ABSTRACT

Title of dissertation: Local Rigidity of Triangle Groups in $\mathrm{Sp}(4, \mathbb{R})$

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This paper studies configurations of Lagrangians in a four dimensional real symplectic vector space. We develop a generalized cross ratio as an invariant for quadruples of Lagrangians. This invariant is then used to study representations of triangle groups into the symplectic group. The main theorem is a local rigidity result for a certain representations factoring through the isometry group of the hyperbolic plane.

LOCAL RIGIDITY OF TRIANGLE GROUPS IN $\mathrm{Sp}(4, \mathbb{R})$

by

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Dedication

I dedicate this work to my grandparents, Joe and Nancy Cote, who made it possible for all of their grandchildren to go to college.

Acknowledgments

Graduate school for me was an endeavor I undertook simply because I enjoyed mathematics. I learned very quickly that my enthusiasm for the subject would not be enough to complete my degree, and this would require substantially more work than I had expected. This is something I never would have been able to do without the help and support of so many people around me.

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Chapter 1

Introduction

This paper studies certain discrete faithful representations of hyperbolic triangle groups into the symplectic group $\mathrm{Sp}_{\pm}(4, \mathbb{R})$. The symplectic group is the isometry group of the *Siegel generalized upper half space* $\mathfrak{S}_n$, which was introduced by Siegel in [19]. The original intent of this research was to study the action of such a representation making use of the geometry of $\mathfrak{S}_n$. It is however more convenient to examine the action on its *Shilov boundary* which identifies with the *Real Lagrangian Grassmannian*. This perhaps should not be very surprising: when studying actions on hyperbolic space one typically studies the action on the boundary and utilizes Poincare extension.

Our approach will be to begin with well known representations of triangle groups into $\mathrm{Isom}(\mathbb{H}^2)$. The classical fact that any two similar triangles in $\mathbb{H}^2$ are congruent implies that such representations are locally rigid. Composing these representations with embeddings $\mathrm{Isom}(\mathbb{H}^2) \hookrightarrow \mathrm{Isom}(\mathfrak{S}_2)$ yields symplectic representations, essentially choosing a component of the representation variety to examine. We can now look for deformations of these representations, i.e. representations which are "close" but not conjugate. In the main case of interest, we show that there are no such deformations, so the specified component of the representation variety is a single point. The main goal is to prove the following

Theorem 1.1 (Rigidity of diagonally embedded triangle groups)**.** *All faithful representations of hyperbolic triangle groups factoring through the diagonal embedding of Isom*$(\mathbb{H}^2)$ *are locally rigid.*

With a mild amount more work we are able to prove that another component is also a single point

Theorem 1.2 (Rigidity of anti-diagonally embedded triangle groups)**.** *All faithful representations of hyperbolic triangle groups factoring the anti-diagonal embedding of Isom*$(\mathbb{H}^2)$ *are locally rigid.*

The proof is accomplished by studying configurations of real Lagrangians. The most fascinating part of the paper is perhaps the development of a *cross ratio* on the Lagrangian Grassmannian. Analogous to the classical cross ratio for quadruples of points in $\mathbb{CP}^1$, this cross ratio yields and invariant for quadruples of Lagrangians. This allows us to translate the relations of a triangle group into equations describing the eigenspaces of the generating involutions. Solving these equations in general has proved quite difficult. We prove theorem 1.1 and 1.2 by showing that in those cases the number of solutions is finite (discrete).

The same techniques seem well equipped to study other components of the representation variety. In §8.5 we investigate triangle groups which factor through the bidisk embedding but not through either the diagonal or the anti-diagonal. The equations obtained are more formidable to work with. We will give these equations describing the deformation space, and justify the following conjecture.

Conjecture 1.3. *Faithful representations of triangle groups which factor through*

the bi-disk embedding Isom$(\mathbb{H}^2) \times$ *Isom*$(\mathbb{H}^2)$ *but not through the diagonal or the anti-diagonal have non-trivial deformations.*

The paper is structured as follows. §1.4 describes $\mathfrak{S}_2$ and the action of $\mathrm{Sp}(4, \mathbb{R})$. Chapter 2 realizes $\mathfrak{S}_2$ as a submanifold of its compact dual, which gives a relatively simple description of the boundary $\partial\mathfrak{S}_2$. Chapter 3 describes $\mathfrak{S}_2$ as a Riemannian manifold and makes use of a few facts tediously obtained in appendix A. With apologies to the readers, we don't make explicit use of much of the material up to this point and it serves mainly to motivate what comes next. A reader willing to accept a few facts about $\mathrm{Sp}(4, \mathbb{R})$ and $\mathfrak{S}_2$ could skip these first chapters and still follow the remainder of the paper.

Chapter 4 describes several embeddings $\mathrm{Isom}(\mathbb{H}^2) \hookrightarrow \mathrm{Isom}(\mathfrak{S}_2)$. These embeddings provide the starting points in our search for deformations. The generators of triangle groups will be Lagrangian involutions and these are described at the end of this chapter. Chapter 5 describes coordinate systems on the Lagrangian Grassmannian and provides tools for visualizing via an identification with the *Einstein universe* (see [6]). Chapter 6 investigates configurations of Lagrangians up to the action of the symplectic group. It develops the generalized cross ratio and gives a visual interpretation of these configurations.

With all these tools, we are now able to investigate triangle groups. Any two of the generators of a triangle group generate a dihedral group, so Chapter 7 uses the cross ratio to describe dihedral groups in $\mathrm{Isom}(\mathfrak{S}_2)$. Chapter 8 first describes triangle groups in $\mathrm{Isom}(\mathbb{H}^2)$. Then we describe how a triangle group in $\mathrm{Isom}(\mathfrak{S}_2)$ gives a

configuration of Lagrangians. The triangle group relations determine equations in the cross ratios of these Lagrangians. In a perfect world, we would simply solve this system and completely characterize all such groups. The world is not perfect, so we focus our attention on a neighborhood of known triangle group representations factoring through $\mathrm{Isom}(\mathbb{H}^2) \times \mathrm{Isom}(\mathbb{H}^2)$. Theorem 1.1 is proven in §8.3.1 by showing that there is a small enough neighborhood which isolates the known solution from any other solution. Theorem 1.2 is proven in a slightly different manner in §8.4.1.

Finally in §8.5 we give a construction of a more exotic representation. **The key difference it seems between this and the previous cases is whether or not the relevant cross ratios have repeated eigenvalues**. We will present evidence that deformations exist for these representations.

There is some supporting Mathematica computations, animations, and interactive notebooks available at *http://www.math.umd.edu/∼rfhoban/ThesisStuff/ThesisStuff.html* ([15]).

1.1 Notation

We will use the following notations:

- $Y > 0$ denotes that the square matrix Y is positive definite
- Y^T denotes the transpose of Y
- $\overline{Z}$ denotes the complex conjugate of Z
- $Z^\dagger$ denotes the conjugate transpose of Z

- $\mathrm{Re}(Z)$ and $\mathrm{Im}(Z)$ denote the real and imaginary parts of Z
- $\mathbb{H}^2$ will denote the hyperbolic plane, usually the upper half plane model
- I_n will denote the $n \times n$ identity matrix
- 0_n will denote the $n \times n$ zero matrix
- For 4×2 matrices U_1 and U_2, we will use the notation $[U_1|U_2]$ to denote the 4×4 matrix whose columns are the columns of U_1 and U_2

1.2 The Symplectic Group

Definition 1.4. A *Symplectic Form* ω on $\mathbb{C}^{2n}$ is a skew symmetric non-degenerate 2-form. Explicitly ω satisfies the following properties:

1. $\omega(u, v) = -\omega(v, u)$ for all $u, v \in \mathbb{C}^{2n}$
2. For all u there is v such that $\omega(u, v) \neq 0$

By choosing a basis for $\mathbb{C}^{2n}$, a symplectic form ω can be given by a non-degenerate skew symmetric matrix Ω, such that for any vectors $u, v \in \mathbb{C}^{2n}$

$$\omega(u, v) = u^T \Omega v.$$

Definition 1.5. *The Symplectic Group*, denoted $\mathrm{Sp}(2n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(2n, \mathbb{C})$ which preserves a symplectic form ω. Explicitly, if ω is defined by the skew symmetric matrix Ω, then the symplectic group is

$$\mathrm{Sp}(2n, \mathbb{C}) = \{M \in \mathrm{GL}(2n, \mathbb{C}) : M^T \Omega M = \Omega\}$$

All symplectic forms are equivalent up to a choice of basis for $\mathbb{C}^{2n}$, so the group $\mathrm{Sp}(2n, \mathbb{C})$ is a well defined subgroup of $\mathrm{GL}(2n, \mathbb{C})$ up to conjugation. The *real symplectic group* $\mathrm{Sp}(2n, \mathbb{R})$ is the subgroup of $\mathrm{Sp}(2n, \mathbb{C})$ with real entries. We will fix a basis so that the symplectic form is given by the skew symmetric matrix

$$\Omega_{2n} = \begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}$$

Our primary focus will be the rank 2 symplectic groups $\mathrm{Sp}(4, \mathbb{C})$ and $\mathrm{Sp}(4, \mathbb{R})$ where the symplectic form is

$$\Omega = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Note that the restriction of ω to either of the planes $P_1 = \mathrm{Span}\{e_1, e_2\}$ or $P_2 = \mathrm{Span}\{e_3, e_4\}$ is identically zero.

Definition 1.6. A subspace W is called *isotropic* if $\omega|_W \equiv 0$, and called *coisotropic* if $W^\perp \subset W$. A subspace which is both isotropic and coisotropic is called *Lagrangian*. If W is Lagrangian, then $\dim(W) = \dim(W^\perp) = n$.

The form defined by Ω thus defines a *Lagrangian splitting* of $\mathbb{C}^4 = P_1 \bigoplus P_2$ as the direct sum of Lagrangian planes. As a result we will refer to this basis for $\mathbb{C}^4$ as a *Lagrangian Basis*. More generally Ω_{2n} defines a splitting of $\mathbb{C}^{2n}$ as the direct sum of n-dimensional Lagrangian subspaces.

It is worth noting that in dimension 2, a non-degenerate skew symmetric 2-form is an area form, so there is an accidental isomorphism $\mathrm{Sp}(2,\mathbb{R}) \approx \mathrm{SL}(2,\mathbb{R})$.

Finally, the following elementary lemma gives some simple relationships between the eigenvalues for a symplectic matrix

Lemma 1.7. *Let $M \in Sp(2n,\mathbb{C})$. If λ is an eigenvalue for M then λ^{-1} is also an eigenvalue for M.*

Proof. If Ω is any non-degenerate, skew-symmetric matrix and if M is symplectic with respect to the bilinear form defined by Ω, then $\Omega M = (M^T)^{-1}\Omega$. Suppose λ is an eigenvalue for M with eigenvector v. Then

$$\Omega M v = (M^T)^{-1}\Omega v$$

$$\lambda(\Omega v) = (M^T)^{-1}(\Omega v)$$

thus λ is an eigenvalue for $(M^T)^{-1}$ with eigenvector Ωv. This implies that λ is an eigenvalue for M^{-1}, thus λ^{-1} is an eigenvalue for M.

□

1.3 Some classical Lie algebras

We will need explicit bases for several of the classical Lie algebras which we briefly develop here. A reader familiar with these Lie algebras should feel free to skip this section and simply refer back to it as needed. For a good treatment of these see [10] or [14].

1.3.1 $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{so}(2)$

The **special linear** Lie algebra is the set of traceless 2×2 matrices

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a, b, c \in \mathbb{R} \right\}$$

In this basis, the *Cartan subalgebra* is 1 dimensional and consists of diagonal matrices

$$\begin{bmatrix} a & 0 \\ 0 & -a \end{bmatrix}$$

The **special orthogonal** Lie algebra is one dimensional and consists of skew symmetric 2×2 matrices

$$\mathfrak{so}(2) = \left\{ \begin{bmatrix} 0 & -\theta \\ \theta & 0 \end{bmatrix} \right\}$$

It is clear that $\mathfrak{so}(2) \subset \mathfrak{sl}(2,\mathbb{R})$.

1.3.2 $\mathfrak{sp}(4,\mathbb{R})$ and $\mathfrak{u}(2)$

The **symplectic** Lie algebra is defined as

$$\mathfrak{sp}(4,\mathbb{R}) := \{X \in \mathrm{Mat}(4,\mathbb{R}) : X^T\Omega + \Omega X = 0_4\}$$

A standard form for X in the chosen Lagrangian basis can be found by writing X as 2×2 block matrices

$$X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

If $X \in \mathfrak{sp}(4, \mathbb{R})$ then

$$0_4 = X^T\Omega + \Omega X = \begin{bmatrix} C^T - C & -A^T - D \\ D^T + A & -B^T + B \end{bmatrix}$$

This implies $B = B^T$, $C = C^T$, and $D = -A^T$. So B and C are symmetric matrices and $X = \begin{bmatrix} A & B \\ C & -A^T \end{bmatrix}$, so a good general form would be:

$$X = \begin{bmatrix} a & a_{12} & b_{11} & b_{12} \\ a_{21} & b & b_{12} & b_{22} \\ c_{11} & c_{12} & -a & -a_{21} \\ c_{12} & c_{22} & -a_{12} & -b \end{bmatrix}$$

It is clear that $\mathfrak{sp}(4, \mathbb{R})$ has dimension 10. The Cartan subalgebra consists of diagonal matrices and has dimension 2.

The **unitary** Lie algebra is

$$\mathfrak{u}(2) := \{X \in \mathrm{Mat}(2, \mathbb{C}) : X^\dagger + X = 0_2\}$$

Letting $X = (x_{ij} + iy_{ij}) \in \mathfrak{u}(2)$, then

$$X^\dagger + X = \begin{pmatrix} 2x_{11} & x_{12} + x_{21} + i\,(y_{12} - y_{21}) \\ x_{12} + x_{21} - i\,(y_{12} - y_{21}) & 2x_{22} \end{pmatrix} = 0_2$$

yields 4 linear equations. These imply that X has the form

$$X = \begin{pmatrix} iy_{11} & x_{12} + iy_{12} \\ -x_{12} + iy_{12} & iy_{22} \end{pmatrix} = \begin{pmatrix} 0 & x_{12} \\ -x_{12} & 0 \end{pmatrix} + i \begin{pmatrix} y_{11} & y_{12} \\ y_{12} & y_{22} \end{pmatrix}$$

So $\mathrm{Re}(X)$ is skew-symmetric, $\mathrm{Im}(X)$ is symmetric and $\mathfrak{u}(2)$ has dimension 4. Somewhat surprisingly, this is a subalgebra of $\mathfrak{sp}(4,\mathbb{R})$ in a very natural way. The embedding is given by:

$$\begin{aligned} \mathfrak{u}(2) &\hookrightarrow \mathfrak{sp}(4,\mathbb{R}) \\ X &\hookrightarrow \begin{bmatrix} Re(X) & Im(X) \\ -Im(X) & Re(X) \end{bmatrix} \end{aligned}$$

In chapter 4 and appendix A we will use the following explicit basis for $\mathfrak{u}(2) \subset \mathfrak{sp}(4,\mathbb{R})$

$$B_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}, B_4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

This embedding of Lie algebras induces an embedding of the corresponding Lie groups

$$\mathrm{U}(2) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$$

given by the same formula. A matrix N is *unitary* if $N^\dagger N = I_2$. Equivalently if $N = A + iB$ then N is unitary if and only if

$$I_2 = N^\dagger N = (A^T - iB^T)(A + iB) = A^T A + B^T B + i(-B^T A + A^T B)$$

The embedding is given by

$$A + iB \hookrightarrow \begin{bmatrix} A & B \\ -B & A \end{bmatrix}$$

and we can check that the image is actually symplectic

$$\begin{bmatrix} A & B \\ -B & A \end{bmatrix}^T \begin{bmatrix} 0_2 & -I_2 \\ I_2 & 0_2 \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} A^T B - B^T A & -A^T A - B^T B \\ B^T B + A^T A & -B^T A + A^T B \end{bmatrix}$$

if and only if $A + iB \in \mathrm{U}(2)$.

1.4 The Siegel Upper Half Space

Definition 1.8. *Siegel's Generalized Upper Half Space* (or simply the Siegel upper half space), denoted $\mathfrak{S}_n$ is the space of $n \times n$ symmetric complex matrices with positive definite imaginary part, that is

$$\mathfrak{S}_n = \{Z \in \mathrm{Mat}(n, \mathbb{C}) : Z = Z^T \text{ and } \mathrm{Im}(Z) > 0\}$$

The rank 1 Siegel upper half space, $\mathfrak{S}_1$ is simply the usual upper half plane model for the hyperbolic plane $\mathbb{H}^2$. This paper we will focus on the rank 2 Siegel space, that is $\mathfrak{S}_2$.

