ABSTRACT

Title of Dissertation: BASE CHANGE FUNDAMENTAL LEMMA FOR CENTRAL ELEMENTS IN DEPTH-ZERO HECKE ALGEBRAS OVER LOCAL FUNCTION FIELDS

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Let G be an unramified group over a local function field F of characteristic $p > 0$. Let E/F be an unramified extension of degree r. Let χ be a depth-zero character on the maximal torus T, which induces a character ρ on an Iwahori subgroup I. Consider the corresponding Hecke algebras of depth-zero principal series block $\mathcal{H}(G(F), \rho)$ and $\mathcal{H}(G(E), \rho_E)$. One can define the base change homomorphism b between the centers of these Hecke algebras, then base change fundamental lemma asks whether the functions ϕ and $b(\phi)$ have the same stable (twisted) orbital integrals for $\phi \in Z(\mathcal{H}(G(E), \rho_E))$, at corresponding semisimple elements.

In this article, we introduce an abstract norm map between the stable twisted conjugacy classes in $G(E)$ and stable conjugacy classes of $G(F)$ in the positive characteristics setting, prove the simple twisted trace formula then apply the process of stabilization of twisted trace formula to show the existence of local data and finally adapt Labesse elementary functions to prove the corresponding base change fundamental lemma for regular semisimple elements.

BASE CHANGE FUNDAMENTAL LEMMA FOR CENTRAL ELEMENTS IN DEPTH-ZERO HECKE ALGEBRAS OVER LOCAL FUNCTION FIELDS

by

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Dedication

To my grandparents.

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Chapter 1: Introduction

1.1 Preliminaries and Main Results

Let F denote a local function field of characteristic $p > 0$, and let $F_r \supset F$ denote the unique degree r unramified extension of F contained in a fixed separable closure F^s of F, and let θ denote a generator of $\mathrm{Gal}(F_r/F)$. Let G denote an unramified connected reductive group over F. The automorphism θ determines an F-automorphism of $G(F_r)$, which is still denoted by θ. To simplify notations and statements, we will assume that $G_{\text{der}} = G_{\text{sc}}$ in the remainder of the introduction.

From the concrete norm map

$$N : G(F_r) \to G(F_r)$$

$$g \mapsto g\theta(g)\ldots\theta^{r-1}(g),$$

Kottwitz [Kot82] was able to define an abstract norm map $\mathcal{N}$ from the set of stable θ-conjugacy classes in $G(F_r)$ to the set of stable conjugacy classes in $G(F)$. We will say that $\gamma \in G(F)$ *is a norm* if γ is stably conjugate to $\mathcal{N}\delta$ for some $\delta \in G(E)$.

Fix a semisimple element $\gamma \in G(F)$ and we let $G_\gamma \subset G$ denote its centralizer in G. For $f \in C_c^\infty(G(F))$, the algebra of locally constant and compactly supported $\mathbb{C}$-valued functions on $G(F)$, we can define the *orbital integral*

$$O_\gamma^{G(F)}(f) = \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\, dg/dt$$

depending on the choice of Haar measures dg and dt on $G(F)$ and $G_\gamma(F)$ respectively. We also consider the *stable orbital integral*

$$SO_{\gamma}^{G(F)}(f) = \sum_{\gamma'} e(G_{\gamma'}) O_{\gamma'}^{G(F)}(f),$$

where γ' sums over the conjugacy classes in $G(F)$ within the stable conjugacy class of γ, and $e(G_{\gamma'}) \in \{\pm 1\}$ is the sign attached by Kottwitz [Kot83] to connected reductive groups. Notice that our assumption that G_{der} is simply connected implies that the centralizers $G_{\gamma'}$ are connected. If γ and γ' are stably conjugate, their centralizers are inner forms of each form, therefore we may require the Haar measures are compatible with each other in the definition of the orbital integrals $TO_{\gamma'}^{G(F)}(f)$ in the sense of [Kot88]: if we fix an inner twisting $\psi : (G_\gamma)_{F^s} \xrightarrow{\sim} (G_{\gamma'})_{F^s}$ and a non-zero invariant top degree differential form ω' on $G_{\gamma'}$ over F, then we say Haar measures μ and μ' on $(G_\gamma)_{F^s}$ and $(G_{\gamma'})_{F^s}$ are *compatible* if there exists $c \in \mathbb{R}^{>0}$ such that $\mu' = c|\omega'|$ and $\mu = c|\psi^*\omega'|$.

Similarly, for an element $\delta \in G(F_r)$ such that $\mathcal{N}\delta$ is semisimple, we define its twisted centralizer to be the connected reductive group $G_{\delta\theta}$ such that

$$G_{\delta\theta}(F) = \{g \in G(F_r) : g^{-1}\delta\theta(g) = \delta\}.$$

Then for $\phi \in C_c^\infty(G(F_r))$, similarly we can define the *twisted orbital integral*

$$TO_{\delta\theta}^{G(F_r)}(\phi) = \int_{G_{\delta\theta}(F)\backslash G(F_r)} \phi(g^{-1}\delta\theta(g))\, dg/dt$$

and its stable version

$$SO_{\delta\theta}^{G(F_r)}(\phi) = \sum_{\delta'} e(G_{\delta'\theta}) TO_{\delta'\theta}^{G(F_r)}(\phi),$$

where the sum is over θ-conjugacy classes δ' in $G(F_r)$ inside the stable θ-conjugacy class of δ, or equivalently (Proposition 2.2.5), whose norm down to $G(F)$ is in the same stable conjugacy class as $N\delta$. If $\gamma \in G(F)$ is stably conjugate to $N\delta$, then $G_{\delta\theta}$ is an inner form of G_γ (Lemma 2.1.5). Therefore we may also require the Haar measures on these groups to be compatible.

We say that $\phi \in C_c^\infty(G(F_r))$ and $f \in C_c^\infty(G(F))$ are *associated* or have *matching orbital integrals* (resp., at regular semisimple elements) if for every (resp., regular) semisimple element $\gamma \in G(F)$ we have

$$SO_\gamma^{G(F)}(f) = \begin{cases} SO_{\delta\theta}^{G(F_r)}(\phi) & \text{if} \quad \gamma = \mathcal{N}\delta \\ 0 & \text{if} \quad \gamma \text{ is not a norm} \end{cases} \tag{1.1.1}$$

The case of spherical Hecke algebras was studied first. Suppose $K_r \subset G(F_r)$ and $K \subset G(F)$ are hyperspecial maximal compact subgroups associated to a hyperspecial vertex in the Bruhat-Tits building $\mathcal{B}(G(F))$ of $G(F)$, and suppose $\phi \in C_c^\infty(G(F_r))$ belongs to the spherical Hecke algebra $\mathcal{H}(G(F_r), K_r)$ of locally constant compactly supported functions on $G(F_r)$ that are bi-invariant under K_r. In particular, the spherical Hecke algebras are commutative. The Satake isomorphism gives rise to a natural algebra homomorphism

$$b_r : \mathcal{H}(G(F_r), K_r) \to \mathcal{H}(G(F), K) \tag{1.1.2}$$

which will be called the *base change homomorphism*. The base change fundamental lemma for spherical functions asserts that ϕ and $b_r(\phi)$ are associated. This was proved in the special cases of GL_2 [Lan80] and GL_n [AC89], which gave rise to global and local applications, and for general unramified reductive groups it was proved by Clozel [Clo90] and Labesse [Lab90].

In [Hai09], an analogous base change fundamental lemma is proved for centers of parahoric

Hecke algebras by Haines, which replaced the hyperspecial maximal compact subgroup K_r (resp., K) above by a general parahoric subgroup J_r (resp., J) defined as intersection of the group of elements in $G(F_r)$ (resp. $G(F)$) that fix a facet in $\mathcal{B}(G(F))$ with the kernel of Kottwitz homomorphism(see section 2 of *loc. cit.*). In particular, when the facet is a hyperspecial vertex, we have $J_r = K_r$. However, since the resulting parahoric Hecke algebras $\mathcal{H}(G(F_r), J_r)$ and $\mathcal{H}(G(F), J)$ are in general no longer commutative, the base change homomorphism will be restricted to the center of those Hecke algebras:

$$b_r : \mathcal{Z}(G(F_r), K_r) \to \mathcal{Z}(G(F), K)$$

From now on and in this article, when we speak of base change fundamental lemma, we refer to the matching of elements in the centers linked by the base change homomorphism. Later in [Hai12], Haines generalize this result to so called *depth-zero* principal series blocks in the Bernstein decomposition.

All the results [Clo90, Lab90, Hai09, Hai12] stated above only work over groups over p-adic fields F of char $F = 0$. Results in positive characteristics are more ad-hoc for specific groups. In his thesis [RD10], Ray-Dulany proved the base change fundamental lemma for Iwahori-Hecke algebra of GL_2 for any local field with characteristic not equal to 2 by explicit calculations. In [Fen20], Feng gave a geometric proof of the base change fundamental lemma for regular semisimple elements for parahoric Hecke algebras of GL_n over local function fields using cohomology of certain moduli stacks of shtukas and the Langlands-Kottwitz method, which generalized the previous results by [Ngô06] for spherical Hecke algebras for GL_n over local function fields.

In this article, we prove the base change fundamental lemma for Bernstein centers of depth-zero principal series blocks of regular semisimple elements in unramified groups G over local function fields, following and generalizing the strategies in [Hai09, Hai12]. To state the theorem, let us fix some more notations. Denote the ring of integers of F by $\mathcal{O}_F$. Let A denote a maximal F-split torus of G and set $T = Z_G(A)$, a maximal torus of G which is defined and unramified over F. We can choose an Iwahori subgroup $I \subset G(F)$ which is in good position relative to T. Let I^+ denote the pro-unipotent radical of I. Furthermore, let $T(F)_1$ denote the maximal compact open subgroup of $T(F)$, and let $T(F)_1^+ = T(F)_1 \cap I^+$ denotes its pro-unipotent radical. We will denote χ to be a character on $T(F)_1/T(F)_1^+$. Such characters are called *depth-zero* characters. Via the canonical isomorphism

$$T(F)_1/T(F)_1^+ \xrightarrow{\sim} I/I^+$$

we see that χ induces a character $\rho := \rho_\chi$ on I, trivial on I^+. Then we can consider the Hecke algebra $\mathcal{H}(G, \rho)$ of our study:

$$\mathcal{H}(G, \rho) := \{f \in C_c^\infty(G) : f(igi') = \rho^{-1}(i) f(g) \rho^{-1}(i'), \forall\, i, i' \in I,\, \forall g \in G\},$$

where the convolution is defined using the Haar measure which gives I volume 1. Write $\mathcal{Z}(G, \rho)$ for $Z(\mathcal{H}(G, \rho))$.

Let $N_r : T(F_r)_1 \to T(F)_1$ denote the naive norm homomorphism. Therefore, the character $\chi_r := \chi \circ N_r$ is a depth-zero character on $T(F_r)_1$, and it gives rise to the character ρ_r on I_r and the Hecke algebra $\mathcal{H}(G_r, \rho_r)$ with center $\mathcal{Z}(G_r, \rho_r)$. Here $G_r := G(F_r)$ and $I_r \subset G_r$ is the Iwahori subgroup corresponding to I.

We will define a *base change homomorphism*

$$b_r : \mathcal{Z}(G_r, \rho_r) \to \mathcal{Z}(G, \rho)$$

Finally we can state the main theorem of this article:

Theorem 1.1.1. *Let F be a local function field of arbitrary positive characteristic. Let G be a connective unramified reductive group defined over F. Denote F_r/F to be an unramified extension of local function fields of degree r. Then for any $\phi \in \mathcal{Z}(G_r, \rho_r)$, ϕ and $b_r(\phi)$ have the matching stable orbital integrals at every* ***regular semisimple element****.*

We write $\mathcal{H}(G(F_r), I_r^+)$ (resp., $\mathcal{H}(G(F), I^+)$)for the Hecke algebra of locally constant compactly supported functions on $G(F_r)$ (resp., $G(F)$) that are bi-invariant under the open compact subgroups I_r^+ (resp., I^+), the pro-unipotent radicals of I_r (resp., I), and we write $\mathcal{Z}(G(F_r), I_r^+)$ (resp., $\mathcal{Z}(G(F), I^+)$) for the center of these Hecke algebras. In [Hai12], §10, a base change homomorphism

$$b_r : \mathcal{Z}(G(F_r), I_r^+) \to \mathcal{Z}(G(F), I^+)$$

was constructed and characterized using the injection

$$\mathcal{Z}(G(F_r), I_r^+) \hookrightarrow \prod_{\chi'} \mathcal{Z}(G(F_r), I_r, \chi'),$$

where χ' sums over the finite set of all depth-zero characters on $T(F_r)_1$. As a result, we are able to show a base change fundamental lemma for pro-p Iwahori level from Theorem 1.1.1:

Corollary 1.1.2. *If $\phi \in \mathcal{Z}(G(F_r), I_r^+)$, then the functions ϕ and $b_r(\phi)$ are associated.*

This article overlaps with [Fen20] in the case when $G = GL_n$. However, we are able to prove the vanishing results when γ is not a norm, which was not done in *loc. cit.*

1.2 Motivations

The original motivation for base change fundamental lemma over nonarchimedean local fields of char 0 was to study the cohomology of a Shimura variety with Iwahori level structure, and in particular to determine the semisimple zeta factor at a place of bad reduction of Iwahori type at p. Base change fundamental lemma serves as a bridge between the counting points formula from the geometric side and trace of functions on automorphic representations from the spectral side.

More specifically, let $Sh_K = Sh(G, h^{-1}, K)$ denote the Shimura variety of Abelian type attached to a connected reductive group G over $\mathbb{Q}$, a family of Hodge structures h, and a compact open subgroup $K \subset G(\mathbb{A}_f)$. For simplicity, we will assume that Sh_K is so called simple Shimura Varieties, in the sense of [Kot92]. In particular, it means that Sh_K is proper and G has no endoscopy.

We denote $\mathbb{A} = \mathbb{A}_{\mathbb{Q}}$ to be the ring of adeles of $\mathbb{Q}$. Fix a prime number p and assume that $K = K_p K^p \subset G(\mathbb{Q}_p) \times G(\mathbb{A}_f^p)$. It is known that Sh_K can be defined over the *reflex field* E, and we assume that it has a suitable integral model over $\mathcal{O}_{E_{\mathfrak{p}}}$ which we still denote by Sh_K, where $\mathfrak{p}$ is a prime ideal of $\mathcal{O}_E$ and $\mathcal{O}_{E_{\mathfrak{p}}}$ is the ring of integers in the $\mathfrak{p}$-adic completion $E_{\mathfrak{p}}$ of E. For simplicity, we assume that $E_{\mathfrak{p}} = \mathbb{Q}_p$ in what follows.

Write $R\Psi := R\Psi^{Sh_K}(\bar{\mathbb{Q}}_l)$ for the complex of nearby cycles on the special fiber of Sh_K over $\mathcal{O}_{E_{\mathfrak{p}}}$, where $l \neq p$ is a fixed prime number. For $r \neq 1$, denote k_r to be the degree r extension of $\mathbb{F}_p$, and consider the semi-simple Lefschetz number defined as the value in $\bar{\mathbb{Q}}_l$ given by the sum

$$\mathrm{Lef}^{ss}(\Phi_{\mathfrak{p}}^r, R\Psi) = \sum_{x \in Sh_K(k_r)} \mathrm{tr}^{ss}(\Phi_{\mathfrak{p}}^r, R\Psi_x),$$

where $\Phi_{\mathfrak{p}}$ denotes the Frobenius morphism on $Sh_K \otimes_{\mathcal{O}_{E_{\mathfrak{p}}}} \bar{\mathbb{F}}_p$. These Lefschetz numbers show up in the definition of the semisimple zeta function at $\mathfrak{p}$, namely,

$$\log(Z^{ss}_{\mathfrak{p}}(s)) = \sum_{r=1}^{\infty} \mathrm{Lef}^{ss}(\Phi^r_{\mathfrak{p}}, R\Psi)\frac{p^{-rs}}{r}$$

The left hand side should take the form of a "counting points formula"

$$\mathrm{Lef}^{ss}(\Phi^r_{\mathfrak{p}}, R\Psi) = \sum_{(\gamma_0;\gamma,\delta)} c(\gamma_0;\gamma,\delta)\, O^{G(\mathbb{A}^p_f)}_{\gamma}(f^p)TO^{G(F_r)}_{\delta\sigma}(\phi_r), \tag{1.2.1}$$

where the sum is over certain equivalent classes of triples $(\gamma_0;\gamma,\delta) \in G(\mathbb{Q}) \times G(\mathbb{A}^p_f) \times G(L_r)$, where L_r is the fraction field of the ring of Witt vectors $W(k_r)$, which is an unramified extension of $\mathbb{Q}_p$ of degree r, and σ denote the Frobenius generator of $\mathrm{Gal}(L_r/\mathbb{Q}_p)$.

The Iwahori subgroup $K_p \subset G(\mathbb{Q}_p)$ also gives rise to an Iwahori subgroup $K_{p^r} \subset G(L_r)$. Any such function ϕ_r making 1.2.1 holds will be called the *test function* and it should be in $\mathcal{Z}(G(L_r), K_{p^r})$ (for details, see [HR12]) Eventually we may simplify the right hand side to be

$$\mathrm{Lef}^{ss}(\Phi^r_{\mathfrak{p}}, R\Psi) = \tau(G)\sum_{\gamma_0} SO^{G(\mathbb{A}^p_f)}_{\gamma_0}(f^p)SO^{G(\mathbb{R})}_{\gamma_0}(f_\infty)SO^{G(L_r)}_{\delta\sigma}(\phi_r) \tag{1.2.2}$$

By the definition of the Kottwitz triple $(\gamma_0;\gamma,\delta)$, we have $\mathcal{N}\delta = \gamma_0$ as stable conjugacy classes in $G(\mathbb{Q}_p)$. The base change fundamental lemma (for all semisimple elements) will allow us to write

$$SO^{G(L_r)}_{\delta\sigma}(\phi_r) = SO^{G(\mathbb{Q}_p)}_{\gamma_0}(b_r(\phi_r)).$$

where $b_r : \mathcal{Z}(G(L_r), K_{p^r}) \to \mathcal{Z}(G(\mathbb{Q}_p), K_p)$ is the base change homomorphism on the center of the corresponding Hecke algebras. Therefore, we can write 1.2.2 as

$$\mathrm{Lef}^{ss}(\Phi_{\mathfrak{p}}^{r}, R\Psi) = \tau(G)\sum_{\gamma_0} SO_{\gamma_0}^{G(\mathbb{A})}(f^p f_r f_\infty), \tag{1.2.3}$$

where $f_r = b_r(\phi_r)$. Use the stabilized global trace formula for $f = f^p f_r f_\infty$, we can write 1.2.3 as

$$\mathrm{Lef}^{ss}(\Phi_{\mathfrak{p}}^{r}, R\Psi) = \sum_{\pi} m(\pi)\mathrm{tr}\,\pi(f^p f_r f_\infty), \tag{1.2.4}$$

where here π ranges over irreducible representations in $L^2(G(\mathbb{Q})A_G(\mathbb{R})^\circ\backslash G(\mathbb{A}))$. Therefore, we can relate the local factor of the zeta function with automorphic representations, eventually the L-function ([HR12], Cor 1.4).

In the function field setting, one can expect a similar counting points formula as 1.2.1 on moduli stacks of Shtukas with bad reductions using Langlands-Kottwitz method. Feng [Fen20] studied parahoric reduction and gave a counting points formula for such moduli stack. In her thesis [Son24], Song investigated the Langlands-Kottwitz approach to proper moduli stack of global Shtukas. With the (full version of) base change fundamental lemma over local function fields, one can hope to stabilize the counting points formula as in the Shimura varieties case.

1.3 Summary of the Paper

Our approach is modeled on that of Haines [Hai12], however, to generalize the results, there are a few obstacles in applying the similar methodology. We will review the process and explain briefly the technical difficulties over function fields.

To first make sense of the identity 1.1.1, one needs to define the abstract norm homomorphism.

This was done by Kottwitz [Kot82] when the base field is perfect, which relied on the fact that a semisimple conjugacy class defined over F has a F-rational point. In Chapter 2, we will define the norm homomorphism over positive characteristic fields (which are not perfect) using the recent progress from Hamacher and Kim [HK22, HK21]. We will then prove that this map is well-defined on the stable conjugacy classes, therefore it is suitable for identities of stable distributions like 1.1.1.

We define the Iwahori subgroups in Chapter 3 and Bernstein center for depth-zero principal series in Chapter 4. Functions inside the center will be the subject of the study in this article.

Another essential ingredient in the formulation of 1.1.1 is the base change homomorphism $b : \mathcal{Z}(G_r, \rho_r) \to \mathcal{Z}(G, \rho)$. For parahoric Hecke algebras, the base change homomorphism can be viewed as the incarnation of the norm map between certain $\mathbb{C}$-algebras, via the Bernstein isomorphisms ([Hai09], §3). For depth-zero Hecke algebras, we have analogous isomorphisms defined by Haines in [Hai12], that will allow us to define and characterize base change homomorphisms.

After being able to formulate the identity 1.1.1, Chapter 6 - Chapter 8 deal with various reductions that will eventually allow us to assume that $G_{\text{der}} = G_{\text{sc}}$ and γ is *strongly elliptic regular*, both in G and in G^{ad}. In Chapter 6 and 7, we define the constant term homomorphism, that will allow us to "descend" the (twisted) orbital integrals onto appropriate Levi subgroups, in which the semisimple elements γ are elliptic. We then make the descent and compare the integrals on G and the Levi subgroup M (7.4.2, 7.4.3). Our reduction steps in Chapter 8 are different from those in [Hai09, Hai12] in that Lemma 7.2.3 in [Hai12] is in general not available[1] without nontrivial modifications when there is a depth-zero character presented in the definition of the Hecke algebras. Therefore, we cannot reduce to adjoint groups. However, we will bypass the technical difficulty of the existence of noncompact center

[1]Lemma 5.3.3 in [Hai09] is valid for parahoric Hecke algebras, as functions inside such algebras are bi-invariant under the parahoric subgroups.

by passing to functions translated under certain unitary characters under the central action (8.4).

After the reduction steps, we will use identities on the representation-theoretic side (or spectral side) to prove identities of the form 1.1.1 on the conjugacy classes of the groups (or geometric side). In order to do so, we will adapt and utilize a twisted version of Arthur-Selberg trace formula in the function field setting. The Arthur-Selberg trace formula is a series of fascinating and deep results that assert identities between two kinds of distributions on the adelic points of the group G, one consisting of conjugacy classes in the group, the other consisting of a sum over certain representations of certain subgroups of the group G. It is a profound subject of its own, however, in our case, after putting various restrictions on the functions, the (twisted) trace formula we get will be the so called *very simple version*, which means that only (strongly) regular elliptic semisimple elements will appear on the geometric side of the trace formula. This will suffice our needs in this article. We will prove a version of simple trace formula when the G has center that is an induced torus, by carefully examining the functions, we will see that, in this case the harmonic analysis on G is essentially the same as the one on G_{ad} (9.3).

The simple (twisted) trace formula we prove in Chapter 9 cannot be put to immediate use, as ultimately we want to use it to compare stable orbital integrals on $G(F)$ and $G(F_r)$. In Chapter 10, we *stabilize* the trace formulas so that they are stable distributions, that is, they are given in terms of stable (twisted) orbital integrals. The stabilization of the twisted trace formula is an involved process that can be basically summarized as the following paradigm:

$$\{\text{rational conjugacy}\} \to \{\text{adelic conjugacy}\} \to \{\text{stable conjugacy}\}$$

each of the steps requires one to prove some kind of vanishing results and to keep track of the numbers of the corresponding conjugacy classes. The results of stabilization (10.4.1 and 10.4.3) will be given in terms of stable adelic integrals, which can be written as products of local stable integrals over places of F, so that we can compare identities 1.1.1 place by place.

Chapter 11 and 12 finish the proof of Theorem 1.1.1. Using the stabilized trace formula, one is able to produce the existence of *local data* (12.1) to turn the identities of integrals on G into identities of characters on G. Following [Lab90], we may define the *elementary functions* in Chapter 11, which in our case are special functions inside the depth-zero Hecke algebras whose (twisted) orbital integrals and whose traces are easy to compute. We finish the proof in the strongly regular elliptic case (12.5) by relating the traces of elementary functions and the traces of the general functions in the center of the depth-zero Hecke algebras (12.5.8).

Chapter 2: Norm Mapping and Conjugacy Completeness

Let F be a field of characteristic $p > 0$. Let G be a connected reductive group over F. We fix once for all a separable closure F^{s} of F inside an algebraic closure $\bar{F}$ of F, and let $\Gamma = \mathrm{Gal}(F^{\mathrm{s}}/F)$ be the absolute Galois group. Consider a conjugacy class C of G. By conjugacy class we mean the conjugacy class in $G(F^s)$ (rather than in $G(\bar{F})$). We denote $C^\sigma := \sigma(C)$ and $x^\sigma := \sigma(x)$ for any $x \in G$. The conjugacy class C is defined over F if and only if $C^\sigma = C$ for all $\sigma \in \Gamma$, and the conjugacy class of an element $x \in G$ is defined over F if and only if x^σ is conjugate to x under $G(F^s)$ for all $\sigma \in \Gamma$.

Let E/F be an unramified extension of degree r. We further assume G to be an unramified connected reductive group over F, let $\theta \in \mathrm{Gal}(E/F)$ be the generator. let $R_{E/F}G_E$ denote the Weil restriction of scalars of $G_E := G \times_F E$ to a group over F. The automorphism θ of E determines an F-automorphism of $R_{E/F}G_E$ and an automorphism on its F-points $G(E)$, which will be also denoted by θ.

Consider the *concrete norm* $N_r : G(E) \to G(E)$ given by

$$N_r\delta = \delta\theta(\delta)\ldots\theta^{r-1}(\delta)$$

It is easy to see that the conjugacy class of $N_r\delta$ in $G(F^s)$ is defined over F since we can calculate that

$$\theta(N_r\delta) = \delta^{-1}(N_r\delta)\delta$$

The goal is to define a well-defined *abstract norm map*.

$$\mathcal{N} : \{\text{stable } \theta\text{-conjugacy classes in } G(E)\} \to \{\text{stable conjugacy classes in } G(F)\} \tag{2.0.1}$$

To make sense of all that, we will have to define *stable θ-conjugacy class* first, then we will show that the conjugacy class of $N_r\delta$ actually has a rational point over F.

2.1 Stable Twisted Conjugacy

We follow [Kot82], section 5. We assume $G_{\text{der}} = G_{\text{sc}}$ for now, later we will remove this assumption by using the so called z-extension of G.

Let $I := R_{E/F}G_E$, hence there is a natural isomorphism $I_E \xrightarrow{\sim} G_E \times \cdots \times G_E$, with the factors indexed by $\mathrm{Gal}(E/F)$. The element $\theta \in \mathrm{Gal}(E/F)$ determines an automorphism $s \in \mathrm{Aut}_F(I)$, which takes the form

$$s : (x_1, \dots, x_r) \mapsto (x_r, x_1, \dots, x_{r-1})$$

on E-valued points. We can identify G with I^s by embedding G into I diagonally. The composition

$$G(E) = I(F) \to I(E) = G(E) \times \cdots \times G(E)$$

is given by

$$x \mapsto (x^{\theta^{r-1}}, \dots, x^{\theta}, x).$$

We can also define the concrete norm map on I by $Nx = x^{s^{r-1}} \dots x^s \cdot x$. It is defined over F and it is the same on $G(E) = I(F)$ as the norm map $N : G(E) \to G(E)$ we defined before. We say that $x, y \in I(F^s)$ are *F^s-θ-conjugate* if there exists an element $g \in I(F^s)$ such that $x = g^{-s}yg$. We say that $x, y \in G(E)$ are *F^s-θ-conjugate* if they are F^s-θ-conjugate as elements $G(E) = I(F) \subset I(F^s)$.

Lemma 2.1.1. *(a) For $g, y \in I(F^s)$, we have $N(g^{-s}yg) = g^{-1}(Ny)g$.*

(b) *Let $x, y \in I(F^s)$. Then x, y are F^s-θ-conjugate if and only if Nx, Ny are conjugate in $I(F^s)$.*

(c) *The embedding $G \hookrightarrow I$ induces a $\Gamma = \mathrm{Gal}(F^s/F)$-equivariant injection from the set of conjugacy classes in $G(F^s)$ to the set of conjugacy classes in $I(F^s)$.*

Proof. The proof of [Kot82], Lemma 5.2 adapts here. Let $g = (g_1, \ldots, g_r)$ and $y = (y_1, \ldots, y_r)$, therefore, by simple calculation, $N(g^{-s}yg) = (g_1^{-1}, \ldots, g_r^{-1}) \cdot (y_2 y_3 \ldots y_r y_1, \ldots, y_1 y_2, \ldots, y_r) \cdot (g_1, \ldots, g_r) = g^{-1}(Ny)g$, thus (a) is proved.

Let $y = g^{-s}xg$ for some $g \in I(F^s)$, by (a) we have $N(y) = g^{-1}(Nx)g$. Conversely, let $x = (x_1, \ldots, x_r)$. We need to find a $g \in I(F^s)$ such that $(g_r^{-1} y_1 g_1, g_1^{-1} y_2 g_2, \ldots, g_{r-1}^{-1} y_r g_r) = (x_1, \ldots, x_r)$. Solving this system of equations in terms of $g_r = 1$, we get

$$\begin{aligned} g_{r-1} &= y_r x_r^{-1}; \\ g_{r-2} &= y_{r-1} g_{r-1} x_{r-1}^{-1}; \\ &\ldots \\ g_1 &= y_2 g_2 x_2^{-1}. \end{aligned} \tag{2.1.1}$$

For the solutions to be consistent, $1 = y_1 g_1 x_1^{-1}$ has to be satisfied. But $g_1 = y_2 \ldots y_r x_r^{-1} \ldots x_2^{-1}$, and $y_1 \ldots y_r = x_1 \ldots x_r$ from the fact that $Nx = Ny$. Therefore (b) is proved.

The Γ-equivariant part is clear since the embedding $G \hookrightarrow I$ is defined over F. Let $x, y \in G(F^s)$, then they embed as $(x, \ldots, x)$ and $(y, \ldots, y)$ in $I(F^s)$. Then it is clear that they are conjugate in $I(F^s)$ if and only if they are conjugate in $G(F^s)$. □

We get an immediate corollary from (b) and (c) of Lemma 2.1.1 above.

Corollary 2.1.2. *Let $x, y \in G(E)$. Then Nx, Ny are conjugate in $G(F^s)$ if and only if x, y are*

F^s-θ-*conjugate.*

To define stable θ-conjugacy, we have to define θ-centralizers.

Definition 2.1.3. *For $x \in G(E) = I(F)$, let $I_{sx} = \{g \in I : g^{-s}xg = x\}$. This group will be called the θ-centralizer of x.*

It is an F-subgroup of I. Since we have $I_E = G_E \times \cdots \times G_E$, where the factors are indexed by elements of $\mathrm{Gal}(E/F)$: $\theta^{r-1}, \ldots, \theta, 1$, let $p : I_E \to G_E$ be the projection onto the last factor: $(x_1, \ldots, x_r) \mapsto x_r$.

Remark 2.1.4. *Consider the F-points of $I_{s\delta}$ for $\delta \in G(E) = I(F)$, this can be given as the F-point of a connected reductive group as follows:*

$$\begin{aligned} I_{s\delta}(F) &= \{g \in I(F) : g^{-s}\delta g = \delta\} \\ &= \{g \in G(E) : g^{-\theta}\delta g = \delta\} \end{aligned}$$

We will denote by $G_{\delta\theta}$ in the future.

Lemma 2.1.5. *Let $x \in G(E)$. Then the projection p induces an isomorphism $(I_{sx})_E \xrightarrow{\sim} (G_E)_{Nx}$. This makes I_{sx} an F-form of the centralizer of Nx in G.*

Proof. We adapt the proof of Lemma 5.4 in [Kot82]. Write $g = (g_1, \ldots, g_r)$ and $x = (x^{\theta^{r-1}}, \ldots, x^\theta, x) \in G(E) = I(F) \subset I(F^s)$. Then $g^{-s} = (g_r^{-1}, g_1^{-1}, \ldots, g_{r-1}^{-1})$. Then the condition that $g \in I_{sx}$, i.e., $g^{-s}xg = x$ translates to equations $g_{r-1} = xg_rx^{-1}, g_{r-2} = x^\theta g_{r-1}x^{-\theta}, \ldots, g_1 = x^{\theta^{r-2}}g_2x^{-\theta^{r-2}}, g_r = x^{\theta^{r-1}}g_1x^{-\theta^{r-1}}$. From the first $r-1$ equations, we can solve for $g_1, \ldots, g_{r-1}$ in terms of g_r and from the last equation, the solutions will be consistent if and only if $g_r = (Nx)g_r(Nx)^{-1}$, that is, if and only if $g_r = p(g) = p(g_1, \ldots, g_r) \in G_{Nx}$. $\square$

Let us assume G is a general quasi-split reductive group defined over F, we need to define yet another subgroup I°_{sx}, just as in the case of stable conjugacy. We define I°_{sx} as the inverse image under the E-isomorphism $p : I_{sx} \to G_{Nx}$ of the subgroup G°_{Nx} of G_{Nx}. We will see that I°_{sx} is defined over F in the following lemma. It is clear that $I^\circ_{sx} = I_{sx}$ when G_{der} is simply connected. In general, we consider a z-extension $\alpha : G' \to G$, whose definitions are given below. We will also get a corresponding z-extension $\gamma : I' \to I$ where $I' = R_{E/F}((G')_E)$.

