

## ABSTRACT

Title of dissertation: Abundance of escaping orbits  
in a family of anti-integrable limits  
of the standard map

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We give quantitative results about the abundance of escaping orbits in a family of exact twist maps preserving Lebesgue measure on the cylinder  $\mathbb{T} \times \mathbb{R}$ ; geometrical features of maps of this family are quite similar to those of the well-known Chirikov-Taylor standard map, and in fact we believe that the techniques presented in this work can be further improved and eventually applied to studying ergodic properties of the standard map itself.

We state conditions which assure that escaping orbits exist and form a full Hausdorff dimension set. Moreover, under stronger conditions we can prove that such orbits are not charged by the invariant measure. We also obtain prove that, generically, the system presents elliptic islands at arbitrarily high values of the action variable and provide estimates for their total measure.

Abundance of escaping orbits in a  
family of anti-integrable limits of the  
standard map

by

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# Chapter 1

## Preliminaries

### 1.1 Introduction

In this dissertation we study dynamical properties of a family of exact twist maps preserving an infinite measure on a non-compact phase space. Our main focus is to establish results about abundance of escaping orbits, i.e. trajectories that eventually leave any compact subset of the phase space. The study of the asymptotic dynamical features of this family is interesting for a variety of reasons. First of all, the maps of the family can serve as a model for the high-energy dynamics of some mechanical problems. Examples include the Fermi-Ulam ping-pong and its generalizations, which have been the starting point for this dissertation, and that will be explained in detail in a following section; some  $n$ -body problems, such as the Sitnikov three body configuration or cometary motions, slow-fast systems and motions close to a resonance also show remarkable similarities with the dynamical system studied in this work.

Another quite interesting feature of the family under consideration is its affinity with the Chirikov-Taylor standard map: in fact both systems share essentially the same geometrical structure. For this reason, most of the difficulties we will encounter in our work will be directly related to corresponding issues for the standard map and we can expect that the techniques we use in our case could also be successfully

applied to the more difficult case of the standard map.

The maps we study are given by transformations of the cylinder  $\mathcal{M} = \mathbb{S}^1 \times \mathbb{R} \ni (x, y)$  onto itself. Fix  $\phi$  to be a smooth real-valued function on  $\mathbb{S}^1$ , let  $\dot{\phi}$  be its derivative and  $Y$  be a function on  $\mathbb{R}$  that will be specified below. Then consider the map given by the following equation:

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + Y(y) \\ y + 2\dot{\phi}(x + Y(y)) \end{pmatrix}. \quad (1.1)$$

From now on, we assume, unless otherwise specified, that:

$$\phi(x) \doteq \frac{A}{2\pi} \sin(2\pi x) \quad (1.2)$$

where  $A > 0$ ; we also define for  $\gamma \geq 1$  the function  $Y$  as follows:

$$Y(y) \doteq \text{Const} \cdot |y|^\gamma. \quad (1.3)$$

The family of maps we study is generated by the two parameters  $A$  and  $\gamma$  in (1.2) and (1.3). As it will be clear later, it turns out that if  $\gamma \notin \{1, 2\}$ , the value of the parameter  $A$  is almost irrelevant for our results; the only significant parameter of the map  $F$  will be the exponent  $\gamma$ . Notice that  $F$  is smooth everywhere except on the circle  $\{y = 0\}$ ; however, since we are only interested in the asymptotic behavior for  $y \rightarrow \infty$  of the map  $F$ , we will effectively neglect the singularity line  $\{y = 0\}$  and treat  $F$  as a smooth (exact) twist map on the cylinder. Furthermore, notice that if  $\gamma \rightarrow 1$  the map  $F$  essentially becomes an unfolding of the standard map:

$$S_k : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + y \\ y + k\dot{\phi}(x + y) \end{pmatrix}.$$

Finally,  $F$  has a symmetry given by  $y \mapsto -y$ ,  $\phi \mapsto -\phi$ , therefore we will restrict ourselves to consider only positive large values of  $y$ .

In the next sections of this introductory chapter we will first explain how the map  $F$  can be obtained as an asymptotic approximation for one of the aforementioned mechanical models, and then state the results we are going to prove in this work.

## 1.2 Fermi acceleration

The acceleration mechanism known by the name of *Fermi acceleration* was originally proposed in 1949 by Enrico Fermi [Fe49] to explain the presence in the universe of high energy particles called cosmic rays. Such particles are believed to gain energy by consecutive interactions with irregularities in an otherwise stationary magnetic field. Naïvely, one would expect a process of thermalization leading to a stationary motion of the particle itself; such a reasoning turns out to be too simplistic and a more refined analysis shows that there is a definite probability of an average gain in energy.

In 1960, Ulam [Ul60] suggested a simple Hamiltonian system to model such statistical acceleration behaviour. The model has been thereafter known as the *Fermi-Ulam ping-pong model* and consists of a particle moving between two infinitely heavy walls that are performing an oscillatory motion; the particle changes its velocity only by elastic collisions with the moving walls and it is not subject to any other force.

The main questions about this problem regarded the existence of trajectories with unbounded energy: such orbits can either be *escaping*, i.e. such that the energy of the particle goes to infinity with time, or *oscillating*, meaning that the lim sup of energy is infinite while the lim inf remains bounded. Fermi and Ulam, supported by some numerical evidence, conjectured the existence of unbounded orbits for their model. In 1977, however, KAM theory provided [Pu77, Pu95] a negative answer to such questions: for sufficiently smooth motions of the walls, all orbits are bounded, because for high energies there are invariant tori that prevent diffusion. It is interesting to note that the smoothness condition is not a mere technical issue, as for less regular motions one can indeed construct unbounded orbits [Zh97]. As an historical remark, it is perhaps worth mentioning that, despite using the most advanced computer machines of their age, Fermi and Ulam were forced to perform very crude approximations in order to obtain numerical results in a reasonable time. In particular, in their simulations, the position of the walls was given by a saw-tooth function of time, which is precisely the function that has been used in the construction of unbounded orbits in the non-smooth case.

A *variazione sul tema* involves a single oscillating wall and introduces a potential  $U(x) = x^\alpha$ ,  $\alpha > 0$  which serves the purpose of bringing the particle back to the wall. By considering different values of the exponent  $\alpha$  one obtains a one-parameter family of models; all such models preserve a measure (Liouville measure) that will be the relevant measure in all results that follow. The case of gravity potential ( $\alpha = 1$ ) has been the first to be investigated [Pu77] and the study yielded the following, indeed quite surprising, result:



**Theorem 1.2.1** (Pustynnikov). *There is an open set of wall motions  $\phi(t)$  (in the space of periodic analytic functions admitting an analytic continuation to a given strip  $|\Im z| < \varepsilon$ ) such that the measure of the escaping orbits is infinite.*

The case of elastic potential ( $\alpha = 2$ ) has been studied in [Or99], [Or02]; abundance of unbounded escaping orbits has been proved under some resonance condition between the motion of the wall and the potential.

In a more general setting we can again use KAM theory [Do08] to prove the following result:

**Theorem 1.2.2** (Dolgopyat). *If  $\alpha > 1$  but  $\alpha \neq 2$  and the motion of the wall is smooth enough, then the set of escaping and oscillatory motions is empty since invariant tori persist for high energies.*

On the other hand, KAM theory do not forbid orbits with unbounded energy for weak potentials. However, it is conjectured that for all potentials weaker than gravity (i.e. for  $\alpha < 1$ ) the measure of escaping motions is zero. The conjecture is substantiated by the following

**Theorem 1.2.3** (Dolgopyat). *If  $\alpha < 1/3$  and the motion of the wall is a sinusoid, then the set of escaping orbits has zero measure.*

The above results leave several open questions regarding the largeness of the following sets:

- The escaping set  $\mathcal{E}$  i.e. the set of orbits such that the energy  $E$  tends to infinity as time  $t$  grows;

- The set of orbits with bounded energy;
- The oscillatory set i.e. the set of orbits such that  $\limsup E(t) = \infty$  and  $\liminf E(t) < \infty$ .

The maps considered in this dissertation can be regarded as the *static wall approximation* of the bouncing ball system. This approximation, described in more detail in section 1.3, is widely used in physics literature. It has the advantage of being given by simpler and more explicit formulae whereas keeping the essential geometrical structure of the complete model.

### 1.3 Static wall approximation of the bouncing ball system

This section closely follows section 3 in [Do08]. Consider the problem of a point mass bouncing vertically on an infinitely heavy horizontal plate which oscillates with period 1 in the vertical direction and interacts with the particle by the law of elastic reflection. The particle is moving in a potential  $U(x) = x^\alpha$ , where  $x$  is the vertical position and  $\alpha$  is some positive real number.

Let  $\phi(t)$  be the vertical position of the plate at time  $t$ , periodic of period 1; for simplicity we will consider the case:

$$\phi(t) = B + \frac{A}{2\pi} \sin(2\pi t). \quad B > \frac{A}{2\pi}.$$

It is natural to associate to the system a discrete time map defined as follows. Let  $t_n$  be the time of the  $n + 1$ st collision between the plate and the particle and  $v_n$  its velocity (pointing upwards) immediately after the collision. Since the position

of the plate is a 1-periodic function of time  $t$ , we can consider  $t_n$  on  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . In this way the phase space is a half cylinder; in fact the velocity immediately after a collision at time  $t$  has to satisfy the inequality  $v \geq \dot{\phi}(t)$ .

An approximation common to this kind of problems is the so-called *static wall approximation*, in which we consider the plate fixed at position  $x_0$  but exchanging momentum with the particle as if it were moving. Notice that, since the displacement of the plate is a bounded quantity, we are neglecting terms of order at most  $1/v$ ; the approximation is hence suitable for studying the high energy dynamics of the mechanical system.

Fix  $x_0 > 0$ , then define  $T(v) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  as the time taken by a ball leaving  $x = x_0$  with upward velocity  $v$  to return on  $x = x_0$  subject to the potential  $U(x)$ . In our case  $T(v) \sim v^\gamma$ , where it is easy to check that  $\gamma = 2/\alpha - 1$ . In fact, let  $E_0$  be the energy of the particle after the collision:

$$T = 2 \int_{x_0}^{x_{\max}} \frac{1}{v(x)} dx = \int_{x_0}^{x_{\max}} \frac{1}{\sqrt{E_0 - x^\alpha}} dx \quad x_{\max} = E_0^{1/\alpha}.$$

Performing the change of variable  $x = x_0^{1/\alpha} y$  we obtain:

$$T = \int_{x_0 E_0^{-1/\alpha}}^1 \frac{1}{E_0^{1/2} \sqrt{1 - y^\alpha}} E_0^{1/\alpha} dy = E_0^{\frac{1}{\alpha} - \frac{1}{2}} \text{Const} \cdot \left( 1 + \mathcal{O}\left(\frac{x_0^\alpha}{E_0}\right) \right) \sim v^{\frac{2}{\alpha} - 1}.$$

Notice that the asymptotic expression is exact for  $x_0 = 0$  or  $\alpha = 1$ .

Using the static wall approximation and considering  $T = C \cdot v^\gamma$ , the map  $F : (t_n, v_n) \mapsto (t_{n+1}, v_{n+1})$  can be written as follows:

$$F : \begin{pmatrix} t \\ v \end{pmatrix} \mapsto \begin{pmatrix} t + T(v) \\ v + 2\dot{\phi}(t + T(v)) \end{pmatrix}$$

which is (1.1) if we let  $(t, v) \rightarrow (x, y)$  and  $T \rightarrow Y$ . In the following, when useful, we will borrow the terminology from the physical problem also when referring to the model map (1.1). In particular we will often refer to the variable  $y$  as the *energy* of the system.

The map  $F$  defined in (1.1) is the composition of two translations, therefore it preserves the Lebesgue measure  $dx dy$ ; by considering  $Y(y)$  rather than  $y$  as the fundamental variable, we would obtain:

$$Y \mapsto C \left( \left( \frac{Y}{C} \right)^{1/\gamma} + 2\dot{\phi}(x + Y) \right)^\gamma \sim Y + Y^{1-1/\gamma} C' \dot{\phi}(x + Y) + \text{h.o.t.}$$

Thus, the map  $F$  appears to be similar to a standard map with perturbation parameter  $k = Y^{1-1/\gamma} C'$  that depends on  $y$ . We can distinguish between the following regimes:

- *weak potentials* ( $\gamma > 1$ ) · the perturbation parameter grows as energy grows; we can expect diffusion to high energies;
- *gravity* or *standard* ( $\gamma = 1$ ) · the system is equivalent to the standard map (unfolded on a semi-cylinder along  $y$ );
- *strong potentials* ( $0 < \gamma < 1$ ) · the perturbation parameter decreases as we increase the value of  $y$ ; there is persistence of invariant tori for large  $y$  and therefore we do not have diffusion to arbitrarily high energies;
- *elastic potential* ( $\gamma = 0$ ) · the function  $Y$  is constant, this is an exceptional case;

- *strong potentials (II)* ( $-1 < \gamma < 0$ ) · large values of  $y$  correspond to small values of  $Y$ ; we recover once more invariant tori bounding energies from above;
- *Fermi-Ulam ping pong* ( $\gamma = -1$ ) · once more we have invariant tori for high energies. Notice that in this case the static wall approximation ceases to be a good approximation because we neglect terms of the same order as  $Y$ .

Notice that, in order for the last two cases to make sense, we have to consider a slightly different function  $Y = \text{Const} \cdot |y - y^*|^\gamma$  for some fixed  $y^* > 0$ . The situation we will study in this dissertation is given by the first case; for each  $\gamma > 1$  we can see the corresponding map as a realization of an anti-integrable limit for the standard map.

## 1.4 Statement of the results

The purpose of this dissertation is to obtain results about abundance of unbounded orbits under iteration of the map  $F$ . Let  $(x_k, y_k) = F^k(x_0, y_0)$ , then the escaping set can be defined as:

$$\mathcal{E} = \{(x_0, y_0) \text{ s.t. } \lim_{n \rightarrow \infty} y_n = \infty\}.$$

The first result ensures that, provided  $\gamma > 1$ , the escaping set is not only non-empty, but it is also topologically large.

**Theorem A.** *If  $\gamma > 1$  then the escaping set  $\mathcal{E}$  has full Hausdorff dimension.*

The theorem is proved in chapter 2 and the proof involves the construction of a full dimensional subset of the escaping set using hyperbolic dynamics. On the other

hand, the conjecture for the bouncing balls system suggests that results analogous to theorem 1.2.3 should be valid for all  $\gamma > 1$ . In fact, theorem 1.2.3 can be easily adapted to our situation; the key ingredient for the proof is showing that the  $x$ -component of most trajectories approaches equidistribution. In the proof, valid up to  $\gamma > 5$ , the equidistribution estimates are obtained by bounding the expansion and distortion rates after a single iteration of  $F$  outside a so-called *critical set*. The idea to improve the condition on  $\gamma$  is to consider further iterates of the map: this allows to obtain stronger estimates outside a smaller critical set and leads us to prove:

**Theorem B.** *Let  $\gamma > 5/2$ , then the escaping set  $\mathcal{E}$  has zero Lebesgue measure.*

The proof of the theorem, given in chapter 3, relies on establishing two-step estimates and defining a suitable critical set. Establishing good equidistribution bounds in this case is considerably more complicated than in the one-step case. Nevertheless, we believe we can further optimize the process and be able to obtain better estimates by considering  $n$ -step estimates along with smaller critical sets. An obvious obstruction to equidistribution, however, is given by the presence of elliptic islands. Our last result deals with abundance of elliptic islands inside the critical set.

**Theorem C.** *Let  $\gamma > 1$ . Then:*

- a) *for almost all values of the parameter  $A$  there are elliptic islands of period 2 for arbitrarily high energies. Moreover, if  $\gamma > 2$  the same result holds for all values of  $A$ ;*

b) *the total measure of such islands is infinite if  $\gamma > 4/3$  and finite if  $\gamma \leq 4/3$ .*

This theorem is proved in chapter 4; we find such islands near homoclinic tangencies which enjoy particular symmetries and for which we can find relations in the parameter space that ensure their existence. This argument involves parameter exclusion techniques that show connections with the work of Young-Wang [YW08].

Theorems A and C will soon be published as a paper [D09].

## 1.5 Remarks

The techniques we developed in order to achieve our objective could be used to answer further natural questions which arise in the model, for instance to study abundance of oscillatory motions, or more optimistically, to establish the presence an ergodic component of infinite measure. In fact, as suggested by B. Fayad, one should be able to adapt the proof of theorem A to prove that the set of oscillatory orbits has also full Hausdorff dimension. Moreover it is likely that, either by a direct application of the results, or by applying the same techniques to the systems which are modeled by the transformation  $F$ , one could prove similar results for the concrete examples mentioned earlier.

The finite-step mixing bounds turn out to be the most sophisticated estimates we obtain in this work; however, they can be probably improved, but substantially more work has to be done in order to achieve better bounds. One problem that will surely arise is the given by abundance of elliptic islands, which prevents a priori good equidistribution estimates; in theorem C we only consider islands of period 2,

whereas we would need to control islands of higher period and more complicated combinatorics. We believe that this task can be accomplished by a suitable adaptation of the techniques developed by Gorodetsky-Kaloshin [GK07]. Such effort could be rewarded with a quite deep understanding of the dynamics of the standard map in the anti-integrable regime. In fact, as it is well known, in spite of all efforts, the existence of a positive measure set of orbits with positive entropy for the standard map has so far eluded all attempts of a proof. Improving the techniques developed in this work could possibly shed some light upon this very *resistant* problem. Finally, we have shown that questions about abundance of elliptic islands are related to certain questions of Diophantine approximations and it would be interesting to further explore this connection. For instance, in the case  $\gamma = 2$ , Elkies-McMullen[EM04] found and investigated a striking relation with flows on homogeneous spaces.

I wish to express my gratitude to my thesis advisor Dmitry Dolgopyat, who introduced me to the problem and followed me through the development of this work with interest and curiosity. I also want to thank Bassam Fayad and Carlangelo Liverani for their most precious comments and suggestions.



## Chapter 2

### Hausdorff Dimension of the escaping set

#### 2.1 Main definitions

We recall the definition of Hausdorff dimension of a metric space. First we define the *Hausdorff  $s$ -measure* of a metric space  $E$  as:

$$H^s(E) \doteq \limsup_{\delta \rightarrow 0} \inf_{\substack{\mathcal{A}=\{A_i\} \\ \delta\text{-covering of } E}} \left\{ \sum_i \text{diam}(A_i)^s \right\}.$$

Then we define the *Hausdorff dimension* of  $E$  as that critical  $s$  such that:

$$\dim_H E \doteq \inf\{s \text{ s.t. } H^s(E) = 0\} = \sup\{s \text{ s.t. } H^s(E) > 0\}.$$

It can be actually proved that if  $s < \dim_H(E)$  then  $H^s(E) = \infty$ ; moreover, Hausdorff dimension is a bilipschitz invariant of metric spaces.

We recall the definition of the escaping set; let  $(x_k, y_k) \doteq F^k(x_0, y_0)$  and define

$$\mathcal{E} \doteq \{(x_0, y_0) \text{ s.t. } y_n \rightarrow \infty \text{ as } n \rightarrow \infty\}.$$

**Theorem A.** *Assume  $\gamma > 1$ , then  $\dim_H \mathcal{E} = 2$ .*

The proof will be given in the next two sections. In the first one we prove an auxiliary result for a model system given by a sequence of expanding map on the circle. In the second part we reduce the proof of theorem A to the previously established result.

## 2.2 Model system

Fix  $\vartheta \in (0, 1)$  and let  $J_0 \in \mathbb{S}^1$  be a closed interval of length  $\vartheta$ . Fix two increasing sequences of positive real numbers  $\{\underline{m}_n\}$  and  $\{\overline{m}_n\}$  such that:

$$\forall n \in \mathbb{N} \quad 2\vartheta^{-1} < \underline{m}_n < \overline{m}_n; \quad \underline{m}_n, \overline{m}_n \nearrow \infty.$$

**Definition 2.2.1.** A continuous function  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is said to be  $n$ -admissible if a lift  $\hat{f} : \mathbb{S}^1 \rightarrow \mathbb{R}$  satisfies the following inequalities for all  $x, x' \in \mathbb{S}^1$ :

$$\underline{m}_n d(x, x') \leq |\hat{f}(x) - \hat{f}(x')| \leq \overline{m}_n d(x, x'),$$

where  $d$  is the standard Euclidean distance on  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ .

Then we can prove the following

**Lemma 2.2.2.** Let  $J_n \subset \mathbb{S}^1$  be a decreasing sequence of sets and let  $F_n$  be a sequence of continuous functions  $F_n : J_{n-1} \rightarrow \mathbb{S}^1$ . Assume for convenience  $F_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  to be the identity map and that  $\forall n \in \mathbb{N}$ :

- $J_n = \bigsqcup_k J_{n,k}$  where each  $J_{n,k}$  is a closed interval such that the restriction  $F_n : J_{n,k} \rightarrow \mathbb{S}^1$  is one to one;
- $F_{n+1}|_{J_{n,k}} = f_{n,k} \circ F_n$  where  $f_{n,k}$  is a  $n$ -admissible map
- $J_{n+1} = F_{n+1}^{-1}(J_0)$

and finally let

$$J = \bigcap_n J_n;$$

If there exists  $C \in \mathbb{R}^+$  such that for all large enough  $n$  we have  $\underline{m}_n \leq C\overline{m}_n$ , then  $\dim_H J = 1$ .

*Proof.* In order to compute  $\dim_H J$  we construct a subset  $J'$  obtained as a limit of a decreasing sequence of sets  $J'_n \subset J_n$  that we define as follows:  $J'_n \doteq \bigsqcup'_k J_{n,k}$  where the union ranges *only* on those  $k$  such that  $F_n : J_{n,k} \rightarrow J_0$  is one-to-one and onto.

We now introduce inductively what we will refer to as the *natural indexing* for the sets  $J_{n,k}$  contained in  $J'_n$ . Let  $J'_1 = \bigsqcup_k J_{1,k}$ ; we arbitrarily define  $J'_{[j_1]} = J_{1,k}$  for each  $J_{1,k} \subset J'_1$ . Then suppose we have already defined a natural indexing  $J'_{[j_1 \dots j_n]}$  for  $J'_n$ ; we label all  $J'_{n+1,k} \subset J'_{[j_1 \dots j_n]}$  as  $J'_{[j_1 \dots j_n j_{n+1}]}$  by arbitrarily choosing the index  $j_{n+1}$ . Notice that we purposefully avoided to specify a range for the  $j_k$ s; in fact each  $j_k$  ranges on an index set which depends on the previous choice of  $j_1 \dots j_{k-1}$ . Finally we let  $J' = \bigcap_n J'_n$ . Define now  $\underline{k}_n \in \mathbb{N}$  according to the relation:

$$\underline{k}_n + 1 \leq \underline{m}_n \vartheta < \underline{k}_n + 2 \quad \underline{k}_n \geq 1. \quad (2.1)$$

For each  $n \in \mathbb{N}$  let  $K_n$  be the number of intervals  $J'_{[j_1 \dots j_n]}$  in  $J'_n$ ; by definition of  $n$ -admissible function we have:

$$K_n \geq K_{n-1} \cdot \underline{k}_n \geq \prod_{j=1}^n \underline{k}_j \doteq \underline{K}_n.$$

We will now show that the set  $J'$  has Hausdorff dimension  $s = 1$ . First of all it is obvious that  $s \leq 1$  since  $J \subset \mathbb{S}^1$ , therefore it suffices to show that for all  $s < 1$  we have  $\dim_H(J') > s$ . To simplify the notation we will from now on write  $J'_{j_1 \dots j_n}$  for  $J'_{[j_1 \dots j_n]}$ .

**Definition 2.2.3.** The *running Hausdorff dimension* of  $\{j_1 \dots j_n\}$  is the real number  $s_{j_1 \dots j_n}$  satisfying

$$|J'_{j_1 \dots j_n}|^{s_{j_1 \dots j_n}} = K_n^{-1}.$$

The running Hausdorff dimension can be bounded using  $\{\underline{m}_n, \overline{m}_n\}$  and  $\vartheta$  according to the following estimate:

**Lemma 2.2.4.** *Suppose that the following holds for all large enough  $n$ :*

$$\overline{m}_n < C\underline{m}_n.$$

*Then we obtain the following lower bound for the running Hausdorff dimension:*

$$s_{j_1 \dots j_n} > 1 + \frac{\log \vartheta + n (\log \frac{\vartheta}{3} - \log C)}{\log \overline{M}_n - \log \vartheta} \doteq 1 - \varepsilon_n \nearrow 1 \text{ as } n \rightarrow \infty. \quad (2.2)$$

*Proof.* Using (2.1) we obtain the following estimates:

$$\underline{k}_n \leq \vartheta \underline{m}_n \leq \underline{k}_n + 2 \leq 3\underline{k}_n$$

$$\underline{K}_n \leq \vartheta^n \underline{M}_n \leq 3^n \underline{K}_n$$

Now since we know that  $|J'_{j_1 \dots j_n}| > \vartheta / \overline{M}_n$ , for all possible choices of  $j_1 \dots j_n$ , we can write:

$$\underline{K}_n (\vartheta / \overline{M}_n)^{s_{j_1 \dots j_n}} < K_n (\vartheta / \overline{M}_n)^{s_{j_1 \dots j_n}} < K_n |J'_{j_1 \dots j_n}|^{s_{j_1 \dots j_n}} = 1,$$

and taking logarithms we establish the following inequality:

$$\log \underline{K}_n - s_{j_1 \dots j_n} (\log \overline{M}_n - \log \vartheta) < 0.$$

Therefore we obtain the bound:

$$s_{j_1 \dots j_n} > \frac{\log \underline{M}_n + n \log \frac{\vartheta}{3}}{\log \overline{M}_n - \log \vartheta}.$$

Now by hypothesis we know that eventually  $\overline{M}_n < C^n \underline{M}_n$ , thus  $\log \overline{M}_n < n \log C + \log \underline{M}_n$  that in turn implies:

$$s_{j_1 \dots j_n} > \frac{\log \overline{M}_n + n (\log \frac{\vartheta}{3} - \log C)}{\log \overline{M}_n - \log \vartheta},$$

that yields estimate (2.2) provided that we show that  $\varepsilon_n$  is going to 0. In fact

$$\varepsilon_n \sim \frac{n}{\log \bar{M}_n} \rightarrow 0,$$

as  $\log \bar{M}_n/n$  is the average of the diverging sequence  $\log \bar{m}_n$ .  $\square$

To obtain a lower bound on the Hausdorff dimension of  $J'$  we are going to use the following two propositions:

**Proposition 2.2.5.** *Suppose there exists a probability measure  $\mu$  on a metric space  $X$  on the  $\sigma$ -algebra of Borel sets such that for all sufficiently small balls  $B$  we have:*

$$\mu(B) < C \text{diam}(B)^s, \quad (2.3)$$

then  $\dim_H X > s$ .

**Proposition 2.2.6.** *There exists a probability measure  $\mu$  on  $J'$  satisfying (2.3) for all  $s < 1$ .*

Propositions 2.2.5 and 2.2.6 imply that  $\dim_H J' > s$  for all  $s < 1$ , which concludes the proof of lemma 2.2.2.  $\square$

Proposition 2.2.5 is a classical result, hence the proof will be omitted (see e.g. [Fa86]).

*Proof of proposition 2.2.6.* We first build a probability measure  $\mu$  on  $J'$ , and then check that  $\mu$  satisfies (2.3) for every  $s < 1$ . For each  $n$  and choice of a natural index  $j_1 \cdots j_n$ , fix a point  $x_{j_1 \cdots j_n} \in J'_{j_1 \cdots j_n}$ . Then define the following sequence of positive functionals acting on  $\mathcal{C}(\mathbb{S}^1, \mathbb{R})$ :

$$\forall \varphi \in \mathcal{C}(\mathbb{S}^1, \mathbb{R}) \quad \Phi_n(\varphi) \doteq \sum_{j_1, \dots, j_n} \frac{1}{K_n} \varphi(x_{j_1 \cdots j_n}).$$

We now argue that this sequence of functionals has a weak limit for  $n \rightarrow \infty$ . In fact any continuous function  $\varphi$  on  $\mathbb{S}^1$  is also uniformly continuous; therefore  $\forall \epsilon \exists \delta$  such that  $d(x, x') < \delta$  implies  $|\varphi(x) - \varphi(x')| < \epsilon$ . Now take  $n$  such that  $\max_{j_1, \dots, j_n} |J'_{j_1, \dots, j_n}| < \delta$ . Then for each  $m > n$ :

$$\begin{aligned}
|\Phi_n(\varphi) - \Phi_m(\varphi)| &= \left| \sum_{j_1, \dots, j_n} \frac{1}{K_n} \varphi(x_{j_1 \dots j_n}) - \sum_{j_1, \dots, j_m} \frac{1}{K_m} \varphi(x_{j_1 \dots j_m}) \right| = \\
&= \left| \sum_{j_1, \dots, j_n} \left( \frac{1}{K_n} \varphi(x_{j_1 \dots j_n}) - \sum_{j_{n+1}, \dots, j_m} \frac{1}{K_m} \varphi(x_{j_1 \dots j_m}) \right) \right| \leq \\
&\leq \sum_{j_1, \dots, j_m} \frac{1}{K_m} |\varphi(x_{j_1 \dots j_n}) - \varphi(x_{j_1 \dots j_m})| \leq \\
&\leq \sum_{j_1, \dots, j_m} \frac{1}{K_m} \epsilon = \epsilon.
\end{aligned}$$

In the above inequalities we used the fact that by definition

$$\sum_{j_{n+1}, \dots, j_m} K_n / K_m = 1$$

and that  $J'_{j_1 \dots j_m} \subset J'_{j_1 \dots j_n}$  which implies  $x_{j_1 \dots j_n}, x_{j_1 \dots j_m} \in J'_{j_1 \dots j_n}$ , therefore  $d(x_{j_1 \dots j_n}, x_{j_1 \dots j_m}) < \delta$ . The sequence  $\Phi_n$  weakly converges to a positive functional  $\Phi$  i.e. to a Borel measure  $\mu$  via the Riesz representation theorem. Moreover  $\mu$  is a probability measure (it suffices to compute the limit of  $\Phi_n$  against the function  $\varphi \equiv 1$ ).