The rank 1 symplectic group, $\mathrm{Sp}(2, \mathbb{R})$, is the isomorphic to $\mathrm{SL}(2, \mathbb{R})$ and acts on the upper half plane, $\mathfrak{S}_1$, by linear fractional transformations. Analogously, $\mathrm{Sp}(4, \mathbb{R})$ acts on $\mathfrak{S}_2$ by generalized linear fractional transformations as follows. Let $M \in \mathrm{Sp}(4, \mathbb{R})$ and $Z = X + iY \in \mathfrak{S}_2$. Write M as blocks of 2×2 real matrices:

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The action $\mathrm{Sp}(4, \mathbb{R}) \times \mathfrak{S}_2 \to \mathfrak{S}_2$ is given by

$$M(Z) = (AZ + B)(CZ + D)^{-1}.$$

Does this even make sense? In order for this to be well defined, we must check that $(CZ + D)^{-1}$ exists and that $\mathfrak{S}_2$ is invariant. The following series of lemmas establishes these and other important facts about this action.

Lemma 1.9. *$CZ + D$ is nonsingular*

Proof. (Following the proof in [11]):

First note the following useful fact:

$$Z - Z^\dagger = X + iY - (X^T - iY^T) = X - X^T + i(Y + Y^T)$$

Since Z is symmetric we see that $X - X^T = 0_2$ and $Y + Y^T = 2Y$, so $\mathrm{Im}(Z)$ is:

$$\mathrm{Im}(Z) = Y = \tfrac{1}{2i}(Z - Z^\dagger)$$

Further note that

$$
\begin{aligned}
Z - Z^\dagger &= \begin{bmatrix} Z^\dagger & I_2 \end{bmatrix} \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} Z \\ I_2 \end{bmatrix}^\dagger \Omega \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} Z \\ I_2 \end{bmatrix}^\dagger M^\dagger \Omega M \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix}^\dagger \Omega \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} \\
&= (CZ^\dagger + D)(AZ+B) - (AZ^\dagger + B)(CZ+D)
\end{aligned}
$$

Y is a real positive definite matrix by assumption and the above computations show that

$$
Y = \tfrac{1}{2i}\left[(CZ^\dagger + D)(AZ+B) - (AZ^\dagger + B)(CZ+D)\right]
$$

Now for any nonzero $v \in \mathbb{C}^2$ we must have that

$$
\begin{aligned}
0 &< v^\dagger Y v \\
&< \left[\frac{1}{2i} v^\dagger (CZ^\dagger + D)(AZ+B)v - \frac{1}{2i} v^\dagger (AZ^\dagger + B)(CZ+D)v\right]
\end{aligned}
$$

and so $(CZ+D)v$ is nonzero. □

Lemma 1.10. *The Siegel Space is invariant under this action.*

Proof. (Again expanding on the proof in [11]):

We must show that $M(Z) = M(Z)^T$ and $M(Z) > 0$. Using the fact the Z is symmetric, we first show that the image $M(Z)$ is also symmetric.

$$
\begin{aligned}
0_2 &= Z - Z^T \\
&= \begin{bmatrix} Z \\ I_2 \end{bmatrix}^T \Omega \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} Z \\ I_2 \end{bmatrix}^T M^T \Omega M \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix}^T \Omega \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} \\
&= -(AZ+B)^T(CZ+D) + (CZ+D)^T(AZ+B) \\
&= (CZ+D)^T \left[-((CZ+D)^{-1})^T (AZ+B)^T + (AZ+B)(CZ+D)^{-1} \right] (CZ+D) \\
&= (CZ+D)^T \left[-((AZ+B)(CZ+D)^{-1})^T + (AZ+B)(CZ+D)^{-1} \right] (CZ+D) \\
&= (CZ+D)^T \left[-M(Z)^T + M(Z) \right] (CZ+D)
\end{aligned}
$$

Since $CZ + D$ is nonsingular, $-M(Z)^T + M(Z) = 0_2$ so $M(Z)$ is symmetric.

We must now show that $\mathrm{Im}(M(Z)) > 0$. To that end note that

$$\mathrm{Im}(M(Z)) = \tfrac{1}{2i}(M(Z) - M(Z))^\dagger.$$

Further a bit of computation yields

$$
\begin{aligned}
M(Z) - M(Z)^\dagger &= (AZ+B)(CZ+D)^{-1} - [AZ+B)(CZ+D)^{-1}]^\dagger \\
&= (AZ+B)(CZ+D)^{-1} - [(CZ+D)^{-1}]^\dagger (AZ+B)^\dagger \\
&= [(CZ+D)^{-1}]^\dagger \left[(CZ+D)^\dagger (AZ+B) - (AZ+B)^\dagger (CZ+D) \right] (CZ+D)^-
\end{aligned}
$$

To simplify notation, let $W = (CZ + D)^{-1}$ and note that the expression in square brackets was computed in the proof of lemma 1.9 to be $Z - Z^\dagger$. So we have

$$M(Z) - M(Z)^\dagger = W^\dagger[Z - Z^\dagger]W$$

$$\frac{1}{2i}[M(Z) - M(Z)^\dagger] = W^\dagger \left[\frac{1}{2i}[Z - Z^\dagger]\right] W$$

$$\mathrm{Im}(M(Z)) = W^\dagger Y W$$

Let $v \in \mathbb{C}^2$ be any nonzero vector. Then

$$v^\dagger \mathrm{Im}(M(Z))v = v^\dagger W^\dagger Y W v$$

$$= (Wv)^\dagger Y (Wv) > 0$$

which is positive since Y is positive definite. Hence the imaginary part of $M(Z)$ is positive definite and $\mathfrak{S}_2$ is invariant. □

Lemma 1.11. *The action of $Sp(4, \mathbb{R})$ on $\mathfrak{S}_2$ is transitive.*

Proof. Choosing iI_2 as a basepoint, it suffices to show that for any point $Z = X + iY \in \mathfrak{S}_2$ there is $M \in \mathrm{Sp}(4, \mathbb{R})$ such that $M(iI_2) = Z$. Since Y is positive definite, it has a positive definite square root, i.e. there is a matrix $Y^{1/2}$ such that $(Y^{1/2})^2 = Y$. Let $M = \begin{bmatrix} Y^{1/2} & XY^{-1/2} \\ 0 & Y^{-1/2} \end{bmatrix}$. It is easy to check that $M^T \Omega M = \Omega$ so $M \in \mathrm{Sp}(4, \mathbb{R})$ and that $M(iI) = Z$ as desired. □

Lemma 1.12. *Any point $Z \in \mathfrak{S}_2$ is a nonsingular matrix*

Proof. Suppose $X + iY \in \mathfrak{S}_2$ and $v = v_1 + iv_2 \in \mathbb{C}^2$ where $v_1, v_2 \in \mathbb{R}^2$ such that $(X + iY)v = 0$. Expanding we have $0 = (Xv_1 - Yv_2) + i(Yv_1 + Xv_2)$ which occurs if and only if

$$Xv_1 = Yv_2 \text{ and } Yv_1 = -Xv_2.$$

Since $Y > 0$ it is invertible it is possible to solve the first equation for v_2 and substituting that solution into the second equation we obtain

$$Yv_1 = -XY^{-1}Xv_1$$

Multiplying both sides on the left by v_1^T and using the fact that $Y > 0$ we obtain:

$$\begin{aligned} 0 &< v_1^T Y v_1 && (Y > 0) \\ &= -v_1^T X Y^{-1} X v_1 \\ &= -(Xv_1)^T Y^{-1}(Xv_1) && (\text{Since } X = X^T) \\ &< 0 && (\text{Since } Y^{-1} > 0) \end{aligned}$$

So $v_1 = 0$, and hence $v_2 = 0$, so $X + iY$ is invertible. □

Lemma 1.13. *The isotropy subgroup of $Sp(4, \mathbb{R})$ acting on $\mathfrak{S}_2$ is isomorphic to the unitary group $U(2)$*

Proof. If M stabilizes the basepoint $iI_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix} \in \mathfrak{S}_2$, then

$$\begin{aligned} (AiI_2 + B)(CiI_2 + D)^{-1} &= iI_2 \\ iA + B &= i(iC + D) \\ iA + B &= iD - C \\ i(A - D) + (B + C) &= 0 \end{aligned}$$

This immediately implies that $A = D$ and $B = -C$. So the isotropy subgroup consists of matrices of the form

$$M = \begin{bmatrix} A & B \\ -B & A \end{bmatrix}.$$

Symplecticity of M means

$$M^T \Omega M = \Omega$$

$$\begin{bmatrix} A^T & -B^T \\ B^T & A^T \end{bmatrix} \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} -B^T A + A^T B & -B^T B - A^T A \\ A^T A + B^T B & A^T B - B^T A \end{bmatrix} = \begin{bmatrix} 0 & -I_2 \\ I_2 & 0 \end{bmatrix}$$

This yields 2 the equations: $A^T B - B^T A = 0_2$ and $A^T A + B^T B = I_2$ which are the defining equations for $\mathrm{U}(2)$ described in §1.3. $\square$

Recapping, the above lemmas established that $\mathrm{Sp}(4, \mathbb{R})$ acts transitively on the $\mathfrak{S}_2$ and the isotropy subgroup is isomorphic to $\mathrm{U}(2)$. Thus the Siegel upper half space is a homogeneous space and there is a diffeomorphism

$$\boxed{\mathrm{Sp}(4, \mathbb{R})/\mathrm{U}(2) \xrightarrow{\approx} \mathfrak{S}_2}$$

This diffeomorphism is given by evaluation at any point in $\mathfrak{S}_2$. A convenient basepoint to choose is $iI_2 = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, so for a matrix $M \in \mathrm{Sp}(4, \mathbb{R})$, the coset $M/\mathrm{U}(2)$ identifies with the point $M(iI_2) \in \mathfrak{S}_2$.

Chapter 2

The Dual Manifold

When studying hyperbolic geometry, it is often convenient to realize the upper half plane as embedded in the complex projective line $\mathbb{CP}^1$. The ideal boundary of $\mathbb{H}^2$ can easily be described as a circle in $\mathbb{CP}^1$, and the linear fractional action of $\mathrm{Isom}(\mathbb{H}^2) \approx \mathrm{SL}(2, \mathbb{R})$ extends naturally to an action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{CP}^1$. In addition $\mathbb{CP}^1$ contains a bounded model for hyperbolic geometry, the *Poincare unit disk*. There is a mapping of $\mathbb{CP}^1$ called the *Cayley Map*, which maps the upper half plane to the Poincare unit disk.

Analogously, the Siegel upper half space is contained in a larger space called its *Compact Dual*. The action of $\mathrm{Sp}(4, \mathbb{R})$ on $\mathfrak{S}_2$ extends to an action of $\mathrm{Sp}(4, \mathbb{C})$ on the compact dual and many computations become easier by first lifting to this larger space. The compact dual contains a bounded model for the geometry of the Siegel space as well. There is an $\mathrm{Sp}(4, \mathbb{C})$ mapping called the Cayley map from $\mathfrak{S}_2$ to this bounded model.

To motivate the construction of the compact dual, we first recall the construction of $\mathbb{CP}^1$ and its relationship to the upper half plane. Then we will construct the compact dual to $\mathfrak{S}_2$ in an analogous manner. We will see that the dual manifold identifies with the *Complex Lagrangian Grassmannian*, i.e. the subspace of the Grassmannian of 2-planes in $\mathbb{C}^4$ consisting of Lagrangian 2-planes.

2.1 $\mathbb{H}^2$ and $\mathbb{CP}^1$

The nonzero complex numbers $\mathbb{C}^*$ act on the vector space $\mathbb{C}^2$ by scalar multiplication and define an equivalence relation on $\mathbb{C}^2$ by $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \sim \begin{bmatrix} \lambda z_1 \\ \lambda z_2 \end{bmatrix}$ for every $\lambda \in \mathbb{C}^*$. The Complex Projective Line $\mathbb{CP}^1$ is defined as the quotient space $(\mathbb{C}^2 - \{\vec{0}\})/\sim$. The group $\mathrm{GL}(2, \mathbb{C})$ acts on $\mathbb{C}^2$ preserving equivalence classes, so the action passes down to a well defined action on $\mathbb{CP}^1$. The upper half plane is then naturally embedded by

$$j : \mathbb{H}^2 \hookrightarrow \mathbb{CP}^1$$
$$j(z) = \begin{bmatrix} z \\ 1 \end{bmatrix}$$

A standard exercise in hyperbolic geometry is to check that the subgroup $\mathrm{SL}(2, \mathbb{R})$ leaves the image of j invariant, and under this embedding corresponds to the linear fractional action of $\mathrm{SL}(2, \mathbb{R})$ on $\mathbb{H}^2$.

The Dual manifold to $\mathbb{H}^2$ is $\mathbb{CP}^1$. For comparison with the dual manifold constructed in the next section, note that a line in $\mathbb{C}^2$ is a Lagrangian subspace of $\mathbb{C}^2$ and hence $\mathbb{CP}^1$ can be thought of as the space of Lagrangian subspaces of $\mathbb{C}^2$.

2.2 $\mathfrak{S}_2$ and $\mathrm{Lag}(\mathbb{C}^4)$

Consider the space of 4×2 complex matrices U of rank 2 satisfying $U^T \Omega U = 0_2$. There are two convenient ways of viewing this space corresponding to 2 natural ways of breaking up a 4×2 matrix.

- We can view elements as pairs of 2×2 block matrices $U = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$. In this description this space takes the form:

$$\left\{ \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} : \text{rank} \left(\begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} \right) = 2 \text{ and } Z_2^T Z_1 - Z_1^T Z_2 = 0_2 \right\}$$

  In this description the condition $U^T \Omega U = 0_2$ sort of generalizes the notion of "symmetric". In particular note that if $Z_2 = I_2$, this is precisely the condition that Z_1 is symmetric.

- Consider elements as pairs of column vectors $U = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$. In this description, U having rank 2 means that $\text{Span}\{c_1, c_2\}$ is a 2-plane in $\mathbb{C}^4$. The condition $U^T \Omega U = 0_2$ means that this plane is **Lagrangian**. In this viewpoint the space takes the form:

$$\{[c_1, c_2] : c_1, c_2 \text{ are linearly independent, and } \omega(c_1, c_2) = 0\}.$$

  The space thus identifies with the space of *Lagrangian 2-Frames* in $\mathbb{C}^4$.

  Define an equivalence relation on Lagrangian 2-Frames by

$$U \sim Ug \text{ for every } g \in \text{GL}(2, \mathbb{C}).$$

Definition 2.1. The Compact Dual Manifold is the space of equivalence classes of Lagrangian frames under the above equivalence relation. For a Lagrangian plane

in $\mathbb{C}^4$, the right action of $\mathrm{GL}(2,\mathbb{C})$ simply changes the basis for that plane. The compact dual can then be seen as

$$\begin{aligned}\text{Compact Dual} &= \{U_{4\times 2} : rk(U) = 2 \text{ and } U^T\Omega U = 0_2\}/\mathrm{GL}(2,\mathbb{C}) \\ &= \{\text{Lagrangian 2-Frames}\}/\text{Change of basis} \\ &= \text{Lagrangian Planes} \\ &=: \mathrm{Lag}(\mathbb{C}^4)\end{aligned}$$

$\mathrm{Lag}(\mathbb{C}^4)$ is called the *Complex Lagrangian Grassmannian*, and is the dual manifold to the Siegel upper half space. It is naturally a subspace of the Grassmannian of all 2-planes in $\mathbb{C}^4$ and is endowed with the subspace topology.

The description of this space as equivalence classes of 4×2 matrices is analogous to the description of homogeneous coordinates on $\mathbb{CP}^1$. So at the risk of creating some confusion, these will be referred to as *Siegel Homogeneous Coordinates* for $\mathrm{Lag}(\mathbb{C}^4)$. There are other natural coordinate systems which are described in chapter 5, but we will primarily use these coordinates.

A collection of facts about $\mathrm{Lag}(\mathbb{C}^4)$:

- <u>$\mathrm{Sp}(4,\mathbb{C})$ action:</u> The group $\mathrm{Sp}(4,\mathbb{C})$ acts on $\mathrm{Lag}(\mathbb{C}^4_\omega)$ by left multiplication. For $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathrm{Sp}(4,\mathbb{C})$ written as 2×2 blocks and a Lagrangian Frame $U = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix}$,

$$MU = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} AZ_1 + BZ_2 \\ CZ_1 + DZ_2 \end{bmatrix}$$

Since M is nonsingular the image also has rank 2. Further since M is symplectic,

$$(MU)^T\Omega(MU) = U^T(M^T\Omega M)U = U^T\Omega U = 0_2$$

hence the image of U is Lagrangian.

Finally, since the $\mathrm{Sp}(4,\mathbb{C})$ action is on the left, and the equivalence relation is defined via a right action, the action of $\mathrm{Sp}(4,\mathbb{C})$ is well defined on equivalence classes and passes down to an action on $\mathrm{Lag}(\mathbb{C}^4)$. This action is simply the natural action of $\mathrm{GL}(4,\mathbb{C})$ on the Grassmannian of 2-planes in $\mathbb{C}^4$, restricted to the subgroup $\mathrm{Sp}(4,\mathbb{C})$ on the subspace of Lagrangian planes.

