ABSTRACT

Title of dissertation: METASTABILITY IN NEARLY-HAMILTONIAN SYSTEMS

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We characterize the phenomenon of metastability for a small random perturbation of a nearly-Hamiltonian dynamical system with one degree of freedom. We use the averaging principle and the theory of large deviations to prove that a metastable state is, in general, not a single state but rather a nondegenerate probability measure across the stable equilibrium points of the unperturbed Hamiltonian system. The set of all possible "metastable distributions" is a finite set that is independent of the stochastic perturbation. These results lead to a generalization of metastability for systems close to Hamiltonian ones.

METASTABILITY IN NEARLY-HAMILTONIAN SYSTEMS

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2009

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In memory of my mother

Parvathi Mani Athreya

November 7, 1949—May 2, 1975

Acknowledgments

First and foremost, I wish to thank my advisor, Dr. Mark Freidlin, for his guidance during every stage of my doctoral program. I am humbled by the depth and breadth of his knowledge and by his patience and generosity. He answered every one of my questions, gave me much-needed encouragement when I was stuck, and helped me fix error after error in my proofs. Indeed, he provided all the key ideas and machinery used in this thesis. It has been an extraordinary privilege to learn from him.

I also found his own life story inspiring: as a Refusenik in the former Soviet Union, Dr. Freidlin remained committed to mathematics in particular, and to intellectual and political freedom in general, in the face of crushing restrictions imposed on him and his family by Soviet authorities for more than a decade. The conditions under which he labored are scarce imaginable to someone like me, who grew up cocooned in the comparative luxury of the West. His struggles have heightened my own appreciation of the freedom and security that, far too often, I simply take for granted.

I am indebted to my professors at the University of Maryland for their expertise as teachers and scholars, and I wish to thank the members of my committee—Dr. Sandra Cerrai, Dr. Leonid Koralov, Dr. Eric Slud, and Dr. Alexander Roytburd—for their advice and suggestions. Dr. Jonathan Rosenberg and Dr. William Goldman provided generous departmental GAANN and VIGRE funding, for which I am very grateful. I would also like to thank Dr. Michael Jakobson for his lively course in ordinary differential equations; Dr. Matei Machedon for the care and attention he gave to every proof in partial differential equations; Dr. Stuart Antman and Dr. David Levermore for their enthusiasm during the RIT in Applied PDE; and Dr. Slud for teaching a rich and detailed statistics class. My fellow students Hyejin Kim, Kostas Spiliopoulous, Christopher Manon, David Shoup, Jeff Heath, Chris Flake, and Eleni Agathocleous listened to me when I practiced talks, gallantly proofread drafts, and buoyed my spirits.

My parents and grandparents gave me their unfailing love and support. My father first nurtured my interest in probability theory and was ever willing to discuss mathematics with me—in short, he made my academic ventures possible. My sister and brothers, my extended family, and my friends offered good advice and endless mirth. My husband Ryan and daughter Leela kept me happier than I have any right to be, and I could never have done this without them.

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Chapter 1

Introduction

1.1 Overview

Consider a Hamiltonian system with one degree of freedom

$$\dot{X}(t) = \overline{\nabla} H(X(t)), \quad X(0) = x_0 \in \mathbb{R}^2. \tag{1.1}$$

where the Hamiltonian $H(x) = H(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function with bounded second derivatives, and $\overline{\nabla} H$ represents the skew-gradient, that is,

$$\overline{\nabla} H(X(t)) = \left[\frac{\partial H}{\partial x_2}, -\frac{\partial H}{\partial x_1} \right].$$

An oscillator is a typical example of such a system:

$$\ddot{q}(t) + f(q(t)) = 0, \quad q(0) = q_0, \quad \dot{q}(0) = p(0) = p_0, \tag{1.2}$$

where $X(t) = (q(t), p(t)) \in \mathbb{R}^2$. The Hamiltonian of this system is

$$H(q, p) = \frac{p^2}{2} + F(q), \tag{1.3}$$

where $F(q) = \int_0^q f(u)du$ is the potential and $p = \dot{q}$. In addition to the assumptions of smoothness and bounded second derivatives, we impose the following restrictions on H: for $x = (x_1, x_2) \in \mathbb{R}^2$, we assume that $\lim_{|x| \to \infty} H(x) = \infty$; we assume H is a generic smooth function with a finite number of critical points, all of which are nondegenerate; and we assume that there exist constants K_1 and K_2 such that for

$x = (x_1, x_2)$ with $|x|$ sufficiently large, $K_1|x| < |\nabla H(x)| < K_2|x|$, so the gradient of H grows linearly for $|x|$ sufficiently large.

The Hamiltonian in Figure 1 has four minima, at O_1, O_3, O_5, and O_7, and three saddle points, at O_2, O_4, and O_6. The corresponding phase portrait for the system is also shown. Except for the separatrix trajectories, all trajectories are periodic closed curves, and each of them forms a connected component of a level set of H. If

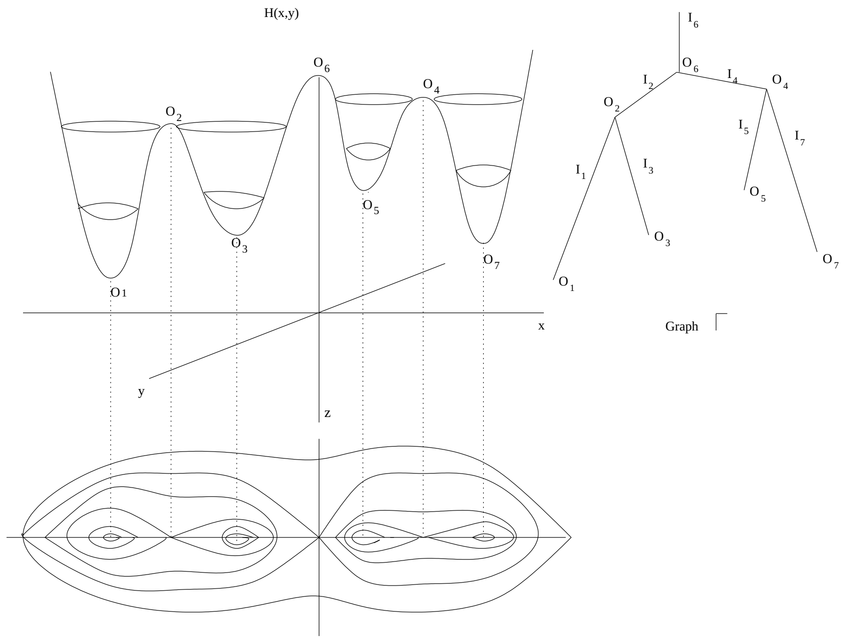

Figure 1.1: $H(x_1, x_2)$ and the Graph Γ

we identify all the points of each connected component of each level set, we get a set, Γ, homeomorphic to a graph (see Figure 1). The vertices of Γ correspond to critical points of H: exterior vertices to minima, and interior vertices to saddle points (see [16], [18], [19]). Each edge of Γ is indexed by a number, $I_1, I_2, \ldots I_m$, and each point y on Γ is indexed by the pair (z, i), where z is the value of the Hamiltonian on the level set corresponding to y, and i is the edge number containing y. The pair (z, i)

forms a global coordinate system on Γ.

Let $Q : \mathbb{R}^2 \to \Gamma; Q(x_1, x_2) = (H(x_1, x_2), i(x_1, x_2))$ be the projection onto Γ of a point (x_1, x_2) in $\mathbb{R}^2$. We denote the images in Γ of the critical points O_r under Q as simply O_r, and we write $I_k \sim O_r$ if O_r lies at the boundary of an edge I_k. We endow Γ with the natural topology, so a set U is open in Γ if and only if $Q^{-1}(U)$ is open in $\mathbb{R}^2$.

Now, consider a small deterministic perturbation of system (1.2):

$$\dot{X}^\epsilon(t) = \overline{\nabla} H(X^\epsilon(t)) + \epsilon B(X^\epsilon(t)),\ X(0) = x,\ 0 < \epsilon \ll 1. \tag{1.4}$$

We assume that B is a smooth vector-valued function on $\mathbb{R}^2$ with bounded derivatives, and that $\mathrm{div}(B(x)) < 0$ for all $x \in \mathbb{R}^2$. The assumption of negative divergence is analogous to the case of classical friction:

$$\ddot{q}^\epsilon(t) + f(q^\epsilon(t)) = -\epsilon \dot{q}(t) \tag{1.5}$$

For any finite time interval $[0, T]$, $X^\epsilon(t)$ converges uniformly to $X(t)$ as $\epsilon \to 0$. Significant deviations between the perturbed and unperturbed trajectories occur only on much longer time intervals, say of order ϵ^{-1}. To investigate the behavior of $X^\epsilon(t)$ on intervals of such order, it is convenient to rescale time: let $\tilde{X}^\epsilon(t) = X^\epsilon(t/\epsilon)$, so that equation (3) becomes

$$\dot{\tilde{X}}^\epsilon(t) = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}^\epsilon(t)) + B(\tilde{X}^\epsilon(t)),\ \tilde{X}^\epsilon(0) = x \tag{1.6}$$

Since H is a first integral for the unperturbed system (1.1), for ϵ small, the value of H changes slowly in time. As a result, the deterministically-perturbed and rescaled system (1.6) has two components: first, a "fast" component which is, roughly, motion along the unperturbed trajectories with speed of order ϵ^{-1} as $\epsilon \downarrow 0$; and a "slow" component which characterizes motion in the direction transverse to the unperturbed trajectories. The slow component has speed of order 1 and can be described by the evolution of the map $Q(\tilde{X}^\epsilon(t))$. As a result, the slow component corresponds to motion on the graph Γ. We can use the averaging principle in this situation to describe the long-time evolution of the slow motion $Q(\tilde{X}^\epsilon(t))$.

The behavior of the slow component is very sensitive to small changes in ϵ, and the slow component $Q(\tilde{X}^\epsilon(t))$ has no limit when $\epsilon \downarrow 0$ and t is sufficiently large. It is reasonable to consider small random perturbations of (1.6). Such perturbations exist naturally in any physical system.

In particular, we can add a white-noise-type perturbation to the system (1.6). For $\kappa > 0$, define $\tilde{X}^{\epsilon,\kappa}(t)$ as the diffusion process in $\mathbb{R}^2$ governed by the operator

$$\mathcal{L}^{\epsilon,\kappa}(u(x)) = \frac{\kappa}{2}\text{div}(a(x)\nabla u(x)) + B(x) \cdot \nabla u(x) + \frac{1}{\epsilon}\overline{\nabla} H(x) \cdot \nabla u(x) \tag{1.7}$$

The diffusion matrix $a(x)$, $x \in \mathbb{R}^2$, is a uniformly positive definite 2×2 matrix with bounded smooth coefficients. The arguments in [3], [15], [16] demonstrate that the slow component of (1.7) converges, first as ϵ converges to zero and then as κ converges to zero, to a *stochastic* process on the graph Γ. This limiting stochastic process is *independent* of the choice of random perturbation characterized by the diffusion matrix $a(x)$, provided that $a(x)$ is nondegenerate, and the stochasticity of the limiting process is concentrated at the interior vertices of Γ. In Theorems (2.1.4), (2.1.6), and (2.1.10) of Chapter 2, we prove certain results about averaging for the processes under study and we summarize the relevant background.

We show in (3.1.1) that under our assumptions on $B(x)$, for sufficiently small ϵ, the equilibrium points of the deterministically-perturbed system (1.6) are in one-to-one correspondence with the equilibrium points of the unperturbed Hamiltonian system (1.1). The equilibrium points corresponding to the minima of H (the centers of the Hamiltonian system) become asymptotically stable equilibrium points in the system (1.6); the saddle points remain saddle points. Moreover, all non-separatrix trajectories are attracted to one of the asymptotically stable equilibrium points.

For fixed ϵ sufficiently small, put $F^\epsilon = \frac{1}{\epsilon}\overline{\nabla} H + B$, and define $Z^\kappa(t)$ to be the solution to the stochastic differential equation

$$\dot{Z}^\kappa(t) = F^\epsilon(Z^\kappa(t)) + \sqrt{\kappa}\sigma(Z^\kappa(t))\dot{W}_t, \quad Z(0) = z \tag{1.8}$$

where $\kappa > 0$ is a small parameter. The process Z^κ is a white-noise perturbation of a deterministic dynamical system with finitely many asymptotically stable fixed points which are attractors for all non-separatrix trajectories. When such a system, with a small but fixed deterministic perturbation of size ϵ, is further perturbed by white noise, then for each initial position of the randomly-perturbed trajectory, there exists a particular stable equilibrium point near which the trajectory remains, with overwhelming probability, on a given timescale. Such an equilibrium position is called a *metastable state* corresponding to the given initial position z and timescale λ. Formally, let $\lambda > 0$ and $T = T(\kappa)$ be such that

$$\lim_{\kappa \to 0} \kappa \ln T(\kappa) = \lambda \tag{1.9}$$

An equilibrium point $K_{(z,\lambda)}$ is a metastable state for the initial condition z and timescale λ if for any $\delta > 0$ and $A > 0$,

$$\lim_{\kappa \downarrow 0} P_z\{\Lambda\{t \in [0, A] : \rho(Z^\kappa(tT(\kappa)), K_{(z,\lambda)}) > \delta\}\} \to 0. \tag{1.10}$$

where Λ denotes Lesbegue measure in $\mathbb{R}^2$.

Metastability is a consequence of large deviations for the process $\tilde{X}^{\epsilon,\kappa}(t)$. The process $\tilde{X}^{\epsilon,\kappa}(t)$ makes transitions from one neighborhood of an asymptotically stable equilibrium to another, and with probability close to one, each of these transitions takes an exponentially long time. The asymptotics as $\kappa \downarrow 0$ of the position of $\tilde{X}^{\epsilon,\kappa}(t)$ at times of order $T(\kappa)$ depend on how rapidly $T(\kappa)$ grows as κ becomes small. In the first section of Chapter 3, we provide a brief overview of the Freidlin-Wentzell theory of large deviations and metastability.

We then show that because of the sensitivity of the deterministically-perturbed system (1.6) to values of $\epsilon \ll 1$, certain *distributions* between the asymptotically stable equilibrium points should be considered as the "final states" of such a system. This leads to a modification of metastability for systems that are close to Hamiltonian ones. In particular, certain probability distributions across asymptotically stable equilibrium points are metastable.

1.2 Outline of results

We outline some of the specific results in this thesis. In Chapter 2, we prove results about the averaging principle for Hamiltonian systems in which the Hamiltonian has a single well (so the unperturbed system has no saddle points, only stable equilibrium points). In such single-well Hamiltonian systems, each non-extremal level set

$$C(z) = \{x \in \mathbb{R}^2 : H(x) = z\}$$

is a simple closed curve, and there exists a unique normalized invariant measure μ_z on $C(z)$ defined by

$$\mu_z(A) = \frac{1}{T(z)} \oint_A \frac{1}{\nabla H(x)} dl \tag{1.11}$$

where A is a measurable subset of $C(z)$, $T(z)$ is the period of the trajectory on level set z, and x is in $\mathbb{R}^2$.

1. Theorem (2.1.4). We show that if $\tilde{X}_t^\epsilon$ satisfies (1.6), namely

$$\dot{\tilde{X}}_t^\epsilon = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}_t^\epsilon) + B(\tilde{X}^\epsilon(t)), \quad \tilde{X}_0^\epsilon = x_0 \tag{1.12}$$

then on any finite time interval, $H(\tilde{X}_t^\epsilon)$ converges uniformly to $\overline{Y}_t$, where $\overline{Y}_t$ is the solution to

$$\dot{\overline{Y}}_t = \overline{B}(\overline{Y}_t) \tag{1.13}$$

and $\overline{B}(z)$ is defined as

$$\overline{B}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{B(x) \cdot \nabla H(x)}{|\nabla H(x)|} dl \tag{1.14}$$

2. Theorem (2.1.6) Suppose X_t^ϵ satisfies the stochastic differential equation

$$\dot{X}_t^\epsilon = \frac{1}{\epsilon}\overline{\nabla} H(X_t^\epsilon) + B(X_t^\epsilon) + \sigma(X_t^\epsilon)\dot{W}_t, \quad X_0^\epsilon = x_0 \tag{1.15}$$

for a smooth bounded matrix $\sigma(x)$ with $a(x) = \sigma(x)\sigma^T(x)$ uniformly positive definite, and initial condition x_0 which is not the minimum of H. Let Y_t be a process which satisfies the stochastic differential equation

$$\dot{Y}_t = \overline{B}(Y_t) + \overline{LH}(Y_t) + \sqrt{\overline{A}(Y_t)}\dot{W}_t, \quad Y_0 = H(x_0) \tag{1.16}$$

with $\overline{B}$ given in (2.37), and $\overline{LH}$ and $\overline{A}$ defined by

$$\overline{LH}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{\sum_{i,j} a_{ij}(x) \frac{\partial^2 H(x)}{\partial x_i \partial x_j}}{|\nabla H(x)|} dl \tag{1.17}$$

$$\overline{A}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{a(x)\nabla H(x) \cdot H(x)}{|\nabla H(x)|} dl \tag{1.18}$$

Then there exists a process Y^ϵ equivalent to Y such that for any fixed time interval $[0, T]$ and for any $\delta > 0$,

$$\lim_{\epsilon \to 0} P\{\sup_{0 \le t \le T} |H(X_t^\epsilon) - Y_t^\epsilon| > \delta\} = 0 \tag{1.19}$$

As a useful corollary, the process $H(X_t^\epsilon)$ converges weakly in C_{0T} to Y_t.

3. Theorem (2.1.10) We consider a single-well Hamiltonian system written in action-angle coordinates, with fast and slow motion separated, in which only the fast component has a stochastic perturbation. Let $(I^\epsilon, \phi^\epsilon)$ satisfy

$$\dot{I}^\epsilon = \beta_1(I^\epsilon, \phi^\epsilon) \tag{1.20}$$

$$\dot{\phi}^\epsilon = \frac{1}{\epsilon}\omega(I^\epsilon) + \frac{1}{\sqrt{\epsilon}}\sigma(I^\epsilon, \phi^\epsilon)\dot{W}_t + \beta_2(I^\epsilon, \phi^\epsilon) \tag{1.21}$$

and assume the coefficients ω, β_1, β_2, and σ are smooth and bounded and that $\sigma\sigma^T$ is uniformly positive definite. We explicitly solve for the invariant density $m_I(\phi)$ on each level set I and prove that on any finite time interval,

$$\lim_{\epsilon \to 0} P\{\sup_{0 \le t \le T} |I_t^\epsilon - \overline{I}_t| > \delta\} = 0 \tag{1.22}$$

where $\overline{I}_t$ is the solution to the equation

$$\dot{\overline{I}}_t = \overline{\beta}(\overline{I}_t), \quad \text{with} \quad \overline{\beta}_1(I) = \int_0^{2\pi} \beta_1(I, \phi) m_I(\phi) d\phi \tag{1.23}$$

More general results concerning stochastically-peturbed systems were proved by Khasminskii (see [22]), in which the fast and slow components are separated.

For Hamiltonians with multiple wells, the structure of the graph corresponding to the level sets is no longer a single interval, and as noted above, there are two first integrals for the unperturbed system: (H, i). In this case, in order to prove that the graph-valued slow component $Q(\tilde{X}^{\epsilon,\kappa}(t))$ of the perturbed process actually converges to a limiting process (see [16], [19]), Freidlin and Wentzell use martingale methods (see [27]) and they prove that the limiting process on the graph can be uniquely determined by generators on each edge and gluing conditions—which are restrictions on the domains of the generator—at interior vertices. We conclude Chapter 2 by summarizing these results and those of Freidlin and Brin (see [3]) on the convergence of the graph-valued slow component $Q(\tilde{X}^{\epsilon,\kappa}(t))$ to a limiting process. A key result, on which we rely heavily and which is proved in [3], is the following:

Theorem ([3]). Let $\tilde{X}^{\epsilon,\kappa}(t)$ be the two-dimensional diffusion processes defined by (1.7). The slow component $Q(\tilde{X}^{\epsilon,\kappa}(t))$ converges weakly in $C_{0T}(\Gamma)$, first as $\epsilon \downarrow 0$, to a stochastic process Q^κ, defined by generators L_i^κ along each edge of Γ and gluing conditions at the interior vertices. Next, as $\kappa \downarrow 0$, Q^κ converges weakly to a process $Q(t)$ which consists of deterministic motion along each edge of Γ and stochastic branching at the interior vertices, with probabilities of branching that depend only on B and not on the diffusion coefficients $a(x)$.

In Chapter 3, we review the Freidlin-Wentzell theory of large deviations and metastability and define metastability for deterministic systems subject to white-noise perturbations. We show that for a nearly-Hamiltonian system perturbed by noise, in which the Hamiltonian H and associated graph Γ have the structure of Figure (1.1), the notion of metastability must be generalized to include probability measures concentrated on stable equilibrium points.

1. In Lemma (3.1.1) We prove that for sufficiently small ϵ, the deterministic system in (1.6) consists of separatrix trajectories and trajectories converging to an asymptotically stable equilibrium. Hence the ω-limit sets for any initial condition have a simple structure.

2. In Example (3.1.4) We give a motivating example of a one-dimensional diffusion with potential drift in which metastability corresponds to a nondegenerate probability distribution over equilibrium points. Suppose X_t^ϵ satisfies

$$\dot{X}_t^\epsilon = -U'(X_t^\epsilon) + \sqrt{\epsilon}\dot{W}_t \tag{1.24}$$

where $U(a_1) = U(a_2)$ and $U(x) > U(a_i)$ for all $x \notin \{a_1, a_2\}$, $U'(x) \neq 0$ except at a_1 and a_2, and a_1 and a_2 are nondegenerate critical points with $U''(a_1) \neq U''(a_2)$. Suppose that for each ϵ,

$$\int_{\mathbb{R}} \exp\left[-\frac{2U(x)}{\epsilon}\right] dx = C^\epsilon < \infty \tag{1.25}$$

Then there exists an invariant measure μ^ϵ on $\mathbb{R}$ for this process which converges as $\epsilon \downarrow 0$ to a probability measure concentrated on a_1 and a_2, and for certain initial conditions and timescales, this limiting measure will be a metastable distribution.

3. In Section (3.2), we analyze large deviations for the process $Q^\kappa(t)$ on the graph Γ. Recall that Q^κ is the weak limit as $\epsilon \downarrow 0$ of the projection onto the graph

Γ of the two-dimensional process $\tilde{X}^{\epsilon,\kappa}(t)$. Q^{κ} is defined through second-order differential operators L_i^{κ} along each edge I_i of Γ, and these operators have degeneracies at interior and exterior vertices. To prove estimates for probabilities of large deviations, we analyze the behavior of $Q^{\kappa}(t)$ in small neighborhoods of exterior and interior vertices separately. We define the quasipotential $\bar{V}_{x,y} = \bar{V}(x,y)$ for any two points (x,y) along an edge I_i of Γ and show that for $x < y$, it can be computed as

$$\bar{V}_{x,y} = \int_x^y \frac{-2\tilde{B}_i(s)}{A_i(s)} ds \tag{1.26}$$

where $\tilde{B}$ is defined as

$$\tilde{B}_i(s) = \oint_{C_i(s)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl \tag{1.27}$$

and $C_i(s)$ is the connected component of the level set $H = s$ corresponding to edge i on Γ.

4. In Theorem (3.2.5), we prove the following: Let I_{k_i} be an exterior edge with exterior vertex O_{k_i} and interior vertex O_j in Γ. Suppose the three edges I_{k_1}, I_{k_2}, and I_j meet at interior vertex O_j. Put $\bar{V}_{ij}^{\max} = \max\{\bar{V}_{k_1 j}, \bar{V}_{k_2 j}\}$ where $\bar{V}_{k_i j} = \bar{V}(O_{k_i}, O_j)$. Let $\tau_z^{\kappa} = \inf\{t > 0 : Q^{\kappa}(t) = z\}$. For any $\alpha > 0$ there exists $\delta > 0$ sufficiently small such that if $y \in I_{k_i}$, $|y - H(O_{k_i})| < \delta$, $y \neq O_{k_i}$, and $z \in I_j$, $|z - H(O_j)| < \delta$, $z \neq O_j$, then

$$\lim_{\kappa \downarrow 0} P_y \left\{ \exp\left[\frac{\bar{V}_{ij}^{\max} - \alpha}{\kappa}\right] < \tau_z^{\kappa} < \exp\left[\frac{\bar{V}_{ij}^{\max} + \alpha}{\kappa}\right] \right\} = 1 \tag{1.28}$$

5. In Theorem (3.2.7) we show that there exist initial conditions z and timescales λ such that for any fixed $t > 0$, $\delta > 0$, and $\theta > 0$, there exists κ_0 sufficiently

small such that if F_{θ,O_i} is the neighborhood $F_{\theta,O_i} = \{y \in I_i : V(O_i, y) < \theta\}$, $i \in \{1, 3, 5, 7\}$, of exterior vertex O_i, and $\kappa < \kappa_0$, then

$$|P\{Q^\kappa(tT(\kappa)) \in F_{\theta,O_i}\} - \tilde{p}_i| < \delta \tag{1.29}$$

for probabilities $\tilde{p}_i \in (0, 1)$ which can be explicitly calculated and depend only on B.

6. In Theorem (3.2.1) we show that for any initial condition $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and all but finitely many timescales λ, the process $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$ converges weakly in the space $C_{0T}(\mathbb{R}^2)$, first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$, to a probability measure concentrated on the stable equilibrium points of the unperturbed Hamiltonian system. In particular, there exist certain initial conditions $w = (x_1(0), x_2(0))$ and time scales λ such that $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$ converges weakly to a nondegenerate probability distribution $\mu_{w,\lambda}$ concentrated on the stable equilibrium points $\{O_1, O_3, O_5, O_7\}$ of the unperturbed Hamiltonian system, with weights $\tilde{p}_i(w, \lambda) = \mu_{w,\lambda}(O_i), i \in \{1, 3, 5, 7\}$ that can be explicitly computed and depend only on B.

Chapter 2

The averaging principle in Hamiltonian systems

2.1 Auxiliary results and background on the averaging principle

2.1.1 Examples of the averaging principle in deterministic systems

In this chapter, we examine the limiting behavior of $Q(\tilde{X}^{\epsilon,\kappa}(t))$, where $\tilde{X}^{\epsilon,\kappa}(t)$ is the diffusion process in $\mathbb{R}^2$ governed by the operator $L^{\epsilon,\kappa}$ in (1.7), namely:

$$\mathcal{L}^{\epsilon,\kappa}(u(x)) = \frac{\kappa}{2}\text{div}(a(x)\nabla u(x)) + B(x)\cdot\nabla u(x) + \frac{1}{\epsilon}\overline{\nabla}H(x)\cdot\nabla u(x),$$

in which the matrix $a(x) = \sigma(x)\sigma^T(x)$ is smooth, bounded, and uniformly positive definite. The diffusion process $X_t^{\epsilon,\kappa}$ is the solution to the stochastic differential equation

$$\dot{X}^{\epsilon,\kappa}(t) = \frac{1}{\epsilon}\overline{\nabla}H(\tilde{X}^{\epsilon,\kappa}(t)) + B(\tilde{X}^{\epsilon,\kappa}(t)) + \frac{\kappa}{2}\begin{bmatrix} \frac{\partial a_{11}(\tilde{X}^{\epsilon,\kappa}(t))}{\partial x_1} + \frac{\partial a_{21}(\tilde{X}^{\epsilon,\kappa}(t))}{\partial x_2} \\ \frac{\partial a_{12}(\tilde{X}^{\epsilon,\kappa}(t))}{\partial x_1} + \frac{\partial a_{22}(\tilde{X}^{\epsilon,\kappa}(t))}{\partial x_2} \end{bmatrix} \tag{2.1}$$

$$+\sqrt{\kappa}\sigma(\tilde{X}^{\epsilon,\kappa}(t))\dot{W}_t, \quad \tilde{X}^{\epsilon,\kappa}(0) = (x_1(0), x_2(0)) \tag{2.2}$$

and $Q : \mathbb{R}^2 \to \Gamma$ is the projection of any point $x = (x_1, x_2)$ in the plane to the corresponding point $(H(x), i(x)) = (H(x_1, x_2), i(x_1, x_2))$ on the graph Γ associated to the Hamiltonian H. We assume that B is a smooth vector-valued function with bounded derivatives and negative divergence, and that $\nabla H(x)\cdot B(x) < 0$. Also, we impose the following restrictions on H: first, H is a smooth function with bounded second derivatives and a finite number of nondegenerate critical points; second, there exist K_1 and K_2 such that for $|x|$ sufficiently large, $K_1|x| < |\nabla H(x)| < K_2|x|$; and third, $\lim_{|(x)|\to\infty} |H(x)| = \infty$. We examine the limiting behavior of $Q^{\epsilon,\kappa} = Q(\tilde{X}^{\epsilon,\kappa}(t))$ first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$. When both ϵ and κ are small, the process $\tilde{X}^{\epsilon,\kappa}(t)$ represents a nearly-Hamiltonian dynamical system with a small random perturbation.