Definition 2.1.6. *Given a connected reductive group G over F, we say that a homomorphism α : $G' \to G$ is a z-extension of G if*

1. *G' is also a connected reductive group over F, whose derived group is simply connected.*

2. $\ker(\alpha) \subset Z(G')$ *and is isomorphic to a product of tori of the form* $\mathrm{Res_{L/F}}\mathbb{G}_\mathrm{m}$, *where each L is a finite separable extension of F.*

3. *α is surjective.*

The existence of z-extensions are proved in [DMO+82] Prop 3.1.

Lemma 2.1.7. *For any $y \in G'(E)$ such that $\alpha(y) = x$, we have $\gamma(I'_{sy}) = I^\circ_{sx}$. In particular, I°_{sx} is defined over F.*

Proof. It is quite easy to see that $\gamma(I'_{sy}) \subset I_{sx}$. Therefore, the composition $p \circ \gamma : (I'_{sy})_E \to G_{Nx}$ makes sense. Since I'_{sy} is connected, we have $p(\gamma(I'_{sy})) \subset G^\circ_{Nx}$, therefore $\gamma(I'_{sy}) \subset I^\circ_{sx}$. Conversely, we use the fact that the map $\alpha : G'_{Ny} \to G^\circ_{Nx}$ is surjective, therefore, the pullback to the isomorphic groups $\gamma : I'_{sy} \to I^\circ_{sx}$ should still be surjective. □

Definition 2.1.8. *We say that $x, y \in G(E) = I(F)$ are stably θ-conjugate if there exists $g \in I(F^s)$ such that $g^{-s}xg = y$ and $g^\tau g^{-1} \in I^\circ_{sx}$ for all $\tau \in \Gamma$.*

Now since $x, y \in G(E) = I(F)$, we have $(g^{-s}xg)^\tau = g^{-\tau s}xg^\tau = g^{-s}xg$, hence $g^\tau g^{-1} \in I_{sx}$ for all $\tau \in \Gamma$. Therefore stable θ-conjugacy and F^s-θ-conjugacy coincide whenever $I^\circ_{sx} = I_{sx}$, in particular whenever G_{der} is simply connected.

We introduce a special kind of z-extension $\alpha : G' \to G$.

Definition 2.1.9. *We will say that the z-extension $\alpha : G' \to G$ is adapted to E (or E/F) if the norm map $\ker(\alpha)(E) \to \ker(\alpha)(F)$ is surjective.*

Lemma 2.1.10. *(a) Let G be a connected quasi-split reductive group. There exists a z-extension $\alpha : G' \to G$ adapted to E.*

(b) Let $\alpha : G' \to G$ be a z-extension adapted to E. Then $x, y \in G(E)$ are stably θ-conjugate if and only if there exist F^s-θ-conjugate elements $x', y' \in G'(E)$ such that $\alpha(x') = x$ and $\alpha(y') = y$.

Proof. The proof of Lemma 5.6 [Kot82] can be adapted to the equal characteristic situation. Note that Kottwitz uses Proposition 3.1 of [DMO+82], which can be also adapted to the equal characteristic case.

For (a), we just simply choose L to be a Galois extension of F large enough such that it contains E and splits G. There exists a z-extension $\alpha : G' \to G$ with $\ker(\alpha)$ isomorphic to a product of copies of $\mathrm{Res}_{L/F}\mathbb{G}_m$. Therefore, it suffices to prove that the norm map $N : \mathrm{Res}_{L/F}\mathbb{G}_m(E) = (L \times \cdots \times L)^\times \to \mathrm{Res}_{L/F}\mathbb{G}_m(F) = L^\times$ is surjective, which is immediate.

For (b), "$\Leftarrow$" is immediate from the definitions and Lemma 1.3. While "$\Rightarrow$" is the same diagram chasing as in the proof of Lemma 5.6 in [Kot82].

2.2 Definition of the Norm Homomorphism

To show that for any $\delta \in G(E)$, $N_r(\delta)$ is stably conjugate to an element in $G(F)$, therefore defines a stable conjugacy class in $G(F)$, we will have to use the classical results from [Kot85] and recent progress from [HK22].

Proposition 2.2.1. *Let $\delta \in G(F^s)$ be G-regular and basic in M (in the sense of [Kot85]), lying inside a F-Levi subgroup M. Then we have a pair (J, u) such that*

1. *J is an inner twist of M;*
2. *u is an F^s-isomorphism $J_{F^s} \xrightarrow{\sim} M_{F^s}$;*
3. *$u(\theta(x)) = b \cdot \theta(u(x)) \cdot b^{-1}$.*
4. *The inner twist u identifies $J(F)$ with the set*

$$\{g \in G(F^s) : g \text{ commutes with } \delta\theta\}$$

Proof. This is from [Kot85] Remark 6.5 and 5.2. □

Remark 2.2.2. *The δ we consider in the norm map might not be G-regular and M-basic, however, using Proposition 6.2 in [Kot85], any $\delta \in G(F^s)$ is σ-conjugate to a G-regular and M-basic element, and by the previous results, their norms are in the same stable conjugacy class, we can assume δ is G-regular and M-basic.*

Now we use Lemma 8.1 from [HK22].

Lemma 2.2.3. *Let G^* be a quasi-split reductive group over an infinite field F with a simply connected derived subgroup, and H^* an F-subgroup of G^* containing a maximal torus. Let H be an inner form of H^*, and fix an inner twist $H_{F^s} \xrightarrow{\sim} H^*_{F^s}$, hence we can view $H(F)$ as a subgroup of $G^*(F^s)$.*

Then for any semisimple element $a \in H(F)$, the $G^*(F^s)$-conjugacy class of a contains an element $\gamma \in G^*(F)$.

Remark 2.2.4. *Hamacher and Kim's proof in loc. cit. relies on their earlier results (Theorem A.1.1) in [HK21], which states that any **regular** semisimple conjugacy class defined over F has a rational point over F. The lemma above, however, does not show that any semisimple conjugacy class has the same property, but it suffices for the purpose of defining the abstract norm map.*

Now we consider a semisimple G-regular M-basic element $\delta \in G(E)$ and its concrete norm $N_r(\delta)$. Since $\delta\,\theta(N_r\delta) = (N_r\delta)\delta$, by the above proposition, with $G^* = G$, $H^* = M$ and $H = J$. We see that $N_r\delta \in J(F)$, and the stable conjugacy class of $N_r\delta$ contains an element of $G(F)$. Combined with Corollary 2.1.2, this finishes the definition of the abstract norm map 2.0.1 in the case when G_{der} is simply connected.

For the next step, we just need to follow the argument after Lemma 5.1 in [Kot82] to define the abstract norm map for all quasi-split reductive groups. To be specific, we find a z-extension $\alpha : G' \to G$ adapted to E. Then we can define the abstract norm map on $G'(E)$ by the previous results, and we have the following commutative diagram

$$\begin{tikzcd}
G'(E) \arrow[r, "\mathcal{N}_r"] \arrow[d, two heads, "\alpha"] & G'(F)/\sim \arrow[d, "\alpha"] \\
G(E) \arrow[r, dashed, "\mathcal{N}_r"] & G(F)/\sim
\end{tikzcd}$$

and we claim that there exists a unique map, still denoted by $\mathcal{N}_r$, mapping stable-θ-conjugate classes in $G(E)$ to stable conjugate classes in $G(F)$. Indeed, uniqueness of the map comes from the fact that $\alpha : G'(E) \to G(E)$ is surjective. For existence, we have to show that if $x, y \in G'(E)$ and $\alpha(x) = \alpha(y)$, then $\alpha(\mathcal{N}_r(x)) = \alpha(\mathcal{N}_r(y))$ as stable classes. We can find $z \in \ker(\alpha)(E)$ such that $x = yz$. Since $\ker(\alpha)$ is central, we have $N_r(x) = N_r(y)N_r(z)$ with $N_r(z) \in \ker(\alpha)(F)$. Hence $\alpha(N_r(x)) = \alpha(N_r(y))$ as desired.

Next, we show that the definition of $\mathcal{N}_r$ does not depend on the choice of the z-extension. As in the [Kot82], this reduces to show that the following diagram is commutative

$$\begin{tikzcd} G_1(E) \arrow[r, "\mathcal{N}_r"] \arrow[d] & G_1(F)/\sim \arrow[d] \\ G_2(E) \arrow[r, "\mathcal{N}_r"] & G_2(F)/\sim \end{tikzcd}$$

for G_1, G_2 two z-extensions of G. But this comes from the definition of $\mathcal{N}_r$ when the derived group is simply connected.

This completes the definition of $\mathcal{N}_r$ for **all semisimple elements in a quasi-split reductive group** G. We are only left to show the following proposition.

Proposition 2.2.5. *Let $x, y \in G(E)$. Then x, y are stably θ-conjugate if and only if $\mathcal{N}_r(x) = \mathcal{N}_r(y)$.*

Proof. We adapt the proof of [Kot82] Proposition 5.7 here. We first assume that G_{der} is simply connected. From Corollary 2.1.2, we know that this is immediate.

For the general quasi-split reductive group G over F, we choose a z-extension $\alpha : G' \to G$ that is adapted to E. Choose $x', y' \in G'(E)$ such that $\alpha(x') = x$ and $\alpha(y') = y$. By Lemma 1.4, we know that x and y are stably θ-conjugate if and only if there exists $z' \in \ker(\alpha)(E)$ such that x' is stably θ-conjugate to $z'y'$, or equivalently, $\mathcal{N}_r(x') = (N_r(z'))\mathcal{N}_r(y')$. Since $N_r : \ker(\alpha)(E) \to \ker(\alpha)(F)$ is surjective by assumption, we see that x, y are stably θ-conjugate if and only if there exists $z \in$

$\ker(\alpha)(F)$ such that $\mathcal{N}_r(x') = z \cdot \mathcal{N}_r(y')$. Hence $\alpha(\mathcal{N}_r(x')) = \alpha(\mathcal{N}_r(y'))$ and by the construction of the abstract norm map above, we know that $\mathcal{N}_r(x) = \mathcal{N}_r(y)$, as desired. □

Conclusion 2.2.6. *Let G be a quasi-split reductive group defined over F and let E/F be an unramified cyclic extension of degree r. For elements $\delta \in G(E)$ such that $N_r\delta$ is semisimple, we can define an abstract norm map*

$$\mathcal{N} : \{\textit{stable } \theta\textit{-conjugacy classes in } G(E)\} \to \{\textit{stable conjugacy classes in } G(F)\}.$$

□

Chapter 3: Iwahori subgroups

3.1 Basic Notions

Let F denote an nonarchimedean local field of characteristics $p \geq 0$. Let $\mathcal{O}_F$ denote the ring of integers in F, and let $\varpi \in \mathcal{O}_F$ be a uniformizer. Let $q = p^n$ denote the cardinality of the residue field of F. Fix an algebraic closure $\overline{F}$ for F and a separable closure $F^s \subset \overline{F}$, and let L denote the completion of the maximal unramified extension of F inside F^s. Let $\sigma \in \mathrm{Aut}(L/F)$ denote the Frobenius automorphism of L over F. Let $\mathcal{O}_L$ denote the ring of integers in L. The valuation $\mathrm{val}_F : F^\times \to \mathbb{Z}$ is normalized such that $\mathrm{val}_F(\varpi) = 1$. Define $|x|_F := q^{-\mathrm{val}_F(x)}$ for $x \in F^\times$.

Let G denote a connected reductive group that is defined and unramified over F, which means that G is quasi-split over F and splits over an unramified extension of F. Let A denote a maximal F-split torus in G and set $T := \mathrm{Cent}_G(A)$, a maximal torus in G defined over F and split over L. We will use the symbol G to denote the group $G(F)$ of F-points.

Let E/F be an unramified extension of degree r contained in L. We fix a generator $\theta \in \mathrm{Gal}(E/F)$ and we use the same symbol θ to denote the induced automorphisms of groups of E-points $T(E), G(E)$, etc.

We consider the Bruhat-Tits building $\mathcal{B}(G(L))$ (resp., $\mathcal{B}(G)$) for $G(L)$ (resp., $G(F)$). The Bruhat-Tits buildings associated to the semisimple F-groups G_{ad} and G_{der} can be canonically identified and are denoted $\mathcal{B}_{ss}(G)$. The group $G(L) \rtimes \mathrm{Aut}(L/F)$ (resp., $G(F)$) acts on $\mathcal{B}(G(L))$ (resp., $\mathcal{B}(G)$). Via this action, we can identify $\mathcal{B}(G)$ with the σ-fixed subset $\mathcal{B}(G(L))^\sigma \subset \mathcal{B}(G(L))$.

Let $\mathcal{A}^L$ (resp., $\mathcal{A}$) denote the apartment of $\mathcal{B}(G(L))$ (resp., $\mathcal{B}(G)$) corresponding to the torus T (resp., A). Then $\mathcal{A}^L$ (resp., $\mathcal{A}$) is endowed with a family of hyperplanes given by the vanishing of the affine roots $\Phi_{\mathrm{aff}}(G, T, L)$ (resp., $\Phi_{\mathrm{aff}}(G, A, F)$). Under the identification $\mathcal{B}(G) = \mathcal{B}(G(L))^\sigma$, the apartment $\mathcal{A}$ is identified with $(\mathcal{A}^L)^\theta$. Moreover, the affine roots $\Phi_{\text{aff}}(G, A, F)$ are nonconstant restrictions to $\mathcal{A} = (\mathcal{A}^L)^\sigma$ of the affine roots $\Phi_{\text{aff}}(G, T, L)$. These affine roots determine the notions of alcoves, facets and Weyl chambers used throughout this article.

3.2 Parahoric Subgroups

We fix once for all a σ-invariant alcove $\mathbf{a} \in \mathcal{A}^L$, moreover, within $\bar{\mathbf{a}}$ we fix a σ-invariant facet $\mathbf{a}_J$ and a σ-invariant hyperspecial vertex $\mathbf{a}_0$. Kottwitz defines a functorial surjective homomorphism

$$\kappa_G : G(L) \longrightarrow X^*(\hat{Z}(G)^I)$$

where $I = \mathrm{Gal}(L^s/L)$ denotes the inertia group. We denote $G(L)_1 := \ker \kappa_G$ by convention.

The group $G(L)$ acts on the building $\mathcal{B}_{\text{ss}}(G(L))$ and moreover, let $V_G = X_*(Z(G))_I \otimes \mathbb{R}$, we have the following decompositions:

$$\mathcal{B}(G(L)) = \mathcal{B}_{\text{ss}}(G(L)) \times V_G$$

$$\mathcal{A}^L = \mathcal{A}^L_{\text{ss}} \times V_G$$

$$\mathbf{a}_J = \mathbf{a}^{\text{ss}}_J \times V_G,$$

where $\mathcal{A}^L_{\text{ss}}$ denotes the apartment of $\mathcal{B}_{\text{ss}}(G(L))$ corresponding to the maximal L-split torus $T_{\text{der}} \subset G_{\text{der}}$, and $\mathbf{a}^{\text{ss}}_J$ is the facet in $\mathcal{B}_{\text{ss}}(G(L))$ determined by $\mathbf{a}_J$.

Definition 3.2.1. *A parahoric subgroup of $G(L)$ associated to an arbitrary facet F of $\mathcal{B}_{ss}(G(L))$ is a subgroup of the form*

$$K_F = \mathrm{Fix(F)} \cap \mathrm{G(L)}_1,$$

where $\mathrm{Fix(F)} \leq \mathrm{G(L)}$ *is the subgroup of elements which fix* F *pointwise. An Iwahori subgroup of* $G(L)$ *is the parahoric subgroup associated to an alcove of* $\mathcal{B}_{ss}(G(L))$. *A parahoric subgroup of* G *will be a subgroup of* G *of the form* $K_F \cap G$.

Therefore, the facets $\mathbf{a}_0, \mathbf{a}$ and $\mathbf{a}_J$ give rise to parahoric subgroups of $G(L)$: a hyperspecial maximal parahoric subgroup $K(L)$, an Iwahori subgroup $I(L)$ and a general parahoric subgroup $J(L)$. Moreover, we have that $K(L) \supset I(L) \subset J(L)$. Moreover, since the facets are σ-invariant, we get the corresponding parahoric subgroups K, I and J of $G(F)$ with similar relations.

Remark 3.2.2. *This definition is taken from [HR08], which is different from the classical definition of Bruhat-Tits. However, they are equivalent by Prop. 3 in loc. cit.*

Remark 3.2.3. *When* $G = T$ *is a torus, then there is exactly one parahoric subgroup* K *of* $T(L)$, *namely,*

$$K = \mathcal{T}^0(\mathcal{O}_L),$$

where $\mathcal{T}^0$ *is the identity component of the Neron model of* T. *Moreover, we have* $T(L)_1 = \mathcal{T}^0(\mathcal{O}_L)$.

Chapter 4: Bernstein center for depth zero principal series

We assume G is an unramified connected reductive group over F. All the representations in this article will be on complex vector spaces. We denote $\mathfrak{R}(G)$ to be the category of smooth representations of $G(F)$ on $\mathbb{C}$-vector spaces. The **Bernstein center** $\mathfrak{Z}(G)$ of the group G is defined as the ring of endomorphisms of the identity functor on $\mathfrak{R}(G)$. If G contains an F-rational parabolic subgroup P with F-Levi factor M and unipotent radical N (such that $P = MN$), we define the modulus function $\delta_P : M(F) \to \mathbb{R}_{>0}$ by

$$\delta_P(m) := |\det(\mathrm{Ad}(m); \mathrm{Lie}(N(F)))|_F,$$

where $|\cdot|_F$ is the normalized absolute value on F. For $\sigma \in \mathfrak{R}(M)$, we will often consider the normalized induced representation

$$i_P^G(\sigma) = \mathrm{Ind}_{P(F)}^{G(F)}(\delta_P^{1/2}\sigma),$$

where $\delta_P^{1/2}(m)$ means the positive square-root of the positive real number $\delta_P(m)$. The normalization will preserve unitary representations.

4.1 General case

In this subsection, G can be any connected reductive group over F. We recall some basic facts of the Kottwitz homomorphism ([Kot97], [KP23] 11.5.) Recall that L is the completion of the maximal unramified extension F^{un} of F, and let $L^s \supset F^s$ denote an separable closure of L. Let $I = \mathrm{Gal}(L^s/L) \cong \mathrm{Gal}(F^s/F^{\mathrm{un}})$ denote the inertia group. Let $\Phi \in \mathrm{Aut}(L/F)$ be the inverse of the Frobenius automorphism σ. In [Kot97], a functorial surjective homomorphism is defined for any connected reductive F-group G

$$\kappa_G : G(L) \twoheadrightarrow X^*(Z(\hat{G}))_I \cong \pi_1(G)_I,$$

and it remains surjective upon taking Φ-fixed points:

$$\kappa_G : G(F) \twoheadrightarrow X^*(Z(\hat{G}))_I^{\Phi} \cong \pi_1(G)_I^{\Phi}.$$

We define

$$G(L)_1 = \ker(\kappa_G);$$
$$G(F)_1 = \ker(\kappa_G) \cap G(F).$$

We also define $G(F)^1 \supseteq G(F)_1$ to be the kernel of the composition map $G(F) \twoheadrightarrow \pi_1(G)_I^{\Phi} \to \pi_1(G)_I^{\Phi}/\text{torsion}$.

We define $X(G) := \mathrm{Hom}_{\mathrm{gp}}(G(F)/G(F)^1, \mathbb{C}^{\times})$, the group of *unramified characters* on $G(F)$, and also $X^{\mathrm{w}}(G) := \mathrm{Hom}_{\mathrm{gp}}(G(F)/G(F)_1, \mathbb{C}^{\times})$ and call it the group of *weakly unramified characters* on $G(F)$.

An irreducible smooth representation π of G is called *supercuspidal* if for all proper parabolic subgroups P with Levi subgroup M and all irreducible smooth representations σ of M, the representation π is not a subrepresentation of $\mathrm{Ind}_P^G\sigma$. Supercuspidal representations are the "building blocks" of (smooth) representations of p-adic groups.

A *cuspidal pair* (M, σ) consists of an F-Levi subgroup $M \subseteq G$ and a supercuspidal representation σ of M. We denote $(M, \sigma)_G$ to be the G-conjugacy class of the cuspidal pair (M, σ). We say that

(M, σ) and (L, τ) are ***(inertially) equivalent*** (write $(M, \sigma) \sim (L, \tau)$) if there exists a $g \in G$ such that $gMg^{-1} = L$ and ${}^g\sigma = \tau \otimes \chi$ for some $\chi \in X(L)$. Let $[M, \sigma]_G$ denote the equivalence class of $(M, \sigma)_G$.

If π is an irreducible representation of G, then the *supercuspidal support* of π is the unique element $(M, \sigma)_G$ such that π is a subquotient of the induced representation $i_P^G(\sigma)$, where P is any F-parabolic subgroup having M as a Levi subgroup. Let $\mathfrak{X}_G$ denote the set of all supercuspidal supports $(M, \sigma)_G$, and denote by the symbol $\mathfrak{s} = [M, \sigma]_G$ a typical inertial class.

For an fixed inertial class $\mathfrak{s} = [M, \sigma]_G$, we define the set $\mathfrak{X}_{\mathfrak{s}} = \{(L, \tau)_G | \, (L, \tau) \sim (M, \sigma)\}$. We have

$$\mathfrak{X}_G = \coprod_{\mathfrak{s}} \mathfrak{X}_{\mathfrak{s}},$$

where $\mathfrak{s}$ ranges over all inertial classes of cuspidal pairs of G. We will see below that $\mathfrak{X}_G$ has a natural structure of an algebraic variety and the *Bernstein components* $\mathfrak{X}_{\mathfrak{s}}$ are the connected components.

We first consider the weakly unramified character group $X^{\mathrm{w}}(M)$. From Kottwitz homomorphism, we have an isomorphism

$$M(F)/M(F)_1 \cong X^*(Z(\hat{M})^I)^{\Phi} = X^*((Z(\hat{M})^I)_{\Phi}).$$

Hence

$$X^{\mathrm{w}}(M)(\mathbb{C}) = \mathrm{Hom}_{\mathrm{gp}}(M(F)/M(F)_1, \mathbb{C}^{\times}) = \mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[X^*((Z(\hat{M})^I)_{\Phi})], \mathbb{C})$$

Therefore we can view $X^{\mathrm{w}}(M)$ as $(Z(\hat{M})^I)_{\Phi}$, which is a complex torus.

Now we can apply the same argument to $X(M)$. We have by definition

$$X(M)(\mathbb{C}) = \mathrm{Hom}_{\mathrm{gp}}(M(F)/M(F)^1, \mathbb{C}^\times).$$

Since $M(F)/M(F)^1$ is the quotient of $M(F)/M(F)_1$ by its torsion ([KP23], Corollary 11.7.6), we have

$$X(M) = ((Z(\hat{M})^I)_\Phi)^\circ,$$

the neutral component of $X^{\mathrm{w}}(M)$.

Now, we consider the variety structure on $\mathfrak{X}_\mathfrak{s}$. We fix a cuspidal pair (M, σ) representing $\mathfrak{s}_M = [M, \sigma]_M$ and $\mathfrak{s} = [M, \sigma]_G$, and denote the corresponding Bernstein components by $\mathfrak{X}_{\mathfrak{s}_M}$ and $\mathfrak{X}_\mathfrak{s}$. As sets, we have, by the definition above, $\mathfrak{X}_{\mathfrak{s}_M} = \{(M, \sigma\chi)_M\}$ and $\mathfrak{X}_\mathfrak{s} = \{(M, \sigma\chi)_G\}$, where the index is over $\chi \in X(M)$. We first consider the variety structure of $\mathfrak{X}_{\mathfrak{s}_M}$.

The complex torus $X(M)$ acts on $\mathfrak{X}_{\mathfrak{s}_M}$ by $\chi \cdot (M, \sigma)_M = (M, \sigma\chi)_M$. This is a transitive action with the stabilizer (of $(M, \sigma)_M$) $\mathrm{stab}_\sigma := \{\chi \,|\, \sigma \cong \sigma\chi\}$. Notice that here by "$\cong$" we mean that there exists an $m \in M$ such that ${}^m\sigma = \sigma\chi$. Therefore, it is easy to see that stab_σ belongs to the kernel of the map $X(M) \to X(Z(M)^\circ)$, $\chi \mapsto \chi|_{Z(M)^\circ}$. Hence stab_σ is a finite subgroup of $X(M)$, which makes $\mathfrak{X}_{\mathfrak{s}_M}$ a torsor under the complex torus $X(M)/\mathrm{stab}_\sigma$, and thus has the structure of an affine variety over $\mathbb{C}$.

Now we consider the variety structure of $\mathfrak{X}_\mathfrak{s}$. There is a well-defined surjective map

$$\mathfrak{X}_{\mathfrak{s}_M} \to \mathfrak{X}_\mathfrak{s}$$

$$(M, \sigma\chi)_M \mapsto (M, \sigma\chi)_G.$$

Define $N_G([M,\sigma]_M) := \{n \in N_G(M) | {}^n\sigma \cong \sigma\chi \text{ for some } \chi \in X(M)\}$. Then the fibers of $\mathfrak{X}_{\mathfrak{s}_M} \to \mathfrak{X}_{\mathfrak{s}}$ are the orbits of the finite group $W^G_{[M,\sigma]_M} := N_G([M,\sigma]_M)/M$ on $\mathfrak{X}_{\mathfrak{s}_M}$. We have

$$\mathfrak{X}_{\mathfrak{s}} = W^G_{[M,\sigma]_M} \backslash \mathfrak{X}_{\mathfrak{s}_M}, \tag{4.1.1}$$

hence the set $\mathfrak{X}_{\mathfrak{s}}$ acquires the structure of an irreducible affine variety over $\mathbb{C}$. This structure does not depend on the choice of the cuspidal pair (M,σ) up to isomorphism.

An element $z \in \mathfrak{Z}(G)$ determines a regular function on $\mathfrak{X}_G$ in the following way: for a point $(M,\sigma)_G \in \mathfrak{X}_{\mathfrak{s}}$ (recall that $\mathfrak{X}_G = \coprod_{\mathfrak{s}} \mathfrak{X}_{\mathfrak{s}}$), z acts on $i_P^G(\sigma)$ by a scalar $z(\sigma)$. The function $(M,\sigma)_G \mapsto z(\sigma)$ is a regular function on $\mathfrak{X}_G$. By [BD+84], Theorem 2.13, we have an isomorphism

$$\mathfrak{Z}(G) \cong \mathbb{C}[\mathfrak{X}_G].$$

4.2 Depth zero case

Now we assume G is a connected reductive group which is unramified over F. Hence T is a maximal unramified F-torus, which means that T splits over an unramified extension of F. In this case $T(F)_1 = T(F)^1$ ([KP23], Lemma 2.5.18) is the unique maximal compact open subgroup of $T(F)$, and therefore we have $X(T) = X^{\mathrm{w}}(T)$.

Let $T(F)_1$ denote the maximal compact open subgroup of $T(F)$, and let $T(F)_1^+ := T(F)_1 \cap I^+$ denote its pro-unipotent radical. Let χ denote a *depth-zero* character on $T(F)_1$, which means that χ factors through the quotient $T(F)_1/T(F)_1^+$. Let $\tilde{\chi}$ denote any extension of χ to a character $\tilde{\chi}$:

$T(F) \to \mathbb{C}^\times$. Consider the inertial class

$$\mathfrak{s} = \mathfrak{s}_\chi = [T(F), \tilde{\chi}]_G.$$

The inertial class $\mathfrak{s}$ depends only on the $W(F)$-orbit of χ.

Let N denote the norm homomorphism $T(E) \to T(F)$. It maps $T(E)_1 \to T(F)_1$ and $T(E)_1^+ \to T(F)_1^+$. Therefore we will use the same notation to denote the norm map on the quotients $N : T(E)_1/T(E)_1^+ \to T(F)_1/T(F)_1^+$.

Let $\chi_N := \chi \circ N$. This will give a depth-zero character on $T(E)_1$. The set of all smooth characters on $T(F)$ carries a left action under the Weyl group $W(F) = N_G T(F)/T(F)$. Let W_χ denote the subgroup of $W(F)$ which fixes χ. Similarly we define W_{χ_N} in the Weyl group $W(E)$ of $G(E)$.

Recall that $\mathfrak{R}(G)$ is the category of smooth representations of $G(F)$. Let $\mathfrak{R}_\mathfrak{s}(G)$ denote the Bernstein component indexed by $\mathfrak{s}$, in other words, this is the full subcategory of $\mathfrak{R}(G)$ whose objects have the property that each of their subquotients is a subquotient of a principal series representation $i_B^G(\tilde{\chi}\eta)$ for some unramified character η of $T(F)$. Sometimes we denote $\mathfrak{R}_\mathfrak{s}(G)$ by $\mathfrak{R}_\chi(G)$.

Now fix χ and $\mathfrak{s} = \mathfrak{s}_\chi$ as above. We have

$$\mathfrak{X}_\mathfrak{s} = \{(T, \xi)_G\}$$

of supercuspidal supports $(T, \xi)_G$ of irreducible representations π in the category $\mathfrak{R}_\chi(G)$. Here $\xi : T(F) \to \mathbb{C}^\times$ is a smooth character extending some $W(F)$-conjugate of χ.

Remark 4.2.1. *There exists at least one W_χ-invariant extension $\tilde{\chi}$ of χ, by using the canonical* *isomorphism* $X_*(T) \xrightarrow{\sim} T(F)/T(F)_1$ *given by* $\nu \mapsto \varpi^\nu$*, where* ϖ *is a choice of uniformizer of* F*. Hence the extension* $\tilde{\chi}^\varpi$ *is defined by the formula*

$$\tilde{\chi}^\varpi(\varpi^\nu t_0) = \chi(t_0), \forall \nu \in X_*(T), t_0 \in T(F)_1.$$

Fix one such extension $\tilde{\chi}$, we have a bijection

$$\hat{A}/W_\chi \xrightarrow{\sim} \mathfrak{X}_\mathfrak{s}$$
$$\eta \mapsto (T, \tilde{\chi}\eta)_G$$

where $\eta \in \hat{A}$ can be viewed as an unramified character on $T(F)$ as in [Hai09], Lemma 2.4.2. This provides an easier way to endow $\mathfrak{X}_\mathfrak{s}$ with the structure of an affine algebraic variety than 4.1.1. Up to isomorphism, this structure does not depend on the choice of χ in its $W(F)$-orbit, nor on the choice of the extension $\tilde{\chi}$ of χ. We have

$$\mathfrak{X}_\mathfrak{s} = \mathrm{Spec}(\mathbb{C}[X_*(A)]^{W_\chi})$$

We have a isomorphism $I/I^+ \xrightarrow{\sim} T(F)_1/T(F)_1^+$, hence the depth zero character $\chi : T(F)_1/T(F)_1^+ \to \mathbb{C}^\times$ induces

$$\rho = \rho_\chi : I/I^+ \to \mathbb{C}^\times.$$

We have the following proposition. By definition, (I, ρ) is a *Bushnell-Kutzko type* for $\mathfrak{R}_\chi(G)$ means that an irreducible representation $\pi \in \mathfrak{R}(G)$ belongs to $\mathfrak{R}_\chi(G)$ if and only if $\rho \subset \pi|_I$.

Proposition 4.2.2. *The pair* (I, ρ) *is a Bushnell-Kutzko type for* $\mathfrak{R}_\chi(G)$*.*

Proof. The proof of Theorem 3.0.2 in [Hai] works for depth-zero case and any characteristic. $\square$

Given Proposition 4.2.2, the category $\mathfrak{R}_\chi(G)$ is equivalent to the category of $\mathcal{H}(G,\rho)$-modules. ([Hai], Prop. 2.0.3). Let $\mathcal{Z}(G,\rho) := Z(\mathcal{H}(G,\rho))$, this also means that there is a canonical algebra isomorphism

$$\beta : \mathbb{C}[\mathfrak{X}_\mathfrak{s}] \xrightarrow{\sim} \mathcal{Z}(G,\rho), \tag{4.2.1}$$

which can be characterized as follows. For an extension ξ of some W-conjugate of χ, we consider the space $i_B^G(\xi)^\rho$ of functions $f \in i_B^G(\xi)$ such that

$$f(gi) = f(g)\rho(i)$$

for all $g \in G$ and $i \in I$. Then

$$z \in \mathcal{Z}(G,\rho) \text{ acts on the left on } i_B^G(\xi)^\rho \text{ by the scalar } [\beta^{-1}(z)](\xi),$$

where $\beta^{-1}(z)$ is viewed as a regular function on the variety $\mathfrak{X}_\mathfrak{s}$ and by ξ we mean the point $(T,\xi)_G \in \mathfrak{X}_\mathfrak{s}$, and the action is convolution of functions on G with suitable Haar measure.