- <u>$\mathfrak{S}_2$ embedding:</u> $\mathfrak{S}_2$ embeds into $\mathrm{Lag}(\mathbb{C}^4)$ by

$$j : Z \hookrightarrow \begin{bmatrix} Z \\ I_2 \end{bmatrix}$$

If $M \in \mathrm{Sp}(4,\mathbb{R}) \subset \mathrm{Sp}(4,\mathbb{C})$, then M acts on the image of j by

$$\begin{aligned} \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} Z \\ I_2 \end{bmatrix} &= \begin{bmatrix} AZ+B \\ CZ+D \end{bmatrix} \\ &\sim \begin{bmatrix} (AZ+B)(CZ+D)^{-1} \\ I_2 \end{bmatrix} \\ &= j((AZ+B)(CZ+D)^{-1}) \end{aligned}$$

so this embedding is equivariant with respect to the generalized linear fractional action of $\mathrm{Sp}(4,\mathbb{R})$ on $\mathfrak{S}_2$. By lemma 1.10, the image of j is invariant under $\mathrm{Sp}(4,\mathbb{R})$.

Conversely, consider the subspace of $\mathrm{Lag}(C^4_\omega)$ consisting of those U such that

$$\tfrac{1}{2i}U^\dagger \Omega U > 0.$$

First note that this in fact makes sense: For any $v \in \mathbb{C}^4$,

$$\overline{\omega(\overline{v}, v)} = \overline{(\overline{v})^T \Omega(v)} = (v)^T \Omega(\overline{v}) = \omega(v, \overline{v}) = -\omega(\overline{v}, v)$$

which implies that $\omega(\overline{v}, v)$ is purely imaginary. Then

$$v^\dagger(\tfrac{1}{2i}U^\dagger \Omega U)v = \tfrac{1}{2i}(Uv)^\dagger \Omega(Uv) = \tfrac{1}{2i}\omega(\overline{Uv}, Uv)$$

is in fact real. Further

$$\tfrac{1}{2i}U^\dagger \Omega U = \tfrac{1}{2i}(Z_2^\dagger Z_1 - Z_1^\dagger Z_2) > 0$$

implies that Z_1 and Z_2 are nonsingular. Acting on the positive definite matrix $\frac{1}{2i}U^\dagger \Omega U$ by Z_2^{-1} we have:

$$\begin{aligned}
0 &< (Z_2^{-1})^\dagger(\frac{1}{2i}U^\dagger \Omega U)(Z_2^{-1}) \\
&= \frac{1}{2i}(UZ_2^{-1})^\dagger \Omega(UZ_2^{-1}) \\
&= \frac{1}{2i}[(Z_1Z_2^{-1})^\dagger \ I]\Omega \begin{bmatrix} Z_1Z_2^{-1} \\ I \end{bmatrix} \\
&= \frac{1}{2i}(Z_1Z_2^{-1} - (Z_1Z_2^{-1})^\dagger) \\
&= \mathrm{Im}(Z_1Z_2^{-1})
\end{aligned}$$

So the complex Lagrangian planes U satisfying $\frac{1}{2i}U^\dagger \Omega U > 0$ correspond to points in the Siegel upper half space.

2.3 The bounded model and the Cayley transform

The Poincare unit disk model of the hyperbolic plane is

$$\{z \in \mathbb{C} : 1 - z\overline{z} > 0\}$$

Since $\mathbb{C}$ naturally identifies with an affine patch of $\mathbb{CP}^1$, the Poincare disk can be seen as contained in $\mathbb{CP}^1$ via the embedding $z \hookrightarrow \begin{bmatrix} z \\ 1 \end{bmatrix}$. There is a conformal automorphism of $\mathbb{CP}^1$ given by the $\mathrm{SL}(2, \mathbb{C})$ matrix

$$\tfrac{\sqrt{2}}{2} \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$$

which maps the upper half plane to the Poincare unit disk. This mapping is an isometry with respect to the respective hyperbolic metrics on the upper half plane and the Poincare disk, and is called the *Cayley map* or *Cayley transformation*.

Similarly, there is a bounded model for $\mathfrak{S}_2$. Let

$$\mathfrak{D}_2 = \{Z \in \mathrm{Mat}(2, \mathbb{C}) : Z = Z^T \text{ and } I_2 - Z^\dagger Z > 0\}$$

This embeds into $\mathrm{Lag}(\mathbb{C}^4)$ in the obvious way: $Z \hookrightarrow \begin{bmatrix} Z \\ I_2 \end{bmatrix}$. In Siegel homogeneous coordinates the bounded model is described by

$$\mathfrak{D}_2 = \left\{ U_{4\times 2} : U^T \Omega U = 0_2 \text{ and } U^\dagger \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} U > 0 \right\}$$

The Cayley transform is given by the $\mathrm{Sp}(4, \mathbb{C})$ matrix:

$$\mathrm{Cay} = \tfrac{\sqrt{2}}{2} \begin{bmatrix} I_2 & -iI_2 \\ -iI_2 & I_2 \end{bmatrix}$$

Lemma 2.2. *The Cayley transform maps* $\mathfrak{S}_2 \to \mathfrak{D}_2$.

Proof. Let $\begin{bmatrix} Z \\ I_2 \end{bmatrix} \in \mathfrak{S}_2$. Then

$$\mathrm{Cay}\left(\begin{bmatrix} Z \\ I_2 \end{bmatrix}\right) = \tfrac{\sqrt{2}}{2}\begin{bmatrix} Z - iI \\ -iZ + I \end{bmatrix}.$$

Since the Cayley transform is in $\mathrm{Sp}(4, \mathbb{C})$ the image of a Lagrangian plane is a Lagrangian plane and thus $U^T \Omega U = 0_2$. For the other condition we compute

$$\begin{aligned} U^\dagger \begin{bmatrix} -I_2 & 0 \\ 0 & I_2 \end{bmatrix} U &= \frac{1}{2}[(Z^\dagger + iI) \quad (iZ^\dagger + I)] \begin{bmatrix} -Z + iI \\ -iZ + I \end{bmatrix} \\ &= \frac{1}{2}[(Z^\dagger + iI)(-Z + iI) + (iZ^\dagger + I)(-iZ + I)] \\ &= \frac{1}{i}(Z - Z^\dagger) \end{aligned}$$

Now $\mathrm{Im}(Z) = \frac{1}{2i}(Z - Z^\dagger) > 0$ by assumption, so $U \in \mathfrak{D}_2$. $\square$

Chapter 3

The Riemannian structure of $\mathfrak{S}_2$

In this section we describe $\mathfrak{S}_2$ as a Riemannian manifold. Much of this geometry was described by Siegel in [19]. We will take a slightly different approach using the geometry of the Lie group $\mathrm{Sp}(4,\mathbb{R})$, and make use of results tediously obtained in appendix A. The main tools we will use are lemma A.1, the bilinear form given at the end of §A.2 and corollary A.6.

The bilinear form given at the end of §A.2 is a positive definite inner product on $T_{iI}\mathfrak{S}_2$ and is invariant under the isotropy subgroup $U(2)$ by construction. The idea will be to use this form together with the transitivity of the $\mathrm{Sp}(4,\mathbb{R})$ action (lemma 1.11) to induce an invariant metric on $\mathfrak{S}_2$.

3.1 The Riemannian Metric

The tangent space at the base point $T_{iI}\mathfrak{S}_2$ identifies with all 2×2 complex symmetric matrices. If $v = Ai + B \in T_{iI}\mathfrak{S}_2$ then lemma A.2 yields an identification of $T_{iI}\mathfrak{S}_2$ with a subspace $\mathfrak{p} \subset \mathfrak{sp}(4,\mathbb{R})$ given by

$$T_{iI}\mathfrak{S}_2 \longleftrightarrow \mathfrak{p}$$

$$v = Ai + B \longleftrightarrow V = \frac{1}{2}\begin{bmatrix} A & B \\ B & -A \end{bmatrix}$$

Restricting the trace form on $\mathfrak{sp}(4,\mathbb{R})$ to $\mathfrak{p}$ (see A.2) induces a positive definite bilinear form

$$B(V,V) = 2\mathrm{tr}(V^2) = \mathrm{tr}(v\overline{v})$$

which defines a U(2) invariant inner product on $T_{iI}\mathfrak{S}_2$ by

$$< v, v >_{iI} = \mathrm{tr}(v\overline{v})$$

This inner product can be extended to a Riemannian metric on $\mathfrak{S}_2$. Let $X + iY \in \mathfrak{S}_2$, and consider the automorphism of $\mathfrak{S}_2$ that is $f : Z \to YZ + X$. The fact that $Y > 0$ implies that Y has a square root, i.e. there is a matrix $\sqrt{Y} > 0$ such that $\sqrt{Y}^2 = Y$. The map f is then given by the symplectic matrix:

$$M = \begin{bmatrix} \sqrt{Y} & X\sqrt{Y}^{-1} \\ 0 & \sqrt{Y}^{-1} \end{bmatrix} \in \mathrm{Sp}(4,\mathbb{R})$$

Now $f(iI) = X + iY$ and the differential of this map is simply Y, so we define the inner product at $X+iY$ in such a way that f is an isometry. So for $v \in T_{X+iY}\mathfrak{S}_2$,

$$\begin{aligned} < v, v >_{X+iY} &=< (df)^{-1}v, (df)^{-1}v >_{iI} \\ &=< Y^{-1}v, Y^{-1}v >_{iI} \\ &= \mathrm{tr}(Y^{-1}v\overline{Y^{-1}v}) \\ &= \mathrm{tr}(Y^{-1}vY^{-1}\overline{v}) \end{aligned}$$

Letting dZ denote the differential of Z, the Riemannian metric on $\mathfrak{S}_2$ is given by

$$ds^2 = \mathrm{tr}(Y^{-1}dZY^{-1}\overline{dZ})$$

$\mathrm{Sp}(4,\mathbb{R})$ now acts by isometries with respect to this metric.

3.2 The Generalized Cross Ratio

In [19], Siegel defines a *Generalized Cross Ratio* for points in $Z, Z_1 \in \mathfrak{S}_2$ by

$$R(Z, Z_1) = (Z - Z_1)(Z - \overline{Z_1})^{-1}(\overline{Z} - \overline{Z_1})(\overline{Z} - Z_1)^{-1}$$

In §6.6 we will extend his definition in a natural way to the dual manifold $\mathrm{Lag}(\mathbb{C}^4)$, and see that this definition generalizes the familiar cross ratio for quadruples of points in $\mathbb{CP}^1$. Siegel's primary use for this definition is the following theorem which generalizes a similar result for the upper half plane:

Theorem 3.1 (Siegel, Theorem 2 from [19])**.** *For any points $Z, Z_1, W, W_1 \in \mathfrak{S}_2$ there is a transformation $M \in Sp(4, \mathbb{R})$ such that $M(Z) = W$ and $M(Z_1) = W_1$ if and only if the cross ratios $R(Z, Z_1)$ and $R(W, W_1)$ have the same eigenvalues.*

Proof. ($\Rightarrow$) Suppose there is $M \in \mathrm{Sp}(4, \mathbb{R})$ written as 2×2 blocks. Then

$$\begin{aligned}
Z - Z_1 &= \begin{bmatrix} Z_1 \\ I_2 \end{bmatrix}^T \Omega \begin{bmatrix} Z \\ I_2 \end{bmatrix} = \begin{bmatrix} Z_1 \\ I_2 \end{bmatrix}^T M^T \Omega M \begin{bmatrix} Z \\ I_2 \end{bmatrix} \\
&= \begin{bmatrix} AZ_1 + B \\ CZ_1 + D \end{bmatrix}^T \Omega \begin{bmatrix} AZ + B \\ CZ + D \end{bmatrix} \\
&= (CZ_1 + D)^T(AZ + B) - (AZ_1 + B)^T(CZ + D) \\
&= (CZ_1 + D)^T \left[(AZ + B)(CZ + D)^{-1} - ((CZ_1 + D)^T)^{-1}(AZ_1 + B)^T\right](CZ + D) \\
&= (CZ_1 + D)^T[M(Z) - M(Z_1)^T](CZ + D) \\
&= (CZ_1 + D)^T[W - W_1](CZ + D)
\end{aligned}$$

Similarly we can compute that:

$$Z - \overline{Z_1} = (C\overline{Z_1} + D)^T (W - \overline{W_1})(CZ + D)$$

$$\overline{Z} - \overline{Z_1} = (C\overline{Z_1} + D)^T (\overline{W} - \overline{W_1})(C\overline{Z} + D)$$

$$\overline{Z} - Z_1 = (CZ_1 + D)^T (\overline{W} - W_1)(C\overline{Z} + D)$$

Observe that $Z - \overline{Z_1} = (X - X_1) + i(Y + Y_1)$, so this is a point in $\mathfrak{S}_2$. By lemma 1.12 it is invertible, hence $(Z - \overline{Z_1})^{-1}$ (and inverse of all the other differences above) make sense. Then computing $R(Z, Z_1)$ we have:

$$\begin{aligned}(Z - Z_1)(Z - \overline{Z_1})^{-1} &= (CZ_1 + D)^T (W - W_1)(CZ + D)\ [(C\overline{Z_1} + D)^T (W - \overline{W_1})(CZ + D)]^{-1} \\ &= (CZ_1 + D)^T (W - W_1)(W - \overline{W_1})^{-1}((C\overline{Z_1} + D)^T)^{-1}\end{aligned}$$

and

$$\begin{aligned}(\overline{Z} - \overline{Z_1})(\overline{Z} - Z_1)^{-1} &= (C\overline{Z_1} + D)^T (\overline{W} - \overline{W_1})(C\overline{Z} + D)[(CZ_1 + D)^T (\overline{W} - W_1)(C\overline{Z} + D)]^{-1} \\ &= (C\overline{Z_1} + D)^T (\overline{W} - \overline{W_1})(\overline{W} - W_1)^{-1}((CZ_1 + D)^T)^{-1}\end{aligned}$$

Multiplying the 2 previous products together we obtain:

$$R(Z, Z_1) = (CZ_1 + D)^T (W - W_1)(W - \overline{W_1})^{-1}(\overline{W} - \overline{W_1})(\overline{W} - W_1)^{-1}((CZ_1 + D)^T)^{-1}$$

$$\boxed{R(Z, Z_1) = (CZ_1 + D)^T R(W, W_1)((CZ_1 + D)^T)^{-1}}$$

So the cross ratios $R(Z, Z_1)$ and $R(W, W_1)$ are conjugate matrices, hence have the same eigenvalues.

($\Leftarrow$) Suppose that $R(Z, Z_1)$ and $R(W, W_1)$ have the same eigenvalues. Since the $\mathrm{Sp}(4, \mathbb{R})$ action is transitive (lemma 1.11) we may assume Z_1 is the basepoint iI. Using Corollary A.6, we can apply a matrix in the stablizer of iI to assume

that $Z = i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ for some $1 \leq y_1 \leq y_2 \in \mathbb{R}$. As proved above, both of these transformations leave the eigenvalues of the cross ratios invariant.

Now it's easy to compute that

$$R(Z, Z_1) = \begin{bmatrix} \left(\frac{y_1-1}{y_1+1}\right)^2 & 0 \\ 0 & \left(\frac{y_2-1}{y_2+1}\right)^2 \end{bmatrix}$$

so the eigenvalues are

$$\lambda_i = \left(\frac{y_i-1}{y_i+1}\right)^2 \text{ for } i = 1, 2.$$

Solving for y_i we obtain that

$$y_i = \frac{1+\sqrt{\lambda_i}}{1-\sqrt{\lambda_i}} \text{ for } i = 1, 2.$$

hence

$$Z = i \begin{bmatrix} \frac{1+\sqrt{\lambda_1}}{1-\sqrt{\lambda_1}} & 0 \\ 0 & \frac{1+\sqrt{\lambda_2}}{1-\sqrt{\lambda_2}} \end{bmatrix}$$

is entirely determined by the eigenvalues of $R(Z, Z_1)$. We could apply the same argument to the pair (W, W_1), and since the image of W depends only upon the eigenvalues of $R(W, W_1)$, W_1 would map to the base point Z_1 and W would map to Z as desired. $\square$

3.3 Geodesics in $\mathfrak{S}_2$

In [19], Siegel computes geodesics directly from the metric by minimizing the length of an arbitrary curve between 2 points. It is much simpler to take a Lie theory approach following the ideas of [7]. For any unit tangent vector v, $\gamma(t) = \exp(tv)$

is a geodesic parameterized at unit speed. By transitivity of the $\mathrm{Sp}(4,\mathbb{R})$ action (lemma 1.11), we need only consider geodesics through the base point iI_2. We can restrict our attention further to geodesics contained in a specified *flat*.

Definition 3.2. A *Flat* in $\mathfrak{S}_2$ is a subspace isometric to Euclidean space. The subspace

$$\left\{ i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix} : y_1, y_2 > 0 \right\}$$

is called the *Standard Flat*.

The standard flat is the image under the exponential map of

$$\left\{ i \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} : a, b \in \mathbb{R} \right\} \subset T_{iI_2}\mathfrak{S}_2$$

Lemma A.1 identifies this with the Cartan subalgebra of $\mathfrak{sp}(4,\mathbb{R})$. Corollary A.6 says that every element in $\mathfrak{sp}(4,\mathbb{R})$ can be mapped to this Cartan subalgebra by a unitary matrix. Thus every $v \in T_{iI}\mathfrak{S}_2$ can be mapped by a unitary matrix (stabilizing iI) to one in the tangent space to the standard flat.