The central result of [3] is that one can associate to $\tilde{X}^{\epsilon,\kappa}(t)$ a stochastic process on the graph Γ which converges, first as ϵ and then as κ tend to zero, to a stochastic process $Q(t)$ on the graph. The limiting process $Q(t)$ is independent of the choice of diffusion coefficients $a(x)$. The convergence to a stochastic process is a consequence of the classical averaging principle and instability near saddle points of a Hamiltonian system in which H has multiple wells.

We summarize the relevant background on the averaging principle and diffusion processes on graphs from [3], [18], and we prove certain results on the averaging principle for our particular case. For a more complete treatment of the classical averaging principle in Hamiltonian systems, see [1, §6]. For full details on the averaging principle for multiwell Hamiltonian systems, see [18, §8], [3], and [16].

First we state a version of the averaging principle applicable to deterministic systems. Let ϵ be a small positive parameter, and suppose ψ_t is a continuous real-valued function. For $x \in \mathbb{R}^n$ and $y \in \mathbb{R}$, let $b(x,y) = (b^1(x,y), \cdots, b^n(x,y)) : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be a bounded, continuous vector-valued function satisfying a Lipschitz condition independent of y: $|b(x_1,y) - b(x_2,y)| \leq K|x_1 - x_2|$. Assume also that there exists a bounded continuous function $\overline{b}$ such that for any T the following limit exists uniformly in $t_0 \in \mathbb{R}$ and $x \in \mathbb{R}^n$:

$$\lim_{\epsilon \to 0} \left| \int_{t_0}^{t_0+T} [b(x, \psi_{\frac{s}{\epsilon}}) - \overline{b}(x)] ds \right| = 0 \tag{2.3}$$

Let X_t^ϵ satisfy the differential equation

$$\dot{X}_t^\epsilon = b(X_t^\epsilon, \psi_{t/\epsilon}); \quad X_0^\epsilon = x \tag{2.4}$$

and suppose $\overline{x}_t$ solves the ordinary differential equation

$$\dot{\overline{x}}_t = \overline{b}(x_t); \quad \overline{x}_0 = x \tag{2.5}$$

Theorem 2.1.1. *Let X_t^ϵ and $\overline{x}_t$ be defined as in* (2.4) *and* (2.5). *Then*

$$\lim_{\epsilon \downarrow 0} \left[\sup_{0 \le t \le T} |X_t^\epsilon - \overline{x}_t| \right] = 0 \tag{2.6}$$

Proof. It is clear that $\overline{b}$ is Lipschitz continuous with the same Lipschitz constant as b. Indeed, let x and y be given, and let $T > 0$ and $\delta > 0$ be arbitrary and positive. By (2.3), given any arbitrary $\delta > 0$, we can find ϵ sufficiently small so that

$$\left| \int_0^T [\overline{b}(x) - b(x, \psi_{\frac{s}{\epsilon}})] ds \right| < \frac{\delta}{2} \text{ and } \left| \int_0^T [b(y, \psi_{\frac{s}{\epsilon}}) - \overline{b}(y)] ds \right| < \frac{\delta}{2} \tag{2.7}$$

and by the Lipschitz continuity of b, there exists K such that for any x and y,

$$\left| \int_0^T [b(x, \psi_{\frac{s}{\epsilon}}) - b(y, \psi_{\frac{s}{\epsilon}})] ds \right| \le \int_0^T K|x - y| ds \tag{2.8}$$

So we conclude that

$$T|\bar{b}(x)-\bar{b}(y)| = \Bigg|\int_0^T [\bar{b}(x)-b(x,\psi_{\frac{s}{\epsilon}})]ds - \int_0^T [b(x,\psi_{\frac{s}{\epsilon}})-b(y,\psi_{\frac{s}{\epsilon}})]ds \tag{2.9}$$

$$+\int_0^T [b(y,\psi_{\frac{s}{\epsilon}})-\bar{b}(y)]ds\Bigg| \tag{2.10}$$

$$\leq \left|\int_0^T [\bar{b}(x)-b(x,\psi_{\frac{s}{\epsilon}})]ds\right| + \left|\int_0^T [b(x,\psi_{\frac{s}{\epsilon}})-b(y,\psi_{\frac{s}{\epsilon}})]ds\right| \tag{2.11}$$

$$+\left|\int_0^T [b(y,\psi_{\frac{s}{\epsilon}})-\bar{b}(y)]ds\right| \tag{2.12}$$

$$\leq \delta + TK|x-y| \tag{2.13}$$

Hence (2.5) has a unique solution. Also, if (2.3) holds, then

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T b(x,\psi_s)ds = \bar{b}(x) \tag{2.14}$$

and we regard $\bar{b}$ as the "long-run" average of $b(x,y)$ over the second component y for each fixed x. Taking the difference between X_t^ϵ and $\overline{x}_t$, we get

$$X_t^\epsilon - \overline{x}_t = \int_0^t \left[b(X_s^\epsilon,\psi_{\frac{s}{\epsilon}}) - b(\overline{x}_s,\psi_{\frac{s}{\epsilon}})\right] ds + \int_0^t \left[b(\overline{x}_s,\psi_{\frac{s}{\epsilon}}) - \bar{b}(\overline{x}_s)\right] ds \tag{2.15}$$

$$\Longrightarrow \sup_{0\leq t_1\leq t} |X_{t_1}^\epsilon - \overline{x}_{t_1}| \leq \int_0^t K \sup_{0\leq u\leq s} |X_u^\epsilon - \overline{x}_u| ds + \sup_{0\leq t_1\leq t}\left|\int_0^t b(\overline{x}_s,\psi_{\frac{s}{\epsilon}}) - \bar{b}(\overline{x}_s)ds\right| \tag{2.16}$$

Applying Gronwall's inequality (see [24, §2.5]), we get

$$\sup_{0\leq t\leq T} |X_t^\epsilon - \overline{x}_t| \leq \exp(KT)\left[\sup_{0\leq t\leq T}\left|\int_0^t b(\overline{x}_s,\psi_{\frac{s}{\epsilon}}) - \bar{b}(\overline{x}_s)ds\right|\right] \tag{2.17}$$

Observe that

$$\int_0^t \left[b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_s)\right] ds = \sum_{k=0}^{n-1} \int_{\frac{kt}{n}}^{\frac{(k+1)t}{n}} \left[b(\overline{x}_{\frac{kt}{n}}, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_{\frac{kt}{n}})\right] ds \tag{2.18}$$

$$+ \sum_{k=0}^{n-1} \int_{\frac{kt}{n}}^{\frac{(k+1)t}{n}} \left[b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - b(\overline{x}_{\frac{kt}{n}}, \psi_{\frac{s}{\epsilon}})\right] ds \tag{2.19}$$

$$+ \sum_{k=0}^{n-1} \int_{\frac{kt}{n}}^{\frac{(k+1)t}{n}} \left[\overline{b}(\overline{x}_{\frac{kt}{n}}) - \overline{b}(\overline{x}_s)\right] ds \tag{2.20}$$

Lipschitz continuity, boundedness of $\overline{b}$ and b, and the mean value theorem imply

$$\int_0^t \left[b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_s)\right] ds \leq \sum_{k=0}^{n-1} \int_{\frac{kt}{n}}^{\frac{(k+1)t}{n}} \left[b(\overline{x}_{\frac{kt}{n}}, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_{\frac{kt}{n}})\right] ds + \rho_{n,t}^{\epsilon} \tag{2.21}$$

where $|\rho_{n,t}^{\epsilon}| < \frac{C}{n}$ and C is a constant depending on T and on the Lipschitz constant for b and $\overline{b}$. Hence $|\rho_{n,t}^{\epsilon}|$ can be made arbitrarily small for all ϵ by choosing n large.

Note that

$$\left|\sum_{k=0}^{n-1} \int_{\frac{kt}{n}}^{\frac{(k+1)t}{n}} \left[b(\overline{x}_{\frac{kt}{n}}, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_{\frac{kt}{n}})\right] ds\right| \tag{2.22}$$

converges to zero for any fixed n sufficiently large as $\epsilon \downarrow 0$ by (2.3). Therefore

$$\lim_{\epsilon \downarrow 0} \left\{ \sup_{0 \leq k \leq n} \left| \int_0^{Kt/n} [b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_s)] ds \right| \right\} = 0 \tag{2.23}$$

Since $G(t) = \int_0^t [b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_s)] ds$ is continuous, its supremum on the interval $[0, T]$ is attained at some point t^*, and we can choose n sufficiently large and ϵ small to guarantee that for any $\eta > 0$

$$\sup_{0 \leq t \leq T} G(t) = \int_0^{\frac{kT}{n}} [b(\overline{x}_s, \psi_{\frac{s}{\epsilon}}) - \overline{b}(\overline{x}_s)] ds + \eta \tag{2.24}$$

for some $k \in \{0, 1, \cdots, T/n\}$. □

In the above analysis, we consider $\psi_{\frac{t}{\epsilon}}$ the "fast" motion and X_t^{ϵ} the "slow" motion for small ϵ. The averaging principle implies that the slow component converges uniformly on finite time intervals to the solution of a differential equation involving the average of a function with respect to the fast motion.

Now consider the case of a Hamiltonian system with a *single*-well potential, $F(x_1)$, with $H(x) = H(x_1, x_2) = F(x_1) + \frac{x_2^2}{2}$, as shown in the figure, and let X_t satisfy $\dot{X}_t = \overline{\nabla} H(X_t)$. Without loss of generality we assume that the unique minimum O is such that $H(O) = 0$.

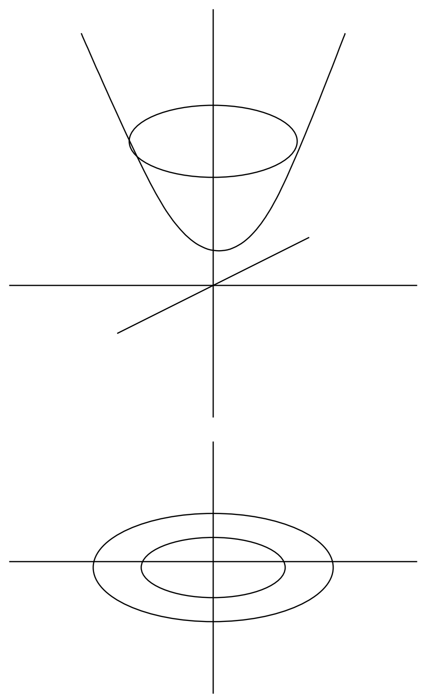

Figure 2.1: A single-well Hamiltonian and phase portrait

The phase portrait consists of periodic trajectories along level sets of H. These level sets are simple closed curves. Let $C(z) = \{x \in \mathbb{R}^2 : H(x) = z\}$ denote the closed curve for level set $H = z$. Each trajectory X_t on $C(z)$ has a finite period $T(z)$. Since $\text{div}(\overline{\nabla} H) = 0$, the flow is area-preserving; that is, if Λ is Lebesgue measure in $\mathbb{R}^2$ and Φ_t is the flow, then for any $t > 0$ and any measurable A,

$$\Lambda(A) = \Lambda[\Phi_t^{-1}(A)] \tag{2.25}$$

As a consequence, there exists a unique invariant measure μ concentrated on each level set $C(z)$. For any measurable subset A of $C(z)$, $\mu_z(A)$ is given by

$$\mu_z(A) = \frac{1}{T(z)} \oint_A \frac{dl}{|\nabla H(x)|} \tag{2.26}$$

where dl is the length element and $T(z)$ is the period of the trajectory concentrated on $C(z)$. The measure μ_z of a set A is the ratio of the occupation time of X_t in A during a single rotation to the period of the trajectory on the curve $C(z)$. The periodicity of each trajectory X_t on the level set $C(z)$ guarantees the equality of time- and space-averages on level sets.

Theorem 2.1.2 (Equality of time- and space-averages on level sets). *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous and let X_t be a solution to $\dot{X}_t = \overline{\nabla} H(X_t)$ with initial condition $X_0 = x_0$; let $H(x_0) = z$. Then*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{T(z)} \oint_C (z) \frac{f(x)}{|\nabla H(x)|} dl \tag{2.27}$$

Proof. Let $T(z) = T$ be the period of X_t on $C(z)$. For any t large, we can find $n \in \mathbb{N}$ such that $t = nT + \gamma$ where $0 \leq \gamma < T$. Since $f(X_s)$ is periodic, we have

$$\frac{1}{t} \int_0^t f(X_s)ds = \frac{1}{nT+\gamma} \left\{ \int_0^{nT} f(X_s)ds + \int_{nT}^{nT+\gamma} f(X_s)ds \right\} \tag{2.28}$$

$$= \left[\frac{nT}{nT+\gamma} \right] \left\{ \frac{1}{T} \int_0^T f(X_s)ds + \frac{1}{nT} \int_0^\gamma f(X_s)ds \right\} \tag{2.29}$$

$$\to \frac{1}{T} \int_0^T f(X_s)ds \quad \text{as} \quad n \to \infty \tag{2.30}$$

Over a single period T, we parameterize $C(z)$ by the curve X_t and immediately derive that

$$\int_0^T f(X_s)ds \ = \oint_{C(z)} \frac{f(x)}{|\overline{\nabla} H(x)|} dl \tag{2.31}$$

□

As in (1.4), consider a small deterministic perturbation of the system $\dot{X}_t = \overline{\nabla} H(X_t)$:

$$\dot{X}_t^\epsilon = \overline{\nabla} H(X_t^\epsilon) + \epsilon B(X_t^\epsilon), \quad X_0^\epsilon = (x_1(0), x_2(0)).$$

We continue to assume that B is smooth, has bounded derivatives and negative divergence, and that $\nabla H(x) \cdot B(x) < 0$. As above, let X_t denote the solution to the unperturbed Hamiltonian system. Fix an initial point $w = (x_1(0), x_2(0))$; the trajectory through $(x_1(0), x_2(0))$, denoted $X_{t,w}$, forms a closed curve in whose interior lies the single minimum of H—that is, exactly one stable center. Since the Hamiltonian is a first integral, the motion of $X_{t,w}$ can be expressed through action-angle coordinates, i.e. the value of a first integral I, the *action*, and an angular coordinate $\phi \in [0, 2\pi]$, the *angle* (see [1]):

$$\dot{I} = 0 \tag{2.32}$$

$$\dot{\phi} = \omega(I) \tag{2.33}$$

with initial conditions $\phi(0) = \theta_0$ and $I(0) = I_0$.

We rescale by time, so let $\tilde{X}_t^\epsilon = X_{t/\epsilon}^\epsilon$. In these local coordinates, the rescaled perturbed system (1.6) can be written:

$$\dot{I}^\epsilon = \beta_1(I^\epsilon, \phi^\epsilon) \tag{2.34}$$

$$\dot{\phi}^\epsilon = \frac{1}{\epsilon}\omega(I^\epsilon) + \beta_2(I^\epsilon, \phi^\epsilon) \tag{2.35}$$

with the same initial conditions $\phi^\epsilon(0) = \theta_0$ and $I^\epsilon(0) = I_0$). In this instance we can separate the fast and slow motion. For small ϵ, near the given energy level $z = H(x_1(0), x_2(0))$, the fast motion for $\tilde{X}_t^\epsilon$ is characterized by the invariant measure concentrated on the closed curve $C(z)$ that forms the level set $H^{-1}(z)$. The slow motion satisfies the equation

$$H(\tilde{X}_t^\epsilon) - H(\tilde{X}_0^\epsilon) = \int_0^t \nabla H(\tilde{X}_s^\epsilon) \cdot B(\tilde{X}_s^\epsilon) ds \tag{2.36}$$

On a time interval $[t, t+\Delta]$, where Δ is independent of ϵ, $H(\tilde{X}_t^\epsilon)$ changes by an amount of order Δ, uniformly in ϵ. The number of "revolutions" made by the fast component along $C(z)$ is of order $\Delta\epsilon^{-1}$. This is precisely the type of situation in which the averaging principle applies.

Because $T(z)$ is the period of the trajectory of the unperturbed system $\dot{X}_t = \overline{\nabla} H(X_t)$ on $C(z)$, we have $T(z) = \oint_{C(z)} \frac{dl}{|\nabla H(x)|}$. Let $\overline{B}(z)$ be defined as

$$\overline{B}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl \tag{2.37}$$

We state the following useful lemma from [18]:

Lemma 2.1.3. *Let $f(x)$ be a function that is continuously differentiable in the interval $\{x \in \mathbb{R}^2 : 0 < z_1 \leq H(x) \leq z_2\}$. Then for any $z \in (z_1, z_2)$,*

$$\frac{d}{dz}\left[\oint_{C(z)} f(x)|\nabla H(x)| dl\right] = \oint_{C(z)} \left[\frac{\nabla f(x) \cdot \nabla H(x)}{|\nabla H(x)|} + f(x)\frac{\Delta H(x)}{|\nabla H(x)|}\right] dl \tag{2.38}$$

Proof. See [18], §8. □

Applying this lemma to the functions $f_1(x) = \frac{B(x)\cdot\nabla H(x)}{|\nabla H(x)|^2}$ and $f_2(x) = \frac{1}{|\nabla H(x)|^2}$, we get that $\overline{B}(z)$ is $(k-1)$-times continuously differentiable in the interval $z : z > 0$ if H is k-times continuously differentiable. Hence $\overline{B}(z)$ is Lipschitz continuous on compact sets.

Let $\tilde{B}(z)$ denote

$$\tilde{B}(z) = \oint_{C(z)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl \tag{2.39}$$

By the divergence theorem, we can write

$$\tilde{B}(z) = \int_{G(z)} \text{div}(B(x))dx \tag{2.40}$$

where $G(z)$ is the closed, simply connected region bounded by $C(z)$. Then $\overline{B}(z) = \tilde{B}(z)/T(z)$. In (2.2.1) we show that $\lim\limits_{z\to H(O)} T(z) = C > 0$, and therefore $\overline{B}(H(O)) = 0$.

Theorem 2.1.4 (An averaging principle for a deterministic perturbation of a single-well Hamiltonian system)**.** *Let $\tilde{X}_t^\epsilon$ satisfy*

$$\dot{\tilde{X}}_t^\epsilon = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}_t^\epsilon) + B(\tilde{X}_t^\epsilon), \quad \tilde{X}_0^\epsilon = (x_1(0), x_2(0)) = w \tag{2.41}$$

Then for any finite time interval $[0, T]$, the slow component $Y_t^\epsilon = H(\tilde{X}_t^\epsilon)$ converges uniformly as $\epsilon \downarrow 0$ to the solution $\overline{Y}_t$ of the averaged system

$$\dot{\overline{Y}}_t = \overline{B}(\overline{Y}_t), \quad \overline{Y}_0 = H(w). \tag{2.42}$$

where

$$\overline{B}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl \tag{2.43}$$

Proof. We will show that for each fixed T, there exists a constant M_T such that

$$\max_{0\le t\le T} |H(\tilde{X}_t^\epsilon) - \overline{Y}_t| \le M_T\epsilon. \tag{2.44}$$

We apply the Newton-Leibniz formula to $H(X_t^\epsilon)$:

$$H(\tilde{X}_t^\epsilon) - H(w) = \int_0^t \nabla H(\tilde{X}_s^\epsilon)\cdot \dot{\tilde{X}}_s^\epsilon ds \tag{2.45}$$

$$= \int_0^t \nabla H(\tilde{X}_s^\epsilon)\cdot \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}_s^\epsilon)ds + \int_0^t \nabla H(\tilde{X}_s^\epsilon)B(\tilde{X}_s^\epsilon)ds \tag{2.46}$$

$$=0 + \int_0^t \nabla H(\tilde{X}_s^\epsilon)B(\tilde{X}_s^\epsilon)ds \tag{2.47}$$

Since $\nabla H(x)\cdot B(x) < 0$, $H(\tilde{X}_t^\epsilon) \le H(w)$ for all $0 \le t \le T$ and for all $0 \le \epsilon \le \epsilon_0$. From the assumption that $\lim_{|x|\to\infty} H(|x|) = \infty$, this implies that there exists a compact set $N \in \mathbb{R}^2$ such that $|\tilde{X}_t^\epsilon| \in N$ for all $0 \le t \le T$ and for all $0 < \epsilon < \epsilon_0$.

We next establish a claim that is essential to this and subsequent proofs.

Claim 2.1.5. *Given a smooth function $g(x) = g(x_1, x_2) : \mathbb{R}^2 \to \mathbb{R}$, the first-order partial differential equation*

$$\nabla u(x)\cdot \overline{\nabla} H(x) = g(x) \tag{2.48}$$

has a solution u if and only if on each level set $H = z$, the integral of g with respect to the invariant density $m_z(x) = \frac{1}{T(z)}\frac{1}{\overline{\nabla}H(x)}$ vanishes:

$$\frac{1}{T(z)}\int_{C(z)} \frac{g(x)}{|\nabla H(x)|}dl = 0 \tag{2.49}$$

Furthermore, if this condition is satisfied, the solution u is twice-continuously differentiable.

Proof. Suppose first that a solution u to (2.48) exists. Let $X_t \in \mathbb{R}^2$ be the solution to $\dot{X}_t = \overline{\nabla} H(X_t)$ with initial condition $X_0 = w_0$, and let T denote the period of X_t on the level set $H = H(w_0)$. We get

$$0 = u(X_T) - u(w_0) = \int_0^T \nabla u(X_s) \cdot \overline{\nabla} H(X_s) ds = \int_0^T g(X_s) ds \tag{2.50}$$

Since (2.1.2) implies

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t g(X_s) ds = \frac{1}{T(z)} \int_{C(z)} \frac{g(x)}{|\nabla H(x)|} dl, \tag{2.51}$$

(2.50) forces necessity. For sufficiency, consider the ordinary differential equation

$$\dot{F}_t = \nabla H(F_t); F_0 = a = (a_1, a_2) \in \mathbb{R}^2 \tag{2.52}$$

where a is not the unique minimum of H. The trajectory F_t intersects each level set of H precisely once, at some point $f(z)$, and $\lim_{t\to\infty} F_t = \infty$, $\lim_{t\to-\infty} F_t = O$. For any point $x = (x_1, x_2)$ on $f(z)$, define

$$u(x) = \int_{f(z)}^{x} \frac{g(y)}{|\nabla H(y)|} dl \tag{2.53}$$

where the line integral is taken along $C(z)$ in the direction of the vector field $\overline{\nabla} H(x)$. Since the integral of g with respect to the invariant density vanishes, and since g is smooth, u is twice-continuously differentiable and solves (2.48) with initial condition $u(f(z)) = 0$. □

Now put $g(x) = \nabla H(x) \cdot B(x) - \overline{B}(H(x))$. By construction the integral of g with respect to the invariant density on each level set vanishes, and therefore there exists a solution u to the partial differential equation

$$\nabla u(x) \overline{\nabla} H(x) = g(x) \tag{2.54}$$

Evaluating the function $u(x)$ along the perturbed trajectories $\tilde{X}_t^\epsilon$, we get

$$u(\tilde{X}_t^\epsilon) - u(w) = \frac{1}{\epsilon}\int_0^t \nabla u(\tilde{X}_t^\epsilon)\cdot \overline{\nabla} H(\tilde{X}_s^\epsilon)ds + \int_0^t \nabla u(\tilde{X}_s^\epsilon)\cdot B(\tilde{X}_s^\epsilon)ds$$

Define the functions R and A as follows:

$$\begin{aligned}
R(\tilde{X}_t^\epsilon) &= \int_0^t \nabla u(\tilde{X}_s^\epsilon)\cdot \overline{\nabla} H(\tilde{X}_s^\epsilon)ds = \int_0^t g(\tilde{X}_s^\epsilon)ds \\
A(\tilde{X}_t^\epsilon) &= \int_0^t \nabla u(\tilde{X}_s^\epsilon)\cdot B(\tilde{X}_s^\epsilon)ds
\end{aligned}$$

We deduce that

$$\begin{aligned}
R(\tilde{X}_t^\epsilon) =& \epsilon u(\tilde{X}_t^\epsilon) - \epsilon u(w) - \epsilon A(\tilde{X}_t^\epsilon) \\
\Rightarrow \int_0^t g(\tilde{X}_s^\epsilon)ds =& \epsilon[u(\tilde{X}_t^\epsilon) - u(w) - A(\tilde{X}_t^\epsilon)] \\
\Rightarrow \left| \int_0^t g(\tilde{X}_s^\epsilon)ds \right| \leq & K_t\epsilon \leq K_T\epsilon
\end{aligned}$$

The final implication holds because u and A are continuous and X_t^ϵ lies in the compact set N for all $0 \leq t \leq T$ and all $\epsilon < \epsilon_0$. Note that

$$\begin{aligned}
H(\tilde{X}_t^\epsilon) - H(w) &= \int_0^t \nabla H(\tilde{X}_s^\epsilon)\cdot B(\tilde{X}_s^\epsilon)ds \\
&= \int_0^t \overline{B}(H(\tilde{X}_s^\epsilon))ds + \int_0^t \left(\nabla H(\tilde{X}_s^\epsilon)\cdot B(\tilde{X}_s^\epsilon) - \overline{B}(H(\tilde{X}_s^\epsilon))\right) ds
\end{aligned}$$

The second integral is bounded:

$$\left| \int_0^t \left(\nabla H(\tilde{X}_s^\epsilon)\cdot B(\tilde{X}_s^\epsilon) - \overline{B}(H(\tilde{X}_s^\epsilon))\right) ds \right| = \left| \int_0^t g(\tilde{X}_s^\epsilon)ds \right| \leq K_t\epsilon$$

Let $Y_t^\epsilon = H(\tilde{X}_t^\epsilon)$. Because of the previous bound,

$$Y_t^\epsilon - Y_0^\epsilon = \int_0^t \overline{B}(Y_s^\epsilon)ds + \rho_\epsilon(t)$$

where $|\rho_\epsilon(t)| \leq K_t\epsilon$. From the Lipschitz continuity of $\overline{B}$, we conclude that

$$\begin{aligned} Y_t^\epsilon - \overline{Y}_t =& \int_0^t \left(\overline{B}(Y_s^\epsilon) - \overline{B}(\overline{Y}_s)\right) ds + \rho_\epsilon(t) \\ \Rightarrow |Y_t^\epsilon - \overline{Y}_t| \leq& C\int_0^t |Y_s^\epsilon - \overline{Y}_s| ds + K_t\epsilon \end{aligned}$$

So by Gronwall's inequality, we obtain

$$\max_{0\leq t\leq T} |Y_t^\epsilon - \overline{Y}_t| \leq \exp[(C_f T)] K_T \epsilon,$$

as required. □

2.1.2 Examples of the averaging principle in single-well Hamiltonian systems with stochastic perturbations

We next consider a stochastic perturbation. Suppose X_t^ϵ is the solution to the following stochastic differential equation:

$$\dot{X}_t^\epsilon = \frac{1}{\epsilon}\overline{\nabla} H(X_t^\epsilon) + B(X_t^\epsilon) + \sigma(X_t^\epsilon)\dot{W}_t, \quad X_0^\epsilon = x_0 \tag{2.55}$$

where B and H are the same functions from (2.1.4); $\dot{W}_t$ represents white noise; and $a(x) = \sigma(x)\sigma^T(x)$ is a smooth, bounded, positive definite 2×2 matrix. Define the partial differential operator L as follows: for any function u,

$$Lu = \frac{1}{2}\sum_{i,j} a_{ij}(x)\frac{\partial^2 u(x)}{\partial x_i \partial x_j}, \tag{2.56}$$

and for any level set $H = z$, $z \neq H(O)$, let $\overline{Lu}(z)$ denote

$$\overline{Lu}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{Lu(x)}{|\nabla H(x)|} dl \tag{2.57}$$

Define $\overline{A}(z)$ and $\overline{B}(z)$ as

$$\overline{A}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{a(x)\nabla H(x) \cdot H(x)}{|\nabla H(x)|} dl \tag{2.58}$$

$$\overline{B}(z) = \frac{1}{T(z)} \oint_{C(z)} \frac{B(x) \cdot \nabla H(x)}{|\nabla H(x)|} dl \tag{2.59}$$

Again, from Lemma (2.1.3) $\overline{A}(z)$, $\overline{B}(z)$ and $\overline{LH}(z)$ are $(k-1)$-times continuously differentiable in the interval $0 < z$ if H is k-times continuously differentiable.

