called an *ellipsoidal distribution* (Jensen (1988)) or *elliptically symmetric distribution* (McGraw and Wagner(1968)). We shall say that a stochastic process is an ellipsoidal process if its finite-dimensional distributions are all elliptically symmetric.

Theorem 2.1 (He and Kedem (1989)) Let $\{Z_k\}$ for $k = 0, \pm 1, \pm 2, \cdots$ be a strictly stationary ellipsoidal sequence with mean 0, variance 1, and autocorrelation sequence ρ_k. Then the cosine formula (2.2) holds.

Proof: If the joint density $f(x, y)$ of (Z_k, Z_{k-1}) is

$$f(x,y) = |\Sigma|^{-\frac{1}{2}}\psi\left((x\ y)\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right)$$

where

$$\mathbf{\Sigma} = c\begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$$

for some $c > 0$, then the correlation coefficient $\rho_1 = \rho$ (see McGraw and Wagner eq. 28) and the orthant probability is given by

$$\Pr(Z_k \geq 0, Z_{k-1} \geq 0) = |\Sigma|^{-\frac{1}{2}} \int_0^\infty \int_0^\infty \psi\left((x\ y)\Sigma^{-1}\begin{pmatrix} x \\ y \end{pmatrix}\right) dx\, dy.$$

By switching to polar coordinates, $x = r\cos\theta$, $y = r\sin\theta$, we find

$$\begin{aligned}
\Pr(Z_k \geq 0, Z_{k-1} \geq 0) &= \int_0^\infty \int_0^\infty \frac{1}{\sqrt{c^2(1-\rho_1^2)}}\psi\left(\frac{x^2+y^2-2\rho_1 xy}{c(1-\rho_1^2)}\right) dx\, dy \\
&= \int_0^{\pi/2} \int_0^\infty \frac{1}{\sqrt{c^2(1-\rho_1^2)}}\psi\left(\frac{r^2(1-\rho_1\sin 2\theta)}{c(1-\rho_1^2)}\right) rdr\, d\theta \\
&= \sqrt{1-\rho_1^2}\int_0^{\pi/2} \frac{d\theta}{2(1-\rho_1\sin 2\theta)}\int_0^\infty \psi(u)du \\
&= (\frac{\pi}{4} + \frac{1}{2}\sin^{-1}\rho_1)\int_0^\infty \psi(u)du.
\end{aligned}$$

To evaluate the remaining integral note that

$$\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{1}{\sqrt{c^2(1-\rho_1^2)}}\psi\left(\frac{x^2+y^2-2\rho_1 xy}{c(1-\rho_1^2)}\right) dx\, dy = 1$$

and by switching to polar coordinates as above

$$\int_0^\infty \psi(u)du = \left[\sqrt{1-\rho_1^2}\int_0^{2\pi}\frac{d\theta}{2(1-\rho_1\sin 2\theta)}\right]^{-1} = \frac{1}{\pi}.$$

The last integral is evaluated by recognizing that

$$\int_0^{2\pi}\frac{d\theta}{2(1-\rho_1\sin 2\theta)} = \int_0^{\pi/2}\frac{d\theta}{1-\rho_1\sin 2\theta} + \int_0^{\pi/2}\frac{d\theta}{1+\rho_1\sin 2\theta}.$$

Collecting our results we obtain

$$\begin{aligned}
\Pr(Z_k \geq 0, Z_{k-1} \geq 0) &= (\frac{\pi}{4} + \frac{1}{2}\sin^{-1}\rho_1)/\pi \\
&= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\rho_1. \qquad \square
\end{aligned}$$

Therefore, from the definition of D_1,

$$
\begin{aligned}
E[D_1] &= (N-1)[1 - 2\Pr(Z_k \geq 0, Z_{k-1} \geq 0)] \\
&= (N-1)\left(\frac{1}{2} - \frac{1}{\pi}\sin^{-1}\rho_1\right)
\end{aligned}
$$

or

$$
\cos\frac{\pi E[D_1]}{N-1} = \cos\left(\frac{\pi}{2} - \sin^{-1}\rho_1\right) = \rho_1.
$$

□

Rice's formula can now be obtained as a corollary. We need only sufficiently smooth sample paths and the finiteness of $\rho''(0)$ (Ylvisaker (1965)).

More precisely, let $\{Z(t)\}$ for $-\infty < t < \infty$ be a strictly stationary ellipsoidal process possessing a correlation function $\rho(t)$ that is twice differentiable at 0. Assume that, for $\Delta > 0$, the probability of more than a single crossing in $(t, t+\Delta)$ is negligible (i.e., goes to zero) as $\Delta \to 0$. Define the sampled time series $Z_k \equiv Z((k-1)\Delta)$ for $k = 1, 2, \cdots, N$ in such a way that

$$
(N-1)\Delta = 1. \tag{2.4}
$$

By using the expected sign-change count of the Δ-sampled process, that is, by sampling at $t = k\Delta$, the expected zero-crossing rate in continuous-time can be obtained in the limit $\Delta \to 0$. The sampled process $\{Z_k\}$ is strictly stationary with correlation sequence ρ_k, say. Note that

$$
\rho_1 = \rho(\Delta). \tag{2.5}
$$

A corresponding binary time series we will use in defining $D_{1,N}$ is

$$
X_k = \begin{cases} 1 & \text{if } Z_k \geq 0 \\ 0 & \text{if } Z_k < 0. \end{cases}
$$

In terms of X_k, $D_{1,N}$ can be expressed as

$$D_{1,N} = \sum_{t=2}^{N}[X_t - X_{t-1}]^2, \tag{2.6}$$

i.e., 0 is treated as positive. By the cosine formula, monotone convergence, and l'Hôpital's rule,

$$\begin{aligned} E[D_c] &= \lim_{N\to\infty} E[D_{1,N}] \\ &= \lim_{\Delta\to 0} \frac{1}{\pi\Delta} \cos^{-1} \rho(\Delta) \\ &= \frac{1}{\pi}\sqrt{-\rho''(0)} \end{aligned} \tag{2.7}$$

and so Rice's formula (2.1) holds. In deriving (2.7) the cosine formula has been used in its inverse form

$$E[D_{1,N}] = \frac{N-1}{\pi} \cos^{-1} \rho_1$$

where we substituted $\rho(\Delta)$ for ρ_1 and $1/\Delta$ for $N-1$.

The foregoing scheme above for deriving a zero-crossing formula in continuous-time from its discrete-time counterpart is repeated often in this dissertation.

2.2.1 Zero-Crossing Rate of a Pure Sinusoid

Next we extend the cosine formula to processes that are purely sinusoidal.

Theorem 2.2 (Kedem 1994, pg. 118-119) If $\{Z_k\}$ for $k = 0, \pm 1, \pm 2, \cdots$, is a wide-sense stationary (i.e., stationary up to order 2) sampled random sinusoid:

$$Z_k = A\cos\omega k + B\sin\omega k \tag{2.8}$$

where A, B are uncorrelated random variables with zero means and each has variance σ^2, ρ_k is the autocorrelation sequence of $\{Z_k\}$, and if for *all* k

$$\Pr(Z_k \geq 0) = \frac{1}{2} \tag{2.9}$$

and

$$\Pr(Z_k \geq 0 | Z_{k-1} \geq 0) = \lambda_1, \tag{2.10}$$

then the cosine formula holds for a time series from $\{Z_k\}$

$$Z_1, Z_2, Z_3, \cdots, Z_N$$

where D_1 is the number of sign-changes or zero crossings, as defined in (2.6),

$$\rho_1 = \cos \frac{\pi E[D_1]}{N-1}. \tag{2.11}$$

Proof: Observe that, from (2.6), (2.9), and (2.10), the expected sign-change or zero-crossing rate

$$\frac{E[D_1]}{N-1} = 1 - \lambda_1$$

is *independent* of N. Also note that, for a random wide-sense or strict-sense stationary sinusoid

$$\rho_1 = \cos \omega.$$

If D_1 zero-crossings are observed in the time-series, and if ω is in the interval $[0, \pi]$, then ω is bounded as

$$\frac{\pi D_1}{N} \leq \omega \leq \frac{\pi D_1}{N} + \frac{2\pi}{N} \leq \frac{\pi D_1}{N-1} + \frac{2\pi}{N}$$

and, by subtraction and simple manipulation,

$$\frac{-2\pi}{N} \leq \omega - \frac{\pi D_1}{N-1} \leq \frac{2\pi}{N}.$$

Therefore with probability 1, as $N \longrightarrow \infty$,

$$\frac{\pi D}{N-1} \longrightarrow \omega.$$

By bounded convergence, then, as $N \longrightarrow \infty$,

$$E[\frac{\pi D_1}{N-1}] = \frac{\pi E[D_1]}{N-1} \longrightarrow \omega.$$

But the zero-crossing rate is independent of N, and it follows that

$$\frac{\pi E[D_1]}{N-1} = \omega$$

and so, we finally have

$$\rho_1 = \cos\omega = \cos\frac{\pi E[D_1]}{N-1}$$

□

In the random sinusoid (2.8), wide-sense stationarity is guaranteed if $E[A] = 0 = E[B]$, Var(A) =Var(B) and $E[AB] = 0$. Furthermore, in this case, A and B may assume *any* distribution, symmetric or not. Since the autocorrelation function of a continuous-time stationary random sinusoid is $\rho(t) = \cos\omega t$, the expected zero-crossing rate may be obtained, as in (2.7), and is given by

$$E[D] = \frac{1}{\pi}\sqrt{-\rho''(0)} = \frac{\omega}{\pi},$$

which says, the number of zero-crossings per unit time is two times the frequency f of the random sinusoid (where $2\pi f = \omega$), a result which is not surprising .

2.3 Two methods for generating zero-crossing formulas

In this section we present methods for generating zero-crossing formulas for some non-Gaussian processes. The underlying general idea is to transform a stationary Gaussian process by a monotone transformation that preserves the zero-crossing count (by fixing the origin) but gives a different correlation or, equivalently, spectral structure.

Let $\varphi(x)$ be a strictly monotone real-valued-function defined over the real line. Let $\{Z(t)\}$ for $-\infty < t < \infty$ be a zero-mean stationary process with unit variance and autocorrelation $\rho_z(t)$. Define a new process $\{Y(t)\}$ for $-\infty < t < \infty$, with autocorrelation $\rho_y(t)$, as

$$Y(t) = \varphi(Z(t)) - \varphi(0). \tag{2.12}$$

$\{Y(t)\}$ is not necessarily Gaussian, and its mean need not be 0. *A zero-crossing occurs in $\{Y(t)\}$ if and only if a zero-crossing occurs in $\{Z(t)\}$.* That is, the zero-crossing count in $\{Y(t)\}, t \in [0,1]$, is equal to the zero-crossing count in $\{Z(t)\}, t \in [0,1]$, with probability one. If $\varphi(x)$ is nonlinear, the finite-dimensional distributions of $\{Y(t)\}$ are different from those of $\{Z(t)\}$. This implies,in particular, that the correlation structures in the two processes are different, and hence we can expect different zero-crossing formulas if they are to depend on correlation.

The same applies to the corresponding discrete-time processes $\{Z_k\}$ and $\{Y_k\}$ defined as in the previous section by sampling at $t = k\Delta$ the continuous-time processes.

We describe two methods for deriving zero-crossing formulas for $\{Y(t)\}$. By the first method, we first derive "cosine formulas" in discrete time and then let the sampling interval Δ approach 0. The second method bypasses the cosine formulas by providing a general formula for the direct evaluation of the zero-crossing rate of $\{Y(t)\}$. In both methods, Price's (not Rice's) theorem (Price (1958)) comes in very handy.

In what follows it is assumed that all relevant derivatives exist and are finite. In particular we assume the existence of the second-order spectral moment

$$-\rho_y''(0) = \int_{-\infty}^{\infty} \lambda^2 dF(\lambda) < \infty \tag{2.13}$$

where $F(\lambda)$ is the normalized spectral distribution function. Later we obtain an equation that relates the second-order spectral moments of $\{Y(t)\}$ and $\{Z(t)\}$.

2.3.1 Method I for generating zero-crossing formulas

The first method for generating new zero-crossing formulas derives the zero-crossing rate in continuous time from the one in discrete time and is given by the following algorithm. We note that the steps in the algorithm may not be easy to follow when $\varphi(x)$ is not "nice", but the three examples that we consider indicate that in general the task may be quite tractable nonetheless. The advantage of this method is that it provides expressions for the expected sign-change or zero-crossing rate in sampled signals.

Algorithm for zero-crossing formulas

- Partition the unit interval $[0, 1]$ into $N - 1$ intervals each of size $\Delta > 0$, with $(N-1)\Delta = 1$. Let $\{Z(t)\}$ be a stationary Gaussian process. Let

$$Z_1 = Z(0), Z_2 = Z(\Delta), \cdots, Z_N = Z((N-1)\Delta).$$

- Denote the sign-change or zero-crossing count in $Z_1, \cdots, Z_N$ by $D_{1,N}$, as in (2.6).

- Define

$$Y_k = \varphi(Z_k) - \varphi(0),\ for\ k = 1, \cdots, N.$$

The zero-crossing count in $Y_1, \cdots, Y_N$ also equals $D_{1,N}$.

- Apply Price's theorem:

$$\sigma_y^2 \frac{\partial \rho_y(\Delta)}{\partial \rho_z(\Delta)} = \frac{\partial E[Y_k Y_{k-1}]}{\partial \rho_z(\Delta)} = E\left(\frac{\partial^2 Y_k Y_{k-1}}{\partial Z_k \partial Z_{k-1}}\right) \tag{2.14}$$

to obtain the functional relationship between $\rho_y(\Delta)$ and $\rho_z(\Delta)$ by integration. That is, $\rho_y(\Delta)=H(\rho_z(\Delta))$, for some function H, which is the solution to (2.14), with the initial condition $\rho_y(\Delta) = 0$ iff $\rho_z(\Delta) = 0$.

- Substitute into the solution of (2.14)

$$\cos \frac{\pi E[D_{1,N}]}{N-1}$$

for $\rho_z(\Delta)$. This gives the new "cosine formula" that relates $\rho_y(\Delta)$ to the expected zero-crossing count in $Y_1, \cdots, Y_N$

$$\rho_y(\Delta) = H(\rho_z(\Delta)) = H\left(\cos \frac{\pi E[D_{1,N}]}{N-1}\right). \tag{2.15}$$

- Solve the new "cosine formula" (2.15) for $E[D_{1,N}]$ in terms of $\rho_y(\Delta)$, substitute $N - 1 = 1/\Delta$, and let $\Delta \to 0$.

- The last step yields the expected zero-crossing rate of $Y(t)$ in continuous time.

A symmetric uniform process

Let $\Phi(x)$ be the distribution function of the standard normal distribution, and define

$$\varphi(x) = \Phi(x).$$

Then, $\varphi(0) = 1/2$, and

$$Y(t) = \Phi(Z(t)) - \frac{1}{2}. \tag{2.16}$$

For each t, $Y(t)$ is uniformly distributed on the interval $[-1/2, 1/2]$. Write Z_2, Z_1 for Z_k, Z_{k-1}, respectively. Observe that, because $\{Z(t)\}$ is Gaussian, $Z_1 + Z_2$ is stochastically independent of $Z_1 - Z_2$. Then Price's theorem yields

$$\frac{\partial E[Y_2 Y_1]}{\partial \rho_z(\Delta)} = E\left(\frac{\partial^2 Y_2 Y_1}{\partial Z_2 \partial Z_1}\right) = E[\Phi^{'}(Z_2)\Phi^{'}(Z_1)]$$

so

$$\frac{\partial \rho_y(\Delta)}{\partial \rho_z(\Delta)} = \frac{6}{\pi} E[\exp(-\frac{1}{2}(Z_1^2 + Z_2^2))]. \tag{2.17}$$

To evaluate the right-hand side of (2.17) it is helpful to note that

$$\frac{(Z_1 + Z_2)^2}{2(1 + \rho_z(\Delta))} \text{ and } \frac{(Z_1 - Z_2)^2}{2(1 - \rho_z(\Delta))}$$

are independent χ^2 (chi-square) random variables. Thus, we can rewrite (2.17)

$$\begin{aligned}\frac{\partial \rho_y(\Delta)}{\partial \rho_z(\Delta)} &= \frac{6}{\pi} E[\exp(-\frac{1}{2}((1 + \rho_z(\Delta))\frac{(Z_1 + Z_2)^2}{2(1 + \rho_z(\Delta))} + (1 - \rho_z(\Delta))\frac{(Z_1 - Z_2)^2}{2(1 - \rho_z(\Delta))}))] \\ &= \frac{6}{\pi} M_{\chi^2}\left(\frac{-1 - \rho_z(\Delta)}{2}\right) M_{\chi^2}\left(\frac{-1 + \rho_z(\Delta)}{2}\right)\end{aligned}$$

$$= \frac{6}{\pi}\frac{1}{\sqrt{2^2 - \rho_z^2(\Delta)}} \tag{2.18}$$

where $M_{\chi^2}(t) = 1/\sqrt{1-2t}$ is the moment-generating function of the chi-square distribution with one degree of freedom. By integrating (2.18), we get

$$\rho_y(\Delta) = \frac{6}{\pi}\sin^{-1}(\rho_z(\Delta)/2) + C_0,$$

where the constant of integration C_0 equals 0 because $\rho_y(\Delta) = 0$ if and only if $\rho_z(\Delta) = 0$. Thus, by substituting for $\rho_z(\Delta)$, we obtain a new "cosine formula" for the foregoing uniform process:

$$\rho_y(\Delta) = \frac{6}{\pi}\sin^{-1}\left(\frac{1}{2}\cos\frac{\pi E[D_{1,N}]}{N-1}\right). \tag{2.19}$$

It is interesting to observe that, by series expansion of $\sin^{-1} x$, the new formula (2.19) is close to the cosine formula (2.2). Indeed we have

$$\rho_y(\Delta) \approx \frac{3}{\pi}\cos\frac{\pi E[D_{1,N}]}{N-1}.$$

To obtain the continuous time version, we let $\Delta \to 0$ in (2.19). Using l'Hôpital's rule, we obtain the expected zero-crossing rate per unit time for the foregoing uniform process,

$$E[D_c] = \frac{1}{\sqrt{2\pi\sqrt{3}}}\sqrt{-\rho_y''(0)}. \tag{2.20}$$

As in the discrete time case (2.19), we find (2.20) is close to (2.1) and we have

$$E[D_c] \approx \frac{1}{\pi}\sqrt{-\rho_y''(0)}$$

Because the expected zero-crossing rate is the same for $\{Z(t)\}$ and $\{Y(t)\}$, this approximation means that the second-order spectral moment was not altered much ($\rho_y(\Delta) = \frac{3}{\pi}\rho_z(\Delta) + O(\rho_z^3(\Delta))$) by the transformation (2.16).

A shifted lognormal process

Let

$$\varphi(x) = \exp(x)$$

and consider the process

$$Y(t) = \exp(Z(t)) - 1. \tag{2.21}$$

For each t, $Y(t)$ has a shifted lognormal distribution with mean $e^{1/2} - 1$ and variance $e(e-1)$. Observe that the mean of $Y(t)$ is *not* 0. Price's theorem yields

$$\begin{aligned}\frac{\partial E[Y_k Y_{k-1}]}{\partial \rho_z(\Delta)} &= E[\exp(Z_k + Z_{k-1})] \\ &= \exp(1 + \rho_z(\Delta)).\end{aligned}$$

The constant of integration is 0, and we have

$$\rho_y(\Delta) = \frac{\exp(\rho_z(\Delta)) - 1}{e - 1}.$$

Replacing $\rho_z(\Delta)$ by $\cos(\pi E[D_{1,N}]/(N-1))$ leads to a new "cosine formula"

$$\rho_y(\Delta) = \frac{\exp\left(\cos \frac{\pi E[D_{1,N}]}{N-1}\right) - 1}{e - 1} \tag{2.22}$$

or

$$E[D_{1,N}] = \frac{1}{\pi}\frac{1}{\Delta}\cos^{-1}[\log((e-1)\rho_y(\Delta) + 1)].$$

Now let $\Delta \to 0$ to obtain the continuous time expected zero-crossing rate

$$E[D_c] = \frac{1}{\pi}\sqrt{\frac{e-1}{e}}\sqrt{-\rho_y''(0)}. \tag{2.23}$$

Because $(e-1)/e \approx 0.63$ (i.e. less than 1), the transformation (2.21) leads to an increase of about 59% in the second spectral moment. Thus, a direct application of Rice's original formula (2.1) as an approximation for the expected zero-crossing rate in $\{Y(t)\}$, would be somewhat erroneous.

The cube of a Gaussian Process

In the previous example one may get the impression that the discrepancy between the zero-crossing rates of $\{Z(t)\}$ and $\{Y(t)\}$ is due to the fact that the distribution of $Y(t)$ is asymmetric. However, this is not exactly the case. The distribution of $Y(t)$ can be symmetric, and yet have a sizable discrepancy between the zero-crossing rates as is shown by the next example.

Let $\varphi(x) = x^3$, and consider the process

$$Y(t) = Z^3(t). \tag{2.24}$$

Observe that, for each t, $Y(t)$ has a symmetric probability density function. By Price's theorem,

$$\begin{aligned} \frac{\partial E[Y_k Y_{k-1}]}{\partial \rho_z(\Delta)} &= 9E[Z_k^2 Z_{k-1}^2] \\ &= 9(2\rho_z^2(\Delta) + 1). \end{aligned}$$

The constant of integration is 0, and

$$\rho_y(\Delta) = \frac{2}{5}\rho_z^3(\Delta) + \frac{3}{5}\rho_z(\Delta).$$

The "cosine formula" now takes the form

$$\rho_y(\Delta) = \frac{9}{10}\cos\frac{\pi E[D_{1,N}]}{N-1} + \frac{1}{10}\cos\frac{3\pi E[D_{1,N}]}{N-1}. \tag{2.25}$$

By implicit differentiation, in the limit as $\Delta \to 0$, we have

$$E[D_c] = \frac{1}{\pi}\sqrt{\frac{5}{9}}\sqrt{-\rho_y''(0)}, \tag{2.26}$$

and so, going from $\{Z(t)\}$ to $\{Y(t)\}$, we find an 80% increase in the second spectral moment.