Let $v = i \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in T_{iI_2}\mathfrak{S}_2$ be a unit tangent vector, so $a^2 + b^2 = 1$. This corresponds in the Cartan subalgebra to the matrix

$$V = \begin{bmatrix} \frac{a}{2} & 0 & 0 & 0 \\ 0 & \frac{b}{2} & 0 & 0 \\ 0 & 0 & \frac{-a}{2} & 0 \\ 0 & 0 & 0 & \frac{-b}{2} \end{bmatrix}$$

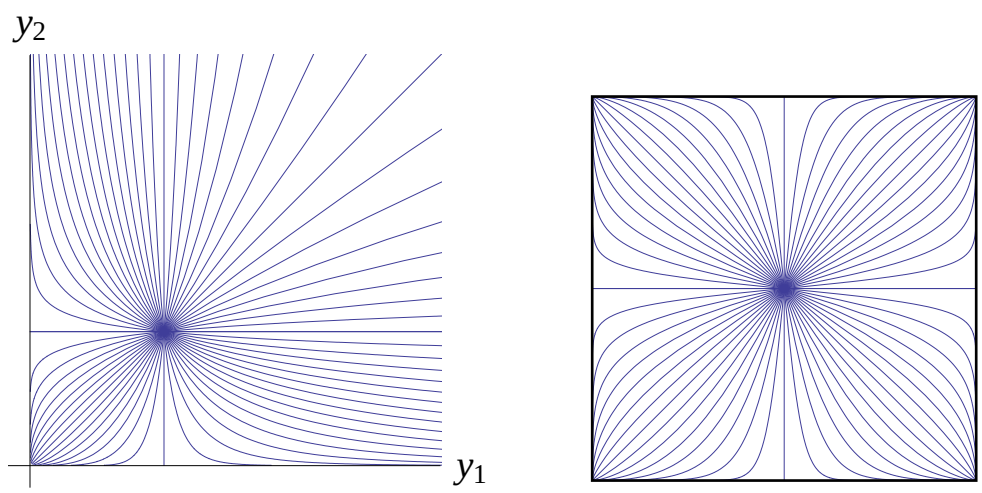
Then

$$\gamma(t) = \exp(tV)(iI) = \begin{bmatrix} ie^{at} & 0 \\ 0 & ie^{bt} \end{bmatrix}$$

is a unit speed geodesic through $\gamma(0) = iI$. Applying the Cayley transform from §2.3 to these geodesics, we obtain geodesics in the standard flat of the bounded model given by:

$$(\text{Cay})\gamma(t) = i \begin{bmatrix} \tanh\left(\frac{at}{2}\right) & 0 \\ 0 & \tanh\left(\frac{bt}{2}\right) \end{bmatrix}$$

The figure below shows these geodesics in the standard flat of the Siegel upper half space on the left and the bounded model on the right.



3.4 The Metric Distance

We can now compute the distance between $Z, Z_1 \in \mathfrak{S}_2$. By Corollary A.6 we may apply $\text{Sp}(4, \mathbb{R})$ transformations to assume that $Z = iI$ and $Z_1 = i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ where $1 \leq y_1 \leq y_2$. $\text{Sp}(4, \mathbb{R})$ acts by isometries so these transformations preserve

$\mathrm{dist}(Z, Z_1)$. In the proof of Theorem 3.1 we saw that y_1 and y_2 could be computed explicitly as functions of the eigenvalues of the cross ratio $R(Z, Z_1)$. Let λ_1, λ_2 be the eigenvalues of $R(Z, Z_1)$. Then

$$y_i = \tfrac{1+\sqrt{\lambda_i}}{1-\sqrt{\lambda_i}} = \coth\left(\tfrac{\ln \lambda_i}{4}\right) \quad \text{for } i = 1, 2$$

Suppose $\gamma(t) = i \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix}$ is the unit speed geodesic connecting these points. We can find the arc length by solving for t:

$$\gamma(t) = iZ_1$$

$$i \begin{bmatrix} e^{at} & 0 \\ 0 & e^{bt} \end{bmatrix} = i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$$

$$\Rightarrow \; e^{at} = y_1 \qquad e^{bt} = y_2$$

$$t = \frac{\ln(y_1)}{a} \qquad t = \frac{\ln y_2}{b}$$

$$\frac{\ln y_1}{\ln y_2} = \frac{a}{b}$$

Since $a^2 + b^2 = 1$ and $1 \leq y_1 \leq y_2$, we have that

$$\left(b\tfrac{\ln y_1}{\ln y_2}\right)^2 + b^2 = 1 \Rightarrow b = \tfrac{\ln y_2}{\sqrt{\ln^2 y_1 + \ln^2 y_2}}$$

Similarly $a = \frac{\ln y_1}{\sqrt{\ln^2 y_1 + \ln^2 y_2}}$. We now can compute

$$t = \tfrac{\ln y_2}{b} = \sqrt{\ln^2 y_1 + \ln^2 y_2}$$

Hence we have $\mathrm{dist}(Z, Z_1)$ as a function of the eigenvalues of $R(Z, Z_1)$

$$\mathrm{dist}(Z, Z_1)^2 = \ln^2 y_1 + \ln^2 y_2 = \ln^2 \coth\left(\tfrac{\ln \lambda_1}{4}\right) + \ln^2 \coth\left(\tfrac{\ln \lambda_2}{4}\right)$$

Chapter 4

Embeded Hyperbolic Planes

This section describes several different embeddings of $\mathrm{SL}(2,\mathbb{R})$ into $\mathrm{Sp}(4,\mathbb{R})$. An embedding of Lie groups induces an embedding of the corresponding homogeneous spaces, i.e. an embedding of the hyperbolic plane into the Siegel space. We will show that these are then equivariant with respect to the natural actions of the groups. The hyperbolic planes are then totally geodesic submanifolds of the Siegel space. In chapter 8 we will examine representations of triangle groups which factor through these embeddings.

The full group of isometries $\mathrm{Isom}(\mathbb{H}^2)$ consists of both orientation preserving and reversing isometries, and involutions generating triangle groups are in fact orientation reversing. Similarly $\mathrm{Isom}(\mathfrak{S}_2)$ consists of both symplectic and *anti-symplectic* transformations and the involutions of interest will in fact be anti-symplectic. For this reason we will need to examine to what extent the embeddings of $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$ extend to embeddings of $\mathrm{Isom}(\mathbb{H}^2) \hookrightarrow \mathrm{Isom}(\mathfrak{S}_2)$. This is the topic of §4.5 and will be used in a very crucial way in §8.2.1.

4.1 The bidisk $\mathbb{H}^2 \times \mathbb{H}^2$

Definition 4.1. The *Bidisk embedding* is the homomorphism

$$\Phi_{bidisk} : \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}(4,\mathbb{R})$$

$$\Phi_{bidisk}\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}\right) = \begin{bmatrix} a_{11} & 0 & a_{12} & 0 \\ 0 & b_{11} & 0 & b_{12} \\ a_{21} & 0 & a_{22} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{bmatrix}$$

In a basis different from the one we have been using this homomorphism is simply given by the direct sum of the two $\mathrm{SL}(2,\mathbb{R})$ matrices. It is straightforward to check that the image is actually symplectic.

The homogeneous space for $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$ is $\mathbb{H}^2 \times \mathbb{H}^2$. Using the Poincare disk model this is simply the product of two disks, hence the name bidisk embedding. The embedding of Lie groups induces an embedding of the bidisk into $\mathfrak{S}_2$ given in upper half plane coordinates by

$$\phi_{bidisk} : \mathbb{H}^2 \times \mathbb{H}^2 \hookrightarrow \mathfrak{S}_2$$

$$\phi_{bidisk}(x_1 + iy_1, x_2 + iy_2) = \begin{bmatrix} x_1 + iy_1 & 0 \\ 0 & x_2 + iy_2 \end{bmatrix}$$

This embedding is equivariant with respect to the appropriate actions, so the following diagram commutes:

$$\begin{tikzcd} \mathbb{H}^2 \times \mathbb{H}^2 \arrow[r, "\mathrm{SL}(2,\mathbb{R})\times\mathrm{SL}(2,\mathbb{R})"] \arrow[d, "\phi_{bidisk}"] & \mathbb{H}^2 \times \mathbb{H}^2 \arrow[d, "\phi_{bidisk}"] \\ \mathfrak{S}_2 \arrow[r, "\mathrm{Sp}(4,\mathbb{R})"] & \mathfrak{S}_2 \end{tikzcd}$$

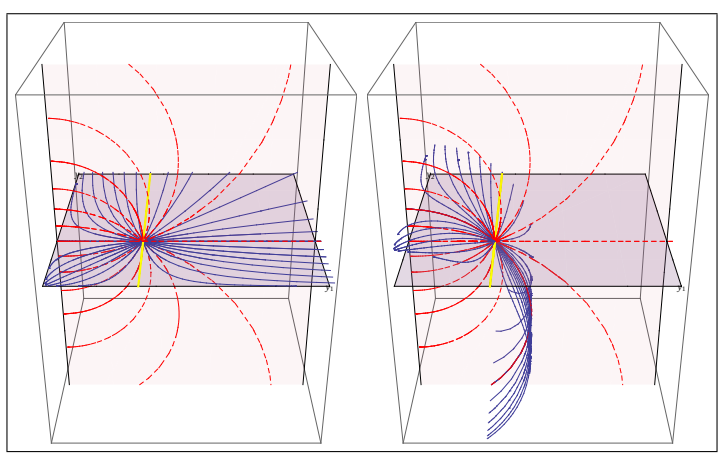
Let $R_\theta = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$. The image $\Phi_{bidisk}(R_\theta, I_2)$ is in

the isotropy subgroup of the base point U(2). For points $Z = i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ in the standard flat, these rotations act by

$$\Phi_{bidisk}(R_\theta, I_2)(Z) = \begin{bmatrix} \frac{\sin(\theta)+i\cos(\theta)y_1}{\cos(\theta)-i\sin(\theta)y_1} & 0 \\ 0 & iy_2 \end{bmatrix}$$

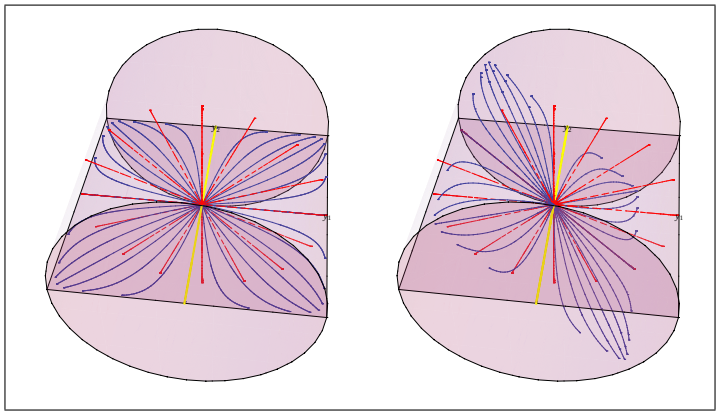
This action of fixes $\{i\} \times \mathbb{H}^2$ point-wise and acts as an elliptic element in the first factor of the bidisk. All geodesics through i in the first factor of the bidisk are images of the singular geodesic $\gamma(t) = i \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$.

The graphic below shows the 3 dimensional subspace of $\mathfrak{S}_2$ consisting of $\begin{bmatrix} x_1 + iy_1 & 0 \\ 0 & iy_2 \end{bmatrix}$. The y_1y_2-plane is the standard flat, and the second frame show the image of this flat under $\Phi_{bidisk}(R_{\frac{\pi}{8}}, I_2)$. The red curves are all images of the geodesic $i \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix}$ and are geodesics through i in the first factor of the embedded bidisk. The geodesic highlighted in yellow is fixed by the action.

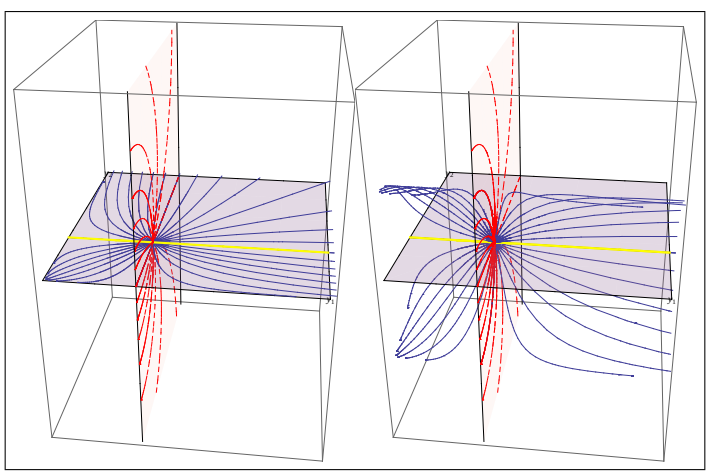


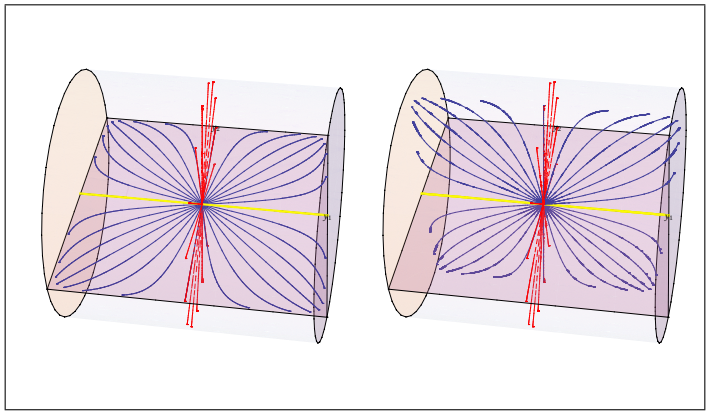
The picture is perhaps a bit less confusing in the bounded model. Here the rotation looks like a Euclidean Rotation and the embedded Poincare Unit Disk is

more apparent.



Similarly $\Phi_{bidisk}(I_2, R_\theta)$ fixes the embedded $\mathbb{H}^2 \times \{i\}$ point-wise and acts as an elliptic transformation in the second factor in the bidisk. The pictures shown below are in the 3 dimensional subspace $\begin{bmatrix} iy_1 & 0 \\ 0 & x_2 + iy_2 \end{bmatrix}$. All geodesics in this embedded $\mathbb{H}^2$ are images of the singular geodesic $\gamma(t) = i \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix}$.





Flash Animations of this action (and all remaining similar pictures) are available online at [15].

4.2 The diagonal in the bidisk

If we compose the diagonal embedding of $\mathrm{SL}(2,\mathbb{R})$ with the bidisk embedding we obtain an embedding

$$\Phi_{diag} : \mathrm{SL}(2,\mathbb{R}) \hookrightarrow SL(2,\mathbb{R}) \times SL(2,\mathbb{R}) \hookrightarrow Sp(4,\mathbb{R})$$

$$\Phi_{diag}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \Phi_{bidisk}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}, \begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{bmatrix}$$

This induces the diagonal embedding of $\mathbb{H}^2$

$$\phi_{diag} : \mathbb{H}^2 \hookrightarrow \mathfrak{S}_2$$

$$\phi_{diag}(x+iy) = \begin{bmatrix} x+iy & 0 \\ 0 & x+iy \end{bmatrix}$$

The matrix $\Phi_{diag}(R_\theta)$ acts on this embedded $\mathbb{H}^2$ by an elliptic transformation preserving i. The base point is its only fixed point in the standard flat. The embedded $\mathbb{H}^2$ is the image under this one parameter family of rotations of the singular geodesic

$$\gamma(t) = i \begin{bmatrix} e^{\frac{\sqrt{2}}{2}t} & 0 \\ 0 & e^{\frac{\sqrt{2}}{2}t} \end{bmatrix}$$

4.3 An embedded $\mathrm{GL}(2, \mathbb{R})$

Let $M \in \mathrm{GL}(2, \mathbb{R})$ and define the embedding

$$\Phi_{PDS} : \mathrm{GL}(2, \mathbb{R}) \hookrightarrow \mathrm{Sp}(4, \mathbb{R})$$
$$\Phi_{PDS}(M) = \begin{bmatrix} M & 0_2 \\ 0_2 & (M^T)^{-1} \end{bmatrix}$$

It is straightforward to check that the image is symplectic. Let $PDS(2, \mathbb{R})$ denote the cone of positive definite symmetric real matrices. This is a homogeneous space for $\mathrm{GL}(2, \mathbb{R})$. $\mathrm{GL}(2, \mathbb{R})$ acts transitively on $PDS(2, \mathbb{R})$ by transpose conjugation, and choosing $I_2 \in PDS(2, \mathbb{R})$ as a base point, the stabilizer is the orthogonal group $\mathrm{O}(2)$. Evaluation at the base point yields a diffeomorphism

$$\mathrm{GL}(2, \mathbb{R})/\mathrm{O}(2) \to PDS(2, \mathbb{R})$$

which identifies the coset $M/\mathrm{O}(2)$ with $MM^T \in PDS(2, \mathbb{R})$. Φ_{PDS} induces an equivariant embedding

$$\phi_{PDS} : PDS(2, \mathbb{R}) \hookrightarrow \mathfrak{S}_2$$

$$\phi_{PDS}(Y) = iY$$

The natural inclusion $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{GL}(2,\mathbb{R})$ induces an embedding $\mathbb{H}^2 \hookrightarrow PDS$. To compute this explicitly, recall that $x+iy \in \mathbb{H}^2$ is the image of i under the $\mathrm{SL}(2,\mathbb{R})$ matrix $M = \begin{bmatrix} y^{\frac{1}{2}} & xy^{-\frac{1}{2}} \\ 0 & y^{-\frac{1}{2}} \end{bmatrix}$, and hence the coset $M/SO(2)$ identifies with $x+iy$.