At this point for any Borel set  $E$  and  $n \in \mathbb{N}$  we can write:

$$\begin{aligned}
\mu(E) &= \lim_{m \rightarrow \infty} \sum_{j_1, \dots, j_m} \frac{1}{K_m} \chi_E(x_{j_1 \dots j_m}) \\
&= \sum_{j_1, \dots, j_n} \lim_{m \rightarrow \infty} \sum_{j_{n+1}, \dots, j_m} \frac{1}{K_m} \chi_E(x_{j_1 \dots j_n j_{n+1} \dots j_m}) \\
&\leq \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap E \neq \emptyset}} \frac{1}{K_n} \lim_{m \rightarrow \infty} \sum_{j_{n+1}, \dots, j_m} \frac{K_n}{K_m} = \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap E \neq \emptyset}} \frac{1}{K_n},
\end{aligned}$$

By definition of running Hausdorff dimension this implies the following estimate for  $\mu(E)$ :

$$\mu(E) \leq \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap E \neq \emptyset}} |J'_{j_1 \dots j_n}|^{s_{j_1 \dots j_n}}. \quad (2.4)$$

In order to obtain estimate (2.3), fix a ball  $B$  of radius  $r = \rho\theta$ . Then  $2\rho \in [\underline{M}_n^{-1}, \underline{M}_{n-1}^{-1})$  for some  $n$ . Now subdivide  $[\underline{M}_n^{-1}, \underline{M}_{n-1}^{-1})$  in  $\underline{m}_n - 1$  intervals each of length  $\underline{M}_n^{-1}$ . Let  $l > 0$  be such that  $2\rho \in [l\underline{M}_n^{-1}, (l+1)\underline{M}_n^{-1})$ . This means that

$$\frac{1}{l+1}2\rho \leq \underline{M}_n^{-1} < \frac{1}{l}2\rho.$$

Using the previous estimate we know that each interval of  $J'_n$  contains a ball of radius  $\rho/(l+1)$ , therefore a ball of radius  $\rho$  can intersect at most  $l+1+2$  such intervals. Using (2.2) and (2.4) we obtain:

$$\begin{aligned} \mu(B) &\leq \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap B \neq \emptyset}} |J'_{j_1 \dots j_n}|^{s_{j_1 \dots j_n}} < \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap B \neq \emptyset}} |J'_{j_1 \dots j_n}|^{1-\varepsilon_n} \\ &\leq \sum_{\substack{j_1, \dots, j_n \\ J'_{j_1 \dots j_n} \cap B \neq \emptyset}} (\vartheta \underline{M}_n)^{-(1-\varepsilon_n)} \leq (l+3)(\vartheta \underline{M}_n)^{-(1-\varepsilon_n)} \\ &\leq (l+3) \left(\frac{2r}{l}\right)^{1-\varepsilon_n} = \frac{l+3}{l^{1-\varepsilon_n}} |B|^{1-\varepsilon_n}. \end{aligned}$$

If  $n$  is large enough, the fraction equals  $l^{\varepsilon_n} + \mathcal{O}(1)$ , which a priori may be not bounded; but we have:

$$l^{\varepsilon_n} < \underline{m}_n^{\varepsilon_n} < \underline{M}_n^{\varepsilon_n},$$

and:

$$\begin{aligned}
\underline{M}_n^{\varepsilon_n} &= \exp \left( \frac{\log \underline{M}_n \log \vartheta + n \left( \log \frac{\vartheta}{3} - \log C \right)}{\log \overline{M}_n - \log \vartheta} \right) \\
&= \exp \left( \frac{\log \underline{M}_n - \log \vartheta + \log \vartheta}{\log \overline{M}_n - \log \vartheta} \left( \log \vartheta + n \left( \log \frac{\vartheta}{3} - \log C \right) \right) \right) \\
&\leq \exp \left( \left( 1 + \frac{\log \vartheta}{\log \overline{M}_n - \log \vartheta} \right) \left( \log \vartheta + n \left( \log \frac{\vartheta}{3} - \log C \right) \right) \right) \\
&= \exp \left( \log \underline{M}_{n-1} \left( 1 + \frac{\log \vartheta}{\log \overline{M}_n - \log \vartheta} \right) \frac{\log \vartheta + n \left( \log \frac{\vartheta}{3} - \log C \right)}{\log \underline{M}_{n-1}} \right) \\
&\doteq \underline{M}_{n-1}^{\varepsilon'_n},
\end{aligned}$$

with  $\varepsilon'_n \rightarrow 0$  as  $n \rightarrow \infty$ . Then

$$l^{\varepsilon_n} < \underline{M}_{n-1}^{\varepsilon'_n} < (2\rho)^{-\varepsilon'_n}.$$

So we finally obtain  $\mu(B) < C|B|^{1-\varepsilon_n-\varepsilon'_n}$ . This estimate still depends on  $n$  therefore on  $|B|$ , but notice that  $\varepsilon_n + \varepsilon'_n$  is monotone decreasing to 0, therefore if we fix  $n$  the inequality will hold for all  $B$  such that  $|B| < \underline{M}_{n-1}^{-1}$ . At this point it is easy to see that  $\forall \epsilon > 0 \exists \bar{\delta}$  such that any  $\delta$ -ball  $B$  with  $\delta < \bar{\delta}$  will satisfy inequality (2.3):

$$\mu(B) < C \text{diam}(B)^{1-\epsilon}.$$

□

### 2.3 Reduction to the model system

In this section we show that lemma 2.2.2 can be applied to our dynamical system to prove theorem A. We first build a set such that the dynamics of orbits that never leave this set is hyperbolic; a quite more elaborate version of this construction



will be introduced in the next chapter. For  $a > 0$  small we define the set

$$B_a \doteq \left\{ (x, y) \text{ s.t. } |\ddot{\phi}(x)| > a \right\};$$

we write for reference the differential  $dF$ :

$$dF = \begin{pmatrix} 1 & Y'(y) \\ 2\ddot{\phi}(x + Y(y)) & 1 + 2Y'(y)\ddot{\phi}(x + Y(y)) \end{pmatrix}.$$

**Lemma 2.3.1.** *If  $y$  is large enough and  $F(x, y) \in B_a$  then  $dF$  is hyperbolic.*

*Proof.* It suffices to check that  $\text{Tr}(dF) = 2(1 + Y'(y)\ddot{\phi}(x + Y(y))) > 2$ , but for large enough  $y$ ,  $Y'(y) \gg 1$ , therefore, since by hypothesis  $|\ddot{\phi}(x + Y(y))| > a$  we have hyperbolicity.  $\square$

We now want to find an invariant cone field. In order to do so we consider the direction corresponding to the expanding eigenvector in the limit  $Y' \rightarrow \infty$ , corresponding to high energies. We claim that a small cone around this direction is invariant for large enough  $y$ . In fact the eigenvectors of  $dF$  in the above limit are

$$V_+ = (\delta x, \delta y) = (1, 2\ddot{\phi}(x + Y(y)))$$

$$V_- = (\delta x, \delta y) = (1, 0)$$

Therefore, having fixed a small  $0 < c < a$ , the cone field defined by the following expression:

$$\mathcal{C}_{(x,y)} \doteq \left\{ (\delta x, \delta y) \text{ s.t. } \left| \frac{\delta y}{\delta x} - 2\ddot{\phi}(x) \right| < c \right\} \quad (2.5)$$

is invariant on  $F^{-1}B_a$  for large enough  $y$  because  $V_+$  and  $V_-$  are well separated on  $F^{-1}B_a$  and the expanding eigenvalue grows arbitrarily large. This means that if we

take a curve whose tangent vectors lie in  $\mathcal{C}$  and we apply  $F$  we are going to obtain (on the hyperbolic set) a curve whose tangent vectors again lie in the cone field; moreover since the vectors tangent to the curve are close to the expanding direction of the map, the dynamics along the curve will also be expanding. For such reasons we now define

$$\Gamma_C = \{(x, y) = (x, C + 2\dot{\phi}(x))\}.$$

By the cone condition, orbits that never leave the set  $B_a$  are hyperbolic; moreover each curve  $\Gamma_C$  will be transversal to the stable direction at each point of the hyperbolic set. Now define the following set:

$$A_\varepsilon \doteq \{(x_n, y_n) \text{ s.t. } y_n - y_{n-1} > \varepsilon\}.$$

Since  $y_n - y_{n-1} = 2\dot{\phi}(x_n)$ , we have that  $\{2\dot{\phi}(x) > \varepsilon\} = A_\varepsilon$ . We can therefore select values of  $a$ ,  $\varepsilon$  and  $\bar{y}$  such that there exists an interval  $J_0 \subset \mathbb{S}^1$  satisfying:

$$J_0 \times \{y \geq \bar{y}\} \subset A_\varepsilon \cap B_a,$$

and  $\bar{y}$  is large enough for lemma 2.3.1 to hold true and for the cone field in (2.5) to be invariant for any  $y \geq \bar{y}$ . We now define a sequence of sets  $J_n$  and of functions  $F_n$  satisfying the hypotheses of lemma 2.2.2. Let  $\pi : \mathcal{M} \rightarrow \mathbb{S}^1$  be the projection onto the  $x$ -component, fix  $\Gamma_C = \{x, \psi(x)\}$  and let:

$$F_n \doteq \pi \circ F^n \circ \psi : x_0 \mapsto x_n$$

$$J_n \doteq \bigcap_{k=0}^n F_k^{-1} J_0.$$

**Lemma 2.3.2.** *For large enough  $C$  there exist positive constants  $\tilde{C}_1, \tilde{C}_2, \underline{C}, \overline{C}$  such that:*

$$\forall x \in J_k \quad \tilde{C}_1 (\underline{C} + \varepsilon k)^{\gamma-1} < \left| \frac{F'_{k+1}(x)}{F'_k(x)} \right| < \tilde{C}_2 (\overline{C} + 3Ak)^{\gamma-1}.$$

*Proof.* We know that  $x_{k+1} = x_k + Y(y_k)$ , thus:

$$\frac{dx_{k+1}}{dx_k} = 1 + Y'(y_k) \frac{dy_k}{dx_k}.$$

If  $x \in J_k$  and  $C$  is large enough, we know that the cone field  $\mathcal{C}$  is invariant i.e.

$(1, \frac{dy_k}{dx_k}) \in \mathcal{C}$ , therefore

$$\left| \frac{dy_k}{dx_k} - 2\ddot{\phi}(x_k) \right| < c,$$

that implies

$$\frac{dx_{k+1}}{dx_k} = \mathcal{O}(1) + \text{Const } y_k^{\gamma-1} (\ddot{\phi}(x_k) + \mathcal{O}(c)).$$

and since  $(x_k, y_k) \in B_a$  and  $c < a$  we can find positive  $\tilde{C}_1$  and  $\tilde{C}_2$  such that

$$\tilde{C}_1 y_k^{\gamma-1} < \left| \frac{dx_{k+1}}{dx_k} \right| < \tilde{C}_2 y_k^{\gamma-1}.$$

Now since each  $(x_k, y_k) \in A_\varepsilon$  we have the following bounds on  $y_k$ :

$$\underline{C} + \varepsilon k < y_k < \overline{C} + 3Ak,$$

where  $\underline{C}$  and  $\overline{C}$  are respectively the minimum and the maximum  $y$  of the curve  $\Gamma$

and by (1.2),  $A$  is the maximum of  $\dot{\phi}(x)$ . Therefore

$$\tilde{C}_1 (\underline{C} + \varepsilon k)^{\gamma-1} < \left| \frac{dx_{k+1}}{dx_k} \right| < \tilde{C}_2 (\overline{C} + 3Ak)^{\gamma-1}$$

which is the required inequality. □

The previous lemma shows that each function  $f : x_k \rightarrow x_{k+1}$  is  $k$ -adapted with respect to the sequences  $\underline{m}_k, \overline{m}_k$  defined as:

$$\underline{m}_k \doteq \tilde{C}_1 (\underline{C} + \varepsilon k)^{\gamma-1} \quad \overline{m}_k \doteq \tilde{C}_2 (\overline{C} + 3Ak)^{\gamma-1}.$$

Finally we can take  $C$  big enough so that we have  $\underline{m}_n |J_0| > 2$  and notice that eventually  $\overline{m}_k < \tilde{C}_2 / \tilde{C}_1 (\frac{4A}{\varepsilon})^{\gamma-1} \cdot \underline{m}_k$ . Thus we can apply lemma 2.2.2 to  $J$  and conclude that for large enough  $C$  all curves  $\Gamma_C$  are such that

$$\dim_H \Gamma_C \cap \bigcap_n F^{-n}(A_\varepsilon \cap B_a) = 1.$$

On the other hand we have that orbits belonging to  $\Gamma_C \cap \bigcap_n F^{-n}(A_\varepsilon \cap B_a)$  are uniformly hyperbolic and escaping. Since  $\Gamma_C$  is transversal to the stable direction we conclude that  $\dim_H \mathcal{E} = 2$ .

## Chapter 3

### Measure of the escaping set

In this section we will prove that the Lebesgue measure of the set of escaping points is zero under some assumptions on the parameter  $\gamma$ . In what follows, to simplify the notation, we again let  $(x_k, y_k) \doteq F^k(x_0, y_0)$ . Recall that:

$$\phi(x) \doteq \frac{A}{2\pi} \sin(2\pi x) \tag{1.2}$$

and that the escaping set is defined as :

$$\mathcal{E} \doteq \{(x_0, y_0) \text{ s.t. } \lim_{n \rightarrow \infty} y_n = \infty\}.$$

The main result of this chapter is the following

**Theorem B.** *Let  $\gamma > 5/2$ ; then the escaping set  $\mathcal{E}$  has zero Lebesgue measure.*

The idea behind the proof is to consider the system as a slow-fast system, the slow variable being the  $y$ -coordinate. Since the map  $F$  is exact, sufficiently fast equidistribution along the  $x$ -coordinate should imply recurrence by comparison with a one-dimensional random walk. However, we cannot obtain good equidistribution estimates on orbits passing through a so-called *critical set*. For  $\gamma$  big enough we can define a finite measure critical set which contains all points of the phase space with small enough energy. Orbits that land on the critical set infinitely many times can be thus recurrent by means of the Poincaré recurrence theorem. We then prove that such orbits form a full-measure set in the phase space by using the previous

comparison argument. To obtain good equidistribution estimates we decompose the invariant Lebesgue measure into suitably defined measures over expanding curves. Such objects are called *standard pairs*; studying the induced dynamics on standard pairs outside the critical set allows us to prove equidistribution bounds along the  $x$  coordinate for  $F$ .

The strategy of the proof of theorem B closely follows the one used in [Do08]; several estimates, however, need to be quite substantially improved. In particular we need to establish two-step equidistribution estimates (lemma 3.4.3); we believe that such bounds can be the first step to obtain sharper  $n$ -step estimates that should in principle allow us to prove theorem B for smaller values of  $\gamma$ .

### 3.1 Induced dynamics

It is convenient to define the set of *basic pairs*, which are a pair of a curve of length  $\mathcal{O}(1)$  and a probability density over the curve.

**Definition 3.1.1.** A *basic curve* is a curve  $\Gamma \subset \mathcal{M}$  which is a graph of a smooth function  $\psi(x)$  over an interval  $I \subset \mathbb{S}^1$  of length  $\delta < |I| < 2\delta$  for some  $\delta > 0$  to be fixed. A *basic pair* is given by a basic curve  $\Gamma$  and a strictly positive smooth probability density  $\rho(x)$  on  $\Gamma$ ; we denote a basic pair by  $\ell = (\Gamma, \rho)$ .

Given a real valued Borel measurable function  $\mathcal{A}(x, y)$  we define:

$$\mathbb{E}_\ell(\mathcal{A}) \doteq \int_\Gamma \mathcal{A} \cdot \rho \, dx = \int_I \mathcal{A}(x, \psi(x)) \rho(x) \, dx,$$

and given a Borel measurable set  $E$ :

$$\mathbb{P}_\ell(E) \doteq \mathbb{E}_\ell(1_E).$$

A dot will denote differentiation with respect to the variable  $x$ ; the slope of a basic curve will be denoted by  $h(x) \doteq \dot{\psi}(x)$  and for convenience we will consider  $h \in \mathbb{R} \cup \{\infty\}$ ; the logarithmic derivative of  $\rho$  will be denoted by  $r(x) \doteq \rho^{-1}(x)\dot{\rho}(x)$ . Finally, given a basic pair  $\ell = (\Gamma, \rho)$  we define:

$$\hat{y}_\ell = \hat{y}_\Gamma \doteq \inf_{(x,y) \in \Gamma} y.$$

We require  $2\delta$  to be smaller than the minimum distance between the critical points of  $\dot{\phi}$ . In our case (1.2) implies that it is enough to require  $\delta \leq 1/10$ .

For future reference, we introduce here the formulae for the push-forward  $F^*$  at the point  $(x, y)$  of all relevant quantities associated to a basic pair; they can be readily computed from definition (1.1).

$$\rho \mapsto \frac{1}{\mathcal{L}} \rho \tag{3.1a}$$

$$r \mapsto \frac{r}{\mathcal{L}} - \frac{\dot{h}Y'}{\mathcal{L}^2} - \frac{Y''}{Y'^2} \left(1 - \frac{1}{\mathcal{L}}\right)^2 \tag{3.1b}$$

$F_{(x,y)}^*$  :

$$h \mapsto 2\ddot{\phi}(x+Y) + \frac{1}{Y'} \left(1 - \frac{1}{\mathcal{L}}\right) \tag{3.1c}$$

$$\dot{h} \mapsto 2\ddot{\phi}(x+Y) + \frac{\dot{h}}{\mathcal{L}^3} - \frac{Y''}{Y'^3} \left(1 - \frac{1}{\mathcal{L}}\right)^3 ; \tag{3.1d}$$

in the above expressions (3.1), we consider  $Y = Y(y)$  (similarly for  $Y'$  and  $Y''$ ) and  $\mathcal{L} = \mathcal{L}(x, y)$  defined as  $1 + h(x, y)Y'(y)$ . Notice that if  $\Gamma$  is a curve of slope  $h$  we have:

$$\mathcal{L}(x_0, y_0) = \left. \frac{dx_1}{dx_0} \right|_{\Gamma} (x_0);$$

for convenience we also define the *adapted slope*  $\tilde{h}$ :

$$\tilde{h}(x, y) = h(x, y) + 1/Y'(y)$$

so that we can conveniently write  $\mathcal{L}(x, y) = \tilde{h}(x, y)Y'(y)$ . We will consider (3.1b-d) as the defining equations for an induced map on the fiber bundle over  $\mathcal{M}$  given by  $\mathcal{F} \ni (x, y, r, h, \dot{h})$ ; this observation allows us to use a natural and convenient geometrical terminology in what follows. In particular we will be able to define “nice” pairs by specifying “nice” sections of the bundle  $\mathcal{F}$ . Notice that we do not include the density  $\rho$  as a coordinate in the bundle  $\mathcal{F}$  because it is a non local quantity; on the other hand  $r, h, \dot{h}$  are all local quantities associated respectively to distortion, slope and curvature.

We now proceed to the definition of *standard curves*, which will be given in order for them to enjoy good averaging and invariance properties.

Let  $V$  be the constant unit vertical vector field and  $h_0$  be the associated slope field, i.e.

$$V(x, y) \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad h_0(x, y) \equiv \infty;$$

fix  $D_0 \in \mathbb{R}^+$  to be specified later and let  $\mathcal{C}_0$  be the cone field around  $V$  defined as follows:

$$\mathcal{C}_0(x, y) = \left\{ v \in \mathbf{T}_{(x,y)}\mathcal{M} \text{ s.t. } v \propto \begin{pmatrix} 1 \\ h \end{pmatrix}, |\tilde{h}| > D_0 Y'(y)^{-1/2} \right\};$$

with a slight abuse of notation, we will often consider the set  $\mathcal{C}_0(x, y)$  as the intersection of the following subset of  $\mathcal{F}$ :

$$\mathcal{C}_0 = \left\{ (x, y, r, h, \dot{h}) \in \mathcal{F} \text{ s.t. } |\tilde{h}| > D_0 Y'(y)^{-1/2} \right\}$$



with the fiber of  $\mathcal{F}$  over  $(x, y)$ .

For each  $k \in \mathbb{Z}$  define a slope field  $h_k$  and a cone field  $\mathcal{C}_k$ , given at any point  $(x, y) = (x_k, y_k)$  by the following expressions:

$$h_k(x_k, y_k) = F_{(x_0, y_0)}^{*k} h_0(x_0, y_0)$$

$$\mathcal{C}_k(x_k, y_k) = F_{(x_0, y_0)}^{*k} \mathcal{C}_0(x_0, y_0).$$

By definition it is clear that, for  $l > k$ , we have the following compatibility condition:

$$\mathcal{C}_l(x_k, y_k) \subset \mathcal{C}_k(x_k, y_k) \iff \mathcal{C}_{l-k}(x_0, y_0) \subset \mathcal{C}_0(x_0, y_0).$$

Given a cone  $\mathcal{C}$ , define its *cone width* at  $(x, y)$  as:

$$|\mathcal{C}(x, y)| = \sup_{h, h' \in \mathcal{C}(x, y)} |h - h'|.$$

Notice that, by definition,  $|\mathcal{C}_0| = \infty$ ; moreover from the formula for the induced slope (3.1c) we obtain, for  $k > 0$ :

$$\begin{aligned} |\mathcal{C}_{k+1}(x_1, y_1)| &= \frac{1}{Y'(y_0)^2} \sup_{h, h' \in \mathcal{C}_k(x_0, y_0)} \left| \frac{1}{\tilde{h}} - \frac{1}{\tilde{h}'} \right| \\ &\leq \underline{\mathcal{L}}^{-2} \cdot |\mathcal{C}_k(x_0, y_0)| (1 + \mathcal{O}(|\mathcal{C}_k(x_0, y_0)|/\underline{\mathcal{L}})), \end{aligned} \quad (3.2)$$

where

$$\underline{\mathcal{L}} = \inf_{h \in \mathcal{C}_k(x_0, y_0)} |\tilde{h} Y'(y_0)|$$

is the least expansion rate at the point  $(x_0, y_0)$  of a curve compatible with  $\mathcal{C}_k$  in  $(x_0, y_0)$ ; by direct computation we have  $|\mathcal{C}_1(x_1, y_1)| \leq 2(D_0 Y'(y_0)^{3/2})^{-1}$ .

**Definition 3.1.2.** We define a *reference section* as a section on  $\mathcal{F}$ :

$$\sigma : (x, y) \mapsto (x, y, r, h, \dot{h})$$

given by the following equations:

$$h(x_0, y_0) \doteq h_1(x_0, y_0) = 2\ddot{\phi}(x_0) + \frac{1}{Y'(y_{-1})} \quad (3.3a)$$

$$\dot{h}(x_0, y_0) \doteq \dot{h}_1(x_0, y_0) = 2\ddot{\phi}(x_0) - \frac{Y''(y_{-1})}{Y'^3(y_{-1})} \quad (3.3b)$$

$$r(x_0, y_0) \doteq 0 \quad (3.3c)$$

A *reference curve* is defined as an integral curve of the reference slope field  $h_1$ , i.e it can be written as  $(x, \psi_1(x))$  where:

$$\psi_1(x) \doteq 2\dot{\phi}(x) + Y^{-1}(x + c)$$

for some  $c \in \mathbb{R}^+$ . A *reference pair* is a basic pair given by a reference curve  $\Gamma$  over  $I$  and the uniform probability density  $\rho(x) \equiv |I|^{-1}$  over  $\Gamma$ .

We now can proceed to define *standard sections*; standard sections are to the reference section as the unstable cone is to the unstable direction in a uniformly hyperbolic setting.

**Definition 3.1.3.** A section  $\sigma$  on  $\mathcal{F}$  is said to be *standard* if it is close to the the reference section in the following sense:

$$\sigma(x, y) \in \left\{ (x, y, r, h, \dot{h}) \in \mathcal{C}_1 \text{ s.t. } \left| \dot{h} - \dot{h}_1 \right| < \frac{A}{10}, |r| < 1 \right\}$$

A *standard pair* is a basic pair  $\ell = (\Gamma, \rho)$  such that  $(x, y, h, \dot{h}, r)$  is a standard section over  $\Gamma$ , which will be called a *standard curve*. The next lemma ensures that standard curves are globally close to reference curves.

**Lemma 3.1.4.** Let  $\Gamma = (x, \psi(x))$  be a curve over  $I$  such that  $|I| < 1$ , let  $\bar{\Gamma} = (x, \bar{\psi}(x))$  be a reference curve over  $I$  which intersects  $\Gamma$ . Define  $\|\Delta h\|_\Gamma = \sup_{x' \in I} |h(x') -$

$h_1(x', \psi(x'))$ ); then:

$$\forall x \in I \quad |\psi(x) - \bar{\psi}(x)| < 2\|\Delta h\|_{\Gamma}|I|.$$

*Proof.* Let  $\hat{y} = \min\{\hat{y}_{\Gamma}, \hat{y}_{\bar{\Gamma}}\}$  and let

$$\mu = \left| \frac{\partial h_1}{\partial y}(\hat{y}) \right|;$$

by definition we have  $\mu \leq \text{Const} \cdot Y''(\hat{y})/Y'(\hat{y})^2$ , moreover  $\forall y > \hat{y}$  we have  $\left| \frac{\partial h_1}{\partial y}(y) \right| \leq \mu$  and we can write:

$$\begin{aligned} \left| \frac{d}{dx} (\psi(x) - \bar{\psi}(x)) \right| &\leq |h(x) - h_1(x, \psi(x))| + |h_1(x, \psi(x)) - h_1(x, \bar{\psi}(x))| \\ &\leq \|\Delta h\|_{\Gamma} + \mu|\psi(x) - \bar{\psi}(x)|. \end{aligned}$$

Let  $J \subset I$  be the connected component of the set  $\{|\psi(x) - \bar{\psi}(x)| < 2\|\Delta h\|_{\Gamma}\}$  containing the  $x$ -coordinate of a point in  $\Gamma \cap \bar{\Gamma}$ ; for all  $x \in J$  we have then:

$$\left| \frac{d}{dx} (\psi(x) - \bar{\psi}(x)) \right| \leq (1 + 2\mu)\|\Delta h\|_{\Gamma} \leq 2\|\Delta h\|_{\Gamma}$$

for large enough  $\hat{y}$ . This implies that  $J = I$ , thus:

$$|\psi(x) - \bar{\psi}(x)| \leq 2\|\Delta h\|_{\Gamma}|I|$$

which concludes the proof. □

### 3.2 Critical sets

We want to establish results regarding invariance properties of standard sections; in order to do so we need to obtain good geometrical and distortion bounds (to control  $r$ ,  $h$  and  $\dot{h}$ ) for the map  $F$ . Such bounds cannot be established everywhere;

points where this is not possible will belong to a set that we will call *critical set*. Clearly the definition of the critical sets depends on what we consider as a “good” bound, and therefore it is far from being unique. We define a *first* and a *second* critical set as follows:

$$\begin{aligned}\mathbf{C}_1 &\doteq \left\{ (x_0, y_0) \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < K_1 Y'(y_0)^{-1/2} \right\} \\ \mathbf{C}_2 &\doteq \mathbf{C}_1 \cap \left\{ (x_0, y_0) \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < K_2 |\tilde{h}_1(x_1, y_1) Y'(y_0)^{1/2} Y'(y_1)^{1/2}|^{-1} \right\}\end{aligned}$$

where we require  $K_1 > 2D_0$  and  $K_2 > 2K_1^2$ . Notice that by definition, if a standard curve lies outside  $\mathbf{C}_1$ , then it is compatible with  $\mathcal{C}_0$ . Now let  $K_3 < D_0^2/A$  and define:

$$\bar{\mathbf{C}}_2 \doteq \left\{ (x_0, y_0) \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < K_3 Y'(y_0)^{-1} \right\}.$$

For convenience we will often use the following notation:

$$\mathbf{C}(y^*) = \mathbf{C} \cap \{y = y^*\}, \text{ for } \mathbf{C} = \mathbf{C}_1, \mathbf{C}_2, \bar{\mathbf{C}}_2.$$

The following observation

$$Y'(y_k) = Y'(y_0) (1 + \mathcal{O}(|k|y_0^{-1}))$$

implies that, given  $k$ :

$$\forall \varepsilon > 0 \exists \bar{y} \text{ s.t. } y_0 > \bar{y} \Rightarrow (1 - \varepsilon)Y'(y_0) < Y'(y_k) < (1 + \varepsilon)Y'(y_0). \quad (3.4)$$

This simple but useful remark allows us, for instance, to show that for large enough  $y$ :

$$\bar{\mathbf{C}}_2(y) \subset \mathbf{C}_2(y)$$

in fact this easily follows from (3.4) and the fact that  $|\tilde{h}|$  is bounded by  $3A$  for large enough  $y$ . We require that all standard curves are expanded outside  $\mathbf{C}_2$  for large

enough  $y$ , therefore we assume  $K_3 > 2$ . The set  $\bar{\mathbf{C}}_2$  we just defined will be called the *core* of the critical set  $\mathbf{C}_2$ . Moreover, from the definitions it is easy to check that  $\mathbf{C}_1 \cap F^{-1}\mathbf{C}_1 \subset \mathbf{C}_2$ . Finally notice that  $\forall y_*$  there exists a  $\bar{D}$  such that if  $D_0 > \bar{D}$  we have

$$\mathbf{C}_1 \supset \{y < y_*\}, \quad \mathbf{C}_2 \supset \{y < y_*\}.$$

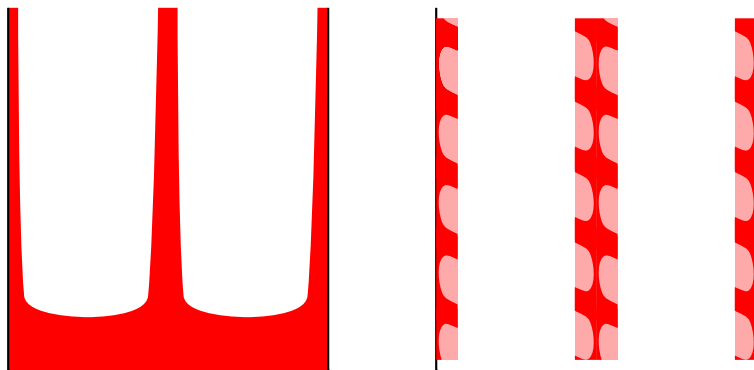


Figure 3.1: On the left picture we see  $\mathbf{C}_1$  as the shaded set; on the right picture we see for large enough  $y$  a detail of  $\mathbf{C}_1$  (the light-shaded set) and the structure of  $\mathbf{C}_2$  (dark-shaded set).