2.3.2 Method II for generating zero-crossing formulas

Our second method for generating zero-crossing formulas is a direct method. For the general transformation (2.12), Price's theorem gives,

$$\frac{\partial \rho_y(\Delta)}{\partial \rho_z(\Delta)} = \frac{1}{\sigma_y^2} E[\varphi^{'}(Z(t))\varphi^{'}(Z(t+\Delta))].$$

Observe that, as $\Delta \to 0$,

$$E[\varphi^{'}(Z(t))|Z(t+\Delta)] \to \varphi^{'}(Z(t))$$

and

$$E[\varphi^{'}(Z(t+\Delta))\varphi^{'}(Z(t))|Z(t+\Delta)] \to \varphi^{'}(Z(t))^2$$

since

$$E[\varphi^{'}(Z(t+\Delta))\varphi^{'}(Z(t))|Z(t+\Delta)] = \varphi^{'}(Z(t+\Delta))E[\varphi^{'}(Z(t))|Z(t+\Delta)].$$

Therefore, by double expectation (i.e., $E[E[Y|X]] = E[Y]$) as $\Delta \to 0$,

$$\frac{d\rho_y}{d\rho_z} \to \frac{E[\varphi^{'}(Z(t))^2]}{\sigma_y^2}. \tag{2.27}$$

Now, by the chain rule,

$$\frac{d^2\rho_y}{d\Delta^2} = \frac{d^2\rho_y}{d\rho_z^2}\left(\frac{d\rho_z}{d\Delta}\right)^2 + \frac{d\rho_y}{d\rho_z}\frac{d^2\rho_z}{d\Delta^2}$$

and by (2.27), as $\Delta \to 0$, we obtain an equation that relates the second-order spectral moments,

$$\rho_y^{''}(0) = \frac{E[\varphi^{'}(Z(t))^2]}{\sigma_y^2}\rho_z^{''}(0). \tag{2.28}$$

From this and Rice's formula (2.1), we obtain the expected zero-crossing rate for $Y(t)$ in (2.12),

$$E[D_c] = \frac{1}{\pi}\sqrt{\frac{\mathrm{Var}[\varphi(Z(t))]}{E[\varphi^{'}(Z(t))^2]}}\sqrt{-\rho_y^{''}(0)}. \tag{2.29}$$

The previously considered three special cases (uniform, lognormal, and the cube of a Gaussian) easily follow from this formula.

A Gaussian process raised to an odd power

To illustrate the use of general formula (2.29), let $\varphi(x) = x^n$, where n is a fixed positive odd integer, and consider the process

$$Y(t) = \varphi(Z(t)).$$

Observe that, for all t, $Y(t)$ has a symmetric distribution about zero and

$$\mathrm{Var}[Y(t)] = 1 \cdot 3 \cdot 5 \cdots (2n-3)(2n-1).$$

Since

$$\varphi'(Z(t)) = nZ(t)^{n-1}$$

$$E[\varphi'(Z(t))^2] = n^2 \cdot 1 \cdot 3 \cdot 5 \cdots (2n-3),$$

thus, using (2.29), we obtain the expected zero-crossing rate for $Y(t)$,

$$E[D_c] = \frac{1}{\pi}\sqrt{\frac{2n-1}{n^2}}\sqrt{-\rho_y''(0)}. \tag{2.30}$$

From (2.30) we see that n can be chosen so that in going from $Z(t)$ to $Y(t)$ the increase in the second spectral moment can be made arbitrarily large.

2.4 Summary

In this chapter we have shown that the "cosine formula" (2.2), and hence Rice's formula (2.1) can be extended to the class of strictly ellipsoidal processes (theorem 2.1) and to the class of wide-sense-stationary purely sinusoidal processes (theorem 2.2). Two distinct methods were presented for generating new formulas for the expected zero-crossing rates of some non-Gaussian processes: those that are smooth, monotone, nonlinear transformations of a Gaussian. The first method provides an algorithm for obtaining new cosine formulas for discrete-time series. These new cosine formulas are then used to obtain new formulas like Rice's by taking an appropriate limit (see eq. (2.7)) To help explain the method, three examples were included for which explicit zero-crossing formulas were calculated. The examples also illustrate that zero-crossing rates of distorted Gaussian processes can be quite different from those given by Rice's original formula, even when the underlying univariate density of the process is symmetric. The second method is a direct method that yields a general formula (2.29) for the average zero-crossing rate of a transformed Gaussian process. This general formula is Rice's original expression multiplied by a constant which is the square root of a ratio of second-moments.

Chapter 3

Zero-Crossing Rates of Mixtures and Products of Gaussian Processes

3.1 Introduction

Formulas for the expected zero-crossing rate of non-Gaussian mixtures and products of Gaussian processes are obtained. The approach we take is to first derive the expected zero-crossing rate in discrete time and then obtain the rate in continuous time by an appropriate limiting argument. The processes considered, which are non-Gaüssian but derived from Gaussian processes, serve to illustrate the variability of the zero-crossing rate in terms of the normalized autocorrelation function, $\rho(t)$, of the process. For Gaussian processes, Rice's formula gives the expected zero-crossing rate in continuous time as $\frac{1}{\pi}\sqrt{-\rho''(0)}$. We show processes exist with expected zero-crossing rates given by $\frac{\kappa}{\pi}\sqrt{-\rho''(0)}$ with either $\kappa \gg 1$ or $\kappa \ll 1$. Consequently, such processes can have an arbitrarily large or small zero-crossing rate as compared to a Gaussian process with the same autocorrelation function.

As usual we will consider a zero-mean, stationary Gaussian process $\{X(t)\}$, $-\infty < t < \infty$, with autocovariance $R(t)$ and autocorrelation function $\rho(t)$. We assume throughout that the variance of the underlying Gaussian process $\{X(t)\}$ is one so that $R(0) = \rho(0) = 1$. And as before we assume $\{X(t)\}$ is mean square differentiable, that is, if $\rho''(0)$ exists and is finite, so that the expected number of zero-crossings per unit time is given by Rice's formula (Rice (1944), Ylvisaker (1965))

$$E[D] = \frac{1}{\pi}\sqrt{-\rho''(0)} \tag{3.1}$$

where D is the number of zero-crossings of $\{X(t)\}$ for t in the unit interval $[0,1]$, and $\rho''(0)$ is the second derivative of the autocorrelation function of $\{X(t)\}$ at 0. Throughout this chapter we shall use D to denote the zero-crossing rate in continuous time regardless of the process.

Again for reference, the analogous formula for a discrete-time, zero-mean, unit variance, stationary Gaussian sequence $\{X(k)\}$, $k = 0, \pm 1, \pm 2 \cdots$ is given by (McFadden (1956), Ylvisaker (1965), Kedem (1980)) the cosine formula

$$\rho_1 = \cos \frac{\pi E[D_1]}{N-1} \tag{3.2}$$

where, D_1 is the number of sign-changes or zero-crossings in $\{X(1), \cdots, X(N)\}$, $\rho_k = E[X(k+j)X(j)]$ is the correlation sequence of $\{X(k)\}$, and $E[D_1]/(N-1)$ is the expected zero-crossing rate in discrete time.

In this chapter we present extensions of Rice's formula of the form $\frac{\kappa}{\pi}\sqrt{-\rho''(0)}$ where $\kappa \le 1$ or $\kappa \ge 1$, and $\rho(t)$ is the autocorrelation function of the process in question. Hence, given a non-Gaussian process and a Gaussian process, both stationary with the same autocorrelation, the Gaussian process may have less or more or an equal number of zero-crossings on the average.

Our approach is to first derive the expected zero-crossing rate in discrete-time (to obtain a cosine formula) and by an appropriate limiting argument arrive at the zero-crossing rate in continuous time. In particular, we derive analogues of the "cosine formula" and "Rice's formula" for a scaled-time mixture of a Gaussian process, for general mixtures of Gaussian processes, and for products of Gaussian processes.

To motivate our investigation, we first discuss a formal "orthant probability formula" for random processes satisfying mild stationarity requirements. Using a formal "cosine formula", a formal "orthant probability formula" is obtained from which we argue that, in general,

$$E[D] = \frac{\kappa}{\pi}\sqrt{-\rho''(0)} \tag{3.3}$$

for sufficiently smooth processes. Moreover, the fact that κ may be quite different than one in (3) serves as a warning that Rice's formula, (1), may not be indiscriminately applied in the non-Gaussian case (e.g. Chang *et al.* (1951) pg. 149, Ito and Donaldson (1971) pg. 236, Ou and Herrmann (1990) pg. 1398).

3.2 A Formal Orthant Probability Formula

Let $\{X(t)\}$, $-\infty < t < \infty$, be a stochastic process consisting of continuous random variables with mean zero and satisfying the "stationarity" requirement:

$$\begin{aligned} \Pr[X(t) \geq 0] &= \frac{1}{2} \\ \Pr[X(t) \geq 0, X(s) \geq 0] &= g(|t-s|) \end{aligned}$$

for some function $g(\cdot)$. For $t \in [0,1]$ and for a positive integer $N > 2$ we define the discrete time process

$$X_k \equiv X((k-1)\Delta),\ k = 1, 2, \cdots, N$$

such that

$$(N-1)\Delta = 1. \tag{3.4}$$

The interval $(0,1]$ is now partitioned into $N-1$ subintervals each of length Δ so that $\{X_k\}$ is simply $\{X(t)\}$ evaluated at the endpoints of the subintervals. Define the indicator,

$$d_k = I_{[\text{Sign change in } X_k, X_{k-1}]}.$$

Then

$$D_1 = \sum_{k=2}^{N} d_k$$

the number of sign changes in $X_k,\ k = 1, 2, \cdots, N$ approximates the number of zero-crossings of $\{X(t)\}$ for $t \in [0,1]$. Clearly D_1 depends on Δ and satisfies: $0 \leq D_1 \leq N-1$.

We are interested in a "correlation-like" quantity. If we define

$$r(\Delta) = \cos\left(\frac{\pi E[D_1]}{N-1}\right),$$

then,

$$\begin{aligned}
\Pr[X(t) \geq 0, X(t-\Delta) \geq 0] &= \frac{1}{2}\left[1 - \frac{E[D_1]}{N-1}\right] \\
&= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1}\left[\sin\left(\frac{\pi}{2} - \frac{\pi E[D_1]}{N-1}\right)\right] \\
&= \frac{1}{4} + \frac{1}{2\pi}\sin^{-1} r(\Delta).
\end{aligned}$$

Thus, $r(\Delta)$ acts to some extent as a correlation coefficient between $X(t)$ and $X(t-\Delta)$, regardless of whether or not $X(t)$ has moments of any order. In the stationary Gaussian case, however, $r(\Delta)$ becomes precisely the correlation coefficient between $X(t)$ and $X(t-\Delta)$. For more details see He and Kedem (1989), and Kedem (1991).

Remark: Note that in the preceding argument we could have used any monotone function defined on $[-\frac{\pi}{2}, \frac{\pi}{2}]$, not just the sine function, but our choice leads naturally to a convenient "cosine formula".

So far our construction uses minimal assumptions. Now assume that $E[d_t]$ is essentially proportional to Δ as Δ approaches 0:

$$E[d_t] = h(\Delta),$$

where, as $\Delta \to 0$, we have $h(\Delta) \to 0, h'(\Delta) \to \mu, h''(\Delta) \to c$, μ and c constants with $\mu > 0$. An example is $h(\Delta) = \mu\Delta + \Delta^3$. This is a reasonable assumption since Korolyook's theorem (Cramér and Leadbetter (1967), pg. 56) guarantees that the probability of a sign change between X_k, X_{k-1} satisfies

$$E[d_t] = \mu\Delta + o(\Delta)$$

as $\Delta \to 0$, if the stream of zero-crossings with intensity μ is both stationary and regular. When Korolyook's theorem holds, the intensity μ coincides with the expected number of zero-crossings per unit time.

Since

$$r(\Delta) = \cos \pi E[d_t],$$

it follows that $r(0) = 1$, and

$$\lim_{\Delta \to 0} r'(\Delta) = 0$$

and

$$\lim_{\Delta \to 0} r''(\Delta) = -(\pi\mu)^2$$

so that

$$E[D_c] \equiv \lim_{\Delta \to 0} E[D_1] = \mu$$

or

$$E[D_c] = \frac{1}{\pi}\sqrt{-r''(0)}.$$

Clearly, $r(\cdot)$ is not necessarily an autocorrelation function. Suppose however that the true autocorrelation, denoted by $\rho(\cdot)$, satisfies for some positive constant κ and sufficiently small Δ,

$$\rho(\kappa\Delta) = r(\Delta).$$

Then we would have

$$E[D_c] = \frac{\kappa}{\pi}\sqrt{-\rho''(0)}. \tag{3.5}$$

This heuristic argument points to the possibility that there may be processes for which the zero-crossing rate is given by (3.5) with κ greater or smaller than 1. We shall show this is in fact true.

3.2.1 Monotone transformations of a Gaussian process

Here we present examples of non-Gaussian processes with zero-crossing rates given by (3.5) with $\kappa \leq 1$. Smooth monotone transformations of Gaussian processes investigated in Chapter 1 provide such examples.

In the previous chapter the cosine formula (3.2) and Rice's formula (3.1) were extended to memoryless monotone transformations of a Gaussian process. There it was shown that for any real-valued function $\varphi(z)$ defined on $(-\infty, \infty)$, which is differentiable, strictly monotone, and for which $E[\varphi'(X(t))^2] < \infty$, the expected zero-crossing rate in continuous time of $\{Y(t)\}$ for $Y(t) = \varphi(X(t))$ is

$$E[D_c] = \frac{1}{\pi}\sqrt{\frac{\mathrm{Var}[\varphi(X(t))]}{E[\varphi'(X(t))^2]}}\sqrt{-\rho_Y''(0)}. \tag{3.6}$$

Note that in (3.6) $\rho_Y(t)$ is the normalized autocorrelation function of the transformed process $\{Y(t)\}$.

By Chernoff's inequality (Chernoff (1981), Houdre and Kagan (1995))

$$\mathrm{Var}[\varphi(X(t))] \quad \leq \quad E[\varphi'(X(t))^2] \tag{3.7}$$

we need only assume $E[\varphi'(X(t))^2] < \infty$ to guarantee $\mathrm{Var}[\varphi(X(t))] < \infty$ in (3.6). Moreover, from (3.7) we see that (3.6) is of the form (3.5) with $\kappa \leq 1$ (with equality iff φ is an affine mapping). This observation motivates the following definition.

Property 1: We say that a random process satisfies Property 1 if its expected zero-crossing rate per unit time, $E[D]$, satisfies

$$E[D] \leq \frac{1}{\pi}\sqrt{-\rho''(0)} \tag{3.8}$$

where $\rho(t)$ is the normalized autocorrelation of the process.

By (3.6) and (3.7) we see that Property 1 holds for monotone transformations of a Gaussian process. We next show it holds for a scaled-time mixture of a Gaussian process.

3.3 Scaled-Time Mixture of a Gaussian Process

Let $\{X(t)\}$ be a continuous time, zero-mean, variance one, stationary Gaussian process with autocorrelation function $\rho_X(t)$. In the sequel we assume $\{X(t)\}$ is mean square differentiable so $-\rho_X''(0) < \infty$, or equivalently

$$\rho_X(t) = 1 - \frac{\lambda_2}{2}t^2 + o(t^2)$$

as $t \to 0$ where λ_2 is the second-spectral-moment of the process (Leadbetter et al. (1983) pg. 151). All processes are separable with continuous sample paths (with probability one) and all random variables and processes are assumed to be real-valued.

For any random variable ξ, independent of $\{X(t)\}$, with or without finite moments, we define the scaled-time mixture of $\{X(t)\}$ by ξ to be the process $\{M(t)\}$, where $M(t) = X(\xi \cdot t)$.

The scaled-time mixture $\{M(t)\}$ is a strictly stationary process which is, in general, non-Gaussian. Note however, that, although the univariate distribution of $\{M(t)\}$ is standard normal, the finite-dimensional joint distributions are, in general, non-Gaussian. Thus, the mean and variance of $\{M(t)\}$ are zero and one respectively. Although we do not require the existence of moments in the above definition, we will assume ξ has a finite second-moment which guarantees a finite expected zero-crossing rate for $\{M(t)\}$.

The normalized autocorrelation function of $\{M(t)\}$, denoted $\rho_M(t)$, can be obtained by conditioning and is given by,

$$\rho_M(t) = E_\xi[E[X(\xi \cdot (t_0 + t))X(\xi \cdot t_0) \mid \xi]] = E_\xi[\rho_X(\xi \cdot t)] = \int_{-\infty}^{\infty} \rho_X(\xi \cdot t)dF_\xi(\xi) \quad (3.9)$$

where $F_\xi(\xi)$ is the probability distribution function of the random variable ξ.

To determine the zero-crossing rate in discrete time of $\{M(t)\}$ we again use the orthant probability of the pair of random variables $\{M(t_1), M(t_2)\}$

$$\Pr[M(t_1) \geq 0, M(t_2) \geq 0]. \tag{3.10}$$

As remarked earlier, (3.10) has a simple closed form for random variables which are jointly Gaussian. For that case, assuming $X(t_1), X(t_2)$ are both zero-mean and unit variance, the orthant probability is (see Ch. 4, Kedem 1995)

$$\Pr[X(t_1) \geq 0, X(t_2) \geq 0] = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho, \tag{3.11}$$

where ρ is the correlation coefficient of $X(t_1), X(t_2)$, so $\rho = E[X(t_1)X(t_2)] = \rho(t_2 - t_1)$.

Now for any continuous, zero-mean, symmetric random variables $M(t_1), M(t_2)$, the probability of a sign-change or zero-crossing can be expressed in terms of the orthant probability

$$2 \cdot \Pr[M(t_1) \leq 0, M(t_2) \geq 0] = 1 - 2 \cdot \Pr[M(t_1) \geq 0, M(t_2) \geq 0],$$

since $\Pr[M(t) \leq 0] = \frac{1}{2}$. By stationarity, the expected number of sign-changes or zero-crossings $E[D_1]$, in a sequence $\{M(1), \cdots, M(N)\}$ is

$$E[D_1] = (N-1)(1 - 2 \cdot \Pr[M(1) \geq 0, M(2) \geq 0]). \tag{3.12}$$

Recall that $\frac{E[D_1]}{N-1}$ is the normalized zero-crossing rate in discrete time and is independent of N. Using (3.11) in the above formula (i.e. take $X(t) = M(t)$) we arrive at (3.2) for the Gaussian case.

3.3.1 Cosine Formula for a Scaled-Time Mixture

Let $\{M(t)\}$ be a scaled-time mixture of a Gaussian process. Consider the discrete time process $\{M(k)\}$, obtained by sampling $\{M(t)\}$, where, $M(k) = X(\xi \cdot k)$ for $k = 0, \pm 1, \pm 2, \cdots$. The orthant probability (3.10) is obtained by conditioning,

$$\Pr[M(k) \geq 0, M(k+1) \geq 0] = E[I_{[X(\xi \cdot k) \geq 0, X(\xi \cdot (k+1)) \geq 0]}] \tag{3.13}$$

and using double expectation,

$$E[I_{[X(\xi \cdot k) \geq 0, X(\xi \cdot (k+1)) \geq 0]}] = E_\xi[E[I_{[X(\xi \cdot k) \geq 0, X(\xi \cdot (k+1)) \geq 0]} | \xi]]. \tag{3.14}$$

Using (3.11)

$$E[I_{[X(\xi \cdot k) \geq 0, X(\xi \cdot (k+1)) \geq 0]} | \xi] = \frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho_X(\xi).$$

By (3.12), the zero-crossing rate in discrete time for the process $\{M(k)\}$ is obtained,

$$\frac{E[D_1]}{N-1} = \frac{1}{2} - \frac{1}{\pi} \int_{-\infty}^{\infty} \sin^{-1} \rho_X(\xi) dF_\xi(\xi) \tag{3.15}$$

or

$$\frac{E[D_1]}{N-1} = \frac{1}{\pi} \int_{-\infty}^{\infty} \cos^{-1} \rho_X(\xi) dF_\xi(\xi). \tag{3.16}$$

3.3.2 Rice's Formula for a Scaled-Time Mixture

Dividing the interval $(0, 1]$ into subintervals of size δ, and then applying the same limiting argument used in the derivation of (3.1), it can be shown rigorously that the zero-crossing rate (per unit time) of a continuous time Gaussian process $E[D]$ is the limit as $\delta \to 0^+$ (from above) of

$$E[D] = \lim_{\delta \to 0^+} \frac{1}{\pi} \frac{\cos^{-1} \rho_X(\delta)}{\delta}. \tag{3.17}$$

(see Ylvisaker (1965), Kedem (1995, p. 129))

This same limiting procedure may be used for non-Gaussian processes by substituting the appropriate discrete time zero-crossing rate (or "cosine formula") in (3.17). Consequently, the zero-crossing rate in continuous time for the scaled-time mixture is obtained by,

$$\lim_{\delta\to 0^+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos^{-1}\rho_X(\delta\xi)}{\delta} dF_\xi(\xi). \tag{3.18}$$

Note that, if ξ has an atom at 0 it (i.e. the atom) does not contribute to the integral in (3.18). Thus, we assume, without loss of generality, that $\Pr[\xi = 0] = 0$. Furthermore, since $\rho_X(t)$ is an even function, we may rewrite (3.18) as

$$\lim_{\delta\to 0^+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos^{-1}\rho_X(\delta\xi)}{\delta} dF_\xi(\xi) = \lim_{\delta\to 0^+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos^{-1}\rho_X(\delta|\xi|)}{\delta|\xi|}|\xi| dF_\xi(\xi).$$

Since

$$\lim_{\delta\to 0^+} \frac{\cos^{-1}\rho_X(\delta)}{\delta} = \sqrt{-\rho_X''(0)} \tag{3.19}$$

and when $\delta \geq 1$ the ratio $|\frac{\cos^{-1}\rho_X(\delta)}{\delta}|$ is bounded by π, $|\frac{\cos^{-1}\rho_X(\delta)}{\delta}|$ is bounded for all $\delta > 0$. Therefore, by the bounded convergence theorem we may interchange the limiting operations and write,

$$E[D] = \lim_{\delta\to 0^+} \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos^{-1}\rho_X(\delta\xi)}{\delta} dF_\xi(\xi) = \frac{1}{\pi}\int_{-\infty}^{\infty} \lim_{\delta\to 0^+} |\xi| \frac{\cos^{-1}\rho_X(\delta|\xi|)}{\delta|\xi|} dF_\xi(\xi).$$

Taking the limit,

$$E[D] = \frac{1}{\pi}\int_{-\infty}^{\infty} \sqrt{-\rho_X''(0)}|\xi| dF_\xi(\xi) = \frac{1}{\pi}E[|\xi|]\sqrt{-\rho_X''(0)}. \tag{3.20}$$

Thus, rescaling time by the random variable ξ, we can speed up or slow down $E[D]$ by the factor $E[|\xi|]$. A good example is a pure stationary Gaussian sinusoid with frequency ω, for which time is rescaled by a positive constant $\xi = k$ (with probability one). Then clearly, $E[D]$ changes from $\frac{\omega}{\pi}$ to $\frac{k\omega}{\pi}$. But this is exactly what (3.20) gives.