The inclusion $\mathrm{SL}(2,\mathbb{R}) \hookrightarrow \mathrm{GL}(2,\mathbb{R})$ induces a map on the coset spaces $\mathrm{SL}(2,\mathbb{R})/\mathrm{SO}(2) \to \mathrm{GL}(2,\mathbb{R})/\mathrm{O}(2)$ sending $M/\mathrm{SO}(2) \to M/\mathrm{O}(2)$. This coset identifies with $MM^T \in PDS$. So we have the mapping

$$\mathbb{H}^2 \hookrightarrow PDS$$

$$x+iy \to MM^T = \begin{bmatrix} y+x^2y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{bmatrix}$$

We summarize all the maps discussed in this section with the following commutative diagram:

$$\begin{tikzcd}
U(2) \arrow[r, hook] & Sp(4,\mathbb{R}) \arrow[r] & Sp(4,\mathbb{R})/U(2) \arrow[r, "\approx"] & \mathfrak{S}_2 \\
O(2) \arrow[r, hook] \arrow[u, hook, "\Phi_{PDS}"] & GL(2,\mathbb{R}) \arrow[r] \arrow[u, hook, "\Phi_{PDS}"] & GL(2,\mathbb{R})/O(2) \arrow[r, "\approx"] \arrow[u, hook, "\phi_{PDS}"] & PDS \arrow[u, hook, "\phi_{PDS}"] \\
SO(2) \arrow[r, hook] \arrow[u, hook] & SL(2,\mathbb{R}) \arrow[r] \arrow[u, hook] & SL(2,\mathbb{R})/SO(2) \arrow[r, "\approx"] \arrow[u, hook] & \mathbb{H}^2 \arrow[u, hook]
\end{tikzcd}$$

In particular the embedding induced by restriction $\mathbb{H}^2 \hookrightarrow \mathfrak{S}_2$ is:

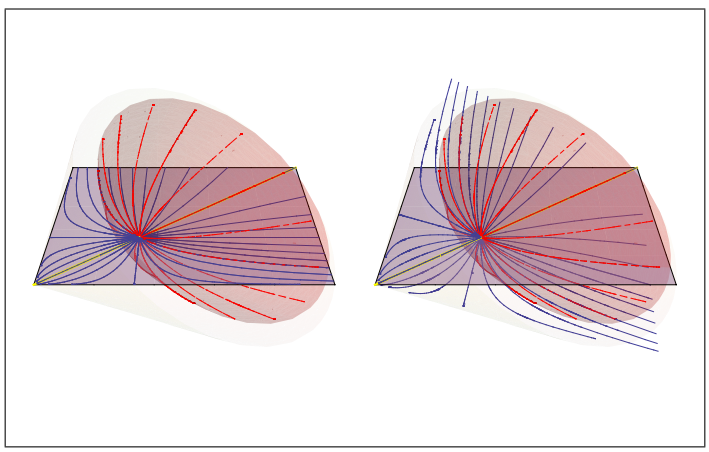
$$\phi_{PDS} : x+iy \to i \begin{bmatrix} (x^2+y^2)y^{-1} & xy^{-1} \\ xy^{-1} & y^{-1} \end{bmatrix}$$

The action of the image of Φ_{PDS} leaves the imaginary subspace of $\mathfrak{S}_2$ invariant. In particular, $\Phi_{PDS}(R_\theta) = \begin{bmatrix} R_\theta & 0_2 \\ 0_2 & R_\theta \end{bmatrix}$ fixes the singular geodesic

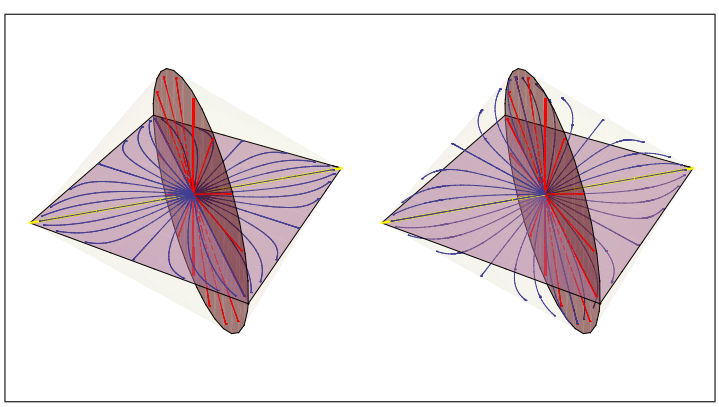
$$i \begin{bmatrix} e^{\frac{\sqrt{2}}{2}t} & 0 \\ 0 & e^{\frac{\sqrt{2}}{2}t} \end{bmatrix}$$

and acts as an elliptic transformation fixing i on this embedded $\mathbb{H}^2$.

The images below show the purely imaginary subspace of $\mathfrak{S}_2$. The standard flat is shown as well as its image under the action of $\Phi_{PDS}\left(R_{\frac{\pi}{8}}\right)$. The geodesic shown in yellow is fixed and the embedded $\mathbb{H}^2$ is a hyperboloid in this cone. All geodesics through the base point in this embedded $\mathbb{H}^2$ are images of the singular geodesic $\gamma(t) = i \begin{bmatrix} e^{\frac{\sqrt{2}}{2}t} & 0 \\ 0 & e^{-\frac{\sqrt{2}}{2}t} \end{bmatrix}$.



Again the picture is perhaps clearer in the bounded model where the embedded $\mathbb{H}^2$ is a copy of the Poincare disk.



4.3.1 A conjugate embedding of $\mathrm{GL}(2,\mathbb{R})$

There is an interesting embedding of $\mathrm{GL}(2,\mathbb{R})$ which is $\mathrm{Sp}(4,\mathbb{R})$ conjugate to Φ_{PDS} given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a & 0 & 0 & b \\ 0 & \frac{a}{ad-bc} & \frac{b}{ad-bc} & 0 \\ 0 & \frac{c}{ad-bc} & \frac{d}{ad-bc} & 0 \\ c & 0 & 0 & d \end{bmatrix}$$

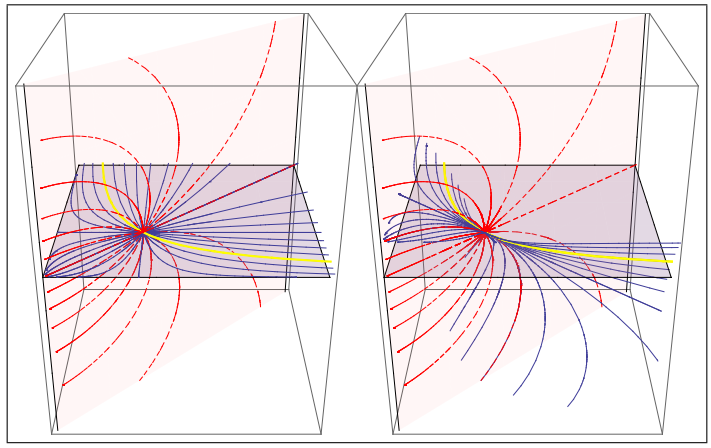
This induces an embedding $PDS \to \mathfrak{S}_2$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & \frac{ac+bd}{c^2+d^2} \\ \frac{ac+bd}{c^2+d^2} & 0 \end{bmatrix} + i \begin{bmatrix} \frac{(bc-ad)^2}{c^2+d^2} & 0 \\ 0 & \frac{1}{c^2+d^2} \end{bmatrix}$$

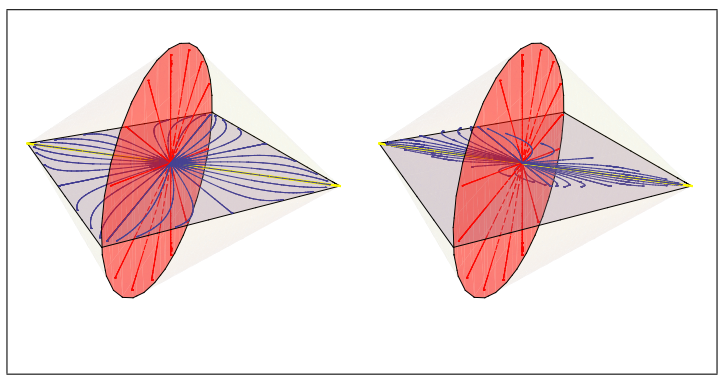
and by restriction we obtain an embedding of $\mathbb{H}^2$:

$$x + iy \to \begin{bmatrix} 0 & x \\ x & 0 \end{bmatrix} + i \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}$$

As in the previous cases, the rotation matrix $R_\theta \in \mathrm{SL}(2,\mathbb{R})$ embeds to obtain a one parameter group of elements in the isotropy subgroup. This group fixes one of the singular geodesics pointwise and the embedded $\mathbb{H}^2$ is the image of another singular geodesic under this group. The graphic below shows the subspace $\begin{bmatrix} iy_1 & x_{12} \\ x_{12} & iy_2 \end{bmatrix}$.



Once again the rotations look more Euclidean in the bounded model



4.4 The irreducible 4-dimensional embedding of SL$(2, R)$

Let $\mathbb{R}^2_\mu$ be the 2 dimensional vector space with basis $\{x, y\}$ equipped with the standard symplectic form μ given by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Let $v_1 \otimes v_2 \otimes v_3$ and $w_1 \otimes w_2 \otimes w_3$ be simple tensors in $(\mathbb{R}^2_\mu)^{\otimes 3}$. μ induces a bilinear form $<,>$ on $(\mathbb{R}^2_\mu)^{\otimes 3}$ defined for simple tensors by

$$< v_1 \otimes v_2 \otimes v_3, w_1 \otimes w_2 \otimes w_3 >:= \mu(v_1, w_1)\mu(v_2, w_2)\mu(v_3, w_3)$$

and extended to $(\mathbb{R}^2_\mu)^{\otimes 3}$ linearly.

Let $\mathcal{P}(x,y)$ denote the four dimensional vector space of homogeneous cubic polynomials in x and y. $\mathcal{P}(x,y)$ has a basis consisting of the cubic monomials $\{x^2, x^2y, xy^2, y^3\}$, and can be can be obtained as a quotient of $(\mathbb{R}^2_\mu)^{\otimes 3}$ by symmetrizing. More precisely, $(\mathbb{R}^2_\mu)^{\otimes 3}$ has a basis of simple tensors in x and y, and we can define a mapping $(\mathbb{R}^2_\mu)^{\otimes 3} \to \mathcal{P}(x,y)$ by specifying the images of the simple tensors:

$$x \otimes x \otimes x \to x^3,$$

$$\{x \otimes x \otimes y, x \otimes y \otimes x, y \otimes x \otimes x\} \to x^2y$$

$$\{x \otimes y \otimes y, y \otimes x \otimes y, y \otimes y \otimes x\} \to xy^2$$

$$y \otimes y \otimes y \to y^3$$

This mapping induces a bilinear form $<,>_{\mathcal{P}}$ on $\mathcal{P}(x,y)$ by averaging over the simple tensors in the preimage. Since μ is skew symmetric and each term will have 3 factors, it is clear that $<,>_{\mathcal{P}}$ is skew symmetric. It suffices to define $<,>_{\mathcal{P}}$ for the pairs of basis elements, most of which will be zero. The only nonzero pairings are:

$$< x^3, y^3 >_{\mathcal{P}} =< x \otimes x \otimes x, y \otimes y \otimes y >= \mu(x,y)\mu(x,y)\mu(x,y) = -1$$

$$\begin{aligned} < x^2y, xy^2 >_{\mathcal{P}} &= \frac{1}{9}\left(\mu(x,y)\mu(x,y)\mu(y,x) + \mu(x,y)\mu(y,x)\mu(x,y) + \mu(y,x)\mu(x,y)\mu(x,y)\right) \\ &= \frac{1}{3} \end{aligned}$$

The bilinear form $<,>_{\mathcal{P}}$ is then a symplectic form given by the skew symmetric matrix

$$\mathcal{J} := \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & -\frac{1}{3} & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The linear action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{R}^2_\mu$ preserves μ and this induces an action on the tensor product $(\mathbb{R}^2_\mu)^{\otimes 3}$ preserving $<,>$. Composing with the above mapping induces an action on $\mathcal{P}$ preserving $\mathcal{J}$.

If $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{R})$, the images of the basis elements x and y under M are $ax + cy$ and $bx + dy$ respectively. M then acts on the basis elements for $\mathcal{P}(x,y)$:

$$x^3 \to (ax+cy)^3 = a^3x^3 + 3a^2cyx^2 + 3ac^2y^2x + c^3y^3$$

$$x^2y \to (ax+cy)^2(bx+dy) = a^2bx^3 + 2abcyx^2 + a^2dyx^2 + bc^2y^2x + 2acdy^2x + c^2dy^3$$

$$xy^2 \to (ax+cy)(bx+dy)^2 = ab^2x^3 + b^2cyx^2 + 2abdyx^2 + ad^2y^2x + 2bcdy^2x + cd^2y^3$$

$$y^3 \to (bx+dy)^3 = b^3x^3 + 3b^2dyx^2 + 3bd^2y^2x + d^3y^3$$

The result is a linear transformation and defines a representation of $\mathrm{SL}(2,\mathbb{R})$ given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \begin{bmatrix} a^3 & a^2b & ab^2 & b^3 \\ 3a^2c & da^2 + 2bca & cb^2 + 2adb & 3b^2d \\ 3ac^2 & bc^2 + 2adc & ad^2 + 2bcd & 3bd^2 \\ c^3 & c^2d & cd^2 & d^3 \end{bmatrix}$$

which is symplectic with respect to $\mathcal{J}$. To make use of this representation in studying $\mathfrak{S}_2$, we need to express this representation with respect to the Lagrangian basis. We obtain

Definition 4.2. The *irreducible 4 dimensional representation* is

$$\Phi_{irr} : \mathrm{SL}(2,\mathbb{R}) \to \mathrm{Sp}(4,\mathbb{R})$$

$$\Phi_{irr}\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{pmatrix} a^3 & -a^2b & b^3 & 3ab^2 \\ -3a^2c & da^2+2bca & -3b^2d & 3\left(-cb^2-2adb\right) \\ c^3 & -c^2d & d^3 & 3cd^2 \\ ac^2 & \frac{1}{3}\left(-bc^2-2adc\right) & bd^2 & ad^2+2bcd \end{pmatrix}$$

This induces an embedding $\phi_{irr} : \mathbb{H}^2 \hookrightarrow \mathfrak{S}_2$ given by

$$\phi_{irr}(x+iy) = \begin{pmatrix} -2x^3 & 3x^2 \\ 3x^2 & -6x \end{pmatrix} + i \begin{pmatrix} y^3+x^2y & -xy \\ -xy & y \end{pmatrix}$$

Of interesting note here is that all geodesics in this embedded $\mathbb{H}^2$ are regular geodesics.

4.5 Anti-Symplectic Mappings and Lagrangian Involutions

The full isometry group of $\mathbb{H}^2$ contains both orientation preserving and reversing transformations and is isomorphic to

$$\mathrm{SL}_{\pm}(2,\mathbb{R}) := \{M \in \mathrm{GL}(2,\mathbb{R}) : \det(M) = \pm 1\}$$

A transformation $M \in \mathrm{SL}(4,\mathbb{R})$ is called *Anti-Symplectic* if $M^T\Omega M = -\Omega$. Let

$$\mathrm{Sp}_{\pm}(4,\mathbb{R}) := \left\{M \in \mathrm{SL}(4,\mathbb{R}) : M^T\Omega M = \pm\Omega\right\}$$

An anti-symplectic mapping M acts on $\mathfrak{S}_2$ by generalized linear fractional action composed with complex conjugation: $Z \to M(\overline{Z})$.

A *Lagrangian Involution* is an involution in $\mathrm{Sp}_{\pm}(4,\mathbb{R})$ conjugate to

$$\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The eigenspaces of a Lagrangian involution are Lagrangian planes. To what extent do the $\mathrm{SL}(2,\mathbb{R})$ embeddings discussed thus far extend to embeddings of $\mathrm{SL}_{\pm}(2,\mathbb{R})$ in $\mathrm{Sp}_{\pm}(4,\mathbb{R})$? The $\mathrm{SL}(2,\mathbb{R})$ embeddings factoring through $\mathrm{GL}(2,\mathbb{R})$ obviously extend and their image still contained in $\mathrm{Sp}(4,\mathbb{R})$. We now address the extent to which the other embeddings extend.