Let Y_t be the one-dimensional process satisfying

$$\dot{Y}_t = \overline{B}(Y_t) + \overline{LH}(Y_t) + \sqrt{\overline{A}(Y_t)}\dot{W}_t, \quad Y_0 = H(x_0) \tag{2.60}$$

Theorem 2.1.6 (An averaging principle for a stochastic perturbation of a single-well Hamiltonian system)**.** *Let* X_t^ϵ *be as given in* (2.55)*. For any fixed time interval* $[0, T]$ *and for any* $\delta > 0$*, there exists a process* Y_t^ϵ *identical in distribution to* Y_t *such that*

$$\lim_{\epsilon \downarrow 0} P\left\{ \sup_{0 \le t \le T} |H(X_t^\epsilon) - Y_t^\epsilon| > \delta \right\} = 0 \tag{2.61}$$

Proof. First, we show that for any $T > 0$ and $\eta > 0$, there exists a compact set N_η and $\epsilon_\eta > 0$ such that for any $0 < \epsilon < \epsilon_\eta$,

$$P\left\{ \sup_{0 \le t \le T} |X_t^\epsilon| \in N_\eta \right\} > 1 - \eta \tag{2.62}$$

Since we assume $\lim_{|x| \to \infty} H(x) = \infty$ and that H has a single nonnegative minimum, to prove (2.62) it suffices to prove that for every $\delta > 0$ and $T > 0$, there exists H_0 and ϵ_0 such that

$$P\left\{ \max_{0 \le t \le T} H(X_t^\epsilon) > H_0 \right\} < \delta \tag{2.63}$$

for all $\epsilon < \epsilon_0$.

To establish (2.63), we apply the Ito formula to H:

$$H(X_t^\epsilon) - H(x) = \int_0^t \nabla H(X_s^\epsilon) \cdot B(X_s^\epsilon) ds + \frac{1}{2} \int_0^t \sum_{ij} a_{ij}(X_s^\epsilon) \frac{\partial^2 H(X_s^\epsilon)}{\partial x_1 \partial x_2} ds \tag{2.64}$$

$$+ \int_0^t \nabla H(X_s^\epsilon) \sigma(X_s^\epsilon) dW_s \tag{2.65}$$

By the boundedness assumptions on both $a(x)$ and the second derivatives of H, we can find a constant A_1 such that

$$|\frac{1}{2} \int_0^t \sum_{ij} a_{ij}(X_s^\epsilon) \frac{\partial^2 H(X_s^\epsilon)}{\partial x_1 \partial x_2} ds| \leq A_1 t \tag{2.66}$$

Since $\nabla H(x) \cdot B(x) < 0$ and the expectation of the stochastic integral term is zero,

$$E[H(X_t^\epsilon)] < H(x_0) + A_1 t \tag{2.67}$$

By our assumptions on $\nabla H(x)$, there exists a constant C_1 such that for $|x|$ sufficiently large, $H(x) > C_1 |x|^2$. As a result, (2.67) implies that there exist constants C_2, C_3 and ϵ_0 such that for all $\epsilon < \epsilon_0$,

$$E[|X_t^\epsilon|^2] < C_2 + C_3 t \tag{2.68}$$

Let $H_0 > H(x_0) + A_1 T$. Applying the Kolmogorov-Doob inequality (see [25], §3.2), we get

$$P\{\max_{0 \leq t \leq T} H(X_t^\epsilon) > H_0\} \tag{2.69}$$

$$\leq P\{\sup_{0 \leq t \leq T} |\int_0^t \nabla H(X_s^\epsilon) \sigma(X_s^\epsilon) ds| > H_0 - H(x_0) - A_1\} \tag{2.70}$$

$$\leq \frac{E\left[\int_0^T |\nabla H(X_s^\epsilon) \sigma(X_s^\epsilon)|^2 ds\right]}{(H_0 - H(x_0) - A_1 T)^2} \tag{2.71}$$

$$\leq \frac{E\left[\int_0^T A_2 |\nabla H(X_s^\epsilon)|^2 ds\right]}{(H_0 - H(x_0) - A_1 T)^2} \tag{2.72}$$

By assumption, there exists a constant K_1 such that for all $|x|$ sufficiently large, $|\nabla H(x)| > K_1|x|$. By (2.68), this implies that there exist constants A_3 and A_4 for which

$$E\left[\int_0^T A_2|\nabla H(X_s^\epsilon)|^2 ds\right] < A_3 + A_4 T \tag{2.73}$$

for all $\epsilon < \epsilon_0$. This ensures that for any fixed T and $\delta > 0$, we can choose H_0 sufficiently large to guarantee (2.63).

In light of (2.62), to prove the theorem, it suffices to prove that for any $N < \infty$,

$$\lim_{\epsilon\to 0} P\left\{\sup_{0\le t\le t}|H(X_t^\epsilon) - Y_t^\epsilon| > \delta,\ \sup_{0\le t\le T}|X_t^\epsilon| < N\right\} = 0 \tag{2.74}$$

Let f be a twice continuously differentiable function $f : \mathbb{R}^2 \to \mathbb{R}$, and let $\hat{f}$ denote the average value of f on each level set $C(z)$ of H:

$$\hat{f}(z) = \frac{1}{T(z)}\oint_{C(z)} \frac{f(x)}{|\nabla H(x)|} dl \tag{2.75}$$

We prove the following claim:

Claim 2.1.7. *Let f satisfy the aforementioned assumptions and let $\hat{f}$ be defined as in* (2.75)*. For any $N < \infty$, define the event $A_N^\epsilon = \left\{\omega : \sup_{0\le t\le T}|X_t^\epsilon| < N\right\}$. Then the following limits hold:*

$$\lim_{\epsilon\downarrow 0} P\left\{\sup_{0\le t\le T}\left|\int_0^t [f(X_s^\epsilon) - \hat{f}(H(X_s^\epsilon))]ds\right| > \delta,\ \sup_{0\le t\le T}|X_t^\epsilon| < N\right\} = 0 \tag{2.76}$$

and also

$$\lim_{\epsilon\downarrow 0}\left\{\sup_{0\le t\le T} E\left[\left(\left(\int_0^t [f(X_s^\epsilon) - \hat{f}(H(X_s^\epsilon))]ds\right) 1_{A_N^\epsilon}\right]^2\right\} = 0 \tag{2.77}$$

Proof. To justify this, let u be a solution of the partial differential equation

$$\overline{\nabla} H(x)\nabla u(x) = f(x) - \hat{f}(H(x)) \tag{2.78}$$

This equation is solvable for u because again, by construction, the right hand side integrates to zero with respect to the invariant density on each level set. The solution u is also twice continuously differentiable. By the Ito formula for u, we have

$$u(X_t^\epsilon)-u(x) = \frac{1}{\epsilon}\int_0^t (\nabla u \cdot \overline{\nabla} H)(X_s^\epsilon)ds + \int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s + \frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds \quad (2.79)$$

which implies

$$\int_0^t (\nabla u \cdot \overline{\nabla} H)(X_s^\epsilon)ds = \epsilon \left[u(X_t^\epsilon) - u(x) + \int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s + \frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds \right] \quad (2.80)$$

$$\int_0^t (f(X_s^\epsilon) - \hat{f}(H(X_s^\epsilon))ds = \epsilon \left[u(X_t^\epsilon) - u(x) + \frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds + \int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s \right] \quad (2.81)$$

and this implies

$$\int_0^t (f(X_s^\epsilon) - \hat{f}(H(X_s^\epsilon))ds = \epsilon \left[u(X_t^\epsilon) - u(x) + \frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds \right] + \epsilon \left[\int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s \right] \quad (2.82)$$

Let $\rho_t^\epsilon(f)$ denote the right-hand side of this expression:

$$\rho_t^\epsilon(f) = \epsilon \left[u(X_t^\epsilon) - u(x) + \frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds + \int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s \right] \quad (2.83)$$

On the event $A_N^\epsilon = \left\{ \sup_{0\le t\le T} |X_t^\epsilon| < N \right\}$, the continuity of the integrand implies that the Riemann integral on the right-hand side is bounded. By the Kolmogorov-Doob inequality, for any $\eta > 0$,

$$P\left\{ \sup_{0\le t\le T} \left| \int_0^t \nabla u \cdot \sigma(X_s^\epsilon)dW_s \right| > \eta, A_N^\epsilon \right\} \le \frac{1}{\eta^2} E\left[\left(\int_0^T \nabla u \cdot \sigma(X_s^\epsilon)dW_s \right) 1_{A_N^\epsilon} \right]^2 \quad (2.84)$$

and by the Ito isometry,

$$E\left[\left(\int_0^T \nabla u\cdot\sigma(X_s^\epsilon)dW_s\right)1_{A_N^\epsilon}\right]^2 = E\left[\left(\int_0^T (\nabla u\cdot\sigma(X_s^\epsilon))^2\,ds\right)1_{A_N^\epsilon}\right] \tag{2.85}$$

and the right-hand side of the above equality is again bounded since ∇u and σ are continuous. Therefore

$$P\left\{\sup_{0\le t\le T}\left|\int_0^t (f(X_s^\epsilon)-\hat{f}(H(X_s^\epsilon))ds\right| > \delta, A_N^\epsilon\right\}\to 0 \tag{2.86}$$

as $\epsilon\downarrow 0$. Also, squaring both sides and taking expectations in (2.82), we get

$$E\left[\left(\int_0^t (f(X_s^\epsilon)-\hat{f}(H(X_s^\epsilon))ds\right)1_{A_N^\epsilon}\right]^2 \le 2\epsilon^2 E\left[\left(u(X_t^\epsilon)-u(x)+\frac{1}{2}\int_0^t Lu(X_s^\epsilon)ds\right)1_{A_N^\epsilon}\right]^2 \tag{2.87}$$

$$+2\epsilon^2 E\left[\left(\int_0^t \nabla u\cdot\sigma(X_s^\epsilon)dW_s\right)1_{A_N^\epsilon}\right]^2 \tag{2.88}$$

and again by the Ito isometry and the smoothness of u and σ, both the expectations on the right-hand side are uniformly bounded for $0\le t\le T$. Hence the right-hand side converges to 0 uniformly in $0\le t\le T$ as $\epsilon\downarrow 0$. This completes the proof of the claim. □

Continuing now with the proof of the theorem, applying the Ito formula to the process $H(X_t^\epsilon)$, we get

$$H(X_t^\epsilon)-H(x) = \int_0^t \nabla H(X_s^\epsilon)\cdot B(X_s^\epsilon)ds + \frac{1}{2}\int_0^t \sum_{ij} a_{ij}(X_s^\epsilon)\frac{\partial^2 H(X_s^\epsilon)}{\partial x_1\partial x_2}ds \tag{2.89}$$

$$+\int_0^t \nabla H(X_s^\epsilon)\sigma(X_s^\epsilon)dW_s \tag{2.90}$$

By the random-time change formula and the self-similarity of the Wiener process (see [25], §8.5), the stochastic integral can be written as

$$\tilde{W}^\epsilon \left[\int_0^t a(X_s^\epsilon) \nabla H(X_s^\epsilon) \cdot \nabla H(X_s^\epsilon) ds \right] \tag{2.91}$$

where $\tilde{W}^\epsilon$ is a one-dimensional Wiener process.

We have the following formula for the evolution of $H(X_t^\epsilon)$:

$$H(X_t^\epsilon) = H(x) + \int_0^t \overline{B}(H(X_s^\epsilon)) + \int_0^t \overline{LH}(H(X_s^\epsilon)) ds \tag{2.92}$$

$$+\tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon)) ds + \int_0^t a(X_s^\epsilon) \nabla H(X_s^\epsilon) \cdot \nabla H(X_s^\epsilon) ds - \int_0^t \overline{A}(H(X_s^\epsilon)) ds \right] \tag{2.93}$$

$$-\tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon)) ds \right] + \tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon)) ds \right] \tag{2.94}$$

$$+ \int_0^t [\nabla H(X_s^\epsilon) B(X_s^\epsilon) - \overline{B}(H(X_s^\epsilon))] ds + \int_0^t [LH(X_s^\epsilon) - \overline{LH}(H(X_s^\epsilon))] ds \tag{2.95}$$

Define the random variable η_t^ϵ as

$$\eta_t^\epsilon = \tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon)) ds + \int_0^t a(X_s^\epsilon) \nabla H(X_s^\epsilon) \cdot \nabla H(X_s^\epsilon) ds - \int_0^t \overline{A}(H(X_s^\epsilon)) ds \right] \tag{2.96}$$

$$-\tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon)) ds \right] \tag{2.97}$$

By the uniform continuity of the Wiener process and (2.1.7),

$$\sup_{0 \le t \le T} |\eta_t^\epsilon| 1_{A_N^\epsilon} \to 0 \text{ in probability, and } \sup_{0 \le t \le T} E\left[\left(|\eta_t^\epsilon|^2 \right) 1_{A_N^\epsilon} \right] \to 0 \tag{2.98}$$

as $\epsilon \downarrow 0$. We have

$$H(X_t^\epsilon) = H(x) + \int_0^t \overline{B}(H(X_s^\epsilon)) + \int_0^t \overline{LH}(H(X_s^\epsilon))ds \tag{2.99}$$

$$+ \eta_t^\epsilon + \tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon))ds\right] \tag{2.100}$$

$$+ \int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(H(X_s^\epsilon))]ds + \int_0^t [LH(X_s^\epsilon) - \overline{LH}(H(X_s^\epsilon))]ds \tag{2.101}$$

By the random-time change formula, there exists another Wiener process $\tilde{\tilde{W}}^\epsilon$ such that

$$\tilde{W}^\epsilon \left[\int_0^t \overline{A}(H(X_s^\epsilon))ds\right] = \int_0^t \sqrt{\overline{A}(H(X_s^\epsilon)}d\tilde{\tilde{W}}_s^\epsilon \tag{2.102}$$

Let Y_t^ϵ be the process defined by

$$Y_t^\epsilon - H(x_0) = \int_0^t \overline{B}(Y_s^\epsilon) + \overline{LH}(Y_s^\epsilon)ds + \int_0^t \sqrt{\overline{A}(Y_s^\epsilon)}d\tilde{\tilde{W}}_s^\epsilon \tag{2.103}$$

This process is identical in distribution to Y_t for each $\epsilon > 0$. Let $\overline{D} = \overline{B} + \overline{LH}$, and put $Z_t^\epsilon = H(X_t^\epsilon)$. Then

$$Z_t^\epsilon - Y_t^\epsilon = \int_0^t [\overline{D}(Z_s^\epsilon) - \overline{D}(Y_s^\epsilon)]ds \tag{2.104}$$

$$+\eta_t^\epsilon + \int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon \tag{2.105}$$

$$+ \int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(Z_s^\epsilon)]ds + \int_0^t [LH(X_s^\epsilon) - \overline{LH}(Z_s^\epsilon)]ds \tag{2.106}$$

By Lipschitz continuity of $\overline{D}$, we get

$$|Z_t^\epsilon - Y_t^\epsilon| \leq \int_0^t K_1 |Z_s^\epsilon - Y_s^\epsilon| ds \tag{2.107}$$

$$+|\eta_t^\epsilon| + \left|\int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon\right| \tag{2.108}$$

$$+ \left|\int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(Z_s^\epsilon)]ds\right| + \left|\int_0^t [LH(X_s^\epsilon) - \overline{LH}(Z_s^\epsilon)]ds\right| \tag{2.109}$$

In what follows, we assume all expectations to be taken over the set $A_N^\epsilon = \{\omega : \sup_{0\le t\le T} |X_t^\epsilon| < N\}$. (For notational convenience, we suppress the repeated use of the indicator function $1_{A_N^\epsilon}$ within the expectation.)

By Claim (2.1.7), the terms

$$\sup_{0\le t\le T} E[\rho_t^\epsilon(B)]^2 = \sup_{0\le t\le T} E\left[\int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(H(X_s^\epsilon))]ds\right]^2 \quad \text{and} \tag{2.110}$$

$$\sup_{0\le t\le T} E[\rho^\epsilon(LH)]^2 = \sup_{0\le t\le T} E\left[\int_0^t [LH(X_s^\epsilon) - \overline{LH}(H(X_s^\epsilon))]ds\right]^2 \tag{2.111}$$

converge to zero as $\epsilon \downarrow 0$.

Let K_2 be the Lipschitz constant for $\sqrt{\overline{A}}$. Squaring and taking expectations, we get:

$$E|Z_t^\epsilon - Y_t^\epsilon|^2 \le 8E\left[\int_0^t K_1|Z_s^\epsilon - Y_s^\epsilon|ds\right]^2 \tag{2.112}$$

$$+8E[|\eta_t^\epsilon|^2] + 8E\left[\int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon\right]^2 \tag{2.113}$$

$$+8E\left[\int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(Z_s^\epsilon)]ds\right]^2 \tag{2.114}$$

$$+8E\left[\int_0^t [LH(X_s^\epsilon) - \overline{LH}(Z_s^\epsilon)]ds\right]^2 \tag{2.115}$$

From the Ito isometry, Fubini's theorem, and Lipschitz continuity, we derive

$$E\left[\int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon\right]^2 = E\int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right]^2 ds \tag{2.116}$$

$$\le \int_0^t K_2^2 E\left[|Z_s^\epsilon - Y_s^\epsilon|^2\right] ds \tag{2.117}$$

Put $m_\epsilon(s) = \sup_{0\le u\le s} E[|Z_u^\epsilon - Y_u^\epsilon|^2]$. From Holder's inequality and (2.107), (2.112), and (2.116), we deduce that

$$m_\epsilon(t) \leq 8TK_1^2 \int_0^t m_\epsilon(s)ds + 8K_2^2 \int_0^t m_\epsilon(s)ds$$
$$+8 \sup_{0\leq t\leq T} E[(\eta_t^\epsilon)^2] + 8 \sup_{0\leq t\leq T} E[(\rho_t^\epsilon(B))^2] + 8 \sup_{0\leq t\leq T} E[(\rho_t^\epsilon(LH))^2]$$

By Gronwall's inequality, $m_\epsilon(t) \to 0$ as $\epsilon \downarrow 0$.

From (2.107), we have

$$|Z_t^\epsilon - Y_t^\epsilon| \leq \int_0^t K_1|Z_s^\epsilon - Y_s^\epsilon|ds + |\eta_t^\epsilon| \tag{2.118}$$
$$+ \left|\int_0^t [\nabla H(X_s^\epsilon)B(X_s^\epsilon) - \overline{B}(Z_s^\epsilon)]ds\right| + \left|\int_0^t [LH(X_s^\epsilon) - \overline{LH}(Z_s^\epsilon)]ds\right| \tag{2.119}$$
$$+ \left| \int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon \right| \tag{2.120}$$

We have established that the second, third, and fourth terms of the right-hand side of the above inequality converge to zero uniformly in probability. To estimate the stochastic integral, we once again apply Kolmogorov's inequality:

$$P\left\{\sup_{0\leq t\leq T}\left| \int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon \right| > \delta\right\} \tag{2.121}$$
$$\leq \frac{E\left[\int_0^T \sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)} d\tilde{\tilde{W}}_s^\epsilon\right]^2}{\delta^2} \leq \frac{1}{\delta^2}\int_0^T K_2 E|Z_s^\epsilon - Y_s^\epsilon|^2 ds \tag{2.122}$$

Since

$$\sup_{0\leq t\leq T} E|Z_t^\epsilon - Y_t^\epsilon|^2 \to 0 \tag{2.123}$$

as $\epsilon \downarrow 0$, we conclude that as $\epsilon \downarrow 0$,

$$P\left\{\sup_{0\leq t\leq T}\left| \int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)}\right] d\tilde{\tilde{W}}_s^\epsilon \right| > \delta\right\} \to 0. \tag{2.124}$$

Let $R^\epsilon(t)$ be defined as

$$R^\epsilon(t) = |\eta_t^\epsilon| + \left| \int_0^t [\nabla H(X_s^\epsilon) B(X_s^\epsilon) - \overline{B}(Z_s^\epsilon)] ds \right| + \left| \int_0^t [LH(X_s^\epsilon) - \overline{LH}(Z_s^\epsilon)] ds \right| \tag{2.125}$$

$$+ \left| \int_0^t \left[\sqrt{\overline{A}(Z_s^\epsilon)} - \sqrt{\overline{A}(Y_s^\epsilon)} \right] d\tilde{\tilde{W}}_s^\epsilon \right| \tag{2.126}$$

We have established that $\sup_{0 \le t \le T} R^\epsilon(t)$ converges to zero in probability as $\epsilon \downarrow 0$. Put $r^\epsilon(t) = \sup_{0 \le t \le T} |Z_t^\epsilon - Y_t|$. We conclude from the above analysis that

$$r^\epsilon(t) \le \int_0^t K_1 r^\epsilon(s) ds + \sup_{0 \le t \le T} R^\epsilon(t) \tag{2.127}$$

So by Gronwall's inequality, we conclude that

$$\lim_{\epsilon \downarrow 0} P \left\{ \sup_{0 \le t \le T} |H(X_t^\epsilon) - Y_t^\epsilon| > \delta, A_N^\epsilon \right\} = 0 \tag{2.128}$$

as required. From (2.74), this proves the theorem. □

Corollary 2.1.8. *The process $H(X_t^\epsilon)$ converges weakly in C_{0T} to the process Y_t.*

We also address the case when the fast motion is stochastic. Consider a single-well Hamiltonian system, with both deterministic and stochastic perturbations, written in action-angle coordinates:

$$\dot{\tilde{I}}^\epsilon = \epsilon(\beta_1(\tilde{I}^\epsilon, \tilde{\phi}^\epsilon)) \tag{2.129}$$

$$\dot{\tilde{\phi}}^\epsilon = \omega(\tilde{I}^\epsilon) + \sigma(\tilde{I}^\epsilon, \tilde{\phi}^\epsilon) \dot{\tilde{W}}_t + \epsilon \beta_2(\tilde{I}^\epsilon, \tilde{\phi}^\epsilon) \tag{2.130}$$

We assume smoothness, boundedness, and non-degeneracy of the diffusion coefficient σ and smoothness and boundedness of the drift coefficients β_1 and β_2. Rescaling time by the transform $t \to t/\epsilon$, we get the system

$$\dot{I}^\epsilon = \beta_1(I^\epsilon, \phi^\epsilon) \tag{2.131}$$

$$\dot{\phi}^\epsilon = \frac{1}{\epsilon}\omega(I^\epsilon) + \frac{1}{\sqrt{\epsilon}}\sigma(I^\epsilon, \phi^\epsilon)\dot{W}_t + \beta_2(I^\epsilon, \phi^\epsilon) \tag{2.132}$$

Consider the one-dimensional family of diffusions parameterized by I on each level set:

$$\dot{\phi} = \omega(I) + \sigma(I, \phi)\dot{W}_t \tag{2.133}$$

Lemma 2.1.9. *On each level set I, there exists a unique invariant density m_I.*

Proof. Let L be the second-order differential operator associated to the one-dimensional diffusion in (2.133). The invariant density is the appropriately normalized kernel of the forward Kolmogorov operator, which corresponds to the formal adjoint of L. The invariant density m must therefore satisfy

$$\frac{1}{2}\frac{d^2}{d\phi^2}(\sigma^2 m) - \omega\frac{dm}{d\phi} = 0 \tag{2.134}$$

and since ω is a solely a function of I, we get

$$\frac{d}{d\phi}\left(\frac{\sigma^2}{2}m\right) - \omega m = C_1 \tag{2.135}$$

Setting $y = (\sigma^2 m)/2$ gives the equation $\frac{\sigma^2}{2}y' - \omega y = C_1\frac{\sigma^2}{2}$. From the non-degeneracy conditions, this equation has solution

$$y(\phi) = y(0)\exp\left(\int_0^\phi \frac{2\omega}{\phi^2}ds\right) + C_1\int_0^\phi \exp\left(\int_\tau^\phi \frac{2\omega}{\sigma^2}ds\right)d\tau \tag{2.136}$$

Because the trajectories on each level set are periodic, we must choose $y(0) = y(2\pi)$, which implies that

$$y(0) = \frac{C_1\int_0^{2\pi}\exp\left(\int_\tau^{2\pi}\frac{2\omega}{\sigma^2}ds\right)d\tau}{1 - \exp\left(\int_0^{2\pi}\frac{2\omega}{\sigma^2}ds\right)} \tag{2.137}$$

and replacing this into the equation for $y(\phi)$ and recalling that $y = \frac{\sigma^2 m}{2}$, we get

$$m(\phi) = C_1 \frac{2}{\sigma^2} \left[\left(\frac{C_1 \int_0^{2\pi} \exp\left(\int_\tau^{2\pi} \frac{2\omega}{\sigma^2} ds\right) d\tau}{1 - \exp\left(\int_0^{2\pi} \frac{2\omega}{\sigma^2} ds\right)} \right) \exp\left(\int_0^\phi \frac{2\omega}{\phi^2} ds\right) + \int_0^\phi \exp\left(\int_\tau^\phi \frac{2\omega}{\sigma^2} ds\right) d\tau \right] \tag{2.138}$$

where C_1 is uniquely chosen to satisfy $\int_0^{2\pi} m(\phi) d\phi = 1$. □

Theorem 2.1.10 (An averaging principle for stochastic fast motion)**.** *Define*

$$\overline{\beta}_1(I) = \int_0^{2\pi} \beta_1(I, \phi) m_I(\phi) d\phi \tag{2.139}$$

where $m_I(\phi)$ is the invariant density on level set I, and $\overline{I}_t$ satisfies the differential equation

$$\dot{\overline{I}}_t = \overline{\beta}_1(\overline{I}_t). \tag{2.140}$$

Then for all $T < \infty$ and $\delta > 0$,

$$\lim_{\epsilon \to 0} P \left\{ \sup_{0 \le t \le T} |I_t^\epsilon - \overline{I}_t| > \delta \right\} = 0 \tag{2.141}$$

Proof. Since the initial conditions for I_t^ϵ and $\overline{I}_t$ are the same, we have

$$|I_t^\epsilon - \overline{I}_t| = | \int_0^t \beta_1(I_s^\epsilon, \phi_s^\epsilon) - \overline{\beta}_1(\overline{I}_s) ds| \tag{2.142}$$

$$= | \int_0^t \beta_1(I_s^\epsilon, \phi_s^\epsilon) - \overline{\beta}_1(I_s^\epsilon) ds + \int_0^t \overline{\beta}_1(I_s^\epsilon) - \overline{\beta}_1(\overline{I}_s) ds| \tag{2.143}$$

Put $m(t) = m^\epsilon(t) = \sup_{0 \le s \le t} |I_s^\epsilon - \overline{I}_s|$. By the Lipschitz continuity of $\overline{\beta}_1$, we find

$$m(t) \le K \int_0^t m(s) ds + \sup_{0 \le t_1 \le t} | \int_0^{t_1} \beta_1(I_s^\epsilon, \phi_s^\epsilon) - \overline{\beta}_1(I_s^\epsilon) ds| \tag{2.144}$$

By Gronwall's inequality, this implies

$$m(T) \le \exp KT \left[\sup_{0 \le t_1 \le t} | \int_0^{t_1} \beta_1(I_s^\epsilon, \phi_s^\epsilon) - \overline{\beta}_1(I_s^\epsilon) ds| \right] \tag{2.145}$$

So it suffices to prove that

$$\sup_{0 \le t_1 \le t} \left| \int_0^{t_1} \beta_1(I_s^\epsilon, \phi_s^\epsilon) - \overline{\beta}_1(I_s^\epsilon) ds \right| \to 0 \tag{2.146}$$

in probability as $\epsilon \downarrow 0$. As in the previous proof, our conditions on the drift and diffusion coefficents guarantee that for any $\eta > 0$, there exists a compact set K_η and a positive real number ϵ_0 such that for all $0 \le t \le T$ and $\epsilon < \epsilon_0$,

$$P\{(I_t^\epsilon, \phi_t^\epsilon) \notin K_\eta, \quad 0 \le t \le T\} < \eta \tag{2.147}$$

so we can again restrict our attention to the case when the trajectories $(I_t^\epsilon, \phi_t^\epsilon)$ belong to a compact set in $\mathbb{R}^2$ for $0 \le t \le T$, $\epsilon < \epsilon_0$.