In (3.20) the zero-crossing rate of $\{M(t)\}$ is given in terms of the autocorrelation function of the process $\{X(t)\}$. We wish to write (3.20) in terms of the autocorrelation function of the mixture process $\{M(t)\}$, itself. Using the representation (3.9) we see,

$$\rho_M(t) = E_\xi[\rho_X(\xi \cdot t)] \tag{3.21}$$

so that $\rho_M(0) = \rho_X(0) = 1$.

Assuming sufficient regularity conditions on $\rho_X(t)$ to justify the interchange of differentiation and expectation ($\rho_X''(t)$ bounded in a neighborhood of $t = 0$ is sufficient) and taking the limit as $t \to 0$,

$$\rho_M''(0) = \rho_X''(0)E[\xi^2]. \tag{3.22}$$

Thus, Rice's formula for the scaled-time mixture process $\{M(t)\}$ is

$$E[D] = \frac{1}{\pi}\frac{E[|\xi|]}{\sqrt{E[\xi^2]}}\sqrt{-\rho_M''(0)}. \tag{3.23}$$

By the Cauchy-Schwarz inequality we see that the *zero-crossing scaling factor* κ is

$$\frac{E[|\xi|]}{\sqrt{E[\xi^2]}},$$

which is strictly less than one except when $\xi = \xi_0$ (with probability one) for some constant ξ_0 in which case equality holds. Thus, scaled-time mixtures satisfy Property 1, eq. (3.8).

In the next section we derive the average zero-crossing rates in discrete time and continuous time for a general mixture of Gaussian processes.

3.4 Mixtures of Gaussian Processes

In this section we derive the cosine formula and Rice's formula for a process which is a mixture of Gaussian processes. We show for a certain subset of the class of Gaussian mixtures, those with suitable integrability conditions on the autocorrelation function, that the expected zero-crossing rate in continuous time satisfies Property 1 in (3.8).

Consider a denumerable collection of independent random processes $\{X_i(t)\}$, indexed on I, with $i \in I$ and $t \in (-\infty, \infty)$. We assume that each member of the collection $\{X_i(t)\}$, is defined on the same probability space and that all processes are stationary, mean square differentiable, and have continuous sample paths with probability one.

For each i, let $\{F_n^i\}$ denote the n-dimensional joint distribution function of the process $\{X_i(t)\}$. For any collection $\{p_i\}$, such that $p_i > 0$ and $\sum_{i\in I} p_i = 1$, define the mixture process $\{M(t)\}$ as the random process whose n-dimensional joint distribution functions are defined by,

$$F_{M,n}(\cdot) = \sum_{i\in I} p_i F_n^i(\cdot). \tag{3.24}$$

Kolmogorov's existence theorem (see Doob (1953), Billingsly (1986), Wise, et al. (1977)) guarantees the existence of a separable process defined on the same probability space as $\{X_i(t)\}$ with the above specified finite-dimensional joint distributions. This is the process we take as the mixture $\{M(t)\}$.

For our purposes we will assume that $\{X_i(t)\}$ is a countable collection of independent mean-zero, unit variance, mean-square differentiable Gaussian processes with normalized autocorrelation functions $\{\rho_i(t)\}$. We now find the zero-crossing rates for a Gaussian mixture $\{M(t)\}$, defined by the collection $\{X_i(t)\}$ and the so-called mixing probabilities $\{p_i\}$.

3.4.1 Cosine Formula for Gaussian Mixtures

We proceed as before and first find the orthant probability for the pair of variables $\{M(1), M(2)\}$. By conditioning and applying the Stieltjes-Sheppard arcsine law,

$$\Pr[M(1) \geq 0, M(2) \geq 0] = \sum_{i \in I} p_i \cdot [\frac{1}{4} + \frac{1}{2\pi} \sin^{-1} \rho_i(1)].$$

Again using stationarity, the expected number of zero-crossings, $E[D_1]$, in the sequence $\{M(1), \cdots, M(N)\}$ is

$$E[D_1] = (N-1)(1 - 2 \cdot \Pr[M(1) \geq 0, M(2) \geq 0]), \tag{3.25}$$

so that from the above expression for the orthant probability we obtain the cosine formula for the mixture,

$$\frac{E[D_1]}{N-1} = \frac{1}{\pi} \sum_{i \in I} p_i \cos^{-1} \rho_i(1). \tag{3.26}$$

Since the series in (3.26) is absolutely summable, the discrete time zero-crossing rate is obtained without any uniform smoothness conditions on the collection of autocorrelation functions $\{\rho_i(t)\}$. However, to obtain the continuous time zero-crossing rate we'll impose uniform smoothness conditions on the family $\{\rho_i(t)\}$ in order to guarantee a differentiable process and a finite zero-crossing rate.

3.4.2 Rice's Formula for Gaussian Mixtures

The zero-crossing rate in continuous time is obtained using the limiting argument as in (3.17) and (3.18),

$$E[D] = \lim_{\delta \to 0^+} \frac{1}{\pi} \sum_{i \in I} p_i \frac{\cos^{-1} \rho_i(\delta)}{\delta}. \tag{3.27}$$

Assuming we can interchange the limit and sum,

$$E[D] = \frac{1}{\pi} \sum_{i \in I} p_i \lim_{\delta \to 0^+} \frac{\cos^{-1} \rho_i(\delta)}{\delta} = \frac{1}{\pi} \sum_{i \in I} p_i \sqrt{-\rho_i''(0)}. \tag{3.28}$$

Equation (3.28) is not unexpected. Intuitively the average rate of zero-crossings for the mixture should be the weighted average of the zero-crossing rates for the individual processes, $\{X_i(t)\}$. This can also be seen by considering realizations of the mixture process. Realizations of $\{M(t)\}$ can be constructed by selecting realizations from the family $\{X_i(t)\}$; that is, we select a realization of the process $\{X_i(t)\}$ with probability p_i.

To show that Property 1 eq. (3.8) is satisfied, consider the autocorrelation function of the mixture $\rho_M(t)$,

$$\rho_M(t) = \sum_{i \in I} p_i \rho_i(t). \tag{3.29}$$

If the family $\{\rho_i(t)\}$ is sufficiently well behaved, (i.e. if the collection $\{\rho_i''(t)\}$ is uniformly bounded in a neighborhood of $t = 0$) we may interchange the limits so that in a neighborhood of zero,

$$\rho_M''(t) = \sum_{i \in I} p_i \rho_i''(t) \tag{3.30}$$

and in particular at $t = 0$,

$$\rho_M''(0) = \sum_{i \in I} p_i \rho_i''(0). \tag{3.31}$$

Finally, by the convexity of the square root function,

$$E[D] = \frac{1}{\pi}\sum_{i\in I} p_i\sqrt{-\rho_i''(0)} \leq \frac{1}{\pi}\sqrt{-\sum_{i\in I} p_i\rho_i''(0)}, \tag{3.32}$$

so Gaussian mixtures satisfy Property 1.

Since monotone transformations and mixtures of Gaussian processes both satisfy Property 1, it becomes of interest to determine if all processes derived from a Gaussian satisfy Property 1. This question is answered next.

3.5 Products of Gaussian Processes

In this last section we derive the expected zero-crossing rate for products of independent Gaussian processes.

Consider a collection of zero-mean, unit variance independent Gaussian processes $\{X_i(t)\}$ indexed over the positive integers ($\mathbf{X}^+$) with our usual assumptions. Again denote the respective normalized autocorrelation function of $\{X_i(t)\}$ by $\rho_i(t)$. For $M \in \mathbf{X}^+$ define the product process $\{W_M(t)\}$ by

$$W_M(t) = \prod_{i=1}^{M} X_i(t) \tag{3.33}$$

The product process $\{W_M(t)\}$ is stationary, with mean zero and unit variance. It is non-Gaussian for all $M > 1$.

Our motivation for using a product of Gaussians is illustrated by the following example. For the moment consider the process $W_2(t) = X_1(t) \cdot X_2(t)$ which is a product of two independent Gaussian processes $X_i(t)$, $i = 1, 2$. Since the $X_i(t)$ are independent, intuitively we would expect the location of the zeros of any two sample realizations from $X_1(t)$ and $X_2(t)$ to be independent. One then might guess that on average, the number of zeros of a sample realization of the product process $\{W_2(t)\}$ for $t \in [0,1]$ is the sum of the average number of zeros of the sample realizations of $X_1(t)$ and $X_2(t)$ for $t \in [0,1]$. As we show below this is indeed the case.

3.5.1 Cosine Formula for a Product of Gaussians

We first derive the cosine formula for the case M=2, then for arbitrary $M \in \mathbf{X}^+$ by using a recursive equation expressing the probability of a sign-change in the process $\{W_{M+1}(k)\}$ in terms of the probability of a sign-change in the process $\{W_M(k)\}$ and the probability of a sign-change in the process $\{X_{M+1}(k)\}, k = 0, \pm 1, \pm 2, \cdots$.

Take $\{X_1(k)\}$ and $\{X_2(k)\}$ to be independent Gaussian processes in discrete time as above. Define the product process, $\{W_2(k)\}$, in discrete time, by $W_2(k) = X_1(k) \cdot X_2(k)$. Now consider the probability of a sign-change or zero-crossing for the pair $\{W_2(k), W_2(k+1)\}$. Denote this probability by $\Pr[XC \quad W_2(k)]$. Similarly denote the probability of no sign-change or zero-crossing by $\Pr[\sim XC \quad W_2(k)]$. Then, by conditioning (or directly) we have $\Pr[XC \quad W_2(k)] =$

$$\Pr[XC \quad X_1(k)] \cdot \Pr[\sim XC \quad X_2(k)] + \Pr[XC \quad X_2(k)] \cdot \Pr[\sim XC \quad X_1(k)],$$

or equivalently,

$\Pr[XC \quad W_2(k)] =$

$$\Pr[XC \quad X_1(k)] + \Pr[XC \quad X_2(k)] - 2 \cdot \Pr[XC \quad X_1(k)] \cdot \Pr[XC \quad X_2(k)]. \quad (3.34)$$

Using the cosine formula (3.2) and (3.34) the expected zero-crossing rate in discrete time for $\{W_2(1), \cdots, W_2(N)\}$ is obtained as

$$\frac{E[D_1]}{N-1} = \frac{1}{\pi} \cos^{-1} \rho_1(1) + \frac{1}{\pi} \cos^{-1} \rho_2(1) - \frac{2}{\pi^2} \cos^{-1} \rho_1(1) \cdot \cos^{-1} \rho_2(1) \quad (3.35)$$

In particular if $\rho_1(t) = \rho_2(t)$, then (3.35) simplifies to

$$\frac{E[D_1]}{N-1} = \frac{2}{\pi^2}\cos^{-1}(\rho_1(1)) \cdot \cos^{-1}(-\rho_1(1)). \tag{3.36}$$

Now observe that we may write a recursive equation equating the probability of a sign-change in $\{W_M(k)\}$ in terms of the probability of a sign-change in $\{W_{M-1}(k)\}$ and the probability of a sign-change in $\{X_M(k)\}$ by conditioning and using the recursive representation, $W_M(t) = W_{M-1}(t) \cdot X_M(t)$. From (3.34) we see that,

$$\Pr[XC \quad W_M(k)] =$$

$$\Pr[XC \quad W_{M-1}(k)] + \Pr[XC \quad X_M(k)] - 2 \cdot \Pr[XC \quad W_{M-1}(k)] \cdot \Pr[XC \quad X_M(k)]. \tag{3.37}$$

Using the cosine formula for $\Pr[XC \quad X_j(k)] \quad j \in 1, ..., M$ we solve (3.37) and obtain

$$\Pr[XC \quad W_M(k)] =$$

$$\sum_{i=1}^{M-1} \left[\frac{1}{\pi}\cos^{-1}\rho_i(1) \cdot \prod_{j=i+1}^{M} (1 - \frac{2}{\pi}\cos^{-1}\rho_j(1)) \right] + \frac{1}{\pi}\cos^{-1}\rho_M(1) \tag{3.38}$$

so that the discrete time zero-crossing rate for $\{W_M(1), \cdots, W_M(N)\}$ is

$$\frac{E[D_1]}{N-1} = \sum_{i=1}^{M-1} \left[\frac{1}{\pi}\cos^{-1}\rho_i(1) \cdot \prod_{j=i+1}^{M} (1 - \frac{2}{\pi}\cos^{-1}\rho_j(1)) \right] + \frac{1}{\pi}\cos^{-1}\rho_M(1).$$

In particular, if all the autocorrelation functions are the same $\rho_i(t) = \rho(t)$, (3.38) simplifies to

$$\Pr[XC \quad W_M(k)] = \frac{1 - (1 - \frac{2}{\pi}\cos^{-1}\rho(1))^M}{2} \tag{3.39}$$

and the cosine formula becomes,

$$\frac{E[D_1]}{N-1} = \frac{1 - (1 - \frac{2}{\pi}\cos^{-1}\rho(1))^M}{2}.$$

For M an odd integer, $0 \leq \frac{E[D_1]}{N-1} \leq 1$ as $-1 \leq \rho(1) \leq 1$. For even M, $0 \leq \frac{E[D_1]}{N-1} \leq \frac{1}{2}$ as $0 \leq \rho(1) \leq 1$. It is also worth noting that for all processes considered in this Chapter it can be shown that the expected zero-crossing rate $\frac{E[D_1]}{N-1}$ is inversely proportional to the correlation coefficient ρ_1, that is, as ρ_1 increases $\frac{E[D_1]}{N-1}$ decreases and vice versa. It is believed, in general, that $\frac{E[D_1]}{N-1}$ is inversely related to ρ_1, however this is still an open problem.

3.5.2 Rice's Formula for Products of Gaussians

Lastly, we obtain the continuous time expected zero-crossing rate of the product process $W_M(t) = \prod_{i=1}^{M} X_i(t)$ by using the same limiting argument as before applied to (3.38). That is, the average zero-crossing rate per unit time is

$$E[D] = \lim_{\delta \to 0^+} \frac{1}{\delta} \sum_{i=1}^{M-1} \left[\frac{1}{\pi} \cos^{-1} \rho_i(\delta) \cdot \prod_{j=i+1}^{M} (1 - \frac{2}{\pi} \cos^{-1} \rho_j(\delta)) \right] + \lim_{\delta \to 0^+} \frac{1}{\delta} \frac{1}{\pi} \cos^{-1} \rho_M(\delta)$$

$$= \sum_{i=1}^{M-1} \left[\lim_{\delta \to 0^+} \frac{1}{\pi} \frac{\cos^{-1} \rho_i(\delta)}{\delta} \cdot \prod_{j=i+1}^{N} (1 - \frac{2}{\pi} \cos^{-1} \rho_j(\delta)) \right] + \frac{1}{\pi} \sqrt{-\rho_M''(0)}$$

$$= \sum_{i=1}^{M-1} \left[\lim_{\delta \to 0^+} \frac{1}{\pi} \frac{\cos^{-1} \rho_i(\delta)}{\delta} \cdot \lim_{\delta \to 0^+} \prod_{j=i+1}^{M} (1 - \frac{2}{\pi} \cos^{-1} \rho_j(\delta)) \right] + \frac{1}{\pi} \sqrt{-\rho_M''(0)},$$

but

$$\lim_{\delta \to 0^+} \prod_{j=i+1}^{M} (1 - \frac{2}{\pi} \cos^{-1} \rho_j(\delta)) = 1,$$

so

$$E[D] = \sum_{i=1}^{M} \frac{1}{\pi} \sqrt{-\rho_i''(0)}. \tag{3.40}$$

As remarked above, (3.40) verifies our intuition regarding the expected zero-crossing rate of the product $\{W_M(t)\}$ as simply the sum of the expected zero-crossing rates of $\{X_i(t)\}$ for $i = 1, ..., M$.

If we again suppose that $\rho_i(t) = \rho(t)$ for all i, then

$$E[D] = \frac{M}{\pi}\sqrt{-\rho''(0)}. \tag{3.41}$$

By direct calculation the autocorrelation function of $\{W_M(t)\}$ is simply the product of the $\{\rho_i(t)\}$ $i \in 1, ..., M$

$$\rho_{W_M}(t) = \prod_{i=1}^{M} \rho_i(t). \tag{3.42}$$

From (3.42) the second-spectral-moment of $\{W_M(t)\}$ is obtained

$$\lambda_2 = -\rho''_{W_M}(0) = -\sum_{i=1}^{M} \rho''_i(0). \tag{3.43}$$

Thus, if $\rho_i(t) = \rho(t)$, for all i, then from (3.41) and (3.42)

$$E[D] = \frac{M}{\pi}\sqrt{-\rho''(0)} = \frac{\sqrt{M}}{\pi}\sqrt{-M\rho''(0)} = \frac{\sqrt{M}}{\pi}\sqrt{-\rho''_{W_M}(0)}.$$

However, a Gaussian process with the same spectrum as $\{W_M(t)\}$ has an expected rate

$$\frac{1}{\pi}\sqrt{-\rho''_{W_M}(0)} \tag{3.44}$$

which is strictly less than $\frac{\sqrt{M}}{\pi}\sqrt{-\rho''_{W_M}(0)}$ $(M > 1)$ by a factor of $\sqrt{M}$. Therefore, $\{W_M(t)\}$ (for large M) is an example of a random process with an expected zero-crossing rate given by $\frac{\kappa}{\pi}\sqrt{-\rho''(0)}$ where $\kappa \gg 1$, and consequently Property 1 is not satisfied.

3.6 Summary

By the celebrated formula of Rice (1944) we know that the expected zero-crossing rate per unit time of a stationary, mean-square differentiable Gaussian process is given by

$$\frac{1}{\pi}\sqrt{-\rho''(0)}.$$

We have shown that there are non-Gaussian processes for which the expected zero-crossing rate per unit time is

$$\frac{\kappa}{\pi}\sqrt{-\rho''(0)}$$

with $\kappa \leq 1$ or $\kappa \geq 1$.

For monotone transformations and mixtures of a Gaussian process $\kappa \leq 1$, for products $\kappa \geq 1$. Moreover, these examples show that non-Gaussian processes exist which can have quite different zero-crossing rates-arbitrarily larger or smaller-than a Gaussian process with the same spectral density as that of the non-Gaussian process.

Chapter 4

Radar Detection via Level-Crossings of the Envelope Process

A radar system generally transmits a waveform which is both amplitude and phase modulated in a deterministic fashion. The transmitted signal, $S_T(t)$, is given by $S_T(t) = A(t)\cos[\omega_c t + \theta(t)]$, where the amplitude, $A(t)$, and the phase, $\theta(t)$, are known deterministic functions. The carrier frequency of the radar transmitter, ω_c, is a known constant. For a simple radar transmitter the amplitude and phase functions, $A(t)$ and $\theta(t)$, are slowly varying relative to the carrier frequency ω_c. This condition will indeed be met if ω_c is much greater than the largest frequency components in the spectra of $A(t)$ and $\theta(t)$. For this case, as we shall see, it is reasonable to identify $A(t)$ as the "envelope" of the signal $S_T(t)$.