**Lemma 3.2.1.** *Recall that  $Y \propto y^\gamma$  and let  $\beta = \frac{1}{2}(\gamma - 1)$ ; then we have:*

- (a) *Leb( $\mathbf{C}_1$ ) is finite if and only if  $\beta > 1$ ;*
- (b) *Leb( $\mathbf{C}_2$ ) is finite if and only if  $\beta > 1/2$ .*

*Proof.* To prove (a) notice that by definition:

$$\left| \tilde{h}_1(x_0, y_0) \right| = \left| 2\ddot{\phi}(x_0) + 1/Y'(y_{-1}) + 1/Y'(y_0) \right|;$$

using (3.4) we can write, for large enough  $y$ :

$$\begin{aligned} \mathbf{C}_1(y) &\subset \left\{ (x, y) \text{ s.t. } \left| 2\ddot{\phi}(x) \right| < 2K_1 Y'(y)^{-1/2} \right\} \\ &\subset \left\{ |x| < \text{Const} \cdot Y'(y)^{-1/2} \right\} \cup \left\{ |x - 1/2| < \text{Const} \cdot Y'(y)^{-1/2} \right\}. \end{aligned}$$

Denote the two sets that appear in the last expression by  $\mathbf{C}_1^{(0)}$  and  $\mathbf{C}_1^{(1)}$  respectively; the Lebesgue measure of  $\mathbf{C}_1^{(i)}$  is finite if the function  $Y'^{-1/2}$  is integrable at  $\infty$ , i.e. if  $\beta > 1$ . In the same way we can obtain a lower bound, so that if  $\beta \leq 1$  then the measure  $\text{Leb}(\mathbf{C}_1) = \infty$ .

In order to prove (b), first define, for  $i \in \{0, 1\}$  and  $n \in \mathbb{N}$ :

$$\mathbf{C}_2^{(i,n)} \doteq \mathbf{C}_2 \cap \mathbf{C}_1^{(i)} \cap \{(x, y) \text{ s.t. } x + Y(y) \in [n/2, (n+1)/2]\};$$

also let  $\hat{y}_n = \inf_{(x,y) \in \mathbf{C}_2^{(i,n)}} y \sim n^{1/\gamma}$ . Then, for each  $\mathbf{C}_2^{(i,n)}$ , consider the following decomposition (see also figure 3.2):

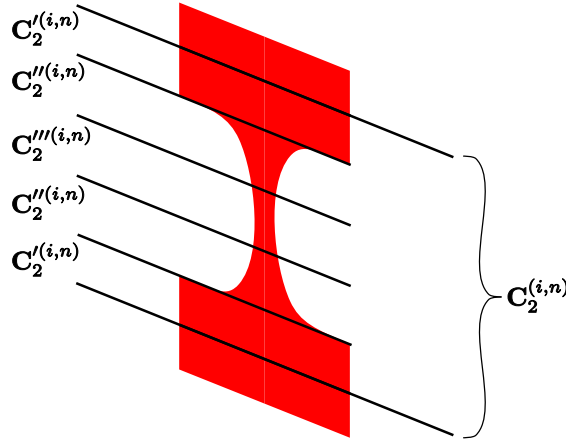


Figure 3.2: Decomposition of  $\mathbf{C}_2^{(i,n)} = \mathbf{C}_2^{\prime(i,n)} \cup \mathbf{C}_2^{\prime\prime(i,n)} \cup \mathbf{C}_2^{\prime\prime\prime(i,n)}$ .

$$\begin{aligned}
\mathbf{C}_2^{(i,n)} &\doteq \left\{ (x_0, y_0) \in \mathbf{C}_2^{(i,n)} \text{ s.t. } \left| \tilde{h}_1(x_1, y_1) \right| < (K_2/K_1)Y'(y_1)^{-1/2} \right\} \\
\mathbf{C}_2''^{(i,n)} &\doteq \left\{ (x_0, y_0) \in \mathbf{C}_2^{(i,n)} \text{ s.t. } \left| \tilde{h}_1(x_1, y_1) \right| < A \right\} \setminus \mathbf{C}_2^{(i,n)} \\
\mathbf{C}_2'''^{(i,n)} &\doteq \mathbf{C}_2^{(i,n)} \setminus \left( \mathbf{C}_2^{(i,n)} \cup \mathbf{C}_2''^{(i,n)} \right).
\end{aligned}$$

First consider  $(x, y) \in \mathbf{C}_2'''^{(i,n)}$ ; by definition we have:

$$\left| \tilde{h}_1(x, y) \right| < \frac{2K_2}{AY'(y)}$$

which is a bound for  $x$  of order  $\mathcal{O}(y^{-2\beta})$ , so that:

$$\text{Leb}(\mathbf{C}_2'''^{(i,n)}) \leq \text{Const} \cdot \hat{y}_n^{-4\beta}.$$

The measure of  $\mathbf{C}_2^{(i,n)}$  and  $\mathbf{C}_2''^{(i,n)}$  can be estimated using the following change of variables:

$$(x_0, y_0) \mapsto (\xi, \eta) = \left( \tilde{h}_1(x_0, y_0), \tilde{h}_1(x_1, y_1) \right);$$

this map is an invertible diffeomorphism and its Jacobian determinant is of order  $Y'(\hat{y}_n)$ ; for convenience let  $Y'_n = Y'(\hat{y}_n)$ . Therefore, for  $\mathbf{C}_2^{(i,n)}$  we obtain:

$$\text{Leb}(\mathbf{C}_2^{(i,n)}) \leq \frac{2}{Y'_n} \int_{-2(K_2/K_1)Y'_n^{-1/2}}^{+2(K_2/K_1)Y'_n^{-1/2}} \int_{-2K_1Y'_n^{-1/2}}^{+2K_1Y'_n^{-1/2}} d\xi d\eta = \mathcal{O}(\hat{y}_n^{-4\beta})$$

and for  $\mathbf{C}_2''^{(i,n)}$ :

$$\text{Leb}(\mathbf{C}_2''^{(i,n)}) \leq \frac{1}{Y'_n} \int_{\frac{1}{2}(K_2/K_1)Y'_n^{-1/2}}^A \int_{-2K_2/\eta Y'_n}^{+2K_2/\eta Y'_n} d\xi d\eta = \mathcal{O}(\hat{y}_n^{-4\beta} \log \hat{y}_n).$$

Therefore we finally have:

$$\text{Leb}(\mathbf{C}_2^{(i,n)}) \leq \text{Const} \cdot \hat{y}_n^{-4\beta} \log \hat{y}_n$$

and summing over  $i$  and  $n$  we obtain

$$\text{Leb}(\mathbf{C}_2) < \infty \text{ if } \sum_n n^{-\frac{4\beta}{\gamma}} \log n < \infty,$$

where the series converges if  $\beta > 1/2$ .

To conclude, notice that if  $\beta \leq 1/2$  then  $\bar{\mathbf{C}}_2$  has infinite measure; since for large enough  $y$  we have  $\mathbf{C}_2(y) \supset \bar{\mathbf{C}}_2(y)$ , statement (b) follows.  $\square$

For convenience we will now define neighbourhoods of the critical sets; first fix  $\delta' > 0$  small, to be determined later and define the following neighbourhood of  $\mathbf{C}_1$ :

$$\hat{\mathbf{C}}_1 \doteq \left\{ (x_0, y_0) \text{ s.t. } d((x_0, y_0), \mathbf{C}_1) < \delta' Y'(y_0)^{-1/2} \right\}$$

where  $d$  is the standard Euclidean distance. Let us define  $\hat{K}_1$  such that the following inclusion holds:

$$\hat{\mathbf{C}}_1 \subset \{ |\tilde{h}_1(x_0, y_0)| < \hat{K}_1 Y'(y_0)^{-1/2} \}. \quad (3.5)$$

We now extend  $\mathbf{C}_2$  to  $\hat{\mathbf{C}}_1$ :

$$\mathbf{C}_2^* \doteq \hat{\mathbf{C}}_1 \cap \left\{ (x_0, y_0) \text{ s.t. } |\tilde{h}_1(x_0, y_0)| < K_2 |\tilde{h}_1(x_1, y_1) Y'(y_0)^{1/2} Y'(y_1)^{1/2}|^{-1} \right\},$$

Notice that, as we did before, we require that  $K_2 > 2K_1 \hat{K}_1$ , so that the inclusion  $F^{-1}\mathbf{C}_1 \cap \hat{\mathbf{C}}_1 \subset \mathbf{C}_2^*$  holds true.

To define the corresponding neighbourhood for  $\mathbf{C}_2$  we need to be more careful; fix  $\delta'' > 0$  small, also to be determined later:

$$\begin{aligned} \hat{\mathbf{C}}_2 \doteq \hat{\mathbf{C}}_1 \cap \{ (x, y) : \exists (x', y') \in \mathbf{C}_2^* \text{ and } \Gamma \text{ standard s.t.} \\ (x, y), (x', y') \in \Gamma \text{ and } |x - x'| < \delta'' Y'(y)^{-1} \} \end{aligned}$$

From the definition of  $\mathbf{C}_1$  and  $\hat{\mathbf{C}}_1$ , and using lemma 3.1.4 we can easily prove the following:

$$\hat{\mathbf{C}}_2 \subset \hat{\mathbf{C}}_1 \cap \{ (x, y) \text{ s.t. } d^*((x, y), \mathbf{C}_2^*) < \delta'' Y'(y)^{-1} \} \quad (3.6)$$



where  $d^*$  is an adapted distance defined as:

$$d^*((x, y), (x', y')) = \max(|x - x'|, 2(\hat{K}_1 Y'(y)^{-1/2})^{-1}|y - y'|)$$

As we mentioned at the beginning of this section, on critical sets we lack good geometrical and distortion estimates that can instead be achieved on the complementary set. In particular, outside  $\mathbf{C}_1$  standard pairs will be mapped to standard pairs; pieces of standard pairs passing through the first critical set will possibly be mapped to non-standard pairs. However, pieces of standard pairs that lie in  $\mathbf{C}_1 \setminus \mathbf{C}_2$  will be standard after one more iteration. In the following lemma we prove the previous statements and moreover we establish some expansion bounds which will be crucial for proving equidistribution properties of  $F$  along the horizontal direction.

**Lemma 3.2.2** (Invariance). *Let  $\ell = (\Gamma, \rho)$  be a standard pair; let  $\hat{y} = \hat{y}_\ell$ ,  $Y = Y(\hat{y})$  and similarly for  $Y'$ . Then for large enough  $\hat{y}$ :*

(a) *there exist positive constants  $D_1$ ,  $D_2$  and  $D_3$  such that:*

$$\left| \frac{dx_1}{dx_0} \right| > D_1 Y'^{1/2} \quad \text{if } (x_0, y_0) \notin \mathbf{C}_1 \quad (\text{a1})$$

$$\left| \frac{dx_1}{dx_0} \right| > D_2 > 1 \quad \text{if } (x_0, y_0) \notin \mathbf{C}_2 \quad (\text{a2})$$

$$\left| \frac{dx_2}{dx_0} \right| > D_3 Y' \quad \text{if } (x_0, y_0) \in \mathbf{C}_1 \setminus \mathbf{C}_2 \quad (\text{a3})$$

(b) *we have the following almost Markov decomposition in respectively standard, stand-by and invalid pairs:*

$$F\ell = \bigcup_j \ell_j \cup \bigcup_k \tilde{\ell}_k \cup Z,$$

where  $\ell_j$  are standard pairs and each  $\tilde{\ell}_k$  is a pair such that we have  $F\tilde{\ell}_k = \bigcup_l \ell_{k,l}$  where  $\ell_{k,l}$  are standard pairs.

*Proof.* To simplify notation we will write  $h' = F^*h$ ,  $h'' = F^{*2}h$  (and similarly for  $\dot{h}$  and  $r$ ). We will once more use (3.4); additionally we consider  $\hat{y}$  to be large enough for all the estimates below to be true. Now, recall that by definition:

$$\begin{aligned}\frac{dx_1}{dx_0} &= \frac{\partial x_1}{\partial x_0} + \frac{dy_0}{dx_0} Y'(y_0) = 1 + h(x_0) Y'(y_0) \simeq \tilde{h}(x_0) Y' \\ \frac{dx_2}{dx_0} &\simeq \tilde{h}'(x_1) \tilde{h}(x_0) Y'^2\end{aligned}$$

If  $(x_0, y_0) \in \Gamma \setminus \mathbf{C}_1$  we have  $|\tilde{h}_1(x_0, y_0)| \geq K_1 Y'(y_0)^{-1/2}$ ; since  $\ell$  is standard,  $|h(x_0) - h_1(x_0, y_0)| \leq |\mathcal{C}_1(x_0, y_0)| = \mathcal{O}(Y'^{-3/2})$ , hence we obtain:

$$|\tilde{h}(x_0)| > 3/4 K_1 Y'^{-1/2},$$

which implies (a1) with  $D_1 = 2/3 K_1$ .

Similarly, if  $(x_0, y_0) \in \Gamma \setminus \mathbf{C}_2$  we have  $|\tilde{h}_1(x_0, y_0)| \geq K_3 Y'(y_0)^{-1}$  and, since  $\ell$  is standard,  $|\tilde{h}(x_0)| \geq 3/4 K_3 Y'^{-1}$  so estimate (a2) follows with  $D_2 = 2/3 K_3 > 1$ .

Using (a2) and (3.1c) we now obtain:

$$h'(x_1) = h_0(x_1, y_1) + \Delta h, \quad |\Delta h| \leq (D_2 Y')^{-1}; \quad (3.7)$$

hence, by definition of standard pair and of  $\mathbf{C}_2$  we obtain the following bound:

$$|\tilde{h}(x_0) \tilde{h}'(x_1)| \geq 3/4 K_2 Y'^{-1}$$

which implies (a3) with  $D_3 = 2/3 K_2$ .

To prove part (b), first let us define:

$$\Gamma_0^* \doteq \Gamma \setminus \mathbf{C}_1.$$

First of all we want to decompose  $F\Gamma_0^*$  into basic curves; by definition of  $\mathbf{C}_1$  we have that  $\Gamma_0^*$  is compatible with  $\mathcal{C}_0$ , therefore  $F\Gamma_0^*$  will be compatible with  $\mathcal{C}_1$  which, in particular, implies that  $F\Gamma_0^*$  is locally the graph of a function of  $x$ . Then we need to decompose the image  $F\Gamma_0^*$  in curves of the required length; in doing so we are possibly left with a piece of curve that is shorter than  $\delta$ ; by requiring  $\delta' = 2\delta D_1^{-1}$ , this piece will necessarily be the image of a portion of curve  $\hat{\Gamma}_0 \subset \Gamma_0^*$  which lies in  $\hat{\mathbf{C}}_1$ . We now define:

$$\Gamma_0 \doteq \Gamma_0^* \setminus \hat{\Gamma}_0 \quad \rho_0 \doteq \rho|_{\Gamma_0} \quad \ell_0 \doteq (\Gamma_0, \rho_0)$$

Finally we prove that the basic pairs in which we decomposed  $F\ell_0$  are indeed standard pairs. In fact, we already established that  $h'$  is compatible with  $\mathcal{C}_1$ ; furthermore, equations (a1) and (3.1b,d) give:

$$|\dot{h}'(x_1) - \dot{h}_1(x_1, y_1)| \leq 3AD_1^{-1}Y'^{-3/2}$$

$$|r'(x_1)| \leq 3AD_1^{-2} + \mathcal{O}(Y'^{-1/2}).$$

Therefore, by taking  $D_1^2 > 6A$ , we can decompose  $F\ell_0$  in standard pairs.

Next, let  $\Gamma_1^* \doteq (\Gamma \setminus \Gamma_0) \setminus \mathbf{C}_2^*$ ; once more, we try to decompose  $F^2\Gamma_1^*$  in basic curves. By (3.7), since  $(x_1, y_1) \notin \mathbf{C}_1$ , we have that  $h' \in \mathcal{C}_0(x_1, y_1)$ , so that  $h'' \in \mathcal{C}_1(x_2, y_2)$  and  $F^2\Gamma_1^*$  is locally the graph of a function of  $x$ . We now need to cut the curve into pieces of the required length. By estimate (a3), and requiring  $\delta'' =$

$2\delta D_3^{-1}$ , we can conclude that pieces that are too small will be image of  $\hat{\Gamma}_1 \subset \Gamma_1^*$  which belongs to  $\hat{\mathbf{C}}_2$  by definition. Now let:

$$\Gamma_1 \doteq \Gamma_1^* \setminus \hat{\Gamma}_1^*, \quad \rho_1 \doteq \rho|_{\Gamma_1} \quad \ell_1 \doteq (\Gamma_1, \rho_1).$$

We now claim that  $F^2\ell_1$  can in fact be decomposed as a union of standard pairs. We already obtained that  $h''$  is compatible with  $\mathbf{C}_1$ ; indeed, since any point of  $\ell_1$  is outside  $\mathbf{C}_2$ , (a2) and (3.1d) yield:

$$\begin{aligned} |\dot{h}'(x_1) - \dot{h}_1(x_1, y_1)| &\leq 3A(D_2 Y')^{-1} \\ |\dot{h}''(x_2) - \dot{h}_1(x_2, y_2)| &\leq 3AD_1^{-1} Y'^{-3/2}. \end{aligned}$$

Iterating equation (3.1b) we obtain:

$$r''(x_2) = \frac{\dot{h}(x_0)}{\tilde{h}^2(x_0)\tilde{h}'(x_1)Y'^2} + \frac{\dot{h}'(x_1)}{\tilde{h}'(x_1)Y'} + \mathcal{O}(Y'^{-1}) \quad (3.8)$$

Since  $F^{-1}\mathbf{C}_1 \cap \hat{\mathbf{C}}_1 \subset \mathbf{C}_2^*$  and  $D_1^2 > 6A$ , the second term is smaller than  $1/2$ ; moreover using the definition of  $\mathbf{C}_2$  and (a2) we have that

$$\tilde{h}^2(x_0)\tilde{h}'(x_1)Y'^2 > D_2\tilde{h}(x_0)\tilde{h}'(x_1)Y' > 3/4K_2D_2$$

Therefore taking  $K_2$  large enough we can make the first term in (3.8) smaller than  $1/4$ . thus we have that  $F\ell_1$  can be written as a union of curves satisfying the requirements, which concludes the proof of (b) by letting  $Z$  be the image of  $\Gamma \setminus (\Gamma_0 \cup \Gamma_1)$ .  $\square$

The following lemma introduces some measure estimates which will be crucial for our result.

**Lemma 3.2.3.** *We have:*

$$\text{Leb}(\hat{\mathbf{C}}_1) < \infty \text{ if } \beta > 1 \quad \text{Leb}(\hat{\mathbf{C}}_2) < \infty \text{ if } \beta > 3/4.$$

Moreover, let  $\ell = (\Gamma, \rho)$  be a standard pair and  $\hat{y} = \hat{y}_\ell$ , then:

$$\mathbb{P}_\ell(\hat{\mathbf{C}}_1) \leq \text{Const} \cdot \hat{y}^{-\beta} \quad \mathbb{P}_\ell(\hat{\mathbf{C}}_2) \leq \text{Const} \cdot \hat{y}^{-\frac{4}{3}\beta}$$

*Proof.* Given (3.5), the estimate for  $\text{Leb}(\hat{\mathbf{C}}_1)$  can be obtained in the same way as for  $\text{Leb}(\mathbf{C}_1)$  in proposition 3.2.1 and will be omitted. On the other hand, to estimate  $\text{Leb}(\hat{\mathbf{C}}_2)$ , we use a more elaborate construction; define the following sets:

$$\begin{aligned} \check{\mathbf{C}}'_2 &\doteq \hat{\mathbf{C}}_1 \cap \{|\tilde{h}_1(x_1, y_1)| < \check{k}Y'^{-1/3}\} \\ \check{\mathbf{C}}''_2 &\doteq \{|\tilde{h}_1(x_0, y_0)| < 2(K_2/\check{k})Y'^{-2/3}\}. \end{aligned}$$

By (3.6) we can take  $\check{k}$  to be large enough so that  $\hat{\mathbf{C}}_2 \subset \check{\mathbf{C}}'_2 \cup \check{\mathbf{C}}''_2$ . Proceeding as in proposition 3.2.1 we obtain that:

$$\text{Leb}(\check{\mathbf{C}}'_2) \sim \sum_n n^{-10\beta/3\gamma} \quad \text{Leb}(\check{\mathbf{C}}''_2) \sim \int_1^\infty y^{-4/3\beta} dy;$$

it is easy to check that for  $\beta > 3/4$  both measures are finite.

Let now  $I$  be the domain of  $\Gamma$ ; and  $I_1 \subset I$  the domain of  $\Gamma \cap \hat{\mathbf{C}}_1$ . Then, since  $\ell$  is standard and we have good control on  $\dot{h}$ , (3.5) implies that

$$|I_1| < \text{Const} \cdot Y'^{-1/2}$$

which in turn gives  $\mathbb{P}_\ell(\hat{\mathbf{C}}_1) \leq \text{Const} \cdot \hat{y}^{-\beta}$ .

Similarly, let us define  $I'_2$  and  $I''_2$  as the domain of  $\Gamma \cap \check{\mathbf{C}}'_2$  and  $\Gamma \cap \check{\mathbf{C}}''_2$  respectively.

The estimate of  $|I_2''|$  is similar to the previous one and yields the expected result.

To estimate  $|I_2|$  notice that:

$$\left| \frac{dh_1}{dx_0}(x_0) \right| \geq \frac{1}{2} \left| \dot{h}_1(x_1, y_1) \tilde{h}(x_0) Y' \right|.$$

Since  $\dot{h}_1(x_1, y_1)$  is bounded below in  $\check{C}'_2$ , we can write

$$\left| \frac{dh_1}{dx_0} \right| \geq \frac{2}{3} A |\tilde{h}(x_0)| Y'$$

so that  $|I_2'' \cap \{|\tilde{h}| > Y'^{-2/3}\}| \leq \text{Const} \cdot Y'^{-2/3}$  and as for the remaining part we have the bound:

$$\frac{2}{3} A Y' \int_{I_2''} \tilde{h}(x) dx < \text{Const} \cdot Y'^{-1/3}$$

which implies  $|I_2''| < \text{Const} \cdot Y'^{-2/3}$ , that in turn yields the required estimate.  $\square$

We conclude this section with the definition of *critical time*, which gives the maximum number  $\bar{n}$  such that, by iterating the decomposition in lemma 3.2.2,  $F^n x$  belongs to a non-invalid curve for all  $n \leq \bar{n}$ .

**Definition 3.2.4.** Let  $\ell = (\Gamma, \rho)$  be a standard pair. The *critical time* is a function  $\tau : \Gamma \rightarrow \mathbb{N} \cup \{\infty\}$  obtained by means of the following procedure. Define a decreasing sequence of sets:

$$\Gamma = \Gamma_{[0]} \supset \Gamma_{[1]} \supset \dots \supset \Gamma_{[k]} \supset \dots$$

by induction as follows. Suppose that  $F^n(\Gamma_{[n]}, \rho)$  can be decomposed exclusively in standard pairs and stand-by pairs:

$$F^n(\Gamma_{[n]}, \rho) = \bigcup_j \ell_j \cup \bigcup_k \tilde{\ell}_k.$$

For each standard pair  $\ell_j$  in the decomposition of we can apply lemma 3.2.2b and obtain:

$$F(\ell_j) = \bigcup_l \ell_{j,l} \cup \bigcup_m \tilde{\ell}_{j,m} \cup Z_j.$$

We then define

$$\Gamma_{[n+1]} = \Gamma_{[n]} \setminus \left( \bigcup_j F^{-(n+1)} Z_j \right),$$

which satisfies the inductive hypothesis. Finally we let:

$$\tau(x) \doteq \sum_{j=1}^{\infty} 1_{\Gamma_{[j]}}(x).$$

### 3.3 Riemann sum lemma

In what follows we will very often approximate integrals with Riemann sums over partitions which are highly non-uniform. Most elements of the partition have very small size and a much smaller portion have a size which is orders of magnitude larger. The naïve bound on Riemann sums, which is optimal in the case of uniform partitions, can be improved in our case. The following lemma allows us to obtain a much better bound which will be crucial in all our estimates.

**Lemma 3.3.1.** *Let  $\mathcal{J}$  be a finite index set and  $\{\delta_i\}_{i \in \mathcal{J}}$  and  $\{X_i\}_{i \in \mathcal{J}}$  be sequences of positive real numbers such that there exist real numbers  $a_1, a_2 > 0$ ,  $0 < \lambda < 1$  and  $0 < \alpha \leq 1$  satisfying the following properties:*

- $a_1 \cdot \lambda < X_i < a_1 \quad \forall i \in \mathcal{J}$

- for  $\kappa \in (0, 1)$ , define the set  $\Xi_\kappa = \{i \in \mathcal{J} \text{ s.t. } X_i > a_1 \cdot \lambda^{1-\kappa}\}$ ; then:

$$\sum_{i \in \Xi_\kappa} \delta_i < a_2 \cdot \lambda^{\alpha\kappa}.$$

Then  $\forall \varepsilon > 0$  sufficiently small there exists  $C_\varepsilon \sim a_1 a_2 \cdot \varepsilon^{-1}$  such that:

$$\sum_i X_i \delta_i < C_\varepsilon \cdot \lambda^{\alpha-\varepsilon}$$

In the previous statement one should think of  $\delta_k$  as the length of the  $k$ -th interval in the partition used to compute the Riemann sum;  $X_k$  should instead be thought as a bound for the error given by considering the integrating function constant on the  $k$ -th interval.

*Proof.* Fix  $n > 0$  and consider any decreasing sequence  $1 = \kappa_0 > \kappa_1 > \dots > \kappa_n = 0$ , so that:

$$\emptyset = \Xi_{\kappa_0} \subset \Xi_{\kappa_1} \subset \dots \subset \Xi_{\kappa_n} = \mathcal{J}$$

Define  $\tilde{\Xi}_j = \Xi_{\kappa_j} \setminus \Xi_{\kappa_{j-1}}$  for  $j = 1, \dots, n$ ; thus we obtain

$$a_1 \lambda^{1-\kappa_j} < X_i < a_1 \lambda^{1-\kappa_{j-1}} \text{ for } i \in \tilde{\Xi}_j, \quad \sum_{i \in \tilde{\Xi}_j} \delta_i < a_2 \lambda^{\alpha\kappa_j},$$

moreover:

$$\sum_{i \in \mathcal{J}} X_i \delta_i = \sum_{j=1}^n \sum_{i \in \tilde{\Xi}_j} X_i \delta_i < \sum_{j=1}^n a_1 a_2 \lambda^{1-\kappa_{j-1} + \alpha\kappa_j}.$$

If  $\alpha < 1$  we choose  $\kappa_j$  satisfying the following relations:

$$\kappa_j = \frac{\alpha^{j-n} - 1}{\alpha(\alpha^{j-n-1} - 1)} \kappa_{j-1},$$

in such a way that  $\kappa_{j-1} - \alpha\kappa_j = \kappa_j - \alpha\kappa_{j+1}$ ; we therefore obtain:

$$1 - \kappa_0 + \alpha\kappa_1 = \frac{\alpha^{1-n} - 1}{\alpha^{1-n} - 1} = \alpha - \varepsilon,$$



with  $\varepsilon \sim 1/n$  which can therefore be taken arbitrarily small.

The case  $\alpha = 1$  can be obtained as a limit for  $\alpha \rightarrow 1$ ; in this setting we choose  $\kappa_j$  as follows:

$$\kappa_j = 1 - \frac{j}{n}$$

hence, again we obtain  $\kappa_{j-1} - \alpha\kappa_j = \kappa_j - \alpha\kappa_{j+1} = 1/n$ , which implies

$$1 - \kappa_0 + \alpha\kappa_1 = 1 - 1/n = \alpha - \varepsilon.$$

□

### 3.4 Equidistribution on standard pairs

The invariance lemma [3.2.2](#) states that the image of a standard pair can be partitioned in standard pairs, stand-by pairs and invalid pairs. Due to the large expansion rate in the standard cones, if we had good distortion bounds, the density on most standard pairs in the image would be very close to the uniform density. However, close to the critical set we lack good distortion bounds and the density could have strong dependence on the position.

In this section we will prove a first and a second equidistribution lemma; they provide an estimate for the expectation of a class of observables on a standard pair after respectively one and two iterates of  $F$ . The observables we consider are functions of the fast variable  $x$  only, constant on the  $y$  direction; for convenience of definition, with a slight abuse of notation we consider them as being functions on  $\mathbb{S}^1$ . Moreover notice that the observables need not to enjoy particularly strong

smoothness requirements; indeed the Lipschitz property is enough to prove both our results.