4.5.1 Extending the bidisk embedding

Direct computation shows that

$$\Phi_{bidisk}(A,B)^T\Omega\Phi_{bidisk}(A,B) = \begin{bmatrix} 0 & 0 & -\det(A) & 0 \\ 0 & 0 & 0 & -\det(B) \\ \det(A) & 0 & 0 & 0 \\ 0 & \det(B) & 0 & 0 \end{bmatrix}$$

thus $\Phi_{bidisk}(A,B)$ is symplectic if and only if $\det(A) = \det(B) = 1$, and anti-symplectic if and only if $\det(A) = \det(B) = -1$. So Φ_{bidisk} extends not to all of $\mathrm{SL}_{\pm}(2,\mathbb{R}) \times \mathrm{SL}_{\pm}(2,\mathbb{R})$ but only to the subgroup

$$\{(A, B) : \det(A) = \det(B)\} \subset \mathrm{SL}_{\pm}(2, \mathbb{R}) \times \mathrm{SL}_{\pm}(2, \mathbb{R})$$

In particular the diagonal embedding Φ_{diag} extends to an embedding

$$\Phi_{diag} : \mathrm{SL}_{\pm}(2, \mathbb{R}) \hookrightarrow \mathrm{Sp}_{\pm}(4, \mathbb{R}).$$

4.5.2 Extending the irreducible representation

Direct computation checking the symplecticity of the image of Φ_{irr} shows

$$\Phi_{irr}(A)^T \Omega \Phi_{irr}(A) = \begin{bmatrix} 0 & 0 & -\det(A) & 0 \\ 0 & 0 & 0 & -\det(A) \\ \det(A) & 0 & 0 & 0 \\ 0 & \det(A) & 0 & 0 \end{bmatrix}$$

so $\Phi_{irr}(A)$ is symplectic if $\det(A) = 1$ and anti-symplectic if $\det(A) = -1$. Thus the irreducible representation extends to an embedding $\Phi_{irr} : \mathrm{SL}_{\pm}(2, \mathbb{R}) \hookrightarrow \mathrm{Sp}_{\pm}(4, \mathbb{R})$.

Chapter 5

Coordinates on the $\mathrm{Lag}(\mathbb{C}^4)$ and the Einstein Universe

In §2.2 we described how the dual manifold to $\mathfrak{S}_2$ identified with the complex Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{C}^4)$ and described *Siegel homogeneous coordinates* for that manifold. This is just one of several different coordinate systems which are convenient for describing this manifold, so here we will describe other coordinate systems and the relationships between them.

We will then give a brief overview of the geometry of the real Lagrangian Grassmannian $\mathrm{Lag}(\mathbb{R}^4)$. This is a proper three dimensional submanifold of the topological boundary of $\mathfrak{S}_2$ which is invariant under $\mathrm{Isom}(\mathfrak{S}_2) \approx \mathrm{Sp}(4, \mathbb{R})$. It is the Shilov boundary for $\mathfrak{S}_2$ and identifies with the three dimensional *Einstein Universe* $\mathrm{Ein}^{2,1}$. The geometry of $\mathrm{Ein}^{2,1}$ will allow us to visualize objects in $\mathrm{Lag}(\mathbb{R}^4)$, and in §5.4 we will give a brief overview of the objects of interest in our study. For a full treatment of the Einstein universe see [6]. Lastly, we will describe how the boundaries of the embedded hyperbolic planes intersect this.

5.1 Siegel "Homogeneous" Coordinates

Given a Lagrangian plane in $\mathbb{C}^4$, choose a basis $\{c_1, c_2\}$ for that plane. These vectors form a Lagrangian frame, that is, a pair of linearly independent vectors spanning a Lagrangian plane. Let L denote the 4×2 matrix whose columns are

these vectors

$$L = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix}$$

The space of Lagrangian frames is $\{L_{4\times 2} : \mathrm{rk}(L) = 2 \text{ and } L^T \Omega L = 0_2\}$. The Lagrangian Grassmannian is obtained as a quotient of the space of Lagrangian frames by considering two frames equivalent if they span the same plane. Two frames L, L' span the same plane if and only if there is a change of basis matrix $g \in \mathrm{GL}(2, \mathbb{C})$ such that $L' = Lg$. So

$$\begin{aligned} \mathrm{Lag}(\mathbb{C}^4) &= \text{Lagrangian Frames/Change of Basis} \\ &= \{L_{4\times 2} : rk(L) = 2 \text{ and } L^T \Omega L = 0_2\}/\mathrm{GL}(2, \mathbb{C}) \end{aligned}$$

In these coordinates the action of $\mathrm{Sp}_{\pm}(4, \mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^4)$ is simply left matrix multiplication. Restricting all matrices to have real entries yields Siegel homogeneous coordinates on $\mathrm{Lag}(\mathbb{R}^4)$.

5.2 Homogeneous Skew Symmetric Matrices

One effective way to represent elements in any Grassmannian is by certain elements in the exterior algebra. We will use an equivalent formulation to this using skew symmetric matrices that is more computationally effective.

Let $\mathrm{Mat}_{skew}(4, \mathbb{C})$ denote the vector space of 4×4 skew symmetric matrices and define a bilinear mapping, called the *Exterior Outer Product*, by

$$\text{ExtOuter} : \mathbb{C}^4 \times \mathbb{C}^4 \to \text{Mat}_{skew}(4, \mathbb{C})$$

$$(u, v) \to uv^T - vu^T$$

where $u, v \in \mathbb{C}^4$ are column vectors. $\text{ExtOuter}(u, v)$ is a skew symmetric matrix whose column space is $\text{Span}(u, v)$. We note that this mapping is equivalent to the wedge product when working in the exterior algebra.

This induces a mapping from Siegel homogeneous coordinates to the projectivization of $\text{Mat}_{skew}(4, \mathbb{R})$ given by

$$\text{Lag}(\mathbb{C}^4) \to \mathbb{P}(\text{Mat}_{skew}(4, \mathbb{C}))$$

$$\begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \to \text{ExtOuter}(c_1, c_2)$$

The following lemma establishes that this mapping is well defined

Lemma 5.1. *Suppose L is a 4×2 matrix of rank 2 and let $g \in GL(2, \mathbb{C})$. Then*

$$\textit{ExtOuter}(Lg) = \det(g)\textit{ExtOuter}(L)$$

hence the above mapping is well defined.

Proof. L and Lg are Siegel homogeneous coordinates for the same Lagrangian plane. Suppose

$$L = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \qquad \text{and} \qquad g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Then $Lg = \begin{bmatrix} | & | \\ c_1 & c_2 \\ | & | \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} | & | \\ ac_1 + cc_2 & bc_1 + dc_2 \\ | & | \end{bmatrix}$

The image of Lg is then:

$$\begin{aligned} Lg \to & \begin{bmatrix} ac_1 + cc_2 \end{bmatrix} \begin{bmatrix} bc_1 + dc_2 \end{bmatrix}^T - \begin{bmatrix} bc_1 + dc_2 \end{bmatrix} \begin{bmatrix} ac_1 + cc_2 \end{bmatrix}^T \\ &= (abc_1c_1^T + adc_1c_2^T + bcc_2c_1^T + cdc_2c_2^T) - (abc_1c_1^T + bcc_1c_2^T + adc_2c_1^T + cdc_2c_2^T) \\ &= adc_1c_2^T + bcc_2c_1^T - bcc_1c_2^T - adc_2c_1^T \\ &= (ad - bc)(c_1c_2^T - c_2c_1^T) \\ &= \det(g)\mathrm{ExtOuter}(c_1, c_2) \end{aligned}$$

□

Definition 5.2. Let $A \in \mathrm{Mat}_{skew}(4, \mathbb{C})$. The *Pfaffian of* A is the quadratic polynomial $\mathrm{Pfaff}(A) = \sqrt{\det(A)}$. Suppose

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & a_{14} \\ -a_{12} & 0 & a_{23} & a_{24} \\ -a_{13} & -a_{23} & 0 & a_{34} \\ -a_{14} & -a_{24} & -a_{34} & 0 \end{bmatrix}$$

It is straightforward to compute

$$\det(A) = (a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34})^2$$

$$\mathrm{Pfaff}(A) = a_{14}a_{23} - a_{13}a_{24} + a_{12}a_{34}$$

The Pfaffian can be used to describe the image of ExtOuter:

Theorem 5.3. *A nonzero skew symmetric matrix A is in the image of ExtOuter if and only if Pfaff$(A) = 0$.*

Proof. ($\Longrightarrow$) Suppose $A = \text{ExtOuter}(u, v)$. Since the image of A is the span of u and v, the rank $rk(A) \leq 2$ so $\det(A) = 0$ and $\text{Pfaff}(A) = 0$.

($\Longleftarrow$) Suppose $\text{Pfaff}(A) = 0$. Then $\det(A) = 0$ hence $rk(A) < 4$. The rank of any skew symmetric matrix is even, and since $A \neq 0_4$, we know $rk(A) = 2$. Hence the image of A is a 2 dimensional plane. Choose a basis $\{u, v\}$ for the image of A. Then $\text{ExtOuter}(u, v)$ and A differ by a scalar, hence A is in the image of ExtOuter. $\square$

The Pfaffian can be used to define a bilinear form of signature (3,3) on $\text{Mat}_{skew}(4, \mathbb{C})$. For $A, B \in \text{Mat}_{skew}(4, \mathbb{R})$, the bilinear form is given by

$$\text{Mat}_{skew}(4, \mathbb{C}) \times \text{Mat}_{skew}(4, \mathbb{C}) \to \mathbb{C}$$

$$(A, B) \to \text{Pfaff}(A + B) - \text{Pfaff}(A) - \text{Pfaff}(B)$$

We note that this is equivalent to the wedge product of bivectors when working in the exterior algebra. It is straightforward to compute that $\text{ExtOuter}(A, A) = 2\text{Pfaff}(A)$, thus the above theorem implies that A corresponds to plane if and only if A is null with respect to this bilinear form.

In fact the mapping $\text{Lag}(\mathbb{C}^4) \to \mathbb{P}(\text{Mat}_{skew}(4, \mathbb{C}))$ extends to a mapping from the entire Grassmannian of two planes in $\mathbb{C}^4$: notice that in this section we have yet to use the assumption that plane of interest in Lagrangian. So the collection of skew symmetric matrices whose Pfaffian is zero actually provides homogeneous coordinates for the entire Grassmannian of two planes.

Restricting now to the Lagrangians, let L be a Lagrangian plane whose Siegel homogeneous coordinates are given by

$$L = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix}$$

(We are abusing language here slightly and referring to L as both the Lagrangian itself and the Siegel homogeneous coordinates for the Lagrangian). The condition that L is Lagrangian is given by:

$$\begin{aligned} 0_2 &= L^T \Omega L \\ &= \begin{bmatrix} 0 & a_{12}a_{31} - a_{11}a_{32} + a_{22}a_{41} - a_{21}a_{42} \\ -a_{12}a_{31} + a_{11}a_{32} - a_{22}a_{41} + a_{21}a_{42} & 0 \end{bmatrix} \end{aligned}$$

L maps to the skew symmetric matrix:

$$\begin{bmatrix} 0 & a_{11}a_{22} - a_{12}a_{21} & a_{11}a_{32} - a_{12}a_{31} & a_{11}a_{42} - a_{12}a_{41} \\ a_{12}a_{21} - a_{11}a_{22} & 0 & a_{21}a_{32} - a_{22}a_{31} & a_{21}a_{42} - a_{22}a_{41} \\ a_{12}a_{31} - a_{11}a_{32} & a_{22}a_{31} - a_{21}a_{32} & 0 & a_{31}a_{42} - a_{32}a_{41} \\ a_{12}a_{41} - a_{11}a_{42} & a_{22}a_{41} - a_{21}a_{42} & a_{32}a_{41} - a_{31}a_{42} & 0 \end{bmatrix}$$

so the assumption that L is Lagrangian implies that the $(1,3)$-entry and the $(2,4)$-entry sum to zero. So the Lagrangian Grassmannian can be described in this coordinate system as:

$$\mathrm{Lag}(\mathbb{C}^4) = \{A_{4\times 4} : A + A^T = 0_4 \text{ and Pfaff(A)} = 0 \text{ and } a_{13} + a_{24} = 0\}/\mathbb{C}^*$$

The action of $\mathrm{Sp}_{\pm}(4,\mathbb{C})$ is given by transpose conjugation. For $M \in \mathrm{Sp}_{\pm}(4,\mathbb{C})$ and $A \in \mathrm{Mat}_{skew}(4,\mathbb{C})$, the action is

$$(M, A) \to MAM^T$$

Since $\det(M) = \pm 1$, the action preserves the $\mathrm{Pfaff}(A)$ and the subspace where $a_{13} + a_{24} = 0$ is invariant.

5.3 Homogeneous Coordinates for Null Lines

The bilinear form on $\mathrm{Mat}_{skew}(4,\mathbb{C})$ has signature $(3,3)$, thus this space identifies with $\mathbb{C}^{3,3}$. The matrices corresponding to Lagrangians are null with respect to this form and lie in the 5 dimensional subspace where the $(1,3)$ and $(2,4)$ entries sum to zero. The restriction of the bilinear form to this subspace has signature $(3,2)$. Let $\mathbb{C}^{3,2}$ denote the $\mathbb{C}^5$ equipped with the signature $(3,2)$ bilinear form given by $X^2 + Y^2 - Z^2 - UV$. Then we have an identification of this subspace of $\mathrm{Mat}_{skew}(4,\mathbb{C})$ with $\mathbb{C}^{3,2}$ given by

$$\begin{pmatrix} 0 & -U & Y & Z-X \\ U & 0 & -X-Z & -Y \\ -Y & X+Z & 0 & V \\ X-Z & Y & -V & 0 \end{pmatrix} \longleftrightarrow \begin{bmatrix} X \\ Y \\ Z \\ U \\ V \end{bmatrix}$$

Thus the subspace of $\mathbb{P}(\mathrm{Mat}_{skew}(4,\mathbb{C}))$ identifies with $\mathbb{P}(\mathbb{C}^{3,2})$. Further the condition $\mathrm{Pfaff}(A) = 0$ is equivalent to the corresponding vector being null. We can thus describe Lagrangians as

$$\mathrm{Lag}(\mathbb{C}^4) = \{[X, Y, Z, U, V] \in \mathbb{C}^{3,2} : X^2 + Y^2 - Z^2 - UV = 0\}/\mathbb{C}^*$$

The action of $\mathrm{Sp}(4, \mathbb{C})$ on $\mathrm{Lag}(\mathbb{C}^4)$ induces an action on $\mathbb{C}^{3,2}$. Restricting to real Lagrangians, we have an action of $\mathrm{Sp}(4, \mathbb{R})$ on $\mathbb{R}^{3,2}$. This results in a homomorphism $\mathrm{Sp}(4, \mathbb{R}) \to \mathrm{SO}(3, 2)$. We summarize all of these coordinate systems:

- **Siegel Homogeneous Coordinates**

  $\mathrm{Lag}(\mathbb{C}^4) = \{L_{4\times 2} : rk(L) = 2 \ \& \ L^T \Omega L = 0_2\}/\mathrm{GL}(2, \mathbb{C})$

- **Skew symmetric matrices**

  $\mathrm{Lag}(\mathbb{C}^4) = \{A_{4\times 4} : A + A^T = 0_4 \ \& \ \mathrm{Pfaff(A)} = 0 \ \& \ a_{13} + a_{24} = 0\}/\mathbb{C}^*$

- **Null Lines in $\mathbb{C}^{3,2}$**

  $\mathrm{Lag}(\mathbb{C}^4) = \{[X, Y, Z, U, V] \in \mathbb{C}^{3,2} : X^2 + Y^2 - Z^2 - UV = 0\}/\mathbb{C}^*$

5.4 Visualizing in the Einstein Universe $\mathrm{Ein}^{2,1}$

The projectivization of null lines in $\mathbb{R}^{3,2}$ identifies with the *Einstein Universe* (see [6]), denoted $\mathrm{Ein}^{2,1}$. The identification $\mathrm{Lag}(\mathbb{R}^4) \approx \mathrm{Ein}^{2,1}$ enables us to visualize the local geometry of $\mathrm{Lag}(\mathbb{R}^4)$ in an affine patch of $\mathrm{Ein}^{2,1}$, which identifies with *Minkowski space-time*. The action of $\mathrm{Sp}(4, \mathbb{R})$ can then be visualized using the local isomorphism $\mathrm{Sp}(4, \mathbb{R}) \approx \mathrm{SO}(3, 2)$.

Let $\mathbb{E}^{2,1}$ denote the affine space of $\mathbb{R}^{2,1}$ called *Minkowski space-time*. Fix a basis for $\mathbb{R}^{2,1}$ where the quadratic form is given by $x^2 + y^2 - z^2$, and a basis for $\mathbb{R}^{3,2}$ where the quadratic form is $X^2 + Y^2 - Z^2 - UV$. There is an embedding $\mathbb{E}^{2,1} \hookrightarrow \mathrm{Ein}^{2,1}$ given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \hookrightarrow \begin{bmatrix} x \\ y \\ z \\ x^2+y^2-z^2 \\ 1 \end{bmatrix}$$

The inverse of this map will serve as an affine chart referred to as the *standard Minkowski patch* where we can visualize objects. For a Lagrangian L_i given in Siegel homogeneous coordinates, the corresponding point in $\mathrm{Ein}^{2,1}$ will be denoted p_i. In particular fix notation for the following Lagrangians:

$$L_0 := \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_1 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_{-1} := \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad L_\infty := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We have used the notation here to be suggestive of homogeneous coordinates for the points $0, 1, -1$ and ∞ in $\mathbb{RP}^1$. Under the identification in the previous sections these correspond to

$$p_0 := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad p_1 := \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad p_{-1} := \begin{bmatrix} 0 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix} \quad p_\infty := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Remark 5.4. We may abuse notation and language slightly by referring to L_i as both a Lagrangian and as Siegel homogeneous coordinates for that Lagrangian. We will have to use caution when doing this as most computations require a choice of basis. For instance the plane L_0 could be specified by $\begin{bmatrix} 0_2 \\ A \end{bmatrix}$ where A is any nonsingular 2×2 matrix, though most often we will want to refer to it using the 4×2 matrix L_0 above.