Let L be the second-order differential operator $L = \omega(I)\frac{d}{d\phi} + \frac{1}{2}\sigma^2(I,\phi)\frac{d^2}{d\phi^2}$, and let u solve the ODE

$$Lu = \left(\omega(I)\frac{d}{d\phi} + \frac{1}{2}\sigma^2(I,\phi)\frac{d^2}{d\phi^2}\right) u = \beta_1(I,\phi) - \overline{\beta}_1(I) \tag{2.148}$$

where ϕ is any point on the circle and I is viewed as a parameter. Note that a solution u exists by the Fredholm alternative (see [7]): the right-hand side is orthogonal to the solution space of $L^* m = 0$ because L^* is precisely the forward Kolmogorov operator and the invariant density $m_I(\phi)$ is the unique normalized element of its kernel. Since the coefficients of L are smooth, u is smooth, and we apply Ito's formula to u:

$$u(I_t^\epsilon, \phi_t^\epsilon) - u(I_0^\epsilon, \phi_0^\epsilon) = \frac{1}{\sqrt{\epsilon}} \int_0^t \frac{\partial u}{\partial \phi} \sigma dW_s + \frac{1}{\epsilon} \int_0^t \frac{\sigma^2}{2} \frac{\partial^2 u}{\partial \phi^2} + \omega \frac{\partial u}{\partial \phi} ds \tag{2.149}$$

$$+ \int_0^t \frac{\partial u}{\partial I} \beta_1 + \frac{\partial u}{\partial \phi} \beta_2 ds \tag{2.150}$$

which implies

$$\int_0^t \beta_1(I_s^\epsilon,\phi_s^\epsilon) - \overline{\beta}_1(\overline{I}_s)ds = \epsilon\left(u(I_t^\epsilon,\phi_t^\epsilon) - u(I_0^\epsilon,\phi_0^\epsilon) - \left[\int_0^t \frac{\partial u}{\partial I}\beta_1 + \frac{\partial u}{\partial \phi}\beta_2 ds\right]\right) \tag{2.151}$$

$$-\sqrt{\epsilon}\left(\int_0^t \frac{\partial u}{\partial \phi}\sigma dW_s\right) \tag{2.152}$$

By the smoothness of u, β_1, and β_2,

$$\sup_{0\le t\le T}\left(u(I_t^\epsilon,\phi_t^\epsilon) - u(I_0^\epsilon,\phi_0^\epsilon) - \left[\int_0^t \frac{\partial u}{\partial I}\beta_1 + \frac{\partial u}{\partial \phi}\beta_2 ds\right]\right) \tag{2.153}$$

is bounded with probability one by some constant C_T. By Kolmogorov's inequality and the Ito isometry, we deduce

$$P\left\{\sup_{0\le t\le T}\left(\int_0^t \frac{\partial u}{\partial \phi}\sigma dW_s\right) > \delta\right\} \le \frac{1}{\delta^2}E\left[\int_0^T |\frac{\partial u}{\partial \phi}\sigma|^2 ds\right] \tag{2.154}$$

and therefore

$$\sup_{0\le t\le T}\left[|\int_0^t \beta_1(I_s^\epsilon,\phi_s^\epsilon) - \overline{\beta}_1(I_s^\epsilon)ds|\right] \to 0 \tag{2.155}$$

in probability as $\epsilon \downarrow 0$. □

2.2 The Freidlin-Wentzell generalization of the averaging principle for multiwell Hamiltonians

In this section, we consider deterministic and stochastic perturbations of a Hamiltonian system in which $H(x_1, x_2)$ has *multiple* wells. Suppose H has the form shown in Figure (1.1), reproduced below.

Here Γ represents the graph obtained by identifying all points of every connected component of each level set of the Hamiltonian, as described in Chapter 1.

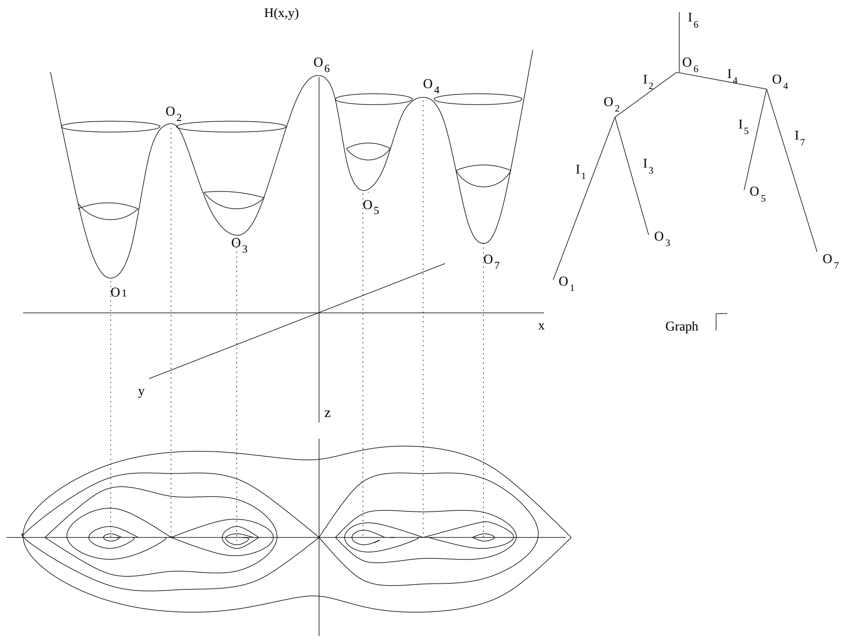

Figure 2.2: $H(x_1, x_2)$ and the Graph Γ

The vertices of Γ correspond to critical points of H: exterior vertices to minima (or maxima of H, if they exist), and interior vertices to saddle points. Each edge of Γ is indexed by a number, $I_1, I_2, \ldots I_m$, and each point y on Γ is indexed by the pair (z, i), where z is the value of the Hamiltonian on the level set corresponding to y, and i is the edge number containing y. The pair (z, i) forms a global coordinate system on Γ.

Let $x \in \mathbb{R}^2$ denote $x = (x_1, x_2)$. Let $Q : \mathbb{R}^2 \to \Gamma; Q(x) = (H(x), i(x))$ be the projection onto Γ of a point x in $\mathbb{R}^2$. We denote the images in Γ of the critical points O_r under Q as simply O_r, and we write $I_k \sim O_r$ if O_r lies at the boundary of an edge I_k. We endow Γ with the natural topology, so a set U is open in Γ if and only if $Q^{-1}(U)$ is open in $\mathbb{R}^2$.

Let $X(t)$ with initial condition $X_0 = x_0$ denote the solution to the unperturbed

Hamiltonian system (1.2) and let $\tilde{X}^\epsilon(t)$ denote the solution to (1.6), the rescaled Hamiltonian system with a small deterministic perturbation. Again, since H is a first integral for the unperturbed system (1.2), the non-separatrix trajectories of $X(t)$ consist of periodic motion around closed curves. Now, however, non-separatrix level sets can have multiple connected components. Let $C(z) = \{x \in \mathbb{R}^2 : H(x) = z\}$ be the level set corresponding to $H = z$, and let $C_i(z)$ be the connected components of $C(z)$, so $C(z) = \bigcup_i C_i(z)$. There exists a unique invariant measure $\mu_{z,i}$ for the dynamical system (1.2) concentrated on each connected component $C_i(z)$ of every non-separatrix level set z, given as before by

$$\mu_{z,i}(A) = \frac{1}{T_i(z)} \oint_A \frac{dl}{|\nabla H(x)|} \tag{2.156}$$

where dl is the length element and $T_i(z)$ is the period of the trajectory concentrated on $C_i(z)$. The evolution of $H(\tilde{X}^\epsilon(t))$ along any given edge I of the graph Γ is identical to what we described in the previous section, namely:

$$H(\tilde{X}^\epsilon(t)) - H(\tilde{X}^\epsilon(0)) = \int_0^t \nabla H(\tilde{X}^\epsilon(s)) \cdot B(\tilde{X}^\epsilon(s))ds \tag{2.157}$$

Near a level set z, the fast motion can be approximated by averaging with respect to the invariant measure concentrated over the trajectory $C_i(z)$, where $i = i(x_0)$.

We denote by $T_i(z)$ the period of the trajectory on $C_i(z)$, so again we have

$$T_i(z) = \oint_{C_i(z)} \frac{dl}{|\nabla H(x)|}.$$

and we denote by $G_i(z)$ the region bounded by $C_i(z)$. Let $S_i(z) = \text{Area}[G_i(z)]$. Put

$$\overline{B}_i(z) = \frac{1}{T_i(z)} \oint_{C_i(z)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl \tag{2.158}$$

and again by the divergence theorem, we define $\tilde{B}_i(z)$ as

$$\tilde{B}_i(z) = \oint_{C_i(z)} \frac{\nabla H(x) \cdot B(x)}{|\nabla H(x)|} dl = \int_{G_i(z)} [\text{div} B(x)] dx_1 dx_2 \tag{2.159}$$

Fix any initial point $x = (x_1(0), x_2(0))$ and let $\tilde{X}^\epsilon(t)$ be the solution of (1.6) with $\tilde{X}^\epsilon(0) = x = (x_1(0), x_2(0))$. Let $Q(X(0)) = (z, i)$ where $z = H(X(0))$ and i is the edge number corresponding to $X(0)$. For any finite time interval $[0, T]$ such that $\tilde{X}^\epsilon(t)$ does not intersect the level set of a saddle point, Theorem (2.1.4) guarantees that the slow component $H(\tilde{X}^\epsilon(t))$ converges uniformly as $\epsilon \downarrow 0$ to the solution $\overline{H}_i(t)$ of the averaged system

$$\dot{\overline{H}}_i(t) = \overline{B_i}(H_i(t)) \tag{2.160}$$

where $\overline{H_i}(0) = H(X(0)) = z_0$.

We apply the averaging principle *inside* certain edges of the graph Γ to describe the limiting slow motion $H(\tilde{X}^\epsilon(t))$ as $\epsilon \downarrow 0$. Since we assume $\text{div}(B(x)) < 0$, $\tilde{B}_i(z)$ is negative and does not change sign, so the limiting slow motion within each edge is monotone. Note that $\lim_{z \to H(O_j)^+} \tilde{B}_i(z) = 0$ for any saddle point O_j and $\lim_{z \to H(O_k)^+} \tilde{B}_i(z) = 0$ for any stable fixed point O_k, where $I_i \sim O_j$ and $I_i \sim O_k$. By linearizing in a small neighborhood of any saddle point O_j, we see that if $H(x) > H(O_j)$, the averaged trajectory $\overline{H}_{i(x)}(t)$ beginning at $(H(x), i(x))$ will reach $H(O_j)$ in finite time.

Lemma 2.2.1. *Let $T_i(z)$, $\tilde{B}_i(z)$, and $A_i(z)$, and $G_i(z)$ be defined as above. Then*

- *For any non-saddle level set, $S_i'(z) = T_i(z)$;*
- *If O is a saddle point of H and $z_0 = H(O)$, there exists a constant C such*

that

$$\lim_{z\to z_0^+} \frac{T_i(z)}{\ln(|z-z_0|)^{-1}} = C \tag{2.161}$$

- *If O_e is a minimum of H, there exists a constant $D > 0$ such that $T_i(z) \to D$ as $z \to H(O_e)$.*

Proof. For (i), let x be some point on the level set $H = z$, and let $C_i(z)$ be the corresponding closed curve containing x; similarly let $C_i(z + \Delta z)$ be the closed curve corresponding to the level set $H = z + \Delta z$. We assume for simplicity that $G_i(z) \subset G_i(z + \Delta z)$. Let Δl be the arc length element along $C_i(z)$. Let Δx be the width of the annular region $G_i(z + \Delta z) \setminus G_i(z)$ at the point x. We have

$$\Delta z = |\nabla H(x)|\Delta x + o(\Delta z) \tag{2.162}$$

$$\Rightarrow \Delta S_i(z) = \oint_{C_i(z)} \frac{dl}{|\nabla H(x)|} \cdot \Delta(z) + o(\Delta z) \tag{2.163}$$

$$\Rightarrow S_i'(z) = \oint \frac{dl}{|\nabla H(x)|} \tag{2.164}$$

More generally, this argument can be applied to show that if

$$F(z) = \int_{G(z)} f(x) dx_1 dx_2 \tag{2.165}$$

for a continuous function f, then

$$F'(z) = \oint_{C_i(z)} \frac{f(x)}{|\nabla H(x)|} dl \tag{2.166}$$

For (ii), let N_a be a small neighborhood of the saddle point O and note that $T_i(z)$ can be separated into two pieces: $T_{i,N_a}(z)$, i.e. the time the trajectory spends in N_a, and $T_{i,N_a^C}(z)$, the time the trajectory spends outside of N_a. $T_{i,N_a^C}(z)$ is bounded uniformly in the region $z - z_0$ because $|\nabla H(x)| > \delta$ for some positive δ in N_a^C, and

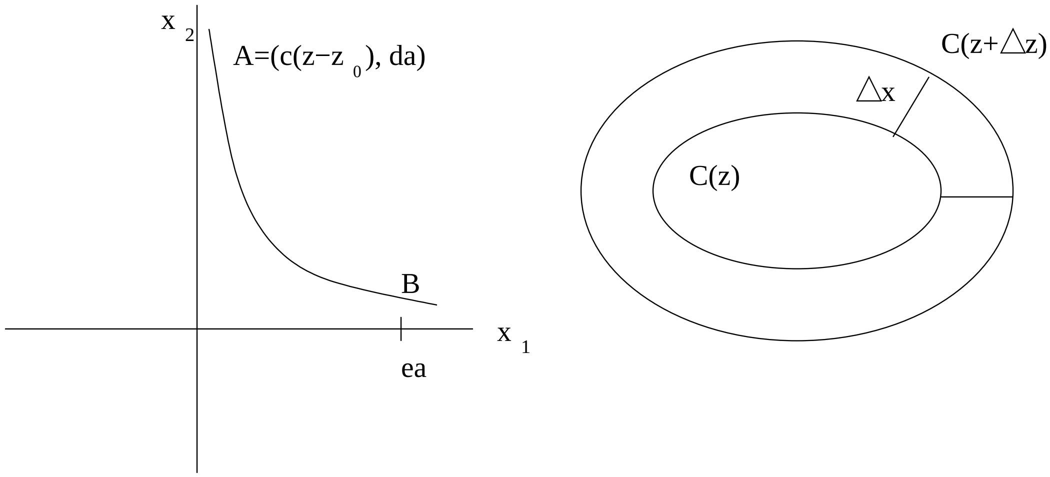

Figure 2.3: Local coordinates near a saddle point and annular regions

the length of the curve $C_k(z)$ is bounded uniformly in z. Since saddle points are hyperbolic fixed points, there exists a smooth, nondegenerate change of coordinates within N_a such that the dynamical system $\dot{X} = \overline{\nabla} H(X)$ can be written

$$\dot{x_1} = \lambda x_1 + x_1 g(x_1, x_2) \tag{2.167}$$

$$\dot{x_2} = -\lambda x_2 + x_2 h(x_1, x_2) \tag{2.168}$$

In these coordinates, the separatrices of a saddle point become the coordinate axes, the saddle point O becomes the origin, and g and h are continuously differentiable and satisfy $g(0,0) = h(0,0) = 0$. We bound the time T_{AB} for the trajectory to travel from $A = (c(z - z_0), da)$ to B where B has x_1-coordinate ea and c, d, and e are constants depending on $z - z_0$, but with finite limits $\overline{c}, \overline{d}$, and $\overline{e}$ as $z - z_0 \to 0$. For every $0 < b << 1$, we can choose a sufficiently small that within N_a, we have

$$(\lambda(1 - b))x_1(t) \leq \dot{x}_1(t) \leq (\lambda(1 + b))x_1(t) \tag{2.169}$$

and we deduce

$$-\frac{\ln(|z-z_0|)+\ln\bar{c}-\ln\bar{e}a}{1+b} \leq \lambda T_{AB}(z-z_0) \leq -\frac{\ln(|z-z_0|)+\ln\bar{c}-\ln\bar{e}a}{1-b} \quad (2.170)$$

so that

$$\frac{1}{\lambda(1+b)} \leq \underline{\lim}_{z\to z_0^+}\frac{T_{AB}(z-z_0)}{\ln(|z-z_0|^{-1})} \leq \overline{\lim}_{z\to z_0^+}\frac{T_{AB}(z-z_0)}{\ln(|z-z_0|^{-1})} \leq \frac{1}{\lambda(1-b)} \quad (2.171)$$

For (iii), let $\left[\frac{\partial^2 H(x)}{\partial x_1 \partial x_2}\right]$ denote the Hessian matrix of partial derivatives of H at x. Observe that for values z near z_0, the Hamiltonian can be approximated by a quadratic form because $\nabla H = 0$ and the Hessian is nondegenerate. We claim

$$\oint_{C_i(z)} \frac{1}{|\nabla H(x)|} dl \to \frac{C}{\sqrt{\det\left[\frac{\partial^2 H(0)}{\partial x_1 \partial x_2}\right]}} > 0 \quad (2.172)$$

as $z \to H(O_e)$ with O_e a minimum of H. Without loss of generality, we can take $H(O_e) = 0$. Since O_e is a nondegenerate mininum, the Hessian of H at O_e is symmetric and positive definite. For any sufficiently small δ-neighborhood of O_e, there exists a change of coordinates so that $H(x_1, x_2)$ can be written

$$H(x_1, x_2) = Ax_1^2 + Bx_2^2 + o(\delta^2) \quad (2.173)$$

for $|(x_1, x_2) - O_e| < \delta$, with

$$\sqrt{AB} = \sqrt{\left[\frac{\partial^2 H(O_e)}{\partial x_1 \partial x_2}\right]} \quad (2.174)$$

For E a nonzero constant, parameterizing the ellipses $Ax_1^2 + Bx_2^2 = E$ by

$$x_1 = \frac{\sqrt{E}}{\sqrt{A}}\cos t, \quad x_2 = \frac{\sqrt{E}}{\sqrt{B}}\sin t \quad (2.175)$$

and computing the line integral, we get

$$\lim_{z\to H(O_e)} T_i(z) = \frac{2\pi}{\sqrt{\left[\frac{\partial^2 H(O_e)}{\partial x_1 \partial x_2}\right]}} > 0 \quad (2.176)$$

□

The behavior of the slow component $Q(\tilde{X}^\epsilon(t))$ is very sensitive to small changes in ϵ, and as described in [3], $Q(\tilde{X}^\epsilon(t))$ does not have a limit as $\epsilon \downarrow 0$ for t large enough. Also, since interior vertices are accessible for $H(\tilde{X}^\epsilon(t))$, the behavior of the process at each interior vertex must be specified. In [3], it is proved that in a certain sense, $Q(\tilde{X}^\epsilon(t))$ tends to a stochastic process on the graph Γ as $\epsilon \to 0$. To give this a rigorous meaning, we let $\tilde{X}^{\epsilon,\kappa}(t)$ be a two-dimensional diffusion with generator $\mathcal{L}^{\epsilon,\kappa}$, as in (1.7):

$$\mathcal{L}^{\epsilon,\kappa}(u(x)) = \frac{\kappa}{2}\text{div}(a(x)\nabla u(x)) + B(x) \cdot \nabla u(x) + \frac{1}{\epsilon}\overline{\nabla} H(x) \cdot \nabla u(x)$$

We assume that $a(x)$ is a smooth, uniformly positive definite 2×2 diffusion matrix.

Fix an initial point $x = (x_1(0), x_2(0))$. Suppose $\tilde{X}^{\epsilon,\kappa}(0) = x$, and let $Q(x) = (H(x), i(x))$ be the projection of x onto the graph Γ. Since $\tilde{X}^{\epsilon,\kappa}(t)$ has a random component, the trajectory can, over time, move from one connected component of a level set to another. As $\epsilon \downarrow 0$, we can still apply the averaging principle to determine the limiting motion of the first component of $Q(\tilde{X}^{\epsilon,\kappa}(t)) = (H(\tilde{X}^{\epsilon,\kappa}(t)), i(\tilde{X}^\epsilon(t)))$, but only *within* each edge of Γ; that is, as long as $i(\tilde{X}^{\epsilon,\kappa}(t)) = i(x)$.

Fix an edge I_i on Γ. We apply the Ito formula to $H(\tilde{X}^{\epsilon,\kappa}(t))$ and average with respect to the invariant measure concentrated on $C_i(z)$ for each level set $H = z$:

$$H(\tilde{X}^{\epsilon,\kappa}(t)) - H(\tilde{X}^{\epsilon,\kappa}(0)) = \int_0^t \nabla H(\tilde{X}^{\epsilon,\kappa}(s)) \cdot B(\tilde{X}^{\epsilon,\kappa}(s))ds \tag{2.177}$$

$$+\int_0^t \frac{\kappa}{2} \begin{bmatrix} \frac{\partial a_{11}(\tilde{X}^{\epsilon,\kappa}(s))}{\partial x_1} + \frac{\partial a_{21}(\tilde{X}^{\epsilon,\kappa}(s))}{\partial x_2} \\ \frac{\partial a_{12}(\tilde{X}^{\epsilon,\kappa}(s))}{\partial x_1} + \frac{\partial a_{22}(\tilde{X}^{\epsilon,\kappa}(s))}{\partial x_2} \end{bmatrix} \nabla H(\tilde{X}^{\epsilon,\kappa}(s))ds \tag{2.178}$$

$$+\int_0^t \frac{\kappa}{2} \left[\sum_{i,j=1}^{2} a_{ij}(\tilde{X}^{\epsilon,\kappa}(s)) \frac{\partial H(\tilde{X}^{\epsilon,\kappa}(s))}{\partial x_i \partial x_j} \right] ds \tag{2.179}$$

$$+\tilde{W} \left[\int_0^t \kappa a(\tilde{X}^{\epsilon,\kappa}(s)) \nabla H(\tilde{X}^{\epsilon,\kappa}(s)) \cdot \nabla H(\tilde{X}^{\epsilon,\kappa}(s))ds \right] \tag{2.180}$$

where $\tilde{W}$ is a Wiener process.

So we get

$$H(\tilde{X}^{\epsilon,\kappa}(t)) - H(\tilde{X}^{\epsilon,\kappa}(0)) = \int_0^t \nabla H(\tilde{X}^{\epsilon,\kappa}(s)) \cdot B(\tilde{X}^{\epsilon,\kappa}(s))ds \tag{2.181}$$

$$+\int_0^t \frac{\kappa}{2} \mathrm{div} \left[a(\tilde{X}^{\epsilon,\kappa}(s)) \cdot \nabla H(\tilde{X}^{\epsilon,\kappa}(s)) \right] ds \tag{2.182}$$

$$+\tilde{W} \left[\int_0^t \kappa a(\tilde{X}^{\epsilon,\kappa}(s)) \nabla H(\tilde{X}^{\epsilon,\kappa}(s)) \cdot \nabla H(\tilde{X}^{\epsilon,\kappa}(s))ds \right] \tag{2.183}$$

We compute the average value of each of the integrands over a level set $H = z$. Define $\tilde{B}(z)$ as before, and put

$$A_i(z) = \oint_{C_i(z)} \frac{a(x)\nabla H(x) \cdot \nabla H(x)}{|\nabla H(x)|} dl \tag{2.184}$$

$$= \int_{G_i(z)} \mathrm{div}(a(x)\nabla H(x))dx_1 dx_2 \tag{2.185}$$

so that $A_i'(z)$is given by

$$A_i'(z) = \oint_{C_i(z)} \frac{\mathrm{div}(a(x)\nabla H(x))}{|\nabla H(x)|} dl \tag{2.186}$$

Along any edge I_i of Γ, $Q(\tilde{X}^{\epsilon,\kappa}(t))$ can be approximated as $\epsilon \downarrow 0$ by a diffusion process on I_i with generator L_i^κ:

$$L_i^\kappa(u_i(z)) = \frac{\kappa}{2T_i(z)} \frac{d}{dz}\left\{A_i(z)\frac{du_i(z)}{dz}\right\} + \frac{\tilde{B}_i(z)}{T_i(z)}\frac{du_i(z)}{dz} \tag{2.187}$$

Let Q^κ be a process on Γ with generator L^κ such that on each edge I_i, L^κ is given by L_i^κ. To complete the description of Q^κ, we specify certain *gluing conditions* (see §8, [18]) at each interior vertex O_j. These conditions are restrictions are on the domain of the generator L^κ.

For any interior vertex O_j with edges $I_k \sim O_j$, let γ_j^k represent the separatrix curves that meet at O_j, and $G_k(O_j)$ the interior regions bounded by the separatrices γ_j^k, as in the figure below.

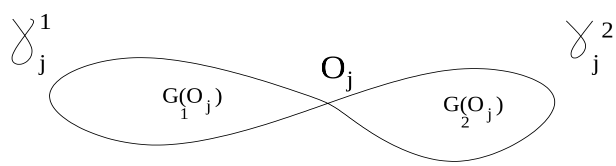

Figure 2.4: Separatrices γ_j^k and interior regions $G_k(O_j)$

Define constants β_{jk} as follows:

$$\beta_{jk} = \oint_{\gamma_j^k} \frac{a(x)\nabla H(x) \cdot \nabla H(x)}{|\nabla H(x)|} dl \tag{2.188}$$

We say that a continuous function $u(z) : \Gamma \to \mathbb{R}$, belongs to the domain of definition of the generator L^κ of diffusion process $Q^\kappa(t)$ if:

1. The function $u(z)$ is smooth on the interior of I_i;

2. At each interior vertex O_j, with corresponding edges I_k meeting at O_j, the

following *gluing condition* is satisfied:

$$\sum_{k:I_k\sim O_j} \pm\beta_{jk} D_k u(O_j) = 0 \tag{2.189}$$

where the (+) or (-) is chosen according to whether the value of H increases or decreases along edge I_k as we approach O_j, and D_k represents the derivative in the direction of the edge I_k.

3. The function $v_i(z) = L_i^\kappa(u_i(z))$ is continuous on Γ.

In §8 of [18] and in [19], it is proved that the generators on each edge and gluing conditions at each interior vertex uniquely determine the process $Q^\kappa(t)$ on Γ, and $Q^\kappa(t)$ is a continuous strong Markov diffusion process on Γ. As we show in the next chapter, exterior vertices are inaccessible for $Q^\kappa(t)$.

For any arbitrary but fixed time interval $[0,T]$, let $C_{0T}(\Gamma)$ be the continuous functions $\phi : [0,T] \to \Gamma$. It is proved in [18] and [16] that $Q(\tilde{X}^{\epsilon,\kappa}(t))$ converges weakly in $C_{0T}(\Gamma)$ to the process $Q^\kappa(t)$ as $\epsilon \downarrow 0$.

An edge $I_k \sim O_j$ is an *exit edge* for O_j if $H(Q^{-1}(z,k))$ increases as (z,k) approaches O_j along I_k; otherwise I_k is an *entrance* edge. For example, edge I_6 on the graph in Figure (2.2) is an entrance edge for O_6, and edges I_4 and I_2 are exit edges for O_6.

For any interior vertex O_j, $G_k(O_j)$ denotes the interior of the region bounded by the separatrix γ_j^k (see previous figure). Define $\tilde{B}_k(O_j)$ according to the formula

$$\tilde{B}_k(O_j) = \int_{G_k(O_j)} \operatorname{div}(B(x))dx \tag{2.190}$$

In [3], it is established that as $\kappa \downarrow 0$, $Q^\kappa(t)$ converges weakly in $C_{[0T]}(\Gamma)$, for any fixed $T > 0$, to a process $Q(t)$, defined as follows:

1. In the interior of any edge I_i of Γ, $Q(t)$ is deterministic motion satisfying

$$\frac{dQ(t)}{dt} = \frac{1}{T_i(Q(t))} \int_{G_i(Q(t))} \operatorname{div}(B(x))dx \tag{2.191}$$

2. If there is only one exit edge for an interior vertex O_j, the process leaves O_j without delay along the exit edge.

3. If there are multiple exit edges $I_{k_s} \sim O_j$, $s \in S$, the process $Q(t)$ leaves O_j without delay along exit edge I_{k_r} with probability

$$p_r = \frac{|\tilde{B}_{k_r}(O_j)|}{\sum_{s\in S} |\tilde{B}_{k_s}(O_j)|} \tag{2.192}$$

independently of the past.