**Lemma 3.4.1** (Equidistribution +1). *Consider a standard pair  $\ell = (\Gamma, \rho)$  with  $\Gamma = (x, \psi(x))$  and let  $\hat{y} = \hat{y}_\ell$ . Then  $\forall \varepsilon > 0$  small there exists  $C_\varepsilon$  such that for all  $\mathcal{A} \in \mathcal{C}(\mathbb{S}^1)$  with zero average:*

$$|\mathbb{E}_\ell(\mathcal{A} \circ F)| \leq C_\varepsilon \cdot \|\mathcal{A}\|_\infty \cdot \hat{y}^{-\beta+\varepsilon} \quad (3.9)$$

Moreover we can prove the following auxiliary results:

- if  $B \in \mathcal{C}^1(\Gamma)$ , then we have:

$$|\mathbb{E}_\ell(B \cdot (\mathcal{A} \circ F))| \leq C_\varepsilon \cdot \|\mathcal{A}\|_\infty \|B\|_* \cdot \hat{y}^{-\beta+\varepsilon} \quad (3.10)$$

where  $\|B\|_* = \max\{\|B\|_\infty, \hat{y}^{-\beta} \|\dot{B}\|_\infty\}$ .

- let  $\bar{\rho}$  be the uniform density on  $\Gamma$  and  $\hat{\rho}(x) = \rho(x) - \bar{\rho}$ ; define  $\hat{\mathbb{E}}_\ell(f) = \int_\Gamma f(x, \psi(x)) \hat{\rho}(x) dx$ , then:

$$|\hat{\mathbb{E}}_\ell(\mathcal{A} \circ F)| \leq C_\varepsilon \cdot \|\mathcal{A}\|_\infty \|r\|_\infty \cdot \hat{y}^{-\beta+\varepsilon}. \quad (3.11)$$

*Proof.* Let  $\Theta(x) = x + Y(\psi(x))$ ; cut  $\Gamma \setminus \mathbf{C}_1$  in curves  $\Gamma_k$  such that the endpoints of  $F\Gamma_k$  lie on two consecutive vertical lines  $x \equiv 0 \pmod{2\pi}$ . In this process we could have some leftover pieces of curve; however, their total measure is small:

$$\mathbb{P}_\ell(\Gamma \setminus \bigcup_k \Gamma_k) = \mathcal{O}(\hat{y}^{-\beta}).$$

In fact by lemma 3.2.3 we have  $\mathbb{P}_\ell(\mathbf{C}_1) = \mathcal{O}(\hat{y}^{-\beta})$  and lemma 3.2.2 guarantees that, outside  $\mathbf{C}_1$ , the expansion rate along  $\Gamma$  is at least  $\mathcal{O}(\hat{y}^\beta)$ . Now define:

$$J_k \doteq \int_{\Gamma_k} \rho(x) \mathcal{A}(\Theta(x)) dx.$$

Then we have

$$\begin{aligned} \mathbb{E}_\ell(\mathcal{A} \circ F) &= \int_{\Gamma} \rho(x_0) \mathcal{A}(x_1) dx_0 = \\ &= \sum_k J_k + \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-\beta}). \end{aligned}$$

where the error term accounts for the leftover pieces and it is compatible with (3.9).

On each  $\Gamma_k$ , we define an inverse function  $x(\theta)$  for  $\Theta$  and push forward the density as  $\rho'_k(\theta) \doteq \rho(x(\theta))/|\dot{\Theta}(x(\theta))|$ . Hence:

$$|J_k| = \left| \int_0^{2\pi} \rho'_k(\theta) \mathcal{A}(\theta) d\theta \right|.$$

Now we write  $\rho'_k = (2\pi)^{-1} c_k + \tilde{\rho}'_k$ , where  $c_k \doteq \mathbb{P}_\ell(\Gamma_k)$ . Since  $\mathcal{A}$  has zero average we obtain:

$$|J_k| \leq \int_0^{2\pi} |\tilde{\rho}'_k(\theta) \mathcal{A}(\theta)| d\theta \leq \|\mathcal{A}\|_\infty \int_0^{2\pi} |\tilde{\rho}'_k(\theta)| d\theta;$$

and since for some  $\theta_0 \in (0, 2\pi)$

$$|\tilde{\rho}'_k(\theta)| \leq \left| \int_{\theta_0}^{\theta} \rho'_k \frac{d \log \rho'_k}{d\theta} ds \right| \leq c_k \left\| \frac{d \log \rho'_k}{d\theta} \right\|_\infty$$

we have

$$|J_k| \leq 2\pi c_k \|\mathcal{A}\|_\infty \left\| \frac{d \log \rho'_k}{d\theta} \right\|_\infty.$$

Recall that we denote by  $h(x)$  the slope  $\dot{\psi}(x)$ ; notice that, since  $\Gamma_k \cap \mathbf{C}_1 = \emptyset$ , we know that  $|h| > \text{Const} \cdot \hat{y}^{-\beta}$  on  $\Gamma_k$ . Using the definition of  $\Theta$  we can therefore write

:

$$\left\| \frac{d \log \rho'_k}{d\theta} \right\|_{\infty} \leq \underbrace{\left\| \frac{r}{\tilde{h}Y'} \right\|_{\Gamma_k}}_{X_k} + \underbrace{\left\| \frac{\dot{h}}{\tilde{h}^2 Y'} \right\|_{\Gamma_k}}_{Y_k} + \underbrace{\left\| \frac{h^2 Y''}{(\tilde{h}Y')^2} \right\|_{\Gamma_k}}_{Z_k}.$$

Since  $\ell$  is standard we know that  $r$  and  $\dot{h}$  are bounded, moreover

$$\frac{h}{\tilde{h}} = \left( 1 - \frac{1}{\tilde{h}Y'} \right)$$

so that if  $|h| > \hat{y}^{-\lambda\beta}$  on  $\Gamma_k$  we obtain:

$$\begin{aligned} X_k &\leq \text{Const} \cdot \hat{y}^{-2\beta+\lambda\beta} \\ Y_k &\leq \text{Const} \cdot \hat{y}^{-2\beta+2\lambda\beta} \\ Z_k &\leq \text{Const} \cdot \hat{y}^{-2\beta-1} (1 + \hat{y}^{-2\beta+\lambda\beta})^2. \end{aligned}$$

Therefore we have  $X_k < \mathcal{O}(\hat{y}^{-\beta})$  and  $Z_k < \mathcal{O}(\hat{y}^{-2\beta-1})$  which are good enough for estimate (3.9). We now use lemma 3.3.1 to bound  $\sum_k c_k Y_k$  taking:

$$\delta_k = c_k, \quad a_1, a_2 = \mathcal{O}(1), \quad \lambda = \hat{y}^{-2\beta} \text{ and } \alpha = 1/2.$$

We conclude that  $\forall \varepsilon > 0$  small there exists a  $C_\varepsilon$  such that:

$$\int_{\Gamma} \rho(x_0) \mathcal{A}(x_1) dx_0 \leq \|\mathcal{A}\|_{\infty} C_\varepsilon \hat{y}^{-\beta+\varepsilon} + \mathcal{O}(\hat{y}^{-\beta}).$$

Notice that we could also apply lemma 3.3.1 to  $\sum_k c_k X_k$  using:

$$\delta_k = c_k, \quad a_1 = \text{Const} \cdot \hat{y}^{-\beta}, \quad a_2 = \mathcal{O}(1), \quad \lambda = \hat{y}^{-\beta} \text{ and } \alpha = 1;$$

in this case we would have obtained the better estimate  $\mathcal{O}(\hat{y}^{-2\beta+\varepsilon})$ , which will indeed be useful later. Also notice that if we had  $h = \mathcal{O}(1)$ , we would obtain  $X_k, Y_k < \mathcal{O}(\hat{y}^{-2\beta})$  so that  $\int_{\Gamma} \rho(x_0) \mathcal{A}(x_1) dx_0 = \mathcal{O}(\hat{y}^{-2\beta})$  if  $\Gamma$  does not intersect a

$\mathcal{O}(1)$ -neighbourhood of the critical set.

To prove the first auxiliary result we proceed in the same way:

$$\int_{\Gamma} B(x_0)\rho(x_0)\mathcal{A}(x_1)dx_0 = \sum_k J'_k + \|\mathcal{A}\|_{\infty}\|B\|_{\infty}\mathcal{O}(\hat{y}^{-\beta})$$

where:

$$J'_k = \int_0^{2\pi} H'_k(\theta)\mathcal{A}(\theta)d\theta$$

and  $H'_k(\theta) = B(x(\theta))\rho(x(\theta))/|\dot{\Theta}(x(\theta))|$ ; as before we separate  $H'_k$  from its constant part so that we can write:

$$|J'_k| \leq \|\mathcal{A}\|_{\infty} \int_0^{2\pi} |\tilde{H}'_k(\theta)|d\theta$$

and then estimate the integral as:

$$\int_0^{2\pi} |\tilde{H}'_k(\theta)|d\theta \leq 2\pi c_k \left[ \left\| \frac{\dot{B}}{\tilde{h}Y'} \right\|_{\Gamma_k} + \|B\|_{\infty} \left\| \frac{d \log \rho'_k}{d\theta} \right\|_{\infty} \right]$$

so that we can directly apply lemma 3.3.1 to the first term, obtaining the bound  $C_{\varepsilon}\|\dot{B}\|_{\infty}\mathcal{O}(\hat{y}^{-2\beta+\varepsilon})$ ; the second term is the same as before, multiplied by  $\|B\|_{\infty}$ , so that we obtain (3.13).

To prove the second auxiliary result, first notice that, as we did before with  $\rho'_k$ , there exists a  $w \in I$  such that  $\rho(w) = \bar{\rho}$ , then we can write:

$$\hat{\rho}(x) = \rho(x) - \bar{\rho} = \int_w^x r(\xi)\rho(\xi)d\xi,$$

so that

$$|\hat{\rho}(x)| < \|r\|_{\infty}$$

consider now:

$$\int_{\Gamma} \hat{\rho}(x_0) \mathcal{A}(x_1) dx_0 = \sum_k J_k'' + \|\mathcal{A}\|_{\infty} \|r\|_{\infty} \mathcal{O}(\hat{y}^{-\beta})$$

where

$$J_k'' = \int_0^{2\pi} \hat{\rho}'_k(\theta) \mathcal{A}(\theta) d\theta$$

and  $\hat{\rho}'_k(\theta) \doteq \hat{\rho}(x(\theta))/|\dot{\Theta}(x(\theta))|$ ; again we separate from  $\hat{\rho}'$  its constant part and we can write:

$$|J_k''| \leq \|\mathcal{A}\|_{\infty} \int_0^{2\pi} |\tilde{\rho}'_k(\theta)| d\theta$$

To estimate the integral notice that:

$$|\tilde{\rho}'_k(\theta)| \leq c_k \left\| \frac{r}{\tilde{h}Y'} \right\|_{\Gamma_k} + \|r\|_{\infty} \left\| \frac{\ddot{\Theta}}{\dot{\Theta}^3} \right\|_{\Gamma_k}$$

The first term is the same as  $c_k X_k$  in the main part; the second term is of the same order as  $c_k(Y_k + Z_k)$  with an additional constant  $\|r\|_{\infty}$  term, so that we finally obtain:

$$\left| \int_I \hat{\rho}(x) \mathcal{A}(\Theta(x)) dx \right| \leq C_{\varepsilon} \cdot \|\mathcal{A}\|_{\infty} \|r\|_{\infty} \mathcal{O}(\hat{y}^{-\beta+\varepsilon}).$$

□

**Corollary 3.4.2.** *Let  $\ell, \mathcal{A}$  and  $B$  be as in lemma 3.4.1, then:*

$$|\mathbb{E}_{\ell}(B \cdot (\mathcal{A} \circ F^k))| \leq C_{k,\varepsilon} \cdot \|\mathcal{A}\|_{\infty} \|B\|_{k*} \hat{y}^{-\beta+\varepsilon} \quad (3.12)$$

where

$$\|B\|_{k*} = \max \left\{ \|B\|_{\infty}, \hat{y}^{-\beta} \left\| \frac{dB}{dx_{k-1}} \right\|_{\infty} \right\}$$

*Proof.* By the invariance lemma 3.2.2 we know that  $F^{k-1}\ell$  can be decomposed in standard pairs and non-standard pairs; by lemma 3.2.3 we can estimate the probability of sitting on a non-standard piece as follows:

$$\mathbb{P}_\ell(F^{k-1}(x, y) \in \text{non-standard pair}) < \text{Const} \cdot (k-1)\hat{y}^{-\beta}.$$

Since this bound is compatible with (3.12), we can neglect non-standard pieces and we conclude by applying lemma 3.4.1 to pairs in the standard part of  $F^{k-1}\ell$ .  $\square$

In fact the  $\varepsilon$  appearing in estimates (3.9), (3.10) and (3.12) could in principle be dropped using a Fresnel-type argument, which also shows that the bound is sharp. This argument, however, is not as robust as the one based on lemma 3.3.1 and it would be less apt to the generalizations we seek.

We now proceed to improve the equidistribution lemma 3.4.1, by directly considering second iterates of a standard pair:

**Lemma 3.4.3** (Equidistribution +2). *Let  $\ell = (\Gamma, \rho)$  be a standard pair and let  $\hat{y} = \hat{y}_\ell$ . Let  $\mathcal{A} \in \text{Lip}(\mathbb{S}^1)$  be a zero-average Lipschitz function (with respect to the standard metric on  $\mathbb{S}^1$ ). Then  $\forall \varepsilon > 0$  small there exists a  $C_\varepsilon$  such that:*

$$|\mathbb{E}_\ell(\mathcal{A} \circ F^2)| \leq C_\varepsilon \|\mathcal{A}\|_{\text{Lip}} (\hat{y}^{-\beta^*} + o(\hat{y}^{-1})) \quad (3.13)$$

where  $\beta^* = \min(2\beta - 1/2 + \varepsilon, 4/3\beta)$  and  $\|\cdot\|_{\text{Lip}}$  is the standard Lipschitz norm  $\|\mathcal{A}\|_{\text{Lip}} = \max\{\|\mathcal{A}\|_\infty, \text{Lip}(\mathcal{A})\}$ ,  $\text{Lip}(\mathcal{A})$  being the Lipschitz constant of  $\mathcal{A}$ .

Notice that the first image of  $\ell$  can be decomposed, by means of the invariance lemma, in a union of invalid pairs, stand-by pairs and standard pairs which are

allowed to intersect the critical set. To improve estimate 3.4.2 we will need to prove the fact that by summing over the aforementioned union of curves, we have some cancellations. First we need to prove a few preliminary results:

**Lemma 3.4.4.** *Fix a zero average function  $\mathcal{A} \in Lip(\mathbb{S}^1)$ .*

(a) *Let  $\ell = (\Gamma, \rho)$  be a standard pair with  $\Gamma = (x, \psi(x))$  on the domain  $I$ . Let  $\Delta h(x) \doteq h(x) - h_1(x, \psi(x))$ . Then for all reference pairs  $\bar{\ell} = (\bar{\Gamma}, \bar{\rho})$  on the same domain  $I$  such that  $\bar{\Gamma} \cap \Gamma \neq \emptyset$ , for all  $\varepsilon$  sufficiently small we can write:*

$$|\mathbb{E}_\ell(\mathcal{A} \circ F) - \mathbb{E}_{\bar{\ell}}(\mathcal{A} \circ F)| \leq C_\varepsilon \|\mathcal{A}\|_{Lip} \left( \|r\|_\infty \hat{y}^{-\beta+\varepsilon} + Y'(\hat{y}) \|\Delta h\|_\infty \right) \quad (3.14)$$

*Consider two reference curves  $\Gamma_1 = \{x, \psi_1(x)\}$  and  $\Gamma_2 = \{x, \psi_2(x)\}$  on the same domain  $I$  such that  $\tilde{h}_1 \neq 0$  on each  $\Gamma_i$ ; let  $\ell_i$  be the reference pairs on  $\Gamma_i$ . Let  $z$  be the endpoint of  $I$  on which  $|\tilde{h}_1|$  attains its minimum value and define  $\delta\eta = \psi_1(z) - \psi_2(z)$ .*

(b) *Assume that  $Y(\psi_1(z)) \equiv Y(\psi_2(z)) \pmod{1}$ ; then for all sufficiently small  $\varepsilon > 0$ :*

$$|\mathbb{E}_{\ell_1}(\mathcal{A} \circ F) - \mathbb{E}_{\ell_2}(\mathcal{A} \circ F)| \leq C_\varepsilon \|\mathcal{A}\|_{Lip} |\delta\eta| \hat{y}^{-1-\beta+\varepsilon} \quad (3.15)$$

(c) *Assume that  $|\delta\eta| \ll 1$ ; then for all sufficiently small  $\varepsilon > 0$ :*

$$|\mathbb{E}_{\ell_1}(\mathcal{A} \circ F) - \mathbb{E}_{\ell_2}(\mathcal{A} \circ F)| \leq C_\varepsilon \|\mathcal{A}\|_{Lip} |\delta\eta| \hat{y}^\beta. \quad (3.16)$$

*Proof.* Fix some  $w \in I$  and define  $\bar{\Gamma} = (x, \bar{\psi}(x))$  as the reference curve over the domain  $I$  passing through the point  $(w, \psi(w))$ . Let  $\Theta(x) = x + Y(\psi(x))$  and



$\bar{\Theta}(x) = x + Y(\bar{\psi}(x))$ , so that  $\Theta(w) = \bar{\Theta}(w)$ ; then we have:

$$\begin{aligned} & \left| \int_I \rho(x) \mathcal{A}(\Theta(x)) dx - \int_I \bar{\rho} \mathcal{A}(\bar{\Theta}(x)) dx \right| \leq \\ & \leq \left| \int_I (\rho(x) - \bar{\rho}) \mathcal{A}(\Theta(x)) dx \right| + \left| \int_I \bar{\rho} (\mathcal{A}(\Theta(x)) - \mathcal{A}(\bar{\Theta}(x))) dx \right| \end{aligned} \quad (3.17)$$

For the first integral we use lemma 3.4.1 to obtain:

$$\left| \int_I (\rho(x) - \bar{\rho}) \mathcal{A}(\Theta(x)) dx \right| \leq C_\varepsilon \|\mathcal{A}\|_\infty \|r\|_\infty \mathcal{O}(\hat{y}^{-\beta+\varepsilon});$$

For the second integral in (3.17), notice that since  $\Gamma$  is standard, lemma 3.1.4 implies

$|\psi(x) - \bar{\psi}(x)| < \text{Const} \cdot \|\Delta h\|_\infty$ , therefore we have:

$$|\Theta(x) - \bar{\Theta}(x)| < 2Y'(\hat{y}) \|\Delta h\|_\infty$$

and since  $\mathcal{A}$  is Lipschitz we can conclude that:

$$\left| \int_I \bar{\rho} (\mathcal{A}(\Theta(x)) - \mathcal{A}(\bar{\Theta}(x))) dx \right| \leq 2 \text{Lip}(\mathcal{A}) Y'(\hat{y}) \|\Delta h\|_\infty$$

which concludes the proof of (a).

To prove (b) and (c), define similarly  $\Theta_i(x) = x + Y(\psi_i(x))$  for  $i = 1, 2$ . For (b) assume without loss of generality that  $\Theta_i(z) = 0$  and that  $|\Theta_1| \geq |\Theta_2|$ , then we can define  $I_* \subset I$  as the maximal domain of the functions  $\xi_1$  and  $\xi_2$  defined in such a way that  $\Theta_1(\xi_1) = \Theta_2(\xi_2)$  (see figure 3.3). Then:

$$\begin{aligned} \int_I \mathcal{A}(\Theta_1(\xi_1)) d\xi_1 &= \int_{I_*} \mathcal{A}(\Theta_1(\xi_1)) d\xi_1 + \mathcal{O}(\hat{y}^{-2\beta}) = \\ &= \int_I \mathcal{A}(\Theta_2(\xi_2)) \frac{\dot{\Theta}_2(\xi_2)}{\dot{\Theta}_1(\xi_1)} d\xi_2 + \mathcal{O}(\hat{y}^{-2\beta}). \end{aligned}$$

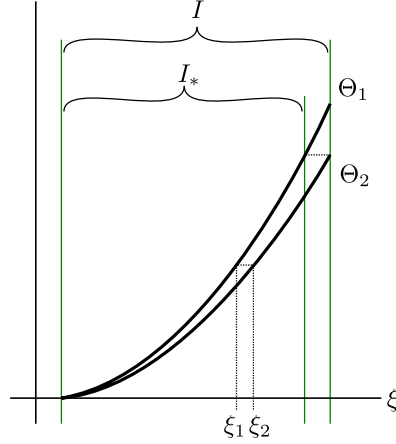


Figure 3.3: Definition of  $\xi_1$  and  $\xi_2$ .

We will prove that  $\|\dot{\Theta}_2(\xi_2)/\dot{\Theta}_1(\xi_1) - 1\|_* = \mathcal{O}(\hat{y}^{-1})$  and then use lemma 3.4.1 to conclude. Notice that by definition of  $\Theta_i$  we can obtain, for  $\delta\Theta = \Theta_2 - \Theta_1$ :

$$\delta\Theta = \nu \cdot \Theta_1 (1 + \mathcal{O}(\hat{y}^{-1}))$$

$$\delta\dot{\Theta} = \nu \cdot \dot{\Theta}_1 (1 + \mathcal{O}(\hat{y}^{-1}))$$

$$\delta\ddot{\Theta} = \nu \cdot \ddot{\Theta}_1 (1 + \mathcal{O}(\hat{y}^{-1}))$$

where

$$\nu = \frac{Y''(\eta_1)}{Y'(\eta_2)} \cdot \delta\eta = \delta\eta \cdot \mathcal{O}(\hat{y}^{-1})$$

and  $\eta_i = \psi_i(z)$ . Again, we neglect the pieces of  $\Gamma_i$  inside  $\mathbf{C}_1$ ; in fact, since  $\delta\Theta < \mathcal{O}(\nu)$  inside  $\mathbf{C}_1$ , the contribution would be  $\text{Lip}(\mathcal{A}) \mathcal{O}(\hat{y}^{-1-\beta})\delta\eta$ , which is compatible with (3.15).

To estimate the remainder we will proceed to obtain an expression for  $\delta\xi = \xi_1 - \xi_2$  by writing the following identity:

$$\Theta_1(\xi_1) - \Theta_1(\xi_2) = \Theta_2(\xi_2) - \Theta_1(\xi_2)$$

Then from the previous estimates and the intermediate value theorem we obtain

$$\dot{\Theta}_1 \delta\xi = \nu\Theta_1 (1 + \mathcal{O}(\hat{y}^{-1}))$$

Next we can estimate:

$$\begin{aligned} |\dot{\Theta}_1(\xi_1) - \dot{\Theta}_2(\xi_2)| &\leq |\dot{\Theta}_1(\xi_1) - \dot{\Theta}_1(\xi_2)| + |\dot{\Theta}_1(\xi_2) - \dot{\Theta}_2(\xi_2)| \\ &\leq \left( |\ddot{\Theta}_1(\xi_1)\delta\xi| + |\nu\dot{\Theta}_1(\xi_1)| \right) (1 + \mathcal{O}(\hat{y}^{-1})) \end{aligned}$$

and in the same way:

$$|\ddot{\Theta}_1(\xi_1) - \ddot{\Theta}_2(\xi_2)| \leq \left( |\ddot{\Theta}_1(\xi_1)\delta\xi| + |\nu\ddot{\Theta}_1(\xi_1)| \right) (1 + \mathcal{O}(\hat{y}^{-1})).$$

The previous estimates allow to obtain a bound for  $\|\dot{\Theta}_2(\xi_2)/\dot{\Theta}_1(\xi_1) - 1\|_*$ ; first,

consider:

$$\left| \frac{\dot{\Theta}_2(\xi_2) - \dot{\Theta}_1(\xi_1)}{\dot{\Theta}_1(\xi_1)} \right| \leq 2 \left( 1 + \left| \frac{\ddot{\Theta}_1(\xi_1)\Theta_1(\xi_1)}{\dot{\Theta}_1(\xi_1)^2} \right| \right) \nu;$$

using the definition of  $\Theta$  we can check that the fraction is bounded on  $I$ , so that we

conclude  $\left\| (\dot{\Theta}_2(\xi_2) - \dot{\Theta}_1(\xi_1))/\dot{\Theta}_1(\xi_1) \right\| \leq C\nu$ .

Next, in order to estimate the derivative:

$$\begin{aligned} &\left| \frac{\ddot{\Theta}_2(\xi_2)\dot{\Theta}_1(\xi_1)^2 - \ddot{\Theta}_1(\xi_1)\dot{\Theta}_2(\xi_2)^2}{\dot{\Theta}_1(\xi_1)^3} \right| \leq \\ &\leq \left| \frac{\ddot{\Theta}_2(\xi_2)\dot{\Theta}_1(\xi_1)^2 - \ddot{\Theta}_1(\xi_1)\dot{\Theta}_1(\xi_1)^2}{\dot{\Theta}_1(\xi_1)^3} \right| + \left| \frac{\ddot{\Theta}_1(\xi_1)\dot{\Theta}_1(\xi_1)^2 - \ddot{\Theta}_1(\xi_1)\dot{\Theta}_2(\xi_2)^2}{\dot{\Theta}_1(\xi_1)^3} \right| \\ &\leq \left( \frac{\ddot{\Theta}_1(\xi_1)}{\dot{\Theta}_1(\xi_1)} (1 + 3C) + \frac{\ddot{\Theta}_1(\xi_1)\Theta_1(\xi_1)}{\dot{\Theta}_1(\xi_1)^2} \right) \nu. \end{aligned}$$

Since we assume the curve does not intersect  $\mathbf{C}_1$  we have  $\ddot{\Theta}_1(\xi_1)/\dot{\Theta}_1(\xi_1) \leq \mathcal{O}(\hat{y}^\beta)$

and  $|\ddot{\Theta}_1(\xi_1)\Theta_1(\xi_1)/\dot{\Theta}_1(\xi_1)^2| \leq C'$ , from which we conclude:

$$\left\| \dot{\Theta}_2(\xi_2)/\dot{\Theta}_1(\xi_1) - 1 \right\|_* \leq C''\nu.$$

The proof of part (c) is similar; first of all notice that we have the a priori bound:

$$|\mathbb{E}_{\ell_1}(\mathcal{A} \circ F) - \mathbb{E}_{\ell_2}(\mathcal{A} \circ F)| \leq \text{Const} \cdot \text{Lip}(\mathcal{A}) \|Y \circ \psi_1 - Y \circ \psi_2\|_I. \quad (3.18)$$

Moreover, by definition of reference curve, and since  $Y''(y)/Y'(y)$  is decreasing we have:

$$\left| \frac{d}{dx}(Y(\psi_1(x)) - Y(\psi_2(x))) \right| \leq \frac{Y''(\hat{y})}{Y'(\hat{y})^2} |Y(\psi_1(x)) - Y(\psi_2(x))|;$$

so that by Grönwall lemma we can conclude for large enough  $\hat{y}$  that

$$\forall x \in I \quad |Y(\psi_1(x)) - Y(\psi_2(x))| < |\delta Y|(1 + \mathcal{O}(\hat{y}^{-1-2\beta})),$$

where  $\delta Y = Y(\psi_1(z)) - Y(\psi_2(z))$ . We can thus assume that  $Y(\psi_2(x)) - Y(\psi_1(x)) = \delta Y$  since the error term would be compatible with (3.16) by the a priori estimate (3.18). We have that  $\Theta_2(x) = \Theta_1(x) + \delta Y$ . First we deal with the portion of the curves inside  $\mathbf{C}_1$ . Let  $I_1$  be the minimal subset of  $I$  such that:

$$\forall x \notin I_1, (x, \psi_i(x)) \notin \mathbf{C}_1 \text{ for } i = 1, 2$$

then, by lemma 3.2.3:

$$\bar{\rho} \int_{I_1} \mathcal{A}(\Theta_1(x)) - \mathcal{A}(\Theta_2(x)) dx \leq \text{Const} \cdot \text{Lip}(\mathcal{A}) \mathcal{O}(\hat{y}^{-\beta}) |\delta Y|,$$

which is compatible with (3.16); we can therefore neglect the pieces of curves over  $I_1$  and assume  $\dot{\Theta}_i \geq \mathcal{O}(\hat{y}^\beta)$  by lemma 3.2.2a. Defining  $\xi_i$  as before we can write:

$$\int_{I \setminus I_1} \mathcal{A}(\Theta_1(\xi_1)) d\xi_1 = \int_{I \setminus I_1} \mathcal{A}(\Theta_2(\xi_2)) \frac{\dot{\Theta}_2(\xi_2)}{\dot{\Theta}_1(\xi_1)} d\xi_2 + \text{Lip}(\mathcal{A}) \mathcal{O}(\hat{y}^{-\beta}) |\delta Y|$$

We will prove that  $\|\dot{\Theta}_2(\xi_2)/\dot{\Theta}_1(\xi_1) - 1\|_* = \mathcal{O}(\delta Y)$  and then use lemma 3.4.1 to conclude. In fact in this case we have:

$$|\delta\xi| = \mathcal{O}(\delta Y/\dot{\Theta}_1)$$

so we can easily see that:

$$\left\| \dot{\Theta}_2(\xi_2)/\dot{\Theta}_1(\xi_1) - 1 \right\|_\infty \leq \text{Const} \cdot \frac{\ddot{\Theta}_1(\xi_1)}{\dot{\Theta}_1(\xi_1)^2} \delta Y \leq \text{Const} \cdot \delta Y$$

and for the derivative:

$$\left\| \frac{\ddot{\Theta}_1(\xi_1)\dot{\Theta}_1(\xi_1) - 2\dot{\Theta}_1(\xi_1)^2}{\dot{\Theta}_1(\xi_1)^3} \delta Y \right\| \leq \text{Const} \cdot \mathcal{O}(\hat{y}^\beta) \delta Y,$$

which concludes the proof.  $\square$

We also need good shadowing estimates for the image of a standard curve outside  $\mathbf{C}_1$ :