Remark 4.2.3. *The fact that $\beta^{-1}(z)$ is well-defined as a function on the class $(T,\xi)_G$ means that $[\beta^{-1}(z)]({}^w\xi) = [\beta^{-1}(z)](\xi)$ for all $w \in W$. In other words, $\beta^{-1}(z)$ is W-invariant as a function of ξ.*

4.3 Right Actions on Principal Series

For the convenience of the later use, we need to rephrase above in terms of the **right** actions of $\mathcal{Z}(G,\rho)$ on principal series representations. Let $\iota : \mathcal{H}(G) \to \mathcal{H}(G)$ be the involution on the Hecke algebra of locally constant compactly-supported functions on G, defined by

$$(\iota h)(g) = h(g^{-1}), h \in \mathcal{H}(G), g \in G.$$

It is clear that $\mathcal{H}(G, \rho^{-1})$ acts on the left on the space $i_B^G(\xi^{-1})^{\rho^{-1}}$, then for $\Psi \in i_B^G(\xi^{-1})^{\rho^{-1}}$ and $h \in \mathcal{H}(G, \rho)$, we can convert the left action of $\mathcal{H}(G, \rho^{-1})$ into right action of $\mathcal{H}(G, \rho)$ on the space $i_B^G(\xi^{-1})^{\rho^{-1}}$, by the formula

$$\iota h \cdot \Psi = \Psi \cdot h \tag{4.3.1}$$

We have a perfect pairing

$$(\cdot, \cdot) : i_B^G(\xi)^{\rho} \times i_B^G(\xi^{-1})^{\rho^{-1}}$$

defined by

$$(\phi_1, \phi_2) := \int_{K_{\text{max}}} \phi_1(k)\overline{\phi_2(k)}\, dk,$$

where $\phi_1 \in i_B^G(\xi)^{\rho}$, $\phi_2 \in i_B^G(\xi^{-1})^{\rho^{-1}}$ and $K_{\text{max}} \leq G$ is a fixed maximal compact open subgroup. (See [GH24], Proposition 8.2.3) Then for $z \in \mathcal{Z}(G, \rho)$, we have the relation

$$(z \cdot \phi_1, \phi_2) = (\phi_1, (\iota z) \cdot \phi_2).$$

Thus

$$\beta^{-1}(z)(\xi) = \beta^{-1}(\iota z)(\xi^{-1}).$$

Combine this with $h = z$ in 4.3.1, we have

Lemma 4.3.1. *Let the Haar measure on $G(F)$ be normalized so that I has measure 1. Then $z \in$*

$\mathcal{Z}(G, \rho)$ *acts via right convolutions on* $i_B^G(\xi^{-1})^{\rho^{-1}}$ *by the scalar*

$$ch_{\xi^{-1}}(z) := \beta^{-1}(z)(\xi).$$

Chapter 5: Base change homomorphism

5.1 Construction of base change homomorphism

We fix a depth-zero character χ on $T(F)_1$ as in the last section, and (I, ρ) be the $\mathfrak{s} = \mathfrak{s}_\chi$-type described above. Let A^E denote the unique maximal E-split torus in G containing A. It is easy to see that $T = Z_G(A^E)$. Consider $\chi_N := \chi \circ N$ as a depth-zero character on $T(E)_1$ and consider the corresponding inertial class $\mathfrak{s}_E := \mathfrak{s}_{\chi_N}$ for $G(E)$. Let (I_E, ρ_E) denote the $\mathfrak{s}_E$-type associated to the character χ_N. We denote the corresponding Hecke algebra and its center by $\mathcal{H}(G(E), \rho_N)$ and $\mathcal{Z}(G(E), \rho_N)$, respectively.

There is a canonical morphism of algebraic varieties

$$N^* : \mathfrak{X}_\mathfrak{s} \to \mathfrak{X}_{\mathfrak{s}_E}$$

$$(T, \xi)_G \mapsto (T(E), \xi_N)_{G(E)},$$

where ξ denotes an extension of some $W(F)$-conjugate of χ as in the last section. This induces an algebra homomorphism

$$N : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_E}] \to \mathbb{C}[\mathfrak{X}_\mathfrak{s}] \tag{5.1.1}$$

Now we can define the *base change homomorphism* $b_E : \mathcal{Z}(G(E), \rho_N) \to \mathcal{Z}(G, \rho)$ to be the unique morphism making the following diagram commute:

$$\begin{tikzcd} \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_E}] \arrow[r, "\beta", "\sim"'] \arrow[d, "N"] & \mathcal{Z}(G(E), \rho_N) \arrow[d, "b"] \\ \mathbb{C}[\mathfrak{X}_\mathfrak{s}] \arrow[r, "\beta", "\sim"'] & \mathcal{Z}(G, \rho) \end{tikzcd} \tag{5.1.2}$$

5.2 Characterization by Actions on Principal Series

We can interpret the above commutative diagram in terms of the actions on principal series in the last section, to get the following lemma:

Lemma 5.2.1. *For any character $\xi : T(F) \to \mathbb{C}^\times$ which extends some $W(F)$-conjugate of χ, define $\xi_N := \xi \circ N$, a character on $T(E)$ which extends some $W(E)$-conjugate of χ_N. Let $z \in \mathcal{Z}(G(E), \rho_E)$. Then $b(z)$ is the unique element in $\mathcal{Z}(G, \rho)$ which acts on every module $i_B^G(\xi)^\rho$ by the same scalar by which z acts on $i_{B(E)}^{G(E)}(\xi_N)^{\rho_N}$. In other words,*

$$[\beta^{-1}(b(z))](\xi) = [\beta^{-1}(z)](\xi_N)$$

We can rephrase the above lemma in terms of right actions: for $z \in \mathcal{Z}(G(E), \rho_E)$, $b(z)$ acts on the right on $i_B^G(\xi^{-1})^{\rho^{-1}}$ by the scalar by which z acts on the right on $i_{B(E)}^{G(E)}(\xi_N^{-1})^{\rho_N^{-1}}$, in other words,

$$ch_{\xi^{-1}}(b(z)) = ch_{\xi_N^{-1}}(z). \tag{5.2.1}$$

5.3 Compatibility with Weyl group conjugation

We fix $w \in W(F)$ and use the same symbol to denote its lift in $N_G(T)$. The character ${}^w\chi$ is defined by ${}^w\chi(t) = \chi(w^{-1}tw)$. Similarly, for any suitable function Φ, we define ${}^w\Phi(\cdot) = \Phi(w^{-1} \cdot w)$. We write ${}^wI := wIw^{-1}$ and ${}^wI^+ := wI^+w^{-1}$. We extend ${}^w\chi$ to the character ${}^w\rho : {}^wI/{}^wI^+ \to \mathbb{C}^\times$ using Iwasawa decomposition of ${}^wI$, and therefore we have ${}^w\rho(wiw^{-1}) = \rho(i)$ for $i \in I$. There is also an isomorphism of algebras $\mathcal{H}(G, I, \rho) \xrightarrow{\sim} \mathcal{H}(G, {}^wI, {}^w\rho)$ given by $h \mapsto {}^wh$.

As usual, we let ξ denote an extension of a $W(F)$-conjugate of χ. We write ${}^wB := wBw^{-1}$, then we have an isomorphism

$$i_B^G(\xi^{-1})^{\rho^{-1}} \xrightarrow{\sim} i_{{}^wB}^G({}^w\xi^{-1})^{{}^w\rho^{-1}}$$

given by $\Psi \mapsto {}^w\Psi$. This isomorphism also intertwines the right actions of $h \in \mathcal{H}(G, I, \rho)$ and ${}^wh \in \mathcal{H}(G, {}^wI, {}^w\rho)$, in the sense that

$${}^w(\Psi \cdot h) = {}^w\Psi \cdot {}^wh.$$

Taking $h = z \in \mathcal{Z}(G, I, \rho)$, we have $\beta^{-1}(z)(\xi) = \beta^{-1}({}^wz)({}^w\xi) = \beta^{-1}({}^wz)(\xi)$. The last equality comes from the Remark 4.2.3. In other words, we have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{X}_\chi] \arrow[r, "\beta", "\sim"'] \arrow[d, "="] & \mathcal{Z}(G, I, \rho) \arrow[d, "z \mapsto {}^wz"] \\
\mathbb{C}[\mathfrak{X}_\chi] \arrow[r, "\beta", "\sim"'] & \mathcal{Z}(G, {}^wI, {}^w\rho)
\end{tikzcd}$$

Combining this diagram with 5.1.2 for G and $G(E)$ respectively, we have the following lemma, which shows the compatibility of $z \mapsto {}^wz$ with the base change homomorphism:

Lemma 5.3.1. *For any $w \in W(F)$, the following diagram is commutative:*

$$\begin{tikzcd}
\mathcal{Z}(G(E), \rho_N) \arrow[r, "z \mapsto {}^wz"] \arrow[d, "b"] & \mathcal{Z}(G(E), {}^w\rho_E) \arrow[d, "b"] \\
\mathcal{Z}(G, \rho) \arrow[r, "z \mapsto {}^wz"] & \mathcal{Z}(G, {}^w\rho)
\end{tikzcd}$$

Chapter 6: Constant term homomorphism

In this section, we define abstractly and concretely the constant term homomorphisms, that will allow us to descent the (twisted) orbital integrals of G to the ones on proper Levi subgroups. We will also show that the constant term homomorphisms are compatible with base change.

6.1 Abstract definition

Let M be an F-Levi subgroup of G containing the torus T. Let $\mathfrak{s}_\chi^M$ denote the inertial class associated to χ but for group M, rather than the group G. In other words, we have $\mathfrak{s}_\chi^M = [T, \xi]_M$. Let $\mathfrak{X}_\chi^M$ denote the corresponding variety of supercuspidal supports $(T, \xi)_M$. We have a canonical surjective morphism of varieties as in 4.1:

$$\mathfrak{X}_\chi^M \to \mathfrak{X}_\chi := \mathfrak{X}_\chi^G$$

$$(T, \xi)_M \mapsto (T, \xi)_G.$$

This induces an algebra homomorphism

$$c_M^G : \mathbb{C}[\mathfrak{X}_\chi] \to \mathbb{C}[\mathfrak{X}_\chi^M]$$

We call this map the *constant term homomorphism*. We will define the map concretely over the next few subsections, and prove that it is compatible with the base change homomorphism.

6.2 Refined Iwasawa decomposition

We say a F-parabolic subgroup $P \subset G$ *standard* if $P \supset B$, and we say M is a *standard* F-Levi subgroup if it is a Levi factor of a standard F-parabolic subgroup P. Fix such M and P, and the Levi decomposition $P = MN$.

Consider the *Kostant representatives* $W^P \subset W(F)$, which is by definition the set of minimal coset representatives for the elements in $W_M(F)\backslash W(F)$, with respect to the Bruhat order on $W(F)$ defined using the Borel subgroup B. We recall that $W_M(F) = N_M(T)(F)/T(F)$. We choose once for all a lift $\dot{w} \in N_G(T)(F) \cap K$ for each $w \in W^P$, but for convenience we will still denote $\dot{w}$ simply by w.

Abbreviate the Weyl group $W(E)$ by W_E and its set of Kostant representatives by W_E^P. The *refined Iwasawa decomposition* states that

$$G(E) = \coprod_{w \in W_E^P} P(E)\, w\, I(E) \tag{6.2.1}$$

We have the following Levi factorization for $P \cap {}^wI$ for $w \in W$:

Lemma 6.2.1. *For each $w \in W$, we have the factorization $P \cap {}^wI = (M \cap {}^wI)(N \cap {}^wI)$.*

Proof. See [Hai09], Lemma 2.9.1 (b) and [Hai12], 11.2. □

6.3 The concrete definition of the constant term

For a locally constant compactly-supported function f on $G(F)$, we define the locally constant compactly-supported function $f^{(P)}$ on $M(F)$ by the formula

$$f^{(P)}(m) = \delta_P^{1/2}(m) \int_{N(F)} f(mn)\, dn = \delta_P^{-1/2}(m) \int_{N(F)} f(nm)\, dn, \tag{6.3.1}$$

where the Haar measure dn is normalized so that $dn(N(F) \cap J) = 1$, where J is an Iwahori subgroup which is $W(F)$-conjugate to I. The definition of $f^{(P)}$ depends on the choice of J and we will allow J to vary.

The character χ gives rise to a smooth character $\rho : J \to \mathbb{C}^\times$ by the isomorphism $J/J^+ \cong T(F)_1/T(F)_1^+$, and (J, ρ) is also a type for $\mathfrak{R}_\chi(G)$ by Proposition 4.2.2. We write the corresponding Hecke algebra as $\mathcal{H}(G, J, \rho)$ and its center as $\mathcal{Z}(G, J, \rho)$.

Denote $J_M = J \cap M$, which is an Iwahori subgroup of M ([Hai09], Lemma 2.9.1). From the isomorphism $J_M/J_M^+ \cong T(F)_1/T(F)_1^+$ we also get the character $\rho : J_M \to \mathbb{C}^\times$ with the same notation, which is the restriction of $\rho : J \to \mathbb{C}^\times$ by construction. The pair (J_M, ρ) is a type for $\mathfrak{R}_\chi(M)$.

We can easily check that $f^{(P)} \in \mathcal{H}(M, J_M, \rho)$ if $f \in \mathcal{H}(G, J, \rho)$, using the fact that $\delta_P(j) = 1$ for all $j \in J_M$. The next proposition is the main goal of this section.

Proposition 6.3.1. *The map $f \mapsto f^{(P)}$ sends $\mathcal{Z}(G, J, \rho)$ to $\mathcal{Z}(M, J_M, \rho)$, and the following diagram commutes:*

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{X}_\chi] \arrow[r, "\beta", "\sim"'] \arrow[d, "c_M^G"] & \mathcal{Z}(G, J, \rho) \arrow[d, "f \mapsto f^{(P)}"] \\
\mathbb{C}[\mathfrak{X}_\chi^M] \arrow[r, "\beta", "\sim"'] & \mathcal{Z}(M, J_M, \rho)
\end{tikzcd}$$

In particular, the operation $f \mapsto f^{(P)}$ is an algebra homomorphism when restricted to $\mathcal{Z}(G, J, \rho)$, and *the map is independent of the choice of P with M as Levi factor.*

6.4 Compatibility with base change

We first notice that the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{C}[\mathfrak{X}_{\chi_N}] & \xrightarrow{c_{M(E)}^{G(E)}} & \mathbb{C}[\mathfrak{X}_{\chi_N}^{M(E)}] \\ \downarrow N & & \downarrow N \\ \mathbb{C}[\mathfrak{X}_{\chi}] & \xrightarrow{c_M^G} & \mathbb{C}[\mathfrak{X}_{\chi}^{M}] \end{array}$$

since they are the same map on the varieties:

$$\mathfrak{X}_{\chi}^{M} \to \mathfrak{X}_{\chi_N}$$

$$(T, \xi)_M \mapsto (T(E), \xi_N)_{G(E)}.$$

Combining this with Proposition 6.3.1, we have the following compatibility of the constant term with base change map:

Corollary 6.4.1. *The following diagram commutes:*

$$\begin{array}{ccc} \mathcal{Z}(G(E), J(E), \rho_N) & \xrightarrow{c_{M(E)}^{G(E)}} & \mathcal{Z}(M(E), J_{M(E)}, \rho_N) \\ \downarrow b & & \downarrow b \\ \mathcal{Z}(G, J, \rho) & \xrightarrow{c_M^G} & \mathcal{Z}(M, J_M, \rho) \end{array}$$

We still need to prove Proposition 6.3.1 in the next subsection.

6.5 Proof of Proposition 6.3.1

This section is a simple adaptation of [Hai12], §5.6. We fix an extension ξ of χ and set $\xi_1 := \delta_P^{1/2}\xi$. We consider the subspace $i_B^G(\xi_1^{-1})^{\rho^{-1}} \subset i_B^G(\xi_1^{-1})$, on which the subgroup J acts on the left by the character ρ^{-1}. Denote $B_M = B \cap M$, we have the following lemma:

Lemma 6.5.1. *Given $\psi \in i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$, there exists $\Psi \in i_B^G(\xi_1^{-1})^{\rho^{-1}}$ such that $\Psi|_M = \psi$.*

Proof. We define Ψ to be the following function, that is supported on PJ and extends ψ:

$$\Psi(mnj) := \psi(m)\rho^{-1}(j),$$

for all $mn \in MN = P$ and $j \in J$. It is clear that $\Psi|_M = \psi$ by definition, it remains to show that Ψ is well-defined and $\Psi \in i_B^G(\xi_1^{-1})$, since it is clear that J translates Ψ by ρ^{-1} by definition.

Indeed, if $mnj = m'n'j'$ for $m, m' \in M$, $n, n' \in N$ and $j, j' \in J$, using the Iwahori factorization of J with respect to $P = MN$: $J \cap P = (J \cap M) \cdot (J \cap N)$, one sees that $\psi(m)\rho^{-1}(j) = \psi(m')\rho^{-1}(j')$, therefore Ψ is well-defined. To prove that Ψ belongs to the desired space, we use the fact that $\delta_B^{1/2}\xi_1^{-1} = \delta_{B_M}^{1/2}\xi^{-1}$, which follows from the Lie algebra decomposition $\mathrm{Lie}(B) = \mathrm{Lie}(N) \oplus \mathrm{Lie}(B_M)$. $\square$

We define the right convolution action $*$ of $\mathcal{H}(M, J_M, \rho)$ on $i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$ using the measure dm such that $dm(J_M) = 1$.

Lemma 6.5.2. *Let $z \in \mathcal{Z}(G, J, \rho)$ and $\psi \in i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$. Then for any $y \in M$, we have the identity*

$$[ch_{\xi_1^{-1}}(z) \cdot \psi](y) = [\psi * (\delta_P^{1/2} z^{(P)})](y)$$

Proof. Let $\Psi \in i_B^G(\xi_1^{-1})^{\rho^{-1}}$ be the extension of ψ from the previous lemma. Then we have

$$\begin{aligned}
[\psi * (\delta_P^{1/2} z^{(P)})](y) &= \int_M \psi(m)\delta_P^{1/2}(m^{-1}y) z^{(P)}(m^{-1}y)\, dm \\
&= \int_M \int_N \Psi(mn^{-1}) z((mn^{-1})^{-1}y)\, dn\, dm \\
&= \int_M \int_N \int_J \Psi(mn^{-1}j) z((mn^{-1}j)^{-1}y)\, dj\, dn\, dm,
\end{aligned}$$

where we let dg (resp. dj) denote the Haar measure on G (resp. J) giving J measure 1. The second line follows from the definition of $z^{(P)}$ and Ψ, while the last line is from the fact that J translates those two functions by ρ^{-1}. Now since Ψ is supported on MNJ, we can use the substitution $g = mn^{-1}j$ to write the above as

$$\begin{aligned}
\int_G \Psi(g) z(g^{-1}y)\, dg &= (\Psi * z)(y) \\
&= ch_{\xi_1^{-1}}(z) \cdot \Psi(y).
\end{aligned}$$

Since $\Psi(y) = \psi(y)$ for all $y \in M$, the lemma follows. □

Now we can state the crucial proposition of this section, which will ultimately imply Proposition 6.3.1:

Proposition 6.5.3. *Let $z \in \mathcal{Z}(G, J, \rho)$. Then for every extension ξ of χ, the element $z^{(P)}$ acts on the right on $i_{B_M}^M(\xi^{-1})^{\rho^{-1}}$ by $ch_{\xi^{-1}}(z)$, the scalar by which z acts on the right on $i_B^G(\xi^{-1})^{\rho^{-1}}$.*

Proof. By definition, we have $\xi^{-1} = \delta_P^{1/2}\xi_1^{-1}$, the previous lemma shows that $\delta_P^{1/2} z^{(P)}$ acts on th right on $i_{B_M}^M(\delta_P^{1/2}\xi_1^{-1})^{\rho^{-1}}$ by $ch_{\xi_1^{-1}}(z)$, the same scalar by which z acts on $i_B^G(\xi_1^{-1})^{\rho^{-1}}$ on the right. Now for any $\psi \in i_{B_M}^M(\delta_P^{1/2}\xi_1^{-1})^{\rho^{-1}}$ and $m \in M$, it is straightforward to check the identity

$$(\psi * \delta_P^{1/2} z^{(P)})(m) = \delta_P^{1/2}(m) \cdot (\delta_P^{-1/2}\psi * z^{(P)})(m),$$

since then $\delta_P^{-1/2}\psi \in i_{B_M}^M(\xi_1^{-1})^{\rho^{-1}}$, we see that $z^{(P)}$ acts on the right on $i_{B_M}^M(\xi_1^{-1})^{\rho^{-1}}$ also by the scalar $\mathrm{ch}_{\xi_1^{-1}}(z)$. As ξ_1 ranges over all extensions of χ as ξ does, the proposition follows. $\square$

Chapter 7: Descent of Twisted Orbital Integrals

Recall that G is a connected reductive group over local field F, quasi-split over F and split over an unramified extension of F. $I \leq G$ is an Iwahori subgroup which is in good position relative to a maximal unramified torus T over F. We have a parabolic subgroup $P = MN$. On these groups G, I and P, we choose once for all Haar measures dg, di and dp such that $dg(I) = di(I) = dp(I \cap P) = 1$. For $w \in W(F)$, let dp_w (resp. dm_w, dn_w) denote the (left) Haar measure on P (resp. M, N) such that $P \cap {}^wI$ (resp. $M \cap {}^wI, N \cap {}^wI$) has volume 1. And we choose similar measures on the groups $G(E), P(E), I(E)$, etc.

7.1 Local cohomology result

For the moment, let F denote any field and let G be any connected reductive group over F. We recall that a F-Levi subgroup $M \subseteq G$ is a Levi factor of a K-rational parabolic subgroup P of G.

Lemma 7.1.1. *We have*

$$\ker[H^1(F, M) \to H^1(F, G)] = \{1\}$$

Proof. Indeed, since $H^1(F, M) = H^1(F, P)$ by Lemme 1.13 in [San81], it suffices to show that $H^1(F, P) \to H^1(F, G)$ is injective, which follows from the fact that $G(F) \to (G/P)(F)$ is surjective, see [CF20] Proposition 7.2.1. $\square$

7.2 Descent preliminaries

The standard Borel subgroup $B = TU$ determines a set of simple positive relative roots Δ_0. In the descent we are given a proper F-Levi subgroup M, which by definition is a Levi factor in an F-parabolic subgroup $P = MN$. Now replacing by a conjugate by elements of $G(F)$, we may assume that M and P are standard, i.e., $M \supseteq T$ and $P \supseteq B$.

Let $B_M := B \cap M$ and $U_M := U \cap M$, so that B_M is a Borel subgroup of M with Levi decomposition $B_M = U_M T$. This corresponds to a subset $\Delta_M \subset \Delta_0$. The relative Weyl group ${}_FW_M := N_M(T)(F)/T(F)$ is a Coxeter subgroup of $({}_FW, \{s_\alpha, \alpha \in \Delta_0\})$, the relative Weyl group of G, with generating set $\{s_\alpha, \alpha \in \Delta_M\}$. For each element $w \in_F W$, we choose a representative in K, denoted by the same symbol. We recall that $K \subset G$ is a hyperspecial maximal parahoric subgroup.

As before, E/F is an unramified extension of degree r, contained in the completion of the maximal unramified extension L of F. We fix a generator $\sigma \in \mathrm{Gal}(E/F)$. For $m \in M(F)$, recall that the *Harish-Chandra function* on $M(F)$ is defined by

$$D_{G(F)/M(F)}(m) = \det(1 - \mathrm{Ad}(m^{-1}); \mathrm{Lie}(G(F))/\mathrm{Lie}(M(F)))$$

Furthermore, we define

$$\Delta_P(m) := \det(1 - \mathrm{Ad}(m^{-1}); \mathrm{Lie}(N(F))),$$

Let $\bar{P}$ denote the parabolic subgroup opposite to P and $\bar{N}$ its unipotent radical. Using the decomposition $\mathrm{Lie}(G(F)) = \mathrm{Lie}(\bar{N}(F)) \oplus \mathrm{Lie}(M(F)) \oplus \mathrm{Lie}(N(F))$, we can show that

$$|D_{G(F)/M(F)}|_F = |\Delta_P|_F^2 \delta_P, \tag{7.2.1}$$

where δ_P is defined in section 3. We can also define $D_{G(E)/M(E)}$ for E by replacing F.

We have the refined Iwasawa decomposition 6.2.1 as

$$G = \coprod_{w \in W^P} P\, w\, I$$

where W^P denotes the set of minimal coset representatives for the elements of $W_M \backslash W$, with respect to the Bruhat order defined by B. We choose a lift for each w in the hyperspecial compact subgroup $K(E) \supset I(E)$, which we continue to denote by the symbol w.

For any integrable smooth function ϕ supported on PwI, we have the integration formula

$$\begin{aligned} \int_G \phi(g) dg &= \mathrm{vol}_{dp}^{-1}(P \cap ({}^w I)) \int_P \int_I \phi(pwi)\, di\, dp. \\ &= \int_P \int_I \phi(pwi)\, di\, dp_w \end{aligned}$$

In the same manner, using the refined Iwasawa decomposition, we have for any integrable smooth function ϕ on G, the integration formula

$$\begin{aligned} \int_G \phi(g) dg &= \sum_{w \in W^P} \mathrm{vol}_{dp}^{-1}(P \cap ({}^w I)) \int_P \int_I \phi(pwi)\, di\, dp. \\ &= \sum_{w \in W^P} \int_P \int_I \phi(pwi)\, di\, dp_w \end{aligned} \tag{7.2.2}$$

7.3 Descent of norm mapping

Let L denote the completion of the maximal unramified extension F^{un} of F inside a chosen separable closure F^s. We use the symbol σ to denote both the Frobenius automorphism in Aut(L/F) and its restriction to $\mathrm{Gal}(E/F)$.

We fix a semisimple element $\gamma \in G(F)$. Let S denote the F-split component of $Z(G_\gamma^\circ)$, which is an F-torus. Let $M := Z_G(S)$, which is a F-Levi subgroup of G. Now it is clear that γ is elliptic in $M(F)$ by the definition of an elliptic semisimple element:

Definition 7.3.1. *Denote A_X to be the maximal F-split component in $Z(X)$. We say that a semisimple* *element γ is elliptic in $G(F)$ if $A_{G^\circ_\gamma} = A_G$.*

In other words, we want the centralizer G_γ to be "as compact as possible", so that $Z(G_\gamma)$ is anisotropic modulo $Z(G)$.

Indeed, in this case, one computes that $A_M = A_{M^\circ_\gamma} = A_{G^\circ_\gamma} = S$, therefore γ is elliptic in M.

It is also clear that γ is elliptic in $G(F)$ if and only if $M = G$: the "if" part is obvious, now if γ is elliptic in $G(F)$, we have S is a subtorus of $Z(G)$, so $M = Z_G(S) \supseteq Z_G(Z(G)) \supseteq G$, as desired.

We will state a crucial lemma used in the reduction step. Suppose that γ is not elliptic in G, so by the discussion above we have $M \subsetneq G$. Since we intend to prove the fundamental lemma by induction on the semisimple rank, we may assume that it is already known for M. If γ is not a norm from $G(E)$, it is not a norm from $M(E)$, then we can deduce that $O^G_\gamma(b\phi) = 0$ by the descent formula in the following subsections. Now suppose that γ is indeed a norm from $G(E)$, the following lemma ensures that it is also a norm from $M(E)$. The desired matching of stable (twisted) orbital integrals is proved by descending to M using the descent formulas (6.4.2) and (6.4.3), along with Lemma 6.3.

Lemma 7.3.2. *Suppose that $\gamma \in G(F)$ is a semisimple element, and suppose that $M \subseteq G$ is an F-Levi subgroup with $G^\circ_\gamma \subseteq M$. Suppose there exists $\delta \in G(E)$ such that $\mathcal{N}\delta = \gamma$. Then there exists an element $\delta_1 \in M(E)$ such that $\mathcal{N}_M(\delta_1) = \gamma$ (we use $\mathcal{N}_M$ to denote the norm mapping restricted to M) and that δ and δ_1 are σ-conjugate under $G(E)$.*

Proof. This is a simply adaptation of [Hai09], Lemma 4.2.1. We also use σ to denote an extension of $\sigma \in \mathrm{Gal}(E/F)$ to $\mathrm{Gal}(F^s/F)$. For $g \in G(F^s)$, set $g^\sigma := \sigma(g)$ and $g^{-\sigma} := \sigma(g^{-1})$.

First we consider the case when the derived group of G is simply-connected. In this case, the connected centralizer $G^\circ_\gamma = G_\gamma$ and stable σ-conjugacy (resp., stable conjugacy) coincides with F^s-σ-conjugacy (resp., F^s-conjugacy). Choose a $h \in G(F^s)$ with $h(N\delta)h^{-1} = \gamma$ (N is the concrete norm). Applying σ to the equation $N\delta = h^{-1}\gamma h$ and using $(N\delta)^\sigma = \delta^{-1}(N\delta)\delta$, we have $h^{-\sigma}\gamma h^\sigma = \delta^{-1}(h^{-1}\gamma h)\delta$, so $h\delta h^{-\sigma} \in G_\gamma \subseteq M$.

For any $\tau \in \mathrm{Gal}(F^s/E)$, by the similar calculation, we can show that $hh^{-\tau} \in G_\gamma \subseteq M$. The cocycle $\tau \mapsto hh^{-\tau}$ is an element of $\ker(H^1(E, M) \to H^1(E, G))$, which is trivial by Lemma 7.1.1. We write $hh^{-\tau} = m^{-1}m^\tau$ for some $m \in M(F^s)$ and every $\tau \in \mathrm{Gal}(F^s/E)$. So we can see that $mh \in G(E)$.

Consider $\delta_1 := (mh)\delta(mh)^{-\sigma} = m(h\delta h^{-\sigma})m^{-\sigma} \in M(E)$. It is σ-conjugate to δ, and since $N\delta_1 = (mh)N\delta(mh)^{-1} = m\gamma m^{-1}$, we have $\mathcal{N}_M\delta_1 = \gamma$.

We now prove the general case using the special case above. We choose a z-extension $\alpha : G' \to G$ adapted to E with $Z = \ker(\alpha)$. By construction, G' is a connected reductive group with simply-connected derived group G'_{der} and we have a short exact sequence.

$$1 \to Z \to G' \to G \to 1.$$

Let $M' = \alpha^{-1}(M)$, an F-Levi subgroup of G'. We choose elements $\gamma' \in G'(F)$ and $\delta' \in G'(E)$ with $\alpha(\gamma') = \gamma$ and $\alpha(\delta') = \delta$. We have $\gamma' \in M'(F)$ and $G'^{\circ}_{\gamma'} = G'_{\gamma'} \subseteq M'$. Furthermore, $\mathcal{N}\delta = \gamma$ could be pulled back to $\mathcal{N}\delta' = \gamma' z$ for some $z \in Z(F)$. Replacing γ' by $\gamma' z$, we may assume that $\mathcal{N}\delta' = \gamma'$. By the previous results applied to G', we know that δ' is σ-conjugate under $G'(E)$ to an element $\delta'_1 \in M'(E)$ with $\mathcal{N}_{M'}(\delta'_1) = \gamma'$. It now follows that δ is σ-conjugate under $G(E)$ to $\delta_1 := \alpha(\delta'_1)$, and $\mathcal{N}_M(\delta_1) = \gamma$ from Lemma 2.1.10. $\square$

7.4 Descent of (twisted) orbital integrals

We recall the setup as the followings: $\gamma \in G(F)$ is a semisimple element, $S = A_{G_\gamma^\circ}$ denotes the F-split component of $Z(G_\gamma^\circ)$, and $M = Z_G(S)$. Suppose that γ is not elliptic in $G(F)$, so that M is a proper Levi subgroup of G (See the second paragraph of 7.2). Choose a standard F-parabolic subgroup $P = MN$ with M as Levi factor.

We have $\gamma \in M(F)$ and $G_\gamma^\circ = M_\gamma^\circ$ by definition. From Lemma 7.3.2, we can assume that $\gamma = \mathcal{N}\delta$ for an element $\delta \in M(E)$.

From section 1, we know that the twisted centralizer $G_{\delta\sigma}$ is an inner form of G_γ, whose group of F-points is

$$G_{\delta\sigma}(F) = \{g \in G(E) |\, g^{-1}\delta\sigma(g) = \delta\}.$$

We have the inclusion $M_{\delta\sigma}^\circ \subseteq G_{\delta\sigma}^\circ$ by definition, and since they are inner forms of $M_\gamma^\circ = G_\gamma^\circ$, we have in fact $M_{\delta\sigma}^\circ = G_{\delta\sigma}^\circ$ for dimension reasons.

Since $G_{\delta\sigma}$ and G_γ are inner forms, we can choose compatible measures $dg_{\delta\sigma}$ on $G_{\delta\sigma}(F)$ and dg_γ on $G_\gamma(F)$ to define the twisted orbital integrals of $\phi \in C_c^\infty(G(E))$

$$TO_{\delta\sigma}^G(\phi) := \int_{G_{\delta\sigma}^\circ \backslash G(E)} \phi(g^{-1}\delta\sigma(g))\, d\bar{g}.$$

Here and in the followings, we denote $G_{\delta\sigma}^\circ(F)$ simply by $G_{\delta\sigma}^\circ$. Since $M_{\delta\sigma}^\circ = G_{\delta\sigma}^\circ$, we can choose the same measure $dg_{\delta\sigma} = dm_{\delta\sigma}$ to define $TO_{\delta\sigma}^M(\psi)$ for $\psi \in C_c^\infty(M(E))$.