When the transmitted signal backscatters off a source (i.e. target), the received signal, $S_R(t)$, is a randomly attenuated and phase distorted version of $S_T(t)$. The phase and amplitude modulation of $S_R(t)$ is, in part, due to radiation propagation effects and source kinematics which can modulate the radar cross-section of the target. Now if the radar receivers' noise characteristics are modeled by a sufficiently regular, ergodic, stationary process, then filtering with an ideal narrow bandpass filter, centered on the carrier frequency, ω_c, essentially converts the receiver noise to a narrow-band Gaussian process (Rosenblatt 1961, Davis 1961). The intuition here is as follows: A wide-band process will necessarily decorrelate "fast"(i.e. have a short decorrelation time). A narrow-band filter has long memory and allows for averaging samples of the input over a long period. Thus, for a wide-band input to a narrow-band filter, the output will contain a component which is a long period averaging of essentially uncorrelated samples. With constraints on the filter weights it should not be unexpected that a central limit theorem holds. Consequently, a reasonable mathematical model for $S_R(t)$ is a narrow-band Gaussian process. That is, $S_R(t) = R(t)\cos[\omega_c t + \phi(t)]$ where $R(t)$ and $\phi(t)$ are jointly stationary random processes with Rayleigh and Uniform marginal densities respectively.

In short, by first conditioning the radar receivers output by pre-filtering with an ideal narrow-band filter, centered on ω_c, we convert the receiver noise to a narrow-band Gaussian noise process, say $N(t)$. As long as the spectrum of the received signal, $S_R(t)$, is contained in the passband of the prefilter, we preserve $S_R(t)$ as well.

The detection problem can now be stated: Determine if a narrow-band Gaussian signal, $S_R(t)$, is present or not in the receiver output $Y(t)$. This detector may be handled as a decision problem, that is, as a hypothesis testing procedure:

H_0: no signal present, noise only $[Y(t) = N(t)]$

H_1: signal plus noise $[Y(t) = S_R(t) + N(t)]$.

If we assume the received signal and noise are statistically independent and jointly Gaussian, the Neyman-Pearson likelihood ratio test is optimal. Here the optimality criterion is maximum power for a fixed size test. Stated in signal processing vernacular, maximum probability of detection for a fixed false alarm rate.

This detection problem and variations of it are known collectively as "incoherent detection" or "partially coherent detection" processing. Optimum detector structures have been derived and investigated by many authors. For a comprehensive and thorough discussion see Van Trees (1968), pp. 333-366.

As an alternative to the optimal procedure for detecting a narrow-band Gaussian signal in narrow-band Gaussian noise, we consider a detector based on level-crossing counts. We detail an approach first proposed by Rainal (1966). His procedure for detecting weak narrow-band signals in narrow-band Gaussian noise uses the sample mean level-crossing counts of the "envelope" of the receiver output as a test statistic for detection processing. This approach, though not optimum, can be less computationally complex than the optimum detector, with apparently little penalty paid in terms of probability of detection performance. In subsequent sections we provide the details of Rainal's detector and formally verify the performance of his detector, he observed, via computer experiments. One assumption made by Rainal, but not rigorously proved, is that the mean level-crossings of the envelope of a Gaussian process are asymptotically normal. For Gaussian processes the level-crossing counts are asymptotically normal (Malevich 1969, Cuzick 1976, Slud 1991). Later, we prove asymptotic normality of the level-crossing count of the envelope of a bandpass Gaussian process and provide an integral expression for the variance of the envelope level-crossing counts, which can be numerically evaluated.

In the next section we define the envelope of a stationary random process and detail some of its properties used in subsequent sections. We shall make frequent use of Hilbert transforms of both deterministic and random functions in the sequel, thus we start with a review the appropriate definitions and properties as well.

4.1 Hilbert Transforms and Envelopes of Functions and Stationary Processes

4.1.1 Hilbert Transforms and Envelopes of Functions

We start with the definition of the Hilbert transform of a deterministic, real-valued, function.

The Hilbert transform of a real-valued function (non-random), $g(t)$, is defined as (see Titchmarsh 1948, pp.119-151),

$$\hat{g}(t) = -\frac{1}{\pi}\int_{0+}^{\infty} \frac{g(t+s) - g(t-s)}{s} ds \tag{4.1}$$

or, equivalently, as a Cauchy Principal Value integral (PV) at $s = t$.

$$\hat{g}(t) = \frac{1}{\pi}\mathrm{PV}\int_{-\infty}^{\infty} \frac{g(s)}{t-s} ds. \tag{4.2}$$

We recall the definition of the PV integral at $s = t$. Let $f(s) \in L^1(-\infty, t - \delta) \cup L^1(t+\delta, \infty,)$ for every $\delta > 0$. Then, the principal value integral of $f(s)$ is defined as (Rudin 1973 pg.165),

$$\mathrm{PV}\int_{-\infty}^{\infty} f(s)ds = \lim_{\delta \to 0+}(\int_{-\infty}^{t-\delta} \quad + \quad \int_{t+\delta}^{\infty})f(s)ds \tag{4.3}$$

when the limit exits. Thus, starting with (4.2)

$$\frac{1}{\pi}PV\int_{-\infty}^{\infty} \frac{g(s)}{t-s} ds = \frac{1}{\pi}(\int_{-\infty}^{t^-} + \int_{t+}^{\infty})\frac{g(s)}{t-s} ds = \frac{1}{\pi}\int_{-\infty}^{t^-} \frac{g(s)}{t-s} ds + \frac{1}{\pi}\int_{t+}^{\infty} \frac{g(s)}{t-s} ds. \tag{4.4}$$

By a change of variables the right hand side of (4.4) becomes

$$\frac{1}{\pi}\int_{0^+}^{\infty}\frac{g(t-s)}{s}ds-\frac{1}{\pi}\int_{0^+}^{\infty}\frac{g(t+s)}{s}ds \tag{4.5}$$

which is (4.1), provided that each individual integral exits. As with the Fourier transform, the Hilbert transform is likewise invertible, with inversion formula given by,

$$g(t)=\frac{1}{\pi}\int_{0}^{\infty}\frac{\hat{g}(t+s)-\hat{g}(t-s)}{s}ds \tag{4.6}$$

or equivalently as a PV integral at $s=t$,

$$g(t)=-\frac{1}{\pi}PV\int_{-\infty}^{\infty}\frac{\hat{g}(s)}{t-s}ds \tag{4.7}$$

We state, without proof, some of the salient properties of the Hilbert transform for later reference (see Bedrosian 1963, Bendat and Piersol 1986, pp. 489-492, Whalen 1971 pp.61-85). Denoting $\hat{g}(t)\doteq\mathcal{H}[g(t)]$,

Hilbert Property 1 *Linearity*:

$$\mathcal{H}[\alpha g_1(t)+\beta g_2(t)]=\alpha\mathcal{H}[g_1(t)]+\beta\mathcal{H}[g_2(t)]$$

Hilbert Property 2 *Parseval's*:

$$\int_{-\infty}^{\infty}g^2(t)dt=\int_{-\infty}^{\infty}\hat{g}^2(t)dt$$

Hilbert Property 3 *Convolution*:

$$\mathcal{H}[g_1(t)*g_2(t)]=\hat{g}_1(t)*g_2(t)=g_1(t)*\hat{g}_2(t)$$

Hilbert Property 4 *Modulation*:

$$\mathcal{H}[g(t)\cos\omega_o t]=g(t)\sin\omega_o t$$

provided that $g(t)$ is a bandlimited function and $\omega_o > 0$ is outside the support of the spectrum of $g(t)$.

From Hilbert Property 1 we see that the Hilbert transform is a linear operation. Moreover, in fact, it is a linear operator or filter on the space of $L^2(-\infty,\infty)$ functions into itself. That is, if $g(t) \in L^2(-\infty,\infty)$ then $\hat{g}(t) \in L^2(-\infty,\infty)$ (Titchmarsh pg. 122) so that $g \xrightarrow{\mathcal{H}} \hat{g}$ is an isometry. Using the equivalent definition given by (4.4), $\hat{g}(t)$ can be written as a convolution,

$$\hat{g}(t) = \mathcal{H}[g(t)] = \frac{1}{\pi} g(t) * \frac{1}{t}$$

where the transfer function of the linear operator or filter,

$$H(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} e^{-i\omega t} t^{-1} dt$$

is given by,

$$H(\omega) = \begin{cases} -i & \omega > 0 \\ 0 & \omega = 0 \\ i & \omega < 0 \end{cases} \tag{4.8}$$

Now to see that $H(\omega)$ is as given in (4.8), consider the sequence of $L^2(-\infty,\infty)$ functions $\{h_n(t)\}$, $n = 1, 2, 3, ...$, where

$$h_n(t) = \begin{cases} \frac{1}{\pi} t^{-1} & \frac{1}{n} < |t| < n \\ 0 & \text{else} \end{cases} \tag{4.9}$$

Now $h_n \to h$ pointwise, where $h(t) = \frac{1}{t}$. Also, for each n, the Fourier transform of $h_n(t)$, denoted by $\tilde{h}_n(\omega)$, is

$$\tilde{h}_n(\omega) = \int_{-\infty}^{\infty} h_n(t) e^{-i\omega t} dt = \frac{-2i}{\pi} \int_{\frac{1}{n}}^{n} \frac{\sin \omega t}{t} dt = \frac{-2i}{\pi} \int_{\frac{\omega}{n}}^{n\omega} \frac{\sin t}{t} dt.$$

For a fixed $\omega > 0$, $\tilde{h}_n(\omega) \to \frac{-2i}{\pi}\frac{\pi}{2} = -i$. Similarly, for $\omega < 0$, $\tilde{h}_n(\omega) \to i$. At $\omega = 0$, $\tilde{h}_n(0) = 0$. Now, $\{\tilde{h}_n\}$ are uniformly bounded so for any $g \in L^2(-\infty,\infty)$

we have

$$g * \tilde{h}_n \xrightarrow{L^2} \hat{g}.$$

We next define the envelope of a function.

The "envelope" of the function, $g(t)$, which we denote by, $A_g(t)$, is defined as (Bendat and Piersol pg. 487),

$$A_g(t) = [g^2(t) + \hat{g}^2(t)]^{\frac{1}{2}}. \tag{4.10}$$

If $g(t)$ is a narrow-band function, the above definition conforms to our intuitive understanding or notion of what the envelope of the function $g(t)$ should be. For example, let $g(t) = A(t)\cos\omega_0 t$. Using Hilbert Property 3: *Modulation*, $\hat{g}(t) = A(t)\sin\omega_0 t$, provided $\omega_0 > 0$ is outside the support of the spectrum of $A(t)$. Thus, our intuition is verified in this case by the fact that the envelope of $g(t)$ is $A(t)$. The following table of transform pairs and envelope functions further illustrates our intuitive notion of the envelope:

$g(t)$	$\hat{g}(t)$	$[g^2(t) + \hat{g}^2(t)]^{\frac{1}{2}}$
$A\cos(\omega t)$	$A\sin(\omega t)$	$\lvert A\rvert$
$A\sin(\omega t)$	$-A\cos(\omega t)$	$\lvert A\rvert$
$\dfrac{\sin(t)}{t}$	$\dfrac{1-\cos(t)}{t}$	$\left\lvert\dfrac{\sin(\frac{t}{2})}{\frac{t}{2}}\right\rvert$
$\dfrac{1}{1+t^2}$	$\dfrac{t}{1+t^2}$	$\dfrac{1}{\sqrt{1+t^2}}$
$\sin(\omega t)J_n(\omega_1 t)$	$\cos(\omega t)J_n(\omega_1 t)$	$\lvert J_n(\omega_1 t)\rvert$

where $J_n(\cdot)$ is the Bessel function of order $n = 0, 1, 2, \cdots$ and $0 < \omega < \omega_1$.

4.1.2 The Hilbert Transform of a Stationary Process

Let $\{X(t)\}$ be a zero-mean weakly stationary process. Using the spectral representation for real-valued processes (Cramér and Leadbetter 1967, pg. 137) one can write $X(t)$,

$$X(t) = \int_{0+}^{\infty} \cos(\omega t)\xi_1(d\omega) + \int_{0+}^{\infty} \sin(\omega t)\xi_2(d\omega) \tag{4.11}$$

where $\xi_1(d\omega)$ and $\xi_2(d\omega)$ are orthogonal random measures, $\xi_1(d\omega)$ is even and $\xi_2(d\omega)$ is odd. We assume the spectral distribution function, $F_X(\omega)$, of the process $\{X(t)\}$, is continuous and normalized so that

$$\int_0^{\infty} dG_X(\omega) = 1$$

where $G_X(\omega) = 2F_X(\omega)$. It then follows from (4.11) that the autocorrelation function, $\rho_X(\tau)$, of $\{X(t)\}$ is,

$$\rho_X(\tau) = \int_0^{\infty} \cos(\omega\tau) dG_X(\omega).$$

The "Hilbert Transform", $\hat{X}(t)$, of $X(t)$ can be defined as (Cramér and Leadbetter (1967), pg. 142),

$$\hat{X}(t) = \int_{0+}^{\infty} \sin(\omega t)\xi_1(d\omega) - \int_{0+}^{\infty} \cos(\omega t)\xi_2(d\omega). \tag{4.12}$$

From definition (4.12) it follows that the Hilbert transform (defined for stationary processes) is a linear operator or filter which maps stationary processes to stationary processes. The Hilbert transform can also be defined, equivalently, in the frequency domain, by its transfer function, $H(\omega)$ (Cramér and Leadbetter (1967), pp. 141-142),

$$H(\omega) = \begin{cases} -i & \omega > 0 \\ 0 & \omega = 0 \\ i & \omega < 0 \end{cases} \tag{4.13}$$

The pre-envelope of the process $\{X(t)\}$ is <u>defined</u> as the complex random process, $\{W(t)\}$, with real part $X(t)$ and and imaginary part $\hat{X}(t)$,

$$W(t) = X(t) + i\hat{X}(t).$$

The envelope of the stationary process $\{X(t)\}$, which we denote by $R(t)$, is <u>defined</u> analogously as in the non-random case. The envelope process is given by,

$$R(t) = [X^2(t) + \hat{X}^2(t)]^{\frac{1}{2}} = |W(t)|.$$

It is worth noting that the above definition for the envelope process, which appears to be different than that given by Rice (Rice 1944, pp. 81-82) is in fact the same. Rice shows that the underlying Gaussian process, which he writes as I(t), in our notation $X(t)$, can in fact be written as

$$I(t) = I_c(t) \cos \omega_c t - I_s(t) \sin \omega_c t = X(t) \tag{4.14}$$

where $I_c(t)$ and $I_s(t)$ are the so-called in-phase and quadrature components.

To see this, make the following change of variables for any $Y(t)$

$$I_c(t) = X(t) \cos \omega_c t + Y(t) \sin \omega_c t \tag{4.15}$$

$$I_s(t) = Y(t) \cos \omega_c t - X(t) \sin \omega_c t \tag{4.16}$$

Then (4.14) holds. If $Y = \hat{X}$, then $I_c(t)$ and $I_s(t)$ are uncorrelated. Rice then defines the envelope by,

$$R(t) = [I_c^2(t) + I_s^2(t)]^{\frac{1}{2}}.$$

However, upon using the Hilbert transform we obtain

$$R(t) = [I_c^2(t) + I_s^2(t)]^{\frac{1}{2}} = [X^2(t) + \hat{X}^2(t)]^{\frac{1}{2}},$$

and this is yet another justification for the use of $\hat{X}$ in defining the envelope. For an interesting survey paper on different definitions one may use for defining an envelope of narrow-band signals see Rice 1982.

The Hilbert transform, $\{\hat{X}(t)\}$, is obtained via a linear operation on $\{X(t)\}$, so we have immediately, that, if $\{X(t)\}$ is a stationary Gaussian process, so is $\{\hat{X}(t)\}$. Generally, if $\{X(t)\}$ is zero-mean and has a continuous spectral distribution function, then $\{\hat{X}(t)\}$ is zero-mean as well, and moreover, since ξ_1 and ξ_2 are uncorrelated, has exactly the same spectrum and autocorrelation function as $\{X(t)\}$ (Cramér and Leadbetter (1967) pg. 142). So, in particular,

$$\rho_{\hat{X}}(\tau) = \rho_X(\tau).$$

We assume in the sequel that all spectral distribution functions considered are continuous unless noted otherwise.

The cross-correlation function of $\{X(t)\}$ and $\{\hat{X}(t)\}$, $\rho^*(\tau)$, is also of interest and is given by (Cramér and Leadbetter (1967) pg. 142)

$$\rho^*(\tau) = E[X(t)\hat{X}(t+\tau)] = \int_0^\infty \sin(\omega\tau)dG_X(\omega).$$

The above integral expression for the cross-correlation is in fact just the Hilbert transform of the autocorrelation function, $\rho_X(\tau)$, (Zakai 1960, pg.556, eq. 3) so that,

$$\rho^*(\tau) = \hat{\rho}_X(\tau) = \mathcal{H}[\rho_X(\tau)]. \tag{4.17}$$

Since the Hilbert transform of an even-function is an odd-function, $\rho^*(\tau)$ is odd and in particular $\rho^*(0) = 0$. Thus, if $\{X(t)\}$ is Gaussian, the pair of random variables, $(X(t), \hat{X}(t))$, are independent for all t.

We next consider the envelope and the squared envelope of a Gaussian process.

4.1.3 The Envelope of a Gaussian Process

We first detail the derivation originally given by Cramér and Leadbetter (1967, pp. 248-255), which is based on the work of Rice (1944, pp. 81-84), for the joint density of the envelope, $\{R(t)\}$, and its mean-square derivative, $\{R'(t)\}$. We will assume the underlying process $\{X(t)\}$ is Gaussian and for convenience mean-zero with variance one.

The main results used later are: the marginal distributions of the envelope, $R(t)$, and its mean-square derivative, $R'(t)$, are respectively, Rayleigh and Gaussian. For each t, $R(t)$ and $R'(t)$ are independent, and hence by Rice's formula

$$\int_{-\infty}^{\infty} |\dot{r}| p_{R,R'}(u,\dot{r}) d\dot{r}$$

the expected u ($u > 0$) level-crossing rate per unit time of $R(t)$ is,

$$ED_u = (\frac{2\Delta}{\pi})^{\frac{1}{2}} u e^{-\frac{u^2}{2}}$$

where Δ is the variance of $R'(t)$ and $D_u = N_u[0,1]$ is the number of u level-crossings in the unit interval. Lastly, the variance of the number of level-crossings per unit time is given by the

$$Var(D_u) = E[D_u] - (E[D_u])^2 + 2\int_{0^+}^{1} (1-\tau)\psi(\tau) d\tau$$

where,

$$\psi(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{r_1}\dot{r_2}| p_{R_1,R_1',R_2,R_2'}(u,\dot{r_1},u,\dot{r_2}) d\dot{r_1} d\dot{r_2}$$

and $p_{R_1,R_1',R_2,R_2'}$ is the joint density of $(R(0), R'(0), R(\tau), R'(\tau))$. The formula above is sometimes called the Bendat-Rice formula (Bendat 1958, pg. 396, eq. 10-121) but has been given by many authors.

4.1.4 The joint Density of $R(t)$ and $R'(t)$

The joint density of $R(t)$ and its mean-square derivative, $R'(t)$, can be obtained as the limit as $\tau \to 0$ of the joint density of $R(t)$ and $\frac{1}{\tau}[R(t+\tau) - R(t)]$. We start with the derivation of the joint density of $R(t)$ and $R(t+\tau)$. As noted above, this derivation is essentially that given by Rice (1944, pp. 81-84).

Consider the jointly normal random variables $X(t)$, $\hat{X}(t)$, $X(t+\tau)$, $\hat{X}(t+\tau)$. Again for convenience we assume mean-zero and variance one. The covariance matrix is obtained using (4.17) and is

$$\begin{bmatrix} 1 & 0 & \rho & \rho^* \\ 0 & 1 & -\rho^* & \rho \\ \rho & -\rho^* & 1 & 0 \\ \rho^* & \rho & 0 & 1 \end{bmatrix}$$

where $\rho = \rho_X(\tau)$ and $\rho^* = \rho^*(\tau)$. The inverse of the above covariance matrix is easily obtained as

$$A^{-1} \begin{bmatrix} 1 & 0 & -\rho & -\rho^* \\ 0 & 1 & \rho^* & -\rho \\ -\rho & \rho^* & 1 & 0 \\ -\rho^* & -\rho & 0 & 1 \end{bmatrix}$$

where $A = 1-\rho^2-\rho^{*2}$. Hence, the joint density of $X(t)$, $\hat{X}(t)$, $X(t+\tau)$, $\hat{X}(t+\tau)$, which we denote by, $f_{X,\tau}(x_1, x_2, x_3, x_4)$, is

$$\frac{1}{4\pi^2 A} \exp\{-\frac{1}{2A}[(x_1^2+x_2^2+x_3^2+x_4^2)-2\rho(x_1x_3+x_2x_4)-2\rho^*(x_1x_4-x_2x_3)]\}. \quad (4.18)$$

By changing variables,

$$x_1 = R_1\cos\theta_1 \quad x_2 = R_1\sin\theta_1$$

$$x_3 = R_2\cos\theta_2 \quad x_4 = R_2\sin\theta_2$$

and integrating over θ_1 and θ_2, the joint density of $R(t)$ and $R(t+\tau)$ is obtained (with some further coordinate transformations) as

$$\frac{R_1R_2}{\pi A} e^{-(R_1^2+R_2^2)/2A} \int_0^\pi \cosh\{[\frac{R_1R_2(\rho^2+\rho^{*2})^{\frac{1}{2}}}{A}]\cos\phi\}d\phi. \quad (4.19)$$

The integral in (4.19) can be evaluated in terms of the zero-order modified Bessel function of the first kind, $I_0(z)$, (Abramowitz and Stegun, 1972 pp. 374-378) and so finally,

$$p_{R(0),R(\tau)}(R_1, R_2) = \frac{R_1 R_2}{\pi A} e^{-(R_1^2+R_2^2)/2A} I_0\{\frac{R_1 R_2}{A}(\rho^2 + \rho^{*2})^{\frac{1}{2}}\}. \tag{4.20}$$

With (4.20) the joint distribution of $R(t)$ and $R'(t)$ is obtained and given by

$$p_{R(t),R'(t)}(R, R') = (2\pi\Delta)^{-\frac{1}{2}} \exp(-R'^{2/2\Delta}) R \exp^{-R^2/2} \tag{4.21}$$

Since the univariate density of $\{R(t)\}$ is Rayleigh and and $\{R'(t)\}$ is Gaussian, mean zero, variance Δ, we see that $R(t)$ and $R'(t)$ are independent for each t.