**Lemma 3.4.5.** *Let  $\Gamma = (x, \psi(x))$  be a standard curve over  $I$  such that  $\Gamma \cap \mathbf{C}_1 = \emptyset$  and let  $\hat{y} = \hat{y}_\Gamma$ . Then there exists a reference curve  $\bar{\Gamma} = (x, \bar{\psi}(x))$  such that:*

$$\forall (x_1, y_1) \in F\Gamma \exists \bar{y}_1 \text{ s.t. } (x_1, \bar{y}_1) \in F\bar{\Gamma}, |y_1 - \bar{y}_1| = \mathcal{O}(\hat{y}^{-4\beta}) \quad (3.19)$$

*Proof.* Let  $(x_1, y_1) \in F\Gamma$  and consider the vertical line  $\{x = x_1\}$  passing through  $(x_1, y_1)$ ; the preimage  $F^{-1}\{x = x_1\}$  is by definition a curve of slope  $h_{-1}(x_0, y_0) = -Y'(y_0)^{-1}$  and we have:

$$\left. \frac{dy_1}{dx_0} \right|_{h_{-1}} = -Y'(y_0)^{-1}. \quad (3.20)$$

Let  $z$  and  $w$  denote the two endpoints of  $I$ . Without loss of generality we can assume that

$$|\tilde{h}_1(z, \psi(z))| \leq |\tilde{h}_1(w, \psi(w))|.$$

Let  $\bar{\Gamma} = (x, \bar{\psi}(x))$  be the reference curve passing through  $(z, \psi(z))$  and let  $\Pi = F^{-1}\{x = w'\}$  where  $w'$  is the  $x$ -coordinate of  $F(w, \psi(w))$ . Then  $\bar{\Gamma}$  will intersect  $\Pi$  at some point  $(\bar{w}, \bar{\psi}(\bar{w}))$ ; let  $\bar{I}$  be the interval bounded by  $z$  and  $\bar{w}$  (see figure 3.4). We claim that  $\bar{\Gamma}|_{\bar{I}}$  satisfies (3.19). In fact let  $I' = I \cap \bar{I}$ ; lemma 3.1.4 ensures that

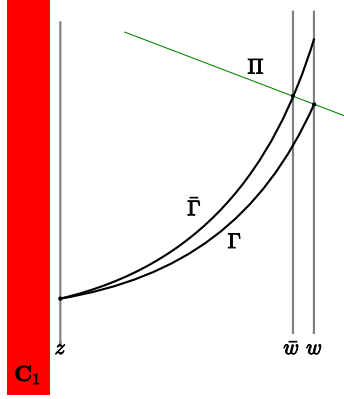


Figure 3.4: Construction of the reference curve  $\bar{\Gamma}$  shadowing  $\Gamma$ .

the vertical distance between  $\Gamma$  and  $\bar{\Gamma}$  is bounded by  $2|I'|\|\Delta h\|_{\Gamma} < \mathcal{O}(y^{-3\beta})$  on  $I'$ , therefore the distance along the slope  $h_{-1}$  is bounded by  $2|I'|\|\Delta h\|_{\Gamma}/\tilde{h}_1 < \mathcal{O}(y^{-2\beta})$  since we are outside  $\mathbf{C}_1$ . Hence, using (3.20), we can conclude that  $F\bar{\Gamma}$  will be  $\mathcal{O}(y^{-4\beta})$ -close to  $F\Gamma$  along the vertical direction.  $\square$

Finally, we will need the following result, which allows us to prove cancellations when integrating over the second image of a standard pair:

**Lemma 3.4.6.** *Fix  $I \subset \mathbb{S}^1$  an interval satisfying the hypothesis for being the domain of a basic curve and such that  $I \cap \bar{\mathbf{C}}_2 = \emptyset$ ; we define the function  $\Psi_I(Y)$  as follows; let  $z$  be the endpoint of  $I$  such that  $|\tilde{h}_1|$  has a minimum, and consider a family of reference pairs  $\{\ell_{I,Y} = (\Gamma_{I,Y}, \bar{\rho})\}$  where  $\Gamma_{I,Y} = (x, \psi_{I,Y}(x))$ , parametrized by*

$Y = Y(\psi_{I,Y}(z))$  and  $\bar{\rho} = 1/|I|$ . Then we define:

$$\Psi_I(Y) \doteq \mathbb{E}_{\ell_{I,Y}}(\mathcal{A} \circ F)$$

Then for large enough  $\hat{y}$ :

$$\int_{Y(\hat{y})}^{Y(\hat{y})+1} \Psi_I(Y) dY = \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-1-2\beta}) \quad (3.21)$$

*Proof.* Let  $\Theta_Y(x) = x + Y(\psi_{I,Y}(x))$ . Applying the definition of  $\Psi_I$  we have

$$\int_Y^{Y+1} \Psi_I(Y) dY = \int_Y^{Y+1} \int_I \bar{\rho} \mathcal{A}(\Theta_Y(x)) dx dY$$

Exchanging the order of integration and changing variables we obtain

$$\int_I \int_{\Theta_Y(x)}^{\Theta_{Y+1}(x)} \mathcal{A}(\theta) J d\theta dx$$

Where  $J$  is the Jacobian of the transformation  $Y \mapsto \Theta$ ; by explicitly computing the holonomy map along the reference foliation we obtain  $J = 1 + \mathcal{O}(\hat{y}^{-1-2\beta})$  and  $\Theta_{Y+1}(x) - \Theta_Y(x) = 2\pi + \mathcal{O}(\hat{y}^{-1-2\beta})$ . This concludes the proof.  $\square$

Once established the previous results, we now proceed to the proof of the second equidistribution lemma.

*Proof of lemma 3.4.3.* By the invariance lemma 3.2.2,  $F\ell$  can be decomposed in a union of standard pairs, stand-by pairs and invalid pairs:

$$F\ell = \underbrace{\bigcup_j \ell'_j}_{\ell'_{\text{std}}} \cup \underbrace{\bigcup_k \tilde{\ell}'_k}_{\ell'_{\text{sb}}} \cup Z.$$

Let us also define  $\ell_{\text{std}} = F^{-1}\ell'_{\text{std}}$  and similarly  $\ell_{\text{sb}} = F^{-1}\ell'_{\text{sb}}$  so that we can write:

$$\mathbb{E}_\ell(\mathcal{A} \circ F^2) = \int_{\Gamma_{\text{std}}} \rho(x_0)\mathcal{A}(x_2)dx_0 + \int_{\Gamma_{\text{sb}}} \rho(x_0)\mathcal{A}(x_2)dx_0 + \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-4/3\beta}) \quad (3.22)$$

where the bound for the last term is obtained using proposition 3.2.3. Now let  $J = \{y = \hat{y} - 4A\} \setminus \bar{\mathbf{C}}_2 \subset \mathbb{S}^1$  and cut  $J$  in intervals  $I_\alpha$  satisfying the hypothesis for being domains of a basic curve (see figure 3.5):

$$J = \bigcup_{\alpha} I_\alpha \quad \delta < |I_\alpha| < 2\delta$$

We then cut  $\ell'_{\text{std}}$  in standard pairs having  $I_\alpha$  as a domain; with abuse of notation

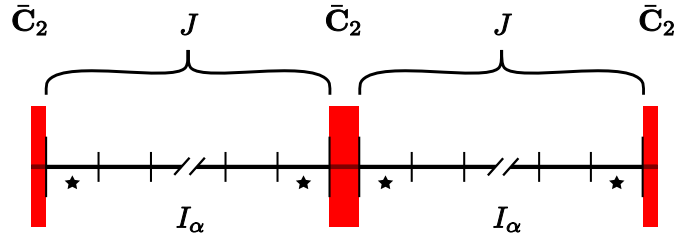


Figure 3.5: Definition of  $J$  and of the intervals  $I_\alpha$ .

we will denote the standard pairs obtained in this way again by  $\ell'_j$ . Notice that by discarding the intersections with  $\bar{\mathbf{C}}_2$  we introduce an error of  $\mathcal{O}(\hat{y}^{-2\beta})$  which is compatible with estimate (3.13). We can also discard pieces of  $\ell'_{\text{std}}$  with domain smaller than any of the  $I_\alpha$ , since lemma 3.2.2 and 3.4.1 imply that their contribution would be at most  $\mathcal{O}(\hat{y}^{-2\beta})$ .

The first term of the right hand side of (3.22) can be therefore written as:

$$\int_{\Gamma_{\text{std}}} \rho(x_0)\mathcal{A}(x_2)dx_0 = \sum_j c_j \mathbb{E}_{\ell'_j}(\mathcal{A} \circ F) + \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-2\beta}), \quad (3.23)$$



where  $c_j = \mathbb{P}_\ell(F^{-1}\Gamma'_j)$ . Define now, for all  $\alpha$ , the index sets

$$\mathcal{J}_\alpha = \{j \text{ s.t. } \text{Domain}(\ell'_j) = I_\alpha\}$$

then we can reorder the sum in (3.23) as follows:

$$\sum_j c_j \mathbb{E}_{\ell'_j}(\mathcal{A} \circ F) = \sum_\alpha \sum_{j \in \mathcal{J}_\alpha} c_j \mathbb{E}_{\ell'_j}(\mathcal{A} \circ F)$$

By the remark at the end of lemma 3.4.1 we have that standard pairs over  $I_\alpha$ s which are not  $\mathcal{O}(1)$ -close to the critical set will contribute with  $\mathcal{O}(\hat{y}^{-2\beta})$  to the sum; thus:

$$\sum_\alpha \sum_{j \in \mathcal{J}_\alpha} c_j \mathbb{E}_{\ell'_j}(\mathcal{A} \circ F) = \sum_{\alpha^*} \sum_{j \in \mathcal{J}_{\alpha^*}} c_j \mathbb{E}_{\ell'_j}(\mathcal{A} \circ F) + \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-2\beta})$$

where the sum in the right hand side ranges only over those  $I_{\alpha^*}$  that are contiguous to  $\bar{\mathbf{C}}_2$  (marked with a  $\star$  in figure 3.5).

Fix now one of such intervals; to simplify notation we will write  $I = I_{\alpha^*}$  and we will re-label  $\{\ell'_j\}_{j \in \mathcal{J}_{\alpha^*}}$  as  $\{\ell'_k\}$ . We consider  $\ell'_k$  to be ordered in such a way that:

$$\|h\|_{F^{-1}\Gamma'_k} < \|h\|_{F^{-1}\Gamma'_{k+1}}$$

Using lemma 3.4.4a and that the average density  $\bar{\rho}'_k = |I|^{-1}$  we have that  $\forall \varepsilon > 0$ :

$$\sum_k c_k \int_{\Gamma'_k} \rho'_k(x_1) \mathcal{A}(x_2) dx_1 = |I|^{-1} \sum_k c_k \int_{\bar{\Gamma}'_k} \mathcal{A}(x_2) dx_1 + \|\mathcal{A}\|_{\text{Lip}} \mathcal{O}(\hat{y}^{-2\beta+\varepsilon}). \quad (3.24)$$

In fact by summing over the error term in lemma 3.4.4a we have:

$$\text{Const} \cdot \|\mathcal{A}\|_{\text{Lip}} \sum_k c_k \left( \|r\|_{\Gamma'_k} \hat{y}^{-\beta+\varepsilon'} + Y'(\hat{y}) \|\Delta h\|_{\Gamma'_k} \right) \quad (3.25)$$

by (3.2) we know that:

$$\|\Delta h\|_{\Gamma'_k} \leq \text{Const} \cdot \hat{y}^{-3\beta} \underline{\mathcal{L}}_k^{-2},$$

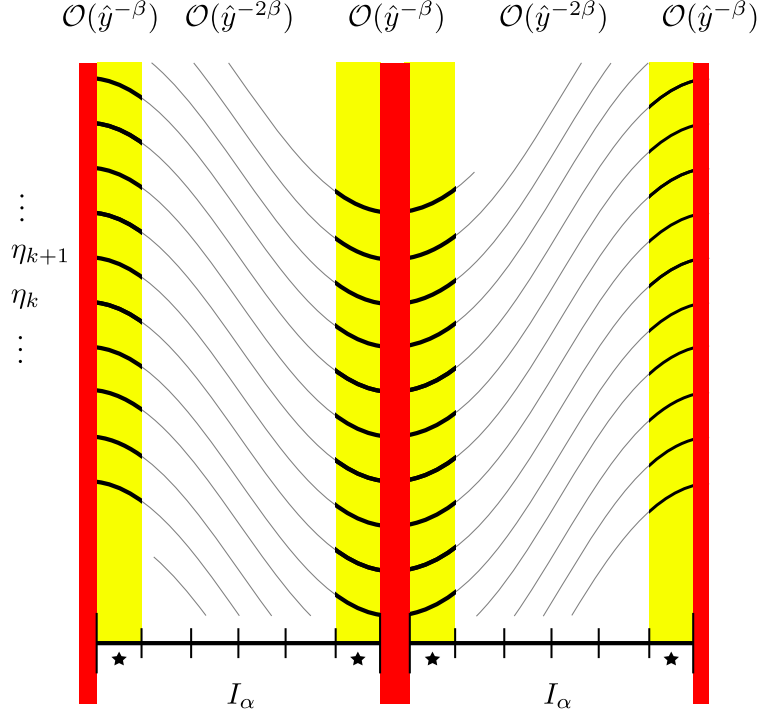


Figure 3.6: Cutting the image  $F\Gamma$  along  $I_\alpha$ .

where

$$\underline{\mathcal{L}}_k = \inf_{\substack{h \in \mathcal{C}_1(x,y) \\ (x,y) \in \Gamma_k}} |\tilde{h}Y'(\hat{y})| \geq \text{Const} \cdot \hat{y}^\beta,$$

which implies that the second part of (3.25) is  $\mathcal{O}(\hat{y}^{-3\beta})$ . To estimate the first term of (3.25) we apply lemma 3.3.1 with:

$$\delta_k = c_k, \quad a_1 = \mathcal{O}(\hat{y}^{-\beta+\varepsilon'}), a_2 = \mathcal{O}(1), \quad \lambda = \hat{y}^{-2\beta} \text{ and } \alpha = +1/2,$$

to obtain the required estimate.

We will now establish a bound for the main term of (3.24):

$$\sum_k c_k \int_{\bar{\Gamma}'_k} \mathcal{A}(x_2) dx_1. \quad (3.26)$$

Let  $z \in I$  be as in lemma 3.4.6 and define  $\eta_k$  such that  $(z, \eta_k) \in \bar{\Gamma}'_k$ . From now on, to fix ideas, we assume that  $\eta_{k+1} > \eta_k$ ; the other case can be treated in the same

way.

By lemma 3.4.5 we know there exists a reference curve  $\bar{\Gamma}$  such that  $F\Gamma$  is  $\mathcal{O}(\hat{y}^{-4\beta})$ -shadowed by  $F\bar{\Gamma}$  along the  $y$  direction. Let  $(z, \bar{\eta}_k) \in F\bar{\Gamma}$  such that  $|\bar{\eta}_k - \eta_k| = \mathcal{O}(\hat{y}^{-4\beta})$  and  $Y_k \doteq Y(\bar{\eta}_k)$ ; using the notation of lemma 3.4.6 and the bound given by lemma 3.4.4c we can write for (3.26):

$$\sum_k c_k \int_{\bar{\Gamma}'_k} \mathcal{A}(x_2) dx_1 = \sum_k c_k \Psi_I(Y_k) + \|\mathcal{A}\|_{\text{Lip}} \mathcal{O}(\hat{y}^{-3\beta}). \quad (3.27)$$

We now proceed to find an expression for  $Y_k = Y(\bar{\eta}_k)$ ; recall that  $\bar{\Gamma} = (x, \bar{\psi}(x))$ ; then we have:

$$\bar{\eta}_k = \bar{\psi}(\xi_k) + 2\dot{\phi}(z) \quad \text{where } \xi_k + Y(\bar{\psi}(\xi_k)) = (K + z) + k$$

for some  $K \in \mathbb{N}$ . Again to fix ideas we assume  $\xi_{k+1} \geq \xi_k$ ; the other case can be treated in a similar manner. Therefore

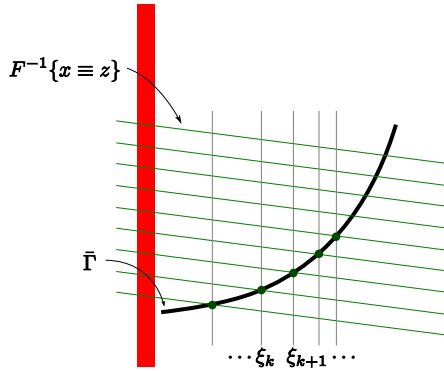


Figure 3.7: Setting for estimating  $Y_k$

$$\begin{aligned} Y_k &= Y(Y^{-1}((K + z) + k - \xi_k) + 2\dot{\phi}(z)) \\ &= Y_0 + (k - (\xi_k - \xi_0)) \left( 1 + \frac{Y_0''}{Y_0'} 2\dot{\phi}(z) + \mathcal{O}(\hat{y}^{-2}) \right) \\ &= Y_0 + k - (\xi_k - \xi_0) + k (\nu + \mathcal{O}(\hat{y}^{-2})) \end{aligned}$$

where  $Y_0 = Y(Y^{-1}(K + z) - \xi_0)$  (and similarly for  $Y'_0$  and  $Y''_0$ ) and

$$\nu = \frac{Y''_0}{Y'_0} 2\dot{\phi}(z) = \mathcal{O}(\hat{y}^{-1}).$$

If  $\nu = 0$  we would have that  $Y_k - Y_0 - k \in [0, 1]$ ; on the other hand, since  $\nu Y' \gg 1$ , we have that  $Y_k - Y_0 - k$  may cover several times the unit interval in  $Y$  (see figure 3.8). In fact define the following sets:

$$\mathbf{K}_i \doteq \{k \text{ s.t. } \lfloor Y_k - Y_0 - k \rfloor = i\};$$

thus we can rewrite the sum in (3.27) as:

$$\sum_k c_k \Psi_I(Y_k) = \sum_i \sum_{k \in \mathbf{K}_i} c_k \Psi_I(Y_k).$$

Let:

$$\hat{\Psi}_I(Y) \doteq \Psi_I(Y_0 + (Y \bmod 1));$$

lemma 3.4.4b ensures that

$$\sum_i \sum_{k \in \mathbf{K}_i} c_k \Psi_I(Y_k) = \sum_i \sum_{k \in \mathbf{K}_i} c_k \hat{\Psi}_I(Y_k - Y_0) + \|\mathcal{A}\|_{\text{Lip}} \mathcal{O}(\hat{y}^{-1-\beta+\varepsilon})$$

and the error is compatible with (3.13). Next let  $\hat{i}$  be the least index such that  $i \geq \hat{i}, \forall k \in \mathbf{K}_i$  we have  $|\tilde{h}| > \hat{y}^{1/2-\beta}$  on  $F^{-1}\Gamma'_k$ . Notice that by definition:

$$\sum_{i < \hat{i}} \sum_{k \in \mathbf{K}_i} c_k \leq \mathbb{P}_\ell \left( |\tilde{h}| < \text{Const} \cdot \hat{y}^{1/2-\beta} \right) = \mathcal{O}(\hat{y}^{1/2-\beta})$$

so that, by lemma 3.4.1, we can assume  $\hat{i} = 0$ , since the error we make is compatible with (3.13). Now for each  $i$ , define  $[c]_i \doteq \sum_{k \in \mathbf{K}_i} c_k$  and for each  $k \in \mathbf{K}_i$  let  $\tilde{c}_k = c_k - [c]_i/|\mathbf{K}_i|$ . We now prove that the contribution given by the constant part is

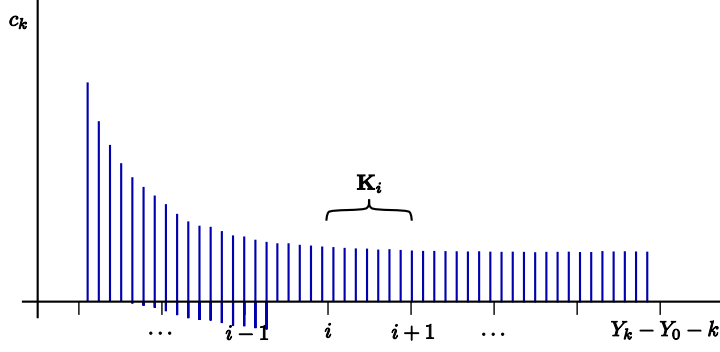


Figure 3.8: Illustration of the distribution of  $c_k$  relative to  $Y_k - Y_0 - k$

negligible; in fact let  $\underline{k}_i = \inf \mathbf{K}_i$ , then  $\forall k \in \mathbf{K}_i$ :

$$\begin{aligned} (Y_k - Y_0) \bmod 1 - \frac{k - \underline{k}_i}{|\mathbf{K}_i|} &\leq \sup_{k', k'' \in \mathbf{K}_i} |\xi_{k'} - \xi_{k''}| + \mathcal{O}(\hat{y}^{-1}) \\ &\leq \mathcal{O}(\hat{y}^{1/2-\beta} + \hat{y}^{-1}) \end{aligned}$$

therefore, using lemma 3.4.4c we have

$$\begin{aligned} [c]_i \sum_{k \in \mathbf{K}_i} \frac{1}{|\mathbf{K}_i|} \hat{\Psi}_I(Y_k - Y_0) &= \\ &= [c]_i \sum_{k \in \mathbf{K}_i} \frac{1}{|\mathbf{K}_i|} \left( \hat{\Psi}_I \left( \frac{k - \underline{k}_i}{|\mathbf{K}_i|} \right) + \|\mathcal{A}\|_{\text{Lip}} \mathcal{O}(\hat{y}^{1/2-2\beta} + \hat{y}^{-1-\beta}) \right) \end{aligned} \quad (3.28)$$

The main term of (3.28) can be seen as a Riemann sum of an integral as in lemma

3.4.6:

$$\begin{aligned} [c]_i \sum_{k \in \mathbf{K}_i} \frac{1}{|\mathbf{K}_i|} \hat{\Psi}_I \left( \frac{k - \underline{k}_i}{|\mathbf{K}_i|} \right) &= [c]_i \left( \int_0^1 \hat{\Psi}_I(Y) dY + \|\mathcal{A}\|_{\infty} \mathcal{O}(\nu \cdot \hat{y}^{-\beta}) \right) \\ &\leq [c]_i \|\mathcal{A}\|_{\infty} \mathcal{O}(\hat{y}^{-1-\beta}) \end{aligned}$$

where we used lemma 3.4.4c to estimate the error and lemma 3.4.6 to bound the integral. We finally need to estimate the terms containing  $\tilde{c}_k$ :

$$\sum_{k \in \mathbf{K}_i} \tilde{c}_k \hat{\Psi}_I(Y_k - Y_0) \leq |\mathbf{K}_i| \|\tilde{c}_k\|_i \cdot \|\mathcal{A}\|_{\infty} \mathcal{O}(\hat{y}^{-\beta})$$

where

$$\|\tilde{c}_k\|_i = \sup_{k \in \mathbf{K}_i} \{|\tilde{c}_k|\}.$$

Let now  $[\Gamma]_i$  be the smallest connected subset of  $\Gamma$  such that

$$F[\Gamma]_i \supset \bigcup_{k \in \mathbf{K}_i} \Gamma'_k$$

then:

$$\|\tilde{c}_k\|_i \leq \left| \int_{F[\Gamma]_i} r' \rho' \right| \leq \|r'\|_{F[\Gamma]_i} \mathbb{P}_\ell([\Gamma]_i).$$

Thus we use lemma 3.3.1 to estimate:

$$\sum_i |\mathbf{K}_i| \underbrace{\|r'\|_{F[\Gamma]_i}}_{X_i} \underbrace{\mathbb{P}_\ell([\Gamma]_i)}_{\delta_i}.$$

Using:

$$a_1, a_2 = \mathcal{O}(1), \quad \lambda = \hat{y}^{1-2\beta} \text{ and } \alpha = +1/2.$$

we obtain:

$$\sum_i |\mathbf{K}_i| \|r'\|_{F[\Gamma]_i} \mathbb{P}_\ell([\Gamma]_i) \leq C_\varepsilon \hat{y}^{1/2-\beta+\varepsilon}.$$

which concludes the proof for the standard part of  $F\Gamma$ .

For the second term of the right hand side of (3.22), we proceed in a way similar to the proof of lemma 3.4.1; let  $\tilde{\Gamma} \doteq \Gamma_{sb}$  and consider  $F^2\tilde{\Gamma}$ . We can once more partition  $\tilde{\Gamma}$  in  $\tilde{\Gamma}_k$  such that the endpoints of  $F^2\tilde{\Gamma}_k$  lie on two consecutive vertical lines  $x \equiv 0 \pmod{2\pi}$ :

$$\int_{\tilde{\Gamma}} \rho(x_0) \mathcal{A}(x_2) dx_0 = \sum_k J_k + \|\mathcal{A}\|_\infty \mathcal{O}(\hat{y}^{-2\beta})$$

Again let  $\Theta(x_0) = x_0 + Y(y_0) + Y(y_1)$ ; on each  $\tilde{\Gamma}_k$  we can define an inverse function for  $\Theta$  and push forward the density as  $\rho_k''(\theta) \doteq \rho(x(\theta))/|\dot{\Theta}(x(\theta))|$ . Again we can separate the constant part:  $\rho_k'' = (2\pi)^{-1}c_k + \tilde{\rho}_k''$  obtaining:

$$|J_k| \leq \|\mathcal{A}\|_\infty \int_0^{2\pi} |\tilde{\rho}_k''(\theta)| d\theta \leq 2\pi c_k \|\mathcal{A}\|_\infty \left\| \frac{d \log \rho_k''}{d\theta} \right\|_\infty$$

So now we need to estimate

$$\left\| \frac{d \log \rho_k''}{d\theta} \right\|_\infty \leq \left\| \frac{r}{\tilde{h}\tilde{h}'Y'^2} \right\|_{\Gamma_k} + \left\| \frac{\dot{h}}{\tilde{h}^2\tilde{h}'Y'^2} \right\|_{\Gamma_k} + \left\| \frac{\dot{h}'}{\tilde{h}'^2Y'} \right\|_{\Gamma_k}$$

By the invariance lemma we have that  $\tilde{h}\tilde{h}'Y'^2 > \mathcal{O}(\hat{y}^{2\beta})$ , so that we can neglect the first term. The second term can be bounded in the following way:

$$\left\| \frac{\dot{h}}{\tilde{h}^2\tilde{h}'Y'^2} \right\|_{\Gamma_k} < \frac{1}{\tilde{h}} \mathcal{O}(\hat{y}^{-2\beta}) = W_k$$

Notice that since  $\hat{y}^{-\beta} < W_k < 1$  and  $W_k > \hat{y}^{-\beta(1-\kappa)}$  for a length  $\sum_k c_k \leq \hat{y}^{-\beta(1+\kappa)}$ , we can use once more lemma 3.3.1 to obtain  $\mathcal{O}(\hat{y}^{-2\beta+\varepsilon})$  for the second term.

The third term is estimated again using lemma 3.3.1 as it is the same as it would be for a uniform density on  $F\tilde{\Gamma}$ ; we can therefore bound it with  $\mathbb{P}_\ell(\tilde{\Gamma}) \cdot \mathcal{O}(\hat{y}^{-\beta+\varepsilon}) \leq \mathcal{O}(\hat{y}^{-2\beta+\varepsilon})$ , which concludes the proof since all bounds we obtained are compatible with (3.13).  $\square$

### 3.5 Reduction to a biased random walk

Given a standard pair  $\ell = (\Gamma, \rho)$ , let once more  $\hat{y} = \hat{y}_\ell$  and define the following sequence for  $k \in \mathbb{Z}$ :

$$R_k \doteq 2^k \hat{y}$$

**Definition 3.5.1.** Let  $\ell = (\Gamma, \rho)$  be a standard pair; we will now give the definition of two functions:

$$\tau' : \Gamma \rightarrow \mathbb{N} \quad \xi : \Gamma \rightarrow \{-1, +1\}$$

As in the definition of the critical time  $\tau$ , define first a decreasing sequence of sets:

$$\Gamma = \Gamma_{[0]} \supset \Gamma_{[1]} \supset \cdots \supset \Gamma_{[k]} \supset \cdots$$

by induction as follows. Suppose that  $F^n(\Gamma_{[n]})$  can be decomposed in standard pairs and stand-by pairs only, i.e.:

$$F^n(\Gamma_{[n]}, \rho) = \bigcup_j \ell_j \cup \bigcup_k \tilde{\ell}_k.$$

For each standard pair  $\ell_j$  in the decomposition of we can apply lemma 3.2.2 and obtain:

$$F(\ell_j) = \bigcup_l \ell_{j,l} \cup \bigcup_m \tilde{\ell}_{j,m} \cup Z_j$$

Then let

$$\Xi = \{j \text{ s.t. } \Gamma_j \subset \{R_{-1} < y < R_{+1}\}\}$$

and define

$$\Gamma_{[n+1]} = \Gamma_{[n]} \setminus \left( \bigcup_{j \notin \Xi} F^{-n} \Gamma_j \cup \bigcup_{j \in \Xi} F^{-(n+1)} Z_j \right),$$

which satisfies the inductive hypothesis. Finally we define

$$\tau'(x) \doteq \sum_{j=1}^{\infty} 1_{\Gamma_{[j]}}(x).$$

and

$$\xi(x) = \begin{cases} +1 & \text{if } \tau'(x) < \tau(x) \text{ and } F^{\tau'(x)}(x) \text{ is close to } R_{+1}; \\ -1 & \text{otherwise.} \end{cases}$$



We will now define two sequences  $\tau_k : \Gamma \rightarrow \mathbb{N}$  and  $\chi_k : \Gamma \rightarrow \mathbb{Z}$ .