We refer to these probabilities of motion along any edge as *limiting edge-access probabilities*. We stress that these probabilities depend only on B and not on the diffusion coefficients $a(x)$.

These results establish the following theorem from [3]:

Theorem 2.2.2. *Let $\tilde{X}^{\epsilon,\kappa}(t)$ be the two-dimensional diffusion processes defined by* (1.7)*. The slow component $Q(\tilde{X}^{\epsilon,\kappa}(t))$ converges weakly in $C_{0T}(\Gamma)$, first as $\epsilon \downarrow 0$, to the stochastic process Q^κ, defined by generators L_i^κ along each edge of Γ and gluing conditions at the interior vertices. Next, as $\kappa \downarrow 0$, Q^κ converges weakly to a process $Q(t)$ which consists of deterministic motion along each edge of Γ and stochastic branching at the interior vertices, with probabilities of branching that depend only on B and not on the diffusion coefficients $a(x)$.*

Proof. See [3]. □

Chapter 3

Metastability and large deviations in certain dynamical systems

3.1 Overview of Freidlin-Wentzell theory of metastability

Metastable states arise in dynamical systems subject to random perturbations. For instance, let Y_t^κ be the diffusion process in $\mathbb{R}^n$ corresponding to the operator D^κ:

$$D^\kappa u^\kappa = \frac{\kappa}{2}\sum_{i,j} a^{ij}(x)\frac{\partial^2 u^\kappa}{\partial x^i \partial x^j} + \sum_i b^i(x)\frac{\partial u^\kappa}{\partial x^i} \tag{3.1}$$

We assume that the coefficients of the operator D^κ are bounded and smooth; the matrix $(a^{ij}(x))$ is uniformly positive definite; and $b(x)$ is Lipschitz continuous and bounded. In studying the behavior of Y_t^κ as both $t \to \infty$ and $\kappa \downarrow 0$, it is natural to assume that that the two paramaters t and κ are connected: $t = t(\kappa)$. Under certain assumptions, the transition probabilities $P^\kappa(t, x, A) = P_x(Y_t^\kappa \in A)$, for some measurable $A \in \mathbb{R}^n$, have different limits as $\kappa \downarrow 0$ for different relationships $t(\kappa)$ and initial points x.

On any fixed time interval $[0, T]$ the diffusion process Y^κ converges uniformly in probability to the solution Y_t of the differential equation $\dot{Y}_t = b(Y_t)$. If, on the other hand, Y^κ has a unique normalized invariant measure μ^κ, then the invariant measure characterizes the long-term behavior of Y^κ in the sense that

$$\lim_{t\to\infty} P^\kappa(t, x, A) = \mu^\kappa(A) \tag{3.2}$$

but this holds only if $t \to \infty$ much faster than $\kappa \to 0$.

In the case of a small white-noise perturbation of deterministic dynamical with finitely many stable equilibrium points, there exists an invariant measure concentrated at each stable equilibrium. Intuitively, we expect the invariant measure μ^κ of the process Y^κ to converge as $\kappa \to 0$ to one of the invariant measures for the unperturbed system, but the question of which one is more delicate. The long-time behavior of the system depends in an essential way on the relationship between t and κ. As in Chapter 1, we consider the case when $t(\kappa) = \exp[\frac{\lambda}{\kappa}]$ for some fixed $\lambda > 0$. For different values λ and different initial conditions y_0, certain *sublimiting distributions* exist, that is, δ-measures $\mu_{K(z,\lambda)}$ concentrated at an equilibrium point $K(z, \lambda)$ such that

$$\lim_{\kappa \downarrow 0} P_z\{Y_{t(\kappa)} \in A\} = 1 \text{ if } K(z,\lambda) \in A, \quad A \text{ open, and} \tag{3.3}$$

$$\lim_{\kappa \downarrow 0} P_z\{Y_{t(\kappa)} \in A\} = 0 \text{ if } K(z,\lambda) \notin A, \quad A \text{ open} \tag{3.4}$$

Such an equilibrium point $K(z, \lambda)$ is called a *metastable state* for the process Y_t^κ with initial condition z and timescale λ.

3.1.1 The qualitative behavior of $\tilde{X}^\epsilon$ for small ϵ

In particular, let us fix the value of ϵ in the nearly-Hamiltonian system (1.6):

$$\dot{\tilde{X}}^\epsilon(t) = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}^\epsilon(t)) + B(\tilde{X}^\epsilon(t)), \ \tilde{X}^\epsilon(0) = (q_0, p_0)$$

We assume that the Hamiltonian H has the form shown in Figure 1.1, so H has four wells. For small ϵ, this is a deterministic dynamical system with finitely many asymptotically stable fixed points and finitely many saddle points.

Lemma 3.1.1. *Assume that $H(x)$ is a generic smooth function with bounded second derivatives. Assume also that $\lim_{|x|\to\infty} H(x) = \infty$ and that there exist K_1 and K_2 such that for all x with $|x|$ sufficiently large, $K_1|x| < \nabla H(x) < K_2|x|$. Suppose that $B(x)$ is smooth, has bounded derivatives, and satisfies div$(B(x)) < 0$ and $\nabla H(x) \cdot B(x) < 0$. Then there exists ϵ_0 such that for all $\epsilon < \epsilon_0$, the equilibrium points of the deterministically-perturbed system (1.6), given by $\dot{\tilde{X}}^\epsilon(t) = \frac{1}{\epsilon}\overline{\nabla} H(\tilde{X}^\epsilon(t)) + B(\tilde{X}^\epsilon(t))$, are in one-to-one correspondence with the equilibrium points of the unperturbed Hamiltonian system (1.1). The minima of H correspond to asymptotically stable fixed points in the perturbed system (1.6) and the saddle points to saddle points. Furthermore, for any compact set K, there exists an $\epsilon_K > 0$ such that for $\epsilon < \epsilon_K$, all trajectories of $\tilde{X}^\epsilon(t)$ with initial values in K are either separatrix trajectories or trajectories attracted to one of the asymptotically stable fixed points.*

Proof. Since the critical points of H are non-degenerate, for ϵ sufficiently small, the fixed points of the Hamiltonian system (1.1) are in one-to-one correspondence with the fixed points of the perturbed system (1.6). For convenience, we denote the fixed points of both systems identically; it is clear from context to which system we refer. Let O_k be a minimum of H. Putting $x = (x_1, x_2)$ in $\mathbb{R}^2$ and linearizing the perturbed system about its perturbed fixed point O_k, we get the matrix

$$\begin{pmatrix} \frac{\partial^2 H}{\partial x_2 \partial x_1} + \epsilon \frac{\partial B_1}{\partial x_1} & \frac{\partial^2 H}{(\partial x_2)^2} + \epsilon \frac{\partial B_1}{\partial x_2} \\ -\frac{\partial^2 H}{(\partial x_1)^2} + \epsilon \frac{\partial B_2}{\partial x_1} & -\frac{\partial^2 H}{\partial x_1 \partial x_2} + \epsilon \frac{\partial B_2}{\partial x_2} \end{pmatrix} \tag{3.5}$$

At minima of H, the Hessian matrix of H is positive definite. Since the divergence $\left[\frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2}\right]$ of $B(x)$ is negative, the eigenvalues of the above matrix have negative real parts at the minima of H. Hence, any minimum O_k of H corresponds to an asymptotically stable fixed point (also denoted O_k) of the perturbed system (1.6).

To prove that all non-separatrix trajectories of the perturbed system are attracted to one of the asymptotically stable equilibrium points, we use Theorem (2.1.4). Let $\tilde{X}^\epsilon(0) = (x_1(0), x_2(0))$ where $(x_1(0), x_2(0))$ is not a saddle point. Let $Q(\tilde{X}^\epsilon(0)) = (z_0, i_0) \in \Gamma$. There exists $T > 0$ and ϵ_1 sufficiently small such that $i(\tilde{X}^\epsilon(t)) = i_0$ for all $t \in [0, T]$ and $\epsilon \leq \epsilon_1$. On such a time interval, the slow motion $H(\tilde{X}^\epsilon(t))$ converges uniformly as $\epsilon \downarrow 0$ to the solution of the averaged system:

$$\dot{\overline{H}}_i(t) = \frac{\tilde{B}_i(\overline{H}_i(t))}{T_i(\overline{H}_i(t))}; \quad H_i(0) = H(\tilde{X}^\epsilon(0)) = z. \tag{3.6}$$

where again $\tilde{B}_i(z) = \int_{G_i(z)} \operatorname{div}(B(x)) dx_1 dx_2$. Fix z sufficiently large that the compact set $F_z = \{x : H(x) \leq z\}$ contains K and all critical points of H. Let $\eta > 0$ and $\delta > 0$ be given; let x be any arbitrary point in F_z, and let $N_\delta(x)$ denote the δ-neighborhood of x. For each fixed finite time interval $[0, T]$, we can find a positive ϵ such that for any point y in $N_\delta(x)$, the perturbed trajectory with initial point y and the averaged trajectory along the corresponding edges, with initial point $H(y)$, differ by less than η in the supremum norm on $C_{0T}(\mathbb{R})$. By compactness we can find $\epsilon_2 > 0$ such that $\epsilon < \epsilon_2$ implies that for all x in F_z, $\sup_{t \in [0,T]} |\overline{H}(t)_{H(x)} - H(\tilde{X}^\epsilon(t))_x| < \eta$, where $\overline{H} = \overline{H}_i$ for the appropriate values of t and edge number i.

Since $\operatorname{div}(B(x)) < 0$, $\overline{H}_i(t)$ is monotone decreasing along each edge $I_i \in \Gamma$. The only fixed points for the averaged system correspond to the minima and saddle points of H, all contained in F_z. Thus for sufficiently small ϵ, all non-separatrix trajectories which originate in a fixed compact set are eventually attracted to one of the asymptotically stable equilibrium points. □

Put $F(z) = \frac{1}{\epsilon}\overline{\nabla} H(z) + B(z)$. Consider the system

$$\dot{Z} = F(z), Z_0 = z_0 \tag{3.7}$$

(Since ϵ is fixed, for notational ease we suppress the dependence on ϵ in what follows in this section.)

We introduce a white-noise-type perturbation to this system: in the equation below, $\dot{W}(t)$ represents white noise. We continue to assume that $\sigma(z) : \mathbb{R}^2 \to \mathbb{R}^2$ is smooth, Lipschitz continuous, and bounded; that $a(z) = \sigma(z)\sigma^T(z)$ is positive definite; and that $0 < \kappa \ll 1$. Let $Z^\kappa(t)$ satisfy the stochastic differential equation

$$\dot{Z}^\kappa(t) = F(Z^\kappa(t)) + \sqrt{\kappa}\sigma(Z^\kappa(t))\dot{W}_t, \quad Z_0 = z_0 \tag{3.8}$$

3.1.2 Metastability for perturbations of a two-dimensional diffusion process

We recall the notion of metastability for the process $Z^\kappa(t)$ as $\kappa \downarrow 0$, following [13], [12], [18, §6]. Qualitatively, Z^κ forms a Markov process which transitions between the basins of attraction for the stable equilibrium points. Beginning at some initial position z_0, the process moves toward the nearest attracting equilibrium. It remains in a small neighborhood of this equilibrium for considerable time before moving to the basin of attraction for another equilibrium. These transitions occur on exponentially large timescales.

Observe that $Z^\kappa(t)$ induces a measure μ^κ on $C_{0T}(\mathbb{R}^2)$, the space of $\mathbb{R}^2$-valued continuous functions on the interval $[0, T]$ endowed with the uniform norm. We are interested in the limiting behavior of these measures both as $T \to \infty$ and as $\kappa \downarrow 0$. It is natural to suppose that T and κ are related: as before, let $\lambda > 0$ be such that

$$\lim_{\kappa \downarrow 0} \kappa \ln T(\kappa) = \lambda > 0. \tag{3.9}$$

We call λ a *timescale*. The logarithmic asymptotics of the measures μ^κ as $\kappa \downarrow 0$ are governed by the *action functional* [18, §3,§5] defined on $C_{0T}(\mathbb{R}^2)$.

Definition 1. A nonnegative functional $S_{0T}(\phi)$ defined on C_{0T} is the action functional for the family of processes Z^κ in C_{0T} with normalizing factor $\frac{1}{\kappa}$ if the following conditions are satisfied:

1. For each $s \geq 0$, the set $\Phi_s = \{\phi : S_{0T}(\phi) \leq s\}$ is compact;

2. For any $\delta > 0$, $\gamma > 0$, and $\phi \in C_{0T}$, there exists κ_0 such that for all $\kappa < \kappa_0$

$$P\{\rho(Z^\kappa, \phi) < \delta\} \geq \exp\left[-\frac{1}{\kappa}(S_{0T}(\phi) + \gamma)\right]; \tag{3.10}$$

3. For all $\delta > 0, \gamma > 0, s > 0$ there exists κ_0 such that

$$P\{\rho(Z^\kappa, \Phi_s) \geq \delta\} \leq \exp\left[-\frac{1}{\kappa}(s - \gamma)\right] \tag{3.11}$$

where ρ denotes the supremum norm on C_{0T}.

The following theorem, proved in [18], §5, gives the explicit form of the action functional for a wide class of diffusion processes. Let $\tilde{a}_{ij}(z)$ denote the inverse of the diffusion coefficient matrix for (3.8): $\tilde{a}_{ij}(z) = [(\sigma(z)\sigma^T(z))_{ij}]^{-1}$.

Theorem 3.1.2. *The* normalized action functional *for the family of processes* Z^κ *is given by* $(1/\kappa)S_{0T}(\phi)$*, where* S *is defined as follows: for absolutely continuous functions* ϕ *which satisfy* $\phi_0 = z_0$*,*

$$S_{0T}(\phi) = \frac{1}{2}\int_0^T \sum_{i,j=1}^{2} \tilde{a}_{ij}(\phi_s^i)(\dot{\phi}_s - F^i(\phi_s))(\dot{\phi}_s^j - F^j(\phi_s))ds \qquad (3.12)$$

For all other $\phi \in C_{0T}$*,* $S_{0T}(\phi) = \infty$*.*

Using the action functional, we define the *quasipotential* V:

Definition 2. The *quasipotential* associated to the dynamical system (3.8) is the function $V : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ given by

$$V(z, y) = \inf\{S_{0T}(\phi) : \phi_0 = z, \phi_T = y, T \geq 0\}. \qquad (3.13)$$

We note that the upper endpoint of $[0, T]$ is not fixed, and the infimum is taken over intervals $[0, T]$ of arbitrary length. We say two points z and y are *equivalent* (denoted $z \sim y$) if $V(z, y) = V(y, z) = 0$.

The quasipotential is the solution to a variational problem, and for many diffusion processes, the quasipotential cannot be explicitly computed. However, when the drift is a potential with a unique minimum O, the quasipotential $V(O, x)$ differs from the potential only by a constant.

Theorem 3.1.3. *Let* X_t^ϵ *be a diffusion process given by*

$$\dot{X}_t^\epsilon = b(X_t^\epsilon) + \epsilon\dot{W}_t \qquad (3.14)$$

where the vector field $b(x) = -\nabla U(x)$ *and* $U(O) = 0$*,* $U(x) > 0$ *for* $x \neq O$*, and* $-\nabla U(x) \neq 0$ *for any* $x \neq O$*. Then the quasipotential* $V(x) = V(O, x) = 2U(x)$*.*

Proof. See 4.3.1 in [18], §4. □

The deterministic system (3.7) has, in general, l asymptotically stable equilibrium points and $l-1$ saddle points (for the Hamiltonian system in Figure 1, $l = 4$). All non-separatrix trajectories have ω-limit sets consisting of a single stable equilibrium. Let $\mathcal{L} = \{K_1, \ldots, K_l\}$ be the set of stable equilibrium points, and for a point z not belonging to a separatrix trajectory, let $K_{i(z)}$ be the stable equilibrium to which the trajectory starting at z is attracted. In the discussion below, we refer to the jth stable equilibrium in $\mathcal{L}$ both as K_j and, for convenience, as simply j, its index in the set $\mathcal{L}$. (Observe that in the notation we have suppressed the dependence of (3.7) on the parameter ϵ; of course, all the points K_j in $\mathcal{L}$ actually depend on ϵ.)

3.1.3 Definitions of metastability

Now consider the randomly perturbed system (3.8), where $0 < \kappa \ll 1$, with the action functional and quasipotential for Z^κ defined as above. We will define metastability for this sytem.

Metastable states can be viewed in two related but different ways. First, the metastable state $K_{(z,\lambda)}$ is the point in a small neighborhood of which Z^κ remains for "most" of the time between $[0, AT(\kappa)]$ for any $A > 0$. Second, the metastable state $K_{(z,\lambda)}$ is the point in a small neighborhood of which, for any fixed t, the process Z^κ at time $tT(\kappa)$, i.e. $Z^\kappa(tT(\kappa))$, is "most" likely to be found. In the current context, these two notions of metastability are equivalent (see [13]).

Definition 3. *(First characterization of a metastable state)* Let $T(\kappa)$ be such that

$\lim_{\kappa\downarrow 0} \kappa \ln T(\kappa) = \lambda > 0$, and let Λ denote Lebesgue measure in $\mathbb{R}$. The *metastable state* for initial condition z and timescale λ, denoted $K_{(z,\lambda)}$, is an asymptotically stable equilibrium point such that for any $\delta > 0$ and $A > 0$,

$$\lim_{\kappa\downarrow 0} P_z\{\Lambda\{t \in [0, A] : \rho(Z^\kappa(tT(\kappa)), K_{(z,\lambda)}) > \delta\}\} \to 0. \tag{3.15}$$

Definition 4. *(Second characterization of a metastable state)* Suppose $K_{(z,\lambda)}$ is the metastable state for (z, λ). Let $\delta > 0, t > 0$, and $\theta > 0$ be arbitrary but fixed. Let $F_{\theta,(z,\lambda)} = \{y \in \mathbb{R}^2 : V(K_{(z,\lambda)}, y) \le \theta\}$. Then there exists $\kappa_0 = \kappa_0(\delta, \theta, t) > 0$ such that for all $\kappa < \kappa_0$, and $T(\kappa)$ as above,

$$P_z\{Z^\kappa(tT(\kappa)) \in F_{\theta,(z,\lambda)}\} > 1 - \delta \tag{3.16}$$

3.1.4 Computing metastable states explicitly

A proof of the existence of a metastable state for each z and λ, and a recipe to find it, are given in [12] and §6 of [18]. We review the concepts here and refer to [12], [13], and [18] for a full discussion.

Put $V_{ij} = V(i, j) = V(K_i, K_j)$, and let $J(i)$ be the index $j \in \mathcal{L}$ such that

$$V_{iJ(i)} = \min\{V_{ik} : k \in \mathcal{L}, k \neq i\}. \tag{3.17}$$

We assume that for each $i \in \mathcal{L}$, the minimum above, and all similar maxima and minima of the quasipotential between a given fixed point K_i and any other fixed point K_k from a finite set M, is achieved at exactly one point $K \in M$. Such a system is called *generic*.

With i and $J(i)$ as above, we say that the point $J(i)$ *follows* i or that i is *followed by* $J(i)$. Once the process Z^κ leaves the basin of attraction for K_i, it moves with overwhelming probability to the basin of attraction for $K_{j(i)}$ [18, p.171]. Define $J^2(i)$ as the index j such that

$$V_{J(i)j} = \min\{V_{J(i)k} : k \in \mathcal{L}, k \neq J(i)\} \tag{3.18}$$

Proceeding inductively, let $J^{k+1}(i) = J(J^k(i))$. Let $m = \min\{k > 0 : J^k(i) = J^n(i), n < k\}$. This enables us to define 0- and 1-cycles.

Definition 5. For a given state $i \in \mathcal{L}$, *the* 0*-cycle containing* i is simply the equilibrium K_i. The equilibrium points (or equilibrium "states") in the collection

$$\mathcal{J} = \{J^n(i), J^{n+1}(i), \ldots, J^m(i) = J^n(i)\} \tag{3.19}$$

form a *cycle of rank 1*, or 1-cycle. If this collection includes i, then we say this is the *1-cycle containing* i. If this collection does *not* include i, we define the point K_i to be both a 0- and 1-cycle.

The cycles of rank zero are simply the points $K_1, \ldots, K_l$ themselves, and a 1-cycle is an ordered collection of states in $\mathcal{L}$. Equivalently, a 1-cycle is an ordered collection of 0-cycles. It is also possible for a single point to be both a 0-cycle and a 1-cycle: suppose there exists a point K_j which is followed by a point K_r, where the 1-cycle beginning with K_r does *not* include K_j. Then the single point K_j forms a 1-cycle.

Within a given 1-cycle and a given time scale, there is, in general, one equilibrium state near which the process principally remains. Accordingly, we define the *main state*, *stationary distribution rate*, *rotation rate*, and *exit rate* for 1-cycles as follows.

Definition 6. For a 1-cycle C,

1. The *main state* of C, $M(C)$, is the state $k^* \in \mathcal{L}$ such that

   $V_{k^*J(k^*)} = \max_{i \in C} V_{iJ(i)}$.

   We assume this maximum is attained at a single point $i = k^* = M(C) \in C$.

2. The *stationary distribution rate* for state i, $m_C(i)$, is given by

   $m_C(i) = V_{iJ(i)} - V_{k^*J(k^*)}$, where k^* is the main state.

3. The *rotation rate* $R(C)$ is defined as $R(C) = \max_{i \in C} V_{iJ(i)}$.

4. The *exit rate*, $\mathcal{E}(C)$, is defined as $\mathcal{E}(C) = \min_{i \in C, j \notin C} (m_C(i) + V_{ij})$,

   where we again assume that the minimum is attained for precisely one value of $i \in C$ and precisely one value of $j \notin C$.

By induction, we can define cycles of higher rank. For example, a cycle of rank 2 consists of transitions between cycles of rank 1, so a cycle of rank 2 is an ordered collection of cycles of rank 1. Formally, assume that for some $r \geq 1$, all the cycles for rank $l \leq r$ are defined. Let $\mathfrak{C}^r$ be the set of all r-cycles. For any r-cycle $C_1 \in \mathfrak{C}^r$, define $C_2 \in \mathfrak{C}^r$ to be the r-cycle containing the point $j^* \in \mathcal{L}$ at which the following minimum is achieved:

$$\min_{i \in C_1, j \notin C_1} [m_{C_1}(i) + V_{ij}] \tag{3.20}$$

As before, we assume that the minimum is attained at precisely one $j^* \in \mathcal{L}$ and precisely one $i^* \in C_1$. Intuitively, K_{j^*} is the "nearest" state in $\mathcal{L}$ external to the cycle C_1: conditional on the process Z^κ exiting the cycle C_1, the basin of attraction for K_{j^*} is the most likely set into which Z^κ will move. We say $i^* = i(C_2)$ is the *exit point* for the cycle C_1 and $j^* = j(C_2)$ is the *entrance point* for C_2. Let $C_2 = J(C_1)$, and consider the ordered sequence of r-cycles

$$C_1, J(C_1), J^2(C_1), \ldots, J^n(C_1), \ldots \tag{3.21}$$

Following our previous notation, let $m^* = \min\{m > 0 : J^m(C_1) = J^n(C_1), n < m\}$. Then the ordered sequence (which we refer to as a *cyclical ordering*) of r-cycles $J^n(C_1), J^{n+1}(C_1), \ldots, J^{m*}(C_1)$ forms an $r+1$-cycle. By induction, we can define exit rates, stationary distribution rates, rotation rates, and main states for $r+1$ cycles. Again, a single r-cycle can also form an $r+1$-cycle.

For any point z, we get a sequence of cycles $C(z)$ containing z:

$$\mathcal{C}(z) : C^{(0)}(z) \subset C^{(1)}(z) \subset \ldots \subset C^{(n)} \tag{3.22}$$

The highest-rank cycle is the unique cycle which contains all the equilibrium points. This cycle is independent of the original point from which the cycles are first constructed.

Since we have defined an $r+1$-cycle as an ordered collection of r-cycles, the elements of an $r+1$-cycle are themselves r-cycles. Nevertheless, all cycles are ultimately composed of equilibrium points from the set $\mathcal{L}$, and we use the notation $i \in C$ to represent any point i in $\mathcal{L}$ which belongs to any of the potentially lower-rank cycles that comprise C.

We can give inductive definitions of main states, rotation rates, and exit rates; these concepts can also be defined directly through i-graphs, which are useful in describing the asymptotic behavior of $Z^\kappa(t)$ on large time intervals [18, p.177].

Definition 7. Let $\mathbf{F}$ be a finite set, and let i be an element of $\mathbf{F}$. A graph consisting of arrows of the form $(m \to n)$, for $n \in \mathbf{F}, m \in \mathbf{F} \setminus \{i\}, n \neq m$, is said to be an *i-graph* if:

1. Every point $m \in \mathbf{F} \setminus \{i\}$ is the initial point of exactly one arrow;

2. For any point $m \in \mathbf{F} \setminus \{i\}$, there exists a sequence of arrows leading from m to the point i.

For each point $i \in \mathcal{L}$, let $G_i(\mathcal{L})$ denote the set of all i-graphs for the finite set of equilibrium points $\mathcal{L}$. Similarly, $G_i(C)$ consists of the set of all possible i-graphs for the finite set of elements within a cycle C.

Suppose inductively that the main state, stationary distribution rate, rotation rate, and exit rate have been defined for all cycles up to and including rank r. We can introduce the corresponding defintions for higher-rank cycles.

Definition 8. For higher-rank cycles,

1. The *main state* $M(C)$ for an r-cycle C is the assumed-unique state $j^*(C)$ that achieves the minimum

$$\min_{j \in C} \min_{g \in G_j(C)} \sum_{(m \to n) \in g} V_{mn}. \tag{3.23}$$

2. The *rotation rate* $R(C)$ for an $r+1$-cycle is defined as

$$R(C) = \max_{i : C_i^r \in C} \mathcal{E}(C_i^r) \tag{3.24}$$

where C_i^r are the r-cycles that form the $r+1$-cycle C and $\mathcal{E}(C_i^r)$ is the exit rate for the r-cycle C_i^r.

3. The *stationary distribution rate* $m_C(i)$ for an $r+1$-cycle C, where $i \in C$, is defined by

$$m_C(i) = \min_{g \in G_i(C)} \sum_{(m \to n) \in g} V_{mn} - \min_{g \in G_{j*}(C)} \sum_{(m \to n) \in g} V_{mn} \tag{3.25}$$

where $j* = M(C)$ is the main state of C defined above.

4. The *exit rate* $\mathcal{E}(C)$ for $C \in \mathcal{C}^{r+1}$ is given by

$$\mathcal{E}(C) = \min_{i \in C, j \notin C} (m_C(i) + V_{ij}), \tag{3.26}$$

where we assume uniqueness of the indices $i^* = i^*(C)$ and $j^* = j^*(C)$ at which the minimum is attained. We call i^* the exit point of C, and j^* the entrance point for the $r+1$-cycle containing K_{j^*}. For the highest-rank cycle C containing all the points of $\mathcal{L} = \{K_1, \ldots, K_l\}$, we define $\mathcal{E}(C) = +\infty$.

In the preceding definitions, we make the genericity assumption that each one of the maxima and minima is attained at a single equilibrium point in $\mathcal{L}$.