By the decomposition $P = MN$, we can write $g = mnwi$ for in each summand in the refined Iwasawa decomposition 6.2.1. For $\phi \in \mathcal{H}(G(E), \rho_E)$, where $\rho_E = \rho \circ N$ for some depth-zero character

ρ on I. By 7.2.2, we have the formula

$$\begin{aligned}
TO_{\delta\sigma}(\phi) &= \int_{G^\circ_{\delta\sigma}\backslash G(E)} \phi(g^{-1}\delta\sigma(g))\,d\bar{g} \\
&= \sum_{w\in {}_EW^P} \int_{M^\circ_{\delta\sigma}\backslash M} \int_N \int_I \phi(i^{-1}w^{-1}n^{-1}m^{-1}\delta\sigma(m)\sigma(n)\sigma(w)\sigma(i))\,di\,dn_w d\bar{m}_w \\
&= \sum_{w\in {}_EW^P} \int_{M^\circ_{\delta\sigma}\backslash M} \int_N \int_I \rho_E^{-1}(i)\phi(w^{-1}n^{-1}m^{-1}\delta\sigma(m)\sigma(n)\sigma(w))\rho_E^{-1}(\sigma(i))\,di\,dn_w d\bar{m}_w
\end{aligned}$$

We notice that $\rho_E \circ \sigma = \rho_E$, therefore we can suppress the integral over I. For each m, we write $m_0 := m^{-1}\delta\sigma(m) \in M(E)$, and we also define the function ${}^{w,\sigma}\phi$ by ${}^{w,\sigma}\phi(g) = \phi(w^{-1}g\sigma(w))$. Then we see that

$$TO_{\delta\sigma}(\phi) = \sum_{w\in {}_EW^P} \int_{M^\circ_{\delta\sigma}\backslash M} \int_N {}^{w,\sigma}\phi(m_0(m_0^{-1}n^{-1}m_0\sigma(n)))dn_w d\bar{m}_w \tag{7.4.1}$$

We hope to make the change of variable $n \mapsto m_0^{-1}n^{-1}m_0\sigma(n)$, which leads us to the calculation of the Jacobian of this transformation:

$$\begin{aligned}
|\det(\theta - \mathrm{Ad}(m_0^{-1}); \mathrm{Lie}\,N(E))|_F &= |\det(1 - \mathrm{Ad}(Nm_0^{-1}); \mathrm{Lie}\,N(F))|_F \\
&= |D_{G(F)/M(F)}(Nm_0)|_F^{1/2}\delta_{P(F)}(Nm_0)^{-1/2} \\
&= |D_{G(F)/M(F)}(N\delta)|_F^{1/2}\delta_{P(E)}(m_0)^{-1/2}.
\end{aligned}$$

Where the second line follows from 7.2.1. Thus by the change of variables formula for p-adic manifolds (see for example, [Igu00], Chapter 7), we can write 7.4.1 as

$$\begin{aligned}
TO_{\delta\theta}(\phi) = &|D_{G(F)/M(F)}(N\delta)|_F^{-1/2} \\
&\times \sum_{w\in {}_EW^P} \int_{M^\circ_{\delta\sigma}\backslash M} \delta_{P(E)}^{1/2}(m_0) \int_N {}^{w,\sigma}\phi(m_0 n)dn_w d\bar{m}_w.
\end{aligned}$$

By the definition of constant term homomorphisms, we have

$$TO_{\delta\sigma}^{G(E)}(\phi) = |D_{G(F)/M(F)}(N\delta)|_F^{-1/2} \sum_{w \in {}_E W^P} TO_{\delta\sigma}^{M(E)}(({}^{w,\sigma}\phi)^{(P(E))}). \tag{7.4.2}$$

Where the twisted orbital integral on $M(E)$ is formed using dm_w and $(\cdot)^{(P(E))}$ is formed using dn_w.

Consider the special case $E = F$ and use the equality $\gamma = N\delta = \delta$, we have a corresponding descent formula for functions in $\mathcal{H}(G, \rho)$. Assume that $\phi \in \mathcal{Z}(G(E), \rho_E)$, and consider the descent formula for the function $b\phi \in \mathcal{Z}(G, \rho)$:

$$O_\gamma^{G(F)}(b\phi) = |D_{G(F)/M(F)}(\gamma)|_F^{-1/2} \sum_{w \in W^P} O_\gamma^{M(F)}(({}^w b\phi)^{(P(F))}).$$

Using the compatibility of b with conjugation by w (5.3) and constant term (6.4), we have

$$O_\gamma^{G(F)}(b\phi) = |D_{G(F)/M(F)}(\gamma)|_F^{-1/2} \sum_{w \in W^P} O_\gamma^{M(F)} b[({}^w\phi)^{(P(E))}]. \tag{7.4.3}$$

Our goal is to compare 7.4.2 with 7.4.3.

7.5 Comparing descent formulas

We compare formulas 7.4.2 and 7.4.3 following the strategy of [Hai09], §4.5. We need the following lemma, and the twisted Kazhdan density in the positive characteristics. The latter one is proved in [CH16] §8.3.

Lemma 7.5.1. *We have $W^P = ({}_E W^P)^\sigma$. Assume $\phi \in \mathcal{Z}(G(E), \rho_E)$, then the summands in 7.4.3 indexed by elements $w \in_E W^P$ with $\sigma(w) \neq w$ are zero.*

Proof. The proof of Lemma 4.5.3 in [Hai09] carries over nearly word-for-word. □

Now if we compare 7.4.2 with 7.4.3, the fundamental lemma for (γ, δ, G) follows from the fundamental lemma for (γ, δ, M). Thus by induction on the semisimple rank of G, we are reduced to the case that γ is elliptic semisimple.

Chapter 8: Various Reductions

In this section, we follow closely the reduction steps of [Hai09], section 5 and [Hai12], section 7 with a few adaptations, and we try to be as detailed as we can.

8.1 Definition of stable twisted orbital integral

We still denote G to be an unramified connected reductive F-group. Let $\delta \in G(E)$ be such that $\mathcal{N}\delta \in G(F)$ (as a stable class) is semisimple. Let $e(\delta) := e(G^\circ_{\delta\sigma})$ denote the sign attached by Kottwitz [Kot83] to the connected reductive F-group $G^\circ_{\delta\sigma}$. (There is no assumption on the characteristics of the ground field F in loc. cit.) And define $a(\delta)$ to be the cardinality of the set

$$\ker[H^1(F, G^\circ_{\delta\sigma}) \to H^1(F, G_{\delta\sigma})]$$

Therefore $a(\delta) = 1$ for those G's with simply-connected derived group. Now for any function $\phi \in C_c^\infty(G(E))$, we can define its *stable* twisted orbital integral by

$$SO_{\delta\sigma}(\phi) := \sum_{\delta'} e(\delta')\, a(\delta') TO_{\delta'\sigma}(\phi) \tag{8.1.1}$$

where $TO_{\delta'\sigma}(\phi)$ is defined as in section 7.4. Here δ' ranges over σ-conjugacy classes in $G(E)$ which are stably σ-conjugate to δ. As a special case when $E = F$ and $\sigma = id$, we have recovered the definition of the stable orbital integral $SO_\gamma(f)$ for a semisimple element $\gamma \in G(F)$ and for $f \in C_c^\infty(G(F))$.

8.2 The case where γ is not a norm

Lemma 8.2.1. *Let $\phi \in \mathcal{Z}(G(E), \rho_r)$. If γ is not a norm from $G(E)$, then $SO_\gamma(b\phi) = 0$.*

Recall that $b\phi \in \mathcal{Z}(G, \rho)$. First we assume γ is elliptic. We will prove the stronger statement that if $\gamma' \in G(F)$ is stably conjugate to γ, then $b\phi(g^{-1}\gamma' g) = 0$ for every $g \in G(F)$. From this we see that $SO_\gamma(b\phi) = 0$.

Consider the canonical map $p : G_{\mathrm{sc}} \to G$ and the abelian group $H^0_{\mathrm{ab}}(F, G) = G(F)/p(G_{\mathrm{sc}}(F))$. Proposition 2.5.3 of [Lab99] shows that an elliptic element γ is a norm from $G(E)$ if and only if its image $p(\gamma)$ in $H^0_{\mathrm{ab}}(F, G)$ is a norm. (Again we notice that Labesse didn't assume the ground field F to be of characteristic 0 in CHAPITRE 1 and 2 in loc. cit.) Now the required vanishing follows from the following lemma.

Lemma 8.2.2. *Let $f = b\phi$ for some $\phi \in \mathcal{Z}(G(E), \rho_r)$. Let $x \in G(F)$ be any element such that, for some character η on the group $H^0_{ab}(F, G)$ which is trivial on the norms, we have $\eta(x) \neq 1$. Then $f(x) = 0$.*

Proof. We can view η as a character $\eta : G(F) \to \mathbb{C}^\times$ by pulling back along the quotient map $G(F) \twoheadrightarrow H^0_{\mathrm{ab}}(F, G)$, thus the condition $\eta(x) \neq 1$ makes sense.

Since η vanishes on $G(F)^\natural := p(G_{\mathrm{sc}}(F))$, it is trivial on $G(F)_1 = G(F)^\natural \cdot T(F)_1$, where $G(F)_1$ is also the kernel of the Kottwitz homomorphism $\kappa_G : G(F) \to \pi_1(G)_I^{\mathrm{Gal}(\breve{F}/F)}$ ([KP23], Definition 2.6.23 and Proposition 11.5.4). Here $\breve{F}$ denotes the maximal unramified extension of F inside a fixed algebraic closure F^s and $I := \mathrm{Gal}(F^s/\breve{F})$, and $T(F)_1$ is the unique maximal compact subgroup of $T(F)$. Hence the restriction of η to $T(F)$ is trivial on $T(F)_1$. Thus we have $f\eta \in \mathcal{H}(G, \rho)$. By examining the right convolution action of $f\eta$ on $i_B^G(\xi^{-1})^{\rho^{-1}}$, we see that $f\eta \in Z(G, \rho)$ (also see the proof of Lemma 8.4.3), and

$$ch_{\xi^{-1}}(f\eta) = ch_{(\eta\xi)^{-1}}(f). \tag{8.2.1}$$

But by formula 5.2.1 in the base change section, this is

$$ch_{(\eta_r\xi_r)^{-1}}(\phi) = ch_{\xi_r^{-1}}(\phi) = ch_{\xi^{-1}}(f),$$

the first equality holds since $\eta_r := \eta \circ N_r$ is trivial by assumption. This implies that $f\eta = f$ since $ch_{\xi^{-1}}(f\eta) = ch_{\xi^{-1}}(f)$ implies that $\beta^{-1}(f\eta) = \beta^{-1}(f)$. Now we have

$$f(x)(\eta(x) - 1) = 0$$

for such x. Since $\eta(x) \neq 1$, we have $f(x) = 0$ as desired. $\square$

Now consider $x := g^{-1}\gamma g$. Since it is not a norm by assumptions, there must exist one such character η satisfying the conditions of Lemma 8.2.2, therefore $f(x) = 0$ as desired. We have proved Lemma 8.2.1 in the case that γ is elliptic, in the general case, we can apply the descent formula 7.4.3 to reduce to the elliptic case.

8.3 The case where the derived group is not simply connected

The strategy is the same as in [Hai09], §5.2 and [Hai12] §7.2. Choose a finite unramified extension $F' \supset F$, which contains E and splits G. Consider a z-extension of F-groups adapted to E/F:

$$1 \longrightarrow Z \longrightarrow H \xrightarrow{p} G \longrightarrow 1,$$

where Z is a finite product of copies of $\mathrm{Res}_{F'/F}\mathbb{G}_m$, $H_{\text{der}} = H_{\text{sc}}$ and p is surjective on E- and F-points since Z is an induced torus. Choose an extension of σ to an element in $\mathrm{Gal}(F'/F)$, which is still denoted σ. Let $Z(E)_1$ (resp. $Z(F)_1$) denote the maximal compact subgroup of $Z(E)$ (resp. $Z(F)$). We endow $Z(E)$ (resp. $Z(F)$) with the Haar measure giving $Z(E)_1$ (resp. $Z(F)_1$) volume 1. The norm homomorphism $N : Z(E) \to Z(F)$ is surjective and determines a measure-preserving isomorphism

$$N : \overline{Z(E)} := (Z(E)/(1-\sigma)(Z(E))) \to Z(F)$$

and $N : Z(E)_1 \twoheadrightarrow Z(F)_1$ is surjective. Here we give the compact subgroup $(1-\sigma)(Z(E)) = (1-\sigma)(Z(E)_1)$ measure 1.

Let $\lambda : Z(F) \to \mathbb{C}^\times$ denote a smooth character, and for $f \in C_c^\infty(H(F))$, we set

$$f_\lambda(h) := \int_{Z(F)} f(hz)\lambda^{-1}(z)\, dz$$

Write $\lambda = 1$ for the trivial character, then we can show that $f_1(hz') = f_1(h)$ for any $h \in H(F)$ and $z' \in Z(F)$. Therefore we can view f_1 as an element in $C_c^\infty(G(F))$, which is denoted $\bar{f}$.

We write λN for the character $\lambda \circ N : Z(E) \to \mathbb{C}^\times$. The depth-zero character χ on $T(F)_1$ determines a depth-zero character χ_H on $T_H(F)_1$, where $T_H := p^{-1}(T)$. Let I_H be the Iwahori subgroup in H corresponding to the Iwahori subgroup I in G, and let $\rho_H : I_H \to \mathbb{C}^\times$ denote the character constructed from χ_H by the isomorphism $I_H/I_H^+ \cong T_H(F)_1/T_H(F)_1^+$. For $\phi \in C_c^\infty(H(E))$, we define analogously the function $\phi_{\lambda N} \in C_c^\infty(H(E))$ by the formula

$$\phi_{\lambda N}(h) := \int_{Z(E)} \phi(hz)\lambda^{-1}(Nz)\, dz,$$

and similarly we use $\bar{\phi}$ when viewing ϕ_1 as an element $C_c^\infty(G(E))$. It is straightforward to check that the functions $\phi_{\lambda N}$ (resp. f_λ) are transformed by $\lambda^{-1}N$ (resp. λ^{-1}) under $Z(E)$ (resp. $Z(F)$). These functions are not longer compactly supported on H, nonetheless we can show that the (twisted) orbital integrals of f_λ (resp. $\phi_{\lambda N}$) exist at (σ)-semisimple element δ by the following simple calculation:

$$\begin{aligned} TO^H_{\delta\sigma}(\phi_{\lambda N}) :&= \int_{H_{\delta\sigma}\backslash H(E)} \phi_{\lambda N}(h^{-1}\delta\sigma(h))\, d\bar{h} \\ &= \int_{H_{\delta\sigma}Z(E)\backslash H(E)} \int_{Z(F)\backslash Z(E)} \phi_{\lambda N}((vh)^{-1}\delta\sigma(hv))\, dv\, d\bar{h} \\ &= \mathrm{vol}(Z(F)\backslash Z(E)) \int_{H_{\delta\sigma}Z(E)\backslash H(E)} \phi_{\lambda N}(\bar{h}^{-1}\delta\sigma(\bar{h}))\, d\bar{h} \\ &= \int_{H_{\delta\sigma}Z(E)\backslash H(E)} \phi_{\lambda N}(\bar{h}^{-1}\delta\sigma(\bar{h}))\, d\bar{h} \end{aligned} \tag{8.3.1}$$

Since $Z(F)\backslash Z(E) \cong (1-\sigma)Z(E)$ which has measure 1 and as a function, $\phi_{\lambda N}(\cdot\, \delta\sigma(\cdot))$ is compactly supported modulo $Z(E)$, the convergence of the (twisted) orbital integral is shown.

We also have the following useful lemma:

Lemma 8.3.1. *Assume $\phi \in \mathcal{H}(H(E), \tilde{K}, \tilde{\rho}_H)$ for some compact open subgroup $\tilde{K} \leq H(E)$ and there exists compact open $K \leq G$ and a character $\rho : K \to \mathbb{C}^\times$ such that*

- *$\tilde{K} \supset (1-\sigma)(Z(E))$, and*
- *$\tilde{\rho}|_{\tilde{K}\cap Z(E)} = \rho \circ N|_{\tilde{K}\cap Z(E)}$.*

Then we have

$$\phi_{\lambda N}(h) = \int_{\overline{Z(E)}} \phi(hz)\lambda^{-1}(Nz)\, d\bar{z}$$

Proof. We have

$$\begin{aligned}
\phi_{\lambda N}(h) &= \int_{Z(E)} \phi(hz)\lambda^{-1}(Nz)\,dz \\
&= \int_{Z(E)/(1-\sigma)Z(E)} \int_{(1-\sigma)Z(E)} \phi(vhz)\lambda^{-1}(N(vz))\,dv\,d\bar{z} \\
&= \int_{Z(E)/(1-\sigma)Z(E)} \int_{(1-\sigma)Z(E)} \phi(hz)\lambda^{-1}(Nz)\,dv\,d\bar{z} \\
&= \int_{\overline{Z(E)}} \phi(hz)\lambda^{-1}(Nz)\,d\bar{z}
\end{aligned}$$

where the second line is [Fol16] Theorem 2.51, the third line follows from that we can write $v = v'\sigma(v'^{-1})$ for some $v' \in Z(E)$ so that $Nv = 1$ and $\phi(vhz) = \phi(hz)\rho(Nv) = \phi(hz)$, and in the last line we use our choice of measure that gives $(1-\sigma)Z(E)$ volume 1. $\square$

Lemma 8.3.2. *Suppose that $\phi \in \mathcal{H}(H(E), \tilde{K}, \tilde{\rho})$ (resp. $f \in \mathcal{H}(H(F), K, \rho)$) for a compact open subgroup $\tilde{K} \subset H(E)$ and a character $\tilde{\rho} : \tilde{K} \to \mathbb{C}^\times$ (resp. $K \subset H$ and $\rho : K \to \mathbb{C}^\times$) such that*

- $N(\tilde{K} \cap Z(E)) = K \cap Z(F)$,
- $\tilde{K} \supset (1-\sigma)Z(E)$,
- $\tilde{\rho}|_{\tilde{K}\cap Z(E)} = \rho \circ N|_{\tilde{K}\cap Z(E)}$.

We have the following statements:

(i) The functions ϕ and f are associated if and only if $\phi_{\lambda N}$ and f_λ are associated for every λ.

(ii) In (i) we only need to consider characters λ such that

$$\lambda|_{K\cap Z(F)} = \rho^{-1}|_{K\cap Z(F)}.$$

(iii) If $\phi \in \mathcal{H}(H(E), \tilde{\rho}_H)$ and $f \in \mathcal{H}(H(F), \rho_H)$, then in (i) we only need to consider characters λ with $\lambda|_{Z(F)_1} = \chi^{-1}|_{Z(F)_1}$.

(iv) The pair (ϕ_1, f_1) are associated if and only if $(\bar{\phi}, \bar{f})$ are associated.

Proof. We follow the proof of Lemma 5.3.1 in [Hai09]. Suppose that $\delta \in H(E)$ (resp. $\gamma \in H(F)$), and we write $\bar{\delta} := p(\delta)$ (resp. $\bar{\gamma} := p(\gamma)$). We have the following formulas for all χ:

$$\begin{aligned} SO^H_{\delta\sigma}(\phi_{\lambda N}) &:= \sum_{\delta'} e(\delta')\, TO^H_{\delta'\sigma}(\phi_{\lambda N}) \\ &= \sum_{\delta'} e(\delta') \int_{H_{\delta'\sigma}\backslash H(E)} \phi_{\lambda N}(h^{-1}\delta'\sigma(h))d\bar{h} \\ &= \sum_{\delta'} e(\delta') \int_{H_{\delta'\sigma}\backslash H(E)} \int_{\overline{Z(E)}} \phi(h^{-1}(\delta' v)\sigma(h))\, \lambda^{-1}(Nv) dv\, d\bar{h} \\ &= \int_{\overline{Z(E)}} \Big[\sum_{\delta'} e(\delta') \int_{H_{\delta'\sigma}\backslash H(E)} \phi(h^{-1}(\delta' v)\sigma(h))\, d\bar{h}\Big] \lambda^{-1}(Nv) dv \\ &= \int_{\overline{Z(E)}} \lambda^{-1}(Nv)\, SO^H_{v\delta,\sigma}(\phi)\, dv. \end{aligned} \tag{8.3.2}$$

and similarly we have

$$SO^H_{\gamma}(f_\lambda) = \int_{Z(F)} \lambda^{-1}(z)\, SO^H_{z\gamma}(f)\, dz \tag{8.3.3}$$

Then it is clear that if ϕ and f are associated, then $\phi_{\lambda N}$ and f_λ are associated for every λ. For the inverse, we apply the Fourier inversion formula to get $SO^H_{v\delta,\sigma}(\phi) = SO^H_{z\gamma}(f)$ for $v \in Z(E)$ and $z \in Z(F)$ such that $Nv = z$. (as functions of v and z, resp.) In particular, let $v = 1$, we have

$$SO^H_{\delta\sigma}(\phi) = SO^H_{\gamma}(f) \tag{8.3.4}$$

For (ii), we assume $f \in \mathcal{H}(H(F), K, \rho)$, then for any $i \in K \cap Z(F)$ and $h \in H(F)$, we have

$$\begin{aligned} f_\lambda(hi) &= \int_{Z(F)} f(hiz)\lambda^{-1}(z)\, dz \\ &= \rho^{-1}(i) f_\lambda(h), \end{aligned}$$

also we have

$$\begin{aligned} f_\lambda(hi) &= \int_{Z(F)} f(hiz)\lambda^{-1}(z)\, dz \\ &= \lambda(i) \int_{Z(F)} f(hu)\lambda^{-1}(u)\, du = \lambda(i) f_\lambda(h) \end{aligned}$$

Therefore either $\lambda|_{Z(F)\cap K} = \rho^{-1}|_{Z(F)\cap K}$ or f_λ vanishes completely on H.

Part (iii) follows from part (ii) by taking $K = I_H, \tilde{K} = \tilde{I}_H$ and $\tilde{\rho} = \tilde{\rho}_H$.

For $\lambda = 1$, we claim that for $\gamma = \mathcal{N}\delta$, we have

$$\begin{aligned} SO^H_{\delta\sigma}(\phi_1) &= SO^G_{\bar{\delta}\sigma}(\bar{\phi}) \\ SO^H_{\gamma}(f_1) &= SO^G_{\bar{\gamma}}(\bar{f}) \end{aligned} \tag{8.3.5}$$

Indeed, we have by definition

$$SO^H_{\delta\sigma}(\phi_1) := \sum_{\delta'} e(\delta')\, TO^H_{\delta'\sigma}(\phi_1)$$

and

$$SO^G_{\bar{\delta}\sigma}(\bar{\phi}) := \sum_{\bar{\delta}'} e(\bar{\delta}')\, a(\bar{\delta}') TO^G_{\bar{\delta}'\sigma}(\bar{\phi})$$

We compare those two equalities term by term. First we notice that $p : H_{\delta'\sigma}(F) \to G^\circ_{\bar{\delta}'\sigma}(F)$

with kernel $Z(E) \cap H_{\delta'\sigma}(F) = Z(F)$, hence $e(\delta') := e(H_{\delta'\sigma}) = e(G_{\bar{\delta}'\sigma}) =: e(\bar{\delta}')$ by the Corollary on Page 295 of [Kot83]. Recall that $a(\bar{\delta}') := |\ker[H^1(F, G^\circ_{\bar{\delta}'\sigma}) \to H^1(F, G_{\bar{\delta}'\sigma})]|$. We claim that p induces a surjective map from the set

$$\{\sigma\text{-conjugacy classes } \delta' \in H(E) \text{ stably } \sigma\text{-conjugate to } \delta\}$$

onto the set

$$\{\sigma\text{-conjugacy classes } \bar{\delta}' \in G(E) \text{ stably } \sigma\text{-conjugate to } \bar{\delta}\}$$

with the fiber over the class of $\bar{\delta}'$ identified with the set

$$\ker[H^1(F, G^\circ_{\bar{\delta}'\sigma}) \to H^1(F, G_{\bar{\delta}'\sigma})]$$

Indeed, we denote $\tilde{G} := R_{E/F}G_E$, and similarly $\tilde{H}$. The set of σ-conjugacy classes in $G(E)$ which are stably σ-conjugate to $\bar{\delta}$ corresponds to the image of

$$\ker[H^1(F, G^\circ_{\bar{\delta}\sigma}) \to H^1(F, \tilde{G})]$$

in

$$\ker[H^1(F, G_{\bar{\delta}\sigma}) \to H^1(F, \tilde{G})].$$

Meanwhile the set of σ-conjugacy classes in $H(E)$ which are stably σ-conjugate to δ corresponds to the set

$$\ker[H^1(F, H_{\delta\sigma}) \to H^1(F, \tilde{H})]$$

since $H_{\delta\sigma} = H^{\circ}_{\delta\sigma}$. This is explained in [Kot82], Pages 805-806. The map p induces a bijection

$$\ker[H^1(F, H_{\delta\sigma}) \to H^1(F, \tilde{H})] \xrightarrow{\sim} \ker[H^1(F, G^{\circ}_{\bar{\delta}\sigma}) \to H^1(F, \tilde{G})].$$

hence the claim is proved. Finally, we have to show that for all σ-conjugacy classes δ' in the fiber over the class of $\bar{\delta}'$,

$$TO^H_{\delta'\sigma}(\phi_1) = TO^G_{\bar{\delta}'\sigma}(\bar{\phi})$$

From the calculations in 8.3.1, we have indeed:

$$\begin{aligned} TO^H_{\delta'\sigma}(\phi_1) &= \int_{(Z(E)H_{\delta'\sigma})\backslash H(E)} \phi_1(h^{-1}\delta'\sigma(h))\, d\bar{h} \\ &= \int_{G^{\circ}_{\bar{\delta}'\sigma}\backslash G(E)} \bar{\phi}(g^{-1}\bar{\delta}'\sigma(g))\, d\bar{g} \\ &= TO^G_{\bar{\delta}'\sigma}(\bar{\phi}). \end{aligned}$$

Where the second equality comes from the fact $Z(E)H_{\delta'\sigma} = p^{-1}(G^{\circ}_{\bar{\delta}'\sigma})$ and $H(E)/Z(E) = G(E)$ since Z is a product of induced tori. Hence we have $SO^H_{\delta\sigma}(\phi_1) = SO^G_{\bar{\delta}\sigma}(\bar{\phi})$, and similarly $SO^H_{\gamma}(f_1) = SO^G_{\bar{\gamma}}(\bar{f})$ (using the analogous $H_{\gamma'}(F) = p^{-1}(G^{\circ}_{\bar{\gamma}'})$), thus (iv) is proved. □

Lemma 8.3.3. *The map $\phi \mapsto \bar{\phi}$ determines a surjective homomorphism $\mathcal{Z}(H(E), \tilde{\rho}_H) \to \mathcal{Z}(G(E), \tilde{\rho})$. It is compatible with the base change homomorphism in the sense that*

$$b(\bar{\phi}) = \overline{b(\phi)}.$$

Proof. The proof is almost the same as in [Hai09] Lemma 5.3.2, but instead of the Bernstein isomorphism B there, we use $\beta : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}^H_E}] \xrightarrow{\sim} \mathcal{Z}(H(E), \tilde{\rho}_H)$ (also $\beta : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_E}] \xrightarrow{\sim} \mathcal{Z}(G(E), \tilde{\rho})$). Here $\mathfrak{s}^H_E := \mathfrak{s}_{\chi_H N}$.

Recall that $\xi : T(F) \to \mathbb{C}^\times$ is a smooth character extending some $W(F)$-conjugate of the depth zero character $\chi : T(F)_1 \to \mathbb{C}^\times$. We can extend ξ to a character ξ_H on $T_H(F)$, which extends χ_H. Therefore the morphism of algebraic varieties

$$p^* : \mathfrak{X}_{\mathfrak{s}_E} \to \mathfrak{X}_{\mathfrak{s}_E^H}$$

$$(T(E), \xi_N)_{G(E)} \mapsto (T_H(E), \xi_H \circ N)_{H(E)},$$

gives rise to the homomorphism $p : \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_E^H}] \to \mathbb{C}[\mathfrak{X}_{\mathfrak{s}_E}]$ that we need as in the proof in loc. cit. □

8.4 The case where the central character is not unitary

In applying Lemma 8.3.2(i) in the following sections, we need the following lemma which will enable us to assume that λ is a unitary character on Z. It is not necessarily the case that λ is unitary in our reduction steps. Nonetheless, we can always twist it by a global character as follows.

Lemma 8.4.1. *If (π, V) is any smooth ω-representation of G, then there exists a unique positive real-valued character χ of G such that the restriction of $\pi \otimes \chi$ to Z is unitary.*

Proof. This is Lemma 5.2.5 in [Cas95]. We notice that by construction in the proof, χ is trivial on the Iwahori subgroup $I \subset G$. □

Lemma 8.4.2. *Assume $\phi \in \mathcal{Z}(H(E), \tilde{\rho})$ and let $f := b\phi \in \mathcal{Z}(H, \rho)$. If $\phi_{\lambda N}$ and f_λ are associated for all unitary central characters $\lambda : Z(F) \to \mathbb{C}^\times$ and all $\phi \in \mathcal{Z}(H(E), \tilde{\rho})$, then ϕ and f are associated.*

Proof. By Lemma 8.3.2 (i) and (iii), the functions ϕ and f are associated if $\phi_{\lambda N}$ and f_λ are associated for every character λ such that $\lambda|_{Z(F)_1} = \chi^{-1}|_{Z(F)_1}$. We claim that it is enough to prove the results for such **unitary** characters λ. Indeed, by the previous lemma, we can find positive characters η on G, such that the set $\{\lambda\eta\}$ for unitary λ contains all the characters we need in Lemma 8.3.2 (i) and (iii).

We consider the function $f_{\lambda\eta}$, where we understand that we actually mean $\lambda\eta|_Z$ in the definition. Now we have

$$\begin{aligned} f_{\lambda\eta}(g) &= \int_{Z(F)} f(gz)\lambda^{-1}(z)\eta^{-1}(z)\, dz \\ &= \eta(g)\int_{Z(F)} f(gz)\eta^{-1}(gz)\lambda^{-1}(z)\, dz \\ &= \eta(g)(\eta^{-1}f)_\lambda(g). \end{aligned}$$

Therefore, we have

$$f_{\lambda\eta} = \eta\cdot(\eta^{-1}f)_\lambda, \tag{8.4.1}$$

Similarly, we have over $H(E)$,

$$\phi_{\lambda\eta N} = (\eta N)\cdot((\eta^{-1}N)\phi)_{\lambda N} \tag{8.4.2}$$

Now we assume $\phi \in \mathcal{Z}(H(E),\tilde{\rho})$ and $f = b\phi \in \mathcal{Z}(H,\rho)$. By the lemma below, $\eta^{-1}f$ and $(\eta^{-1}N)\phi$ are still in the center of respective Hecke algebras and we have $b((\eta^{-1}N)\phi) = \eta^{-1}\cdot b\phi = \eta^{-1}f$. Therefore, by assumption, $(\eta^{-1}f)_\lambda$ and $((\eta^{-1}N)\phi)_{\lambda N}$ are associated. Then by a straightforward calculation, we see that the left hand sides of 8.4.1 and 8.4.2 are associated, as desired. □

Lemma 8.4.3. *Let $\phi \in \mathcal{Z}(H(E),\rho)$ and η be a character on H such that $\eta|_I =$ triv, where $I \subset H$ is the Iwahori subgroup where ρ is defined on. Then $(\eta N)\cdot\phi \in \mathcal{Z}(H(E),\rho)$ and $b((\eta N)\cdot\phi) = \eta\cdot b\phi$.*

Proof. Let $\psi \in \mathcal{H}(H(E),\rho)$. Since η is trivial on I, it is clear that $(\eta N)\cdot\phi \in \mathcal{H}(H(E),\rho)$ then we have

$$
\begin{aligned}
[(\eta N)\cdot\phi]*\psi(h) &= \int_{H(E)} \eta N(g)\phi(g)\psi(g^{-1}h)\,dg \\
&= \eta N(h)\int_{H(E)} \eta N(g^{-1})\phi(hg^{-1})\psi(g)\,dg \\
&= \eta N(h)[\phi*((\eta^{-1}N)\cdot\psi)](h).
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
[(\eta N)\cdot\phi]*\psi &= \eta N\cdot[\phi*((\eta^{-1}N)\psi)] \\
&= \eta N\cdot[((\eta^{-1}N)\psi)*\phi]\ (\phi \text{ is central}) \\
&= \psi*[(\eta N)\cdot\phi]\ (\text{use the calculation above reversely}).
\end{aligned}
$$

The centrality is shown. Using the identities 8.2.1 and 5.2.1, we have

$$
\begin{aligned}
ch_{\xi^{-1}}(b((\eta N)\cdot\phi)) &= ch_{\xi_N^{-1}}((\eta N)\cdot\phi) \\
&= ch_{(\xi\eta)_N^{-1}}(\phi) \\
&= ch_{(\xi\eta)^{-1}}(b\phi) \\
&= ch_{\xi^{-1}}(\eta\cdot b\phi)
\end{aligned}
$$

therefore $b((\eta N)\cdot\phi) = \eta\cdot b\phi$, as desired. $\square$

8.5 The case where the center is not an induced torus

We assume $\gamma \in G(F)$ is a regular elliptic semisimple element, and $G_{\text{der}} = G_{\text{sc}}$. We will show that there is an exact sequence

$$1 \longrightarrow G \longrightarrow G' \longrightarrow Q \longrightarrow 1,$$

where G' is an unramified group over F with the properties that $G'_{\text{der}} = G'_{\text{sc}}$ and $Z(G')$ is an induced torus. We follow the construction in [Clo90], 6.1(b). Indeed, since G is unramified, $Z(G)$ is contained in a maximal unramified F-torus $T \subset G$. The group of characters $X^*(T)$ is a finitely generated $Z[\mathrm{Gal}(F'/F)]$-module, where F' is an unramified extension of F splitting G. Let $P \twoheadrightarrow X^*(T)$ be a free $Z[\mathrm{Gal}(F'/F)]$-module. Then dually we get an embedding $T \hookrightarrow Z'$ where Z' is an induced torus. We have the following exact sequence

$$1 \longrightarrow Z(G_{\text{der}}) \longrightarrow Z(G) \times G_{\text{der}} \longrightarrow G \longrightarrow 1,$$

see [Mil17], Example 19.25. We take G' to be $(G_{\text{der}} \times Z')/Z(G_{\text{der}})$. It is easy to check that $Z(G') = Z'$ and $G'_{\text{der}} = G_{\text{der}}$, therefore G' satisfies the conditions we need.