**Definition 3.5.2.** Let  $\chi_0(\Gamma) \equiv 0$ ,  $\tau_0(\Gamma) \equiv 0$ ; suppose we have defined  $\chi_n$ ,  $\tau_n$  we proceed by induction; if  $\tau_n(x) = \tau(x)$  then we let

$$\tau_{n+1}(x) \doteq \tau_n(x) \quad \chi_{n+1}(x) \doteq \chi_n(x) - 1.$$

Otherwise  $F^{\tau_n(x)}(x)$  will belong to some standard pair  $\ell$ , on which the functions  $\tau'$  and  $\xi$  are defined, so that we let:

$$\tau_{n+1}(x) \doteq \tau_n(x) + \tau'(F^{\tau_n(x)}(x)) \quad \chi_{n+1}(x) \doteq \chi_n(x) + \xi(F^{\tau_n(x)}(x)).$$

Theorem B will then follow from:

**Lemma 3.5.3.** *If  $\beta > 3/4$  ( $\gamma > 5/2$ ) and  $D_0$  is large enough, then:*

$$\text{Leb}(\{(x_0, y_0) \text{ s.t. } (x_n, y_n) \notin \hat{\mathbf{C}}_2 \forall n \in \mathbb{N}\}) = 0.$$

*Proof of theorem B.* Define  $\hat{F} : \hat{\mathbf{C}}_2 \rightarrow \hat{\mathbf{C}}_2$  as the first return map of  $F$  on  $\hat{\mathbf{C}}_2$ ;  $\hat{F}$  is well defined almost everywhere by lemma 3.5.3; since  $\text{Leb}(\hat{\mathbf{C}}_2) < \infty$  by proposition 3.2.3, we can apply Poincaré Recurrence theorem and conclude that almost every point in  $\hat{\mathbf{C}}_2$  is recurrent, which shows that  $\text{Leb}(\mathcal{E} \cap \hat{\mathbf{C}}_2) = 0$ . This implies theorem B by lemma 3.5.3, since almost every point will land on  $\hat{\mathbf{C}}_2$  infinitely many times..  $\square$

In order to prove lemma 3.5.3 it is sufficient to prove that, for every standard pair  $\ell$ , we have the following bound for the critical time:

$$\mathbb{P}_\ell(\tau < \infty) = 1. \tag{3.29}$$

In fact this implies that  $\forall \ell$  standard we have:

$$\mathbb{P}_\ell(\{(x_0, y_0) \text{ s.t. } (x_n, y_n) \notin \hat{\mathbf{C}}_2 \forall n \in \mathbb{N}\}) = 0.$$

The result follows since for all Borel measurable sets  $E$  we can write

$$\text{Leb}(E) = \int \mathbb{P}_{\tilde{\ell}_\alpha}(E) d\lambda(\alpha)$$

where  $d\lambda(\alpha)$  is some factor measure on the set of reference pairs. Given the following lemma we can obtain (3.29) by following the exact same procedure used in [Do08]; for completeness the argument will be given in the next section.

**Lemma 3.5.4.** *Let  $\ell = (\Gamma, \rho)$  be a standard pair and let  $\hat{y} = \hat{y}_\ell$ . If  $\beta > 3/4$  ( $\gamma > 5/2$ ) then:*

(a)  $\mathbb{P}_\ell(\xi = -1) \geq 0.6;$

(b) *There exists  $0 < \theta < 1$  such that  $\mathbb{P}_\ell(\tau' \geq s) \leq C\theta^{s/\hat{y}^2}$ .*

*Proof.* The invariance lemma 3.2.2b implies that:

$$\mathbb{E}_\ell(\mathcal{A} \circ F^n \cdot 1_{\tau' \geq n}) = \sum_j c_{nj} \mathbb{E}_{\ell_{nj}}(\mathcal{A}) + \sum_k \tilde{c}_{nk} \mathbb{E}_{\tilde{\ell}_{nk}}(\mathcal{A}) \quad (3.30)$$

In fact, by iterating the invariance lemma and discarding the pieces of curve which do not satisfy  $\tau' \geq n$  we can write:

$$F^n \ell = \bigcup_j \ell_{nj} \cup \bigcup_k \tilde{\ell}_{nk} \cup \{\tau' < n\},$$

then:

$$\begin{aligned} c_{nj} &\doteq \mathbb{P}_\ell(F^{-n}\Gamma_{nj}) & \rho_{nj} &= F^{*n}\rho/c_{nj} \\ \tilde{c}_{nk} &\doteq \mathbb{P}_\ell(F^{-n}\tilde{\Gamma}_{nk}) & \rho_{nk} &= F^{*n}\rho/\tilde{c}_{nk}. \end{aligned}$$

Clearly we have:

$$\sum_j c_{nj} \leq \mathbb{P}_\ell(\tau' \geq n);$$

moreover we claim that:

$$\sum_k \tilde{c}_{nk} \leq \mathcal{O}(\hat{y}^{-\beta}).$$

In fact, if  $F^n(x)$  belongs to a stand-by pair then, since  $\tau(x) \geq \tau'(x) \geq n$ , we know that  $F^{n-1}(x)$  belongs to some standard pair  $\ell_{(n-1)j}$  and that  $F^{n-1}(x) \in \hat{\mathbf{C}}_1$ . So by proposition 3.2.3 we can conclude that

$$\mathbb{P}_{\ell_{(n-1)j}}(\hat{\mathbf{C}}_1) \leq \text{Const} \cdot \hat{y}^{-\beta},$$

which yields the desired estimate and in turns allows us to write:

$$\sum_j c_{nj} = \mathbb{P}_\ell(\tau' \geq n) + \mathcal{O}(\hat{y}^{-\beta}).$$

Define now

$$\zeta_n \doteq \dot{\phi}(x_{n+2}) \mathbf{1}_{\tau' \geq n}$$

Then, using (3.30), lemmata 3.4.1 and 3.4.3, we obtain:

$$\begin{aligned} \mathbb{E}_\ell(\zeta_n) &= \sum_j c_{nj} \mathbb{E}_{\ell_{nj}}(\dot{\phi}(x_2)) + \sum_k \tilde{c}_{nk} \mathbb{E}_{\tilde{\ell}_{nk}}(\dot{\phi}(x_2)) \\ &\leq \sum_j c_{nj} o(\hat{y}^{-1}) + \sum_k \tilde{c}_{nk} \mathcal{O}(\hat{y}^{-\beta}) \\ &= o(\hat{y}^{-1}), \end{aligned} \tag{3.31}$$

where, to estimate the expectation on stand-by pairs, we just apply lemma 3.4.1 to  $F\tilde{\ell}_{nk}$  which by definition can be cut in standard pairs. Moreover, since  $\beta > 3/4$  we have  $\beta^* > 1$  in the statement of lemma 3.4.3.

The same argument shows that

$$\mathbb{E}_\ell(\zeta_n^2) = 2A^2 \mathbb{P}_\ell(\tau' \geq n) + \mathcal{O}(\hat{y}^{-\beta}) + o(\hat{y}^{-1}). \tag{3.32}$$

Next we consider:

$$\begin{aligned} \mathbb{E}_\ell \left( \zeta_m \sum_{i=0}^{m-1} \zeta_i \right) &= \sum_j c_{mj} \mathbb{E}_{\ell_{mj}} \left( \dot{\phi}(x_2) \sum_{i=-m}^{-1} \dot{\phi}(x_{i+2}) \right) + \\ &+ \sum_k \tilde{c}_{mk} \mathbb{E}_{\tilde{\ell}_{mk}} \left( \dot{\phi}(x_2) \sum_{i=-m}^{-1} \dot{\phi}(x_{i+2}) \right). \end{aligned} \quad (3.33)$$

Let

$$B(x_0) = \sum_{i=-m}^{-1} \dot{\phi}(x_{i+2});$$

fix now  $p > 0$  to be determined later; if  $m > p$  we define:

$$B'(x_0) \doteq \sum_{i=-m}^{-p} \dot{\phi}(x_{i+2}) \quad B''(x_0) \doteq \sum_{i=-p+1}^{-1} \dot{\phi}(x_{i+2}).$$

If otherwise  $m \leq p$ , we just let  $B' = 0$  and  $B'' = B$ . By definition of  $\tau'$  we have  $\|B'\|_\infty \leq \hat{y} + 10A \leq 2\hat{y}$ ; since moreover  $B'$  depends only on  $x_i$  with  $i < -p + 2$ , we have by the invariance lemma 3.2.2a that:

$$\|\dot{B}'\|_\infty \leq \hat{y}^{-(p-3)\beta}.$$

To estimate the contribution of  $B'$ , we write  $B' = \bar{B}' + \tilde{B}'$ , where  $\bar{B}'$  is the constant part of  $B'$  and therefore  $\|\tilde{B}'\|_\infty \leq \hat{y}^{-(p-3)\beta}$ . Therefore we have, by using lemmata 3.4.1 and 3.4.3 that:

$$\begin{aligned} \left| \mathbb{E}_{\ell_{mj}} \left( \bar{B}' \dot{\phi}(x_2) \right) \right| &= o(1) \\ \left| \mathbb{E}_{\tilde{\ell}_{mk}} \left( \bar{B}' \dot{\phi}(x_2) \right) \right| &\leq \mathcal{O}(\hat{y}^{1-\beta}) \end{aligned}$$

By requiring  $p$  large enough, we can make the contribution of  $\tilde{B}'$  as small as we need. Thus consider  $\hat{B}'' = \tilde{B}' + B''$ :

$$\|\hat{B}''\|_\infty \leq (p+1)2A;$$

moreover if  $x_0 \notin \mathbf{C}_1$  we have:

$$\left\| \frac{d\hat{B}''}{dx_1} \right\| = \mathcal{O}(1);$$

on the other hand  $\mathbb{P}_\ell(x_0 \in \mathbf{C}_1) \leq \mathcal{O}(\hat{y}^{-\beta})$  so that we obtain:

$$\begin{aligned} \left| \mathbb{E}_{\ell_{m_j}} \left( \hat{B}'' \dot{\phi}(x_2) \right) \right| &\leq \mathcal{O}(\hat{y}^{-\beta}) \\ \left| \mathbb{E}_{\tilde{\ell}_{m_k}} \left( \hat{B}'' \dot{\phi}(x_2) \right) \right| &\leq \mathcal{O}(\hat{y}^{-\beta}). \end{aligned}$$

By linearity of the expectation we can thus write:

$$\left| \mathbb{E}_\ell \left( \zeta_m \sum_{i=0}^{m-1} \zeta_j \right) \right| = o(1) \tag{3.34}$$

Now using (3.32) and (3.34) we have:

$$\begin{aligned} \mathbb{E}_\ell \left( \left( \sum_{i=0}^N \zeta_i \right)^2 \right) &= \sum_{i=0}^N (2A^2 \mathbb{P}_\ell(\tau' > i) + o(1)) \\ &\geq N \cdot 2A^2 \mathbb{P}_\ell(\tau' > N) + N \cdot o(1). \end{aligned}$$

But since  $\| \sum_{i=0}^N \zeta_i \| < 2\hat{y}$ , by taking  $N = L \cdot \hat{y}^2$  we can write:

$$L \cdot 2A^2 \mathbb{P}_\ell(\tau' > L\hat{y}^2) \leq 4 + L \cdot o(1)$$

which proves (b) by taking  $L$  large enough.

To prove (a), we write:

$$\begin{aligned} \mathbb{E}_\ell \left( \sum_{k=1}^{\infty} \zeta_k \right) &\leq \sum_n \mathbb{P}_\ell(\tau' \geq n) o(\hat{y}^{-1}) \\ &\leq \mathbb{E}_\ell(\tau') o(\hat{y}^{-1}) = o(\hat{y}). \end{aligned}$$

On the other hand:

$$\mathbb{E}_\ell \left( \sum_{k=1}^{\infty} \zeta_k \right) = \hat{y} \cdot \mathbb{P}_\ell(\xi = +1) + \lambda \hat{y} \cdot \mathbb{P}_\ell(\xi = -1),$$

where  $\lambda \in (-1/2, 1)$ ; dividing by  $\hat{y}$  we obtain:

$$\mathbb{P}_\ell(\xi = +1) + \lambda \mathbb{P}_\ell(\xi = -1) = o(1)$$

which implies:

$$\mathbb{P}_\ell(\xi = -1) = \frac{1}{1-\lambda}(1 + o(1)) > 0.6$$

that is (a). □

### 3.6 Conclusion of the proof

In this section we prove that lemma 3.5.4 implies lemma 3.5.3 by a trivial adaptation to our situation of the analogous argument found in sections 6 and 7 of [Do08]. It is described here for the sake of completeness. We first need a few preliminary results about biased random walks.

**Proposition 3.6.1.** *Let  $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n, \dots$  be iid random variables such that  $\tilde{\xi}_k \in \{-1, 1\}$  and  $\mathbb{P}(\tilde{\xi}_n = -1) = p > 1/2$ . Let  $\tilde{\chi}_n = \sum_{k=1}^n \tilde{\xi}_k$ . Then*

- $\mathbb{P}(\tilde{\chi}_n \leq 0 \text{ for all } n) > 0$ ;
- For each  $c > 1 - 2p$  there exist constants  $C > 0$  and  $\theta < 1$  such that:

$$\mathbb{P}(\tilde{\chi}_n > cn) \leq C\theta^n.$$

**Proposition 3.6.2.** *Suppose  $\xi_1, \xi_2, \dots, \xi_n, \dots$  is a random process such that  $\xi_k \in \{+1, -1\}$  and for all  $n$ :*

$$\mathbb{P}(\xi_n = -1 | \xi_1 \xi_2 \cdots \xi_{n-1}) \geq p > 1/2.$$

*Let  $\tilde{\xi}_1, \tilde{\xi}_2, \dots, \tilde{\xi}_n, \dots$  be iid random variables such that  $\tilde{\xi}_k \in \{-1, 1\}$  and  $\mathbb{P}(\tilde{\xi}_n = -1) = p$ ; correspondingly define:*

$$\chi_n = \sum_{k=1}^n \xi_k \quad \tilde{\chi}_n = \sum_{k=1}^n \tilde{\xi}_k$$

*Then for any  $n, m_1, m_2$ :*

$$\mathbb{P}(\max_{k \leq n} \tilde{\chi}_k \leq m_1, \min_{k \leq n} \tilde{\chi}_k \leq m_2) \leq \mathbb{P}(\max_{k \leq n} \chi_k \leq m_1, \min_{k \leq n} \chi_k \leq m_2).$$

The proofs of propositions 3.6.1 and 3.6.2 can be found in section 2 of [Do08] and references therein.

*Proof of lemma 3.5.3.* We will prove (3.29) which, as previously noted, implies lemma 3.5.3; fix a standard pair  $\ell$  and let  $\hat{y} = \hat{y}_\ell$ . Take  $n = Q \cdot \log_2 \hat{y}$  and  $m = -\log_2 \hat{y}$ , for  $Q$  large; then we have:

$$\mathbb{P}_\ell(\Omega) = 2\kappa > 0 \quad \Omega = \{\max_{k \leq n} \chi_k \leq 0, \min_{k \leq n} \chi_k < m\}.$$

Notice that for each standard pair  $\ell^*$  in  $F^{\tau_k} \ell$  we have that  $\chi_k \leq 0$  implies  $\hat{y}_{\ell^*} \leq 2\hat{y}$ ; we then use lemma 3.5.4b with  $s = \hat{y}^{5/2}$  to show that:

$$\forall k \quad \mathbb{P}_\ell(\tau_{k+1} - \tau_k > \hat{y}^{5/2} | \chi_k \leq 0) \leq C \cdot \theta^{\hat{y}^{1/2}}$$

notice that if  $\tau_k = \tau$  for some  $k$  we cannot apply lemma 3.5.4, however the previous inequality still holds true by definition of  $\tau_k$ . Therefore:

$$\mathbb{P}_\ell(\max_{k \leq n}(\tau_{k+1} - \tau_k) > \hat{y}^{5/2} | \Omega) \leq Cn \cdot \theta^{\hat{y}^{1/2}};$$

since  $n$  grows only logarithmically in  $\hat{y}$ , the previous expression implies that:

$$\mathbb{P}_\ell(\tau_k \geq \hat{y}^3 | \Omega) \rightarrow 0 \text{ as } \hat{y} \rightarrow \infty$$

therefore:

$$\mathbb{P}_\ell(\{\tau_n < \hat{y}^3\} \cap \Omega) > \kappa.$$

On the other hand, by our choice of  $m$ , we have that  $\forall (x_0, y_0) \in \Omega$  there exists a  $k \leq n$  such that  $y_k \leq \text{Const}$  and so  $(x_k, y_k) \in \mathbf{C}_2$  by taking  $K_1$  large enough. Hence, on  $\Omega$  we have  $\tau = \tau_n$ , which implies:

$$\mathbb{P}_\ell(\tau < \hat{y}^3) > \kappa.$$

Then for any  $k \in \mathbb{N}$  we can define functions  $n_k(x_0, y_0)$  such that

$$\mathbb{P}_\ell(\tau > n_k) < (1 - \kappa)^k.$$

In fact let  $n_1 = \hat{y}^3$ . Next, if  $\tau(x_0, y_0) < n_k$  we let  $n_{k+1} = n_k$ ; otherwise we have that either  $F^{n_k-1}(x_0, y_0)$  or  $F^{n_k}(x_0, y_0)$  belongs to a standard pair  $\ell_k^*$ . We then define  $n_{k+1} = n_k + \hat{y}_{\ell_k^*}^3$ . Since  $k$  can be taken to be arbitrarily large, we obtain (3.29) which concludes the proof of lemma 3.5.3. □



## Chapter 4

### Existence of elliptic islands for arbitrarily high energy

In this chapter we are going to prove Theorem C:

**Theorem C.** *Assume  $\gamma > 1$ ; then:*

- (a) for almost all values of the parameter  $A$  there are elliptic islands of period 2 for arbitrarily high values of  $y$ . Moreover if  $\gamma > 2$  the same result holds true for all values of  $A$ ;*
- (b) the total measure of such islands is infinite if  $\gamma \leq 4/3$  and finite if  $\gamma > 4/3$ .*

First we recall the definition of elliptic island: if an elliptic fixed point  $p$  for a two-dimensional symplectic map  $F$  is surrounded by a invariant set of closed curves and on each curve the dynamics is conjugated to an irrational rotation on the circle, we say that  $p$  is surrounded by an elliptic island. Such islands are obviously Lyapunov stable.

The outline of the proof of Theorem C is as follows. In section 4.1 and 4.2 we build a reversor map by exploiting a symmetry of the system; we recall that a reversor is an idempotent map that conjugates the dynamics with its inverse. Following a standard technique in the theory of reversible maps (see e.g. [LR98]), we use the locus of fixed points of the reversor map to find a number of periodic orbits; most of them will be hyperbolic but by fine-tuning the amplitude  $A$  we can turn some of them into elliptic periodic orbits. In our case it is quite easy to state the ellipticity

condition (section 4.3) in terms of  $A$ . We can actually state conditions to ensure that the multiplier of such periodic orbits belong to some given sub-interval of  $\mathbb{S}^1$ , which will turn out to be useful to avoid resonances. Such conditions, along with a non-degeneracy requirement on the Birkhoff normal form that we check in section 4.6, are sufficient to establish the presence of an elliptic island around the periodic point (see for instance [La93] or [dlL01]). The ellipticity condition (section 4.4) turns out to be an arithmetic condition on the parameter  $A$ ; a Borel-Cantelli argument (section 4.5) shows that this condition is satisfied by infinitely many periodic points for a set of full measure of  $A$  for all  $\gamma > 1$ . The same proof gives the stronger result that for  $\gamma > 2$  the statement is true for *all* parameters  $A$ .

Notice that a well-known result due to M. Herman ensures that an elliptic periodic point with Diophantine multiplier is surrounded by an elliptic island. Since such multipliers form a full-measure set in  $\mathbb{S}^1$ , as B. Fayad pointed out to the author, one could easily modify the Borel-Cantelli argument to prove existence of infinitely many elliptic islands with Diophantine multiplier for almost all  $A$ . However, this elegant argument does not allow to obtain estimates on the size of the islands; on the other hand, in proposition 4.5.13 we are able to state conditions on  $\gamma$  which guarantee that the Lebesgue measure of the elliptic islands obtained with our construction is either infinite or finite.

## 4.1 Construction of periodic orbits

Recall once more the definition of  $F$ :

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x + Y(y) \\ y + 2\dot{\phi}(x + Y(y)) \end{pmatrix} \quad (1.1)$$

We are going to obtain periodic orbits by building a set with special dynamical properties and then considering intersections with its forward and backward images.

**Remark 4.1.1.** Suppose  $\dot{\phi}$  is odd with respect to some point  $\bar{x}$ :

$$\dot{\phi}(\bar{x} + x) = -\dot{\phi}(\bar{x} - x),$$

then  $F$  has a *reversor* map  $R$  such that

$$R^2 = \text{Id} \quad RFR = F^{-1}.$$

We can explicitly write  $R$  as follows:

$$R : (x, y) \mapsto (2\bar{x} - x - Y(y), y).$$

Squaring the map  $R$  gives the identity map and an easy check shows that  $R$  conjugates  $F$  with its inverse. Notice also that, being defined on the cylinder, if  $\dot{\phi}$  is odd with respect to  $\bar{x}$  it has to be odd also with respect to  $\bar{x} + 1/2$ .

We are going to define the set  $\ell$  of fixed points of  $R$ :

$$\ell \doteq \{(x, y) \text{ s.t. } R(x, y) = (x, y)\}.$$

The set  $\ell$  is the disjoint union of the following two curves:

$$\begin{aligned}\ell_+(y) &= \left(\bar{x} - \frac{1}{2}Y(y), y\right) \\ \ell_-(y) &= \left(\frac{1}{2} + \bar{x} - \frac{1}{2}Y(y), y\right).\end{aligned}\tag{4.1}$$

These curves wind around the cylinder as  $y$  increases. It is more convenient to partition  $\ell_+$  and  $\ell_-$  in pieces that wind just once around the cylinder in order to work with graphs of (single valued) functions of  $x$ . This can be easily done by inverting the 1-1 map  $y \mapsto Y$ ; let this inverse be  $y(Y)$ . Define now:

$$\forall n \in \mathbb{N} \quad \ell_n(x) = (x, y(2(\bar{x} - x) + n)).$$

The curve  $\ell_+$  corresponds to even values of  $n$  while  $\ell_-$  to odd values. Subscripts will always refer to branches and superscripts will always refer to iterates of the set, i.e. for  $k \in \mathbb{Z}$ ,  $\ell_n^k \doteq F^k \ell_n$ .

The important dynamical property of  $\ell$  is that

$$\forall p \in \ell \quad F^k p = RF^{-k}Rp = RF^{-k}p,$$

therefore, if  $F^k p$  also belongs to  $\ell$ , we find  $F^k p = F^{-k} p$  that implies that the orbit of  $p$  is periodic of (possibly not least) period  $2k$ . Hence, points belonging to  $\ell^k \cap \ell$  for  $k \neq 0$  are periodic points. The issue is now to understand whether the corresponding periodic orbits are elliptic or hyperbolic. Taking inspiration from [GL00] we work out from scratch the period 2 case.

## 4.2 Period 2 orbits

First we classify period 2 orbits. This turns out to be quite simple, as the following proposition shows. To fix notations, let  $\{p_1, p_2\}$  be a 2-periodic orbit,

$p_1 = (x_1, y_1)$  and  $p_2 = (x_2, y_2)$ ; since  $y_0 = y_2$ , we have  $\dot{\phi}(x_1) = -\dot{\phi}(x_2)$ .

**Definition 4.2.1.** Being  $\phi$  a sine function, 2-periodic orbits can only be of one of the following types:

- $\ddot{\phi}(x_1) = \ddot{\phi}(x_2)$ , such orbits will be called (+)-orbits;
- $\ddot{\phi}(x_1) = -\ddot{\phi}(x_2)$ , such orbits will be called (-)-orbits;

**Proposition 4.2.2.** Let  $\{p_1, p_2\}$  be a 2-periodic orbit; there can be two cases:

- $p_1, p_2 \in \ell$ , the orbit is a (+)-orbit;
- $Y(y_1) \equiv Y(y_2) \equiv \frac{1}{2} \pmod{1}$ ; the orbit is a (-)-orbit.

*Proof.* Let us write the condition for  $p_1 = (x_1, y_1)$  and  $p_2 = (x_2, y_2)$  to be periodic:

$$x_1 \mapsto x_1 + Y(y_1) = x_2$$

$$x_2 \mapsto x_1 + Y(y_1) + Y(y_2) = x_1$$

$$y_1 \mapsto y_1 + 2\dot{\phi}(x + Y(y)) = y_2$$

$$y_2 \mapsto y_1 + 2\left(\dot{\phi}(x + Y(y_1)) + \dot{\phi}(x + Y(y_1) + Y(y_2))\right) = y_1.$$

So that we have the two conditions:

$$Y(y_1) + Y(y_2) \equiv 0 \pmod{1} \quad \dot{\phi}(x + Y(y_1)) + \dot{\phi}(x) = 0.$$

Since  $\dot{\phi}(x)$  is a cosine function, the second possibility can be true only under one of the two following conditions:

- $x_1 + Y(y_1) = 2x_0 - x_1$  therefore  $p_1 \in \ell$ ; the same is true for  $p_2$
- $x_1 + Y(y_1) = x_1 + \frac{1}{2}$  therefore  $Y(y_1) \equiv Y(y_2) \equiv \frac{1}{2}$ .

Notice moreover that since  $\dot{\phi}$  is odd with respect to  $\bar{x}$ ,  $\ddot{\phi}$  is even with respect to the same point  $\bar{x}$ , therefore orbits of the first kind are (+)-orbits. On the other hand, orbits of the second type satisfy the opposite condition  $\ddot{\phi}(x_1) = -\ddot{\phi}(x_1 + 1/2 = x_2)$  and so they are (-)-orbits.  $\square$

### 4.3 Elliptic locus for period 2 orbits

In this section we find a subset of the phase space such that all 2-orbits that lie in the set are elliptic.

**Proposition 4.3.1.** *Denote the points of the orbit by  $(x_1, y_1)$  and  $(x_2, y_2)$ ; let  $Y'_i \doteq Y'(y_i)$ , and  $\dot{\phi}_1 \doteq \dot{\phi}(x_2)$ ,  $\dot{\phi}_2 \doteq \dot{\phi}(x_1)$  and similarly for  $\ddot{\phi}$  (swapped indices are intended). Notice that proposition 4.2.2 implies  $\ddot{\phi}_1 = \pm\ddot{\phi}_2$ . Let  $\nu_i \doteq \ddot{\phi}_i Y'_i$  and  $-1 \leq c_1 < c_2 \leq 0$ ; define the following sets:*

$$E_{c_1 c_2}^+ \doteq \{(\nu_1, \nu_2) \text{ s.t. } \nu_1 + \nu_2 + \nu_1 \nu_2 \in (c_1, c_2)\}$$

$$E_{c_1 c_2}^- \doteq \{(\nu_1, \nu_2) \text{ s.t. } \nu_1 \nu_2 \in (c_2, c_1)\}.$$

Then  $(\pm)$ -orbits belonging to  $E_{c_1 c_2}^\pm$  are elliptic and the multiplier  $\lambda$  is such that  $\Re\lambda \in (1 + 2c_1, 1 + 2c_2)$ .

*Proof.* We start by computing the differential  $dF$  on each point on the orbit:

$$dF_1 = \begin{pmatrix} 1 & Y'_1 \\ 2\ddot{\phi}_1 & 1 + 2\ddot{\phi}_1 Y'_1 \end{pmatrix} \quad dF_2 = \begin{pmatrix} 1 & Y'_2 \\ 2\ddot{\phi}_2 & 1 + 2\ddot{\phi}_2 Y'_2 \end{pmatrix}.$$

The condition for a matrix  $M$  to be elliptic is that  $|\text{Tr}M| < 2$ ; moreover, for any elliptic matrix  $M$ ,  $\frac{1}{2}\text{Tr}M$  is the real part of its multiplier. By direct computation of

the trace of the product of the two differentials and using the  $(\pm)$  relations on  $\ddot{\phi}$ , we obtain:

$$1/2 \operatorname{Tr} (dF_1 dF_2) = 1 + 2 \left( \ddot{\phi}_1 Y_1' + \ddot{\phi}_2 Y_2' \right) + 2Y_1' Y_2' \ddot{\phi}_1 \ddot{\phi}_2 \quad (+)$$

$$1/2 \operatorname{Tr} (dF_1 dF_2) = 1 + 2Y_1' Y_2' \ddot{\phi}_1 \ddot{\phi}_2. \quad (-)$$

By direct computation we obtain the following conditions in  $\nu_i$  that ensure ellipticity and the supplementary condition on the multiplier:

$$\nu_1 + \nu_2 + \nu_1 \nu_2 \in (c_1, c_2) \quad (+)$$

$$\nu_1 \nu_2 \in (c_1, c_2), \quad (-)$$

that are the defining conditions for the sets  $E_{c_1 c_2}^\pm$ . □

Notice that since  $\ddot{\phi}_1 = \pm \ddot{\phi}_2$  we have that

$$\frac{\nu_1}{\nu_2} = \frac{\ddot{\phi}_1 Y_1'}{\ddot{\phi}_2 Y_2'} = \pm \frac{Y_1'}{Y_2'}.$$

Since  $Y' = Y^{1-1/\gamma}$  and  $\left| Y_2^{1/\gamma} - Y_1^{1/\gamma} \right| \sim A = \max |\dot{\phi}|$  we have:

$$\frac{\nu_1}{\nu_2} \sim \pm \left( \frac{Y_1}{Y_2} \right)^{1-\frac{1}{\gamma}} \rightarrow \pm 1^- \text{ as } Y_1 \rightarrow \infty.$$

Even if the shape of the sets  $E_{c_1 c_2}^\pm$  is not very complicated, it is convenient to state a sufficient condition in terms of just one parameter  $\nu$ . Fix  $\varepsilon$  small, then if we let  $y_1 < y_2$  big enough, we have  $(1 - \varepsilon)|\nu_2| < |\nu_1| < |\nu_2|$ ; a direct calculation yields the following sufficient conditions for 2-orbits to be elliptic and satisfying the required condition on the multiplier:

$$(+) \text{-orbits: } \nu_2 \in (c_1', c_2') \quad (-) \text{-orbits: } |\nu_2| \in (|c_1''|, |c_2''|) \quad (4.2)$$

where  $c_1', c_2'$  and  $c_1'', c_2''$  are  $\varepsilon$ -close respectively to  $c_1$  and  $c_2$ .