Using (4.20) the covariance function of $\{R(t)\}$ can be calculated, with some work (the two-fold integration is a bit involved), and is

$$E[R(t)R(t+\tau)] = \frac{\pi}{2} {}_2F_1(-\frac{1}{2}, -\frac{1}{2}; 1; k_0^2), \tag{4.22}$$

where, $k_0^2 = \rho^2 + \rho^{*2} \leq 1$ and ${}_2F_1(\alpha, \beta; \gamma; x)$ is the Gaussian hypergeometric function (see Middleton 1960, pp.1076-1077) which is represented by the series

$${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{\gamma}\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\gamma(\gamma+1)2!}x^2 + \cdots \quad |x| < 1. \tag{4.23}$$

Equation (4.22), giving the covariance in terms of the hypergeometric function, ${}_2F_1$, was originally given by Uhlenbeck (1943) but is hinted at by Rice (1944 pg. 84 eq. 3.7-13) as well. For our special case, where $\alpha = \beta = -\frac{1}{2}$ and $\gamma = 1$, the covariance is given by a power series in k_0^2 (which depends on τ),

$$E[R(t)R(t+\tau)] = \frac{\pi}{2}(1 + \frac{k_0^2}{4} + \frac{k_0^4}{64} + \cdots). \tag{4.24}$$

The power series clearly converges for all $k_0^2 \leq 1$.

It is interesting to note by (4.24), that if we take the Fourier transform of the autocovariance of the envelope, $E[R(t)R(t+\tau)] - E[R(t)]^2$, noting $E[R(t)]^2 = \frac{\pi}{2}$, to obtain the power spectrum we see that, in general, the envelope process may not be bandlimited. To see this assume that the spectrum of $\{X(t)\}$ has a continuous component. The power spectrum of the envelope, $P_R(\omega)$, is given by the termwise Fourier transform of

$$\frac{\pi}{2}(\frac{k_0^2}{4}+\frac{k_0^4}{64}+\cdots)$$

or

$$P_R(\omega)=\frac{\pi}{2}(\frac{\tilde{k}_0*\tilde{k}_0}{4}+\frac{\tilde{k}_0*\tilde{k}_0*\tilde{k}_0*\tilde{k}_0}{64}+\cdots). \tag{4.25}$$

Assume the spectrum of $\{X(t)\}$ is bandlimited. Then, by considering each successive term in the series for $P_R(\omega)$, we see that each higher-order convolution of $\tilde{\rho}(\omega)=f_X(\omega)$ with itself effectively doubles the bandwidth, thus guaranteeing that the support of the spectrum is unbounded.

An example, and one we will use in the sequel, is provided by the envelope of an ideal bandpass process. Let $\{X(t)\}$ be a Gaussian process with a spectral density function which is constant over the frequency intervals $(-\omega_c-\delta,-\omega_c+\delta)$ and $(\omega_c-\delta,\omega_c+\delta)$ and zero elsewhere. The center frequency is said to be ω_c and the bandwidth is 2δ. Assume the total power of this ideal bandpass process is unity. Then, the autocorrelation function is given by

$$\rho(\tau)=\frac{\sin\ \delta\tau}{\delta\tau}\cos\omega_c\tau$$

. Using Hilbert Property 4 *Modulation* we see

$$\hat{\rho}(\tau)=\frac{\sin\ \delta\tau}{\delta\tau}\sin\omega_c\tau$$

and so k_0^2 is given by,

$$k_0^2\equiv k_0^2(\tau)=\rho^2(\tau)+\hat{\rho}^2(\tau)=(\frac{\sin\ \delta\tau}{\delta\tau})^2.$$

Note that in this example the spectral support of $k_0^2(\tau)$ is $[-2\delta, 2\delta]$. This will, in fact, turn out to be the spectral support of the squared envelope of the ideal bandpass process as we see next.

4.1.5 The Squared Envelope Process

For the purpose of counting mean level-crossings of the envelope process we can use the squared envelope process instead. That is, the u level-crossings of $R(t)$ are, of course, the u^2 level-crossings of $R^2(t)$.

As we saw in the last section the autocorrelation function and spectral density of the envelope process $\{R(t)\}$ are given by infinite series expansions. This is in contrast to the squared envelope process, $\{R^2(t)\}$, whose autocorrelation and spectral density are given by simpler looking expressions which are easily obtained in terms of $\rho_X(\tau)$ and $f_X(\omega)$. We will see that the autocorrelation is,

$$\rho_{R^2}(\tau) = \rho_X^2(\tau) + \hat{\rho}_X^2(\tau) \tag{4.26}$$

and thus, by the convolution theorem, the spectral density is

$$f_{R^2}(\omega) = f_X(\omega) * f_X(\omega) + \tilde{f}_X(\omega) * \tilde{f}_X(\omega) \tag{4.27}$$

where $\tilde{f}_X(\omega)$ is the Fourier transform of $\hat{\rho}_X(\tau)$.

To obtain the autocorrelation of the squared envelope, $\{R^2(t)\}$, we compute the following expectation,

$$\begin{aligned}
E[R^2(t+\tau)R^2(t)] &= E[(X^2(t+\tau) + \hat{X}^2(t+\tau))(X^2(t) + \hat{X}^2(t))] && (4.28)\\
&= E[X^2(t+\tau)X^2(t)] + E[X^2(t)\hat{X}^2(t+\tau)] && (4.29)\\
&\quad + E[X^2(t+\tau)\hat{X}^2(t)] + E[\hat{X}^2(t+\tau)\hat{X}^2(t)] && (4.30)\\
& && (4.31)
\end{aligned}$$

Now using the fact that X and $\hat{X}$ are jointly Gaussian, we assume again mean zero and unit variance, the individual expectations are given by,

$$\begin{aligned}
E[X^2(t+\tau)X^2(t)] &= E^2[X^2(t+\tau)]E^2[X^2(t)] + 2E^2[X(t+\tau)X(t)] && (4.32)\\
&= 1 + 2\rho_X^2(\tau) && (4.33)\\
E[X^2(t)\hat{X}^2(t+\tau)] &= E^2[X^2(t)]E^2[\hat{X}^2(t+\tau)] + 2E^2[X(t)\hat{X}(t+\tau)] && (4.34)\\
&= 1 + 2\hat{\rho}_X^2(\tau) && (4.35)\\
E[X^2(t+\tau)\hat{X}^2(t)] &= E^2[X^2(t+\tau)]E^2[\hat{X}^2(t)] + 2E^2[X(t+\tau)\hat{X}(t)] && (4.36)\\
&= 1 + 2\hat{\rho}_X^2(-\tau) && (4.37)\\
E[X^2(t)\hat{X}^2(t+\tau)] &= E^2[X^2(t)]E^2[\hat{X}^2(t+\tau)] + 2E^2[X(t)\hat{X}(t+\tau)] && (4.38)\\
&= 1 + 2\rho_X^2(\tau). && (4.39)\\
& && (4.40)
\end{aligned}$$

Using $\hat{\rho}_X(-\tau) = -\hat{\rho}_X(\tau)$ and collecting terms,

$$E[R^2(t+\tau)R^2(t)] = 4 + 4\rho_X^2(\tau) + 4\hat{\rho}_X^2(\tau). \tag{4.41}$$

Now

$$E[R^2(t)] = E[X^2(t) + \hat{X}^2(t)] = 2E[R^4(t)] = E[(X^2(t) + \hat{X}^2(t))^2] = 8 \tag{4.42}$$

so $\sigma_{R^2}^2 = 4$, and hence, the autocorrelation, $\rho_{R^2}(\tau)$, of $\{R^2(t)\}$ is,

$$\begin{aligned}
\rho_{R^2}(\tau) &= \frac{E[R^2(t+\tau)R^2(t)] - E[R^2(t+\tau)]E[R^2(t)]}{\sigma_{R^2}^2} && (4.43)\\
&= \rho_X^2(\tau) + \rho_{\hat{X}}^2(\tau) && (4.44)
\end{aligned}$$

If the underlying Gaussian process is an ideal bandpass process, we see from (4.44) that

$$\rho_{R^2}(\tau) = \rho_X^2(\tau) + \rho_{\hat{X}}^2(\tau) = \left(\frac{\sin\,\delta\tau}{\delta\tau}\right)^2 \tag{4.45}$$

and hence the spectral density is

$$f_{R^2}(\omega) = \begin{cases} \frac{1}{4\delta^2}\omega + \frac{1}{2\delta} & -2\delta < \omega < 0 \\ \frac{1}{2\delta} - \frac{1}{4\delta^2}\omega & 0 \leq \omega < 2\delta \\ 0 & \text{else} \end{cases} \tag{4.46}$$

Using (4.25) it is interesting to observe that for this particular example, when $\{X(t)\}$ is an ideal bandpass process, the power spectrum of $\{R^2(t)\}$ is bandlimited while the spectral support of $\{R(t)\}$ is the whole real line.

From (4.20) the joint density of $R^2(t)$ and $R^2(t+\tau)$ is easily obtained by a change of variables and is given

$$p_{R^2(0),R^2(\tau)}(U,V) = \frac{1}{4\pi A}e^{-(U+V))/2A}I_0\{\frac{\sqrt{UV}}{A}(\rho^2 + \rho^{*2})^{\frac{1}{2}}\}. \tag{4.47}$$

To obtain the joint probability density $R^2(t)$ and its mean-square derivative, $2R(t)R'(t)$, we first show that, $2R(t)R'(t)$, is indeed the mean-square derivative, and then by a simple change of variables, obtain the joint density of $(R^2(t), 2R(t)R'(t))$ from the joint density of $(R(t), R'(t))$.

To see that the mean-square derivative of, $\{R^2(t)\}$, at time t is $\{2R(t)R'(t)\}$ consider the following limits,

$$\lim_{\delta\to 0}[R(t+\delta) + R(t)] \overset{L^2}{\to} 2R(t) \tag{4.48}$$

$$\lim_{\delta\to 0}\frac{[R(t+\delta) - R(t)]}{\delta} \overset{L^2}{\to} R'(t) \tag{4.49}$$

Since we assume $\{R(t)\}$ is mean-square differentiable, this implies mean-square continuity (the autocorrelation function is continuous at origin) so both the above limits hold. Now if $X_n \overset{L^2}{\to} X$ and $Y_n \overset{L^2}{\to} Y$ then $X_nY_n \overset{L^2}{\to} XY$ (Yaglom pg. 63) so

$$[R(t+\delta) + R(t)]\frac{[R(t+\delta) - R(t)]}{\delta} = \frac{R^2(t+\delta) - R^2(t)}{\delta} \overset{L^2}{\to} 2R(t)R'(t)$$

The joint density of $R^2(t)$ and $2R(t)R'(t)$ is obtained using the change of variables formula for the transformation,

$$U(t) = R^2(t) \tag{4.50}$$

$$V(t) = 2R(t)R'(t). \tag{4.51}$$

$$\tag{4.52}$$

The Jacobian of the above transformation is $\frac{1}{4U(t)}$. Using (4.21) the joint density of $R^2(t)$ and $2R(t)R'(t)$ is,

$$p_{R^2,2RR'}(U,V) = \frac{1}{\sqrt{32\pi U\Delta}}e^{-(\frac{V^2}{8U\Delta})}e^{-\frac{U}{2}}. \tag{4.53}$$

4.2 Level-Crossing Based Detector

As an alternative to the optimal procedure for detecting a narrow-band Gaussian signal in narrow-band Gaussian noise we consider a detector based on level-crossing counts of the envelope of the observed process.

Following Rainal's (1964) procedure, the envelope of the received signal is obtained and the mean level-crossing counts of the envelope are used as a test statistic for detection processing, that is, to determine whether a narrow-band Gaussian signal is present or not in the narrow-band Gaussian noise.

Rainal assumes, without proof, that the level-crossing counts are asymptotically normal (for observations over a large time interval), and consequently, a test of a given significance level is then determined by the variance (asymptotic) of the mean level-crossing count. Hence, the asymptotic variance of the crossing counts is needed to set an appropriate threshold for a fixed probability of false alarm. This is the usual Neyman-Pearson criterion. Rainal shows experimentally that his less complex level-crossing based detector is competitive with the near-optimal quadratic detector, also known as the square-law detector. Optimal linear-quadratic detectors for Gaussian systems are well known. For a general discussion, including both Gaussian and non-Gaussian systems, see (Picinbono and Duvaut, 1988).

In the next two sections we derive an expression for the variance of the u-level-crossings ($u > 0$) of the envelope process and then prove asymptotic normality of the crossing counts. We start with the general formula for the variance of the crossings.

4.2.1 Variance of the Level-Crossing Count

The formula for the variance of the number of zero-crossings in an interval, $[0, T]$, has its roots in the work of Rice (1944). This formula has been investigated by many authors over the last 40 years with emphasis on necessary and sufficient conditions for a finite second moment.

One of the earliest papers, if not the first, that deals with the variance of the number of zero-crossings is Steinberg et al., 1955. In their paper an explicit formula (equation 40) is given for the mean-square number of zeros in the interval $[0, T]$ of a Gaussian process. The formula for the variance includes expressions which depend on the autocorrelation function and its first two derivatives.

Let $D = N[0, T]$ be the number of zero-crossings of $\{X(t)\}$ in the interval $[0, T]$. Assume $\{X(t)\}$ is a zero-mean sufficiently smooth stationary process. Then,

$$\mathrm{Var}(D) = E[D] - (E[D])^2 + 2\int_{0^+}^{T} (T - \tau)\psi(\tau)d\tau \tag{4.54}$$

where,

$$\psi(\tau) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\dot{x}_1 \dot{x}_2| p_{X_1,X_1',X_2,X_2'}(0,\dot{x}_1,0,\dot{x}_2) d\dot{x}_1 d\dot{x}_2 \tag{4.55}$$

and $p_{X_1,X_1',X_2,X_2'}$ is the joint density of $(X(0), X'(0), X(\tau), X'(\tau))$. To obtain the variance of the u-level-crossings of $\{X(t)\}$, for any u, simply replace $p_{X_1,X_1',X_2,X_2'}(0,\dot{x}_1,0,\dot{x}_2)$ by $p_{X_1,X_1',X_2,X_2'}(u,\dot{x}_1,u,\dot{x}_2)$.

The above formula, (4.54), is in fact a general formula, which is applicable to a wide class of random processes, and may be adapted to non-stationary processes as well. A thorough discussion for the general case including rigorous mathematical formulation and proof is in Cramér H. and M. R. Leadbetter (1967), pp. 202-212. A general treatment of higher order product moments of level-crossing counts for processes with absolutely continuous sample paths can be found in Marcus 1977.

When $\{X(t)\}$ is a mean zero, unit variance, Gaussian process with autocorrelation function $\rho(\tau)$, (4.54) becomes (Bendat 1958, pp. 398-401)

$$\mathrm{Var}(D) = E[D] - (E[D])^2 + \frac{2}{\pi^2} \int_{0^+}^{T} (T-\tau) \frac{1 + g(\tau) \arctan g(\tau)}{1-\rho^2(\tau)} \sqrt{h(\tau)} d\tau, \tag{4.56}$$

where

$$\begin{aligned} h(\tau) &= [1-\rho^2(\tau)][\rho''(0) - \rho''(\tau)] + 2[\rho''(0) - \rho''(\tau)\rho(\tau)]\rho'^2(\tau) + \rho'^4(\tau) \\ g(\tau) &= \frac{[1-\rho^2(\tau)]\rho''(\tau) + \rho'^2(\tau)\rho(\tau)}{\sqrt{(1-\rho^2(\tau))h(\tau)}} \end{aligned}$$

Necessary and sufficient conditions for the variance of the number of zero-crossings of $\{X(t)\}$ to be finite can be found in Geman 1972. The conditions are given in terms of the second derivative of the autocorrelation function, $\rho(\tau)$, of $\{X(t)\}$ and are: (1) $\rho''(0)$ finite and (2)

$$\int_0^{\delta} \frac{\rho''(\tau) - \rho''(0)}{\tau} d\tau < \infty \text{ for some } \delta > 0. \tag{4.57}$$

4.2.2 Variance for the Envelope Process

In this section we obtain an expression for the variance of the u-level-crossing count ($u > 0$) of the envelope of a symmetric bandpass Gaussian process. We assume the underlying process is Gaussian, zero-mean, unit variance, with a one-sided power spectral density, $g_X(\omega)$, which is symmetric about the positive midband frequency, $\omega_c > 0$. That is, for any $\delta \in [0, \omega_c]$ we have $g_X(\omega_c - \delta) = g_X(\omega_c + \delta)$.

Unless otherwise stated, we understand $\{R(t)\}$ to be the envelope of the symmetric bandpass Gaussian process, whose autocorrelation function, $\rho(\tau)$, is given by,

$$\rho(\tau) = \int_0^\infty \cos(\omega\tau) dG_X(\omega) = \int_0^\infty \cos(\omega\tau) g_X(\omega) d\omega. \qquad (4.58)$$

By using (4.14) we can write

$$X(t) = I_c(t)\cos\omega_c t - I_s(t)\sin\omega_c t$$

and

$$R^2(t) = I_c^2(t) + I_s^2(t),$$

where $\{I_c(t)\}$ and $\{I_s(t)\}$ are the so-called in-phase and quadrature components respectively. $\{I_c(t)\}$ and $\{I_s(t)\}$ are, independent, identical, Gaussian processes, zero-mean, unit variance with power spectral density, $g_I(\omega)$, given by

$$g_I(\omega) = \frac{1}{2}\{g_X(\omega - \omega_c) + g_X(\omega + \omega_c)\}.$$

The results obtain in this section on the variance rely heavily on the work of Rice (1958). In his paper entitled, " Duration of Fades in Radio Transmission", Rice is concerned with obtaining the probability density function of the interval length between zero-crossings for a particular class of Gaussian processes. He also considers the probability density function for the interval length between u-level-crossings of the envelope for this same class of Gaussian processes.

Rice approximates the density function for the interval between crossings by considering related conditional probability density functions. Let $p(\tau, u)$ denote the probability density function for the length of the interval when $R(t) < u$ (i.e., $p(\tau, u)d\tau$ is the probability that the interval length is between τ and $\tau + d\tau$). Rice argued that as a "first approximation " to the density function $p(\tau, u)$, one could use the conditional probability that an upcrossing occurs at time, τ, given a downcrossing occurred at time 0, we will denote this conditional probability by $p_1(\tau, u)$. Rice maintained that $p_1(\tau, u)$ should be close to the actual density function, $p(\tau, u)$, especially for small τ. (Finding an expression for $p(\tau, u)$ was known to be difficult, and in fact, it is still an open problem today.)

Since conditional probabilities must be approached with care and depend on the limiting process itself, to be precise, all conditional probabilities are to be understood in the horizontal-window averaging sense (Kac and Slepian, 1959 pg. 1216, eq. 2.1 and Cramér H. and M. R. Leadbetter (1967), pp. 219-223.). That is, we will use the following definition:

The probability of the event $\{R(t) \in S\}$ conditioned on $R(0) = u$, denoted by, $\Pr[R(t) \in S \mid R(0) = u]$ is defined by the following limit,

$$\lim_{\delta \to 0} \Pr[R(t) \in S \mid R(t) = u \text{ for some } t \in [-\delta, 0]], \tag{4.59}$$

provided the limit exits.

The conditioning event A we need for determining $p_1(\tau, u)$ is $A = \{R(t) = u \text{ for some } t \in [-\delta, 0] \text{ and } R'(t) > 0\}$. By Korolyook's theorem the limiting behavior of this probability is given by,

$$\lim_{\delta\to 0}\Pr[R(t)=u \text{ for some } t\in[-\delta,0] \text{ and } R'(t)>0] = (\frac{\Delta}{2\pi})^{\frac{1}{2}} u e^{-\frac{u^2}{2}}\cdot\delta + o(\delta),$$

which is just the average u-level-upcrossing rate per unit time of $R(t)$ times δ plus a $o(\delta)$. Thus,

$$p_1(\tau,u)\cdot(\frac{\Delta}{2\pi})^{\frac{1}{2}} u e^{-\frac{u^2}{2}} = \int_{-\infty}^{0} d\dot{r}_1 \int_0^{\infty} d\dot{r}_2 |\dot{r}_1\dot{r}_2| p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2). \quad (4.60)$$

Rice (1958, pp. 611-613, eq. 97) completes the required integration and shows that,

$$p_1(\tau,u) = \frac{u M_{22} e^{u^2/2}}{(2\pi\Delta)^{\frac{1}{2}}(1-\eta^2(\tau))^2}\int_0^{2\pi} J(\gamma,k)\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \quad (4.61)$$

where

$$\begin{aligned}
\eta(\tau) &= \int_0^{\infty} g_X(\omega)\cos((\omega-\omega_c)\tau)\,d\omega && (4.62)\\
J(\gamma,k) &= \frac{1}{2\pi\sqrt{1-\gamma^2}}\int_k^{\infty} dx\int_k^{\infty}(x-k)(y-k)e^z dy && (4.63)\\
\gamma &= \frac{M_{23}}{M_{22}}\cos\phi && (4.64)\\
k &= \frac{u\eta'(\tau)[\eta(\tau)-\cos\phi]}{1-\eta^2(\tau)}\sqrt{\frac{1-\eta^2(\tau)}{M_{22}}} && (4.65)\\
M_{22} &= -\eta''(0)[1-\eta^2(\tau)]-\eta'^2(\tau) && (4.66)\\
M_{23} &= \eta''(\tau)[1-\eta^2(\tau)]+\eta(\tau)\eta'^2(\tau) && (4.67)\\
z &= -\frac{x^2+y^2-2\gamma xy}{2(1-\gamma^2)}. && (4.68)\\
\Delta &= -\eta''(0) && (4.69)
\end{aligned}$$

Now the first step in obtaining an expression for the variance of the u-level-crossing count of the envelope process, $\{R(t)\}$, is the computation of $\psi(\tau)$,

$$\psi(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\dot{r}_1\dot{r}_2| p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)d\dot{r}_1 d\dot{r}_2. \quad (4.70)$$

Here again $p_{R_1,R_1',R_2,R_2'}$ is the joint density of $(R(0), R'(0), R(\tau), R'(\tau))$. From (4.60) and (4.61) we see that

$$\int_{-\infty}^{0} d\dot{r}_1 \int_{0}^{\infty} d\dot{r}_2 |\dot{r}_1\dot{r}_2| p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2) = \tag{4.71}$$

$$\frac{u^2 M_{22}}{2\pi(1-\eta^2(\tau))^2}\int_0^{2\pi} J(\gamma,k)\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \tag{4.72}$$

And thus, we have a start on evaluating $\psi(\tau)$.