The natural embedding $G \hookrightarrow G'$ induces a natural homomorphism

$$j : \mathcal{Z}(G, \rho) \hookrightarrow \mathcal{Z}(G', \rho')$$

by the following commutative diagram

$$\begin{tikzcd}
\mathbb{C}[\mathfrak{X}_{\mathfrak{s}}] \arrow[r, "\sim", "\beta"'] \arrow[d, hook] & \mathcal{Z}(G,\rho) \arrow[d, hook, "j"'] \\
\mathbb{C}[\mathfrak{X}_{\mathfrak{s}'}] \arrow[r, "\sim", "\beta"'] & \mathcal{Z}(G',\rho')
\end{tikzcd}$$

where the left vertical map is induced by the surjective map

$$\mathfrak{X}_{\mathfrak{s}'} \twoheadrightarrow \mathfrak{X}_{\mathfrak{s}}$$

$$(T',\xi')_{G'} \mapsto (T := T' \cap G, \xi'|_T)_G$$

where $T' \subset G'$ is a maximal F-torus and $\xi' : T'(F) \to \mathbb{C}^\times$ is a smooth character extending some $W'(F)$-conjugate of χ', where W' is the Weyl group of G' and $\chi' : T'(F)_1 \to \mathbb{C}^\times$ is an extension of χ.

The injective morphism j is uniquely determined by the Bernstein isomorphisms and it is the restriction of the map between Hecke algebras

$$\mathcal{H}(G,\rho) \to \mathcal{H}(G',\rho')$$

$$1_{n_w} \mapsto 1'_{n_w},$$

where n_w is the extended affine Weyl group $\tilde{W}_H = X_*(A) \rtimes W_\chi^\circ$ and 1_{n_w} (resp. $1'_{n_w}$) is the function in $\mathcal{H}(G,\rho)$ (resp., $\mathcal{H}(G',\rho')$) supported on $I_r n_w I_r$ (resp., $I'_r n_w I'_r$), whose value at n_w is 1. Such functions 1_{n_w} form a basis of the Hecke algebra $\mathcal{H}(G,\rho)$. Here we are using the notations from §7.3, [Hai12]. Li ([Li26], Lemma 7.2.3) shows that the morphism j on the centers is indeed the restriction of the linear transformation on the whole Hecke algebras, therefore, it suffices to prove the following:

Lemma 8.5.1. *There exists a constant C_T such that, for every $\delta \in G(E)$ with elliptic regular norm in T, and for every $w \in \tilde{W}_E$, we have*

$$SO^{G(E)}_{\delta\sigma}(1_{n_w}) = C_T SO^{G'(E)}_{\delta\sigma}(1'_{n_w}) \tag{8.5.1}$$

Proof. This is proved in [Hai12], Lemma 7.3.1. The proof also works for positive characteristic. □

From 8.5.1, we have the following equalities:

$$\begin{aligned} SO^G_\gamma(f, dt, dg) &= c_T SO^{G'}_\gamma(jf, dt', dg') \\ SO^{G(E)}_{\delta\sigma}(\phi, dt, dg_E) &= C_T SO^{G'(E)}_{\delta\sigma}(j\phi, dt', dg'_E), \end{aligned} \tag{8.5.2}$$

for all functions $f \in \mathcal{Z}(G, \rho)$ and $\phi \in \mathcal{Z}(G(E), \rho_E)$ and for all (σ-)regular (σ-)elliptic elements γ and δ in G whose (σ-)centralizer is the elliptic F-torus T. The proof of Lemma 7.3.1 in *loc. citc* shows that the constants c_T and C_T making 8.5.2 hold depend only on the torus T and choices of measures (so not on the functions f and ϕ, and the choice of $\gamma = \mathcal{N}(\delta)$). Therefore, to force $c_T = C_T$, we use the fact that 1_{I_r} and 1_I are associated [Kot86a] and the stable orbital integrals of 1_I do not vanish identically on any torus T in G.

8.6 Summary of reduction steps

By Lemma 8.2.1, we may assume that γ is a norm, and we write $\gamma = \mathcal{N}\delta$. We assume γ is *regular semisimple* from the beginning. Notice that it is different from the assumptions in [Hai09, Hai12, Clo90] of just being semisimple since we don't have *(twisted) Shalika germs* for local function fields and the homogeneity that were used in the proof of Proposition 7.2 in [Clo90] to reduce to regular semisimple elements. The reduction steps (1) and (2) are the same as in [Hai09], 5.4. However, we need to pass it to the global setup afterwards.

(1) *We may assume that* $G_{\text{der}} = G_{\text{sc}}$. Indeed, for given G, we take H to be a z-extension as in 8.3, and let $\phi \in \mathcal{Z}(H(E), \tilde{\rho}_H)$ with $b\phi \in \mathcal{Z}(H, \rho_H)$. Assume that $(\phi, b\phi)$ are associated, then by Lemma 8.3.2(i) and (iv), $(\overline{\phi}, \overline{b\phi})$ are associated. By Lemma 8.3.3, $(\bar{\phi}, b\bar{\phi})$ are associated and $\bar{\phi}$ ranges over all functions in $\mathcal{Z}(G(E), \tilde{\rho})$, therefore the base change fundamental lemma is proved for G.

(2) *We may assume that* γ *is elliptic.* We work under assumption (1). If γ is not elliptic, we can use Lemma 7.3.2 and the discussion above the lemma to associate to γ a proper F-Levi subgroup M containing $G_\gamma^\circ = G_\gamma$. By induction on semisimple ranks, the fundamental lemma is known for M. We then use the descent formulas 7.4.2 and 7.4.3, and the following Lemma 7.5.1, we see that the fundamental lemma for M implies the fundamental lemma for G.

From here we momentarily pass it to the global setup. We may assume that G is split over an unramified extension K/F such that $E \subset K$. We may also assume G satisfies the conditions (1) and (2). Choose a degree $[K : F]$ cyclic extension of function fields $\underline{K}/\underline{F}$ and a finite place v_0 of $\underline{F}$ such that $\underline{K}_{v_0}$ is a field and $\underline{K}_{v_0}/\underline{F}_{v_0} \cong K/F$. Then there is a degree $r = [E : F]$ cyclic extension $\underline{E}/\underline{F}$ with $\underline{E} \subset \underline{K}$ and $\underline{E}_{v_0}/\underline{F}_{v_0} \cong E/F$.

There is a quasi-split group $\underline{G}$ over $\underline{F}$ with the property that $\underline{G} \times_{\underline{F}} \underline{F}_{v_0} \cong G$. Let θ denote the $\underline{F}$-linear automorphism of $\underline{\tilde{G}}$ from $\mathrm{Gal}(E/F) \cong \mathrm{Gal}(\underline{E}/\underline{F})$.. We may repeat the process of finding a $G \hookrightarrow G'$ at the beginning of 8.4, but globally: a global group $\underline{G'}$ such that $\underline{G'}_{\text{der}} = \underline{G}_{\text{der}}$ and $Z(\underline{G'})$ is an induced torus. Therefore after base change to v_0, we still have the desired properties for G and $G' := \underline{G'}_{v_0}$ as in 8.4. Therefore we have the following reduction step:

(3) *We may assume that $G_{\text{der}} = G_{\text{sc}}$ and $Z(G)$ is an induced torus which comes from a global induced torus.* Indeed we work under assumptions (1) and (2), and Lemma 8.5.1 and the discussion there show that we can assume $Z(G)$ is indeed an globally induced torus.

(4) *We may assume that γ belongs to a specific dense subset in the set of regular elliptic elements.* This is explained in [Clo90] Lemma 6.7 and the discussion followed. Recall that we call a regular semisimple element γ *strongly regular semisimple* if G_γ is a (connected) torus. Later in Lemma 9.3.5, we will see that $\gamma \in G(F)$ will be assumed to be strongly regular semisimple, whose image in the adjoint group is still strongly regular. By Lemma 8.7, we will indeed see that such γ will form a dense subset of the set of regular elliptic elements in $G(F)$. Notice that we don't need to worry about the $a(\delta')$ terms as in the reduction steps of [Hai09] since our group G has simply connected derived group.

The conclusion is that we may assume that $G_{\text{der}} = G_{\text{sc}}$ with $Z(G)$ an globally induced torus, and γ is a strongly regular elliptic element itself and in the adjoint group, and is a norm. That concludes our reduction on the group and the regular semisimple element. We hope to show the associations of functions $\phi \in \mathcal{Z}(H(E), \tilde{\rho})$ and $b\phi$ at all such elements γ over the group G.

The strategy of the proof in the following sections is that, after all the reduction steps, using Lemma 8.3.2 (i) and Lemma 8.4.2, we will show that $\phi_{\lambda N}$ and $(b\phi)_\lambda$ are associated for all unitary central characters and all $\phi \in \mathcal{Z}(H(E), \tilde{\rho})$, at all strongly regular elliptic semisimple elements in H whose images in the adjoint group is still strongly regular.

Using the existence of *local data* adapted to the case with unitary central characters (Chapter 12), we are able to show the desired association. In order to produce the adapted local data, one needs to use the global (twisted) trace formulae (Chapter 9) and one needs to *stabilize* the trace formulae

(Chapter 10) for them to be useful.

Chapter 9: The Simple Trace Formula

9.1 Setup

This section is mostly independent of other sections. We do not make any assumptions on the characteristics of the global field F.

Assume that E/F is a cyclic extension of global fields of degree r, we write $\mathbb{A}$ (resp. $\mathbb{A}_E$) for the adeles of F (resp. E). We denote $\Gamma := \mathrm{Gal}(E/F)$ and $\theta \in \Gamma$ be a generator. Let G be a connected reductive group defined over F, and let $[G]_E := G(E)\backslash G(\mathbb{A}_E)$ denote the usual *adelic quotient* over E. We consider the left action of $G(\mathbb{A}_E)$ on $L^2(G(E)\backslash G(\mathbb{A}_E))$ given by

$$(R(g)f)(x) := f(xg) \tag{9.1.1}$$

for any $g \in G(\mathbb{A}_E)$, $f \in L^2(G(E)\backslash G(\mathbb{A}_E))$ and $x \in G(E)\backslash G(\mathbb{A}_E)$.

Let $Z = Z(G)$ be the center of G. By the reduction steps, we can assume that Z is an induced torus over F, therefore, we have $G(R)/Z(R) = G_{\mathrm{ad}}(R)$ for any F-algebra R, where $G_{\mathrm{ad}} := G/Z$. Fix a **unitary** character $\lambda : Z(\mathbb{A}_F) \to \mathbb{C}^\times$ which is trivial on $Z(F)$, and define $\lambda N : Z(\mathbb{A}_E) \to \mathbb{C}^\times$ in the usual way . We consider the space $C_c^\infty(Z(\mathbb{A}_E)\backslash G(\mathbb{A}_E), \lambda^{-1})$ of functions whose supports are compact modulo $Z(\mathbb{A}_E)$ and transformed by $\lambda^{-1}N$ under the action of the center Z. We will denote it by $C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}_E), \lambda^{-1})$ with $Z(\mathbb{A}_E)$ understood in the definition. Similarly, we consider the space $L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E), \lambda) = L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E), \lambda)$ of functions on $G(E)\backslash G(\mathbb{A}_E)$ that are square-integrable on $G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)$ which transform by λ under Z. Notice that we need λ to be unitary in order to define the integrability over the quotient.

Now for any $\phi \in C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}_E), \lambda^{-1})$, the above action 9.1.1 induces an action of ϕ on

$L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$, given by

$$(R(\phi)f)(x) := \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g)R(g)f(x)\,dg = \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g)f(xg)\,dg. \tag{9.1.2}$$

for any $f \in L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$.

Definition 9.1.1. *We define the cuspidal subspace $L^2_{cusp}(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda) \subset L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$ to be the space of functions $\phi \in L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$ such that, for every parabolic subgroup $P \subset G$ with unipotent radical N, one has*

$$\int_{[N]} \phi(ng)\,dn = 0$$

for all $g \in G(\mathbb{A}_E)$. We call such ϕ cuspidal functions.

We also have the operator

$$\begin{aligned} I_\theta : L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda) &\to L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda),\\ f &\mapsto (x \mapsto f(\theta^{-1}(x))), \end{aligned}$$

that preserves $L^2_{\mathrm{cusp}}(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$.

Proposition 9.1.2. *For each $\phi \in C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}_E),\lambda^{-1})$, the composite*

$$R(\phi)\circ I_\theta : L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda) \to L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)$$

has kernel function

$$K_\phi(g, \theta(h)) := \sum_{\delta \in G_{\mathrm{ad}}(E)} \phi(g^{-1}\delta\theta(h)),$$

In other words, we have

$$\begin{aligned}(R(\phi) \circ I_\theta(f))(g) &= \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} K_\phi(g, \theta(h)) f(h)\, dh \\ &= \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \sum_{\delta \in G_{\mathrm{ad}}(E)} \phi(g^{-1}\delta\theta(h)) f(h)\, dh\end{aligned}$$

For any $f \in L^2(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E), \lambda)$ and $g \in G_{\mathrm{ad}}(\mathbb{A}_E)$.

Proof. Indeed, since

$$\begin{aligned}(R(\phi) \circ I_\theta(f))(g) &= \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(h) I_\theta(f)(gh)\, d\bar{h} \\ &= \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(h) f(\theta^{-1}(g)\theta^{-1}(h))\, d\bar{h} \\ &= \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g^{-1}\theta(h)) f(h)\, d\bar{h} \\ &= \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \sum_{\delta \in G_{\mathrm{ad}}(E)} \phi(g^{-1}\delta\theta(h)) f(h)\, d\bar{h}\end{aligned}$$

where the third line follows from change of variables, and the last line follows from a well-known technique called **unfolding** or **integration in stages** and the fact that f is invariant under the action of elements in $G_{\mathrm{ad}}(E)$. We also use the fact that the set $G_{\mathrm{ad}}(E)$ is θ-stable. For the sake of completeness we include the unfolding lemma below. For a proof, see [GH24] Theorem 3.2.2 and Lemma 9.2.4. □

Lemma 9.1.3. *Suppose that G is a Hausdorff, locally compact, second countable topological group with right Haar measure dg. If $f \in L^1(G)$ and $\Gamma \leq G$ is a discrete subgroup such that the modular* *character of G is trivial on Γ, then we have*

$$\int_{G/\Gamma} \sum_{\gamma\in\Gamma} f(\gamma g)\, dg = \int_G f(g)\, dg.$$

Lemma 9.1.4. *The linear operator $R_{cusp}(\phi) \circ I_\theta$ is of trace class on $L^2_{cusp}(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E), \lambda)$.*

Proof. Since $R_{\mathrm{cusp}}(\phi) := R(\phi)|_{L^2_{\mathrm{cusp}}(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E),\lambda)}$ is of trace class by Theorem 9.1.1 of [GH24], the same is true of $R_{\mathrm{cusp}}(\phi) \circ I_\theta$, since the set of trace class operators forms a two-sided ideal in the algebra of bounded linear operators on $L^2_{\mathrm{cusp}}(G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E), \lambda)$ (see [DE14], Lemma 5.3.4). $\square$

9.2 Proof of the Simple Trace Formula

Let us choose two places v_1, v_2 of F which split completely in E. We put the following assumptions on the *test function* $\phi \in C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}_E), \lambda^{-1})$:

(1) The adelic function ϕ is a (pure) tensor product of local functions $\phi = \prod \phi_v$ over **places** v **of** F, where $\phi_v \in C_c^\infty(G(E_v)/Z(E_v), \lambda^{-1})$. For almost all places v of F, ϕ_v is invariant under $G(\mathcal{O}_{E_v})$ and supported on $Z(E_v)G(\mathcal{O}_{E_v})$, and satisfies

$$\int_{Z(E_v)\backslash Z(E_v)G(\mathcal{O}_{E_v})} \phi(g)\, dg = 1.$$

(2) On $G(E_{v_1}) \cong G(E_{w_1}) \times G(E_{w_2}) \times \cdots \times G(E_{w_r})$, we have $\phi_{v_1} = (\phi^1_{w_1}, \ldots, \phi^1_{w_r})$ where each $\phi^1_{w_i}$ is a matrix coefficient of the same supercuspidal representation π of $G(E_{w_i}) \cong G(F_{v_1})$.

(3) Let $\phi_{v_2} = (\phi^2_{u_1}, \ldots, \phi^2_{u_r})$ be the analogous decomposition at v_2. Let $\Omega_i = \mathrm{Supp}(\phi^2_{u_i})$, then $\Omega_1\Omega_2 \ldots \Omega_r$ is contained in the set of elements of $G(F_{v_2})$ with strongly regular elliptic image in

$G_{\mathrm{ad}}(F_{v_2})$.

We first show that the assumption (2) implies that the image of $R(\phi)$ is in the space of cusp forms.

Lemma 9.2.1. *Let v be a finite place of E, let $f^v \in C_c^\infty(G(\mathbb{A}_E^v))$, and let $f_v \in C_c^\infty(G(E_v))$ be supercuspidal. Let $f(g) = f^v(g^v)f_v(g_v)$, so that $f \in C_c^\infty(G(\mathbb{A}_E))$. Then $R(f)$ has cuspidal image.*

Recall that we say $f_v \in C_c^\infty(G(E_v))$ is *supercuspidal* if

$$\int_{N(E_v)} f_v(gnh)\, dn,$$

for all proper parabolic subgroup $P < G_{E_v}$ with unipotent radical N and for all $g, h \in G(E_v)$. As the name suggests, matrix coefficients of supercuspidal representations are indeed supercuspidal, see [GH24], Lemma 16.4.2.

Proof. Let $P < G$ be a proper parabolic subgroup with unipotent radical N. Let $[N]_E := N(E)\backslash N(\mathbb{A}_E)$. For $\varphi \in L^2([G]_E)$, $R(f)\varphi$ is smooth hence can be integrated over any compact subset. For all

$x \in G(\mathbb{A}_E)$, we have

$$\begin{aligned}
\int_{[N]_E} R(f)\varphi(nx)dn &= \int_{[N]_E}\int_{G(\mathbb{A}_E)} f(g)\varphi(nxg)dgdn \\
&= \int_{[N]_E}\int_{G(\mathbb{A}_E)} f(x^{-1}n^{-1}g)\varphi(g)dgdn \\
&= \int_{[N]_E}\int_{N(E)\backslash G(\mathbb{A}_E)} \sum_{\delta\in N(E)} f(x^{-1}n^{-1}\delta g)\varphi(g)dgdn \\
&= \int_{N(E)\backslash G(\mathbb{A}_E)}\int_{[N]_E} \sum_{\delta\in N(E)} f(x^{-1}n^{-1}\delta g)\varphi(g)dndg \\
&= \int_{N(E)\backslash G(\mathbb{A}_E)}\int_{N(\mathbb{A}_E)} f(x^{-1}n^{-1}g)\varphi(g)dndg \\
&= 0
\end{aligned}$$

Where the third and the fifth equalities come from unfolding, and the last equality comes from the fact that

$$\int_{N(\mathbb{A}_E)} f(x^{-1}n^{-1}g)\varphi(g)dn = 0$$

since f_v is supercuspidal. $\square$

And we have the main theorem of this section. We define the *adelic twisted orbital integral* of $\phi \in C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}_E), \lambda^{-1})$ to be

$$TO_{\delta\theta}^{G_{\mathrm{ad}}(\mathbb{A}_E)}(\phi) = \int_{G_{\mathrm{ad},\delta\theta}(\mathbb{A}_E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g^{-1}\delta\theta(g))\, dg.$$

We notice that the above twisted orbital integral converges since ϕ is compactly supported on $G_{\mathrm{ad}}(\mathbb{A}_E)$.

Theorem 9.2.2. *Under the assumptions (1)(2)(3) above, the operator $R(\phi)$ sends L^2 automorphic* *forms into cusp forms, and*

$$tr\,(R_{cusp}(\phi)\circ I_\theta)=\sum_\delta \tau(G_{\mathrm{ad},\delta\theta})TO_{\delta\theta}^{G_{\mathrm{ad}}(\mathbb{A}_E)}(\phi) \tag{9.2.1}$$

where δ runs over the θ-conjugacy classes of elements of $G_{\mathrm{ad}}(E)$ with strongly elliptic regular norms, and the group $G_{\mathrm{ad},\delta\theta}$ is the θ-centralizer of δ. And $\tau(G_{\mathrm{ad},\delta\theta}) = vol\,(G_{\mathrm{ad},\delta\theta}(E)\backslash G_{\mathrm{ad},\delta\theta}(\mathbb{A}_E))$.

Proof. From the lemma above we know that

$$\mathrm{tr}(R_{\text{cusp}}(\phi)\circ I_\theta)=\mathrm{tr}(R(\phi)\circ I_\theta)$$

and we obtain the latter trace by integrating along the diagonal $K_\phi(g,\theta(g))$ of the kernel associated to $R(\phi)\circ I_\theta$, whence

$$\begin{aligned}\mathrm{tr}\,(R_{\text{cusp}}(\phi)\circ I_\theta) &= \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} K_\phi(g,\theta(g))\,dg\\ &= \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \sum_{\delta\in G_{\mathrm{ad}}(E)} \phi(g^{-1}\delta\theta(g))dg.\end{aligned}$$

Lemma 9.2.3. *Only those δ with $N\delta$ strongly regular elliptic in $G_{\mathrm{ad}}(E)$ will appear in the summation above.*

Proof. At the place v_2, $E_{v_2}\cong F_{v_2}\oplus\cdots\oplus F_{v_2}$ (r factors) with the Galois group acting by cyclic permutations.

Let $\delta\in G(E)$, the image of δ in the completion E_{v_2} is of the form $\delta=(\delta_1,\delta_2,\ldots,\delta_r)$ with

$\delta_i \in G(F_{v_2})$. If $g = (g_1, \ldots, g_r) \in G(E_{v_2})$, we have

$$g^{-1}\delta\theta(g) = (g_1^{-1}\delta_1 g_2, g_2^{-1}\delta_1 g_3, \ldots, g_r^{-1}\delta_1 g_1)$$

Assume that $\phi_{v_2}(g^{-1}\delta\theta(g)) \neq 0$. By assumption (3), we have $g_i^{-1}\delta_1 g_{i+1} \in \Omega_i$, taking the product we have:

$$g_1^{-1}(\delta_1\delta_2 \ldots \delta_r)g_1 \in \Omega_1\Omega_2 \ldots \Omega_r$$

This shows that $N\delta = (\delta_1 \ldots \delta_r, \ldots, \delta_r\delta_1 \ldots \delta_{r-1})$ is strongly regular elliptic as an element in $G_{\mathrm{ad}}(E_{v_2})$, hence it is strongly regular elliptic as a global element. Therefore, in the sum appearing in the expression of $\mathrm{tr}(R(\phi) \circ I_\theta)$, only elements with strongly regular elliptic norms in the adjoint group will appear. $\square$

We still need to swap the order of the integration and the summation.

Lemma 9.2.4. *The function*

$$F(g) = \sum_{\substack{\delta \in G_{\mathrm{ad}}(E),\\ N\delta \text{ strongly elliptic regular}}} |\phi(g^{-1}\delta\theta(g))|$$

is compactly supported on $G(E)Z(\mathbb{A}_E)\backslash G(\mathbb{A}_E)$.

Proof. If $\phi(g^{-1}\delta\theta(g)) \neq 0$, then $g^{-1}\delta\theta(g) \in C = \mathrm{Supp}(\phi) \subset G(\mathbb{A}_E)$, hence $\delta \in gC\theta(g^{-1})$. We first see that the finite for g in a compact set, since those δ's will be inside the intersection of a compact group in $G(\mathbb{A}_E)$ with the discrete subgroup $G(E)$. Taking norms, we get

$$N\delta \in gC\theta(C) \ldots \theta^{r-1}(C)g^{-1},$$

with $C\theta(C)\ldots\theta^{r-1}(C)$ compact. Henniart ([Hen83], Appendice 2) shows that the set of g satisfying this condition for some elliptic regular δ (in place of $N\delta$) is compact modulo $G(E)Z(\mathbb{A}_E)$. This proves the lemma. $\square$

We regroup the summation by θ-conjugacy by $G_{\mathrm{ad}}(E)$:

$$\mathrm{tr}\,(R_{\text{cusp}}(\phi)\circ I_\theta) = \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \sum_{\substack{\delta'\in G_{\mathrm{ad}}(E)\\ \delta'\sim\delta \text{ by } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \phi(g^{-1}\delta'\theta(g))dg,$$

which implies the final result of the theorem by the following manipulations of integration "in stages":

$$\begin{aligned}
\mathrm{tr}\,(R_{\text{cusp}}(\phi)\circ I_\theta) &= \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \sum_{\substack{\delta'\in G_{\mathrm{ad}}(E)\\ \delta'\sim\delta \text{ by } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \phi(g^{-1}\delta'\theta(g))dg \\
&= \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \mathrm{vol}(G_{\mathrm{ad},\delta\theta}(E))^{-1} \int_{G_{\mathrm{ad}}(E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \int_{G_{\mathrm{ad}}(E)} \phi(g^{-1}v^{-1}\delta\theta(v)\theta(g))\,dv\,dg \\
&= \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \mathrm{vol}(G_{\mathrm{ad},\delta\theta}(E))^{-1} \int_{G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g^{-1}\delta\theta(g))dg \\
&= \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \mathrm{vol}(G_{\mathrm{ad},\delta\theta}(E))^{-1}\,\mathrm{vol}(G_{\mathrm{ad},\delta\theta}(\mathbb{A}_E)) \int_{G_{\mathrm{ad},\delta\theta}(\mathbb{A}_E)\backslash G_{\mathrm{ad}}(\mathbb{A}_E)} \phi(g^{-1}\delta\theta(g))dg \\
&= \sum_{\substack{\delta\in G_{\mathrm{ad}}(E)\\ N\delta \text{ strongly ell. reg.}\\ \text{up to } G_{\mathrm{ad}}(E)\text{-}\theta\text{-conjugacy}}} \mathrm{vol}(G_{\mathrm{ad},\delta\theta}(E)\backslash G_{\mathrm{ad},\delta\theta}(\mathbb{A}_E))\, TO_{\delta\theta}^{G_{\mathrm{ad}}(\mathbb{A}_E)}(\phi)
\end{aligned}$$

Thus the theorem is proved. $\square$

In the special case of $E = F$, where $r = 1$ and $\theta = \mathrm{id}$, we denote $r(f)$ to be the representation of $f \in C_c^\infty(G_{\mathrm{ad}}(\mathbb{A}), \lambda^{-1})$ on the space $L^2(G_{\mathrm{ad}}(\mathbb{A}), \lambda)$, and choose f similarly as in the twisted case, by the same process, we will have

$$\mathrm{tr}\,(r_{\text{cusp}}(f)) = \sum_{\gamma} \tau(G_{\mathrm{ad},\gamma}) O_{\gamma}^{G_{\mathrm{ad}}(\mathbb{A})}(f), \tag{9.2.2}$$

where the sum γ is over the set of conjugacy classes of regular elliptic elements in $G_{\mathrm{ad}}(F)$ and $\tau(G_{\mathrm{ad},\gamma}) := \mathrm{vol}(G_{\mathrm{ad},\gamma}(F)\backslash G_{\mathrm{ad},\gamma}(\mathbb{A}))$. Similarly the *adelic orbital integral* is defined to be

$$O_{\gamma}^{G_{\mathrm{ad}}(\mathbb{A})}(f) = \int_{G_{\mathrm{ad},\gamma}(\mathbb{A})\backslash G_{\mathrm{ad}}(\mathbb{A})} f(g^{-1}\gamma g)\, dg$$

In the next section, we will perform the so-called *stabilizations* of the trace formulas 9.2.1 and 9.2.2 in order to compare them effectively.

However, before doing that, we will want to examine further the relation between various objects on the group and adjoint group in this setting.

9.3 Passing Between the Adjoint Group and the Group

The trace formulas are given in terms of (twisted) orbital integrals on the adjoint group G_{ad}, however, since our functions actually are defined on G, later we would like to make statements about integrals on G. Therefore, it seems necessary to relate various integrals on the group and the adjoint group. In 9.3, we assume that G is a connected reductive group defined over a local or global field F with $G_{\mathrm{der}} = G_{\mathrm{sc}}$. We assume that E/F is a cyclic extension of degree r, and let $\theta \in \mathrm{Gal}(E/F)$ be a generator. We use the same θ to denote the automorphism on the group. We assume that $Z(G)$ is an induced torus over F (in particular, it is connected). We also assume the Haar measure dg on $G(F)$ is normalized so that $\mathrm{vol}(Z(E)/Z(F)) = 1$. Denote $G_{\mathrm{ad}} := G/Z$, therefore $G_{\mathrm{ad}}(E) = G(E)/Z(E)$ and $G_{\mathrm{ad}}(F) = G(F)/Z(F)$.

Assume $\lambda : Z(F) \to \mathbb{C}^\times$ is a smooth character trivial on the maximal compact subgroup $Z(F)_1 \subset Z(F)$. Define $\lambda N : Z(E) \to \mathbb{C}^\times$ as before, and we still denote it by λ by abuse of notations. Let $\phi \in C_c^\infty(G_{\mathrm{ad}}(E), \lambda^{-1})$ and $f \in C_c^\infty(G_{\mathrm{ad}}(F), \lambda^{-1})$ with similar definitions as in 8.1. Let $\gamma \in G(F)$ be a semisimple element.

Lemma 9.3.1. *(a) γ is regular in $G(F)$ if and only if $\bar{\gamma}$ is regular in $G_{\mathrm{ad}}(F)$;*

(b) γ is elliptic in $G(F)$ if and only if $\bar{\gamma}$ is elliptic in $G_{\mathrm{ad}}(F)$;

(c) If $\bar{\gamma}$ is strongly regular in $G_{\mathrm{ad}}(F)$, then so is γ in $G(F)$.

Proof. From the isomorphisms $G_\gamma/Z \cong G^\circ_{\mathrm{ad},\bar{\gamma}}$ and $G_\gamma(F)/Z(F) \cong G^\circ_{\mathrm{ad},\bar{\gamma}}(F)$, we know that (a)(b) are immediate as regularity and ellipticity have nothing to do with the connectedness of the centralizers. (c) also follows since $G_\gamma = G_\gamma^\circ$ for every semisimple $\gamma \in G(F)$. However we don't even need to assume $G_{\mathrm{der}} = G_{\mathrm{sc}}$, since $G_\gamma/Z \cong G^\circ_{\mathrm{ad},\bar{\gamma}} = G_{\mathrm{ad},\bar{\gamma}}$, and since Z is connected, G_γ is also connected. □

As an easy corollary, let $\delta \in G(E)$ be a θ-semisimple element. We have the identity $N\bar{\delta} = \overline{N\delta}$. Therefore, we get the same statements by replacing regular (resp., elliptic, strongly regular) with θ-regular (resp., θ-elliptic, θ-strongly regular) and replace γ (resp. F) with δ (resp. E).

Remark 9.3.2. *We remark that it is not necessary for $\bar{\gamma}$ to be strongly regular even if γ is strongly regular. Let F be a field of char $F \neq 2$. Consider the matrix*

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in GL_2(F).$$

It is strongly regular semisimple in $GL_2(F)$. However, one can calculate that $(PGL_2)_\gamma(F) = T \rtimes (\mathbb{Z}/2\mathbb{Z})$, where T is the one-dimensional maximal split torus consists of diagonal matrices in PGL_2 and the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ is represented by the matrix

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Lemma 9.3.3. *The projection map $\pi : G(F) \to G_{\mathrm{ad}}(F)$ restricts to a surjection*

$$\pi : \{\textit{regular elliptic semisimple elements in } G(F)\} \to \{\textit{regular elliptic semisimple elements in } G_{\mathrm{ad}}(F)\}$$

Moreover, the inverse image of

$$\{\textit{strongly regular elliptic semisimple elements in } G_{\mathrm{ad}}(F)\}$$

is dense in the left hand side (in the analytic topology[1] of $G(F)$).