#### 4.4 Description of $\ell^1$ and ellipticity condition

In this and the subsequent sections all pictures and geometric constructions are made keeping in mind the coordinates  $(x, Y(y))$ . In these coordinates  $\ell$  is represented by a straight line, and it is much easier to have geometric intuition about the dynamics. Recall that the simple choice for  $\phi$  given by equation (1.2) implies the following expression for  $\dot{\phi}$ :

$$\dot{\phi}(x) = A \cos(2\pi x).$$

**Proposition 4.4.1.** *Let  $y(Y)$  be the inverse function of  $Y(y)$  and fix  $-1 \leq c_1 < c_2 \leq 0$ . Then there exist real positive numbers  $C_1 < C_2$  such that, for any  $n < m \in \mathbb{N}$  the following condition*

$$y \left( m - \frac{1}{2} - \frac{C_2}{A} \frac{1}{m^{1-1/\gamma}} \right) < y \left( n + \frac{1}{2} \right) + 2A < y \left( m - \frac{1}{2} - \frac{C_1}{A} \frac{1}{m^{1-1/\gamma}} \right) \quad (4.3)$$

*implies the existence of a 2-periodic elliptic orbit oscillating between  $Y = n + 1/2$  and  $Y = m + 1/2$  such that its multiplier satisfies  $\Re \lambda \in (1 + 2c_1, 1 + 2c_2)$ .*

*Proof.* Consider  $\ell_n^1$ : let  $\eta_k(\xi) \doteq y(2(\xi - \bar{x}) + k)$  so that:

$$\ell_n^1(\xi) = \left( \xi, \eta_n(\xi) + 2\dot{\phi}(\xi) \right).$$

So for  $A = 0$  this is just a line in the  $(x, Y(y))$ -plane. As  $A$  increases, the line deforms and presents similarities with the shape of  $\dot{\phi}$ , as we can observe in figure 4.1. As we proved in proposition 4.2.2, 2-orbits obtained by intersecting  $\ell$  and  $\ell^1$  are (+)-orbits; we now claim that the highest energy point  $(x_2, y_2)$  of the orbit lies where  $\dot{\phi} > 0$ . In fact we know that  $\dot{\phi}(x_2) = y_2 - y_1$ ; since we want that  $y_2 > y_1$  we need



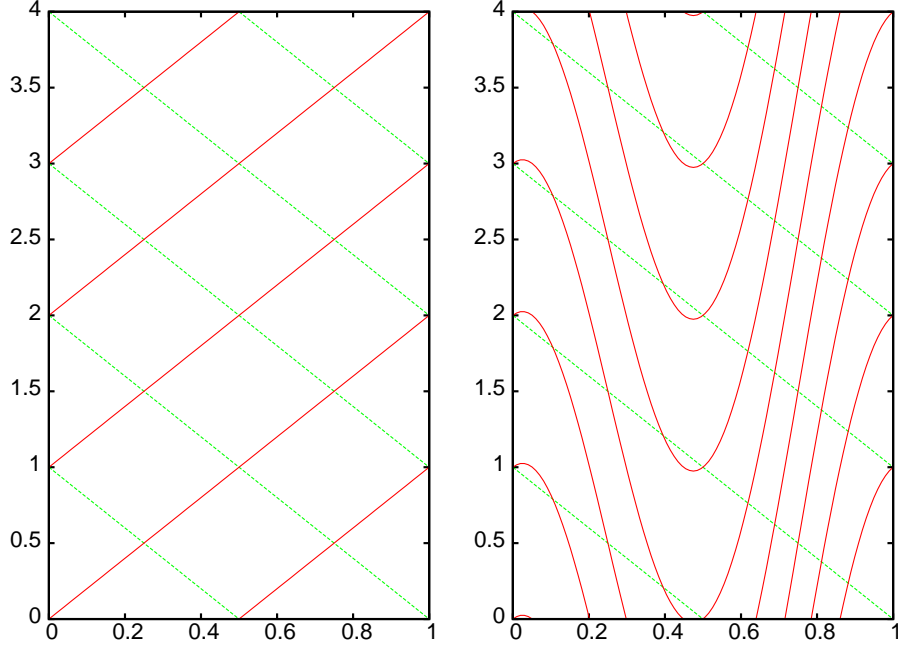


Figure 4.1: On the left the reference picture for  $A = 0$ , on the right the situation for  $A > 0$

such quantity to be positive. Having that fixed,  $\nu_2 = \ddot{\phi}(x_1)Y'(y_2) = \ddot{\phi}(x_2)Y'(y_2)$ , therefore we have that condition (4.2) is satisfied if  $(x_2, y_2) \in \ell \cap \ell^1$  belongs to this set:

$$\tilde{E}_{c'_1 c'_2}^+ \doteq \left\{ (x, y) \text{ s.t. } \frac{c'_1}{Y'(y)} < \ddot{\phi}(x) < \frac{c'_2}{Y'(y)}, \dot{\phi}(x) > 0 \right\}.$$

First notice that this set is an  $\mathcal{O}(1/(A \cdot Y'(y)))$ -thin strip that lies  $\mathcal{O}(1/(A \cdot Y'(y)))$  to the right of the vertical line  $\xi = 0$  (that corresponds to  $c'_2 = -1$ ). By direct inspection we obtain that in  $(x, Y)$  coordinates, each branch of  $\ell^0$  is a straight line with angular coefficient  $-2$  and each branch of  $\ell^1$  near  $\xi = 0$  is approximated by a parabola that intersects  $\xi = 0$  with positive derivative (close to 2); the maximum of

the parabola is given by the equation:

$$\ddot{\phi}(\xi) = -\frac{1}{Y'(y_n(\xi))} < -\frac{1}{Y'(y)} < \frac{c_1}{Y'(y)}.$$

Figure 4.2 illustrates the properties we just described. The key fact to notice is that

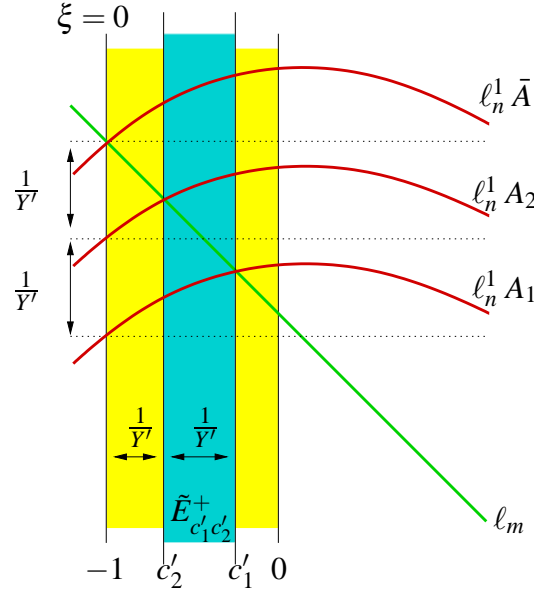


Figure 4.2: Explicit construction, in  $(x, Y)$  coordinates, of values of the parameter  $A$  for which we have an elliptic periodic point of period 2 with given bounds on the multiplier.

the values of  $A$  we are seeking are close to values  $\bar{A}$  of the parameter for which the intersection lies on the vertical line  $\xi = 0$ . Let us compute the intersection of  $\ell_m$  and  $\ell_n^1$  with the vertical line  $\xi = 0$ :

$$\ell_m(0) = (0, y(-1/2 + m)) \tag{4.4}$$

$$\ell_n^1(0) = (0, y(1/2 + n) + 2A) \tag{4.5}$$

Therefore if we want  $\ell$  and  $\ell^1$  to intersect on the line  $\xi = 0$  we need to find  $\bar{A}$  such that points in (4.4) and (4.5) are equal for some  $n, m$ , i.e.:

$$y\left(n + \frac{1}{2}\right) + 2\bar{A} = y\left(m - \frac{1}{2}\right) \quad (4.6)$$

Now it is clear that we can find  $A_1$  and  $A_2$  as in the picture such that the intersection lies on the boundary of  $\tilde{E}_{c'_1 c'_2}^+$ . Using the properties we described above it is also clear that, in  $(x, Y)$  coordinates, the distances between intercepts of  $\ell^1$  corresponding to each  $A_i$  with the vertical  $\xi = 0$  are linear functions of the  $x$  coordinates of the intersections themselves, therefore of order  $\mathcal{O}(1/(A \cdot Y'(y)))$ . More precisely, mimicking equation (4.5) and recalling that  $Y' \sim Y^{1-1/\gamma}$ , we obtain that there exist  $C_1$  and  $C_2$  such that if

$$y\left(m - \frac{1}{2} - \frac{C_2}{A} \frac{1}{m^{1-1/\gamma}}\right) < y\left(n + \frac{1}{2}\right) + 2A < y\left(m - \frac{1}{2} - \frac{C_1}{A} \frac{1}{m^{1-1/\gamma}}\right).$$

then the intersection  $(\ell_m \cap \ell_n^1) \cap \tilde{E}_{c'_1 c'_2}^+ \neq \emptyset$ . □

Condition 4.3 is essentially an arithmetic condition on  $A$  and  $\gamma$ . In the next section we prove that this condition is satisfied for parameters  $A$  as in the statement of (a) in Theorem C. At that point we will be only left with checking the non-degeneracy condition and estimating the measure of the islands.

## 4.5 Arithmetic condition

In this section we are going to prove a result that is of independent interest; for simplicity we state the arithmetic condition in a slightly simplified form with respect to the case in consideration. Namely we drop the  $1/2$  that appears in 4.3

and we rescale by a factor of  $-2$  both  $C_1$  and  $C_2$ . One can easily verify that this does not affect the proof in any sense.

The condition is reminiscent of the Khinchin's theorem on Diophantine approximation [Kh64]. In fact we want to investigate parameters  $\gamma$  and  $a$  such that the following inclusion is true for infinitely many  $n$  and  $m \in \mathbb{N}$ :

$$(n^{1/\gamma} + a)^\gamma \in \left( m + \frac{C_1}{a} m^{-\xi}, m + \frac{C_2}{a} m^{-\xi} \right),$$

for an appropriate (and fixed) choice of  $C_2 > C_1 > 0$  and  $\xi > 0$ . In our case  $a = 2A$  and  $\xi = 1 - 1/\gamma$ .

Let us first introduce some useful definitions:

**Definition 4.5.1.** Let us fix  $\xi > 0$ ,  $\gamma > 1$ ,  $C_2 > C_1 > 0$ . Then

$$\begin{aligned} \mathcal{G}_{a,m} &\doteq \left( m + \frac{C_1}{a} m^{-\xi}, m + \frac{C_2}{a} m^{-\xi} \right); \\ \mathcal{G}_a &\doteq \bigcup_{m \in \mathbb{N}} \mathcal{G}_{a,m} \quad \tilde{\mathcal{G}}_a \doteq \mathcal{G}_a^{1/\gamma}; \\ X_a &\doteq \{ n^{1/\gamma} + a, n \in \mathbb{N} \}. \end{aligned}$$

Using this notation the parameter  $a$  satisfies the arithmetic condition if the cardinality  $|X_a \cap \tilde{\mathcal{G}}_a|$  is infinite.

**Definition 4.5.2.** Let  $n, k \in \mathbb{N}$ :

$$\begin{aligned} \mathcal{A} &\doteq \left\{ a \in \mathbb{R}^+ \text{ s.t. } |X_a \cap \tilde{\mathcal{G}}_a| = \infty \right\}; \\ \mathcal{A}^n &\doteq \left\{ a \in \mathbb{R}^+ \text{ s.t. } (n^{1/\gamma} + a)^\gamma \in \mathcal{G}_a \right\}; \\ \mathcal{A}_k^n &\doteq \left\{ a \in \mathbb{R}^+ \text{ s.t. } (n^{1/\gamma} + a)^\gamma \in \mathcal{G}_{a,(n+k)} \right\}; \end{aligned}$$

Clearly  $\mathcal{A}^n = \bigcup_k \mathcal{A}_k^n$ , moreover if  $\tilde{\mathcal{A}}^{n_0} \doteq \bigcup_{n \geq n_0} \mathcal{A}^n$ , then  $\mathcal{A} = \bigcap_{n_0} \tilde{\mathcal{A}}^{n_0} = \limsup_{n \rightarrow \infty} \mathcal{A}^n$

**Lemma 4.5.3.** *For all  $\gamma > 1$ ,  $\mathcal{A}$  is a  $G_\delta$  set dense in  $\mathbb{R}^+$ .*

*Proof.* Each  $\tilde{\mathcal{A}}^{n_0}$  is open since it is a union of open sets. Moreover it is dense because the distance between endpoints of consecutive intervals belonging to  $\tilde{\mathcal{G}}_a$  goes to 0 as  $m \rightarrow \infty$ ; thus, so do the distances between endpoints of the intervals belonging to  $\mathcal{A}^n$  as  $n \rightarrow \infty$ . As the point  $a = 0$  is accumulated by the left endpoints of the first interval in  $\mathcal{A}^n$ , we conclude that  $\tilde{\mathcal{A}}^{n_0}$  is dense in  $\mathbb{R}^+$ .  $\square$

Let us define the following conditions involving  $\xi$  and  $\gamma$ :

$$\xi \leq 1 \quad (\text{diverging});$$

$$\xi < \frac{1}{\gamma} \quad (\text{overlapping}).$$

Notice that, since  $\gamma > 1$ , the overlapping condition implies the diverging condition.

Now we can state the result as follows:

**Theorem 4.5.4.** *If the diverging condition does not hold, then the measure of  $\mathcal{A}$  is zero. If the diverging condition holds then  $\mathcal{A}$  has full measure in  $\mathbb{R}^+$ ; moreover if the overlapping condition also holds then  $\mathcal{A}$  is the whole  $\mathbb{R}^+$ .*

Notice that, as in Khinchin's theorem, we obtain that the required property is satisfied either by a null set or by a full measure set. This dichotomy seems to be quite common in approximation problems similar to the one we are studying.

*Proof.* The proof will be presented in four steps.

Step one *Reduction to a compact set of parameters*

Consider a partition of  $\mathbb{R}^+$  as follows:

$$\mathbb{R}^+ = \bigcup_k [\alpha_k, \beta_k],$$

such that

$$\forall k \quad \beta_k/\alpha_k < C_2/C_1. \quad (4.7)$$

This implies, for all  $a \in [\alpha_k, \beta_k]$  the middle inequality in the following expression:

$$\frac{C_1}{\beta_k} < \frac{C_1}{a} < \frac{C_1}{\alpha_k} < \frac{C_2}{\beta_k} < \frac{C_2}{a} < \frac{C_2}{\alpha_k}.$$

Hence we can build a superset  $\mathcal{G}_{[k]}^\#$  and a subset  $\mathcal{G}_{[k]}^\flat$  as follows:

$$\begin{aligned} \mathcal{G}_{[k],m}^\# &\doteq \left( m + \frac{C_1}{\beta_k} m^{-\xi}, m + \frac{C_2}{\alpha_k} m^{-\xi} \right) \\ \mathcal{G}_{[k],m}^\flat &\doteq \left( m + \frac{C_1}{\alpha_k} m^{-\xi}, m + \frac{C_2}{\beta_k} m^{-\xi} \right) \end{aligned}$$

then as before:

$$\begin{aligned} \mathcal{G}_{[k]}^\# &\doteq \bigcup_{m \in \mathbb{N}} \mathcal{G}_{[k],m}^\# & \tilde{\mathcal{G}}_{[k]}^\# &\doteq \mathcal{G}_{[k]}^\#{}^{1/\gamma} \\ \mathcal{G}_{[k]}^\flat &\doteq \bigcup_{m \in \mathbb{N}} \mathcal{G}_{[k],m}^\flat & \tilde{\mathcal{G}}_{[k]}^\flat &\doteq \mathcal{G}_{[k]}^\flat{}^{1/\gamma}, \end{aligned}$$

so that if  $a \in [\alpha_k, \beta_k]$ :

$$\tilde{\mathcal{G}}_{[k]}^\# \supset \tilde{\mathcal{G}}_a \supset \tilde{\mathcal{G}}_{[k]}^\flat.$$

Hence, it is enough to prove the result for  $\tilde{\mathcal{G}}_{[k]}^\#$  (to obtain estimates from above) and  $\tilde{\mathcal{G}}_{[k]}^\flat$  (to obtain estimates from below) for all  $k$ . To simplify notation we now fix  $k$ , we let  $\alpha = \alpha_k$  and  $\beta = \beta_k$ ; we then redefine  $\mathcal{A}, \mathcal{A}^n, \mathcal{A}_k^n$  as their intersection with the interval  $[\alpha, \beta]$ ; finally we define  $\mathcal{A}^\#, \mathcal{A}^\flat$  and their components as we did before for  $\mathcal{A}$ , but using in the definition respectively  $\tilde{\mathcal{G}}^\#$  and  $\tilde{\mathcal{G}}^\flat$ .

Step two *Structure of the sets  $\mathcal{A}^n$*

Define  $\delta_k^n, \bar{\Delta}_k^n, \Delta_k^n, I_k^n$  as in figure 4.3. The following lemma provides some useful estimates:

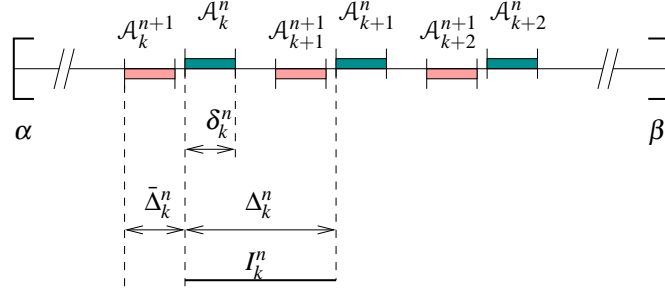


Figure 4.3: Definition of  $\delta_k^n, \bar{\Delta}_k^n, \Delta_k^n, I_k^n$ .

**Lemma 4.5.5.** *Define the following positive quantities:*

$$\ell_+ \doteq \frac{C_2}{\alpha} - \frac{C_1}{\beta} \quad \ell_- \doteq \frac{C_2}{\beta} - \frac{C_1}{\alpha},$$

and the set  $K_n \doteq \{k \in \mathbb{N} \text{ s.t. } \mathcal{A}_k^n \neq \emptyset\}$ . Then:

$$\delta_k^n = \frac{\ell}{\gamma} (n+k)^{-\xi-(1-1/\gamma)} + h.o.t. \text{ for } \ell \in (\ell^+, \ell^-) \quad (\text{E1})$$

$$\Delta_k^n = \frac{1}{\gamma} (n+k)^{-(1-1/\gamma)} + h.o.t. \quad (\text{E2})$$

$$\bar{\Delta}_k^n = \frac{1}{\gamma} \left(1 - \frac{1}{\gamma}\right) kn^{1/\gamma-2} + h.o.t. \quad (\text{E3})$$

$$K_n \sim [\gamma\alpha \cdot n^{1-1/\gamma} + h.o.t., \gamma\beta \cdot n^{1-1/\gamma} + h.o.t.] \cap \mathbb{N} \quad (\text{E4})$$

*Proof.* We first bound the length of the intervals  $\mathcal{A}_k^n$ :

$$\begin{aligned} \delta_k^n &< \left(n+k + \frac{C_2}{\alpha} (n+k)^{-\xi}\right)^{1/\gamma} - \left(n+k + \frac{C_1}{\beta} (n+k)^{-\xi}\right)^{1/\gamma} \\ &= \frac{\ell^+}{\gamma} (n+k)^{-\xi-(1-1/\gamma)} + h.o.t. \end{aligned}$$

The bound from below is similar and yields the expected result. Then we estimate the length of the intervals  $I_k^n$ :

$$\begin{aligned}
\Delta_k^n &= (n+k+1)^{1/\gamma} - n^{1/\gamma} - \left( (n+k)^{1/\gamma} - n^{1/\gamma} \right) \\
&= (n+k+1)^{1/\gamma} - (n+k)^{1/\gamma} \\
&= \frac{1}{\gamma} (n+k)^{1/\gamma-1} + \text{h.o.t.}
\end{aligned}$$

Next the offset of two subsequent  $\mathcal{A}_k^n$ :

$$\begin{aligned}
\bar{\Delta}_k^n &= (n+k+1)^{1/\gamma} - (n+1)^{1/\gamma} - \left( (n+k)^{1/\gamma} - n^{1/\gamma} \right) \\
&= (n+k+1)^{1/\gamma} - (n+k)^{1/\gamma} - \left( (n+1)^{1/\gamma} - n^{1/\gamma} \right) \\
&= \frac{1}{\gamma} \left( (n+k)^{1/\gamma-1} - n^{1/\gamma-1} \right) + \text{h.o.t.} \\
&= -\frac{1}{\gamma} \left( 1 - \frac{1}{\gamma} \right) k \cdot n^{1/\gamma-2} + \text{h.o.t.}
\end{aligned}$$

Finally we estimate  $K_n$ :

$$\begin{aligned}
K_n &= \left[ (n^{1/\gamma} + \alpha)^\gamma - n, (n^{1/\gamma} + \beta)^\gamma - n \right] \cap \mathbb{N} \\
&= \left[ \gamma\alpha \cdot n^{1-1/\gamma} + \text{h.o.t.}, \gamma\beta \cdot n^{1-1/\gamma} + \text{h.o.t.} \right] \cap \mathbb{N};
\end{aligned}$$

notice that

$$|K_n| \sim \gamma n^{1-1/\gamma} \cdot (\beta - \alpha).$$

□

Step three *Overlapping regime*

From the previous estimates we can already obtain the result in the overlapping regime. In fact (E4) implies that  $k$  is  $\mathcal{O}(n^{1-1/\gamma})$ , therefore, by (E3),  $\bar{\Delta}_k^n$  is  $\mathcal{O}(n^{-1})$ .



This implies that if  $-\xi - (1 - 1/\gamma) > -1$  (i.e.  $\xi < 1/\gamma$  that is the overlapping condition)  $\mathcal{A}_k^n$  and  $\mathcal{A}_k^{n+1}$  will eventually overlap because, by (E1), the length of the intervals  $\mathcal{A}_k^n$  goes to zero slower than their offset. Since they overlap and they are moving like  $1/n$ , they will eventually cross the left endpoint so for each fixed  $k$  they are going to cover the whole interval, therefore  $\mathcal{A}^n$  is going to cover  $[\alpha, \beta]$  infinitely many times, and  $\mathcal{A}$  will contain  $[\alpha, \beta]$ .

Step four *Non-overlapping regime*

We will now focus on the strictly non-overlapping regime i.e.  $\xi > 1/\gamma$ ; the critical case  $\xi = 1/\gamma$  will be considered later as it is just a combination of this and the previous situation.

Define, for any Borel set  $E \subset [\alpha, \beta]$ ,  $\mathbb{P}(E) = \text{Leb}(E)/(\beta - \alpha)$  as a probability measure on  $[\alpha, \beta]$ . We are going to prove that the set  $\mathcal{A}$  has either full measure or measure zero using the following strong form of the Borel-Cantelli lemma:

**Lemma 4.5.6** (Borel-Cantelli-Erdős-Rényi [ER59]). *Let  $\{\mathcal{A}_k\}$  be a sequence of events on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$ . If*

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{A}_n) < \infty \quad (\text{convergence}),$$

*then  $\mathbb{P}(\limsup \mathcal{A}_n) = 0$ . If instead*

$$\sum_{n=1}^{\infty} \mathbb{P}(\mathcal{A}_n) = \infty \quad (\text{divergence})$$

*and*

$$\liminf \frac{\sum_{k,l=1}^n \mathbb{P}(\mathcal{A}_k \mathcal{A}_l)}{(\sum_{k=1}^n \mathbb{P}(\mathcal{A}_k))^2} = 1 \quad (\text{weak independence}),$$

then  $\mathbb{P}(\limsup \mathcal{A}_n) = 1$ .

We will obtain the result by verifying one of the above conditions for the sets  $\mathcal{A}^n$ . The following two lemmata deal with the estimates of the convergence/divergence condition and the weak independence condition respectively. They need to be proved for both  $\mathcal{A}^\sharp$  and  $\mathcal{A}^\flat$ ; however to simplify the exposition we will drop the superscript and let  $\ell = \ell_+$  or  $\ell = \ell_-$  according to the case. In this way we simultaneously obtain an upper and lower bound for the measure of  $\mathcal{A}$ , from which we obtain any of the required conditions.

**Lemma 4.5.7.** *Fix  $\bar{n}$  and let  $N$  tend to infinity. Then:*

$$\sum_{n=\bar{n}}^N \mathbb{P}(\mathcal{A}^n) = \frac{1}{1-\xi} \ell N^{1-\xi} + h.o.t. \quad (4.8)$$

*Proof.* Let us first compute:

$$\begin{aligned} \text{Leb}(\mathcal{A}^n) &= \sum_{k \in K_n} \text{Leb}(\mathcal{A}_k^n) = \sum_{k \in K_n} \delta_k^n = \frac{\ell}{\gamma} \sum_{k \in K_n} (n+k)^{-\xi-1+1/\gamma} \\ &= \frac{\ell}{\gamma} \sum_{k \in K_n} n^{-\xi-1+1/\gamma} + h.o.t. = (\beta - \alpha) \ell n^{-\xi} + h.o.t. \end{aligned}$$

Then normalizing and summing on  $n$  yields (4.8):

$$\sum_{n=\bar{n}}^N \mathbb{P}(\mathcal{A}^n) = \ell \sum_{n=\bar{n}}^N n^{-\xi} + h.o.t. = \frac{1}{1-\xi} \ell N^{1-\xi} + h.o.t.$$

□

Observe that the diverging condition implies that (4.8) diverges as  $N \rightarrow \infty$ , while the series converges if the condition is not satisfied.

**Lemma 4.5.8.** Fix  $\bar{n}$ , let  $N$  go to infinity, then:

$$\sum_{n=\bar{n}}^N \sum_{m=\bar{n}}^N \mathbb{P}(\mathcal{A}^n \mathcal{A}^m) = \ell^2 \frac{1}{(1-\xi)^2} N^{2-2\xi} + h.o.t.$$

*Proof.* First of all, by symmetry we can assume  $m > n$  by paying a factor of 2 and some diagonal higher order terms; then we separate again  $\mathcal{A}^n$  in their respective components  $\mathcal{A}_k^n$

$$\begin{aligned} \sum_{n=\bar{n}}^N \sum_{m=\bar{n}}^N \mathbb{P}(\mathcal{A}^n \mathcal{A}^m) &= 2 \sum_{n=\bar{n}}^N \sum_{m=n}^N \mathbb{P}(\mathcal{A}^n \mathcal{A}^m) + h.o.t. = \\ &= 2 \sum_{n=\bar{n}}^N \sum_{k \in K_n} \sum_{m=n}^N \sum_{l \in K_m} \mathbb{P}(\mathcal{A}_k^n \mathcal{A}_l^m) + h.o.t. \end{aligned}$$

Now let us focus on the last two sums, fixing momentarily  $n$  and  $k$ . We can write the following equality:

$$\sum_{m=n}^N \sum_{l \in K_m} \mathbb{P}(\mathcal{A}_k^n \mathcal{A}_l^m) = \sum_{p=1}^{P_k^n} \mathbb{P}(\mathcal{A}_k^n \mathcal{B}_p^{nk}) + \text{error term} \quad (4.9)$$

where we define the sets  $\mathcal{B}_p^{nk}$  as the  $p$ -th wave (see figure 4.4):

$$\mathcal{B}_p^{nk} \doteq \bigsqcup_{r \in \mathbb{N}} \mathcal{A}_{k+p}^r \cap I_k^n,$$

$P_k^n$  is an appropriate number that is estimated by the next lemma and the error term is due to the fact that the last wave could be incomplete.

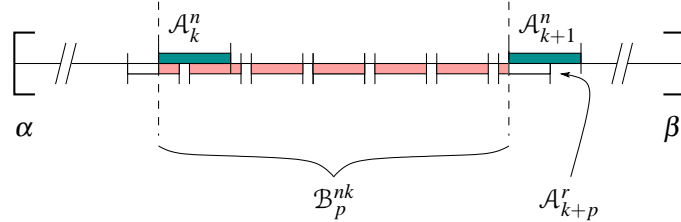


Figure 4.4: Relation between waves  $\mathcal{B}_p^{nk}$  and sets  $\mathcal{A}_k^n$

**Lemma 4.5.9.** *The following estimate holds:*

$$P_k^n = \left( \frac{N^{1-1/\gamma}}{n^{1-1/\gamma}} - 1 \right) \cdot k + \mathcal{O}(1).$$

*Proof.* As it follows from (E3), the offset between  $\mathcal{A}_{k+p}^r$  and  $\mathcal{A}_{k+p}^{r+1}$  is  $\bar{\Delta}_{k+p}^r$ .  $\mathcal{B}_p^{nk}$  is the union over all  $r$  such that  $\mathcal{A}_{k+p}^r$  intersects  $I_k^n$ ; for each fixed  $p$  there is a wave of  $\mathcal{A}_{k+p}^r$  that will be inside  $I_k^n$  for some time and then leave the set.  $P_k^n$  is the number of waves that will pass through  $I_k^n$  in the time  $N$ . This means that:

$$(N + k + P_k^n)^{1/\gamma} - N^{1/\gamma} \sim (n + k)^{1/\gamma} - n^{1/\gamma},$$

that is:

$$N^{1/\gamma-1} (k + P_k^n) \sim n^{1/\gamma-1} k \Rightarrow P_k^n = \left( \frac{N^{1-1/\gamma}}{n^{1-1/\gamma}} - 1 \right) \cdot k + \mathcal{O}(1).$$

□

The error term in (4.9) can be easily bounded by the order of  $\mathbb{P}(\mathcal{A}_k^n)$ , as we miscount of at most 1 wave. We are left with the computation of  $\mathbb{P}(\mathcal{A}_k^n \mathcal{B}_p^{nk})$ . Each  $\mathcal{B}_p^{nk}$  is the union of intervals that are  $\bar{\Delta}_{k+p}^r$  apart and  $\delta_{k+p}^r$  long. Their ratio gives the portion of the interval covered by each wave.