Let D_j be the domain of attraction for K_j. For any cycle C, let $D(C) = \bigcup_{i \in C} D_i$. Let τ_C^κ be the exit time for $Z^\kappa(t)$ from $D(C)$, (where $Z_0 = z_0 = z \in D(C)$); that is,

$$\tau = \inf\{t : Z^\kappa(t) \notin D(C)\} \tag{3.27}$$

We can compute the expected value of τ_C^κ through the exit rate for the cycle C. By Theorem 6.6.2 [18, p.201], we note that

$$\lim_{\kappa \downarrow 0} \ln E_z(\tau_C^\kappa) = \mathcal{E}(C) \tag{3.28}$$

Furthermore, according to [18, p.201], for any $\gamma > 0$,

$$\lim_{\kappa \downarrow 0} P_z\{\exp(\kappa^{-1}[\mathcal{E}(C) - \gamma]) < \tau_C^\kappa < \exp(\kappa^{-1}[\mathcal{E}(C) + \gamma])\} = 1 \tag{3.29}$$

uniformly for all $z \in B$, where B is any compact subset of $D(C)$.

As before, let $T(\kappa)$ be a function such that

$$\lim_{\kappa \downarrow 0} \kappa \ln T(\kappa) = \lambda > 0. \tag{3.30}$$

Let $\{\mathcal{C}(z)\}$ represent the ordered sequence of cycles containing z, as in (3.22), up to the cycle of highest rank $n(z) = n$. Let e_k denote the exit rate for the k^{th}-rank cycle containing z, so $e_k = \mathcal{E}(C^k(z))$. It is clear that the numbers e_k form an increasing sequence, with

$$e_0 = V_{iJ(i)} < e_1 < e_2 < \ldots < e_{n-1} < e_n = \infty. \tag{3.31}$$

Let $r_k = R(C^{(k)}(z))$ denote the rotation rates. The sequence r_k is also increasing and $r_k < e_k$.

Let m^* be a positive integer with $e_{m^*} < \lambda < e_{m^*+1}$.

We conclude by describing how to find the metastable state for (z, λ).

Proposition 1. The *metastable state* $K_{(z,\lambda)}$ for the process Z^κ with initial position z and timescale λ is given as follows.

1. Case 1: $\lambda > r_{m^*+1}$. In this instance, $r_{m^*+1} < \lambda < e_{m^*+1}$, and $K_{(z,\lambda)} = M(C^{(m^*+1)}(z))$, the main state of the cycle $C^{(m^*+1)}(z)$.

2. Case 2: $\lambda < r_{m^*+1}$. Let $C^{(m^*)}(z)$ be the m^*-cycle containing z, and let $C^{(m^*+1)}(z)$ be the m^*+1-cycle generated by $C^{(m^*)}(z)$. Consider the m^*-cycles in $C^{(m^*+1)}(z)$ that follow $C^{(m^*)}(z)$, denoted (in cyclic order)

$$C_1^{m^*}(z), C_2^{m^*}(z), \ldots, C_p^{m^*}(z).$$

Since $\lambda < r_{m*+1} = \max_{C_i^{m^*}(z) \in C^{(m^*+1)}(z)} \mathcal{E}(C_i^{(m^*)})$, there exists at least one cycle $C_i^{(m^*)}$, where $i \in \{1, \ldots, p\}$, for which $\mathcal{E}(C_i^{(m^*)}) > \lambda$. Let i^* be the minimum of those indices i; $C_{i^*}^{(m^*)}(z)$ is the first cycle after $C^{(m^*)}(z)$ for which the exit rate exceeds λ.

If $\lambda > r(C_{i^*}^{(m^*)}(z))$, then the metastable state $K_{(z,\lambda)}$ is the main state of $C_{i^*}^{(m^*)}(z)$. If $\lambda < r(C_{i^*}^{(m^*)}(z))$, then, of the $(m^* - 1)$-rank cycles that comprise the m^*-cycle $C_{i^*}^{(m^*)}(z)$, there exists one $(m^* - 1)$-cycle, denoted $\tilde{C}^{(m^*-1)}(z)$, containing the entrance state for $C_{i^*}^{(m^*)}(z)$. Cyclically ordering the $(m^* - 1)$ cycles that follow $\tilde{C}^{(m^*-1)}(z)$ in $C_{i^*}^{(m^*)}(z)$, there exists a first cycle in the sequence, say $C'^{(m^*-1)}(z)$, whose exit rate exceeds λ. If $\lambda > R(C'(m^* - 1)(z))$, the the metastable state $K_{(z,\lambda)}$ is the main state of $C'(m^* - 1)(z)$. If not, proceed inductively to consider the collection of $(m^* - 2)$-order cycles which comprise $C'(m^* - 1)(z)$, until reaching a cycle C of some nonnegative order for which $\mathcal{E}(C) > \lambda > R(C)$. Such a cycle always exists, because the rotation rates of zero-cycles are, by definition, zero. The metastable state $K_{(z,\lambda)}$ is the main state of the cycle C.

Proof. See [18], §6. □

3.1.5 An example of a metastable state as a probability distribution

For a generic dynamical system subject to white-noise perturbations, a metastable state is a fixed equilibrium point. In the next section, we generalize metastability to a nearly-Hamiltonian system in which the underlying deterministic system is generic, but the limiting dynamical system is stochastic. We conclude that in such a setting, nondegenerate probability distributions across equilibrium points, rather than equilibrium points themselves, serve as metastable states.

In this section, we consider a non-generic dynamical system—specifically, a one-dimensional potential with two wells of identical depth—in which metastability nevertheless corresponds to a probability distribution. This provides a simple illustration of this phenomenon.

Example 3.1.4 (A one-dimensional example of a metastable "state" as a probability distribution)**.**

Let X_t^ϵ be the diffusion process in $\mathbb{R}$

$$\dot{X}_t^\epsilon = -U'(X_t^\epsilon) + \sqrt{\epsilon}\dot{W}_t \tag{3.32}$$

where the potential $U(x)$ has two wells. Let $S_{0T}(\phi)$ and $V(x, y)$ be the action functional and quasipotential, respectively, associated to the process X_t^ϵ. Suppose the two local minima of U are equal, so $U(a_1) = U(a_2)$. Assume that $U(x) > U(a_i)$ for all other x, that $U'(x) \neq 0$ for $x \notin \{a_1, a_2\}$, and that a_1 and a_2 are nondegenerate critical points with $U''(a_1) \neq U''(a_2)$. Finally, assume that

$$\int_{\mathbb{R}} \exp\left[-\frac{U(x)}{\epsilon}\right] dx = C^\epsilon < \infty \tag{3.33}$$

Claim 3.1.5. *Let X_t^ϵ solve* (3.32) *and let U satisfy the above assumptions. Then the process X_t^ϵ has a unique normalized invariant measure μ^ϵ given by*

$$\mu^\epsilon(A) = \frac{1}{C^\epsilon} \int_A \exp\left[-\frac{2U(x)}{\epsilon}\right] dx \tag{3.34}$$

and μ^ϵ converges weakly as $\epsilon \to 0$ to the measure $\mu = \frac{p_1}{p_1+p_2}\delta_{a_1} + \frac{p_2}{p_1+p_2}\delta_{a_2}$ with $p_i = \frac{1}{\sqrt{U''(a_i)}}$ and δ_a the δ-measure at the point a. Furthermore, if $\lambda > V_{12} = V_{21}$, the metastable distribution for any initial condition x_0 with timescale λ is given by this limiting measure.

Proof. A substitution into the forward Kolmogorov equation immediately gives the expression for the density $m(x)$ of the unique invariant measure:

$$L^* m(x) = 0 \Rightarrow \frac{d}{dx}(m(x)U'(x)) + \frac{\epsilon}{2}\frac{d^2 m(x)}{(dx)^2} = 0 \tag{3.35}$$

Define $A_i(\delta) = (a_i - \delta, a_i + \delta)$, $i = 1, 2$, and put $A_3(\delta) = [A_1(\delta) \cup A_2(\delta)]^c$. Put $f(x) = 2U(x)$, $\lambda = 2U(a_1) = 2U(a_2)$ and define $m_i^\epsilon(\delta)$ as

$$m_i^\epsilon(\delta) = \int_{A_i(\delta)} \exp\left[-\frac{f(x)}{\epsilon}\right] dx, \quad i = 1, 2, 3 \tag{3.36}$$

We prove that for $\delta > 0$ sufficiently small,

$$\frac{m_i^\epsilon(\delta)\exp\left[\frac{\lambda}{\epsilon}\right]}{\sqrt{\epsilon}\sqrt{2\pi}} \to p_i, \quad i = 1, 2 \tag{3.37}$$

$$\frac{m_3^\epsilon(\delta)\exp\left[\frac{\lambda}{\epsilon}\right]}{\sqrt{\epsilon}\sqrt{2\pi}} \to 0 \tag{3.38}$$

as $\epsilon \to 0$. The result follows from this and from the fact that for δ sufficiently small,

$$\mu^\epsilon(A_i(\delta)) = \frac{m_i^\epsilon(\delta)}{m_1^\epsilon(\delta) + m_2^\epsilon(\delta) + m_3^\epsilon(\delta)} \tag{3.39}$$

$$= \frac{\frac{m_i^\epsilon(\delta)\exp(\frac{\lambda}{\epsilon})}{\sqrt{\epsilon}\sqrt{2\pi}}}{[m_1^\epsilon(\delta) + m_2^\epsilon(\delta) + m_3^\epsilon(\delta)]\frac{\exp(\frac{\lambda}{\epsilon})}{\sqrt{\epsilon}\sqrt{2\pi}}} \tag{3.40}$$

Note that

$$\frac{m_3^\epsilon(\delta)}{m_1^\epsilon(\delta)} = \frac{\int_{A_3} \exp\left[-\frac{2U(x)}{\epsilon}\right] dx}{\int_{A_1} \exp\left[-\frac{2U(x)}{\epsilon}\right] dx} \tag{3.41}$$

(3.42)

Put $\lambda = \inf_{x\in\mathbb{R}} f(x) = f(a_1) = f(a_2)$. Let $\lambda_3(\delta) = \inf\{f(x) : x \in A_3(\delta)\}$. By hypothesis, $\lambda_3 > \lambda$. Expanding f in a Taylor series around each of the point a_1, we get

$$f(x) - \lambda = f'(a_1)(x - a_1) + f''(a_1)(x - a_1)^2(1 + h((x - a_1)) \tag{3.43}$$

where $h((x - a_1)) < C_1\delta_1^2$ if $|x - a_1| < \delta_1$. We have

$$m_1^\epsilon(\delta) \exp\left[\frac{\lambda}{\epsilon}\right] = \int_{a_1-\delta}^{a_1+\delta} \exp -\left[\frac{f(x) - \lambda}{\epsilon}\right] dx \tag{3.44}$$

$$= \int_{a_1-\delta}^{a_1+\delta} \exp\left[-\frac{f''(a_1)(x - a_1)^2(1 + h(x - a_1))}{2\epsilon}\right] dx \tag{3.45}$$

By the change of variable

$$u = \frac{\sqrt{\epsilon}}{\sqrt{f''(a_1)}}(x - a_1) \tag{3.46}$$

we deduce that

$$m_1^\epsilon(\delta) \exp\left[\frac{\lambda}{\epsilon}\right] = \int_{-\frac{\delta\sqrt{f''(a_1)}}{\sqrt{\epsilon}}}^{\frac{\delta\sqrt{f''(a_1)}}{\sqrt{\epsilon}}} \exp\left[-\frac{u^2}{2}(1 + h(\sqrt{\frac{\epsilon}{f''(a_1)}}u)\right] du \tag{3.47}$$

By the dominated convergence theorem,

$$\lim_{\epsilon\downarrow 0} \frac{m_1^\epsilon(\delta) \exp\left[\frac{\lambda}{\epsilon}\right]}{\sqrt{\epsilon}\sqrt{2\pi}} = \frac{1}{\sqrt{2\pi f''(a_1)}} \int_{-\infty}^{\infty} \exp\left[\frac{-u^2}{2}\right] du = \frac{1}{\sqrt{f''(a_1)}} \tag{3.48}$$

The identical result holds for a_2. Now consider

$$m_3^\epsilon(\delta) = \int_{A_3(\delta)} \exp\left[-\frac{f(x)}{\epsilon}\right] dx \tag{3.49}$$

$$= \exp\left[-\frac{\lambda_3(\delta)}{\epsilon}\right] \int_{A_3(\delta)} \exp\left[-\frac{f(x) - \lambda_3}{\epsilon}\right] \tag{3.50}$$

$$\Rightarrow \frac{m_3^\epsilon(\delta) \exp\left[\frac{\lambda}{\epsilon}\right]}{\sqrt{\epsilon}} = \left[\frac{\exp\left[-\frac{\lambda_3(\delta)-\lambda}{\epsilon}\right]}{\sqrt{\epsilon}}\right] \int_{A_3(\delta)} \exp\left[-\frac{f(x) - \lambda_3(\delta)}{\epsilon}\right] dx \tag{3.51}$$

and for $\epsilon < \epsilon_0$, we have

$$\int_{A_3(\delta)} \exp\left[-\frac{f(x) - \lambda_3(\delta)}{\epsilon}\right] dx \leq \int_{\mathbb{R}} \exp\left[-\frac{f(x) - \lambda_3(\delta)}{\epsilon_0}\right] dx = C < \infty \tag{3.52}$$

and since $\lambda_3(\delta) - \lambda > 0$, L'Hôpital's rule implies

$$\frac{1}{\sqrt{\epsilon}} \exp\left[-\frac{\lambda_3(\delta) - \lambda}{\epsilon}\right] \to 0 \tag{3.53}$$

as $\epsilon \downarrow 0$.

Let V_{12} and V_{21} be the quasipotential between the two minima of U; since the two minima of U are equal, by Theorem (3.1.3), $V_{12} = V_{21}$. Now, for any initial position x_0 belonging to the domain of attraction for a_1, and for a timescale $\lambda < V_{12}$, Proposition (1) ensures that the metastable state is simply a_1; similarly, for any initial position x_0 belonging to the domain of attraction for a_2 and timescale $\lambda < V_{21}$, the metastable state is a_2. However, for any position x_0 and timescale $\lambda > V_{12}$, λ is greater than the rotation rate for the maximal-rank cycle (a cycle of rank 1). Due to the non-genericity of the system, however, this cycle has two main states: a_1 and a_2, and from the convergence of the invariant measures μ^ϵ to μ, the metastable distribution is the limiting probability measure μ. □

3.2 Results on metastable distributions for nearly-Hamiltonian systems

While the previous example illustrates a metastable distribution, the potential U, with two identical minima, is non-generic. Furthermore, the property of identical minima is not preserved under small perturbations of the potential. However, the nearly-Hamiltonian system we consider is both generic and stable under small perturbations, and we prove that in this case probability distributions serve as metastable states.

Recall that $\tilde{X}^{\epsilon,\kappa}(t)$ is the two-dimensional diffusion process with generator

$$\mathcal{L}^{\epsilon,\kappa}(u(x)) = \frac{\kappa}{2}\text{div}(a(x)\nabla u(x)) + B(x)\cdot\nabla u(x) + \frac{1}{\epsilon}\overline{\nabla}H(x)\cdot\nabla u(x),$$

where H is the four-well Hamiltonian with associated graph and phase portrait given in Chapters 1 and 2 and reproduced below.

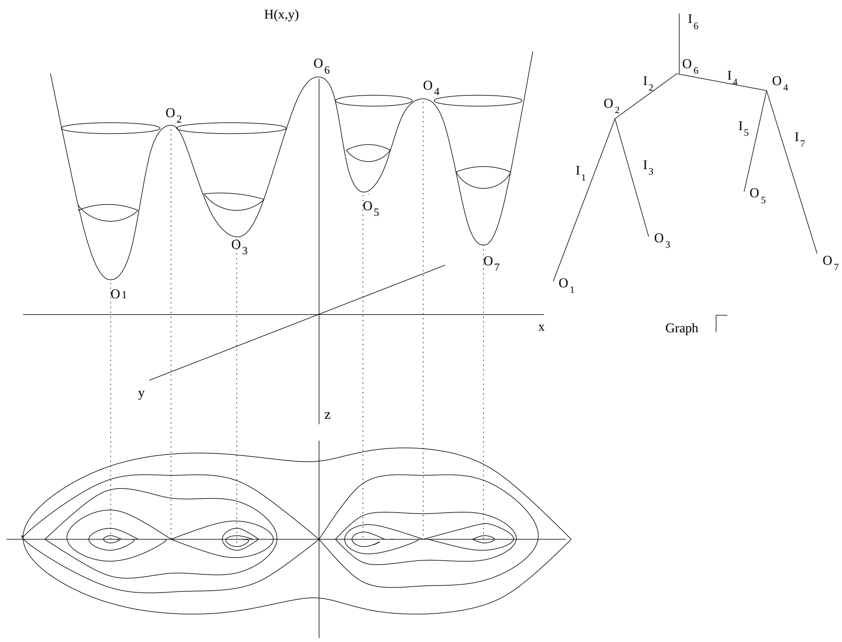

Figure 3.1: $H(x_1, x_2)$ and the Graph Γ

Our goal is to establish the following theorem.

Theorem 3.2.1. *Let $\lambda > 0$ and $T(\kappa)$ be such that*

$$\lim_{\kappa\downarrow 0}\kappa\ln T(\kappa) = \lambda \tag{3.54}$$

For any initial condition $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and all but finitely many timescales λ, the process $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$ converges weakly in the space $C_{0T}(\mathbb{R}^2)$, first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$, to a probability measure concentrated on the stable equilibrium points of the unperturbed Hamiltonian system. In particular, there exist initial conditions

$w = (x_1(0), x_2(0)) \in \mathbb{R}^2$ and timescales λ such that process $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$, converges weakly in the space $C_{0T}(\mathbb{R}^2)$, first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$, to a nondegenerate probability measure $\mu_{w,\lambda}$ concentrated on the stable equilibrium points $\{O_1, O_3, O_5, O_7\}$ of the unperturbed Hamiltonian system, with weights $p_i(w,\lambda) = \mu_{w,\lambda}(O_i), i \in \{1,3,5,7\}$ that can be explicitly computed and depend only on B.

To prove this theorem, we rely upon the results in [3], [18], and [19], and certain large-deviation estimates which we prove below.

As before, let Q be the projection of a point $x = (x_1, x_2) \in \mathbb{R}^2$ onto the graph Γ associated to the Hamiltonian H. We will use the notation O_k to represent both a zero of ∇H in $\mathbb{R}^2$ (a stable equilibrium or saddle point of the unperturbed Hamiltonian system $\dot{X}_t = \overline{\nabla} H(X_t)$ in the plane) and the corresponding interior or exterior vertex on the graph Γ associated to the Hamiltonian. That is, we use the notation O_k to represent both the point O_k in the plane and the point $H(O_k)$ on the graph Γ. It will be clear from context to which we refer.

Recall that the process $Q^\kappa(t)$ on the graph Γ is the weak limit in $C_{0T}(\Gamma)$ of $Q^{\epsilon,\kappa}(t)$ as $\epsilon \downarrow 0$, and Q^κ is defined through generators L^κ_i along each edge and gluing conditions at each interior vertex (see (2.187), (2.188), and (2.189) in Chapter 2).

3.2.1 Estimates for probabilities of large deviations for the process Q^κ on Γ

We estimate probabilities of large deviations for the process Q^κ on the graph Γ. In particular, we determine the logarithmic asympotics of the exit time $\tau^\kappa_i(y)$ for the process Q^κ to leave an exterior edge I_{k_i} starting from some point y in a small neighborhood of the exterior vertex O_{k_i}. Recall from Chapter 2 that Q^κ is defined through second-order ordinary differential operators along each edge. Because these operators have degeneracies at interior and exterior vertices, we analyze the behavior of Q^κ in three parts: first, in closed subintervals of an edge (i.e., away from the vertices); next, in small neighborhoods of exterior vertices; and finally, in small neighborhoods of interior vertices.

Along each edge i, $Q^\kappa(t)$ is a process with infinitesimal generator L_i^κ:

$$L_i^\kappa(u_i(z)) = \left\{ \frac{\kappa A_i'(z)}{2T_i(z)} + \frac{\tilde{B}_i(z)}{T_i(z)} \right\} \frac{du_i(z)}{dz} + \frac{\kappa A_i(z)}{2T_i(z)} \frac{d^2 u_i(z)}{(dz)^2}$$

The drift and diffusion coefficients vanish only at interior and exterior vertices. As $z \to H(O_{j_i})$ for an interior vertex O_{j_i}, $\tilde{B}_i(z)$ and $A_i(z)$ have finite non-zero limits, and from Lemma (2.2.1), $T_i(z) \sim \ln(z - H(O_{j_i}))^{-1}$. By Lemma (2.2.1), as $z \to H(O_k)$ for a minimum O_k of H in the plane (i.e. an exterior vertex of Γ), $T_k(z)$ approaches a constant. At exterior vertices, it is easy to see that $\tilde{B}_k(z)$ and $A_k(z)$ converge to zero linearly: for any minimum O_k of H, the curve $C(H(O_k))$ consists of a single point, so the area of the enclosed region $G(H(O_k))$ is zero. However,

$$A_k(z) = \int_{G_k(z)} \operatorname{div}(a(x)\nabla H(x))dx \tag{3.55}$$

$$\text{and} \quad \tilde{B}_k(z) = \int_{G_k(z)} \operatorname{div} B(x)dx \tag{3.56}$$

and the integrands in each of the above integrals are bounded away from zero. Let $S_k(z) = \text{Area } (G_k(z))$. Then since $S_k'(z) = T_k(z)$ and $T_k(z)$ approaches a constant as $z \to H(O_k)$, the conclusion follows.

Let $E = \{O_{k_1}, \dots O_{k_l}\}$ be the set of exterior vertices.

Lemma 3.2.2. *Any exterior vertex O_{k_i} is inaccessible for $Q^\kappa(t)$.*

Proof. Following Mandl [23], the criterion for inaccessibility of an endpoint for the one-dimensional diffusion process with generator $L^\kappa_{k_i}$ on an exterior edge I_{k_i} is that the integral

$$\int \exp\left[-\int\left\{\frac{\frac{\kappa A'_{k_i}(z)}{2T_{k_i}(z)}+\frac{\tilde{B}_{k_i}(z)}{T_{k_i}(z)}}{\frac{\kappa A_{k_i}(z)}{2T_{k_i}(z)}}\right\}dz\right]dz \tag{3.57}$$

diverge at the exterior vertex O_i; note that

$$\int \exp\left[\int -\frac{A'_{k_i}(z)}{A_{k_i}(z)}dz\right]dz = \int \frac{1}{A_{k_i}(z)}dz \tag{3.58}$$

which diverges at any exterior vertex because $A_{k_i}(z)$ converges to zero linearly as $z \downarrow H(O_{k_i})$, so the singularity at O_{k_i} is not integrable. □

Let I_{k_i} be an exterior edge of Γ with exterior vertex O_{k_i} and corresponding interior vertex O_{j_i}. Without loss of generality we can take $H(O_k) = 0$. In the analysis that follows, we focus on the interval I_{k_i} and the associated differential operator L^κ_i defined on I_{k_i}. Let δ_1 be arbitrary and positive. For any such δ_1, denote by $I_{k_i} \setminus \delta_1$ the subinterval of I_{k_i}

$$I_{k_i} \setminus \delta_1 = \{z \in I_{k_i} : \delta_1 < z < H(O_{j_i}) - \delta_1\} \tag{3.59}$$

so $I_{k_i} \setminus \delta_1$ is the open subinterval of I_{k_i} with δ_1 neighborhoods of each vertex removed.

Using an approximation to the identity (see [7]), we can construct nonzero smooth, bounded functions $A^F_{k_i}(z)$ and $T^F_{k_i}(z)$ defined on the closed interval $\overline{I}_{k_i}$ (including the endpoints O_{j_i} and O_{k_i}) such that $A^F_{k_i}(z)$, $A'^F_{k_i}$, $T^F_{k_i}(z)$ coincide with the functions $A_{k_i}(z)$, $A'_{k_i}(z)$, and $T_{k_i}(z)$ on $I_{k_i} \setminus [\delta_1/4]$.

Let $F^{\kappa}_{k_i}(t)$ be the one-dimensional diffusion process on the positive half-line with generator

$$L^{F,\kappa}_{k_i}(u) = \frac{\kappa}{2} A^F_{k_i}(z) \frac{d^2u}{dz^2} + \left\{ \frac{\kappa}{2} \frac{A'^F_{k_i}(z)}{T^F_{k_i}(z)} + \frac{\tilde{B}_{k_i}(z)}{T^F_{k_i}(z)} \right\} \frac{du}{dz} \tag{3.60}$$

and reflection at the origin.

The drift and diffusion coefficients for $F^{\kappa}_{k_i}$ coincide with the drift and diffusion coefficients for the process $Q^{\kappa}_{k_i}$ on the subinterval $I_{k_i} \setminus [\delta_1/4]$. However, the process $F^{\kappa}_{k_i}$ has a diffusion coefficient that is uniformly nondegenerate on I_{k_i}.

Lemma 3.2.3. *The action functional for the process $F^{\kappa}_{k_i}(t)$ in the space $C_{0T}(I_{k_i})$ is given by:*

$$S^F_{k_i}(\phi) = \frac{1}{2} \int_0^T \left[\dot{\phi}(s) - \frac{\tilde{B}_{k_i}(\phi(s))}{T^F_{k_i}(\phi(s))} \right]^2 \frac{T^F_{k_i}(\phi(s))}{A^F_{k_i}(\phi(s))} ds \tag{3.61}$$

Proof. Let $\tilde{F}^{\kappa}_i$ be the diffusion process with generator

$$\tilde{L}^{F,\kappa}_{k_i}(u(z)) = \frac{\tilde{B}_i(z)}{T^F_{k_i}(z)} \frac{du}{dz} + \frac{A^F_{k_i}(z)}{T^F_{k_i}(z)} \frac{d^2u}{dz^2}$$

Since the diffusion coefficient $\frac{A^F_{k_i}(z)}{T^F_{k_i}(z)}$ is uniformly nondegenerate, Theorem (3.1.2) implies that the the action functional $S^{\tilde{F}_{k_i}}_{0T}$ for the process $\tilde{F}^{\kappa}_{k_i}$ is given by (3.61).

The diffusion process $F^{\kappa}_{k_i}$ defined by $L^{F,\kappa}_{k_i}$ differs from the diffusion process $\tilde{F}^{\kappa}_{k_i}$ defined by $\tilde{L}^{F,\kappa}_{k_i}$ only in the drift, by a term of order κ. The measures induced by the two processes on $C_{0T}(I_{k_i})$ are absolutely continuous with respect to one another; by Girsanov's formula the Radon-Nikodym derivative $\frac{d\mu_{F^{\kappa}_{k_i}}}{d\mu_{\tilde{F}^{\kappa}_{k_i}}}$ is given by

$$\exp\left[\frac{\sqrt{\kappa}}{\sqrt{2}} \left(\int_0^T f(\tilde{F}^{\kappa}_{k_i}(s)) dW_s - \frac{1}{2} \int_0^t |f(\tilde{F}^{\kappa}_{k_i}(s))|^2 ds \right) \right] \tag{3.62}$$

where $f(z) = \frac{A'^F_{k_i}(z)\sqrt{T^F_{k_i}(z)}}{\sqrt{A^F_{k_i}(z)}}$. The Radon-Nikodym derivative can be made arbitrarily close to one for κ sufficiently small, and hence the action functionals for the two processes coincide. □

Lemma 3.2.4. *The quasipotential $V_i^F(z,w)$ between any two points z and w on I_i is given by*

$$V_i^F(z,w) = \int_z^w -2\frac{\tilde{B}_i(s)}{A_i^F(s)}ds \tag{3.63}$$

Proof. Since the quasipotential is defined as

$$\inf\{S^F_{T_1T_2}(\phi), \phi_{T_1} = x, \phi_{T_2} = y\} \tag{3.64}$$

where the infimimum is taken over time intervals of arbitrary length, the quasipotential for a process is invariant with respect to time changes, so we employ the random time-change formula to rewrite the diffusion process F_i^κ. Let X_t be the diffusion process defined by

$$X_t = \int_0^t \frac{\tilde{B}_i(X_s)}{T_i^F(X_s)}ds + \int_0^t \frac{\sqrt{A_i^F(X_s)}}{\sqrt{T_i^F(X_s)}}dW_s \tag{3.65}$$

Define α_t as

$$\alpha_t = \int_0^t \frac{A_i^F(X_s)}{T_i^F(X_s)}ds \tag{3.66}$$

and note that α_t is strictly increasing for almost all ω because the integrand is positive. Hence α_t is invertible, and by the random time change formula there exists a Wiener process $\tilde{W}$ such that X can be written:

$$X_{\alpha^{-1}(u)} - X_0 = \int_0^u \frac{\tilde{B}_i(X_{\alpha^{-1}(s)})}{A_i^F(X_{\alpha^{-1}(s)})}ds + \int_0^u d\tilde{W}_s \tag{3.67}$$

If we put $\tilde{X}_u = X_{\alpha^{-1}(u)}$, then $\tilde{X}$ has unit diffusion, and according to Theorem (3.1.3), the quasipotential is given by

$$V_i^F(z,w) = \int_z^w -2\frac{\tilde{B}_i(s)}{A_i^F(s)}ds \tag{3.68}$$

□

Note that $A_{k_i}^F(z) = A_{k_i}(z)$ on $I_{k_i} \setminus [\delta_1/4]$; for $z, w \in I_{k_i} \setminus [\delta_1/4]$,

$$V_i^F(z,w) = \int_z^w -2\frac{\tilde{B}_{k_i}(s)}{A_{k_i}(s)}ds \tag{3.69}$$

The integrand $\frac{\tilde{B}_{k_i}(s)}{A_{k_i}(s)}$ has a singularity at the exterior vertex O_{k_i}, but because both $\tilde{B}_{k_i}$ and A_{k_i} converge to zero linearly, this singularity is removable, and the integrand is in fact bounded. Thus we define the function $\bar{V} : \Gamma \times \Gamma \to \mathbb{R}$ as follows.