Other Lagrangians of interest which will arise in §6.5 are

$$p_{-1,1} := \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow L_{-1,1} := \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$p_{1,-1} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \leftrightarrow L_{1,-1} := \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

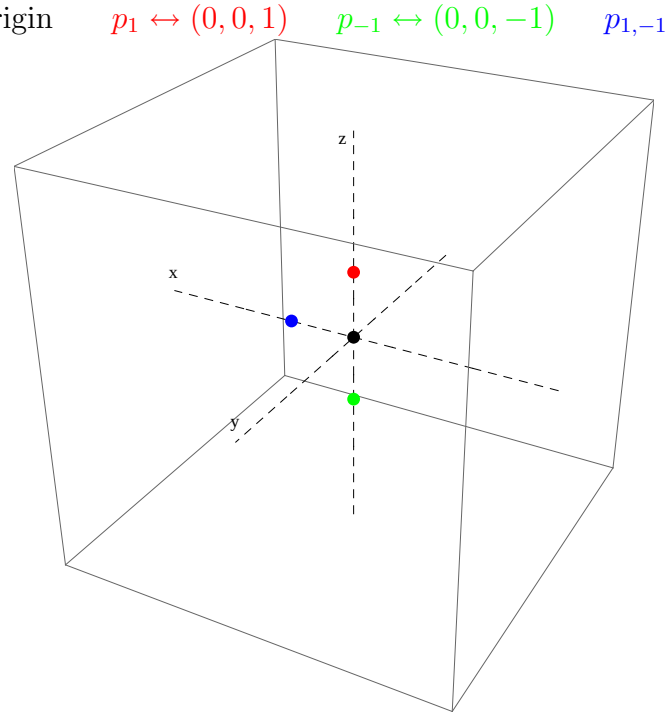
The points in the standard Minkowski patch (the affine points) have the last coordinate $V \neq 0$. All points in this chart may be represented in Siegel homogeneous coordinates where the bottom 2×2 block is nonsingular. We can thus choose a normal form for arbitrary points in $\mathbb{E}^{2,1}$ with the correspondence

$$\mathbb{E}^{2,1} \hookrightarrow \mathrm{Lag}(\mathbb{R}^4)$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \hookrightarrow \begin{bmatrix} x+z & -y \\ -y & -x+z \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Under this correspondence in the standard Minkowski patch

p_0 is the origin $\quad p_1 \leftrightarrow (0,0,1) \quad p_{-1} \leftrightarrow (0,0,-1) \quad p_{1,-1} \leftrightarrow (1,0,0)$



The ideal points come in a few different flavors. All ideal points can be represented in Siegel homogeneous coordinates by matrices whose bottom block is singular. The *improper point* $p_\infty \in \mathrm{Ein}^{2,1}$ corresponds to L_∞. The *generic ideal points* are Lagrangians whose bottom block has rank 1 and whose top block is nonsingular. Those Lagrangians with both top and bottom block having rank 1 form the *ideal circle*.

Distinct Lagrangians can either intersect *transversely* or in a line. Two Lagrangians which do not intersect transversely correspond to points in $\mathrm{Ein}^{2,1}$ which are *incident*, i.e. lie on a photon. Photons are diffeomorphic to $\mathbb{RP}^1$ and intersect $\mathbb{E}^{2,1}$ in a null line. The collection of all null lines through a point p is the *Light Cone at* p and will be denoted $\mathrm{Light}(p)$. Thus $\mathrm{Light}(p_i)$ consists of points corresponding to all Lagrangians which are not transverse to L_i.

For a Lagrangian written in Siegel coordinates as 2×2 blocks $L = \begin{bmatrix} A \\ B \end{bmatrix}$, let $[L_\infty | L]$ denote the 4×4 matrix whose columns are the columns of L_∞ and L

$$[L_\infty | L] = \begin{bmatrix} I_2 & A \\ 0_2 & B \end{bmatrix}$$

If L is transverse to L_∞, then $R^4 = L_\infty \oplus L$ hence $[L_\infty | L]$ is nonsingular. Since $\det([L_\infty | L]) = \det(B)$ we see that the Lagrangians transverse to L_∞ have nonsingular bottom block, thus correspond to the affine points $\mathbb{E}^{2,1}$. All ideal points have a singular bottom block, hence the collection of all ideal points is $\mathrm{Light}(p_\infty)$. We visualize these objects in the Minkowski patch as follows:

- Photons which intersect the Minkowski patch are simply seen as null lines in $\mathbb{E}^{2,1}$. Photons have a single ideal point.

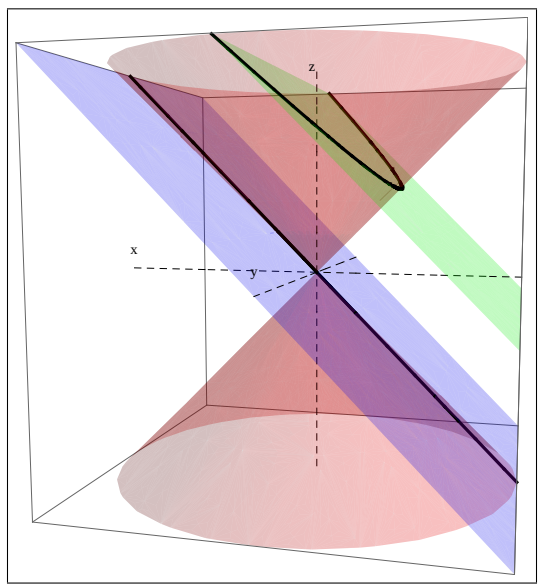
- If p is an affine point, then its light cone intersects the Minkowski patch in an affine light cone. The light cone of p_0 is shown in the picture below in red. Its points correspond to Lagrangians not transverse to L_0.

- If p is a generic ideal point, its light cone intersects the Minkowski patch in

an affine null plane, shown in the picture below in green.

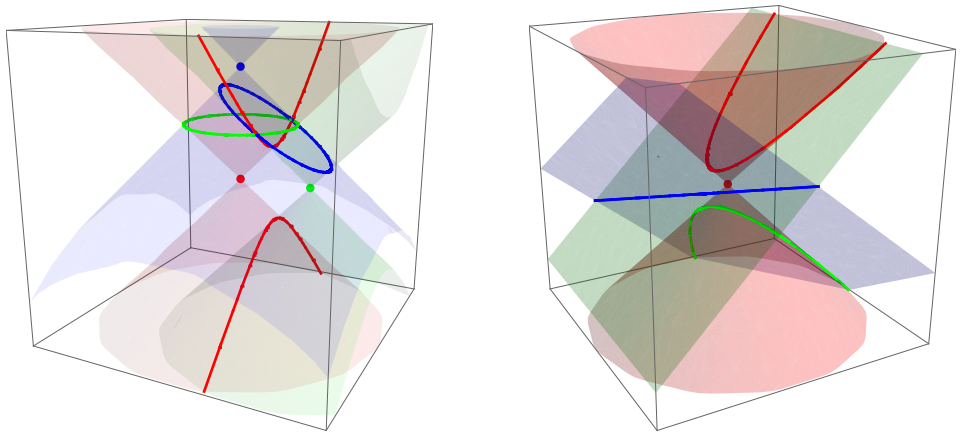
- If p is on the ideal circle, its light cone intersects the Minkowski patch in a null plane passing through the origin in $\mathbb{E}^{2,1}$, shown in the picture below in blue.

- The light cone of the improper point does not intersect the Minkowski patch and contains all the ideal points. All photons on this light cone are ideal photons.



Light cones for points which are incident intersect in a photon containing both of these points, as seen with the red and blue light cones above. Light cones for two non incident points intersect in a *Spacelike Circle*. All spacelike circles are diffeomorphic to $\mathbb{RP}^1$ but their intersection with the Minkowski patch can be any type of conic section. The pictures below illustrate typical spacelike circles in the Minkowski patch. On the left, the three light cones shown are for points in the Minkowski patch, and the intersection of these light cones is seen as either a circle,

ellipse, or hyperbola. On the right, two of the light cones shown are for ideal points and one is for an affine point. The intersections are seen as either a parabola or line.



The spacelike circle determined by non-incident points p_a and p_b consists of those points p_c such that $L_a \cap L_c$ and $L_b \cap L_c$ are both not transverse.

5.5 The intersections $\partial\mathbb{H}^2 \cap \mathrm{Ein}^{2,1}$

Each embedding of $\mathrm{SL}(2,\mathbb{R})$ yields an embedding of $\mathbb{H}^2 \hookrightarrow \mathfrak{S}_2$. The image of the $\partial\mathbb{H}^2$ is then contained in $\mathrm{Lag}(\mathbb{R}^4)$. We describe here the images of $\partial\mathbb{H}^2$ for each of the $\mathrm{SL}(2,\mathbb{R})$ embeddings discussed above.

Composing ϕ_{bidisk} with the embedding of $\mathfrak{S}_2 \hookrightarrow \mathrm{Lag}(\mathbb{C}^4)$ from §2.2 yields a mapping

$$\begin{aligned} \mathbb{H}^2 \times \mathbb{H}^2 &\hookrightarrow \mathrm{Lag}(\mathbb{C}^4) \\ (x_1 + iy_1, x_2 + iy_2) &\to \begin{bmatrix} x_1 + iy_1 & 0 \\ 0 & x_2 + iy_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

The image of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ is a torus contained in $\mathrm{Lag}(\mathbb{R}^4)$. Under the identification $\mathrm{Lag}(\mathbb{R}^4) \approx \mathrm{Ein}^{2,1}$ this torus intersects the Minkowski patch in the xz-plane

$$\partial\mathbb{H}^2 \times \partial\mathbb{H}^2 \hookrightarrow \mathrm{Lag}(R^4) \leftrightarrow \mathbb{E}^{2,1} \subset \mathrm{Ein}^{2,1}$$

$$(x_1, x_2) \hookrightarrow \begin{bmatrix} x_1 & 0 \\ 0 & x_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} \frac{x_1 - x_2}{2} \\ 0 \\ \frac{x_1 + x_2}{2} \end{bmatrix}$$

The image of each factor of $\partial\mathbb{H}^2 \times \partial\mathbb{H}^2$ is a photon in the xz-plane. The image $\phi_{diag}(\partial\mathbb{H}^2)$ is the z-axis.

If we compose $\phi_{PDS}(\mathbb{H}^2)$ with the embedding $\mathfrak{S}_2 \hookrightarrow \mathrm{Lag}(\mathbb{C}^4)$ we obtain

$$x + iy \to \begin{bmatrix} i(y + x^2y^{-1}) & ixy^{-1} \\ ixy^{-1} & iy^{-1} \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} iy & -x \\ 0 & -1 \\ 1 & 0 \\ -x & iy \end{bmatrix}$$

The image of $\phi_{PDS}(\partial\mathbb{H}^2)$ is then

$$\begin{bmatrix} 0 & -x \\ 0 & -1 \\ 1 & 0 \\ -x & 0 \end{bmatrix}$$

which describes the ideal circle. The conjugate $\mathrm{GL}(2, \mathbb{R})$ embedding described above yields a hyperbolic plane embedded by

$$x + iy \to \begin{bmatrix} iy & x \\ x & iy \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

whose boundary is easily seen to intersect $\mathbb{E}^{2,1}$ in the y-axis

$$\begin{bmatrix} 0 & x \\ x & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \leftrightarrow \begin{bmatrix} 0 \\ x \\ 0 \end{bmatrix}$$

These are both spacelike circles in $\mathrm{Ein}^{2,1}$, and can be mapped to the spacelike that intersects $\mathbb{E}^{2,1}$ in the x-axis. This is the boundary of a hyperbolic plane embedded in the *anti-diagonal* in the bidisk.

$$x + iy \to \begin{bmatrix} x + iy & 0 \\ 0 & -x + iy \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

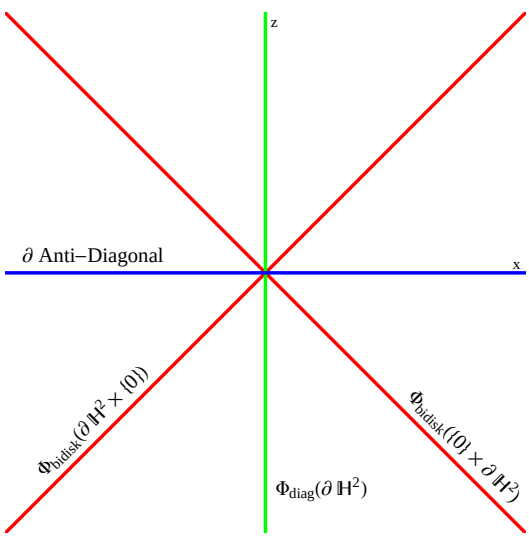
Finally composing $\phi_{irr}(\mathbb{H}^2)$ with $\mathfrak{S}_2 \hookrightarrow \mathrm{Lag}(\mathbb{C}^4)$ yields

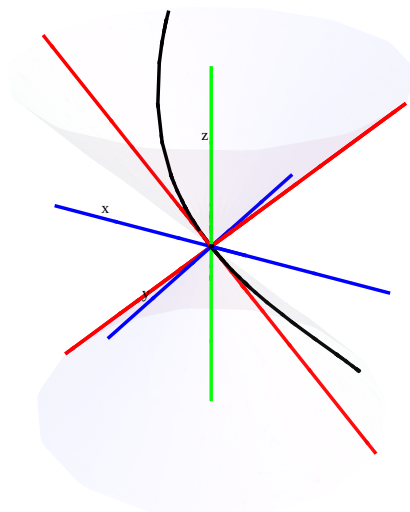
$$x + iy \to \begin{pmatrix} i\left(y^3 + x^2 y\right) - 2x^3 & 3x^2 - ixy \\ 3x^2 - ixy & iy - 6x \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

whose boundary embeds intersects $E^{2,1}$ in a timelike cubic curve

$$\begin{pmatrix} -2x^3 & 3x^2 \\ 3x^2 & -6x \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \leftrightarrow \begin{bmatrix} x\left(x^2-3\right) \\ -3x^2 \\ -x\left(x^2+3\right) \end{bmatrix}$$

The pictures below illustrate the images of each of these boundaries in the Minkowski patch.





Chapter 6

Configurations of Lagrangian Planes

In this section we will investigate configurations of pairwise transverse Lagrangians up to the action of the symplectic group. A set of Lagrangian planes will be called *transverse* if each pair in the set is transverse. We will see that $\mathrm{Sp}(4,\mathbb{C})$ acts triply transitively on $\mathrm{Lag}(\mathbb{C}^4)$ but $\mathrm{Sp}(4,\mathbb{R})$ acts only doubly transitively on $\mathrm{Lag}(\mathbb{R}^4)$. Triples of transverse real Lagrangians have 3 distinct orbits, and the geometry of $\mathrm{Ein}^{2,1}$ provides a nice way of easily visualizing this fact. Finally we develop a *Generalized Cross Ratio* on quadruples of transverse Lagrangians which generalizes the classical cross ratio on $\mathbb{CP}^1$ and extends the cross ratio defined by Siegel (see §3.2). As motivation we begin by recalling some facts about the classical cross ratio.

6.1 The Classical Cross Ratio

It is well known that $\mathrm{SL}(2,\mathbb{C})$ acts triply transitively on $\mathbb{CP}^1$- any triple of distinct points can be mapped by an $\mathrm{SL}(2,\mathbb{C})$ transformation to $(0,1,\infty)$. Quadruples of distinct points in $\mathbb{CP}^1$ are characterized by the cross ratio.

Definition 6.1 (The Classical Cross Ratio)**.** Let $a,b,c,d \in \mathbb{CP}^1$ be distinct and have homogeneous coordinates $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$, etc. The cross ratio of these points is given by:

$$[a,b,c,d] := \frac{\det\begin{bmatrix} c_1 & a_1 \\ c_2 & a_2 \end{bmatrix} \det\begin{bmatrix} b_1 & d_1 \\ b_2 & d_2 \end{bmatrix}}{\det\begin{bmatrix} c_1 & d_1 \\ c_2 & d_2 \end{bmatrix} \det\begin{bmatrix} b_1 & a_1 \\ b_2 & a_2 \end{bmatrix}}$$

There are several variants on the definition of the cross ratio found in the literature. We have chosen a definition such that

$$[0,1,\lambda,\infty] = \lambda$$

The cross ratio determines quadruples up to the action of $\mathrm{SL}(2,\mathbb{C})$ as seen in the following

Lemma 6.2. *Let z_0, z_1, z, z_∞ be distinct points in $\mathbb{CP}^1$. There exists $M \in SL(2,\mathbb{C})$ mapping the quadruple $(z_0, z_1, z, z_\infty) \to (0,1,\lambda,\infty)$ if and only if the cross ratio $[z_0, z_1, z, z_\infty] = \lambda$.*

The action of $\mathrm{SL}_(2,\mathbb{R})$ on $\mathbb{RP}^1 \subset \mathbb{CP}^1$ is not quite triply transitive, but it is once we allow for orientation reversing transformations. The action of $\mathrm{SL}_\pm(2,\mathbb{R})$ on $\mathbb{RP}^1$ is triply transitive. The cross ratio can be used to classify distinct quadruples in $\mathbb{RP}^1$ up to the action of $\mathrm{SL}_\pm(2,\mathbb{R})$.