Proof. The surjection is from Lemma 9.3.1(a)(b). Since we have the surjection statement. For the density statement, we fix a maximal F-torus T and it is clear that we only to prove the density statement for T, which follows from the following lemma. ☐

Lemma 9.3.4. *Let T be a maximal F-torus splits over a finite extension. Then the inverse image of*

$$U = \{\textit{strongly regular elliptic elements in } T_{\mathrm{ad}}(F)\}$$

[1]Density in Zariski topology can be similarly shown as in §2.5 in [Hum95], however density in Zariski topology in general does not automatically imply density in analytic topology.

is dense in the set of regular elliptic elements in $T(F)$, in the analytic topology on $T(F)$.

Proof. Let $X = \{\text{regular elliptic elements in } T(F)\}$ and let $Y = \pi^{-1}(U) \subset X$. We claim that $X\backslash Y$ has no interior point in X, therefore the lemma is proved.

Let $W = N_G(T)/T$ absolute Weyl group and let Φ be the roots relative to T. Both of them are finite. The condition of $y \in Y$ can be described as:

$$\alpha(y) \neq 1 \text{ for all } \alpha \in \Phi$$

$$w(y) \neq y \text{ for all } 1 \neq w \in W$$

$$\alpha(w(y)y^{-1}) \neq 1 \text{ for some } \alpha \in \Phi \text{ and for all } 1 \neq w \in W$$

Therefore, we see that the vanishing locus $X\backslash Y$ is given by a finite number of locally analytic equations. Therefore, locally they are given in terms of a finite number of convergent power series over the non-archimedean field. Therefore they vanish on a set with no interior point, as desired. □

Now we can relate orbital integrals on both sides. Assume $\delta \in G(E)$ is such that $\bar{\delta}$ is θ-strongly regular, θ-elliptic and θ-semisimple in $G_{\text{ad}}(E)$, and $\gamma \in G(F)$ is such that $\bar{\gamma}$ is strongly regular semisimple in $G_{\text{ad}}(F)$.

Lemma 9.3.5. *We have*

$$TO_{\delta\theta}^{G(E)}(\phi) = TO_{\bar{\delta}\theta}^{G_{\text{ad}}(E)}(\phi) \tag{9.3.1}$$

and

$$O_{\gamma}^{G(F)}(f) = O_{\bar{\gamma}}^{G_{\text{ad}}(F)}(f) \tag{9.3.2}$$

Proof. We will just show 9.3.1, as 9.3.2 follows from the same calculations. Recall that $\phi \in C_c^\infty(G_{\text{ad}}(E), \lambda^{-1})$, therefore, the calculation is already done in 8.3.1 given our normalization that

$\mathrm{vol}(Z(E)/Z(F)) = 1.$ $\square$

Lemma 9.3.6. *Similarly, we have*

$$SO_{\delta\theta}^{G(E)}(\phi) = SO_{\bar{\delta}\theta}^{G_{\mathrm{ad}}(E)}(\phi) \tag{9.3.3}$$

and

$$SO_{\gamma}^{G(F)}(f) = SO_{\bar{\gamma}}^{G_{\mathrm{ad}}(F)}(f) \tag{9.3.4}$$

Proof. The arguments are exactly like those in the proof of Lemma 8.3.2 (iv), except it is easier in this case: since δ and $\bar{\delta}$ are θ-strongly regular, we don't have the $a(\delta')$ terms in the stable orbital integrals. They reduce to

$$SO_{\delta\theta}^{G(E)}(\phi) = \sum_{\delta'} e(\delta') TO_{\delta'\theta}^{G(E)}(\phi)$$

$$SO_{\bar{\delta}\theta}^{G_{\mathrm{ad}}(E)}(\phi) = \sum_{\bar{\delta}'} e(\bar{\delta}') TO_{\bar{\delta}'\theta}^{G_{\mathrm{ad}}(E)}(\phi)$$

respectly, where δ' (resp. $\bar{\delta}'$) ranges over θ-conjugacy classes in $G(E)$ (resp. $G_{\mathrm{ad}}(E)$) which are stably θ-conjugate to δ (resp. $\bar{\delta}$). The orbital integrals are equal by the previous lemma, and $e(\delta') = e(\bar{\delta}')$ again from [Kot83]. Finally, as in Lemma 8.3.2 (iv), the surjective map from the set of θ-conjugacy classes in $G(E)$ which are stably θ-conjugate to δ to the set of θ-conjugacy classes in $G_{\mathrm{ad}}(E)$ which are stably θ-conjugate to $\bar{\delta}$ has single fiber

$$\ker[H^1(F, G^{\circ}_{\mathrm{ad},\bar{\delta}'\theta}) \to H^1(F, G_{\mathrm{ad},\bar{\delta}'\theta})]$$

over $\bar{\delta}'$ since $G^{\circ}_{\mathrm{ad},\bar{\delta}'\theta} = G_{\mathrm{ad},\bar{\delta}'\theta}$ in this case, the lemma is proved. $\square$

The upshot of these is that, after stabilization in the next section, we might extract conditional identities of (stable) orbital integrals on G_{ad}, but with the relations we have proved in this subsection, we can regard them as identities on G, which are eventually what we desire.

Chapter 10: Stabilization of the Twisted Trace Formula

We follow [Clo90], section 6.2 closely with some changes along the way.

We fix the notations that will be used in this section. Let E/F be a unramified cyclic extension of global fields of degree r, and let $\theta \in \mathrm{Gal}(E/F)$ be a generator. If G is an F-group, then $\tilde{G} := \mathrm{Res}_{E/F} G_E$ is the F-group obtained by the restriction of scalars. We use the same notation θ to denote the F-linear automorphism of $\tilde{G}$ over F. We denote by $\mathbb{A}$ the adeles of F, and we denote by $\mathbb{A}^s$ the ring of adeles of F^s.

We make the assumption that G is a quasi-split, connected reductive group over F, such that G_{der} is simply connected. By the reduction steps, we also assume that $Z(G)$ is an induced torus over F and let $H := G/Z(G)$ to denote the adjoint group in this section, which we denote by G_{ad} before, to avoid the overflow of the superscripts. We will assume that H splits over an unramified extension K/F such that $E \subset K$. We may also assume that $\tilde{H}$ satisfies the *Hasse principle* for H^1, that is:

$$\ker^1(F, \tilde{H}) := \ker[H^1(F, \tilde{H}) \to \prod_v H^1(F_v, \tilde{H})] = 1.$$

We denote by $\mathcal{N}$ the abstract norm mapping sending regular semisimple elements of $\tilde{H}(F)$ to stable conjugacy classes of regular semisimple elements in $H(F)$, as well as its local versions.

10.1 The Construction of the Obstruction

Let $\gamma \in H(F)$ be a strongly regular semisimple element, then its centralizer T (resp., $\tilde{T}$) is a maximal torus of H (resp., $\tilde{H}$) over F. Assume there exists $\delta \in \tilde{H}(\mathbb{A})$ is such that $\mathcal{N}(\delta_v)$ is equal to the stable conjugacy class of $\gamma \in H(F_v)$ for every place v of F.

Let N denote the map $x \mapsto x\theta(x)\ldots\theta^{r-1}(x)$ from $\tilde{H}$ to itself. The map $N : \tilde{T}(F^s) \to T(F^s)$ is surjective, so we can choose $t \in \tilde{T}(F^s)$ such that $Nt = \gamma$. Since $\mathcal{N}(\delta_v) = \gamma$, for any place v of F, there is an element $g_v \in \tilde{H}(F_v^s)$ such that $g_v^{-1}\delta_v\theta(g_v) = t$. We may even assume that $g = (g_v) \in \tilde{H}(\mathbb{A}^s)$ by the following lemma.

Lemma 10.1.1. *Let G be a (possibly disconnected) reductive group over a global field F, and let $t \in G(F)$ be a semisimple element. Then outside of finitely many places v of F, for every $\delta \in G(\mathcal{O}_v)$ such that δ is conjugate to t under $G^\circ(F_v^s)$, there exists $y \in G^\circ(\mathcal{O}_v^{un})$ such that $yty^{-1} = \delta$.*

Here $\mathcal{O}_v$ is the valuation ring of the completion F_v of F at the place v, and $\mathcal{O}_v^{\mathrm{un}}$ is the valuation ring of the maximal unramified extension F_v^{un} of F_v.

Proof. We adapt the proof of Lemma 5 in [KR00]. Let X denote the conjugacy class of t under G°, and let $i : X \hookrightarrow G$ denote the inclusion, it is a closed immersion since t is semisimple. Let f denote the morphism $g \mapsto gtg^{-1}$ from G° to X. Then f is smooth and surjective. There exists an ideal $I \subset \mathcal{O}_F$, which is the product of finitely many prime ideals, such that G, G°, X, t, i and f come from objects over $\mathcal{O}_F[\frac{1}{I}]$. We may assume that f is smooth and surjective and i is a closed immersion over $\mathcal{O}_F[\frac{1}{I}]$ by replacing I with a suitable multiple.

Now we choose a place v of F outside of the finitely many places of I. By hypothesis $\delta \in G(\mathcal{O}_v) \cap X(F_v)$, which is equal to $X(\mathcal{O}_v)$, since i is a closed immersion. The fiber Y_δ of $f : G^\circ \to X$ over $\delta \in X(\mathcal{O}_v)$ is a smooth scheme of finite type over $\mathcal{O}_v$. The structural morphism $Y_\delta \to \mathrm{Spec}(\mathcal{O}_v)$ is surjective since it is the composite of the following surjective morphisms:

$$Y_\delta \longrightarrow \{\text{conjugacy class of } \delta \text{ under } G^\circ\} \longrightarrow \mathrm{Spec}(\mathcal{O}_v)$$

hence the special fiber Y_δ is non-empty, and it has a point in some finite extension of the residue field of $\mathcal{O}_v$. Therefore by the smoothness of Y_δ, it has a point in the valuation ring of some finite unramified extension of F_v. Hence $Y_\delta(\mathcal{O}_v^{\text{un}})$ is non-empty, which means that there exists $y \in G^\circ(\mathcal{O}_v^{\text{un}})$ such that $y\gamma y^{-1} = \delta$. $\square$

Remark 10.1.2. *Given the Lemma above, we set $G = \tilde{H} \rtimes \langle\theta\rangle$, hence $G^\circ = \tilde{H} \rtimes 1$. We can regard $t = (t, \theta) \in G(F)$ and similarly $\delta \in G^\circ(\mathbb{A})$. Outside of finitely many places, we have $\delta_v \in G^\circ(\mathcal{O}_v)$. In this way, we can translate the fact that δ_v and t are θ-conjugate under $\tilde{H}(F_v^s)$ to that they are conjugate under $G^\circ(F_v^s)$, therefore we can apply the lemma to get $g \in G^\circ(\mathbb{A}^s) = \tilde{H}(\mathbb{A}^s)$ such that*

$$g^{-1}\delta\theta(g) = t. \tag{10.1.1}$$

Consider the map $\tau \mapsto t_\tau = g^{-1}\tau(g) \in \tilde{H}(\mathbb{A}^s)$ for $\tau \in \mathrm{Gal}(F^s/F)$. We claim that t_τ defines a 1-cocycle of $\mathrm{Gal}(F^s/F)$ in $T(\mathbb{A}^s)\tilde{T}(F^s)$. Indeed, apply τ to 10.1.1 we get

$$\tau(g^{-1})\delta\theta(\tau(g)) = \tau(t). \tag{10.1.2}$$

from 10.1.1 and 10.1.2 we get

$$t_\tau\tau(t)\theta(t_\tau)^{-1} = t \tag{10.1.3}$$

Apply the map N on both sides, we get

$$t_\tau N(\tau(t))t_\tau^{-1} = Nt;$$

we have $Nt = \gamma$, and τ commutes with N since τ commutes with θ, hence we have $t_\tau\gamma t_\tau^{-1} = \gamma$. Therefore we have $t_\tau \in \tilde{T}(\mathbb{A}^s)$. Then from 10.1.3 we know that $t_\tau\theta(t_\tau)^{-1} = t\tau(t)^{-1} \in \tilde{T}(F^s)$, and we know that $u = t\tau(t^{-1})$ satisfies $Nu = 1$. Since θ acts on $\tilde{T}(F^s) = T(F^s) \times \cdots \times T(F^s)$ by cyclic permutation, this implies that $u = t_\tau \theta(t_\tau)^{-1} = t\tau(t)^{-1} = v\theta(v)^{-1}$ for some $v \in \tilde{T}(F^s)$. Then $t_\tau v^{-1}$ is fixed by θ, in other words, it belongs to $T(\mathbb{A}^s)$. This shows that $t_\tau \in T(\mathbb{A}^s)\tilde{T}(F^s)$. It is a 1-cocycle since by definition it is a coboundary in $\tilde{H}(\mathbb{A}^s)$.

Definition 10.1.3. *Let* $t_\tau := g^{-1}\tau(g)$ *for* $\tau \in \mathrm{Gal}(F^s/F)$. *We take* x_τ *to be the image of* t_τ *in* $T(\mathbb{A}^s)\tilde{T}(F^s)/\tilde{T}(F^s) = T(\mathbb{A}^s)/T(F^s)$, *and define* $\mathrm{obs}(\delta)$ *to be the class of* (x_τ) *in* $H^1(F, T(\mathbb{A}^s)/T(F^s))$. *This definition does not depend on the choice of* g *or* t.

We have the following important property of obs(δ):

Lemma 10.1.4. $\mathrm{obs}(\delta) \in \mathrm{H}^1(\mathrm{F}, \mathrm{T}(\mathbb{A}^s)/\mathrm{T}(\mathrm{F}^s))$ *is trivial if and only if* δ *is* θ*-conjugate under* $\tilde{H}(\mathbb{A})$ *to an element of* $\tilde{H}(F)$.

Proof. We first show that obs$(\delta) = 1$ under the assumption:

$$x^{-1}\delta\theta(x) = \delta_1 \in \tilde{H}(F) \tag{10.1.4}$$

for some $x \in \tilde{H}(\mathbb{A})$. This and the fact that $g^{-1}\delta\theta(g) = t$ for $g \in \tilde{H}(\mathbb{A}^s)$ imply that t and δ_1 are θ-conjugate under $\tilde{H}(\mathbb{A}^s)$, therefore under $\tilde{H}(F^s)$ by the following lemma since they are rational over F^s. Write $t = y^{-1}\delta_1\theta(y)$ for $y \in \tilde{H}(F^s)$, hence from 10.1.4 we have $t = (xy)^{-1}\delta\theta(xy)$. Let $g = xy$, we have $t_\tau = g^{-1}\tau(g) = y^{-1}\tau(y)$ since x is rational over $\mathbb{A}$. The construction above gives that $t_\tau \in T(\mathbb{A}^s)\tilde{T}(F^s)$ and the fact that y is rational over F^s imply that $t_\tau \in \tilde{T}(F^s)$, therefore its image x_τ is trivial.

For the converse, assume that $g^{-1}\tau(g) = \gamma^{-1}\tau(\gamma)u_\tau$ for some $\gamma \in T(\mathbb{A}^s)$ and $u_\tau \in \tilde{T}(F^s)$. We can replace g by $g\gamma^{-1}$ and assume that $g^{-1}\tau(g) = u_\tau$. Hence the cocycle $\tau \to u_\tau \in \tilde{H}(\mathbb{A}^s)$

is a coboundary in $\tilde{H}(\mathbb{A}^s)$, hence it is also a coboundary in $\tilde{H}(F^s)$ by the Hasse principle for $\tilde{H}$. Therefore $g^{-1}\tau(g) = \eta^{-1}\tau(\eta)$ for some $\eta \in \tilde{H}(F^s)$. Hence $x^{-1} := \eta g^{-1} \in \tilde{H}(\mathbb{A})$. We then have $x^{-1}\delta\theta(x) = \eta g^{-1}\delta\theta(g)\theta(\eta^{-1}) = \eta t\theta(\eta^{-1}) \in \tilde{H}(\mathbb{A}) \cap \tilde{H}(F^s) = \tilde{H}(F)$. $\square$

Lemma 10.1.5. *Let K be a global field and H be a connected reductive group over K. Let θ be an K-linear automorphism of H. Let $a, b \in H(K)$ such that they are strongly θ-regular semisimple and they are θ-conjugate under $H(\mathbb{A}_K)$, where $\mathbb{A}_K$ is the ring of adeles of K, then they are θ-conjugated under $H(K^s)$, where K^s is a fixed separable closure of K.*

Proof. Indeed, the conditions imply that the algebraic variety $X := \{h \in H : h^{-1}a\theta(h) = b\}$ has a point in K_v^s, where v is a place of K and K_v^s is a separable closure of the completion K_v. Since X is defined by algebraic equations over K as a, b are rational over K, this implies that $X(\bar{K}) \neq \emptyset$, where $\bar{K}$ is an algebraic closure containing K^s. Since X is isomorphic to $H_{a\theta}$ as schemes and the latter is geometrically reduced, since $H_{a\theta}$ is a connected torus, we have $X(K^s) \neq \emptyset$, as desired. $\square$

10.2 Pre-stabilization

Now we assume that T is an F-torus of H of the form $T = H_\gamma$ for a strongly regular semisimple element $\gamma \in H(F)$ (recall that γ is strongly regular means that $T = H_\gamma$ is a maximal torus, in particular, connected). We set $D = H/H_{\text{der}}$ and $\tilde{D} = \tilde{H}/\tilde{H}_{\text{der}}$, We have the following commutative diagram

$$\begin{tikzcd} T \arrow[r, hook] \arrow[d, two heads] & \tilde{T} \arrow[d, two heads] \\ D \arrow[r, hook] & \tilde{D} \end{tikzcd}$$

Dually, it becomes

$$\begin{tikzcd}
\hat{\tilde{D}} \arrow[r, hook] \arrow[d, two heads] & \hat{\tilde{T}} \arrow[d, two heads] \\
\hat{D} \arrow[r, hook] & \hat{T}
\end{tikzcd}$$

Those maps are all Γ-equivariant.

We define the finite abelian groups

$$A(T/F) = \pi_0(\hat{T}^\Gamma)/\mathrm{Im}\,\pi_0(\hat{\tilde{D}}^\Gamma), \tag{10.2.1}$$
$$A(T/F_{v_1}) = \pi_0(\hat{T}^{\Gamma_1})/\mathrm{Im}\,\pi_0(\hat{\tilde{D}}^{\Gamma_1}).$$

Where $\Gamma_1 = \mathrm{Gal}(F^s_{v_1}/F_{v_1})$ is the decomposition group at the place v_1 of F.

The finite Abelian groups $\pi_0(\hat{T}^\Gamma)$ and $H^1(F, T(\mathbb{A}^s)/T(F^s))$ are canonically dual by [Kot84]. Let $<,>$ be the pairing between them with value in $\mathbb{C}^\times$. We claim that obs(δ) has trivial image under the composition

$$H^1(F, T(\mathbb{A}^s)/T(F^s)) \xrightarrow{\sim} \pi_0(\hat{T}^\Gamma)^* \to \pi_0(\hat{\tilde{D}}^\Gamma)^*$$

Indeed, since $\pi_0(\hat{\tilde{D}}^\Gamma)^*$ is canonically dual to $H^1(F, \tilde{D}(\mathbb{A}^s)/\tilde{D}(F^s))$, so we only need to find the image of obs(δ) under the map

$$H^1(F, T(\mathbb{A}^s)/T(F^s)) \to H^1(F, \tilde{D}(\mathbb{A}^s)/\tilde{D}(F^s)),$$

since obs(δ) was constructed to be the class of $t_\tau = g^{-1}\tau(g)$ for some $g \in \tilde{H}(\mathbb{A}^s)$, it splits when we project t_τ to $\tilde{D}(\mathbb{A}^s)$. Hence for $\kappa \in A(T/F)$, we can further define the pairing $\langle \mathrm{obs}(\delta), \kappa\rangle \in \mathbb{C}$. We have

$$\mathrm{obs}(\delta) = 1 \Leftrightarrow \langle \mathrm{obs}(\delta), \kappa\rangle = 1 \text{ for all } \kappa \in A(T/F).$$

We now begin the stabilization of the right hand side of the trace formula 9.2.1

$$\sum_{\delta} \tau(H_{\delta\theta}) TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi) \tag{10.2.2}$$

where we recall that the sum is over $\delta \in H(E)$ up to θ-conjugacy under $H(E)$ with $N\delta$ regular elliptic. We also recall that $\phi \in C_c^\infty(H(\mathbb{A}_E), \lambda^{-1})$.

The global twisted orbital integral $TO_{\delta\theta}^{H(\mathbb{A}_E)}$ of ϕ only depends on the $H(\mathbb{A}_E)$-θ-conjugacy class of δ. We have the following lemma which allows us to regroup the elements in the summation:

Lemma 10.2.1. *The number of the terms in the sum that are indexed by the $H(\mathbb{A}_E)$-θ-conjugates of δ is given by*

$$|\ker(\ker^1(F, H_{\delta\theta}) \to \ker^1(F, \tilde{H}))| = |\ker^1(F, H_{\delta\theta})|$$

where we have used the Hasse principle for $\tilde{H}$.

Proof. We just need to look at the following diagram:

$$\begin{tikzcd} H^1(F, H_{\delta\theta}) \arrow[r, "f"] \arrow[d, "g"] & \prod_v H^1(F_v, H_{\delta\theta}) \arrow[d, "g'"] \\ H^1(F, \tilde{H}) \arrow[r, "f'"] & \prod_v H^1(F_v, \tilde{H}) \end{tikzcd}$$

Assume δ' is θ-conjugate to δ under $H(\mathbb{A}_E)$ but not under $H(E)$, then they are θ-conjugate under $H(E^s)$ by Lemma 10.1.5, therefore corresponds to a cocycle $\tau \in \ker(g)$, which image $f(\tau)$ in $\prod_v H^1(F_v, H_{\delta\theta})$ is trivial by assumption, that is to say, $\tau \in \ker(\ker^1(F, H_{\delta\theta}) \to \ker^1(F, \tilde{H}))$. Conversely, if $\tau \in \ker(H^1(F, H_{\delta\theta}) \to H^1(F, \tilde{H}))$, then we can associated a $\delta' \in H(E)$ that is stably θ-conjugate to δ. The condition $f(\tau) = 0$ guarantees that δ' and δ are θ-conjugate under $H(\mathbb{A}_E)$. $\square$

Therefore we can write 10.2.2 as

$$\sum_{\delta}\tau(H_{\delta\theta})|\ker^1(F,H_{\delta\theta})|TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi) \tag{10.2.3}$$

where the sum is over $\delta\in H(E)$ up to θ-conjugacy under $H(\mathbb{A}_E)$ with $N\delta$ strongly elliptic regular.

Now we consider a strongly regular elliptic element $\gamma\in H(F)$ up to stable conjugacy, in other words, conjugacy under $H(F^s)$ since γ is strongly regular. Assume $\delta\in H(\mathbb{A}_E)$ is such that its norm $\mathcal{N}\delta$ is equal to the stable class of γ at every place. The value of

$$\frac{1}{|A(T/F)|}\sum_{\kappa\in A(T/F)}\langle \mathrm{obs}(\delta),\kappa\rangle$$

is 1 when δ is θ-conjugate to an element of $H(E)$ under $H(\mathbb{A}_E)$, by Lemma 10.1.4, and 0 otherwise since $A(T/F)$ is a finite abelian group. Use this, we can write 10.2.3 as

$$\sum_{\substack{\gamma\in H(F)\text{ strongly ell. reg.}\\ \text{up to } H(F^s)-\text{conj}}}\tau(T)|\ker^1(F,T)|\sum_{\substack{\delta\in H(\mathbb{A}_E)\\ \mathcal{N}\delta=\gamma\\ \text{up to } H(\mathbb{A}_E)-\theta-\text{conjugacy}}}[\frac{1}{|A(T/F)|}\sum_{\kappa\in A(T/F)}\langle \mathrm{obs}(\delta),\kappa\rangle]TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi) \tag{10.2.4}$$

The equation $\mathcal{N}\delta=\gamma$ holds for every place v of E. And we write $T=H_\gamma$ for the centralizer for γ, which is isomorphic to $H_{\delta\theta}$ if δ is a global element of norm γ.

In order to swap the order of the outer summations, we claim that the triple sum has only finitely many nonzero terms on any compact subset.

Lemma 10.2.2. *Fix a compact set C of $H(\mathbb{A}_E)$, then there are only finitely many θ-conjugacy classes δ under $H(\mathbb{A}_E)$ that meet C and such that $\mathcal{N}\delta$ is the class of a regular semisimple element of $H(F)$.*

Proof. Following the same proof as in [Kot86b], we can reduce to case that $N\delta$ is stably conjugate to a fixed regular semisimple element γ of $H(F)$. We can find $\delta_0 \in H(E)$ such that $N\delta_0 = \gamma$. Hence δ_0 and δ are stably θ-conjugate. We assume that C is contained in $H(\mathcal{O}_{\mathbb{A}_E^S}) \times H(\prod_{v\in S} E_v)$, where S is a finite set of fixed places of E. We enlarge S to apply Lemma 10.1.1. Now for any $\delta \in C$ such that $\delta_{0,v}$ is stably θ-conjugate to δ_v for any place $v \notin S$, we view δ and δ_0 as (δ, θ) and (δ_0, θ) in the disconnected group $H \ltimes \langle\theta\rangle$, respectively. Hence (δ, θ) is conjugate to (δ_0, θ) under $H(E_v^s) = (H \ltimes \langle\theta\rangle)^\circ(E_v^s)$ in this sense. By Lemma 10.1.1, there exists $g \in H(\mathcal{O}_v^{\mathrm{un}})$ such that δ and δ_0 are conjugate under g. We claim that we can actually assume $g \in H(O_v)$.

Indeed, consider the cocycle $\tau_g : \theta \mapsto g^{-1}\theta(g)$, here we use θ to denote the topological generator of $\mathrm{Gal}(F^{\mathrm{un}}/F)$. It is clear that τ_g is a 1-cocycle in $\mathcal{T}(\mathcal{O}_v^{\mathrm{un}}) := H_\gamma(\mathcal{O}_v^{\mathrm{un}})$, by applying θ on the equation $g^{-1}\delta_v g = \delta_{0,v}$. Here we use the existence of lft Neron model $\mathcal{T}$ associated to $T = H_\gamma$ over $\mathcal{O}_v^{\mathrm{un}}$. Hence $\tau_g \in H^1(\langle\theta\rangle, \mathcal{T}(\mathcal{O}_v^{\mathrm{un}}))$. It is trivial by (7.6.1) of [Kot97], hence there exists $t \in \mathcal{T}(\mathcal{O}_v^{\mathrm{un}})$ such that $g^{-1}\theta(g) = t^{-1}\theta(t)$, in other words, $gt^{-1} \in H(\mathcal{O}_v^{\mathrm{un}}) \cap H(F_v) = H(O_v)$. Replace g by gt^{-1}, the claim is proved.

Now for every θ-conjugacy class δ that meet C and $\mathcal{N}\delta = \gamma$, we know for $v \notin S$, the θ-conjugacy class of δ_v contains $\delta_{0,v}$, and for every place $w \in S$, there are only finitely many θ-conjugacy classes inside a fixed stable θ-conjugacy class. Therefore, there are only finitely many such θ-conjugacy classes δ satisfying those conditions when $C \subset H(\mathbb{A}_E)$. □

Therefore, we can reorder the sum in 10.2.4 as

$$\sum_\gamma \tau(T)\frac{|\ker^1(F,T)|}{|A(T,F)|}\sum_\kappa\sum_\delta\langle \mathrm{obs}(\delta),\kappa\rangle TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi). \tag{10.2.5}$$

Where each index is summing over the same set as in 10.2.4.

10.3 Vanishing of κ-Orbital Integrals

Consider the inner sum in 10.2.5

$$\sum_{\delta} \langle \mathrm{obs}(\delta), \kappa \rangle TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi) \tag{10.3.1}$$

where the summation is over $H(\mathbb{A}_E)$-θ-conjugacy classes of $\delta \in H(\mathbb{A}_E)$ such that $\mathcal{N}\delta = \gamma$ locally everythere, for an elliptic strongly regular $\gamma \in H(F)$ up to conjugacy by $H(F^s)$.

We assume that there is a θ-elliptic, θ-strongly regular $\delta^0 \in H(\mathbb{A}_E)$ such that $\mathcal{N}\delta^0 = \gamma$ holds. Otherwise, the sum in 10.3.1 would be empty. Therefore, the summation 10.3.1 will be over the global θ-conjugacy classes within the global stable θ-conjugacy class of δ^0, by Proposition 2.2.5.

We know that the θ-conjugacy classes within the stable θ-conjugacy class of δ^0_v, the local component of δ^0 at v, are in bijection with

$$\mathcal{D}_\theta(I_v/F_v) := \ker(H^1(F_v, I_v) \to H^1(F_v, \tilde{H})),$$

where I_v is the θ-centralizer of δ^0_v in $\tilde{H}_{F_v}$. We notice that $I_v = I_v^\circ$ by assumptions on δ^0.

The next goal is to write 10.3.1 as a product of local sums. If $\delta = (\delta_v)$ is in the (global) stable θ-conjugacy class of δ_0, by above we can write $(\delta_v) = (x_v \cdot \delta^0_v)$ where $x = (x_v) \in \bigoplus_v \mathcal{D}_\theta(I_v/F_v)$. Let $\mathrm{inv}(\delta, \delta^0) \in H^1(F, T(\mathbb{A}^s)/T(F^s))$ be the image of (x_v) under the maps

$$\bigoplus_v \mathcal{D}_\theta(I_v/F_v) \to \bigoplus_v H^1(F_v, T(F_v)) \to H^1(F, T(\mathbb{A}^s)/T(F^s)),$$

where the first map is given by the canonical isomorphisms between I_v and $T(F_v)$, where $T = H_\gamma$.

From the definitions of $\mathrm{obs}(\delta)$ and $\mathrm{inv}(\delta, \delta^0)$, we see that

$$\mathrm{obs}(\delta) = \mathrm{inv}(\delta, \delta^0)\mathrm{obs}(\delta^0).$$

Therefore, assuming $\phi = \bigotimes_v \phi_v$ (over places of F) is a pure tensor product element in $C_c^\infty(H(\mathbb{A}_E)) \cong C_c^\infty(\tilde{H}(\mathbb{A}_F))$, therefore we can write 10.3.1, when the sum is nonempty, as

$$\langle \mathrm{obs}(\delta^0), \kappa \rangle \prod_v \Big(\sum_{x_v \in \mathcal{D}_\theta(I_v/F_v)} \langle x_v, \kappa_v \rangle TO^{H(E_v)}_{x_v \cdot \delta_v^0, \theta}(\phi_v) \Big) \tag{10.3.2}$$

which we write as

$$\langle \mathrm{obs}(\delta^0), \kappa \rangle \prod_v O^{\kappa_v}_{\delta_v^0, \theta}(\phi_v) \tag{10.3.3}$$

where we denote the κ_v-*twisted orbital integral*,

$$O^{\kappa_v}_{\delta_v^0, \theta}(\phi_v) := \sum_{x_v \in \mathcal{D}_\theta(I_v/F_v)} \langle x_v, \kappa \rangle TO^{H(E_v)}_{x_v \cdot \delta_v^0, \theta}(\phi_v),$$

and $\kappa_v \in \mathcal{D}_\theta(I_v/F_v)^*$ is the local image of κ.

We have arrived at the vanishing results of this section:

Lemma 10.3.1. *Assume $\phi = \bigotimes_v \phi_v$ is a pure tensor product element in $\phi \in C_c^\infty(H(\mathbb{A}_E), \lambda^{-1})$ over places v of F, and at some place v_1 of F, the following two conditions hold:*

(a) $TO^{H(E_v)}_{\delta_{v_1}, \theta}(\phi_{v_1}) = 0$ for all elements $\delta_{v_1} \in \tilde{H}(F_{v_1}) = H(E \otimes F_{v_1})$ that are not θ-elliptic, and for θ-elliptic θ-regular elements δ_{v_1}, the twisted orbital integral is constant on stable θ-conjugacy classes of δ_{v_1}.

(b) There exists a finite Galois extension L of F such that L_{v_1} is a field and L splits $\tilde{D}$.

Then

$$\sum_{\delta} \langle obs(\delta), \kappa \rangle TO_{\delta\theta}^{H(\mathbb{A}_E)}(\phi) = 0 \tag{10.3.4}$$

if κ is nontrivial in $A(T/F)$.

Remark 10.3.2. *If ϕ and H indeed satisfy the conditions in the lemma, the sum 10.2.5 would reduce to its stable part:*

$$\sum_{\gamma} \tau(T) \frac{|\ker^1(F,T)|}{|A(T,F)|} \sum_{\delta} TO_{\delta\theta}(\phi), \tag{10.3.5}$$

which means that the summation over κ only has $\kappa = 1$ left.