Definition 9. Let (z,i) and (z',i') be two points on Γ.

- If $i = i'$ (i.e. the two points lie on the same edge), then

$$\bar{V}((z,i),(w,i)) = \int_z^w -2\frac{\tilde{B}_i(s)}{A_i(s)}ds \tag{3.70}$$

- If $i \neq i'$, there exists a shortest path from (z,i) to (z',i') which intersects any interior vertex at most once; denote this path by

$$(z,i) \to O_{j_1} \to O_{j_2} \to \ldots \to O_{j_M} \to (z',i')$$

  The quasipotential is equal to

$$\bar{V}((z,i),(z,i')) = \bar{V}((z,i),O_{j_1}) + \bar{V}(O_{j_1},O_{j_2}) + \ldots + \bar{V}(O_{j_M},(z',i')) \tag{3.71}$$

For any two points (z, w) belonging to the same edge, it is clear that $\bar{V}(z, w)$ is Lipschitz continuous.

According to our genericity assumption, if $\bar{V}(O_i, O_j) \neq 0$ and $\bar{V}(O_k, O_m) \neq 0$, then $\bar{V}(O_i, O_j) \neq \bar{V}(O_k, O_m)$ for any two different pairs of vertices.

Next, define the functional $S_{0T}^i(\phi)$ for functions $\phi \in C_{0T}(I_i)$ as follows:

$$S_{0T}^i(\phi) = \frac{1}{2} \int_0^T \left[\dot{\phi}(s) - \frac{\tilde{B}_i(\phi(s))}{T_i(\phi(s))} \right]^2 \frac{T_i(\phi(s))}{A_i(\phi(s))} ds \tag{3.72}$$

Since $A_{k_i}(H(O_{k_i})) = 0$, and $A_{k_i}(z)$ converges to zero linearly at O_{k_i}, the functional $S_{0T}^i(\phi)$ is finite for absolutely functions $\phi \in C_{0T}(I_{k_i})$ such that $\phi_t \neq O_{k_i}, t \in [0, T]$. For functions ϕ_t with $\phi_t = O_{k_i}$ for some $t \in [0, T]$, $S_{0T}^i(\phi)$ is finite provided $\dot{\phi}$ decays at O_{k_i} sufficiently quickly.

We prove the following theorem about exit times for the process Q^κ from a small neighborhood of an exterior vertex.

Theorem 3.2.5. *Let I_{k_i} be an exterior edge with exterior vertex O_{k_i} and interior vertex O_j in Γ. Suppose the three edges I_{k_1}, I_{k_2}, and I_j meet at interior vertex O_j. Let $\tau_z^\kappa = \inf\{t > 0 : Q^\kappa(t) = z\}$. Put $\bar{V}_{ij}^{\max} = \max\{\bar{V}(O_{k_i}, O_j), i = 1, 2\}$. For any $\alpha > 0$ there exists $\delta > 0$ sufficiently small such that if $y \in I_{k_i}$, $|y - H(O_{k_i})| < \delta$, $y \neq O_{k_i}$, and $z \in I_j$, $|z - H(O_j)| < \delta$, $z \neq O_j$, then*

$$\lim_{\kappa \downarrow 0} P_y \left\{ \exp \left[\frac{\bar{V}_{ij}^{\max} - \alpha}{\kappa} \right] < \tau_z^\kappa < \exp \left[\frac{\bar{V}_{ij}^{\max} + \alpha}{\kappa} \right] \right\} = 1 \tag{3.73}$$

Proof. From our genericity assumption, $\bar{V}_{ij}^{\max}$ is achieved for only one edge $i = i_{\max}$. Denote the other edge by $i_{\min}$. We first consider the case when $i = i_{\max}$ and $y \in I_{i_{\max}}$.

To prove the lower bound in (3.73), note that by the continuity of $\bar{V}$ in both arguments, for any $\alpha > 0$ there exists a $\delta > 0$ and a modified diffusion process F_t^κ

on the interval $I = I_{k_i} \cup [O_j, O_j + 2\delta]$ such that (a) F^κ_t coincides with Q^κ_t on $I_{k_i} \setminus \frac{\delta}{4}$; (b) F^κ_t has drift B^F_i that coincides with the drift for Q^κ_t; (c) F^κ_t has nondegenerate diffusion $A^F_i((z,i)) \geq \frac{A_i(z)}{T_i(z)}$; and (d) For $(z,j) : |z - O_j| < \delta$, $|V^F(O_{k_i}, z) - \bar{V}^{\max}_{ij}| < \frac{\alpha}{2}$.

Let $\tau^\kappa_z = \inf\{t > 0 : Q^\kappa(t)\} = z$ where (z, O_j) is a point on the interior edge with $|z - H(O_j)| < \delta$, $z \neq H(O_j)$. Similarly, let $\tau^{F,\kappa}_z = \inf\{t > 0 : F^\kappa_t = z\}$. Theorem 4.1.2 in [18] it is proved that for any $\alpha > 0$,

$$\liminf_{\kappa \downarrow 0} P_y \left\{ \tau^{F,\kappa}_z < \exp\left[\frac{V^F(O_{k_i}, z) - \frac{\alpha}{2}}{\kappa} \right] \right\} = 0 \tag{3.74}$$

Since F^κ has the identical drift as Q^κ_t but uniformly greater positive diffusion, we get

$$P_y \left\{ \tau^{F,\kappa}_z < \exp\left[\frac{V^F(O_{k_i}, z) - \frac{\alpha}{2}}{\kappa} \right] \right\} \tag{3.75}$$

$$\geq P_y \left\{ \tau^{\kappa}_z < \exp\left[\frac{V^F(O_{k_i}, z) - \frac{\alpha}{2}}{\kappa} \right] \right\} \tag{3.76}$$

$$\geq P_y \left\{ \tau^{\kappa}_z < \exp\left[\frac{\bar{V}^{\max}_{ij} - \alpha}{\kappa} \right] \right\} \tag{3.77}$$

The lower bound in (3.73) now follows from this and (3.74). It remains to prove the upper bound.

Since the diffusion coefficient $A_{k_i}(O_{k_i}) = 0$ and $\frac{1}{A_{k_i(z)}}$ is not integrable near the exterior vertex O_{k_i}, we first consider the behavior of the process in a small neighborhood of O_{k_i}.

Let $\delta > 0$ be positive and fixed, and sufficiently small that $I_{k_i} \setminus \delta$ contains a nontrivial interval. Without loss of generality we can consider I_{k_i} to be a bounded interval with one endpoint at the origin; the origin corresponds to the exterior vertex O_{k_i}.

Let $x \in (0, \delta)$ be some point in the δ-neighborhood of the origin, and put $\tau^{\kappa}_{x,\delta,k_i} = \inf\{t > 0 : Q^{\kappa}(t) = \delta, Q^{\kappa}(0) = (x, k_i)\}$. That is, $\tau^{\kappa}_{x,\delta,k_i}$ is the first time the process Q^{κ} hits δ after starting at some point $x < \delta$ on edge k_i.

It is well-known (see [10]) that the solution to certain Dirichlet problems can be expressed through the expected values of functionals of associated diffusion processes; conversely, the expected values of exit times from bounded domains for diffusion processes can be expressed through the solutions of corresponding Dirichlet problems. In general, for a domain $D \in \mathbb{R}^n$ with smooth boundary, consider the Dirichlet problem

$$L(u(x)) - c(x)u(x) = f(x); \tag{3.78}$$

$$\lim_{x \in D, x \to y \in \partial D} u(x) = \psi(y); \tag{3.79}$$

where L is the second-order differential operator

$$L = \frac{1}{2} \sum_{i,j} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_i b_i(x) \frac{\partial u}{\partial x_i} \tag{3.80}$$

and the coefficients are smooth, bounded, and (a_{ij}) is positive definite; ψ is bounded and continuous.

Let X_t be the diffusion process corresponding to the operator L, and let $\tau_D = \inf\{t > 0 : X_t \in \partial D, X_0 = x\}$. In [10], §2, it is proved that the solution u to (3.78) is given by:

$$u(x) = - E_x \left[\int_0^{\tau_D} f(X_t) \exp\left\{ - \int_0^t c(X_s) ds \right\} dt \right] \tag{3.81}$$

$$+ E_x \left[\psi(X_{\tau_D}) \exp\left\{ - \int_0^{\tau_D} c(X_s) ds \right\} \right] \tag{3.82}$$

In particular, if we take $c(x) = 0$ and $f(x) = -1$, and $\psi(y) = 0$ for $y \in \partial D$, then the solution $u(x)$ is given by $u(x) = E[\tau_D]$.

To determine $E[\tau^{\kappa}_{x,\delta,k_i}]$, we consider the one-dimensional operator

$$L^{\kappa}_{k_i}u(z) = \left\{ \frac{\kappa A'_{k_i}(z)}{2T_{k_i}(z)} + \frac{\tilde{B}_{k_i}(z)}{T_{k_i}(z)} \right\} \frac{du_(z)}{dz} + \frac{\kappa A_{k_i}(z)}{2T_{k_i}(z)} \frac{d^2u(z)}{(dz)^2} \tag{3.83}$$

and the associated Dirichlet problem

$$L^{\kappa}_{k_i}u = -1, \quad u(\delta) = 0 \tag{3.84}$$

with u defined for $x \in (0, \delta)$. The operator $L^{\kappa}_{k_i}$ has a degeneracy at the point $H(O_{k_i})$; on any closed subinterval of I_k it is positive definite. It is proved in [10] that the minimal positive solution of (3.84) is $u(x) = E[\tau^{\kappa}_{x,\delta,k_i}]$.

For δ sufficiently small, the coefficients $\tilde{B}_{k_i}(z)$ and $A_{k_i}(z)$ are approximately linear, and $\tilde{B}_{k_i}(z) < 0$ and $A'_{k_i}(z) > 0$ for $z \in (0, \delta]$. Also $T_{k_i}(z)$ is nonzero and approximately constant for z near $H(O_{k_i})$, so we consider the following linearized second-order ordinary differential equation:

$$\kappa z u'' + \kappa u' + bzu' = -1 \tag{3.85}$$

Since div$B < 0$ and the diffusion matrix $a(x)$ is uniformly positive definite, b must be strictly negative. Putting $v = zu'$ and integrating, we obtain

$$v = \frac{C_1}{b} \exp\left\{ \frac{-b}{\kappa} z \right\} - \frac{1}{b} \tag{3.86}$$

We require u to be the minimal positive solution to (3.85). There exists a unique bounded solution u of (3.85) which satisfies $u(\delta) = 0$, and this solution u is minimal. Since $u' = \frac{v}{z}$, the constant of integration C_1 is chosen so that $v(0) = 0$. This implies $C = 1$.

We have

$$u(\delta) - u(x) = \int_x^\delta u'(s)ds \tag{3.87}$$

$$\Rightarrow u(x) = \int_x^\delta \left[\frac{-1}{bs}\left\{\exp\left(-\frac{b}{\kappa}s\right) - 1\right\}\right] ds \tag{3.88}$$

For δ small, $u(x)$ is approximately

$$u(x) = \frac{-1}{bx}\left[\exp\left(-\frac{b}{\kappa}(x)\right) - 1\right](\delta - x) \tag{3.89}$$

Hence we conclude

$$\limsup_{\kappa\downarrow 0} \frac{E[\tau^\kappa_{x,\delta,k_i}]}{\left[\exp\left(\frac{d_1}{\kappa}\right)\right]} < C \tag{3.90}$$

where $d_1 > 0$ and $C > 0$ are constants and d_1 can be made arbitrarily small for δ small.

By Chebyshev's inequality, we deduce that

$$P\left\{\tau^\kappa_{x,\delta,k_i} > \exp\left[\frac{3d_1}{\kappa}\right]\right\} < \exp\left[\frac{-d_1}{\kappa}\right] \tag{3.91}$$

for κ sufficiently small. Of course, this bound holds in the neighborhood $|z - H(O_{k_i})| < \delta$ if $H(O_{k_i}) \neq 0$. Furthermore, by the maximum principle, this bound holds for variable coefficients (in our case, $\frac{\tilde{B}_{k_i}(z)}{T_{k_i}(z)}$ and $\frac{A_{k_i}(z)}{T_{k_i}(z)}$) provided the coefficients behave linearly near O_{k_i} (see [10], §3).

Hence we have an exponential bound for the exit time for Q^κ from a small neighborhood of any exterior vertex.

We next consider the behavior of the process near the interior vertex O_j, at which three edges intersect: the two edges I_{k_1} and I_{k_2}, which are exit edges for the vertex O_j, and the edge I_j, which is an entrance edge (see diagram). Recall that an edge I containing a vertex O is an *entrance edge* or an *exit edge* if the value of $H(x)$ decreases or increases, respectively, along that edge as x approaches O. Let $N(O_j, h)$ be the h-neighborhood of O_j in Γ, so $N(O_j, h) = \{(z, l) \in \Gamma : |z - H(O_j)| < h, l = k_1, k_2, j\}$. Again for ease of notation (and without loss of generality) we consider the case when $H(O_j) = 0$, and $z < 0$ for $(z, k_i), i = 1, 2$ and $z > 0$ for (z, j).

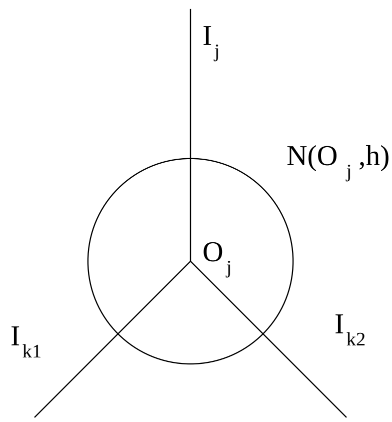

Figure 3.2: An interior vertex O_j, entrance edge I_j, and two exit edges I_{k_1}, I_{k_2}

Q_t^κ has generators $L_{k_i}^\kappa, i = 1, 2$ on each exit edge and generator L_j^κ on the entrance edge, and gluing conditions

$$\pm \sum_{m: I_m \sim O_j} \beta_{jm} D_m(O_j) = 0 \tag{3.92}$$

with

$$\beta_{jm} = \oint_{\gamma_j^m} \frac{a(x) \nabla H(x) \cdot \nabla H(x)}{|\nabla H(x)|} dl \tag{3.93}$$

where γ_j^m, $m = 1, 2$ are the two separatrices that meet at O_j, and $+$ is taken when $H(x)$ increases as x approaches O_j along a given edge (i.e. along edges k_1 and k_2) and $-$ is taken when $H(x)$ decreases as x approaches O_j along a given edge (i.e. along edge I_j).

For any point $(z,l) \in N(O_j,h)$, define τ_h^κ to be

$$\tau_h^\kappa = \inf\{t > 0 : Q^\kappa(t) \notin N(O_j,h)\} \tag{3.94}$$

Define $v_{ki}^\kappa(z,l)$ and $v_j^\kappa(z,l)$ for $i = 1,2$ and $l = k_1, k_2, j$, to be

$$v_{ki}^\kappa(z,l) = P_{(z,l)}\left\{Q^\kappa(\tau_h^\kappa) \in I_{k_i}\right\}, i = 1,2 \tag{3.95}$$

$$v_j^\kappa(z,l) = P_{(z,l)}\left\{Q^\kappa(\tau_h^\kappa) \in I_j\right\} \tag{3.96}$$

We see that $v_{ki}^\kappa(z,l)$ is the probability that, starting from (z,l), the process Q^κ exits $N(O_j,h)$ through edge k_i, and $v_j^\kappa(z,l)$ is the probability that, starting from (z,l), the process Q^κ exits $N(O_j,h)$ through edge I_j. These probabilities can be expressed as solutions to corresponding Dirichlet problems.

Specifically, $v_{ki}^\kappa(z,l)$ is the unique continuous solution of the Dirichlet problem

$$L_l(v_{ki}^\kappa(z,l)) = 0 \tag{3.97}$$

$$\text{with boundary conditions } \; v_{ki}^\kappa(z,k_i) = 1 \;\text{ if }\; (z,k_i) \in \partial N(O_j,h) \tag{3.98}$$

$$\text{and } \; v_{ki}^\kappa(z,l) = 0 \;\text{ if }\; (z,l) \in \partial N(O_j,h), l \neq k_i \tag{3.99}$$

$$\text{and } \; \pm \sum_{m:I_m \sim O_j} \beta_{jm} D_m(v_{ki}^\kappa(O_j)) = 0 \tag{3.100}$$

Note that D_m represents the derivative of v_{ki}^κ in the direction of edge I_{k_i}. Similarly, $v_j^\kappa(z,l)$ is the unique continuous solution of the Dirichlet problem

$$L_l(v_j^\kappa(z,l)) = 0 \tag{3.101}$$

$$\text{with boundary conditions } \; v_j^\kappa(z,j) = 1 \;\text{ if }\; (z,j) \in \partial N(O_j,h) \tag{3.102}$$

$$\text{and } \; v_j^\kappa(z,l) = 0 \;\text{ if }\; (z,l) \in \partial N(O_j,h), l \neq j \tag{3.103}$$

$$\text{and } \; \pm \sum_{m:I_m \sim O_j} \beta_{jm} D_m(v_j^\kappa(O_j)) = 0 \tag{3.104}$$

We wish to estimate the probability of exit through the interior edge j given that $Q^\kappa(0) = (z, i)$ where i is an exterior edge. Since the drift $\frac{\tilde{B}_l(z)}{T_l(z)}$ is negative, the process $Q^\kappa(t)$ will exit with probability close to one through edge k_1 or edge k_2. We give a lower bound on the probability of exit through the interior edge j.

We assert that for any $d_2 > 0$, we can choose $h > 0$ and κ sufficiently small that for any $(z, k_i) : |z - H(O_j)| < h$, $i = 1, 2$, and $z \neq H(O_j)$,

$$P_{(z,i)} \left\{ (Q^\kappa(\tau_h^\kappa) \in I_j \right\} > \exp\left[\frac{-d_2}{\kappa}\right] \tag{3.105}$$

The intuition behind this bound arises by analogy with the one-dimensional case. To see the parallel, consider the homogeneous problem on the interval $[0, h]$:

$$L^\kappa(w^\kappa(z)) = \frac{\kappa}{2}a(z)\frac{d^2w^\kappa(z)}{dz^2} + \frac{\kappa}{2}a'(z)\frac{dw^\kappa(z)}{dz} + b(z)\frac{dw^\kappa(z)}{dz} = 0 \tag{3.106}$$

$$w(0) = 0, \quad w(h) = 1 \tag{3.107}$$

and let Y_t^κ be the diffusion process with generator L^κ. Suppose $b(z) < 0$ and the diffusion coefficient $a(z)$ is positive definite. Then the solution $w^\kappa(z)$ to the above boundary-value problem is precisely the probability that the first exit out of $[0, h]$ for Y_t^κ occurs through h when the drift is negative (directed toward 0). In this instance, $w^\kappa(z)$ is given by

$$w^\kappa(z) = C \int_0^z \frac{1}{a(t)} \exp\left[\int_0^t \frac{-2b(s)}{\kappa a(s)} ds\right] dt \tag{3.108}$$

$$\text{where } \ C = \int_0^h \frac{1}{a(t)} \exp\left[\int_0^t \frac{-2b(s)}{\kappa a(s)} ds\right] dt \tag{3.109}$$

From the expression for C and the fact that b is negative, for any $d_2 > 0$, we can choose $h > 0$ and κ sufficiently small to guarantee that $w^\kappa(z) > \exp\left[\frac{-d_2}{\kappa}\right]$. In particular, for the case when $a(z)$ and $b(z)$ are constants, the bound is straightforward.

The situation at an interior vertex on the graph is similar, except that gluing conditions must be taken into account because of the degeneracy of the drift and diffusion coefficients at O_j. Recall that $\tilde{B}_l$ and $A_l(s)$ are both nonzero at $H(O_j)$, but $T_l(z) \to \infty$ as $z \to O_j$. However, this singularity is integrable.

First, fix the edge $r = j$. Along edge l, where $l = k_1, k_2$, or j, we have the homogeneous equation

$$L_l^\kappa(v_r^\kappa(z,l)) = 0 \tag{3.110}$$

$$\Rightarrow \frac{\tilde{B}_l(z)}{T_l(z)}\frac{dv_r^\kappa}{dz} + \frac{\kappa}{2}\frac{A_l(z)}{T_l(z)}\frac{d^2v_r^\kappa}{dz^2} + \frac{\kappa}{2}\frac{A_l'(z)}{T_l(z)}\frac{dv_r^\kappa}{dz} = 0 \tag{3.111}$$

Solving this equation for $v_r^\kappa(z,l)$ we get

$$v_r^\kappa(z,l) = C_{2,l}^{r,\kappa} + \int_{-h}^{z} \frac{C_{1,l}^{r,\kappa}}{A_l(t)}\left(\exp\left[\int_{-h}^{t}\frac{-2\tilde{B}_l(s)}{\kappa A_l(s)}ds\right]\right)dt, \quad \text{for} \quad l = k_1, k_2, r = j \tag{3.112}$$

$$v_r^\kappa(z,l) = C_{2,l}^{r,\kappa} + \int_{0}^{z} \frac{C_{1,l}^{r,\kappa}}{A_l(t)}\left(\exp\left[\int_{0}^{t}\frac{-2\tilde{B}_l(s)}{\kappa A_l(s)}ds\right]\right)dt, \quad \text{for} \quad l = j, r = j \tag{3.113}$$

(3.114)

and we determine the constants $C_{1,l}^{r,\kappa}$ from the continuity conditions and boundary and gluing conditions in (3.101).

From the boundary conditions $v_j^\kappa(-h,k_1) = v_j^\kappa(-h,k_2) = 0$, we get

$$v_j^\kappa(z,k_1) = \int_{-h}^{z} \frac{C_{1,k_1}^{j,\kappa}}{A_{k_1}(t)}\left(\exp\left[\int_{-h}^{t}\frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)}\right]ds\right)dt \tag{3.115}$$

and similarly for $v_j^\kappa(z,k_2)$.

From the boundary condition $v_j^\kappa(h,j) = 1$, we get

$$C_{2,j}^{j,\kappa} = 1 - C_{1,j}^{j,\kappa}\int_0^h \frac{1}{A_j(t)}\exp\left[\int_0^t \frac{-2\tilde{B}_j(s)}{\kappa A_j(s)}ds\right]dt \tag{3.116}$$

From the continuity condition

$$\lim_{z\to 0} v_j^\kappa(z,k_1) = \lim_{z\to 0} v_j^\kappa(z,k_2) = \lim_{z\to 0} v_j^\kappa(z,j) \tag{3.117}$$

we derive

$$C_{2,j}^{j,\kappa} = C_{1,k_1}^{j,\kappa} \int_{-h}^{0} \frac{1}{A_{k_1}(t)} \exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right] dt \tag{3.118}$$

$$-C_{1,j}^{j,\kappa} = \frac{C_{1,k_1}^{j,\kappa}\left[\int_{-h}^{0} \frac{1}{A_{k_1}(t)} \exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right] dt\right] - 1}{\int_{0}^{h} \frac{1}{A_j(t)} \exp\left[\int_{0}^{t} \frac{-2\tilde{B}_j(s)}{\kappa A_j(s)} ds\right] dt} \tag{3.119}$$

$$C_{1,k_2}^{j,\kappa} = C_{1,k_1}^{j,\kappa} \frac{\int_{-h}^{0} \frac{1}{A_{k_1}(t)} \left(\exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s) ds}\right]\right) dt}{\int_{-h}^{0} \frac{1}{A_{k_2}(t)} \left(\exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right]\right) dt} \tag{3.120}$$

$$\tag{3.121}$$

From the gluing condition

$$\beta_{jk_1} D_{k_1} v_j^\kappa(O_j) + \beta_{jk_2} D_{k_2} v_j^\kappa(O_j) - \beta_{jj} D_j v_j^\kappa(O_j) = 0 \tag{3.122}$$

and the fact that $\beta_{jm} = \lim_{(z,m)\to O_j} A_m(z) = A_m(0)$, we get

$$-C_{1,j}^{j,\kappa} + C_{1,k_1}^{j,\kappa}\left[\exp\left(\int_{-h}^{0} -\frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right)\right] + C_{1,k_2}^{j,\kappa}\left[\exp\left(\int_{-h}^{0} -\frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right)\right] = 0 \tag{3.123}$$

We derive

$$\frac{C_{1,k_1}^{j,\kappa}\left[\int_{-h}^{0} \frac{1}{A_1(t)} \exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right] dt\right] - 1}{\int_{0}^{h} \frac{1}{A_j(t)} \exp\left[\int_{0}^{t} \frac{-2\tilde{B}_j(s)}{\kappa A_j(s)} ds\right] dt} \tag{3.124}$$

$$+C_{1,k_1}^{j,\kappa}\left[\exp\left(\int_{-h}^{0} -\frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right)\right] \tag{3.125}$$

$$+C_{1,k_1}^{j,\kappa} \exp\left[\int_{-h}^{0} \frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right] \frac{\int_{-h}^{0} \frac{1}{A_{k_1}(t)} \left(\exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right]\right) dt}{\int_{-h}^{0} \frac{1}{A_{k_2}(t)} \left(\exp\left[\int_{-h}^{t} \frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right]\right) dt} = 0 \tag{3.126}$$

and this gives

$$C^{r,\kappa}_{1,k_1} = \left[\int_0^h \frac{1}{A_j(t)} \exp\left(\int_0^t \frac{-2\tilde{B}_j(s)}{\kappa A_j(s)} ds\right) dt\right]^{-1} \tag{3.127}$$

$$\times \left\{ \left[\frac{\int_{-h}^0 \frac{1}{A_1(t)} \exp\left[\int_{-h}^t \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right] dt}{\int_0^h \frac{1}{A_j(t)} \exp\left[\int_0^t \frac{-2\tilde{B}_j(s)}{\kappa A_j(s)} ds\right] dt} \right] \right. \tag{3.128}$$

$$+ \exp\left(\int_0^h \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right) \tag{3.129}$$

$$\left. + \left[\exp\left[\int_{-h}^0 \frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right] \frac{\int_{-h}^0 \frac{1}{A_{k_1}(t)} \left(\exp\left[\int_{-h}^t \frac{-2\tilde{B}_{k_1}(s)}{\kappa A_{k_1}(s)} ds\right]\right) dt}{\int_{-h}^0 \frac{1}{A_{k_2}(t)} \left(\exp\left[\int_{-h}^t \frac{-2\tilde{B}_{k_2}(s)}{\kappa A_{k_2}(s)} ds\right]\right) dt}\right] \right\}^{-1} \tag{3.130}$$

Observe that we can choose $h > 0$ sufficiently small that the functions $\tilde{B}_l(s)$ and $A_l(s)$, $l = k_1, k_2, j$, are nonzero in $N(O_j, h)$. Since we seek a lower bound for the probability of exit through edge j and the functions $\tilde{B}_j, \tilde{B}_{k_i}$ are negative, we can replace the functions $\tilde{B}_{k_1}(s)$, $\tilde{B}_{k_2}(s)$, and $\tilde{B}_j$ by a constant $\beta < 0$ such that

$$|\beta| > \max_{l=k_1,k_2,j} \sup_{I_l \cap N(O_j,h)} |\tilde{B}_l(s)| \tag{3.131}$$

and we can similarly replace the functions A_{k_i} by a positive constant α for which

$$|\alpha| < \min_{l=k_1,k_2,j} \inf_{I_l \cap N(O_j,h)} |A_l(s)| \tag{3.132}$$

and the bound in (3.105) follows from the negativity of β and the form of the solution and explicit values of the constants in (3.112).