**Lemma 4.5.10.** *Let us introduce the parameter  $\eta = (k + p)/k$ . Then we have:*

$$\frac{\delta_{k+p}^r}{\bar{\Delta}_{k+p}^r} = \ell \frac{\gamma}{\gamma - 1} \frac{n^{1-\xi}}{k} \eta^{\frac{1-\xi\gamma}{\gamma-1}} + h.o.t. \doteq \lambda_k^n(p).$$

*Proof.* By definitions of  $\delta_k^n$  and  $\bar{\Delta}_k^n$  we have:

$$\frac{\delta_{k+p}^r}{\bar{\Delta}_{k+p}^r} = \ell \frac{\gamma}{\gamma - 1} \frac{(r + k + p)^{-\xi-1+1/\gamma}}{(k + p)(r^{1/\gamma-2})}.$$

We now need an estimate on  $r$ : acting as before for the computation of  $P$  we obtain the following bound:

$$r^{1/\gamma-1}(k+p) \sim n^{1/\gamma-1}k \Rightarrow r = n \left( \frac{k+p}{k} \right)^{\frac{\gamma}{\gamma-1}}.$$

We can rewrite the previous expression as:

$$\ell \frac{\gamma}{\gamma-1} \frac{(n\eta^{\gamma/(\gamma-1)} + \eta k)^{-\xi-1+1/\gamma}}{\eta k \cdot n^{1/\gamma-2} \cdot \eta^{(\gamma/(\gamma-1))(1/\gamma-2)}},$$

that is:

$$\begin{aligned} & \ell \frac{\gamma}{\gamma-1} \frac{\left( (n + \eta^{-1/(\gamma-1)}k) \eta^{\gamma/(\gamma-1)} \right)^{-\xi-1+1/\gamma}}{k \cdot n^{1/\gamma-2} \cdot \eta^{-\gamma/(\gamma-1)}} + \text{h.o.t.} \simeq \\ & \simeq \ell \frac{\gamma}{\gamma-1} \frac{n^{1-\xi}}{k} \eta^{\frac{1-\xi\gamma}{\gamma-1}} \end{aligned}$$

□

Therefore  $\mathbb{P}(\mathcal{A}_k^n \mathcal{B}_p^{nk}) = \lambda_k^n(p) \cdot \mathbb{P}(\mathcal{A}_k^n) + \mathcal{O}(\bar{\Delta}_{k+p}^r / (\beta - \alpha))$ , where the error term comes from the non-uniformity of the set  $\mathcal{B}$ . Therefore:

$$\begin{aligned} \sum_{p=1}^{P_k^n} \mathbb{P}(\mathcal{A}_k^n \mathcal{B}_p^{nk}) &= \mathbb{P}(\mathcal{A}_k^n) \sum_{p=1}^{P_k^n} \lambda_k^n(p) + \text{h.o.t.} \simeq \\ &\simeq \mathbb{P}(\mathcal{A}_k^n) \ell \frac{\gamma}{\gamma-1} \frac{n^{1-\xi}}{k} \sum_{p=1}^{P_k^n} \left( \frac{k+p}{k} \right)^{\frac{1-\xi\gamma}{\gamma-1}} = \\ &\simeq \mathbb{P}(\mathcal{A}_k^n) \ell \frac{\gamma}{\gamma-1} \frac{n^{1-\xi}}{k^{\frac{(1-\xi)\gamma}{\gamma-1}}} \sum_{p=1}^{P_k^n} (k+p)^{\frac{1-\xi\gamma}{\gamma-1}} = \\ &\simeq \mathbb{P}(\mathcal{A}_k^n) \ell \frac{1}{1-\xi} \frac{n^{1-\xi}}{k^{\frac{(1-\xi)\gamma}{\gamma-1}}} \left( \left( (k+P_k^n)^{\frac{(1-\xi)\gamma}{\gamma-1}} \right) - k^{\frac{(1-\xi)\gamma}{\gamma-1}} \right) = \\ &\simeq \mathbb{P}(\mathcal{A}_k^n) \ell \frac{1}{1-\xi} \frac{n^{1-\xi}}{k^{\frac{(1-\xi)\gamma}{\gamma-1}}} \left( \left( \frac{N^{1-1/\gamma}}{n^{1-1/\gamma}} \right)^{\frac{(1-\xi)\gamma}{\gamma-1}} - 1 \right) = \\ &\simeq \mathbb{P}(\mathcal{A}_k^n) \ell \frac{1}{1-\xi} n^{1-\xi} \left( \frac{N^{1-\xi}}{n^{1-\xi}} - 1 \right) \end{aligned}$$

Now we sum over  $n$  and  $k$ :

$$\begin{aligned}
& \sum_{n=\bar{n}}^N \sum_{k \in K_n} \mathbb{P}(\mathcal{A}_k^n) \ell \frac{1}{1-\xi} n^{1-\xi} \left( \frac{N^{1-\xi}}{n^{1-\xi}} - 1 \right) + \text{h.o.t.} = \\
& \simeq \sum_{n=\bar{n}}^N \ell \frac{1}{1-\xi} n^{1-\xi} \left( \frac{N^{1-\xi}}{n^{1-\xi}} - 1 \right) \frac{1}{\beta-\alpha} \sum_{k \in K_n} \delta_k^n = \\
& \simeq \sum_{n=\bar{n}}^N \ell^2 \frac{1}{1-\xi} n^{1-2\xi} \left( \frac{N^{1-\xi}}{n^{1-\xi}} - 1 \right) = \\
& \simeq \ell^2 \frac{1}{1-\xi} \sum_{n=\bar{n}}^N (N^{1-\xi} n^{-\xi} - n^{1-2\xi}) = \\
& \simeq \ell^2 \frac{1}{2(1-\xi)^2} N^{2-2\xi}
\end{aligned}$$

Recalling the factor 2 we had at the beginning of the estimate, the desired result follows.  $\square$

The last two lemmata prove that the weak independence condition is always satisfied regardless of the value of  $\xi$ . Therefore we have only to check the diverging condition. In the diverging regime we can conclude that the set  $\mathcal{A}$  has full measure, whereas in the non-diverging regime we can as well conclude that  $\mathcal{A}$  has zero measure.

Step five *Critical case*  $\xi = 1/\gamma$

For  $\xi = 1/\gamma$  we have that the overlapping condition is satisfied for small enough  $a$ , because  $\delta_k^n$  grows bigger as  $a$  decreases. As we notice from lemma 4.5.10, we can find a critical  $\bar{a}$  such that for  $a < \bar{a}$  we have overlapping and for  $a > \bar{a}$  we have no overlapping.  $\square$

In our case  $\xi = 1 - 1/\gamma$  so that the diverging condition is always satisfied, thus

statement (a) in theorem C is proved up to the non-degeneracy condition.

**Remark 4.5.11.** For this particular value of  $\xi$  the overlapping condition gives  $\gamma < 2$ ; the critical case is therefore  $\gamma = 2$ . In this case it is easy, given a large enough  $a \in \mathbb{N}$  to find  $C_1$  and  $C_2$  such that the arithmetic condition is satisfied for only finitely many  $n, m$ . In general, for  $\gamma > 2$  we still ignore if  $\mathbb{R}^+ \setminus \mathcal{A}$  is non-empty.

**Remark 4.5.12.** The technique we developed can be applied to  $(-)$ -orbits as well. The arithmetic condition relative to such orbits turns out to be more restrictive than the one for  $(+)$ -orbits; more precisely it yields  $\xi = 2 - 2/\gamma$ . This implies that the diverging condition is not anymore guaranteed. In fact it fails for  $\gamma > 2$ , which means that such orbits appear for arbitrarily high energies for almost all parameters  $A$  only for  $\gamma \leq 2$  and for all parameters for  $\gamma < 3/2$ .

Having studied all possible 2-periodic orbits, we notice how the conditions we stated are actually also necessary conditions for the presence of elliptic 2-periodic orbits. This implies the following interesting results:

- if  $\gamma = 2$  we can explicitly check that for  $A = 3$  the system has only finitely many  $(+)$ -elliptic islands of period 2 (no restrictions on the multiplier).
- if  $\gamma > 2$  we have infinitely many  $(-)$ -elliptic islands only for a null-measure set. Notice however that lemma 4.5.3 does not depend on  $\xi$ , therefore such set is non-empty.

Remark 4.5.12 also allows to prove (b) in theorem C, provided the result given in proposition 4.6.3, that states that the measure of each  $(+)$ -elliptic island is of order  $1/Y'^3$ . In fact:

**Proposition 4.5.13.** *If  $\gamma > 4/3$  the total Lebesgue measure of (+)-elliptic islands of period 2 is finite. If  $\gamma \leq 4/3$  the total Lebesgue measure of elliptic islands is infinite.*

*Proof.* We first obtain a rough upper bound to the total measure of elliptic island by summing the measure of a single island over all intersections where an island could appear regardless of the arithmetic condition. As there could be one for each  $Y \sim n + 1/2$  we have the following estimate:

$$\text{Leb}(\text{islands}) < \text{Const} \cdot \sum_{n=1}^{\infty} \frac{1}{Y_n^{\gamma}} = \text{Const} \cdot \sum_{n=1}^{\infty} n^{-3(1-1/\gamma)}.$$

The series converges for  $\gamma > 3/2$ . In order to obtain the sharp estimate we need to take into account that for some of the  $n$  we cannot have an elliptic island. For  $\gamma < 2$  this can be estimated quite easily, because the following expansion holds true:

$$(n^{\frac{1}{\gamma}} + A)^{\gamma} = n + \gamma A n^{1-1/\gamma} + o(1).$$

From the previous expression it is clear that:

$$\left\{ (n^{\frac{1}{\gamma}} + A)^{\gamma} \right\} = \left\{ \gamma A n^{1-\frac{1}{\gamma}} \right\} + o(1).$$

This function has an infinite number of branches, let us index them by  $k$ . Each branch will start at  $n_k \sim k^{\gamma/(\gamma-1)}$ . The arithmetic condition can be expressed in terms of  $k$  in the following way:

$$\left\{ n^{1-\frac{1}{\gamma}} \right\} < \mathcal{O}(k^{-1}).$$

Figure 4.5 illustrates this condition; for each branch  $k$  we have elliptic islands until the fractional part grows too large and the arithmetic condition no longer holds



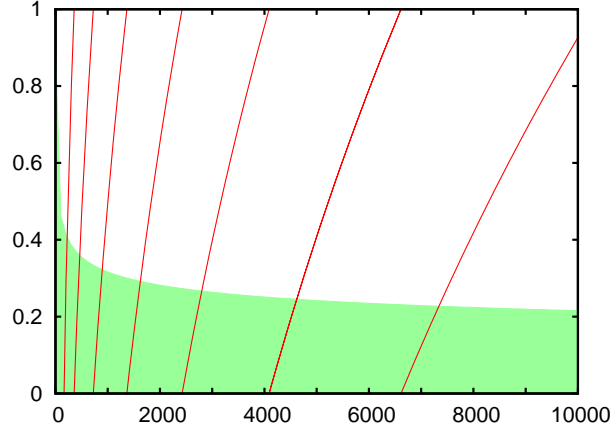


Figure 4.5: The arithmetic condition for  $\gamma < 2$ : we have an elliptic island for all values of  $n$  such that  $\left\{ \gamma A n^{1-\frac{1}{\gamma}} \right\}$  is in the shaded region.

true. Given this fact it is easy to estimate the number of islands belonging to the  $k$ -th branch. We compute the derivative of  $n^{1-\frac{1}{\gamma}}$  for  $n_{k+1}$ , obtaining a linear lower bound on the growth of  $n^{1-\frac{1}{\gamma}}$  in the  $k$ -th branch:

$$\left\{ n^{1-\frac{1}{\gamma}} \right\} > n_{k+1}^{-\frac{1}{\gamma}} \cdot (n - n_k) \sim (k+1)^{-\frac{1}{\gamma-1}} \cdot (n - n_k).$$

The smallest  $n$  for which the arithmetic condition fails can therefore be bound from above by requiring:

$$(k+1)^{-\frac{1}{\gamma-1}} \cdot (n - n_k) < (k+1)^{-1} \text{ i.e. } n - n_k < (k+1)^{\frac{2-\gamma}{\gamma-1}}.$$

therefore for the  $k$ -th branch we have at most  $\mathcal{O}(k^{\frac{2-\gamma}{\gamma-1}})$  elliptic islands. We now multiply this number by the measure of such islands and sum over all branches  $k$  to find the total measure:

$$\text{Leb}(\text{islands}) < C \cdot \sum_{k=1}^{\infty} k^{\frac{2-\gamma}{\gamma-1}} k^{-3},$$

which converges for  $\gamma > 4/3$ . Notice that along the same lines we can obtain as

well a lower bound of the same order, that means that the total measure of islands diverges for  $\gamma \leq 4/3$ . □

## 4.6 Non-degeneracy condition

According to general KAM theory, there exists a stability island around each point of a periodic orbit provided that generic non-resonance and non-degeneracy conditions are satisfied. Following [La93]:

**Definition 4.6.1.** An elliptic fixed point  $p$  of a two-dimensional symplectic diffeomorphism  $f$  is said to be *general elliptic* if:

- the multiplier  $\lambda_p$  is such that  $\lambda_p^k \neq 1$  for  $k = 1, 2, 3, 4$  (non-resonance up to order 4);
- the Birkhoff normal form is non-degenerate, i.e. a quantity that can be written in terms of derivatives up to fourth order is different from zero (see below).

**Theorem 4.6.2 (KAM).** *If  $p$  is general elliptic, then it is stable, i.e. for each neighbourhood  $U$  of  $p$  there exist another neighbourhood  $V$  such that  $\forall k, F^k(V) \subset U$ .*

Stability around the point implies the presence of an elliptic island. As noted before, the construction we described yields elliptic points with multiplier which can be chosen to belong to some prescribed interval; this implies that we can a priori avoid resonances. The non-degeneracy condition can be explicitly computed by the following procedure:

- We perform a linear change of coordinates such that the differential of the map  $dF$  at the fixed point is a rotation in the new coordinate  $(\xi, \eta)$  where  $(\xi = 0, \eta = 0) \mapsto p$ .
- We compute the Taylor expansion coefficients up to order four (excluded) in the coordinates  $u = \xi + i\eta$  and  $\bar{u}$  obtaining the following expression:

$$u \mapsto \lambda_p u + A_3 u^2 + A_4 u \bar{u} + A_5 \bar{u}^2 + A_6 u^3 + A_7 u^2 \bar{u} + A_8 u \bar{u}^2 + A_9 \bar{u}^3 + \mathcal{O}(4).$$

- We compute the following expression:

$$\omega = -i \left\{ i \Im(\bar{\lambda}_p A_7) + 3|A_3|^2 \frac{\lambda_p + 1}{\lambda_p - 1} + |A_5|^2 \frac{\lambda_p^3 + 1}{\lambda_p^3 - 1} \right\}.$$

The non-degeneracy condition requires that  $\omega \neq 0$ . The coefficients  $A_i$  contain derivatives of  $Y$  up to order 3 and derivatives of  $\phi$  up to order 4. As for high energies we have  $Y'_i \gg 1$ , instead of computing all  $A_i$  exactly, we perform an expansion in terms of powers of  $Y'_i$  and compute the highest order non-zero term, taking into account that ellipticity implies  $\ddot{\phi} Y'_i = \nu_i \sim 1$  (i.e. condition (4.2)). We find by direct computation<sup>1</sup> that the highest nonzero term in  $\omega$  is of order  $Y'_i{}^3$ . As a further simplification note that we have  $Y'_2 = Y'_1 + \mathcal{O}(Y'')$ ; this implies that if we compute  $\omega$  by setting  $Y'_1 = Y'_2 \doteq Y'$  and find a quantity bounded away from zero in this limit, it will be bounded away from zero also for all  $Y'_i$  sufficiently large. The coefficient

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<sup>1</sup>Computations were made using the software Mathematica. A printout can be found at <http://www.math.umd.edu/~jacopods/bnf.pdf>

of order 3 turns out to be the following polynomial in  $\nu = \ddot{\phi}Y' \in (-1, 0)$ :

$$\omega_3 = \frac{(2 + \nu)\ddot{\phi}^2}{64D^6/Y'^3} \left[ 2(\nu^2 + 4\nu + 6) + \left( -i\nu(2 + \nu)^2 \left( 3\frac{\lambda_p + 1}{\lambda_p - 1} + \frac{\lambda_p^3 + 1}{\lambda_p^3 - 1}(3 + \nu)^2 \right) \right) \right].$$

where

$$D = \sqrt{2(1 - \operatorname{Re}(\lambda_p \cdot \partial_z(F^2(p))))}.$$

is of order  $Y'^{-1/2}$  and  $\partial_z(F^2(p))$  is the holomorphic derivative of  $F^2$  with respect to  $z = x + iy$ . Notice that the fractions involving the multiplier  $\lambda_p = \exp(i\theta_p)$  give respectively  $i \cdot \cot(\theta_p/2)$  and  $i \cdot \cot(3\theta_p/2)$ . Of course  $\theta_p$  is not independent of  $\nu$ , but recall that since we can control the multiplier, we can assume both cotangent functions to be bounded away from zero and positive. It is easy to check that for a fixed  $\nu \in (-1, 0)$  this polynomial is bounded away from zero, as each term in the sum is positive. This is enough to establish the presence of an elliptic island around each periodic point found with the construction, proving Theorem B. Notice that the expression for  $\omega_3$  does not involve derivatives of  $\phi$  of order higher than 3 and derivatives of order 2 or higher of  $Y$  as such terms appear only in terms of lower order in the expansion in  $Y'$  (see below). We finally prove an estimate regarding the size of the elliptic islands we obtained. This estimate concludes the proof of Theorem B

**Proposition 4.6.3.** *Consider a 2-periodic orbit of type (+), given by the points  $(x_1, y_1)$  and  $(x_2, y_2)$  and such that the multiplier is bounded away from resonances of order up to four; we define  $Y' = (Y'(y_1) + Y'(y_2)) / 2$ . Elliptic islands of type (+) around such points have area of order  $Y'^{-3}$  for large enough  $y$ .*

*Proof.* We consider the map  $F^2$  expressed in terms of the variables  $u, \bar{u}$  defined above, close to a periodic point  $p$ ; for simplicity we assume  $p = 0$ . Recall that the variables  $u$  and  $\bar{u}$  are related to  $z = x + iy$  and  $\bar{z}$  by a linear symplectic transformation, i.e.  $z = b_1 u + b_2 \bar{u}$  and  $b_1 \bar{b}_1 - b_2 \bar{b}_2 = 1$ . In such variables one can write the map as follows:

$$u \mapsto A_1(u, \bar{u})u + A_2(u, \bar{u})\bar{u} \text{ where } A_1(0, 0) = \lambda_p, \quad A_2(0, 0) = 0.$$

One can obtain all terms of the Taylor polynomial of  $F^2$  in such variables by appropriately differentiating the functions  $A_1$  and  $A_2$  with respect to  $u$  and  $\bar{u}$ . We claim that the term of order  $n$  is of order at most  $Y'^{3+(n-1)/2}$ . By direct computation we find that

$$A_3, A_4, A_5 \sim Y'^{3/2} \quad A_6, A_7, A_8, A_9 \sim Y'^3.$$

This, along with the estimate we claim, is sufficient to prove that the area of the elliptic island is of the required order. In fact, one can perform a rescaling  $u \mapsto \Lambda u$ , obtaining the following (symbolic) expression:

$$u \mapsto \lambda_p u + \sum_{n=2}^{\infty} A^{(n)} \Lambda^{n-1} \{u, \bar{u}\}^n.$$

Therefore by choosing  $\Lambda$  such that  $A^{(n)} \Lambda^{n-1} \lesssim 1$  we obtain that the linearized part is dominant in a disk of radius of order  $\Lambda$  around the origin. The explicit computations and the claim allows us to take  $\Lambda \sim Y'^{-3/2}$ ; the result follows by recalling that the map  $z \mapsto u$  is symplectic and therefore it preserves the area form.

We are now left with the proof of the claim, i.e. to prove that  $A^{(n)} \lesssim Y'^{3+(n-1)/2}$ .

First we obtain by direct computation a relation between the coefficients  $a_i$  of the

Taylor expansion in terms of  $z, \bar{z}$  and the coefficients  $A_i$  of the expansion in terms of  $u, \bar{u}$

$$z \mapsto a_1(z, \bar{z})z + a_2(z, \bar{z})\bar{z} \quad z = b_1u + b_2\bar{u}.$$

$$A_1 = b_1\bar{b}_1a_1 + \bar{b}_1\bar{b}_2a_2 - b_1b_2\bar{a}_2 - b_2\bar{b}_2\bar{a}_1 \quad A_2 = \bar{b}_1b_2a_1 + \bar{b}_1\bar{b}_1a_2 - b_2b_2\bar{a}_2 - \bar{b}_1b_2\bar{a}_1.$$

We can obtain all coefficients  $A^{(n)}$  by applying the relative differential operator to the appropriate  $A_i$ ; the key fact to notice is that,  $b_i$  being constant, the differential operator will operate only on the  $a_i$ . One can express  $\partial_u$  and  $\partial_{\bar{u}}$  in terms of  $\partial_x$  and  $\partial_y$  in the following way:

$$\partial_u = \frac{b_1 + \bar{b}_2}{2}\partial_x + i\frac{\bar{b}_2 - b_1}{2}\partial_y \quad \partial_{\bar{u}} = \frac{\bar{b}_1 + b_2}{2}\partial_x - i\frac{b_2 - \bar{b}_1}{2}\partial_y.$$

We are going to explicitly compute the coefficients of  $\partial_x$  and  $\partial_y$  to check that they are respectively of order  $Y'^{1/2}$  and  $Y'^{-(1/2)}$ . Then we find a general expression for the order of arbitrary derivatives of  $a_1$  and  $a_2$ . Explicit calculations provide the following values for  $a_i$ :

$$a_1 = \left(1 + 2\ddot{\phi}(Y'_1 + Y'_2) + 2\ddot{\phi}^2Y'_1Y'_2\right) + i\left(2\ddot{\phi} + 2\ddot{\phi}^2Y'_1 - \frac{Y'_1 + Y'_2}{2} - \ddot{\phi}Y'_1Y'_2\right)$$

$$a_2 = \left(-2\ddot{\phi}Y'_2 - 2\ddot{\phi}^2Y'_1Y'_2\right) + i\left(2\ddot{\phi} + 2\ddot{\phi}^2Y'_1 + \frac{Y'_1 + Y'_2}{2} + \ddot{\phi}Y'_1Y'_2\right)$$

Where recall that we defined  $Y'_i = Y'(y_i)$ ; notice than the coefficient of  $\partial_x$  is of order  $Y'^{1/2}$  and the coefficient of  $\partial_y$  is of order  $Y'^{-(1/2)}$ . This reflects the fact that the symplectic transformation stretches along the  $y$  direction and contracts along the  $x$  direction in order to put the differential in normal form. This in turn implies that the shape of the invariant curves is elongated in the  $x$  direction (as we can notice in figure 4.6). We are left with computing the order of magnitude of derivatives of

$a_1$  and  $a_2$ . It is convenient to notice that  $d(F^2)$  is close to a square of a matrix, i.e. if we write  $Y'_1 = Y'(1 - \delta)$  and  $Y'_2 = Y'(1 + \delta)$  we obtain:

$$\alpha_1 \doteq 1 + \ddot{\phi}Y' + i(\ddot{\phi} - Y'/2) \quad \alpha_2 \doteq -\ddot{\phi}Y' + i(\ddot{\phi} + Y'/2).$$

$$a_1 = (\alpha_1\alpha_1 + \alpha_2\bar{\alpha}_2) - 2\ddot{\phi}^2Y'^2\delta^2 - 2i\ddot{\phi}^2Y'\delta + i\ddot{\phi}Y'^2\delta^2$$

$$a_2 = (\alpha_2\alpha_1 + \alpha_2\bar{\alpha}_1) - 2\ddot{\phi}Y'\delta + 2\ddot{\phi}Y'^2\delta^2 - 2i\ddot{\phi}^2Y'\delta - i\ddot{\phi}Y'^2\delta^2$$

As  $\delta$  is of order 1 (and limiting to 0 as  $Y' \rightarrow \infty$ ), the error term is of order at most  $Y'$ , whereas the main term is of order  $Y'^2$ . Now we differentiate  $\alpha_1$  and  $\alpha_2$  with respect to  $x$  and  $y$ :

$$\partial_x\alpha_1 = \ddot{\phi}(Y' + i) \quad \partial_x\alpha_2 = \ddot{\phi}(-Y' + i)$$

$$\partial_y\alpha_1 = Y'\partial_x\alpha_1 + Y''(\ddot{\phi} - i/2) \quad \partial_y\alpha_2 = Y'\partial_x\alpha_2 + Y''(-\ddot{\phi} + i/2).$$

In order to obtain an upper bound on such derivatives, we will consider  $\phi^{(n)} \sim 1$  regardless of the fact that even derivatives will be of order  $Y'^{-1}$ . To this extent, we observe that all terms containing second (and higher) derivatives of  $Y$  will appear in terms of lower order than the dominant  $Y'$  for  $\partial_x$  and  $Y'^2$  for  $\partial_y$ . By direct inspection, the same statement is true for terms containing  $\delta$  in the expression for  $a_i$  (in fact  $\delta \sim Y''/Y'$ ). Therefore if we restrict to the maximum order:

$$\partial_x^k\partial_y^l\alpha_1|_{max} = Y''\phi^{(k+l+2)}(Y' + i) \quad \partial_x^k\partial_y^l\alpha_2|_{max} = Y''\phi^{(k+l+2)}(-Y' + i).$$

Now recall that we were to compute derivatives with respect to the  $(u, \bar{u})$  variables and as such we should recall that the coefficients in front of  $\partial_x$  and  $\partial_y$  are of order

respectively  $Y'^{1/2}$  and  $Y'^{-1/2}$ . This means that we obtain:

$$\partial_u^k \partial_{\bar{u}}^l \alpha_1 \lesssim Y'^{(k+l)/2+1} \quad \partial_u^k \partial_{\bar{u}}^l \alpha_2 \lesssim Y'^{(k+l)/2+1},$$

which in turn implies:

$$\partial_u^k \partial_{\bar{u}}^l a_1 \lesssim Y'^{(k+l)/2+2} \quad \partial_u^k \partial_{\bar{u}}^l a_2 \lesssim Y'^{(k+l)/2+2}.$$

Therefore we obtain the required estimate for  $A^{(n)}$ , i.e.:

$$A^{(n)} \lesssim Y' \partial^{(n-1)} a_i \lesssim Y'^{3+(n-1)/2},$$

which concludes the proof. □

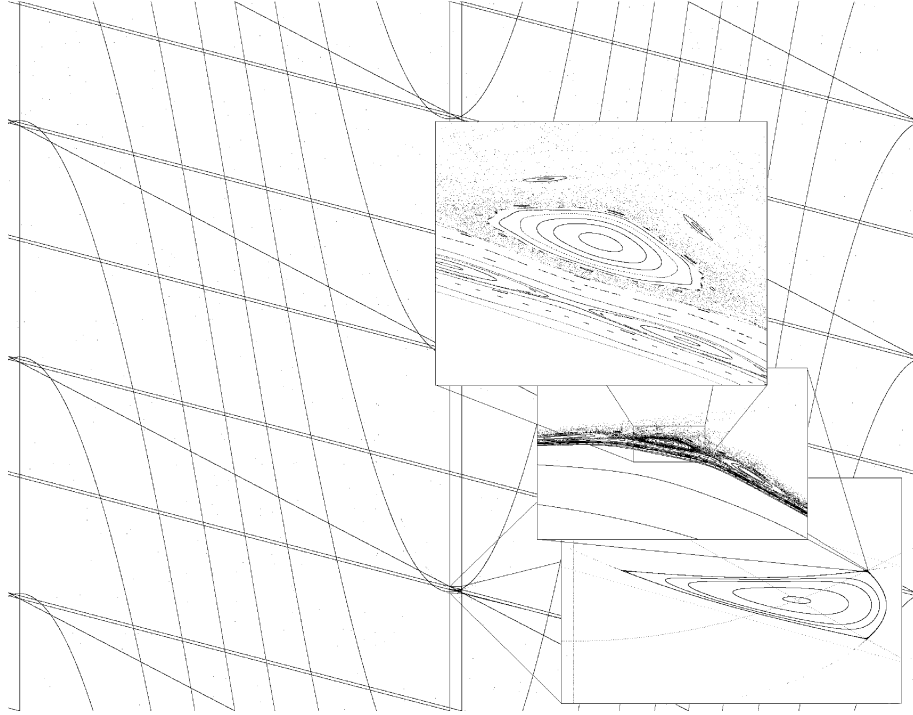


Figure 4.6: Elliptic island of period 2 and type (+); each smaller picture is an enlarged portion of the previous one. In the big picture we see the 2-periodic islands (bottom center) at two suitable intersection points of  $\ell$  and  $\ell^1$ .



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