Proof. By the formula 10.3.2, it suffices to show that the component at v_1

$$\sum_{x_{v_1} \in \mathcal{D}(I_{v_1}/F_{v_1})} \langle x_{v_1}, \kappa \rangle TO_{x_{v_1} \cdot \delta_{v_1}^0, \theta}(\phi_{v_1}) = 0$$

By condition (a), it becomes

$$[\sum_{x_{v_1} \in \mathcal{D}_\theta(I_{v_1}/F_{v_1})} \langle x_{v_1}, \kappa \rangle] TO_{\delta_{v_1}^0, \theta}^{H(E_v)}(\phi_{v_1}) = 0$$

Hence it suffices to show that for $\kappa \neq 1$

$$\sum_{x_{v_1} \in \mathcal{D}_\theta(I_{v_1}/F_{v_1})} \langle x_{v_1}, \kappa \rangle = 0 \tag{10.3.6}$$

Recall that $\mathcal{D}_\theta(I_{v_1}/F_{v_1}) = \ker(H^1(F_{v_1}, I_{v_1}) \to H^1(F_{v_1}, \tilde{H}))$ is canonically dual to the finite abelian group $A(T/F_{v_1}) = \pi_0(\hat{T}^{\Gamma_1})/\mathrm{Im}\, \pi_0(\hat{\tilde{D}}^{\Gamma_1})$. Thus the lemma is proved once we know that the canonical map $A(T/F) \to A(T/F_{v_1})$ is injective.

Lemma 10.3.3. *Suppose there exists a finite Galois extension L of F such that L_{v_1} is a field and L splits $\tilde{D}$. Suppose T is a maximal F-torus in H which is elliptic at v_1, then the canonical map*

$$A(T/F) \to A(T/F_{v_1})$$

is injective.

Proof. As in the proof of Lemma 8.4.1 of [Hai09] and Lemma 6.5 in [Clo90], we simply notice that $\hat{\tilde{D}}^{\Gamma} = \hat{\tilde{D}}^{\Gamma_1} = \hat{\tilde{D}}^{\Gamma'}$, where $\Gamma' = \mathrm{Gal}(L_{v_1}/F_{v_1})$. ☐

The identity 10.3.6 follows from the character relations on the finite abelian group $A(T/F_{v_1})$ and its dual group. ☐

Functions with properties in Lemma 10.3.1(a) are called *generalized Kottwitz functions* (see [Lab99] 3.9 and [Hai09] 8.2). We will discuss them further when we move to the global setup in 11.2. However, the basic idea is pretty simple: we just pullback the generalized Kottwitz functions on the adjoint group $H(E)$ to $G(E)$, with the trivial central character.

10.4 End of Stabilization

From now on, we assume that ϕ satisfies the conditions of Lemma 10.3.1. For any elliptic strongly regular semisimple element $\gamma \in H(F)$, we have $(\hat{T}^{\Gamma})^{\circ} = (\hat{D}^{\Gamma})^{\circ}$, therefore there is an injection

$$\pi_0(\hat{T}^{\Gamma}) \hookrightarrow \pi_0(\hat{D}^{\Gamma}).$$

Thus we have

$$|A(T/F)| = \frac{|\pi_0(\hat{T}^\Gamma)|}{|\mathrm{Im}\,\pi_0(\hat{\hat{D}}^\Gamma)|} = \frac{|\pi_0(\hat{T}^\Gamma)|}{|\pi_0(\hat{D}^\Gamma)|}\frac{|\pi_0(\hat{D}^\Gamma)|}{|\mathrm{Im}\,\pi_0(\hat{\hat{D}}^\Gamma)|}$$

Under the assumptions on H and ϕ, we see that the elliptic regular term of the twisted trace formula 10.3.5 is equal to

$$c(H) \sum_{\substack{\gamma\in H(F) \text{ strongly ell. reg.} \\ \text{up to } H(F^s)-\text{conj}}} \tau(T)|\ker^1(F,T)|\frac{|\pi_0(\hat{D}^\Gamma)|}{|\pi_0(\hat{T}^\Gamma)|} \sum_{\substack{\delta\in H(\mathbb{A}_E) \\ \mathcal{N}\delta=\gamma \\ \text{up to } H(\mathbb{A}_E)-\theta-\text{conjugacy}}} TO_{\delta\theta}(\phi), \tag{10.4.1}$$

with

$$c(H) = \frac{|\mathrm{Im}\,\pi_0(\hat{\hat{D}}^\Gamma)|}{|\pi_0(\hat{D}^\Gamma)|}. \tag{10.4.2}$$

We assume H satisfies the Hasse principle for H^1, and that $f = \bigotimes_v f_v \in C_c^\infty(H(\mathbb{A}), \lambda^{-1})$ satisfies the analogs of conditions (a) and (b) in Lemma 10.3.1. In other words, we regard $E = F$ and $\theta = \mathrm{id}$. Then the elliptic strongly regular part of the trace formula for f, by the same stabilization process, reduces to

$$\sum_{\substack{\gamma\in H(F) \text{ strongly ell. reg.} \\ \text{up to } H(F^s)-\text{conj}}} \tau(T)|\ker^1(F,T)|\frac{|\pi_0(\hat{D}^\Gamma)|}{|\pi_0(\hat{T}^\Gamma)|} \sum_{\substack{\gamma'\in H(\mathbb{A}) \\ \gamma' \text{ stably conjugate to } \gamma \\ \text{up to } H(\mathbb{A})-\text{conjugacy}}} O_{\gamma'}(f). \tag{10.4.3}$$

Since the last sums in 10.4.1 and 10.4.3 are expressed locally over places v of F, as we discussed before, we see that if ϕ and f have matching stable orbital integrals at every place of F, the two expressions 10.4.1 and 10.4.3 agree up to the factor $c(H)$, that is:

Proposition 10.4.1. *Let $\phi = \bigotimes_v \phi_v \in C_c^\infty(H(\mathbb{A}_E), \lambda^{-1})$ and $f = \bigotimes_v f_v \in C_c^\infty(H(\mathbb{A}), \lambda^{-1})$ be functions satisfying the conditions in 9.2 (1)(2)(3) and Lemma 10.3.1 (a). Assume $\gamma \in G(F)$ and the* *image* $\bar{\gamma} \in H(F) = G_{\mathrm{ad}}(F)$ *are strongly elliptic regular in the respective groups,* $\delta \in G(\mathbb{A}_E)$ *such that* $\mathcal{N}\delta = \gamma$ *locally everywhere (therefore we also have* $\mathcal{N}\bar{\delta} = \bar{\gamma}$*, and* δ_v *(resp.* $\bar{\delta}_v$*) is* θ*-strongly regular* θ*-elliptic in* $G(E_v)$ *(resp.* $H(E_v)$*)). If we have for every place* v *of* F

$$SO_{\delta_v\theta}^{G(E_v)} = SO_{\gamma_v}^{G(E_v)}, \tag{10.4.4}$$

for every such $\delta \in G(\mathbb{A}_E)$ *and* $\gamma \in G(F)$ *then there exists a constant* $c > 0$ *such that*

$$\mathrm{tr}(\mathrm{R}_{cusp}(\phi) \circ \mathrm{I}_\theta) = \mathrm{c}\,\mathrm{tr}(\mathrm{r}_{cusp}(\mathrm{f})). \tag{10.4.5}$$

Proof. By the stabilization process in this section, we know that

$$\mathrm{tr}(R_{\text{cusp}}(\phi) \circ I_\theta) = 10.4.1$$

$$\mathrm{tr}(r_{\text{cusp}}(f)) = 10.4.3.$$

And we know that $10.4.1 = c(H) \cdot 10.4.3$ provided

$$SO_{\bar{\delta}_v\theta}^{H(E_v)} = SO_{\bar{\gamma}_v}^{H(E_v)}$$

for every place v of F. By Lemma 9.3.6, this is equivalent to 10.4.4, as desired. ∎

Chapter 11: Elementary Functions of Labesse

11.1 Definition

Following [Hai12], §8, we slightly change the notation in this section. Let F_r be an unramified extension of degree r over F. Denote A^{F_r} (resp., A) to be a maximal F_r-split (resp., F-split) torus in G, with centralizer T, let $B = TU$ be the F-rational Borel subgroup defining the dominant Weyl chamber in $X_*(A^{F_r}) \otimes \mathbb{R}$. Let ρ denote the half-sum of the B-positive roots of G.

Fix any uniformizer ϖ for the field F. Consider a regular dominant cocharacter $\nu \in X_*(A^{F_r})$ and set $u = \nu(\varpi) \in T(F_r)$. Let $\tau = N_r(\nu)$, also a regular dominant cocharacter in $X_*(A)$, and set $t = \tau(\varpi) \in T(F)$. Hence $t = N_r(u)$.

To any element $g \in G(F)$ we can associate an F-parabolic $P_g = M_g N_g$ with unipotent radical N_g and F-rational Levi factor M_g. We may construct the analogous objects associated to $u\theta \in G_r \rtimes \langle\theta\rangle$. The resulting subgroups $M_{u\theta}$ (resp. $P_{u\theta}$) of G can be characterized as the set of elements $g \in G$ such that $(u\theta)^n g (u\theta)^{-n}$ remains bounded as n ranges over all integers (resp. all positive integers). In our situation, since $(u\theta)^r = t$, we have $M_{u\theta} = M_t = T$ and $P_{u\theta} = P_t = B$.

Fix any F-Levi subgroup M. Write G_r (resp. M_r) for the set of F_r-points $G(F_r)$ (resp. $M(F_r)$). Then M_r acts on $G_r \times M_r$ on the left by $m_1(g, m) = (m_1 g, m_1 m \theta(m_1)^{-1})$. Let $[g, m]$ denote the equivalence class of (g, m) in the quotient space $M_r \backslash G_r \times M_r$. There is a natural morphism of analytic manifolds

$$M_r \backslash G_r \times M_r \to G_r$$
$$[g, m] \mapsto g^{-1} m \theta(g).$$

this morphism is known to be generically one-to-one and the normalized absolute value of its Jacobian at the point $[g, m]$ is $|D_{G(F)/M(F)}(\mathcal{N}(m))|_F$.

We apply this to our Levi subgroup $M_{u\theta} = T$. We are now concerned only with the quotient by $T(F_r)_1$ of the compact open subset $I_r \times T(F_r)_1 u \subset G_r \times T_r$. The action of $T(F_r)_1$ on $I_r \times T(F_r)_1 u$ is the restriction of the above action of T_r on $G_r \times T_r$. The map

$$\begin{aligned} T(F_r)_1 \backslash I_r \times T(F_r)_1 u &\to G_r \\ [k, mu] &\mapsto k^{-1} m u \theta(k). \end{aligned} \tag{11.1.1}$$

is injective from the following lemma and has absolute value of its Jacobian everywhere equal to the non-zero number

$$|\mathrm{Jac}_{[k,mu]}|_F = \delta_{B_r}^{-1}(u) = q^{r\langle 2\rho, \nu \rangle}.$$

Thus its image is a compact open subset, which we will denote $\mathfrak{O}_u$.

We define the elementary function ϕ_{u,χ_r} on G_r to be the function supported on $\mathfrak{O}_u$, given by the formula

$$\phi_{u,\chi_r}(k^{-1} m u \theta(k)) := \chi_r^{-1}(m)$$

for $k \in I_r$ and $m \in T(F_r)_1$.

Lemma 11.1.1. *Suppose $g_1, g_2 \in G(F^s)$ and $m_1, m_2 \in T(F_r)_1$. We extend θ to an element of $\mathrm{Gal}(F^s/F)$ and use the same symbol for the induced automorphism of $G(F^s)$. We have the following statements.*

(i) If $g_1^{-1} m_1 u \theta(g_1) = g_2^{-1} m_2 u \theta(g_2)$, then $g_1 \in T(F^s) g_2$.

(ii) The θ-centralizer of $m_1 u$ is T; hence $m_1 u$ is strongly θ-regular.

(iii) If $g_1 \in G_r$ and $g_1^{-1} m_1 u \theta(g_1)$ lies in the support of ϕ_u, then $g_1 \in T_r I_r$.

Proof. Let $g = g_1 g_2^{-1}$. It is enough to show that $g \in M_{u\theta} = T$, hence it suffices to show that $(u\theta)^n g (u\theta)^{-n}$ remains bounded as n ranges over $\mathbb{Z}$. Furthermore, it is enough to let n range only over multiples of r.

Let n be any positive integer. We have

$$g_1^{-1}(m_1 u\theta)^{rn} g_1 = g_2^{-1}(m_2 u\theta)^{rn} g_2,$$

and thus we have

$$[N_r(m_1)N_r(u)\theta^r]^n = g[N_r(m_2)N_r(u)\theta^r]^n g^{-1},$$

since recall that $m_1 u\theta$ and $m_2 u\theta$ are elements in $G_r \rtimes \langle\theta\rangle$. Since u commutes with m_1 and m_2, the terms in each set of brackets pairwise commute, so that we have

$$[N_r(u)\theta^r]^n g [N_r(u)\theta^r]^{-n} = N_r(m_1)^{-n} g N_r(m_2)^n.$$

The left hand side is just $(u\theta)^{rn} g (u\theta)^{-rn}$, the right hand side remains bounded since it is inside $T(F)_1 g T(F)_1$. This shows that $g \in M_{u\theta} = T$ and proves part (i).

Parts (ii) and (iii) follow from part (i). $\square$

We see that ϕ_{u,χ_r} is a smooth compactly-supported function on G_r, and it is invariant under θ-conjugation by elements in I_r by definition. Let $r = 1$ (hence $\theta = \mathrm{id}$ and $u = t$), we get the similar smooth compactly supported function $f_{t,\chi}$ on G, and it is invariant under conjugation by I. We have the following lemma.

Lemma 11.1.2. *The functions $\phi_{u,\chi_r} \in \mathcal{H}(G_r, \rho_r)$.*

Proof. The calculations in the proof of Lemma 8.1.3 in [Hai12] stay valid here. To show that $\mathrm{supp}(\phi_{u,\chi_r}) = \mathfrak{O}_u = I_r u I_r$, the proof in loc. cit. uses results from [Lab95], Prop IV.1.1, namely to show the map 11.1.1 has $I_r u I_r$, we show that the image of the map and $I_r u I_r$ have the same volume. The calculations of the volumes are still valid in positive characteristics. $\square$

11.2 Orbital Integrals of Elementary Functions

We normalize the Haar measure dg on G_r (resp. ds on T_r) so that I_r (resp. $T(F_r)_1$) has measure 1. We use this to define the quotient measures $d\bar{g}$ on $T(F)\backslash G_r$ (resp. $d\bar{s}$ on $T(F)\backslash T_r$).

Proposition 11.2.1. *The following statements hold.*

(1) For $\delta \in G_r$, $TO_{\delta\theta}(\phi_u)$ is non-zero if and only if δ is θ-conjugate in G_r to an element of the form mu where $m \in T(F_r)_1$. For such m we have

$$TO_{mu,\theta}(\phi_{u,\chi_r}) = \chi_r^{-1}(m)$$

(2) For $\gamma \in G$, $O_\gamma(f_t)$ is non-zero if and only if γ is conjugate in G to an element of the form $m_0 t$ where $m_0 \in T(F)_1$. For such m_0 we have

$$O_{m_0 t}(f_{t,\chi}) = \chi^{-1}(m_0)$$

Proof. It suffices to prove (1). If $TO_{\delta\theta}(\phi_{u,\chi_r}) \neq 0$, then δ is θ-conjugate under G_r to an element of the form mu, where $m \in T(F_r)_1$. Hence we can assume $\delta = mu$, in that case by lemma 7.4(ii), we know that $G^\circ_{\delta\theta}(F) = G^\circ_{mu\theta}(F) = T(F)$. Now if $g \in G_r$ is such that $g^{-1}mu\theta(g) \in \mathrm{supp}(\phi_{u,\chi_r})$, then we have $g \in T_r I_r$. Then using our choice of measures, we get

$$\int_{T(F)\backslash G_r} \phi_{u,\chi_r}(g^{-1}mu\theta(g))\, d\bar{g} = \int_{T(F)\backslash T_r} \phi_{u,\chi_r}(s^{-1}mu\theta(s))\, d\bar{s}$$
$$= \int_{T(F)\backslash T_r} \phi^{T_r}_{1,\chi_r}(s^{-1}m\theta(s))\, d\bar{s}$$

where $\phi^{T_r}_{1,\chi_r}$ denotes the elementary function on T_r associated to $u = 1$, which takes value $\chi_r^{-1}(m)$ on the image of the map

$$T(F_r)_1\backslash T(F_r)_1 \times T(F_r)_1 \to T_r$$
$$[k, m] \mapsto k^{-1}m\theta(k).$$

which is just $T(F_r)_1$. Hence $\phi^{T_r}_1$ is simply the characteristic function $1_{T(F_r)_1}$ of $T(F_r)_1$ times the character χ_r^{-1}. Thus we have proved that

$$TO^{G_r}_{mu\theta}(\phi_u) = \chi_r^{-1}(m)\, TO^{T_r}_{m\theta}(1_{T(F_r)_1})$$

We claim that $TO^{T_r}_{m\theta}(1_{T(F_r)_1}) = \mathrm{vol}_{d\bar{s}}(T(F)\backslash T(F)T(F_r)_1)$. Indeed we have

$$TO^{T_r}_{m\theta}(1_{T(F_r)_1}) = \int_{T(F)\backslash T_r} 1_{T(F_r)_1}(s^{-1}m\theta(s))\, d\bar{s}$$
$$= \mathrm{vol}(\{s \in T(F)\backslash T_r : s^{-1}m\theta(s) \in T(F_r)_1\})$$

Since $m \in T(F_r)_1$ we see that $s^{-1}m\theta(s) \in T(F_r)_1$ is equivalent to $s^{-1}\theta(s) \in T(F_r)_1$. Therefore $\tau_s := s^{-1}\theta(s) \in H^1(F_r/F, T(F_r)_1) = 0$, therefore we can find $t \in T(F_r)_1$ such that $st = \theta(st)$, in other words, $st \in T(F)$. We see that $s \in T(F)T(F_r)_1$. The other direction is obvious, hence the claim.

By our choice of measures, this volume equals 1, therefore $TO^{G_r}_{mu\theta}(\phi_u) = \chi_r^{-1}(m)$, as desired.

□

Lemma 11.2.2. *We have the following statements.*

(i) Let $m_0, m_0' \in T(F)_1$. Then m_0t and $m_0't$ are stably-conjugate in G if and only if $m_0 = m_0'$.

(ii) Let $m, m' \in T(F_r)_1$. Then mu and $m'u$ are stably θ-conjugate in G_r if and only if they are θ-conjugate by an element of $T(F_r)_1$.

Proof. We take $r = 1$, hence $\theta = \text{id}$ and $u = t$ in lemma 11.1.1 (i), we get if $g \in G(F^s)$ such that $g^{-1}m_0tg = m_0't$, then $g \in T(F^s)$, hence we have $m_0 = m_0'$ as desired. For part (ii), we have mu is stably θ-conjugate to $m'u$ if and only if $N_r(m)t = N_r(m')t$. From part (i) this holds if and only if $N_r(m) = N_r(m')$. Therefore, there exists a $g \in T(F^s)$ such that $g^{-1}\theta(g)m = m'$. As in the proof in the previous lemma, we can show that $g \in T(F)T(F_r)_1$, but the $T(F)$-part won't contribute to the θ-conjugacy, therefore we can assume that $g \in T(F_r)_1$. □

Corollary 11.2.3. *For every $\delta \in G_r$, $SO_{\delta\theta}(\phi_{u,\chi_r}) = TO_{mu\theta}(\phi_{u,\chi_r})$, where $mu, m \in T(F_r)_1$ is any element of this form which is stably θ-conjugate to δ.*

Proof. If δ is not stably θ-conjugate to an element of the form mu, then both sides are zero. Now we assume that $\delta = mu$, then the non-zero summands in $SO_{\delta\theta}(\phi_{u,\chi_r})$ are of the form $TO_{m'u}(\phi_u)$ where mu and $m'u$ are stably θ-conjugate, and we notice that the coefficients $e(m'u)$ and $a(m'u)$ in the definition 8.1.1 are all equal to 1 since $m'u$ is strongly θ-regular (Lemma 11.1.1 (ii)). Thus from Lemma 11.2.1 (ii), only one term appears in this sum.

□

Proposition 11.2.4. *The functions $f_{t,\chi}$ and ϕ_{u,χ_r} are associated.*

Proof. We first show that $SO_\gamma(f_{t,\chi}) \neq 0$ implies that γ is a norm from G_r. From Proposition 11.2.1 (2), we know that γ is stably-conjugate in G to an element of the form $m_0 t$, $m_0 \in T(F)_1$. By the fact that $N_r : T(F_r)_1 \to T(F)_1$ is surjective ([Hai12], Lemma 8.2.1), we know that $m_0 = N_r(m)$ for some $m \in T(F_r)_1$. Since $t = N_r(u)$, this shows that γ is a norm from G_r.

Next we assume $\gamma = N_r(\delta)$ for an element $\delta \in G_r$. If $SO_{\delta\theta}(\phi_{u,\chi_r}) \neq 0$, by Proposition 11.2.1 (1) δ is stably θ-conjugate to an element of the form mu. Then from the previous lemma and corollary, we have

$$SO_{\delta\theta}(\phi_{u,\chi_r}) = TO_{mu\theta}(\phi_{u,\chi_r}) = \chi_r^{-1}(m).$$

Letting $m_0 = N_r(m)$, then γ is stably-conjugate to the element $m_0 t$ and the same reasoning then gives us

$$SO_\gamma(f_{t,\chi}) = O_{m_0 t}(f_{t,\chi}) = \chi^{-1}(m_0),$$

and since $m_0 = N_r(m)$, we see that $\chi_r^{-1}(m) := \chi^{-1}(N_r(m)) = \chi^{-1}(m_0)$.

If $SO_{\delta\theta}(\phi_{u,\chi_r}) = 0$, then δ is not stably θ-conjugate to an element of the form mu. Now if γ were stably conjugate to $m_0 t$, then writing $m_0 = N_r(m)$ we would have that $N_r(\delta)$ is stably conjugate to $N_r(mu)$. But then this would imply that δ is stably θ-conjugate to mu, which is a contradiction. Hence we have $SO_\gamma(f_{t,\chi}) = 0$ as well. $\square$

11.3 Traces of Elementary Functions

Let Π denote a θ-stable admissible representation of $G(F_r)$, and we fix an intertwiner $I_\theta : \Pi \simeq \Pi^\theta$. Let $\Theta_{\Pi\theta}$ denote the **locally constant** function[1] representing the functional $\phi \mapsto \langle \operatorname{tr} \Pi I_\theta, \phi \rangle$ for

[1] Harish-Chandra [HC06] proved stronger results, namely, he showed that $\Theta_{\Pi\theta}$ is locally integrable and supported on the regular semisimple elements over char 0 fields. Over function fields, similar stronger results exist with restrictions on the residue characteristic, see [AGK+26] for GL_n and [CGH14] for general reductive group G over sufficiently large

$\phi \in C_c^\infty(G(F_r))$, whose existence is shown for any (not necessarily connected) G and any field k by Adler and Korman (see [AK07], Prop 13.1). Thus we have

$$\langle \operatorname{tr} \Pi I_\theta, \phi \rangle = \int_{G_r} \Theta_{\Pi\theta}(g)\phi(g)\, dg.$$

We calculate this trace for $\phi = \phi_{u,\chi_r}$ using the map 11.1.1. Let $d\bar{k}$ denote the quotient measure on $T(F_r)_1 \backslash I_r$. The change of variables formula yields

$$\begin{aligned}
\langle \operatorname{tr} \Pi I_\theta, \phi_{u,\chi_r} \rangle &= q^{r\langle 2\rho,\nu\rangle} \int_{T(F_r)_1 \backslash I_r} \int_{T(F_r)_1} \Theta_{\Pi\theta}(k^{-1} m u \theta(k)) \cdot \phi_{u,\chi_r}(k^{-1} m u \theta(k))\, dm\, d\bar{k} \\
&= q^{r\langle 2\rho,\nu\rangle} \int_{T(F_r)_1} \Theta_{\Pi\theta}(mu)\chi_r^{-1}(m)\, dm \\
&= q^{r\langle 2\rho,\nu\rangle} \int_{T(F_r)_1} \Theta_{\Pi_U\theta}(mu)\chi_r^{-1}(m)\, dm.
\end{aligned} \tag{11.3.1}$$

Here Π_U is the Jacquet module of Π corresponding to the Borel subgroup $B_r = T_r U_r$, and the equality $\Theta_{\Pi\theta}(mu) = \Theta_{\Pi_U\theta}(mu)$ is due to Prop 7.4 in [Rog88], which generalizes Thm 5.2 in [Cas77]. We notice that these results are unconditional of characteristics.

From now on, we write W (resp. W_r) for the relative Weyl group associated to the maximal F-split (resp. F_r-split) torus A (resp. A^{F_r}) in G.

We will continue to compute the trace

$$\langle \operatorname{tr} \Pi I_\theta, \phi_{u,\chi_r} \rangle,$$

for any irreducible θ-stable representation Π of G_r with intertwiner I_θ.

characteristic.

Let Ξ denote the set of characters on $T(F_r)$ which extend some W_r-conjugate of χ_r. Let $\Xi(\chi_r) \subset \Xi$ denote those whose restriction to $T(F_r)_1$ is precisely χ_r. Let Ξ^θ (resp. $\Xi(\chi_r)^\theta$) denote the subset of θ-fixed elements in Ξ (resp. $\Xi(\chi_r)$). From now on, ξ' will denote an arbitrary element of Ξ.

Recall that Π_U denotes the Jacquet module of Π relative to U_r. Because $(\Pi^\theta)_U = (\Pi_U)^\theta$, the intertwiner $I_\theta : \Pi \simeq \Pi^\theta$ induces an intertwiner $I_\theta : \Pi_U \simeq \Pi_U^\theta$. We have that Π_U is a subquotient of

$$(i_{B_r}^{G_r}(\xi'))_U = \delta_{B_r}^{1/2} \bigoplus_{w \in W_r} \mathbb{C}_{{}^w\xi'}$$

where $\mathbb{C}_{{}^w\xi'}$ is the 1-dimensional representation of T_r corresponding to the character ${}^w\xi'$. Thus there is a well-defined subset $\Xi(\Pi) \subset \Xi$ and positive multiplicities $a_{\xi'} := a_{\xi',\Pi}$ for $\xi' \in \Xi(\Pi)$ such that, as T_r-representations,

$$\Pi_U = \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi)} \mathbb{C}_{\xi'}^{a_{\xi'}}. \tag{11.3.2}$$

Since Π_U^θ is isomorphic to Π_U, the set $\Xi(\Pi)$ is fixed by θ.

We set $\Xi(\Pi)^\theta = \Xi^\theta \cap \Xi(\Pi)$, and also $\Xi(\Pi, \chi_r)^\theta := \Xi(\chi_r)^\theta \cap \Xi(\Pi)$. Further, for $\xi' \in \Xi^\theta$, we set

$$\mathrm{tr}(I_\theta, \Pi, \xi') := \langle \mathrm{tr}\, I_\theta ; \delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle$$

For $\xi' \in \Xi(\chi_r)$, since $\xi'|_{T(F_r)_1} = \chi_r$, we can write

$$\xi' = \tilde{\chi}_r \eta'$$

for a unique unramified character η' on T_r, and $\tilde{\chi}_r$ is an extension of χ_r to $T(F_r)$..

Proposition 11.3.1. *In the notation above, we have*

$$\langle \operatorname{tr} \Pi \mathrm{I}_\theta, \phi_{\mathrm{u},\chi_\mathrm{r}} \rangle = \mathrm{q}^{\langle \rho, \tau \rangle} \sum_{\xi' \in \Xi(\Pi, \chi_\mathrm{r})^\theta} \eta'(\mathrm{u}) \operatorname{tr}(\mathrm{I}_\theta, \Pi, \xi'). \tag{11.3.3}$$

Proof. We rewrite the integral in 11.3.1 using 11.3.2. It is

$$\int_{T(F_r)_1} \chi_r^{-1}(m) \langle \mathrm{tr}\, \Pi_U(mu) I_\theta; \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi)} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle \, dm.$$

The isotypical component $\delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}}$ can contribute to the trace of $\Pi_U I_\theta$ only if $\xi' \in \Xi(\Pi)^\theta$ and in that case $\Pi_U I_\theta$ preserves the component. Thus the integral can be further expressed as

$$\sum_{\xi' \in \Xi(\Pi)^\theta} \int_{T(F_r)_1} \chi_r^{-1}(m) \langle \mathrm{tr}\, \Pi_U(mu) I_\theta;\ \delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle \, dm \tag{11.3.4}$$

On $\delta_{B_r}^{1/2} \mathbb{C}_{\xi'}^{a_{\xi'}}$ appearing here, $\Pi_U(mu)$ acts by the scalar

$$\delta_{B_r}^{1/2}(u) \xi'(mu) = q^{-r\langle \rho, \nu \rangle} \xi'(m) \xi'(u) = q^{-\langle \rho, \tau \rangle} \xi'(m) \eta'(u).$$

Where we recall that $\delta_{B_r}(u) := |\det(\mathrm{Ad}(u); \mathrm{Lie}(U_r(F)))|_F = q^{-r\langle 2\rho, \nu \rangle} = q^{-\langle 2\rho, \tau \rangle}$. Thus 11.3.4 becomes

$$q^{-\langle \rho, \tau \rangle} \sum_{\xi' \in \Xi(\Pi)^\theta} \xi'(u) \mathrm{tr}\, (I_\theta, \Pi, \xi') \int_{T(F_r)_1} \chi_r^{-1}(m) \xi'(m) \, dm$$

By the orthogonality of characters on the abelian group $T(F_r)_1$, we know that the integral is 1 if and only if $\xi' = \chi_r$ and 0 otherwise. Therefore, the only ξ''s that survive in the summation are those whose restriction to $T(F_r)_1$ is exactly χ_r, in other words, those $\xi' \in \Xi(\Pi, \chi_r)^\theta$. Therefore the proposition is proved. $\square$

We use e_ρ to denote the idempotent $e_\rho \in \mathcal{Z}(G,\rho)$ with support I, whose value at $k \in I$ is $\rho(k)^{-1}$.

Lemma 11.3.2. *We have*

$$\sum_{\xi' \in \Xi(\Pi)^\theta} \mathrm{tr}\,(\mathrm{I}_\theta, \Pi, \xi') = \langle \mathrm{tr}\,\Pi \mathrm{I}_\theta, \mathrm{e}_{\rho_\mathrm{r}} \rangle \tag{11.3.5}$$

Proof. By definition, we can write the right hand side as

$$\begin{aligned}\int_{I_r} \langle \mathrm{tr}\,\Pi(g) I_\theta, \Pi \rangle e_{\rho_r}(g)\, dg &= \int_{I_r} \langle \mathrm{tr}\,\Pi(g) I_\theta, \Pi^{\rho_r} \rangle e_{\rho_r}(g)\, dg \\ &= \langle \mathrm{tr}\, I_\theta, \Pi^{\rho_r} \rangle\end{aligned}$$

And by the isomorphism $\Pi^{\rho_r} \cong \Pi_U^{\chi_r}$ ([Hai], Prop. 6.0.1) and 11.3.2, this can be written as

$$\langle \mathrm{tr}\, I_\theta, \Pi_U^{\chi_r} \rangle = \langle \mathrm{tr}\, I_\theta, \delta_{B_r}^{1/2} \bigoplus_{\xi' \in \Xi(\Pi, \chi_r)^\theta} \mathbb{C}_{\xi'}^{a_{\xi'}} \rangle$$

$\square$

Corollary 11.3.3. *We have*

$$\langle \mathrm{tr}\,\Pi \mathrm{I}_\theta, \phi_{\mathrm{u},\chi_\mathrm{r}} \rangle|_{\mathrm{u}=1} = \langle \mathrm{tr}\,\Pi \mathrm{I}_\theta, \mathrm{e}_{\rho_\mathrm{r}} \rangle. \tag{11.3.6}$$

In particular, if $r = 1$, we have

$$\sum_{\xi \in \Xi(\pi,\chi)} \dim(\mathbb{C}_\xi^{a_{\xi,\pi}}) = \langle \mathrm{tr}\,\pi, \mathrm{e}_\rho \rangle. \tag{11.3.7}$$

Chapter 12: Proof In the Strongly Regular Elliptic Case

12.1 Local Data

We first give the definition of the local data, which is the necessary bridge between spectral side geometric side of the theory. We assume G is any unramified reductive group defined over local field F. Let E/F be an unramified extension of degree r, denote θ to be a generator of $\mathrm{Gal}(E/F)$ as usual. Let $G_r := G(E)$.

Let $\mathrm{Irr}_{\chi,\lambda}(G)$ (resp. $\mathrm{Irr}^{\theta}_{\chi_r,\lambda_r}(G_r)$) denote the set of irreducible (resp. irreducible θ-stable) admissible representations in $\mathfrak{R}_{\chi}(G)$ (resp. $\mathfrak{R}_{\chi_r}(G_r)$) that transformed by λ (resp. λ_r) under the center action. We define $\mathcal{H}(G_r, \rho_r, \lambda_r^{-1})$ to the algebra of compactly supported functions on $G_r/Z(G_r)$ transformed by ρ_r (resp. λ^{-1}) under the subgroups I_r (resp. $Z(G_r)$), and similarly for $\mathcal{H}(G, \rho, \lambda^{-1})$.