Let I_R denote the integrand $|\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)$. (4.71) gives the integral of I_R over the second quadrant in the $(\dot{r}_1,\dot{r}_2)$ plane. The integral over the fourth quadrant is obtained using Rice's result for the conditional probability of a u-level-downcrossing, at time τ, given a u-level-upcrossing at time 0. Denote this conditional probability by $p_2(\tau,u)$ (taken in horizontal window sense). Then,

$$p_2(\tau,u)\cdot(\frac{\Delta}{2\pi})^{\frac{1}{2}}ue^{-\frac{u^2}{2}} = \int_0^{\infty} d\dot{r}_1\int_{-\infty}^{0} d\dot{r}_2|\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2). \tag{4.73}$$

Rice (1958, pg. 615, eq. 107) shows

$p_2(\tau,u) = p_1(\tau,u)+$

$$\frac{uM_{22}e^{u^2/2}}{(2\pi\Delta)^{\frac{1}{2}}(1-\eta^2(\tau))^2}\int_0^{2\pi}[(\gamma+k^2)\mathrm{erf}(\frac{k}{\sqrt{2}})+\frac{2k}{\sqrt{2\pi}}e^{-k^2/2}]\cdot\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \tag{4.74}$$

where $\mathrm{erf}(x)$ is the error function,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{4.75}$$

Using (4.71) and (4.74) the integral over the second and fourth quadrants (denoted $\int_{II\cup IV}$) is,

$\int_{II\cup IV}|\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)d\dot{r}_1 d\dot{r}_2 =$

$$\frac{u^2 M_{22}}{2\pi(1-\eta^2(\tau))^2}\int_0^{2\pi}\left[2J(\gamma,k)+[(\gamma+k^2)\mathrm{erf}(\frac{k}{\sqrt{2}})+\frac{2k}{\sqrt{2\pi}}e^{-k^2/2}]\right]\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \quad (4.76)$$

Following the same type of analysis as Rice, we obtain the two remaining integrals necessary for determining $\psi(\tau)$. These are, the integral of $|\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)$, over the first and third quadrants of the $(\dot{r}_1,\dot{r}_2)$ plane. In fact, due to symmetry of the integrand, we will see that

$$\int_I |\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)d\dot{r}_1 d\dot{r}_2 = \int_{III} |\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2)d\dot{r}_1 d\dot{r}_2.$$

From Rice (1958, pg. 613, eq. 92) the joint density

$$p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2) = (\frac{u}{2\pi})^2\int_0^{2\pi}(M_{22}-\cos^2\phi M_{23}^2)^{-\frac{1}{2}}e^A d\phi \quad (4.77)$$

where A is a quadratic form in the variables $\{\dot{r}_1,\dot{r}_2\}$. All linear terms in the quadratic expression enter in the form, $\dot{r}_1-\dot{r}_2$, and Rice uses the following change of variables to simplify the exponent A (1958, pg. 613, eq. 96),

$$x = -(\dot{r}_1-a_1)[(1-\eta^2(\tau))/M_{22}]^{\frac{1}{2}} \qquad y = (\dot{r}_2-a_1)[(1-\eta^2(\tau))/M_{22}]^{\frac{1}{2}} \quad (4.78)$$

where,

$$a_1 = \frac{u\eta'(\tau)(\cos\phi-\eta(\tau))}{1-\eta^2(\tau)}. \quad (4.79)$$

With this final change of variables Rice then arrives at (4.71). Key to the right hand side of (4.71) is the function $J(\gamma,k)$, which is obtained by integrating the right side of (4.77) with respect to x and y using (4.78).

Following the same analysis that led to (4.77), the integral of I_R over the first quadrant can be evaluated and is given by,

$$\int_0^\infty d\dot{r}_1\int_0^\infty d\dot{r}_2|\dot{r}_1\dot{r}_2|p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2) = \quad (4.80)$$

$$\frac{u^2 M_{22}}{2\pi(1-\eta^2(\tau))^2}\int_0^{2\pi}J^*(\gamma,k)\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \quad (4.81)$$

where,

$$J^*(\gamma,k) = \frac{1}{2\pi\sqrt{1-\gamma^2}}\int_k^{-\infty} dx \int_k^{\infty} (x-k)(y-k)e^z dy. \tag{4.82}$$

(4.81) is the same form as (4.71) except that $J(\gamma,k)$ is replaced by $J^*(\gamma,k)$.

Finally, the integral of I_R over the third quadrant yields,

$$\int_{-\infty}^{0} d\dot{r}_1 \int_{-\infty}^{0} d\dot{r}_2 |\dot{r}_1\dot{r}_2| p_{R_1,R_1',R_2,R_2'}(u,\dot{r}_1,u,\dot{r}_2) = \tag{4.83}$$

$$\frac{u^2 M_{22}}{2\pi(1-\eta^2(\tau))^2}\int_0^{2\pi} J^{**}(\gamma,k)\exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \tag{4.84}$$

where,

$$J^{**}(\gamma,k) = \frac{1}{2\pi\sqrt{1-\gamma^2}}\int_k^{\infty} dx \int_k^{-\infty} (x-k)(y-k)e^z dy. \tag{4.85}$$

However, recall that

$$z = -\frac{x^2+y^2-2\gamma xy}{2(1-\gamma^2)},$$

so the integrand in (4.85) is symmetric in (x,y), and therefore $J^{**}(\gamma,k) = J^*(\gamma,k)$. Collecting terms from (4.76), (4.81), and (4.84),

$$\psi(\tau) = \frac{u^2 M_{22}}{2\pi(1-\eta^2(\tau))^2}\int_0^{2\pi} \Big[2J(\gamma,k) + 2J^*(\gamma,k) + \tag{4.86}$$

$$[(\gamma+k^2)\mathrm{erf}(\frac{k}{\sqrt{2}}) + \frac{2k}{\sqrt{2\pi}}e^{-k^2/2}]\Big] \cdot \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi.$$

Further simplification of (4.86) occurs using the result of Rainal (1965, eq. 5) which relates $J^*(\cdot)$ to $J(\cdot)$ by

$$J^*(\gamma,k) = J(\gamma,k) + \frac{k}{\sqrt{2\pi}}\exp^{-k^2/2} - \frac{k^2+\gamma}{2}[1-\mathrm{erf}(\frac{k}{\sqrt{2}})], \tag{4.87}$$

and allows us to write (4.86)

$$\psi(\tau) = \frac{2u^2 M_{22}}{\pi(1-\eta^2(\tau))^2}\int_0^{2\pi} \Big[J(\gamma,k) + \frac{k}{\sqrt{2\pi}}e^{-k^2/2} + \tag{4.88}$$

$$\frac{(\gamma+k^2)}{4}[2\mathrm{erf}(\frac{k}{\sqrt{2}})-1]\Big] \cdot \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi.$$

To see that $\psi(\tau)$ is well behaved, assume baseband autocorrelation function, $\eta(\tau)$, is at least four times continuously differentiable (which is certainly true for a symmetric bandpass process). Expanding $\eta(\tau)$ about zero we have

$$\eta(\tau) = 1 - \frac{\Delta\tau^2}{2!} + \frac{\kappa\tau^4}{4!} + o(\tau^4), \tag{4.89}$$

where Δ and κ are the second and fourth spectral moments, respectively, for the baseband spectrum $g_I(\omega)$.

We need to examine the behavior of $\psi(\tau)$ as τ approaches zero from above. Note that as $\tau \to 0^+$, using (4.89), we have the following limits

$$\frac{M_{22}}{(1-\eta^2(\tau))^2} \to \frac{B}{4\Delta} \tag{4.90}$$

$$\gamma \to \cos\phi \tag{4.91}$$

$$k \to \frac{-2u}{\sqrt{B}} \frac{(1 - \Delta\tau^2/2 - \cos\phi)}{\tau^2} \tag{4.92}$$

where $B = \kappa - \Delta^2$.

Thus, for small τ, the contribution to $\psi(\tau)$ from the first term in the integrand of (4.88),

$$\int_0^{2\pi} J(\gamma, k) \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi, \tag{4.93}$$

as $\tau \to 0$ is given by (see Rice pg. 614, 1958 using the change of variable $\phi = \sqrt{\Delta}\tau x$,)

$$\int_0^{2\pi} J(\gamma, k) \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \to \tau \cdot \int_0^{\infty} J[1, \frac{u\Delta}{\sqrt{B}}(1-x^2)] \exp[-u^2x^2/2]dx$$

so that

$$\int_0^{2\pi} J(\gamma, k) \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \to O(\tau). \tag{4.94}$$

Also, for small τ, we see

$$\int_0^{2\pi} \Big[\frac{k}{\sqrt{2\pi}}e^{-k^2/2} + \frac{(\gamma + k^2)}{4}[2\mathrm{erf}(\frac{k}{\sqrt{2}}) - 1]\Big] \cdot \exp[-\frac{u^2(1-\eta(\tau)\cos\phi)}{1-\eta^2(\tau)}]d\phi \quad (4.95)$$

$$\to \tau \int_0^{\infty} \Big[\frac{k_0}{\sqrt{2\pi}}e^{-k_0^2/2} + \frac{(1 + k_0^2)}{4}[2\mathrm{erf}(\frac{k_0}{\sqrt{2}}) - 1]\Big] \cdot \exp[-u^2x^2/2]dx \quad (4.96)$$

$$\to O(\tau^4) \quad (4.97)$$

where $k_0 = \frac{u\Delta}{\sqrt{B}}(1 - x^2)$.

For use later we also need the asymptotic behavior of ψ at the origin when the baseband autocorrelation function is singular. Suppose that $\eta(\tau)$ is twice differentiable but has a discontinuous third derivative at the origin. So that

$$\eta(\tau) = 1 - \frac{\Delta\tau^2}{2!} + \frac{\vartheta|\tau^3|}{3!} + o(\tau^3) \quad (4.98)$$

Then using (4.98) we can shown that

$$\frac{M_{22}\cdot\tau}{(1-\eta^2(\tau))^2} \to \frac{2\vartheta}{3\Delta} \quad (4.99)$$

$$\gamma \to \frac{\cos\phi}{2} \quad (4.100)$$

$$k \to 0 \quad (4.101)$$

So that using (4.88) as before we see that as $\tau \to 0$

$$\psi(\tau) \to \psi_0 > 0. \quad (4.102)$$

So $\psi(\tau)$ does not go to zero but still is well-behaved, i.e. continuous (see Longuet-Higgins, 1962, pgs. 557, 572 on how this relates to the distribution of the interval between crossings).

We evaluated $\psi(\tau)$ for the particular process of interest, namely, the envelope of an ideal narrow-band Gaussian process. For this special case we performed numerical quadrature to evaluate $\psi(\tau)$, and then found the variance using (4.54) and (4.88). These numerical results based on the analytic expression (4.88) were then compared with computer simulations for the envelope of a Gaussian process.

Computer simulations of the envelope of bandlimited white Gaussian noise were obtained by two different methods. The first method used an approximation of continuous time bandlimited white Gaussian noise (BLWGN), which was synthesized by first generating a discrete time sequence of independent, identically distributed, pseudo-Gaussian numbers, and then using the sampling theorem to approximate the continuous time BLWGN. The envelope was then obtained via the Hilbert transform of the continuous time process (4.12). The second method used simply synthesized the in-phase, $I_c(t)$, and quadrature, $I_s(t)$, components of BLWGN and then used Rice's equivalent definition for the envelope, $R(t)$,

$$R(t) = [I_c^2(t) + I_s^2(t)]^{\frac{1}{2}}.$$

Results of computer experiments for the mean and variance of the crossing counts are presented in Tables (4.2.2) and (4.2.2) and in Figure's (4.2.2), (4.2.2), (4.2.2), (4.2.2). We compare theoretical values with the sample statistics obtained from the simulations for both BLWGN and the envelope of BLWGN.

4.3 Asymptotic Normality for the Level-Crossings of the Envelope of a Gaussian Process

In this last section we prove asymptotic normality for the level-crossing counts of the envelope of a narrow-band Gaussian process. Throughout we will assume the underlying Gaussian process, $\{X(t)\}$, is separable, and whose one-sided spectral density, $g_X(\omega)$, is symmetric about the center frequency of the passband ω_c.

Table 4.1: **Sampled crossing rates for an ideal bandpass Gaussian process.**

	Sample Mean $N_X(1000)$	Sample Variance $N_X(1000)$
Theor. value	577.4	284
Run # 1	573.4	285.8
2	573.8	277.7
3	572.9	278.3
4	573.1	278.5
5	572.7	272.3

The approach taken here parallels the proof given by Cuzick 1976 of the asymptotic normality for the zeros of a differentiable Gaussian process. Cuzick's proof is based on the paper of Malevich 1969, whereby Malevich proves asymptotic normality of the zero-crossings with restrictive assumptions on the spectrum.

Both Cuzick's and Malevich's proofs use a sequence of M-dependent processes which converge in mean-square to the underlying Gaussian process $\{X(t)\}$. By using a CLT for M-dependent processes (see Diananda 1953, 1955) the sequence of zero-crossings of the approximating sequence are shown to be asymptotically normal, so it is enough to show uniform convergence of the M-dependent processes to $\{X(t)\}$, in mean-square, and show that the zero-crossing counts, as well, converge uniformly in mean-square.

Cuzick shows that some associated correlation functions and cross-correlation functions between the M-dependent processes and the underlying Gaussian pro-

Table 4.2: **Sampled crossing rates for the envelope of an ideal bandpass Gaussian process.**

	Sample Mean $N_R(1000)$	Sample Variance $N_R(1000)$
Theor. value	413.5	265
Run # 1	411.5	255.5
2	411.8	263.9
3	411.4	256.1
4	411.4	268.6
5	411.7	264.1

cess converge in mean-square and this enables him to prove a CLT under less restrictive conditions than those used by Malevich. However, still even less restrictive assumptions were needed by Slud 1991 (Theorem 3, pg. 353) to prove asymptotic normality of the crossing counts. Using the powerful stochastic calculus of multiple Wiener-Ito expansions and under the least restrictive assumptions to date, ($\rho(\tau) \in L^2(-\infty,\infty)$ and $\rho''(\tau) \in L^2(-\infty,\infty)$ along with the indispensable (4.57)), Slud proves asymptotic normality of the level-crossings and guarantees nondegeneracy of asymptotic variance with a useful positive lower bound.

In the next section we prove a CLT for the level-crossings of the envelope process. The method used is an adaptation of Cuzick's and Malevich's proofs whereby we approximate the in-phase and quadrature components of the underlying Gaussian process by M-dependent Gaussian processes. Since each component is Gaussian we can readily apply a number of results of Cuzick and Malevich

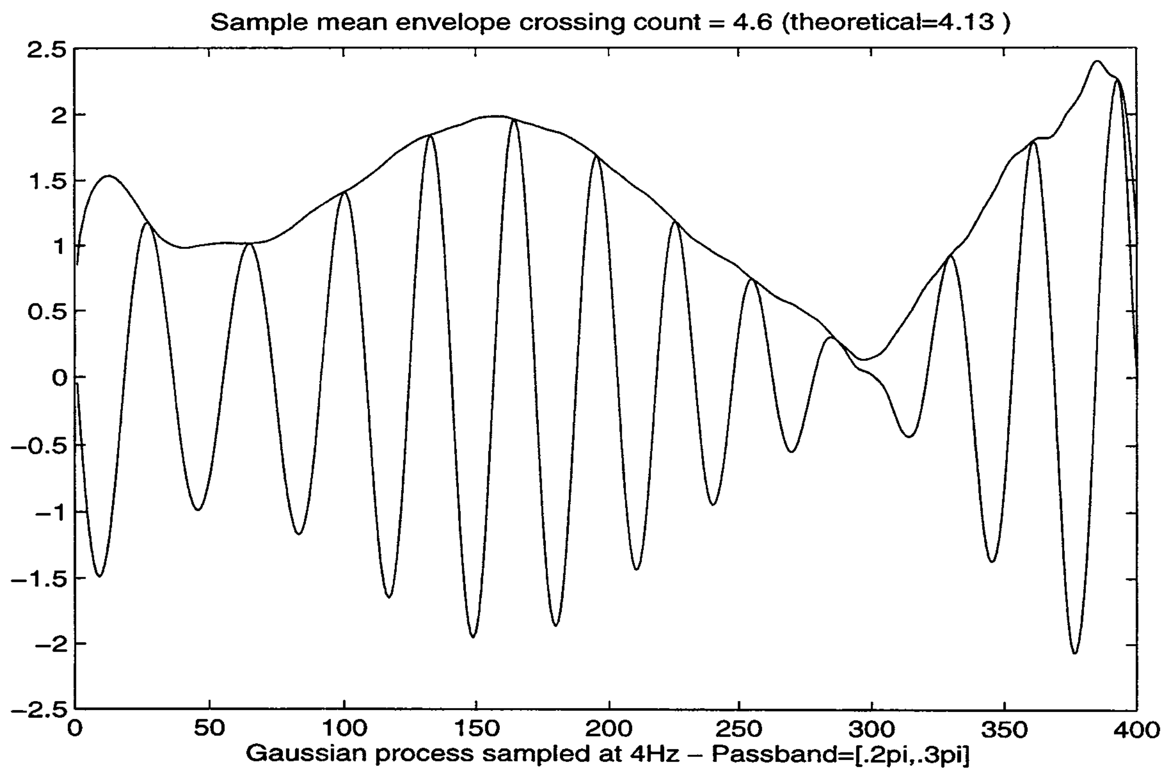

Figure 4.1: **Lowpass Gaussian process and envelope sampled at 4 Hz**

to aid in our proof.