In Lemma 2.3 of [3], it is proved that there exists h_0 and a constant $C > 0$ and $\kappa_0 > 0$ such that for any $h \in (0, h_0]$, $(z, i) \in N(O_j, h)$ and $\kappa < \kappa_0$,

$$E_{(z,i)}\left[\tau_h^\kappa\right] \le Ch|\ln h| \tag{3.133}$$

From this we immediately derive a much weaker bound: namely, for any $d_3 >$

0, we can choose h and κ_0 sufficiently small such that for all $\kappa < \kappa_0$,

$$E\left[\tau_h^\kappa\right] < \exp\left[\frac{d_3}{\kappa}\right] \tag{3.134}$$

Again from Chebyshev's inequality, (3.134) implies that for any positive d_4, d_5, $d_5 < d_4$, we can find h and κ_0 sufficiently small so that for all $\kappa < \kappa_0$,

$$P\left\{\tau_h^\kappa > \exp\left[\frac{d_4}{\kappa}\right]\right\} < \frac{E[\tau_h^\kappa]}{\exp\left[\frac{d_4}{\kappa}\right]} < \exp\left[\frac{-d_5}{\kappa}\right] \tag{3.135}$$

We conclude that for any $d_2 > 0, d_4 > 0, d_5 > 0$ with $d_5 < d_4$, $d_2 < d_5$, we can find $h > 0$ and κ_0 sufficiently small to ensure that for all $(z,i) \in N(O_j, h)$, $i = i_{max}$ and $\kappa < \kappa_0$,

$$P_{(z,i)}\left\{Q^\kappa(\tau_h^\kappa) \in I_j, \tau_h^\kappa < \exp\left[\frac{d_4}{\kappa}\right]\right\} = P_{(z,i)}\left\{Q(\tau_h^\kappa) \in I_j\right\} \tag{3.136}$$

$$- P\left\{Q^\kappa(\tau_h^\kappa) \in I_j, \tau_h^\kappa > \exp\left[\frac{d_4}{\kappa}\right]\right\} \tag{3.137}$$

$$> \exp\left[\frac{-d_2}{\kappa}\right] - \exp\left[\frac{-d_5}{\kappa}\right] \tag{3.138}$$

Since d_2 and d_5 are at our disposal, for any $d_6 > 0$ we can choose $d_2 = d_6/2$ and $d_5 = d_6$ to guarantee that for all κ sufficiently small, we have

$$\exp\left[\frac{-d_2}{\kappa}\right] - \exp\left[\frac{-d_5}{\kappa}\right] > \exp\left[\frac{-d_6}{\kappa}\right] \tag{3.139}$$

From the results of [3],

$$P_{O_j}(Q^\kappa(\tau_h^\kappa) \in I_{k_i}) = p_i^\kappa \tag{3.140}$$

where $p_i^\kappa \to p_i > 0$ as $\kappa \downarrow 0$.

Therefore, the logarithmic asymptotics of the transition time τ_z from any point $y \in I_{k_i}$, whether $i = i_{\min}$ or $i = i_{\max}$, to a point $z \in I_j$, $z \neq H(O_j)$, $|z - H(O_j)| < h$, depend on $\bar{V}_{ij}^{\max}$.

We construct a Markov process with state space consisting of three points: one point on the interior edge O_j, one point on edge $k_{i_{\max}}$, and one point on edge $k_{i_{\min}}$. Let $\delta > 0$ be arbitrary. Fix points the $\Delta_1 = (\delta + H(O_{k_{i_{\max}}}), i_{\max})$, $\Delta_2 = (\delta + H(O_{k_{i_{\min}}}), i_{\min})$. Put $\gamma_1 = (\frac{\delta}{2} + H(O_{k_{i_{\max}}}), i_{\max})$, $\gamma_2 = (\frac{\delta}{2} + H(O_{k_{i_{\min}}}), i_{\min})$. Let z on I_j satisfy $|z - H(O_j)| = \delta/2$. Let x be any point in $I_{k_1} \cup I_{k_2}$ and suppose $Q^\kappa(0) = x$. Following [18], §4, define the sequence of Markov times τ_n: $\tau_0 = 0$, $\sigma_n = \inf\{t > \tau_n : Q_t^\kappa \in \Delta_1 \cup \Delta_2\}$, $\tau_n = \inf\{t > \sigma_{n-1} : Q_t^\kappa \in \gamma_1 \cup \gamma_2 \cup z\}$.

Put $Z_n = Q^\kappa_{\tau_n}$. The Markov chain Z_n is a discrete-time, discrete-state-space Markov chain. For each integer n, Z_n is equal to γ_1, γ_2, or z. We estimate the transition probabilities for this chain. To prove the upper bound in (3.2.5), it suffices to prove that for any $d_7 > 0$ and $\alpha > 0$, we can find $\delta > 0$ such that if γ_i and Δ_i are defined as above,

$$P_{\gamma_i}\left\{Z_1 = z, \tau_z^\kappa < \exp\left[\frac{d_7}{\kappa}\right]\right\} \geq \exp\left[-\frac{\bar{V}_{ij}^{\max} + \alpha}{\kappa}\right], i = 1, 2. \tag{3.141}$$

Now, by continuity of the function $\bar{V}$, for any $\alpha > 0$ we can choose $\delta > 0$ and $h > 0$ such that if $i = i_{\max}$ and $y, y' \in I_{k_i}$ satisfy $\frac{\delta}{2} < |y - H(O_{k_i})| < \delta$ and $|y' - H(O_j)| < h$, then

$$\bar{V}_{ij}^{\max} > \bar{V}(y, y') > \bar{V}_{ij}^{\max} - \frac{\alpha}{4} \tag{3.142}$$

Without loss of generality we can choose $\delta < \alpha$.

We can then find a finite time $T(y') < \infty$ and a smooth function ϕ satisfying

$$\phi(0) = y, \quad \phi(T) = y' + \delta/16, \quad \phi(t) \in [y - \delta/8, y' + \delta/8], 0 \leq t \leq T \tag{3.143}$$

for which $S_{0T}(\phi) < \bar{V}(y, y') + \frac{\delta}{32}$.

Put $\tau_{y'}^{\kappa} = \inf\{t > 0 : Q^{\kappa}(t) = y'\}$. From the nondegeneracy of the process $Q^{\kappa}(t)$ in the interval $[y - \delta/4, y' + \delta/4]$ and Lemma (3.2.3), we conclude that for sufficiently small κ,

$$P_y\left\{\tau_{y'}^{\kappa} < T(y')\right\} \geq P_y\left\{\sup_{0\leq t\leq T(y')} |Q^{\kappa}(t) - \phi(t)| \leq \frac{\delta}{32}\right\} \tag{3.144}$$

$$\geq \exp\left[-\frac{\left(S_{0T}(\phi) + \frac{\delta}{32}\right)}{\kappa}\right] \tag{3.145}$$

$$\geq \exp\left[-\frac{\bar{V}(y, y') + \frac{\alpha}{4}}{\kappa}\right] \tag{3.146}$$

$$\geq \exp\left[-\frac{\bar{V}_{ij}^{\max} + \frac{\alpha}{2}}{\kappa}\right] \tag{3.147}$$

Hence, starting at y, $y \in i = i_{\max}$, the process $Q^{\kappa}(t)$ can hit y' in finite time with probability bounded from below by $\exp\left[-\frac{\bar{V}_{ij}^{\max}+\frac{\alpha}{2}}{\kappa}\right]$.

Thus, to prove (3.141), note that as a consequence of (3.91), (3.136), (3.144), and the strong Markov property for Z_n, for any positive d_1, d_2, d_4, α we can choose $\delta > 0$ and $\gamma = \delta/2$ and κ_0 such that if $|z - H(O_j)| < \delta$, $z \neq O_j$, and $|y - H(O_{k_i})| < \delta$, $y \neq O_k$, and $\kappa < \kappa_0$, then

$$P_{\gamma_{i_{\max}}}\left\{Z_1 = z, \tau_z^{\kappa} < \exp\left[\frac{d_4}{\kappa}\right]\right\} > \exp\left\{\left[\frac{-d_1}{\kappa}\right] + \left[\frac{-d_2}{\kappa}\right] + \left[\frac{-(\bar{V}_{ij}^{\max} + \alpha)}{\kappa}\right]\right\} \tag{3.148}$$

An identical bound holds for the initial point y belonging to the second edge $i = i_{\min}$:

$$P_{\gamma_{i_{\min}}}\left\{Z_1 = z, \tau_z^{\kappa} < \exp\left[\frac{d_4}{\kappa}\right]\right\} > \exp\left\{\left[\frac{-d_1}{\kappa}\right] + \left[\frac{-d_2}{\kappa}\right] + \left[\frac{-(\bar{V}_{i_{\min}j} + \alpha)}{\kappa}\right]\right\} \tag{3.149}$$

Since $V_{ij}^{\max} > \bar{V}_{i_{\min}j}$, this establishes (3.141), from which the theorem follows. □

Corollary 3.2.6. *Let* $\tau_{O_j}^{\kappa} = \inf\{t > 0 : Q^{\kappa}(t) = O_j\}$. *For any point* z *along edge*

k_i, and for any $\alpha > 0$,

$$\lim_{\kappa\to 0} P_z \left\{ \exp\left[\frac{\bar{V}(O_i, O_j) - \alpha}{\kappa}\right] < \tau^{\kappa}_{O_j} < \exp\left[\frac{\bar{V}(O_i, O_j) + \alpha}{\kappa}\right] \right\} = 1 \qquad (3.150)$$

and if (z, j) and (z', j) are two points on the same interior edge O_j with with $z < z'$, then for any α,

$$\lim_{\kappa\to 0} P_z \left\{ \exp\left[\frac{\bar{V}^{\max} - \alpha}{\kappa}\right] < \tau^{\kappa}_{z'} < \exp\left[\frac{\bar{V}^{\max} + \alpha}{\kappa}\right] \right\} = 1 \qquad (3.151)$$

where $\bar{V}^{\max} = \max\{\bar{V}(O_k, z')\}$ and this maximum is taken over all exterior vertices O_k such that $H(O_k) < H(O_j)$ and O_k can be reached from O_j along a path which does not intersect any interior vertex O_r satisfying $H(O_r) > H(O_j)$.

Proof. The corollary follows from (3.149) and the fact that the exterior vertex O_i is an asymptotically stable equilibrium for the limiting process $Q(t)$ along edge I_i whose domain of attraction is the entire edge I_i (see Lemma 2.1, [18]). □

3.2.2 The set of possible metastable distributions

The process $Q^{\kappa}(t)$ converges to the limiting stochastic process $Q(t)$, which consists of deterministic motion along each edge and stochastic branching at each interior vertex. Let p^j_k represent the probability of the process $Q(t)$ branching toward vertex O_k from the interior vertex O_j; for any j we must have $\sum p^j_k = 1$, where k ranges over all edges $I_k \sim O_j$. For ease of notation, we abbreviate these p_k, and the vertex from which the branching occurs is understood to be the vertex $O_j \sim I_k$ with $H(O_j) > H(O_k)$. Let O_h be the vertex with maximal Hamiltonian value, so $H(O_h) > H(O_r)$ for every other (interior or exterior) vertex O_r. In our example, $h = 6$.

Since $Q^\kappa(t)$ converges weakly to a stochastic process, metastability corresponds not to single equilibrium states but to probability distributions μ across exterior vertices. The set of such distributions is finite and independent of the diffusion matrix $a(x)$ for the two-dimensional process $\tilde{X}^{\epsilon,\kappa}(t)$.

We describe the distributions in our case of a four-well Hamiltonian in Figure (3.2); this can be generalized to any finite number of wells.

1. The delta-distributions at each fixed exterior vertex O_k: these are $\mu_k(O_k) = 1$, $\mu_k(O_r) = 0$ for all exterior vertices O_r with $r \neq k$;

2. The distributions over any fixed pair of exterior vertices O_{k_1}, O_{k_2} such that $I_{k_i} \sim O_j$: these are $\mu(O_{k_1}) = p_{k_1}, \mu(O_{k_2}) = p_{k_2}$, and $\mu(O_r) = 0$ for other exterior vertices O_r. (Not all of these probability distributions will necessarily correspond to metastable distributions.)

3. The distributions over any three exterior vertices: $\mu(O_k) = 0$ for some fixed exterior vertex O_k, and if the exit edges I_k, I_r meet at interior vertex O_j, then $\mu(O_r) = p_j$. For all other exterior vertices $O_m, m \neq k, r$, let $\{O_{l_1}, \ldots O_{l_m}\}$ be the interior vertices in the shortest path between O_m and the interior vertex O_h.Then $\mu(O_m) = p_{l_1} p_{l_2} \ldots p_{l_m}$. (Not all of these vertices will necessarily correspond to metastable distributions.)

4. The distributions over all four exterior vertices: For any exterior vertex O_k, let $O_{l_1}, \ldots, O_{l_k}$ be the interior vertices along the shortest path from O_h to O_k. Then $\mu(O_k) = p_{l_1} p_{l_2} \ldots p_{l_k}$.

Theorem 3.2.7. *Let* $\lambda < \min\{\bar{V}(O_1, O_2), \bar{V}(O_3, O_2), \bar{V}(O_5, O_4), \bar{V}(O_7, O_4)\}$. *Let* $T(\kappa)$ *be any time parameter such that*

$$\lim_{\kappa \downarrow 0} \kappa \ln T(\kappa) \leq \lambda.$$

Let $(z, 6)$ *be any point on* Γ *with* $z > H(O_6)$. *Then the metastable state for the initial point* (z, i) *and the timescale* λ *is the nondegenerate probability distribution* μ *across all four exterior vertices* O_1, O_3, O_5, *and* O_7, *with associated weights given by:* $\mu(O_1) = p_2 p_1$; $\mu(O_3) = p_2 p_3$; $\mu(O_5) = p_4 p_5$; $\mu(O_7) = p_4 p_7$.

Proof. Let positive η and α, and $\theta > 0$ be given. By Theorem (3.2.5), we can choose $\delta > 0$ such that the following hold: first,

$$\{x \in I_i : V(O_i, x) < \theta\} \supset \{x \in I_i : |x - O_i| < \delta\} \tag{3.152}$$

and second, if (x, i) and (y, j) are two points along any exterior edge i with exterior vertex O_i and interior vertex O_j such that $|x - H(O_i)| < \delta$, $x \neq H(O_i)$, and $|y - H(O_j)| < \delta$, $y \neq H(O_j)$, then there exists κ_0 such that for all $\kappa < \kappa_0$,

$$P_x \left\{ \exp\left[\frac{\bar{V}_{ij}^{\max} - \alpha}{\kappa}\right] < \tau_y^\kappa < \exp\left[\frac{\bar{V}_{ij}^{\max} + \alpha}{\kappa}\right] \right\} > 1 - \eta \tag{3.153}$$

where $\tau_y^\kappa = \inf\{t > 0 : Q^\kappa(t) = y\}$. let $N_\delta(O_i)$ be the δ-neighborhood of the exterior vertex O_i on the graph Γ. Since λ is the timescale, let $T(\kappa)$ be such that

$$\lim_{\kappa \to 0} \kappa \ln T(\kappa) = \lambda \tag{3.154}$$

The results of [3] imply that for any fixed $t > 0$, there exists a $\kappa_1 < \kappa_0$ such that:

$$|P_{(z,6)}\left\{Q^{\kappa_1}(tT(\kappa_1)) \in N_\delta(O_1)\right\} - p_2 p_i| < \eta, \quad i = 1, 3$$

$$|P_{(z,6)}\left\{Q^{\kappa_1}(tT(\kappa_1)) \in N_\delta(O_5)\right\} - p_4 p_i| < \eta, \quad i = 5, 7$$

and p_i are the probabilities calculated explicitly in (2.192). Recall that the numbers p_i depend only on B.

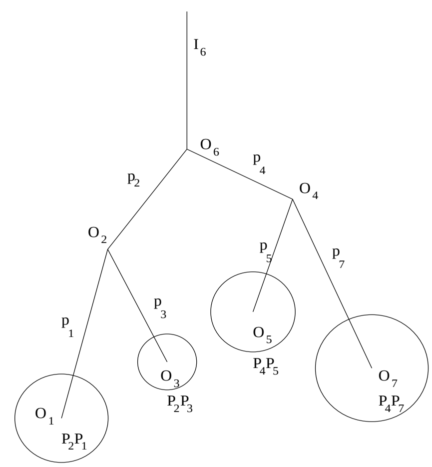

Figure 3.3: Asymptotic probabilities for small neighborhoods of exterior vertices

Let $A_i, i = 1, 3, 5, 7$ be the event $\{Q^{\kappa_1}(tT(\kappa_1) \in N_\delta(O_i)\}$. On each set A_i, for any $x \in N_\delta(O_i)$, (3.153) implies that

$$\lim_{\kappa \to 0} P_x \left\{Q^\kappa(tT(\kappa) \in I_i\right\} = 1 \tag{3.155}$$

and thus $Q^\kappa(tT(\kappa)$ is governed by the one-dimensional operator L_i^κ. There exists a unique invariant measure associated to the process governed by L_i^κ, and this measure converges as $\kappa \downarrow 0$ to the delta measure concentrated at O_i. By the results of Proposition (1), we conclude that the metastable state is O_i, so that if $F_{\theta, O_i} =$

$\{(x, i) : V(O_i, x) < \theta\}$, then

$$\lim_{\kappa \to 0} P_{(z,i)}\{Q^\kappa(tT(\kappa)) \in F_{\theta,O_i}\} = 1 \tag{3.156}$$

which establishes the second characterization of metastability. Hence the metastable distribution $\mu_{(z,6),\lambda}$ is a probability distribution across exterior vertices with weights $p_2 p_i$ for $i = 1, 3$ and $p_4 p_i$ for $i = 5, 7$. □

At times of order $T(\kappa)$, then, Q^κ lies within a small neighborhood of the exterior vertex O_i with asymptotic probabilities $p_j p_i$ above. Rather than the metastable state being a single point, for this initial position and timescale it is a probability distribution across all four exterior vertices.

3.2.3 Metastable states as probability distributions

We stress that the metastable state (or distribution) depends both on initial position and timescale λ. If λ remains as before, but the initial position (z, i) is such that $H(O_4) < z < H(O_6)$, then the metastable distribution for (z, λ) is a distribution between O_5 and O_7, with associated probabilities p_5 and p_7, respectively. An analogous result holds for $H(O_2) < z < H(O_6)$. Finally, if the initial position (z, i) lies on any exit edge I_1, I_3, I_5 or I_7, the metastable state is the single exterior vertex on the corresponding edge.

Theorem 3.2.8. *Let (z, i) be an initial position with $z > H(O_6)$. Assume that $\bar{V}(O_5, O_4) < \bar{V}(O_7, O_4) < \bar{V}(O_3, O_2) < \bar{V}(O_1, O_2)$ and that $\bar{V}(O_2, O_6) > \bar{V}(O_4, O_6)$. Suppose λ satisfies $\bar{V}(O_5, O_4) < \lambda < \bar{V}(O_7, O_4)$. The metastable distribution for this* *initial position and timescale is the probability distribution μ across O_1, O_3, and O_7 with $\mu(O_1) = p_2p_1$; $\mu(O_3) = p_2p_3$; and $\mu(O_7) = p_4$.*

Proof. It follows from [3] that

$$\lim_{\kappa\downarrow 0} P_{O_4}\left\{Q^\kappa(\tau_h^\kappa) \in I_7\right\} = p_7 > 0 \tag{3.157}$$

From Corollary (3.2.6), since $\bar{V}(O_7, O_4) > \lambda > \bar{V}(O_5, O_4)$, we derive that

$$\lim_{\kappa\downarrow 0} P_{O_4}\left\{Q^\kappa(tT(\kappa) \in I_7\right\} = 1 \tag{3.158}$$

and thus, as in Theorem (3.2.7), since $z > H(O_6)$, the metastable distribution is concentrated on the exterior vertices O_7, O_1, and O_3. The associated probabilities are given explicitly in (2.192) and depend only on B. □

The above results illustrate the main steps in finding metastable distributions for any initial condition $(z, i) \in \Gamma$ with $z > H(O_6)$ and for all but finitely many values of λ. Formally, we restate (3.2.1), whose proof follows immediately from the previous results.

Theorem 3.2.9. *Let $\lambda > 0$ and $T(\kappa)_\lambda$ be such that*

$$\lim_{\kappa\downarrow 0} \kappa \ln T(\kappa) = \lambda \tag{3.159}$$

For any initial condition $(x_1(0), x_2(0)) \in \mathbb{R}^2$ and all but finitely many timescales λ, the process $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$ converges weakly in the space $C_{0T}(\mathbb{R}^2)$, first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$, to a probability measure concentrated on the stable equilibrium points of the unperturbed Hamiltonian system. In particular, there exist initial conditions $w = (x_1(0), x_2(0)) \in \mathbb{R}^2$ and timescales λ such that process $\tilde{X}^{\epsilon,\kappa}_{T_\lambda(\kappa)}$, converges weakly *in the space $C_{0T}(\mathbb{R}^2)$, first as $\epsilon \downarrow 0$ and then as $\kappa \downarrow 0$, to a nondegenerate probability measure $\mu_{w,\lambda}$ concentrated on the stable equilibrium points $\{O_1, O_3, O_5, O_7\}$ of the unperturbed Hamiltonian system, with weights $p_i(w, \lambda) = \mu_{w,\lambda}(O_i), i \in \{1, 3, 5, 7\}$ that can be explicitly computed and depend only on B.*

Proof. The proof follows from the weak convergence results of [3], Theorem (3.2.5), Corollary (3.2.6), and Theorems (3.2.7) and (3.2.8). We can determine the metastable distribution for λ such that $\lambda \neq \bar{V}(O_k, O_j)$ for any exterior vertex O_k and interior vertex O_j.

For initial condition $z > H(O_6)$, if there exists λ such that $\bar{V}(O_3, O_2) > \lambda > \bar{V}(O_7, O_6)$, the metastable distribution is the probability measure concentrated at O_3 and O_1 with weights p_3 and p_1; for $\lambda > \bar{V}(O_3, O_2)$, the metastable state is O_1.

For initial condition (z, i) with $z < H(O_4)$, $i = 5$ and $\lambda < \bar{V}(O_5, O_4)$, the metastable state is O_5.

For initial condition $z < H(O_4)$, $i = 7$ and $\lambda < \bar{V}(O_7, O_4)$, the metastable state is O_7.

For initial condition $z < H(O_4)$, $i = 5$ and $\bar{V}(O_5, O_4) < \lambda < \bar{V}(O_7, O_4)$, the metastable state is O_7.

The corresponding results hold for initial conditions $z < H(O_2)$, $i = 1, 3$ and timescales λ satisfying $\lambda < \bar{V}(O_3, O_2)$ and $\bar{V}(O_3, O_2) < \lambda < \bar{V}(O_1, O_2)$.

For initial conditions (z, i) satisfying $H(O_4) < z < H(O_6)$ and $i = 4$, for $\lambda < \bar{V}(O_5, O_4)$ the metastable distribution is concentrated on the two exterior vertices O_7 and O_5 with weights p_7 and p_5, respectively. For $\bar{V}(O_5, O_4) < \lambda < \bar{V}(O_7, O_6)$, the metastable state is O_7. If there exists λ such that $\bar{V}(O_3, O_2) > \lambda > \bar{V}(O_7, O_6)$, the metastable distribution is concentrated on O_3 with probability p_3 and O_1 with probability p_1; if no such λ exists and $\lambda > \bar{V}(O_7, O_6)$ automatically implies $\lambda > \bar{V}(O_3, O_2)$, then the metastable state is O_1.

Corresponding results hold for initial conditions (z, i) satisfying $H(O_2) < z < H(O_6)$ and timescales $\lambda < \bar{V}(O_3, O_2)$, for which the metastable distribution is concentrated on O_3 with probability p_3 and O_1 with probability p_1, and $\bar{V}(O_3, O_2) < \lambda < \bar{V}(O_1, O_6)$, for which the metstable state is O_1.

For any (z, i), if $\lambda > \bar{V}(O_3, O_6)$ (so that, by hypothesis, we automatically have $\lambda > \bar{V}(O_7, O_6)$), the metastable state is O_1. □

3.2.4 Remarks and generalizations

The above results can be generalized to the case of a Hamiltonian with finitely many wells and the same generic structure; namely, with three edges meeting at each interior vertex and the property that the numbers $\bar{V}(O_k, O_i)$ are distinct for any two pairs of exterior and interior vertices with $H(O_k) < H(O_i)$.

We hope to investigate further questions about averaging, large deviations, and metastability for nearly-Hamiltonian systems. Some extensions and generalizations of these results include:

1. The case of weaker assumptions on B: i.e. when $\text{div}(B)$ changes sign. This introduces additional "fixed points" for the limiting process $Q(t)$ on the graph.

2. The construction of an action functional for the process $Q^\kappa(t)$ on the graph.

We hypothesize that the action functional for the process $Q^\kappa(t)$ takes the following form. First we define the action functional along each edge: for absolutely continuous functions $\phi(s) : [0,T] \to I_i$, where $\phi(0) = Q^\kappa(0)$, the *edge action functional* $S^i_{0T}(\phi)$ along edge I_i for the process with generator L^κ_i is given as

$$S_i(\phi) = \frac{1}{2}\int_0^T \left[\dot{\phi}(s) - \frac{\tilde{B}_i(\phi(s))}{T_i(\phi(s))}\right]^2 \frac{T_i(\phi(s))}{A_i(\phi(s))} ds,$$

and $S^i_{0T}(\phi)$ is defined to be infinite for all other functions ϕ. Next, since the process $Q^\kappa(t)$ has no delay at interior vertices—that is, only first-order terms appear in the gluing conditions—the action functional $S_{0T}(\phi)$ on the graph is given as follows: for functions ϕ that are not absolutely continuous along each edge I_i or for functions $\phi(t)$ which intersect the set of interior vertices at an uncountably infinite number of time points $t \in [0,T]$, the action functional $S_{0T}(\phi)$ associated to Q^κ is infinite. For all other $\phi \in C_{0T}(\Gamma)$, let $t_1 < t_2 < \ldots < t_N \ldots$ be the points such that $\phi(t_n) = O_j$, where $1 \leq n$ and O_j is an interior vertex. For $t : t_n < t < t_{n+1}$, $\phi(t)$ lies entirely within an edge I_n. Let $S^n_{[t_n,t_{n+1}]}$ be the edge action functional along edge I_n. We define $S_{0T}(\phi)$ by

$$S_{0T}(\phi) = \sum_n S^n_{[t_n,t_{n+1}]}(\phi). \tag{3.160}$$

If the sum diverges, the action functional is defined to be infinite. Once this result is proved, a shorter proof of Theorem (3.2.1) can be obtained by invoking the results of §4, [18]. The degeneracies of the diffusion and drift coefficients along an exterior edge prevent us from directly applying Theorem 3.2, §5, in [18] to get the form of the action functional along each edge.

3. More precise asymptotics for the behavior of $Q(\tilde{X}^{\epsilon,\kappa}(t))$ in the double limit as ϵ and $\kappa \downarrow 0$. We surmise that if $\epsilon < \sqrt{\kappa}$, then Theorem (3.2.1) still holds, but if $\kappa < \epsilon^{2+\delta}, \delta > 0$, then sublimiting distributions may not exist.

4. The case of Hamiltonians with multiple degrees of freedom. The situation in this instance is fundamentally different from the case of one degree of freedom because of the possible existence of multiple invariant measures on each level set (and consequent difficulties in averaging). Freidlin and Wentzell treat the case of n-independent one-degree-of-freedom oscillators and show that in this case, the slow component converges to a process on an *open book* space, with a description of the behavior of the process along the "binding." We would like to characterize metastability in this setting.

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