We define the *local data adapted to* $\mathrm{Irr}_{\chi,\lambda}(G)$ to consist of data (a), (b) and (c), subject to conditions (1) and (2) below:

(a) An index set $\mathcal{I}_\lambda$, possibly infinite;

(b) A collection of complex numbers $a_i(\pi)$ for $i \in \mathcal{I}_\lambda$ and $\pi \in \mathrm{Irr}_{\chi,\lambda}(G)$;

(c) A collection of complex numbers $b_i(\Pi)$ for $i \in \mathcal{I}_\lambda$ and $\Pi \in \mathrm{Irr}^{\theta}_{\chi_r,\lambda_r}(G_r)$.

(1) For an fixed i, the constants $a_i(\pi)$ and $b_i(\Pi)$ are zero for all but finitely many π and Π.

(2) For $\phi \in \mathcal{H}(G_r, \rho_r, \lambda_r^{-1})$ and $f \in \mathcal{H}(G, \rho, \lambda^{-1})$, the following statements are equivalent:

  (A) For all $i \in \mathcal{I}_\lambda$, we have $\sum_\pi a_i(\pi)\langle \mathrm{tr}\,\pi, f\rangle = \sum_\Pi b_i(\Pi)\langle \mathrm{tr}\,\Pi I_\theta, \phi\rangle$;

  (B) For all strongly regular elliptic semisimple norms $\gamma = \mathcal{N}(\delta)$, where $\gamma \in G$ and $\delta \in G_r$, we have

$$SO_\gamma(f) = SO_{\delta\theta}(\phi).$$

Remark 12.1.1. *Since G might not have a compact center and the functions we consider here are only compactly supported mod centers, the trace $\langle tr\,\pi, f\rangle$ (resp. $\langle tr\,\Pi I_\theta, \phi\rangle$) should be understood to be defined over G/Z (resp. G_r/Z_r) instead of over G (resp. G_r).*

12.2 Global Setup

From now on, we embed the local situation into a suitable global setup, in order to apply the stabilizations of (twisted) simple trace formula to prove the existence of the local data.

We assume $G_{\text{der}} = G_{\text{sc}}$. We may assume that G is split over an unramified extension K/F such that $E \subset K$. We choose a degree $[K : F]$ cyclic extension of global function fields $\underline{K}/\underline{F}$ and a finite place v_0 of $\underline{F}$ such that $\underline{K}_{v_0}$ is a field and $\underline{K}_{v_0}/\underline{F}_{v_0} \cong K/F$. Then there is a degree $r = [E : F]$ cyclic extension $\underline{E}/\underline{F}$ with $\underline{E} \subset \underline{K}$ and $\underline{E}_{v_0}/\underline{F}_{v_0} \cong E/F$.

The Tchebotarev density theorem is still valid in positive characteristics (for example, see [Jar82]), therefore we can find an inert place v_1 of $\underline{F}$, $v_1 \neq v_0$ with $\mathrm{Gal}(\underline{K}_{v_1}/\underline{F}_{v_1}) = \mathrm{Gal}(K/F)$. In addition, we fix two more auxiliary finite places v_2 and v_3 of $\underline{F}$ where $\underline{E}/\underline{F}$ splits completely at these places. We can assume that $v_0 \notin \{v_1, v_2, v_3\}$.

There is a quasi-split group $\underline{G}$ over $\underline{F}$ with the property that $\underline{G} \times_{\underline{F}} \underline{F}_{v_0} \cong G$. We set $\underline{\tilde{G}} = \mathrm{Res}_{\underline{E}/\underline{F}} \underline{G}_{\underline{E}}$. By the reduction steps, we can assume that $Z(\underline{G})$ is an induced torus over F, and denote $\underline{G}^* = \underline{G}/Z(\underline{G})$. Let θ denote the $\underline{F}$-linear automorphism of $\underline{\tilde{G}}$ from $\theta = \mathrm{Gal}(E/F) \cong \mathrm{Gal}(\underline{E}/\underline{F})$.

We may assume that the groups $\underline{G}$ and $\underline{\tilde{G}}$ have simply connected derived subgroups and split over $\underline{K}$. In order to apply the stabilization results in §10, they have to satisfy the Hasse principle for H^1 on $\underline{G}^*$ and $\underline{\tilde{G}}^*$, namely $\ker^1(\underline{F}, \underline{G}^*) = \ker^1(\underline{F}, \underline{\tilde{G}}^*) = 1$. Since v_i splits completely for $i = 2, 3$, we have identifications

$$\underline{G}(\underline{E}_{v_i}) = \underline{G}(\underline{F}_{v_i}) \times \cdots \times \underline{G}(\underline{F}_{v_i})$$

with r factors, and $\mathrm{Gal}(\underline{E}_{v_i}/\underline{F}_{v_i})$ acts by cyclic permutations. Same decompositions and permutations hold for $\underline{G}^*(\underline{E}_{v_i})$.

We write $\underline{\phi} = \phi^{v_0} \otimes \phi_{v_0} = \otimes_v \phi_v$ for a pure tensor element of $C_c^\infty(\underline{G}^*(\mathbb{A}_{\underline{E}}), \lambda^{-1})$ for a global unitary character $\lambda : \underline{Z}(\mathbb{A}_{\underline{E}})/\underline{Z}(\underline{E}) \to \mathbb{C}^\times$ over places of $\underline{F}$ and similarly $\underline{f} = f^{v_0} \otimes f_{v_0} = \otimes_v f_v$. We write ϕ for ϕ_{v_0} and f for f_{v_0}.

We will always use the symbol S_3 to denote a finite set of places of $\underline{F}$ such that $v_3 \in S_3$ and $v_1, v_2 \notin S_3$. Finally, we consider triples $(\phi^{v_0}, f^{v_0}, S_3)$ satisfying the following conditions.

(a) At any place $v \notin S_3 \cup \{v_1, v_2\}$, the group $\underline{G} \times_{\underline{F}} \underline{F}_v$ and the extension $\underline{E}_v/\underline{F}_v$ are unramified.

(b) The function f_{v_1} (resp. ϕ_{v_1}) is (up to a scalar) a pullback of *generalized Kottwitz function* on $\underline{G}^*(\underline{F}_{v_1})$ (resp., $\tilde{\underline{G}}^*(\underline{F}_{v_1})$) satisfying the conditions in Lemma 10.3.1 (a). In particular, $\lambda_{v_1} =$ triv. Moreover, we may assume they are associated, as in §8.2 of [Hai09]. We notice that this construction requires that E/F is inert at v_1.

(c) The function f'_{v_2} is a coefficient of a supercuspidal representation, $\phi_{v_2} = f'_{v_2} \otimes \cdots \otimes f'_{v_2}$ and $f_{v_2} = f'_{v_2} * \cdots * f'_{v_2}$. Thus (ϕ_{v_2}, f_{v_2}) are associated and have nonvanishing (twisted) orbital integrals at (θ)-elliptic strongly (θ)-regular elements which are close to the identity. Notice that these functions are only compactly supported modulo centers.

(d) For any $v \in S_3$, f_v (resp., ϕ_v) is supported on the set of strongly regular elements (resp., elements with strongly regular norms), and (ϕ_v, f_v) are associated. Moreover, the function f_{v_3} (resp., ϕ_{v_3}) is supported on the set of elliptic elements with strongly regular elliptic images in $G^*(\underline{F}_{v_3})$

(resp., elements with elliptic norms).

(e) For any other place $v \notin S_3 \cup \{v_0, v_1, v_2\}$, the function f'_v (resp., ϕ'_v) is the unit element of the spherical Hecke algebra of $\underline{G}^*(\underline{F}_v)$ (resp., $\underline{G}^*(\underline{E}_v)$). Then (ϕ'_v, f'_v) are associated by [Kot86a]. We then form the function ϕ_v (resp., f_v) to be the pullback of ϕ'_v (resp., f'_v) with the trivial central action. Therefore (ϕ_v, f_v) are associated by Lemma 9.3.6.

(f) At every place v of $\underline{F}$ where $\underline{E}/\underline{F}$ splits, we have $\phi_v = f'_v \otimes \cdots \otimes f'_v$ and $f_v = f'_v * \cdots * f'_v$ for an appropriate function f'_v so that (ϕ_v, f_v) are associated.

We note that by the above conditions, the functions f_v and ϕ_v are assumed to be associated at every place $v \neq v_0$.

12.3 Existence of the Local Data

We prove that local data adapted to $\mathrm{Irr}_{\chi,\lambda}(G)$ in this subsection.

By abuse of notations, we use $R(\underline{\phi})$ (resp. $R(\underline{f})$) to denote the action of $\underline{\phi}$ (resp., $\underline{f}$) on the cuspidal spectrum $L^2_{\text{cusp}}(\underline{G}^*(\underline{E})\backslash\underline{G}^*(\mathbb{A}_{\underline{E}}), \lambda)$ (resp., $L^2_{\text{cusp}}(\underline{G}^*(\underline{F})\backslash\underline{G}^*(\mathbb{A}_{\underline{F}}), \lambda)$). Let I_θ denote the intertwining operator on the same space given by $I_\theta(\psi)(x) := \psi(\theta^{-1}(x))$ for $\psi \in L^2_{\text{cusp}}(\underline{G}^*(\underline{E})\backslash\underline{G}^*(\mathbb{A}_{\underline{E}}), \lambda)$ and $x \in \underline{G}^*(\mathbb{A}_{\underline{E}})$. The existence of local data comes from the following proposition. We say that f_{v_0} and ϕ_{v_0} are associated at strongly regular elliptic norms if they are associated for every strongly regular elliptic semisimple norm in the adjoint group $\gamma = \mathcal{N}\delta$ in $\underline{G}(\underline{F}_{v_0})$ whose image $\bar{\gamma}$ is also strongly regular elliptic semisimple in $\underline{G}^*(\underline{F}_{v_0})$.

Proposition 12.3.1. *There is a constant $c \neq 0$, depending only on $(\underline{G}, \underline{E}/\underline{F})$, with the following property: the functions f_{v_0} and ϕ_{v_0} are associated at strongly regular norms if and only we have the* *equality of traces*

$$tr(R(\phi^{v_0} \otimes \phi_{v_0})I_\theta) = c\, tr\, R(f^{v_0} \otimes f_{v_0}), \tag{12.3.1}$$

for every triple $(\phi^{v_0}, f^{v_0}, S_3)$ *satisfying conditions (a)-(f) in §12.2.*

Proof. For the "if" part, at the place v_0, if we are given $\gamma_0 = \mathcal{N}\delta_0$ for (resp. θ-)strongly regular elliptic semisimple element $\gamma_0 \in \underline{G}(\underline{F}_{v_0})$ (resp. $\delta_0 \in \underline{G}(\underline{E}_{v_0})$), we choose an appropriate global element $\delta \in \underline{\tilde{G}}(\underline{F})$ such that:

(1) δ_{v_0} is close to δ_0 at v_0 and close to 1 at v_2;

(2) δ_{v_i} is θ-elliptic and strongly θ-regular at $i = 1, 2, 3$.

Thus δ itself is θ-elliptic and strongly θ-regular. Then we can choose the set S_3 and the associated functions (f^{v_0}, ϕ^{v_0}) such that, the geometric sides of 12.3.1, according to the stabilization process, take the stabilized form of equations 10.4.1 and 10.4.3, which involve only the term indexed by $\gamma := \mathcal{N}\delta$. The second sums in those equations, which are the sum of adelic (twisted) orbital integrals can be written as a product over all places of local stable (twisted) orbital integrals, and at all places except v_0, these are nonzero (by choice) and matching. Then the identity 12.3.1 will force the matching at v_0, namely, $SO^{\underline{G}^*(\underline{E}_{v_0})}_{\delta_{v_0}\theta}(\phi_{v_0}) = SO^{\underline{G}^*(\underline{F}_{v_0})}_{\gamma_{v_0}}(f_{v_0})$. By Lemma 8.9, we then have $SO^{\underline{G}(\underline{E}_{v_0})}_{\delta_{v_0}\theta}(\phi_{v_0}) = SO^{\underline{G}(\underline{F}_{v_0})}_{\gamma_{v_0}}(f_{v_0})$. A continuity argument then forces the desired identity $SO^{\underline{G}(\underline{E}_{v_0})}_{\delta_0\theta}(\phi_{v_0}) = SO^{\underline{G}(\underline{F}_{v_0})}_{\gamma_0}(f_{v_0})$.

For the "only if" part, we just need to use Proposition 10.4.1, since for 12.3.1 to be true, by assumption, the only ingredient we need is simply $SO^{\underline{G}(\underline{E}_{v_0})}_{\delta_{v_0}\theta}(\phi_{v_0}) = SO^{\underline{G}(\underline{F}_{v_0})}_{\gamma_{v_0}}(f_{v_0})$ for every $\gamma = \mathcal{N}\delta$, which holds by assumption. ∎

Proposition 12.3.2. *Local data adapted to* $\mathrm{Irr}_{\chi,\lambda}(G)$ *in the sense of §12.1 exists.*

Proof. On the spectral side, we can rewrite equation 12.3.1 as

$$\sum_{\Pi} \langle \operatorname{tr} \Pi I_\theta, \phi \rangle = c \sum_{\pi} \langle \operatorname{tr} \pi, f \rangle,$$

where the sum is over automorphic cuspidal representation Π (resp., π) in the decomposition of the space $L^2_{\text{cusp}}(\underline{G}^*(\underline{E}) \backslash \underline{G}^*(\mathbb{A}_{\underline{E}}), \lambda)$ (resp., $L^2_{\text{cusp}}(\underline{G}^*(\underline{F}) \backslash \underline{G}^*(\mathbb{A}_{\underline{F}}), \lambda)$). Using the product decomposition of ϕ and f, this is

$$\sum_{\Pi} \prod_{v} \langle \operatorname{tr} \Pi_v I_\theta, \phi_v \rangle = c \sum_{\pi} \prod_{v} \langle \operatorname{tr} \pi_v, f_v \rangle \tag{12.3.2}$$

Therefore the identity 12.3.1 is equivalent to the identity 12.3.2. Fixing the functions ϕ_v, f_v at the places other than v_0, for each we get an identity

$$\sum_{\Pi} \langle \operatorname{tr} \Pi^{v_0} I_\theta, \phi^{v_0} \rangle \langle \operatorname{tr} \Pi_{v_0} I_\theta, \phi_{v_0} \rangle = \sum_{\pi} \langle \operatorname{tr} \pi^{v_0}, f^{v_0} \rangle \langle \operatorname{tr} \pi_{v_0}, f_{v_0} \rangle.$$

Assuming f_{v_0} and ϕ_{v_0} range only over functions bi-ρ-invariant under a fixed Iwahori subgroup, then at v_0, the function is bi-invariant under the pro-unipotent of that fixed Iwahori subgroup, outside of v_0, the functions must be biinvariant under certain open compact subgroup ([GH24], Lemma 5.2.1). That is to say, the level at all places are fixed. Therefore, we can just apply an adapted version of Harish-Chandra's finiteness theorem (Lemma 12.3.3) for cusp forms to see that, the number of representations Π (resp. π) make nonzero contribution to the above identity is finite.

Since the sums that we want ultimately in the local data are over automorphic representations of local component at v_0, we regroup the summation as

$$\sum_{\Pi_{v_0}}[\sum_{\substack{\Pi' \text{ s.t.} \\ \Pi'_{v_0}\cong\Pi_{v_0}}} \langle \mathrm{tr}\, \Pi'^{v_0} I_\theta, \phi^{v_0}\rangle]\langle \mathrm{tr}\, \Pi_{v_0} I_\theta, \phi_{v_0}\rangle = \sum_{\pi_{v_0}}[\sum_{\substack{\pi' \text{ s.t.} \\ \pi'_{v_0}\cong\pi_{v_0}}} \langle \mathrm{tr}\, \pi'^{v_0}, f^{v_0}\rangle]\langle \mathrm{tr}\, \pi_{v_0}, f_{v_0}\rangle. \tag{12.3.3}$$

Where the outer sums are over isomorphic classes of irreducible cuspidal (θ-stable) automorphic representations of $\underline{G}(\underline{E}_{v_0})$ (resp. $\underline{G}(\underline{F}_{v_0})$) that transforms under the central character λ_{r,v_0}^{-1} (resp. $\lambda_{v_0}^{-1}$). In particular, they are in $\mathrm{Irr}^\theta_{\chi_{r,v_0},\lambda_{r,v_0}}(\underline{G}(\underline{E}_{v_0}))$ (resp. $\mathrm{Irr}^\theta_{\chi_{v_0},\lambda_{v_0}}(\underline{G}(\underline{F}_{v_0}))$). Now we set

$$b(\Pi_{v_0}) = \sum_{\substack{\Pi' \text{ s.t.} \\ \Pi'_{v_0}\cong\Pi_{v_0}}} \langle \mathrm{tr}\, \Pi'^{v_0} I_\theta, \phi^{v_0}\rangle$$

$$a(\pi_{v_0}) = \sum_{\substack{\pi' \text{ s.t.} \\ \pi'_{v_0}\cong\pi_{v_0}}} \langle \mathrm{tr}\, \pi'^{v_0}, f^{v_0}\rangle$$

Then 12.3.3 becomes

$$\sum_{\Pi_{v_0}} b(\Pi_{v_0})\langle \mathrm{tr}\, \Pi_{v_0} I_\theta, \phi_{v_0}\rangle = \sum_{\pi_{v_0}} a(\pi_{v_0})\langle \mathrm{tr}\, \pi_{v_0}, f_{v_0}\rangle, \tag{12.3.4}$$

therefore, the implication (B)⇒(A) in the definition of local data is proved.

Assume that 12.3.4 holds for the functions ϕ_{v_0} and f_{v_0}, since we already fix functions at other places v other than v_0, we get 12.3.3, thus 12.3.2, ultimately 12.3.1. Then we get ϕ_{v_0} and f_{v_0} are associated at strongly regular norms by Proposition 12.3.1, hence the implication (A)⇒(B) is also proved. □

Lemma 12.3.3. *(Harish-Chandra's finiteness Theorem for cusp forms over function fields) Let K be an open compact subgroup of $\underline{G}(\mathbb{A}_{\underline{F}})$, let $V_{cusp}(\lambda, K)$ denote the space of cuspidal functions f :*

$\underline{G}(F)\backslash\underline{G}(\mathbb{A}_{\underline{F}})/K \to \mathbb{C}$ *such that* $f(zg) = \lambda(z)f(g)$ *for all* $g \in \underline{Z}(\mathbb{A}_{\underline{F}})$. *Then:*

1. *There exists a compact subset* $C \subset \underline{G}(\mathbb{A}_{\underline{F}})$ *such that every function in* $V_{cusp}(\lambda, K)$ *is supported on* $\underline{Z}(\mathbb{A}_{\underline{F}})\underline{G}(F)C$.

2. $\dim V_{cusp}(\lambda, K) < \infty$.

Proof. For (i), see [Har74] Theorem 1.2.1, also [GH24] Theorem 9.5.1. (ii) follows from (i) by noticing that the value of such $f \in V_{\mathrm{cusp}}(\lambda, K)$ is determined on the finite cosets C/K (we can always enlarge C to contain K). $\square$

12.4 A Further Reduction

In this subsection, we show that the existence of *local data adapted to Irr*$_{\chi,\lambda}(G)$ *for all unitary characters* λ implies the existence of *local data adapted to Irr*$_{\chi}(G)$, so that we are back to the scenario considered in [Hai12]. The local data adapted to $\mathrm{Irr}_{\chi}(G)$ is given analogously as follows:

(a') An indexing set $\mathcal{I}$, possibly infinite;

(b') A collection of complex numbers $a_i(\pi)$ for $i \in \mathcal{I}$ and $\pi \in \mathrm{Irr}_{\chi}(G)$;

(c') A collection of complex numbers $b_i(\Pi)$ for $i \in \mathcal{I}$ and $\Pi \in \mathrm{Irr}^{\theta}_{\chi_r}(G_r)$.

(1') For i fixed, the constants $a_i(\pi)$ and $b_i(\Pi)$ are zero for all but finitely many π and Π.

(2') For $\phi \in \mathcal{H}(G_r, \rho_r)$ and $f \in \mathcal{H}(G, \rho)$, the following statements are equivalent:

(A') For all i, we have $\sum_{\pi} a_i(\pi)\langle \mathrm{tr}\,\pi, f\rangle = \sum_{\Pi} b_i(\Pi)\langle \mathrm{tr}\,\Pi I_{\theta}, \phi\rangle$;

(B') For all strongly regular elliptic semisimple norms $\gamma = \mathcal{N}(\delta)$, we have

$$SO_\gamma(f) = SO_{\delta\theta}(\phi).$$

Proposition 12.4.1. *Assume local data adapted to Irr*$_{\chi,\lambda}(G)$ *exists for all unitary character* λ *on* Z*, then the local data adapted to Irr*$_\chi(G)$ *exists.*

Proof. Indeed, given that local data exists for every unitary character λ, denote $\mathcal{I}_\lambda$ to be the index set in the corresponding local data, we take $\mathcal{I} = \coprod_\lambda \mathcal{I}_\lambda$. For $\pi \in \mathrm{Irr}_\chi(G)$, if $\pi \in \mathrm{Irr}_{\chi,\lambda}(G)$, then we take $a_i(\pi)$ to be the complex number in the local data adapted to $\mathrm{Irr}_{\chi,\lambda}(G)$, otherwise we set $a_i(\pi) = 0$. We set complex numbers $b_i(\Pi)$ analogously. Therefore we have data (a'), (b'), (c'), subject to condition (1').

If (2')(A') is satisfied, for any character λ and all $i \in \mathcal{I}_\lambda$, we have

$$\sum_\pi a_i(\pi)\langle \mathrm{tr}\,\pi, f\rangle = \sum_\Pi b_i(\Pi)\langle \mathrm{tr}\,\Pi I_\theta, \phi\rangle.$$

It is straightforward to check that for $\pi \in \mathrm{Irr}_{\chi,\lambda}(G)$ (resp. $\Pi \in \mathrm{Irr}_{\chi_r,\lambda_r}(G_r)$), we have $\langle \mathrm{tr}\,\pi, f\rangle = \langle \mathrm{tr}\,\pi, f_\lambda\rangle$ (resp. $\langle \mathrm{tr}\,\Pi I_\theta, \phi\rangle = \langle \mathrm{tr}\,\Pi I_\theta, \phi_{\lambda_r}\rangle$) (recall that the latter traces are defined over G/Z). Therefore, we have $(f_\lambda, \phi_{\lambda_r})$ are associated for any unitary character λ on Z. By Lemma 8.3.2 (i) and the further reduction to unitary characters, we can assert that (2')(B') holds.

Similarly if (2')(B') is satisfied, we just reverse the above process to get (2')(A'). Therefore we have the existence of local data adapted to $\mathrm{Irr}_\chi(G)$.

□

12.5 End of Proof

We fix $\phi \in Z(G_r, \rho_r)$ and $f = b(\phi) \in \mathcal{Z}(G, \rho)$. By the various reduction steps, we may assume γ is a strongly regular elliptic semisimple norm, i.e., $\gamma = \mathcal{N}(\delta)$ for some $\delta \in G(E)$ θ-strongly regular θ-elliptic semisimple.

We need to show that for each $i \in \mathcal{I}$, the identity (2')(A') in the definition of the local data holds. Dropping the i, we need to prove that

$$\sum_{\pi} a(\pi)\langle \text{tr}\,\pi, (b\phi)_\lambda\rangle = \sum_{\Pi} b(\Pi)\langle \text{tr}\,\Pi I_\theta, \phi_{\lambda N}\rangle.$$

By Proposition 11.2.4, the equivalent statements (2')(A') and (2')(B') hold for each part of elementary functions $\phi_u := \phi_{u,\chi_r}$ and $f_t := f_{t,\chi}$, that is,

$$\sum_{\pi} a(\pi)\langle \text{tr}\,\pi, (f_t)_\lambda\rangle = \sum_{\Pi} b(\Pi)\langle \text{tr}\,\Pi I_\theta, (\phi_u)_{\lambda N}\rangle. \tag{12.5.1}$$

We can rewrite right hand side as follows:

$$\begin{aligned}
\langle \text{tr}\,\Pi I_\theta, (\phi_u)_{\lambda N}\rangle &= \int_{G_r} \Theta_{\Pi\theta}(g)\,(\phi_u)_{\lambda N}(g)\,dg \\
&= \int_{G_r} \Theta_{\Pi\theta}(g) \int_{Z(E)} \phi_u(gz)\lambda^{-1}N(z)\,dz\,dg \\
&= \int_{Z(E)} \lambda^{-1}N(z) \int_{G_r} \Theta_{\Pi\theta}(g)\phi_u(gz)\,dg\,dz \\
&= \int_{Z(E)} \lambda^{-1}N(z) \int_{G_r} \Theta_{\Pi\theta}(yz^{-1})\phi_u(y)\,dy\,dz \\
&= \int_{Z(E)} \langle \text{tr}\,\Pi I_\theta, \phi_u\rangle dz
\end{aligned}$$

We can rewrite the left hand side using 11.3.3, in the case where $r = 1$, we get

$$\begin{aligned}\sum_{\pi} a(\pi)\langle \operatorname{tr}\pi, f_t\rangle &= q^{\langle\rho,\tau\rangle} \sum_{\pi\in \mathrm{Irr}_\chi(G)} a(\pi) \sum_{\xi\in\Xi(\pi,\chi)} \eta(t)\dim \mathbb{C}_\xi^{a_\xi},\\ &= q^{\langle\rho,\tau\rangle} \sum_{\xi_0\in\Xi(\chi)/W_\chi} \sum_{\pi\in i_B^G(\xi_0)} a(\pi) \sum_{\xi\in\Xi(\pi,\chi)} \eta(t)(\dim \mathbb{C}_\xi^{a_{\xi,\pi}}),\end{aligned} \tag{12.5.2}$$

where ξ_0 ranges over a set of representatives for the W_χ-orbits on the set of $\Xi(\chi)$ of characters on $T(F)$ extending χ on $T(F)_1$, and for the second identity, we use the description of irreducible objects in $\mathfrak{R}_\chi(G)$ as the subquotients of $i_B^G(\tilde{\chi}\eta)$ for some unramified character η of $T(F)$. Since $\eta(t) = \eta(N(u))$, 12.5.2 is a linear combination of characters which are evaluated on $u = \varpi^\nu$.

If we try the same strategy on the right hand side, we get

$$\begin{aligned}\sum_{\Pi} b(\Pi)\langle \operatorname{tr}\Pi I_\theta, \phi_u\rangle &= q^{\langle\rho,\tau\rangle} \sum_{\Pi} b(\Pi) \sum_{\xi'\in\Xi(\Pi,\chi_r)^\theta} \eta'(u)\operatorname{tr}(I_\theta,\Pi,\xi')\\ &= q^{\langle\rho,\tau\rangle} \sum_{\xi_0'\in\Xi(\chi_r)^\theta/W_{\chi_r}} \sum_{\Pi\in i_{B_r}^{G_r}(\xi_0')} b(\Pi) \sum_{\xi\in\Xi(\Pi,\chi_r)^\theta} \eta'(u)\operatorname{tr}(I_\theta,\Pi,\xi').\end{aligned}$$

Since later we want to separate the contributions of those ξ_0' which are norms and are not, this creates difficulty, since that given two different elements in $\Xi(\chi_r)^\theta$ which are W_{χ_r}-conjugate, it could happen that one of them is a norm and the other is not. Therefore we need to change the way of grouping characters, by the following lemma:

Lemma 12.5.1. *If ξ_1', ξ_2' belong to $\Xi(\chi_r)^\theta$ and are W_χ-conjugate. then one is a norm if and only if the other one is.*

Proof. Assume that there exists a $w \in W_\chi$ such that $\xi_2' = \xi_1'^w$, and without loss of generality we may assume $\xi_1'(t) = \xi_1(Nt)$ for some character ξ on $T(F)$ for $t \in T(F_r)$. Then clearly $\xi_2'(t) =$

$\xi_1'(w^{-1}tw) = \xi_1(N(w^{-1}tw)) = \xi_1(w^{-1}Ntw) = \xi_1^w(Nt)$, therefore ξ_2' is a norm as desired. $\square$

Therefore, instead of W_{χ_r}-conjugacy, we group by W_χ to rewrite the right hand side in the following way:

$$\sum_{\Pi} b(\Pi)\langle \operatorname{tr} \Pi I_\theta, \phi_u\rangle = q^{\langle \rho,\tau\rangle} \sum_{\Pi} b(\Pi) \sum_{\xi' \in \Xi(\Pi,\chi_r)^\theta} \eta'(u) \operatorname{tr}(I_\theta, \Pi, \xi')$$
$$= q^{\langle \rho,\tau\rangle} \sum_{\xi_0' \in \Xi(\chi_r)^\theta / W_\chi} \sum_{\substack{\Pi \text{ s.t.} \\ W_\chi \xi_0' \cap \Xi(\Pi,\chi_r)^\theta \neq \emptyset}} b(\Pi) \sum_{\xi' \in W_\chi \xi_0' \cap \Xi(\Pi,\chi_r)^\theta} \eta'(u) \operatorname{tr}(I_\theta, \Pi, \xi') \tag{12.5.3}$$

Let $12.5.3_n$ (resp $12.5.3_{nn}$) denote the contribution to 12.5.3 coming from those ξ_0' which are norms (resp. not norms). By the previous lemma we know that all (resp. none) of the terms η' appearing in $12.5.3_n$ (resp. $12.5.3_{nn}$) are norms.

Now we can regard each side of the identity

$$12.5.2 = 12.5.3_n + 12.5.3_{nn} \tag{12.5.4}$$

as a linear combination of characters on $\nu \in X_*(A^{F_r})^{++}$, the set of regular dominant cocharacters in $X_*(A^{F_r})$, since $u = \nu(\varpi)$. The equation 12.5.4 holds for any regular dominant cocharacter. The linear independence of characters on $X_*(A^{F_r})$ implies the linear independence on $X_*(A^{F_r})^{++}$ if we restrict the characters to the latter set. Thus we have that the equation 12.5.4 must hold for $\nu = 0$, that is for $u = 1$. Then from 12.5.4, we have two identities

$$12.5.2|_{u=1} = 12.5.3_n|_{u=1} \tag{12.5.5}$$

$$0 = 12.5.3_{nn}|_{u=1} \tag{12.5.6}$$

The linear independence gives us a stronger statement: the contributions of each $\xi_0' \in \Xi(\chi_r)^\theta/W_\chi$ to the above equations satisfy corresponding identities. Using Corollary 6.2, for each $\xi_0' \in \Xi(\chi_r)^\theta/W_\chi$, we have the following identities:

$$\begin{gathered}\sum_{\substack{\xi_0\in\Xi(\chi)/W_\chi \\ \text{s.t. } \xi_{0r}=\xi_0'}} \sum_{\pi\in i_B^G(\xi_0)} a(\pi)\langle \operatorname{tr}\pi, e_\rho\rangle = \\ \sum_{\substack{\Pi \text{ s.t.} \\ W_\chi\xi_0'\cap\Xi(\Pi,\chi_r)^\theta\neq\emptyset}} b(\Pi) \sum_{\xi'\in W_\chi\xi_0'\cap\Xi(\Pi,\chi_r)^\theta} \operatorname{tr}(I_\theta,\Pi,\xi')\end{gathered} \tag{12.5.7}$$

Notice that if ξ_0' is not a norm, then the left hand side of 12.5.7 vanishes, and the right hand side also vanishes because of the equality 12.5.6.

Next, we sum these equations over all W_χ-orbits of ξ_0' in $W_{\chi_r}\xi_0' \cap \Xi(\chi_r)^\theta$. The right hand side will simplify in the sum because of the identity 11.3.5. Hence we get an equality

$$\begin{gathered}\sum_{\substack{\xi_0\in\Xi(\chi)/W_\chi \\ \text{s.t. } \xi_{0r}\in W_{\chi_r}\xi_0'}} \sum_{\pi\in i_B^G(\xi_0)} a(\pi)\langle \operatorname{tr}\pi, e_\rho\rangle = \\ \sum_{\Pi\in i_{B_r}^{G_r}(\xi_0')} b(\Pi)\,\langle \operatorname{tr}\Pi I_\theta, e_{\rho_r}\rangle.\end{gathered} \tag{12.5.8}$$

Finally, recall that if $\xi_0' = \xi_{0N}$, then by Lemma 5.2.1, $b(\phi)$ acts on $i_B^G(\xi_0)^\rho$ by the same scalar by which ϕ acts on $i_{B_r}^{G_r}(\xi_0')$. Thus multiply both sides of 12.5.8 by the scalar $ch_{\xi_0}(b(\phi)) = ch_{\xi_0'}(\phi)$, we get an identity of the same form, but wuth e_{ρ_r} (resp. e_ρ) replaced by ϕ (resp. $b(\phi)$). Then summing over all $\xi_0' \in \Xi(\chi_r)/W_{\chi_r}$, we get the desired identity 12.5.1. This completes the proof of Theorem 1.1.1.

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