4.3.1 Preliminaries

Let $\{X(t)\}$ be our standard Gaussian bandpass process, zero-mean, and unit variance, whose one-sided spectral density is symmetric about the midband frequency ω_c. Using Rice's representation, we can write

$$X(t) = I_c(t)\cos\omega_c t - I_s(t)\sin\omega_c t,$$

and as before, the envelope $R(t)$ is

$$R(t) = [I_c^2(t) + I_s^2(t)]^{\frac{1}{2}}.$$

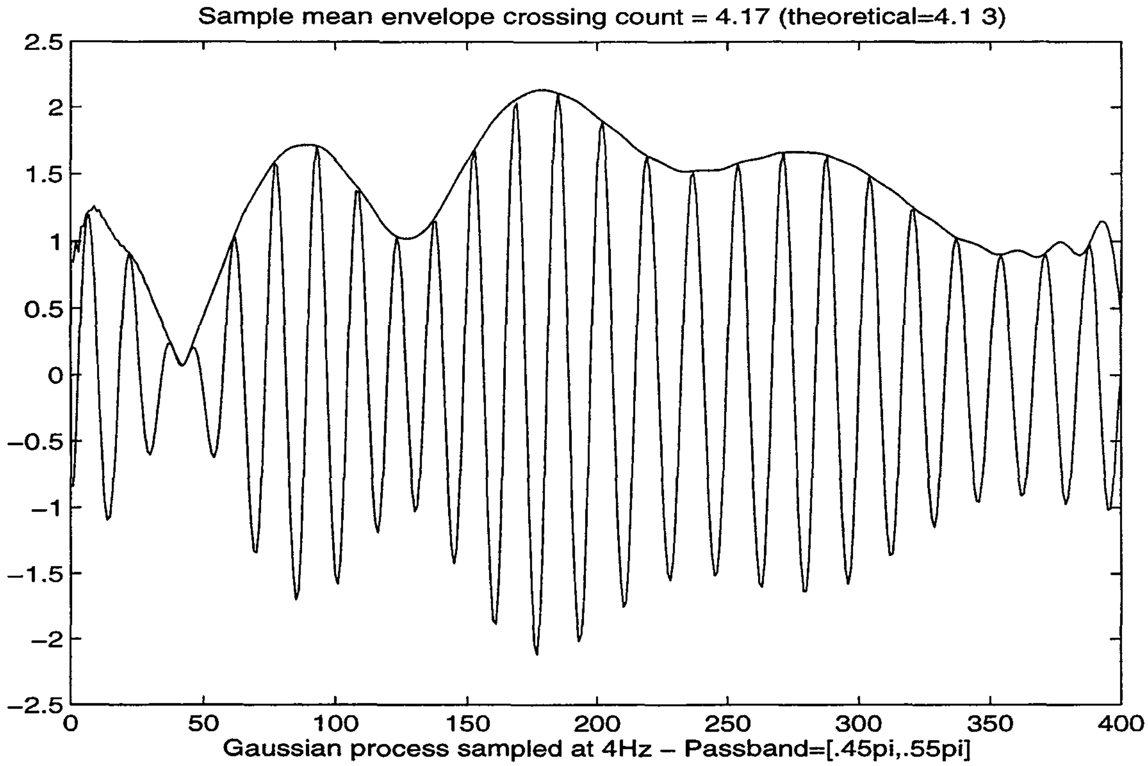

Figure 4.2: **Bandpass Gaussian process and envelope sampled at 4 Hz**

We define M-dependent (i.e. autocorrelation function vanishes for $|\tau| > 4M$) Gaussian processes which are approximations to $\{I_c(t)\}$ and $\{I_s(t)\}$. For the in-phase component $\{I_c(t)\}$ define the M-dependent approximation, $\{I_{c,M}(t)\}$ by

$$I_{c,M}(t) = \int_{-\infty}^{\infty} \cos \omega t[(g_I * P_M)(\omega)]^{\frac{1}{2}} dB_c(\omega) \tag{4.103}$$

where $dB_c(\omega)$ is a Gaussian white noise process, $g_I(\omega)$ the spectral density of $\{I_c(t)\}$ (and $\{I_s(t)\}$ as well), and

$$P_M(\omega) = K \cdot M[\frac{\sin M\omega}{M\omega}]^4. \tag{4.104}$$

Above the $*$ denotes convolution and K is a normalization constant such that

$$\int_{-\infty}^{\infty} P_M(\omega) d\omega = 1.$$

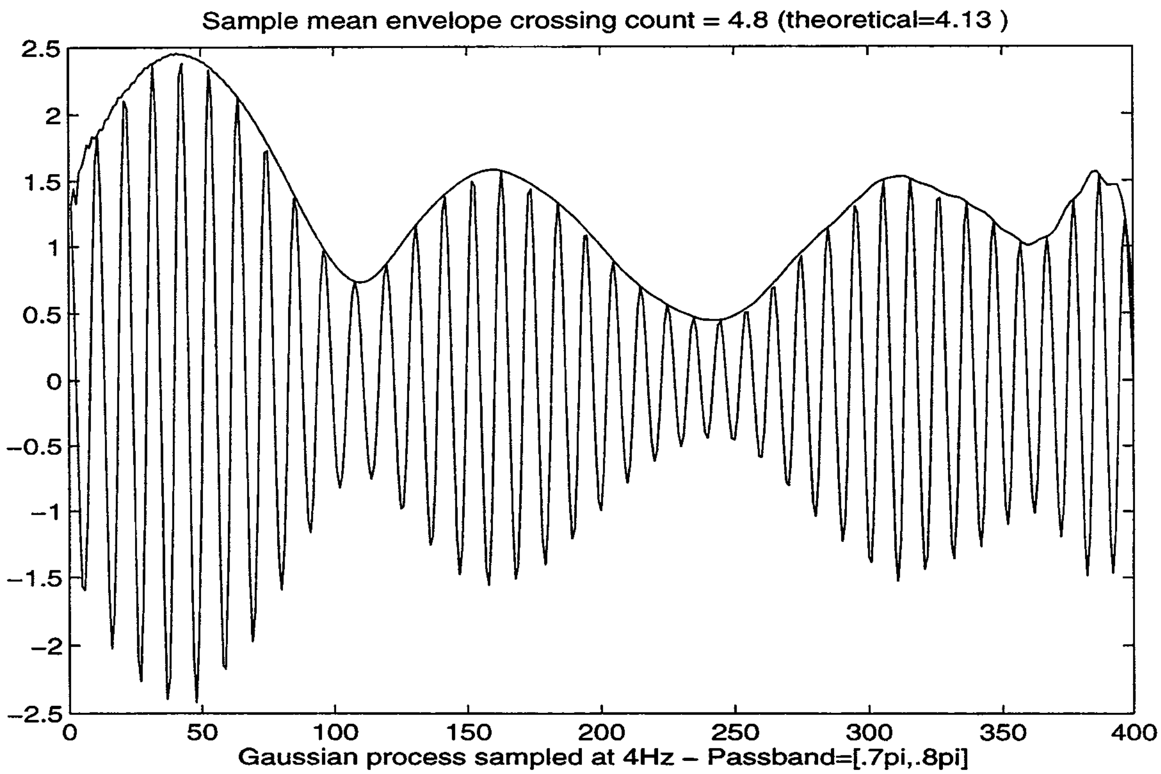

Figure 4.3: **Bandpass Gaussian process and envelope sampled at 4 Hz**

By the convolution theorem the autocorrelation, $\rho_{I,M}(\tau)$, of $\{I_{c,M}(t)\}$ is given by pointwise product

$$\rho_{I,M}(\tau) = \rho_I(\tau) \cdot \tilde{P}_M(\tau), \tag{4.105}$$

where $\rho_I(\tau)$ is the autocorrelation function of $\{I_c(t)\}$ (and $\{I_s(t)\}$), and $\tilde{P}_M(\tau) = \mathcal{F}\{P_M(\omega)\}$ is the Fourier transform of $P_M(\omega)$. It follows that (see pg. 549, Cuzick 1976):

(1) $\tilde{P}_M(\tau)$ is piecewise cubic,

(2) $\tilde{P}_M(\tau) = 1 - (K_0/M^2)\tau^2 + O(|\tau|^3)$ as $\tau \to 0$, with $K_0 > 0$,

(3) $\tilde{P}_M(\tau) = 0$ for $|\tau| > 4M$.

So by (3) above we see that $\{I_{c,M}(t)\}$ is an M-dependent Gaussian process. Now

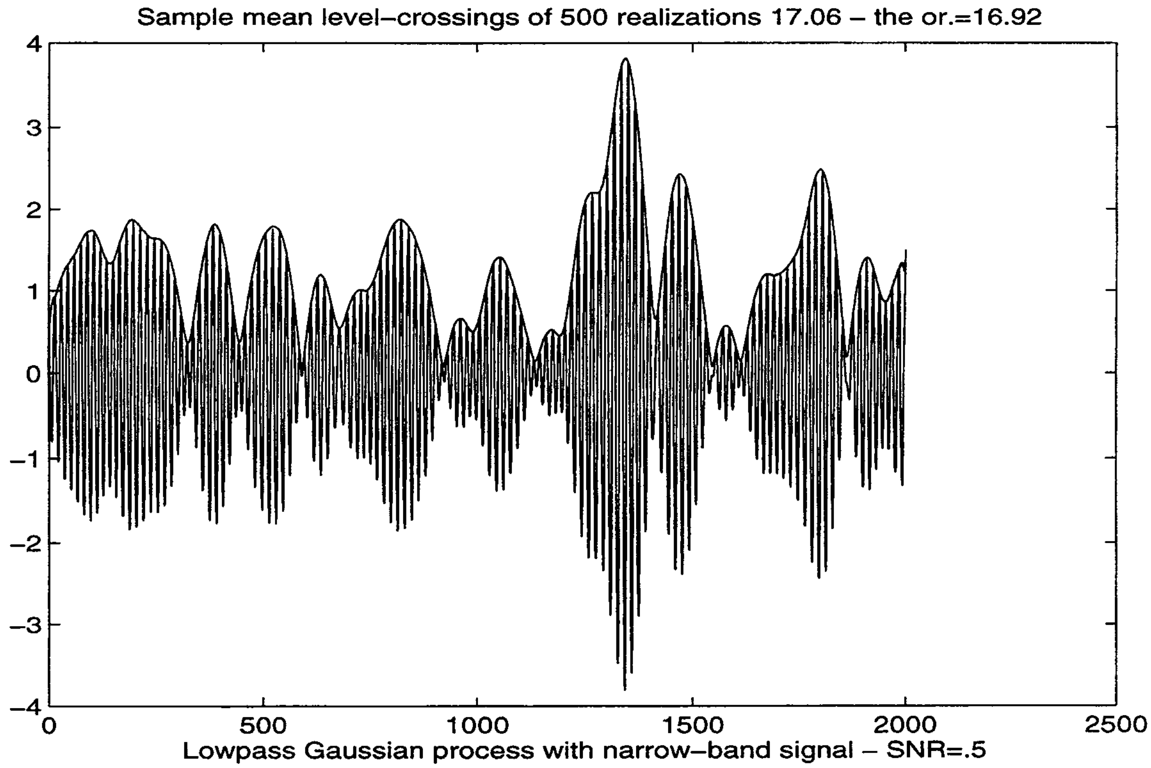

Figure 4.4: **Superposition of two Gaussian process and the envelope sampled at 4 Hz**

take $\{I_{s,M}(t)\}$ to be defined in an analogous fashion

$$I_{s,M}(t) = \int_{-\infty}^{\infty} \cos \omega t[(g_I * P_M)(\omega)]^{\frac{1}{2}} dB_s(\omega),$$

where $dB_s(\omega)$ is again Gaussian white noise, but is independent of $dB_c(\omega)$. This implies that $\{I_{c,M}(t)\}$ and $\{I_{s,M}(t)\}$ are independent random processes.

Define the M-dependent approximation to the envelope by

$$R_M(t) = [I_{c,M}^2(t) + I_{s,M}^2(t)]^{\frac{1}{2}}. \tag{4.106}$$

Observe that since $\{I_{c,M}(t)\}$ and $\{I_{s,M}(t)\}$ are independent processes, and each M-dependent, it follows that $\{R_M(t)\}$ is an M-dependent process as well. This is easily seen to be the case using the expressions (4.22) and (4.24) for the covariance of the envelope

$$E[R_M(t)R_M(t+\tau)] = \frac{\pi}{2}(1 + \frac{k_0^2}{4} + \frac{k_0^4}{64} + \cdots), \tag{4.107}$$

and noting $k_0 = k_0(\tau) = \rho_{I,M}(\tau)$. The squared envelope process, $\{R_M^2(t)\}$ is M-dependent as well by observing,

$$\rho_{R_M^2}(\tau) = \rho_I^2(\tau) \cdot \tilde{P}_M^2(\tau)) = \rho_{I,M}^2(\tau).$$

Next with the help of the following two lemmas from (Cuzick 1976, pg. 549) we prove mean-square convergence of $\{R_M^2(t)\}$ and its derivative $\{2R_M(t)R_M'(t)\}$. We show

$$R_M^2(t) \xrightarrow{L^2} R^2(t) \tag{4.108}$$

$$2R_M(t)R_M'(t) \xrightarrow{L^2} 2R(t)R'(t). \tag{4.109}$$

Lemma 4.1 If $f \geq 0$, $f_n \geq 0$, and $f_n^2 \to f^2$ in $L^1(-\infty,\infty)$ then $f_n \to f$ in $L^2(-\infty,\infty)$.

Lemma 4.2 If f in $L^2(-\infty,\infty)$, then $(f * P_M)^{\frac{1}{2}} \to \sqrt{f}$ in $L^2(-\infty,\infty)$ and $\omega \cdot (f * P_M)^{\frac{1}{2}} \to \omega\sqrt{f}$ in $L^2(-\infty,\infty)$.

Since $g_I \in L^1$, and $P_M \in L^1$, then $g_I * P_M \in L^1$ so by virtue of lemma 4.1 we have

$$(g_I * P_M)^{\frac{1}{2}} \xrightarrow{L^2} \sqrt{g_I}.$$

Consequently,

$$\int_{-\infty}^{\infty} \cos \omega t[(g_I * P_M)(\omega)]^{\frac{1}{2}} dB_c(\omega) \xrightarrow{L^2} \int_{-\infty}^{\infty} \cos \omega\sqrt{g_I(\omega)} dB_c(\omega) \tag{4.110}$$

$$\tag{4.111}$$

so

$$I_{c,M}(t) \xrightarrow{L^2} I_c(t) \tag{4.112}$$

and similarly for $I_{s,M}(t)$, so that

$$I_{s,M}(t) \xrightarrow{L^2} I_s(t). \tag{4.113}$$

The convergence in both cases is uniform in t. From the last two equations we easily obtain (since $E[(I_{\{\cdot\}}(T))^2] < \infty$)

$$I_{c,M}^2(t) \xrightarrow{L^2} I_c^2(t) \tag{4.114}$$

$$I_{s,M}^2(t) \xrightarrow{L^2} I_s^2(t), \tag{4.115}$$

again uniformly in t. Thus, we have mean-square convergence, uniformly in t, of the approximating squared envelope process,

$$R_M^2(t) \xrightarrow{L^2} R^2(t). \tag{4.116}$$

We also have uniform convergence of the approximating sequence of derivatives also. To see this recall from (??) that the mean-square derivative of $\{R^2(t)\}$ is $\{2R(t)R'(t)\}$ and likewise the derivative of $\{R_M^2(t)\}$ is $\{2R_M(t)R_M'(t)\}$. Using lemma 4.2 we get

$$\omega[g_I * P_M]^{\frac{1}{2}} \xrightarrow{L^2} \omega\sqrt{g_I}$$

so that

$$-\int_{-\infty}^{\infty} \omega \ \sin\omega t \ [(g_I * P_M)(\omega)]^{\frac{1}{2}} dB_c(\omega) \xrightarrow{L^2} -\int_{-\infty}^{\infty} \omega \ \sin\omega \ \sqrt{g_I(\omega)} dB_c(\omega) \tag{4.117}$$

and we have convergence uniformly in t

$$I_{c,M}'(t) \xrightarrow{L^2} I_c'(t).$$

Similarly for the quadrature component, $I_s(t)$, we get

$$I'_{s,M}(t) \xrightarrow{L^2} I'_s(t), \tag{4.118}$$

uniformly in t, and since all second-moments are finite,

$$I_{c,M}(t)I'_{c,M}(t) \xrightarrow{L^2} I_c(t)I'_c(t) \tag{4.119}$$

$$I_{s,M}(t)I'_{s,M}(t) \xrightarrow{L^2} I_s(t)I'_s(t). \tag{4.120}$$

Therefore, the sequence of M-dependent derivatives converge uniformly in t

$$2R_M(t)R'_M(t) \xrightarrow{L^2} 2R(t)R'(t). \tag{4.121}$$

Definition 4.1 Denote the number of u^2-level-crossings of $R^2(t)$ for $t \in [0, T]$ by $N_{R^2}(T)$. We define the centered normalized u^2-level-crossings, $Z(T)$, by

$$Z(T) \doteq T^{-\frac{1}{2}}[\, N_{R^2}(T) - E[N_{R^2}(T)]\,].$$

Similarly, we define the u^2-level-crossings of $R^2_M(t)$ for $t \in [0, T]$ by $N_{R^2_M}(T)$ and

$$Z_M(T) \doteq T^{-\frac{1}{2}}[\, N_{R^2_M}(T) - E[N_{R^2_M}(T)]\,].$$

With these preliminary results and definitions we are ready to prove a CLT for the u level-crossings of the envelope of a sufficiently smooth Gaussian process.

Theorem 4.1 Let $I_c(t)$ and $I_s(t)$ be independent, identical, Gaussian processes, mean-zero, variance 1. Suppose their autocorrelation function, $\rho_I(\tau)$, is four times continuously differentiable at the origin, both $\rho_I(\tau)$ and $\rho''_I(\tau)$ are $\in L^2(-\infty, \infty)$

Then for $u > 0$ the u^2-level-crossings of $R^2(t) = I_c^2(t) + I_s^2(t)$ are asymptotically normal. That is,

$$T^{-\frac{1}{2}}[\, N_{R^2}(T) - E[N_{R^2}(T)]\,] \xrightarrow{Law} Normal(0, \sigma^2),$$

where

$$\sigma^2 = E[Z_{R^2}(1)] + 2 \int_0^\infty [\psi(\tau) - (E[Z_{R^2}(1)])^2] d\tau. \tag{4.122}$$

Proof: First note that since $\rho_I(t)$ is four times continuously differentiable at the origin, we have the following expansion about the origin for small τ,

$$\rho_I(\tau) = 1 - \frac{\Delta\tau^2}{2!} + \frac{\kappa\tau^4}{4!} + o(\tau^4)$$

and the indispensable (4.57),

$$\int_0^\delta \frac{\rho_I''(\tau) - \rho_I''(0)}{\tau} d\tau \ < \infty \text{ for some } \delta > 0, \tag{4.123}$$

is satisfied.

Now following Cuzick's argument, to prove asymptotic normality of $Z(T)$ as $T \to \infty$ it is enough to show that

(A) $Z_M(T) \overset{L^2}{\to} Z(T)$ uniformly in T as $M \to \infty$,

(B) $Z_M(T) \overset{Law}{\to} Normal(0, \sigma_M^2)$ for each M as $T \to \infty$,

and

(C) $\lim_{T\to\infty} T^{-1}\text{Var}[N_{R^2}(t)] \to V_0 > 0$.

If we assume our one-sided spectrum is symmetric about a midband frequency ω_c then our expression for the variance of the envelope crossings guarantees that

$$\lim_{T\to\infty} T^{\mp 1}\text{Var}[N_{R^2}(t)] \to V_0 > 0.$$

For the more general case we will assume we have non-degeneracy.

From (A) and (B) we that

$$\sigma_M^2 \to \sigma^2$$

and so by (C)

$$\sigma_M^2 > 0. \tag{4.124}$$

Once we have (4.124) we can use Diananda's (1953, 1955) CLT for M-dependent sequences and obtain the asymptotic normality of $\{Z_M(t)\}$ given as (B). Therefore, we need to show (A)

$$Z_M(T) \xrightarrow{L^2} Z(T) \tag{4.125}$$

uniformly in T as $M \to \infty$, and (C)

$$\lim_{T\to\infty} \mathrm{Var}[N_{R^2}(T)] \to T \cdot V_0, \tag{4.126}$$

with $V_0 > 0$.

To demonstrate (A), $Z_M(T) \xrightarrow{L^2} Z(T)$ uniformly in T as $M \to \infty$, it is enough to show that for any $\epsilon > 0$ there is an M_ϵ such that

$$T^{-1}\Big[E[(N_{R^2}(T) - N_{R^2_M}(T))^2] - (E[N_{R^2}(T) - N_{R^2_M}(T)])^2\Big] < \epsilon \tag{4.127}$$

when $M > M_\epsilon$ and for all $T > T_0$ where T_0 is independent of M_ϵ. The next part of the proof follows Cuzick, at times verbatim, with appropriate modifications to deal with the envelope process.

Let T tend to infinity through the integers and define $\nu = 2^{-n}$ for n a positive integer which will be determined later. By subdividing the interval $[0, T]$ into 2^nT subintervals we can write

$$N_{R^2}(T) = \sum_{i=0}^{2^nT-1} N_{R^2}(i)$$

and

$$N_{R^2_M}(T) = \sum_{i=0}^{2^nT-1} N_{R^2_M}(i)$$

where $N_{R^2}(i)$ is the number of u^2 level-crossings in the interval $[i2^{-n}, (i+1)2^{-n})$ and similarly for $N_{R^2_M}(i)$. We now can write (4.127) as

$$T^{-1} \sum_{|j-i|\le 1, i=0}^{2^nT-1} \mathrm{Cov}[N_{R^2}(i) - N_{R^2_M}(i), N_{R^2}(j) - N_{R^2_M}(j)] + \tag{4.128}$$

$$T^{-1}\sum_{|j-i|\geq 2,i=0}^{2^nT-1}\mathrm{Cov}[N_{R^2}(i)-N_{R^2_M}(i),N_{R^2}(j)-N_{R^2_M}(j)]. \tag{4.129}$$

Using stationarity and the Cauchy-Schwarz inequality, the first term in the above sum, which only contains terms about the diagonal, is less than

$$\frac{3}{\nu}E[(N_{R^2}(\nu)-N_{R^2_M}(\nu))^2+\frac{3}{\nu}|E[N_{R^2}(\nu)-N_{R^2_M}(\nu)]|. \tag{4.130}$$

As $M\to\infty$,

$$|E[N_{R^2}(\nu)-N_{R^2_M}(\nu)]|\to 0$$

by the fact that $\{\rho''_{I,M}(0)\}\to\rho''_I(0)$.

The first term,

$$\frac{3}{\nu}E[(N_{R^2}(\nu)-N_{R^2_M}(\nu))^2]$$

is less than

$$\frac{9}{\nu}(E[(N_{R^2}(\nu)-\chi_\nu)^2]+E[(\chi_\nu-\chi_\nu^M)^2]+E[(N_{R^2_M}(\nu)-\chi_\nu^M)^2]$$

where χ_ν is the indicator random variable on $\{[R^2(0)-u^2]\cdot[R^2(\nu)-u^2]<0\}$ and χ_ν^M is the indicator on $\{[R^2_M(0)-u^2]\cdot[R^2_M(\nu)-u^2]<0\}$. Observe that,

$$\begin{aligned}
E[(N_{R^2}(\nu)-\chi_\nu)^2] &\leq \sum_{k=2}^{\infty}k^2\Pr[N_{R^2}(\nu)=k] && (4.131)\\
&\leq 2\sum_{k=2}^{\infty}(k^2-k)\Pr[N_{R^2}(\nu)=k] && (4.132)\\
&= 2E[N^2_{R^2}(\nu)-N_{R^2}(\nu)]. && (4.133)
\end{aligned}$$

Using expressions (4.54), (4.88) for the variance of the envelope level-crossings we have

$$\frac{1}{\nu}E[(N_{R^2}(\nu)-\chi_\nu)^2]\leq\frac{4}{\nu}\int_0^\nu(\nu-\tau)\psi(\tau)d\tau \tag{4.134}$$

where $\psi(\tau)$ is given by (4.86) with $\eta(\tau)$ replaced by $\rho_I(\tau)$. From (4.94) and (4.97) and the fact that ρ_I is four-times differentiable, we know that $\psi(\tau)$ is continuous, and in this case, $\psi(0) = 0$. Thus, as $\nu \to 0$

$$\frac{1}{\nu}\int_0^{\nu}(\nu - \tau)\psi(\tau)d\tau \to 0$$

so that

$$\lim_{\nu \to 0}\frac{1}{\nu}E[(N_{R^2}(\nu) - \chi_\nu)^2] \to 0.$$

In the same way we can show that

$$\lim_{\nu \to 0} E[(N_{R_M^2}(\nu) - \chi_\nu^M)^2] \to 0 \tag{4.135}$$

provided that the function $\psi(\tau)$ obtained from (4.86), when we replace $\eta(\tau)$ by $\rho_{I,M}(\tau)$, is continuous at the origin. This is indeed the case from (4.102). To see that

$$\lim_{M \to \infty} E[(\chi_\nu - \chi_\nu^M)^2] \to 0$$

for any ν, note that

$$E[(\chi_\nu - \chi_\nu^M)^2] = \Pr[\chi_\nu \neq \chi_\nu^M)]$$

and since $R_M^2 \xrightarrow{L^2} R^2$ (which implies $R_M^2 \xrightarrow{prob} R^2$) the probability of the set where the indicator random variables differ tends to zero, for any fixed ν, as $M \to \infty$.

We next show that the second sum in (4.129) vanishes, uniformly for $T > T_0$ as $M \to \infty$. To this end we use the fact that $E[N_{R^2}(i)N_{R_M^2}(j)]$ can be expressed as

$$E[N_{R^2}(i)N_{R_M^2}(j)] = \tag{4.136}$$

$$\int_{i\nu}^{(i+1)\nu} ds \int_{j\nu}^{(j+1)\nu} dt E[|2R(s)R'(s)| \cdot |2R_M(t)R_M'(t)|\,|R^2(s) = u^2 = R_M^2(t)]p_{s,t}(u^2) \tag{4.137}$$

where $p_{s,t}(u^2)$ is the joint density of $\{R^2(s), R_M^2(t)\}$ evaluated at the point (u^2, u^2). The covariance of $N_{R^2}(i), N_{R_M^2}(j)$ is then

$$\mathrm{Cov}[N_{R^2}(i)N_{R_M^2}(j)] = \tag{4.138}$$

$$E[N_{R^2}(i)N_{R_M^2}(j)] - \frac{2ij}{\pi}(\Delta\Delta_M)^{\frac{1}{2}}u^2 \exp -u^2 \tag{4.139}$$

where Δ_M is $-\rho''_{I,M}(0)$. Since $R^2_M(t) \xrightarrow{L^2} R^2(t)$ and $2R_M(t)R'_M(t) \xrightarrow{L^2} 2R(t)R'(t)$ we have uniform convergence, in i, of $N_{R^2_M}(i) \xrightarrow{L^2} N_{R^2}(i)$. Therefore we only need consider the case for $|i-j| > 2^n T_0$, with T_0 independent of $M > M_0$. Using stationarity we can write the first summand in (4.129) as

$$\mathrm{Cov}[N_{R^2}(i)N_{R^2}(j)] - 2\mathrm{Cov}[N_{R^2}(i)N_{R^2_M}(j)] + \mathrm{Cov}[N_{R^2_M}(i)N_{R^2_M}(j)]. \quad (4.140)$$

We show that $\mathrm{Cov}[N_{R^2}(i)N_{R^2_M}(j)]$ tends to zero as $T_0 \to \infty$ independent of $M > M_0$. The other two covariances can be bounded exactly in the same fashion. We will estimate the covariance $\mathrm{Cov}[N_{R^2}(i)N_{R^2_M}(j)]$ indirectly, by first conditioning with the set of random variables

$$\mathcal{C} = \{I_c(s), I_s(s), I_{c,M}(t), I_{s,M}(t), R^2(s), R^2_M(t)\}.$$

By conditioning the derivatives on the above set of variables we will see that the conditional derivatives, $2R(s)R'(s)|\mathcal{C}$ and $2R_M(t)R'_M(t)|\mathcal{C}$ are in fact a pair of jointly Gaussian random variables whose conditional bivariate joint density can be obtained in terms of the underlying autocorrelation function $\rho_I(\tau)$ and the cross-correlation function of I_c and $I_{c,M}$, $\gamma_M(\tau)$,

$$\gamma_M(\tau) \doteq E[I_c(t)I_{c,M}(t+\tau)]. \quad (4.141)$$

Since

$$2R(s)R'(s) = 2I_c(s)I'_c(s) + 2I_s(s)I'_s(s)$$

and

$$2R_M(t)R'_M(t) = 2I_{c,M}(t)I'_{c,M}(t) + 2I_{s,M}(t)I'_{s,M}(t)$$

we use the statistical independence of the I_c's and the I_s's and the conditional joint density of the pair $\{I_c(s),\ I_{c,M}(t)|\mathcal{C}\}$ (which is the same as the conditional joint density of the pair $\{I_s(s),\ I_{s,M}(t)|\mathcal{C}\}$ to obtain the conditional joint density of $2R(s)R'(s),\ 2R_M(t)R'_M(t)|\mathcal{C}$.

Using Anderson 1984, pg. 37, Theorem 2.5.1 which gives the conditional density of any q components of an n-dimensional multivariate normal vector conditioned on the remaining n-q variables we arrive at the following conditional bivariate normal joint density for $2R(s)R'(s),\ 2R_M(t)R'_M(t)|\mathcal{C}$ (see also Sharpe 1978, pg. 379 eq.'s 4.7 and 4.8) with

$$\text{mean} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} = -\frac{2\gamma'_M}{1-\gamma_M^2} \begin{bmatrix} \sum_{i=1}^2 x_{i1}x_{i2} - \gamma_M u^2 \\ \gamma_M u^2 - \sum_{i=1}^2 x_{i1}x_{i2} \end{bmatrix}$$

and covariance matrix

$$\frac{4}{1-\gamma_M^2} \begin{bmatrix} [\Delta(1-\gamma_M^2) - \gamma_M'^2] & -[\gamma''_M(1-\gamma_M^2) + \gamma_M\gamma_M'^2]\sum_{i=1}^2 x_{i1}x_{i2} \\ -[\gamma''_M(1-\gamma_M^2) + \gamma_M\gamma_M'^2]\sum_{i=1}^2 x_{i1}x_{i2} & [\Delta_M(1-\gamma_M^2) - \gamma_M'^2] \end{bmatrix}$$

where γ_M and its derivatives are evaluated at $t - s = \tau$ and the particular conditioning values are given by

$$\mathcal{C} = \{I_c(s) = x_{11}, I_s(s) = x_{21}, I_{c,M}(t) = x_{12}, I_{s,M}(t) = x_{22}, R^2(s) = u^2 = R_M^2(t)\}.$$

Now using stationarity and (4.137) we are done if we can show, independent of the conditioning values, that as $T_0 \to \infty$

$$\int_{T_0}^{\infty} \operatorname{Cov}(|Y_1(t)|, |Y_2(t)|)p_t(u^2)dt \tag{4.142}$$

and

$$\int_{T_0}^{\infty} \left|E[|Y_1(t)|]E[|Y_2(t)|]p_t(u^2) - \frac{\sqrt{\Delta\Delta_M}}{2\pi}u^2\exp(-u^2)\right|dt \tag{4.143}$$

vanish uniformly for $M > M_0$. The variables $Y_1(t)$ and $Y_2(t)$ above are, respectively, the conditional derivatives $2R(s)R'(s)|\mathcal{C}$ and $2R_M(t)R'_M(t)|\mathcal{C}$. $p_\tau(u^2)$

is the joint density of $\{R^2(0)\}$ and $\{R_M^2(t)\}$ evaluated at the diagonal point, (u^2, u^2),

$$p_\tau(u^2) = \frac{e^{-u^2/(1-\gamma_M^2(\tau))}}{4\pi(1-\gamma_M^2(\tau))} I_0[\frac{u^2|\gamma_M(\tau)|}{1-\gamma_M^2(\tau))}]. \tag{4.144}$$

Since (Abromowitz and Stegun, pg. 375)

$$I_0(x) = 1 + \frac{1}{4}x^2 + O(x^4)$$

for all x and

$$\frac{|\gamma_M(\tau)|}{1-\gamma_M^2(\tau))} = O(\frac{1}{\tau})$$

for large τ, we see that the asymptotic behavior is

$$I_0(\frac{u^2}{\tau}) = 1 + \frac{u^4}{4\tau^2} + O(\frac{1}{\tau^4}). \tag{4.145}$$

Therefore, it is sufficient to show that

$$\int_{T_0}^{\infty} \mathrm{Cov}(|Y_1(t)|, |Y_2(t)|)dt \tag{4.146}$$

and

$$\int_{T_0}^{\infty} \left| E[|Y_1(t)|]E[|Y_2(t)|] - \frac{\sqrt{\Delta\Delta_M}}{2\pi} u^2 \exp(-u^2) \right| dt \tag{4.147}$$

both vanish uniformly for $M > M_0$.

Observe that

$$|\mu_i| \leq \frac{6u^2|\gamma_M'(\tau)|}{1-\gamma_M'^2(\tau)}$$

and by direct calculation,

$$E[|Y_i(t)|] = \sqrt{\frac{2}{\pi}\sigma_{Mi}^2} \exp^{-\mu_i^2/2\sigma_{Mi}^2} + 2|\mu_i|[\Phi_{\sigma_{Mi}^2}(\mu_i) - \frac{1}{2}] \tag{4.148}$$

where $\Phi_{\sigma_{Mi}^2}(\cdot)$ is the cumulative distribution of a normal mean-zero, variance σ_{Mi}^2 random variable and

$$\sigma_{M1}^2 = \frac{4}{1-\gamma_M^2}[\Delta(1-\gamma_M^2) - \gamma_M'^2] \tag{4.149}$$

$$\sigma_{M2}^2 = \frac{4}{1-\gamma_M^2}[\Delta_M(1-\gamma_M^2) - \gamma_M'^2]. \tag{4.150}$$

Now, without loss of generality, take $\sigma_{Mi}^2 = 1$ and write $Y_1(t) = X - \mu$ and $Y_2(t) = Y + \mu$ where X and Y are normal mean-zero, variance one, random variables with correlation ρ. Using the Hermite polynomial expansion for bivariate normal variables we can express $E[|Y_1(t)Y_2(t)|]$ by (see Cuzick, pg. 550)

$$f(\rho) = E[|Y_1(t)Y_2(t)|] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{\rho^n}{n!}|x-\mu||y+\mu|H_n(x)H_n(y)e^{-(x^2+y^2)/2}dxdy \tag{4.151}$$

where $\{H_n(x)\}$ are the Hermite polynomials. Note that $f(0) = (E[|Y_1(t)|])^2$, since $E[|Y_1(t)|] = E[|Y_2(t)|]$ and that

$$f(1) = E[|X^2 - \mu^2|] = \sum_{n=0}^{\infty}(-1)^n[\frac{1}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty}|x-\mu|H_n(x)e^{-x^2/2}dx]^2 \tag{4.152}$$

$$f(-1) = E[(X-\mu)^2] = \sum_{n=0}^{\infty}[\frac{1}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty}|x-\mu|H_n(x)e^{-x^2/2}dx]^2 = 1+\mu^2.$$

Thus, the covariance $E[|Y_1(t)Y_2(t)|] - E[|Y_1(t)|]E[|Y_2(t)|]$ is

$$\text{Cov}(|Y_1(t)|,|Y_2(t)|) = \sum_{n=1}^{\infty}(-\rho)^n[\frac{1}{\sqrt{2\pi n!}}\int_{-\infty}^{\infty}|x-\mu|H_n(x)e^{-x^2/2}dx]^2 \tag{4.153}$$

Note that the first term in the above sum is bounded above by

$$\rho[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x-\mu|xe^{-x^2/2}dx]^2 \le |\rho|[1-2\Phi_1(\mu)] \tag{4.154}$$

so that

$$\text{Cov}(|Y_1(t)|,|Y_2(t)|) \le \rho^2[1+\mu^2] + |\rho|[1-2\Phi_1(\mu)]. \tag{4.155}$$

Since γ_M and γ_M'' are both L^2, this implies that γ_M' is L^2 as well. Consequently, μ is L^2 since

$$|\mu_i| \le \frac{6u^2|\gamma_M'(\tau)|}{1-\gamma_M'^2(\tau)}.$$

Therefore $|\rho|[1-2\Phi_1(\mu)]$ is L^1 in light of ρ^2 being L^2 and so

$$\int_{T_0}^{\infty}\text{Cov}(|Y_1(t)|,|Y_2(t)|)dt \tag{4.156}$$

can be made uniformly small for $M > M_0$. Finally, we see that

$$\int_{T_0}^{\infty} \left| E[|Y_1(t)|]E[|Y_2(t)|] - \frac{\sqrt{\Delta\Delta_M}}{2\pi} u^2 \exp(-u^2) \right| dt \tag{4.157}$$

vanishes uniformly in $M > M_0$ if each of the following

$$\int_{T_0}^{\infty} \left| E[|Y_1(t)|] - \sqrt{\frac{\Delta}{2\pi}} u \exp(-u^2/2) \right| dt \tag{4.158}$$

$$\int_{T_0}^{\infty} \left| E[|Y_2(t)|] - \sqrt{\frac{\Delta_M}{2\pi}} u \exp(-u^2/2) \right| dt \tag{4.159}$$

These both follow, exactly as in Cuzick pg. 553, using (4.148) and this completes the proof.

The asymptotic variance is obtained as follows. Recall the expression for the variance of $Z_{R^2}(T)$ (use the fact that the u^2-level-crossings of $R^2(t)$ are the same as the u-level-crossings of $R(t)$,

$$\mathrm{Var}[Z_{R^2}(T)] = E[Z_{R^2}(T)] - (E[Z_{R^2}(T)])^2 + 2 \int_{0+}^{T} (T-\tau)\psi(\tau)d\tau \tag{4.160}$$

where,

$$\psi(\tau) = \frac{2u^2 M_{22}}{\pi(1-\rho_I^2(\tau))^2} \int_0^{2\pi} \Big[J(\gamma, k) + \frac{k}{\sqrt{2\pi}} e^{-k^2/2} + \tag{4.161}$$
$$\frac{(\gamma+k^2)}{4}[2\mathrm{erf}(\frac{k}{\sqrt{2}}) - 1]\Big] \cdot \exp[-\frac{u^2(1-\rho_I(\tau)\cos\phi)}{1-\rho_I^2(\tau)}]d\phi.$$

Using $E[Z_{R^2}(T)] = T \cdot E[Z_{R^2}(1)]$ and (4.160) and by the above analysis the limit exists and is finite we have

$$\lim_{T\to\infty} \frac{\mathrm{Var}[Z_{R^2}(T)]}{T} = E[Z_{R^2}(1)] + 2 \int_{0+}^{\infty} [\psi(\tau) - (E[Z_{R^2}(1)])^2] d\tau. \tag{4.162}$$

Normal probability plots from computer simulations are given in Figures (4.3.1) and (4.3.1).

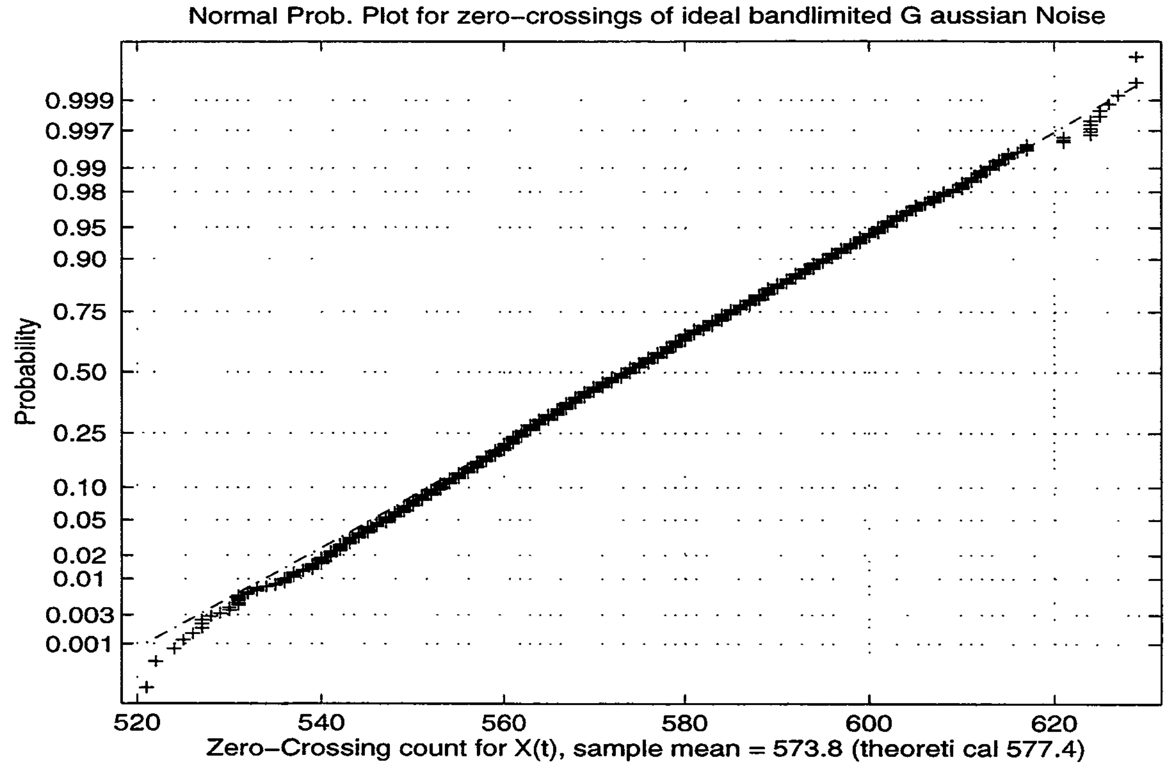

Figure 4.5: **Normal Probability Plot for the zero-crossings of an ideal bandpass Gaussian process**

4.4 Summary

In this last chapter we derived an expression for the variance of the level-crossings of the envelope of a Gaussian process possessing a symmetric one-sided spectral density. The integral expression obtained for the variance was evaluated numerically for the case of an ideal bandpass process and the results were compared with computer simulations. The theoretical values were found to be in good agreement with the monte carlo computer experiments.

Lastly, we proved for sufficiently smooth Gaussian processes, that the level-crossings of the envelope are asymptotically normal. This was fact was assumed

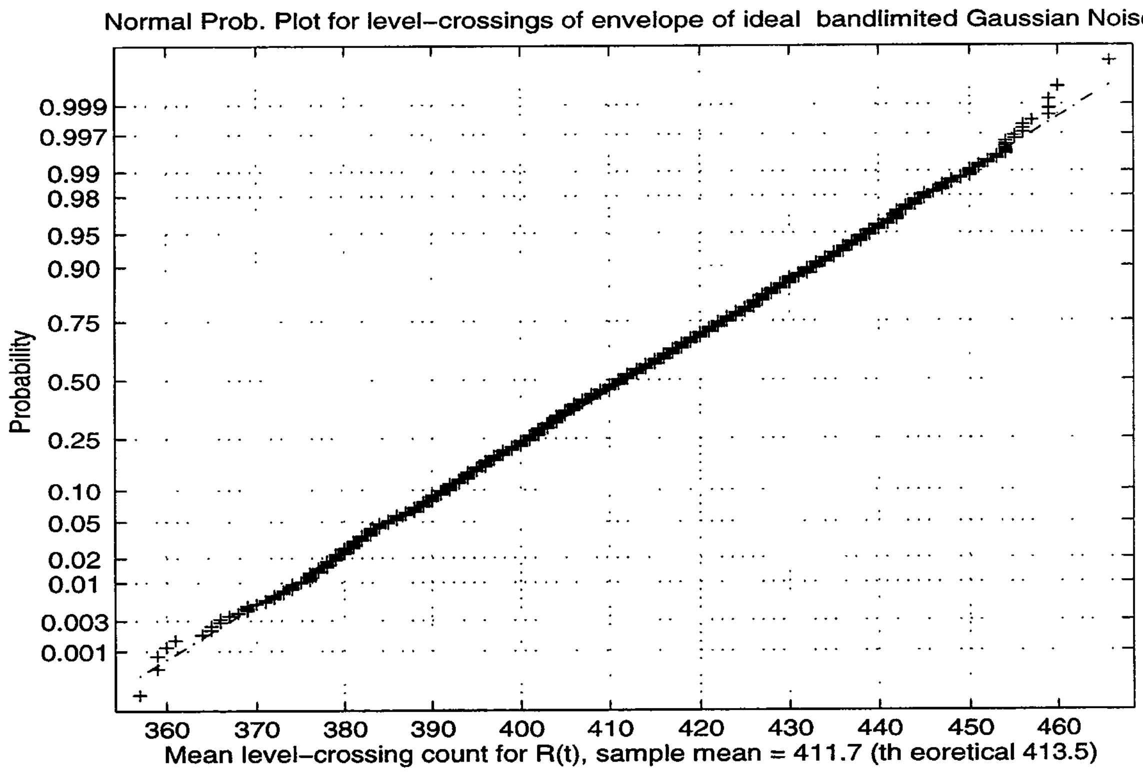

Figure 4.6: **Normal Probability Plot for the mean-level- crossings of the envelope of an ideal bandpass Gaussian process**

in Rainal (1966) and then used to devise a radar detector based on the mean-level-crossing counts of the envelope. The advantage of the level-crossing detector is that it is within 1dB of the square law detector but less computationally complex and immune to gain fluctuations. The central limit theorem for the envelope level-crossings formalizes the computer analysis by Rainal and verifies his intuition.

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