Lemma 6.3. *Let x_0, x_1, x, x_∞ be distinct points in $\mathbb{RP}^1$. There exists $M \in SL_\pm(2,\mathbb{R})$ mapping the quadruple $(x_0, x_1, x, x_\infty) \to (0,1,\lambda,\infty)$ if and only if the cross ratio $[x_0, x_1, x, x_\infty] = \lambda$*

6.2 Pairs of Transverse Lagrangians

We will show that $\mathrm{Sp}(4,\mathbb{C})$ act doubly transitively on transverse pairs in $\mathrm{Lag}(\mathbb{C}^4)$. Exactly the same argument shows that $\mathrm{Sp}(4,\mathbb{R})$ act doubly transitively on $\mathrm{Lag}(\mathbb{R}^4)$. We will prove that every pair of transverse Lagrangians is in the orbit of the pair (L_0, L_∞).

Lemma 6.4 (Sp(4, ℂ) acts doubly transitively:)**.** *Let U, V be a pair of transverse Lagrangians. Then there is $M \in Sp(4,\mathbb{C})$ such that $ML_0 = U$ and $ML_\infty = V$.*

Proof. Suppose U and V have Siegel homogeneous coordinates

$$U = \begin{bmatrix} | & | \\ u_3 & u_4 \\ | & | \end{bmatrix} \qquad V = \begin{bmatrix} | & | \\ v_1 & v_2 \\ | & | \end{bmatrix}$$

The reason for the strange choice of indices for these column vectors will be clear in a moment. Since ω is non-degenerate we can choose u_2 such that $\omega(u_2, u_3) = 0$ and $\omega(u_2, u_4) = -1$. Now $u_2^\perp \cap u_4^\perp$ is a 2-plane intersecting $u_3^\perp$ in a line. So we can choose $u_1 \in u_2^\perp \cap u_4^\perp$ such that $\omega(u_1, u_3) = -1$. Let M be the matrix whose columns are u_1, u_2, u_3, u_4:

$$M_0 = \begin{bmatrix} | & | & | & | \\ u_1 & u_2 & u_3 & u_4 \\ | & | & | & | \end{bmatrix}$$

Then $M_0^T \Omega M_0 = \Omega$ by construction so $M_0 \in \mathrm{Sp}(4,\mathbb{C})$. Further $M_0 L_0 = U$ as desired. Applying M_0 to both U and V we may now assume without loss of

generality that $U = L_0$. Since V intersects $U = L_0$ transversely we can write $\mathbb{C}^4 = V \oplus L_0$, and hence the matrix

$$[V|L_0] = \begin{bmatrix} | & | & 0 & 0 \\ v_1 & v_2 & 0 & 0 \\ | & | & 1 & 0 \\ | & | & 0 & 1 \end{bmatrix}$$

is nonsingular. It is straightforward to compute that the determinant of this matrix is equal to the determinant of the top block of V, hence the top block of V is nonsingular.

The isotropy subgroup of L_0 consists of symplectic matrices of the form:

$$\mathrm{Stab}(L_0) = \begin{bmatrix} A & 0_2 \\ C & D \end{bmatrix}.$$

Let $v_3 = \begin{bmatrix} 0 \\ 0 \\ z \\ w \end{bmatrix}$. Since the top block of V is nonsingular the equations $\omega(v_1, v_3) = -1$ and $\omega(v_2, v_3) = 0$ are non singular and hence there is a solution for z, w. Choose v_3 satisfying those equations. Now choose v_4 such that $\omega(v_1, v_4) = 0$, $\omega(v_2, v_4) = -1$ and $\omega(v_3, v_4) = 0$. Once again such a solution exists since the top block of V is nonsingular and ω is non-degenerate. Let M_∞ be the matrix whose columns are v_1, v_2, v_3, v_4:

$$M_\infty = \begin{bmatrix} | & | & | & | \\ v_1 & v_2 & v_3 & v_4 \\ | & | & | & | \end{bmatrix}$$

Then by construction $M_\infty \in \mathrm{Sp}(4, \mathbb{C})$, $M_\infty L_0 = L_0$ and $M_\infty V = L_\infty$. Now the matrix $M_\infty M_0$ is a symplectic transformation taking $U \to L_0$ and $V \to L_\infty$ as desired. $\square$

6.3 Triples of Lagrangian Planes

Suppose now L is a Lagrangian transverse to both L_0 and L_∞ with coordinates

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ l_{31} & l_{32} \\ l_{41} & l_{42} \end{bmatrix}$$

Since $L \cap L_\infty$ is transverse, we can then write $\mathbb{C}^4 = L_\infty \oplus L$ and hence the matrix

$$[L_\infty | L] = \begin{bmatrix} 1 & 0 & l_{11} & l_{12} \\ 0 & 1 & l_{21} & l_{22} \\ 0 & 0 & l_{31} & l_{32} \\ 0 & 0 & l_{41} & l_{42} \end{bmatrix}$$

is nonsingular. It is easy to see that $\det[L_\infty | L] = l_{31}l_{42} - l_{32}l_{41}$, hence the bottom 2×2 block for L is nonsingular. Thus we may choose a basis for L such that

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Similarly since $L \cap L_0$ is transverse we can write $\mathbb{C}^4 = L_0 \oplus L$ and the matrix

$$[L_0|L] = \begin{bmatrix} 0 & 0 & l_{11} & l_{12} \\ 0 & 0 & l_{21} & l_{22} \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

is nonsingular. This implies that $\det[L_0|L] = l_{11}l_{22} - l_{12}l_{21}$ is nonzero and hence the top 2×2 block of L is nonsingular. Since L is assumed to be Lagrangian, it satisfies $L^T \Omega_4 L = 0_2$. As noted in §2.2 this implies that the top block of L is a symmetric matrix. So L has coordinates:

$$L = \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The discussion thus far applies equally to real and complex Lagrangians. There is now a difference due to the fact that $\mathrm{GL}(2, \mathbb{C})$ acts transitively on non-degenerate symmetric bilinear forms, but $\mathrm{GL}(2, \mathbb{R})$ preserves the signature of the form.

6.4 Triples of Complex Lagrangians

The following lemma allows us to further simplify the coordinates for L in the complex case.

Lemma 6.5. *$GL(2,\mathbb{C})$ acts transitively on non-degenerate symmetric bilinear forms. More precisely, for any 2×2 nonsingular symmetric matrix l there exists $g \in GL(2,\mathbb{C})$ such that $g^T l g = I_2$.*

Proof. Since l is nonsingular there exists $v \in \mathbb{C}^2$ such that $v^T l v \neq 0$. Rescaling v by $\frac{1}{\sqrt{v^T l v}}$, which exists since we are working over $\mathbb{C}$, we see that

$$\tfrac{v^T}{\sqrt{v^T l v}} l \tfrac{v}{\sqrt{v^T l v}} = \tfrac{v^T l v}{v^T l v} = 1$$

So without any loss of generality we may assume $v^T l v = 1$. Choose $w \in v^\perp$ so that $w^T l w \neq 0$. Again by rescaling we may assume $w^T l w = 1$. Then $\{v, w\}$ is an orthonormal basis for $\mathbb{C}^2$ with respect to the bilinear form defined by l. Let $g = [v|w]$. It is now easy to check that:

$$g^T l g = \begin{bmatrix} v^T l v & v^T l w \\ w^T l v & w^T l w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

It is straightforward to check that the stabilizer of (L_0, L_∞) is image of Φ_{PDS} (see §4.3). Specifically,

$$\mathrm{Stab}(L_0, L_\infty) = \left\{ \begin{bmatrix} A & 0_2 \\ 0_2 & (A^T)^{-1} \end{bmatrix} : A \in \mathrm{GL}(2,\mathbb{C}) \right\}$$

If L is a Lagrangian in $\mathbb{C}^4$ which is transverse to L_0 and L_∞, the above argument says that L has coordinates $L = \begin{bmatrix} l \\ I_2 \end{bmatrix}$ where l is a nonsigular symmetric matrix. Lemma 6.5 yields $g \in \mathrm{GL}(2, \mathbb{C})$ such that $g^T l g = I_2$. The image of g^T under Φ_{PDS} is

$$\begin{bmatrix} g^T & 0_2 \\ 0_2 & g^{-1} \end{bmatrix}$$

This stabilizes L_0 and L_∞ and acts on L by

$$\begin{bmatrix} g^T & 0_2 \\ 0_2 & g^{-1} \end{bmatrix} \begin{bmatrix} l \\ I_2 \end{bmatrix} = \begin{bmatrix} g^T l \\ g^{-1} \end{bmatrix} \sim \begin{bmatrix} g^T l g \\ I_2 \end{bmatrix} = \begin{bmatrix} I_2 \\ I_2 \end{bmatrix}$$

Thus we may assume that the third plane is

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We thus conclude

Lemma 6.6 ($\mathrm{Sp}(4, \mathbb{C})$ acts triply transitively)**.** *Any transverse triple in Lag*$(\mathbb{C}^4)$ *is* $Sp(4, \mathbb{C})$ *equivalent to*

$$(L_0, L_1, L_\infty) = \left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right)$$

6.5 Triples of Real Lagrangians

If we restrict our attention to $\mathrm{Sp}(4,\mathbb{R})$ and the real Lagrangian Grassmannian, lemma 6.5 does not apply and $\mathrm{Sp}(4,\mathbb{R})$ does not act transitively on triples of transverse Lagrangian planes. The argument in §6.3 still applies: any triple of transverse Lagrangians is $\mathrm{Sp}(4,\mathbb{R})$ equivalent to the triple

$$L_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, L_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, L = \begin{bmatrix} l_{11} & l_{12} \\ l_{12} & l_{22} \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The top block for L is now a nonsingular real symmetric matrix. By the **spectral theorem** for real symmetric matrices there is an orthogonal matrix P which diagonalizes the top block of L. Now the matrix

$$\Phi_{PDS}(P) = \begin{bmatrix} P & 0_2 \\ 0_2 & P \end{bmatrix}$$

stabilizes L_0 and L_∞, and acts on L by

$$\begin{bmatrix} P & 0_2 \\ 0_2 & (P^T)^{-1} \end{bmatrix} \begin{bmatrix} l \\ I_2 \end{bmatrix} = \begin{bmatrix} Pl \\ P \end{bmatrix} \sim \begin{bmatrix} PlP^T \\ I_2 \end{bmatrix}$$

So we may assume that the top block of L is diagonal and the diagonal entries are nonzero. We may then apply a (real) diagonal matrix to L to assume that the diagonal entries are ± 1. Thus there are 3 possibilities for L:

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \; L_{-1} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } L_{-1,1} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

We conclude that there are 3 orbits of transverse Lagrangian triples in $\mathbb{R}^4$ under the action of $\mathrm{Sp}(4, \mathbb{R})$.

Lemma 6.7 (Orbits of transverse triples in $\mathrm{Lag}(\mathbb{R}^4)$)**.** *Any triple of transverse Lagrangians in Lag*$(\mathbb{R}^4)$ *is Sp*$(4, \mathbb{R})$ *equivalent to one of the following*

- (L_0, L_1, L_∞)
- (L_0, L_{-1}, L_∞)
- $(L_0, L_{1,-1}, L_\infty)$

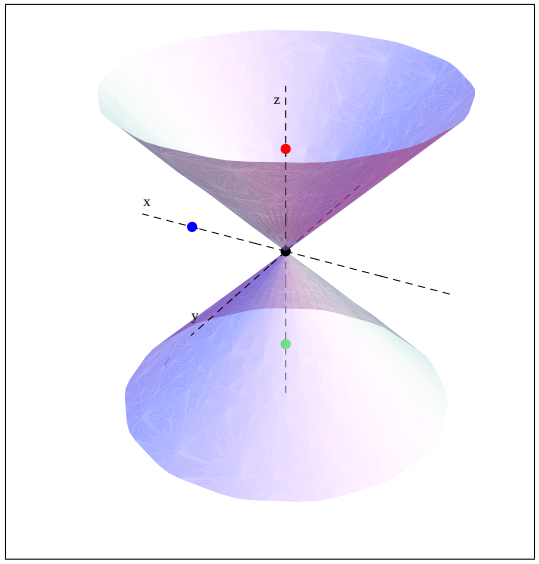
The identification $\mathrm{Lag}(\mathbb{R}^4) \approx \mathrm{Ein}^{2,1}$ yields an easy way of visualizing these orbits. L_0 and L_∞ correspond to the origin p_0 and the improper point p_∞ respectively. The light cones $\mathrm{Light}(p_0)$ and $\mathrm{Light}(p_\infty)$ consist of points corresponding to non-transverse Lagrangians. The complement

$$\mathrm{Ein}^{2,1} - \{\mathrm{Light}(p_0) \cup \mathrm{Light}(p_\infty)\}$$

consists of points corresponding to Lagrangians transverse to both. It is easy to see that this is equal to the complement

$$\mathbb{E}^{2,1} - \{\mathrm{Light}(p_0)\}$$

and this set has 3 components. The picture below shows $\mathrm{Light}(p_0)$ and a point corresponding each of L_1, L_{-1} and $L_{1,-1}$.



Viewing these triples in this manner also makes apparent that there is an involution in Minkowski space which maps the triple $(L_0, L_1, L_\infty) \to (L_0, L_{-1}, L_\infty)$. This is the Lagrangian involution

$$\begin{bmatrix} -I_2 & 0_2 \\ 0_2 & I_2 \end{bmatrix}$$

thus any transverse triple is $\mathrm{Sp}_\pm(4, \mathbb{R})$ equivalent to either (L_0, L_1, L_∞) or $(L_0, L_{-1,1}, L_\infty)$. For obvious reasons we will refer to these orbits as the *Definite triple* or *Indefinite triple*. Further it is clear that the subgroup stabilizing a triple is nontrivial. Specifically,

Theorem 6.8 (Stabilizers of Triples)**.**

1. *The subgroup of $Sp(4, \mathbb{R})$ stabilizing the definite triple is isomorphic to the orthogonal group $O(2)$. The isomorphism is given by restricting Φ_{PDS} to $O(2) \subset GL(2, \mathbb{R})$:*

$$Stab(L_0, L_1, L_\infty) = \Phi_{PDS}(O(2)) = \left\{ \begin{bmatrix} P & 0_2 \\ 0_2 & P \end{bmatrix} : P \in O(2) \right\}$$

2. *The subgroup stabilizing the indefinite triple is isomorphic to $O(1,1)$. The isomorphism is again given by restricting Φ_{PDS}:*

$$Stab(L_0, L_{-1,1}, L_\infty) = \Phi_{PDS}(O(1,1)) = \left\{ \begin{bmatrix} P & 0_2 \\ 0_2 & P \end{bmatrix} : P \in O(1,1) \right\}$$

6.6 Extending Siegel's Generalized Cross Ratio

Here we will generalize the cross ratio defined by Siegel and discussed in §3.2. This is an invariant for configurations of quadruples of transverse Lagrangians in $\mathbb{C}^4$. We then present a series of lemmas establishing the properties expected from any reasonably defined cross ratio. Specifically the last few theorems in this section generalize lemma 6.2.

First we define the Gram matrix for pairs of 4×2 matrices.

Definition 6.9. Let U_1, U_2 be 4×2 matrices. The *Gram Matrix* for the symplectic pairings of the columns of U_1 and U_2 is

$$Gr(U_1, U_2) := U_1^T \Omega U_2$$

It is clear that the Gram matrix is invariant under the left action of $\mathrm{Sp}(4, \mathbb{C})$.

The Gram matrix is not well defined for pairs of Lagrangian planes, only for pairs of Lagrangian frames. Changing basis for a plane will change the Gram matrix by either right or left multiplication by some matrix in $\mathrm{GL}(2, \mathbb{C})$.

Lemma 6.10. *If U_1 and U_2 are 4×2 matrices spanning transverse Lagrangian planes, then $Gr(U_1, U_2)$ is nonsingular.*

Proof. By lemma 6.4, we may apply symplectomorphisms to assume that the planes are $U_1 = L_0$ and $U_2 = L_\infty$. These transformations preserve $Gr(U_1, U_2)$. We may then change basis to make the coordinates $L_0 = \begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}$ and $L_\infty = \begin{bmatrix} I_2 \\ 0_2 \end{bmatrix}$, and this change of basis multiplies the Gram matrix by a nonsingular matrix. Now direct computation yields that $Gr(L_0, L_\infty) = I_2$, hence $Gr(U_1, U_2)$ is nonsingular. □

The cross ratio can now be defined for transverse quadruples. It also depends upon a choice of basis for each plane, but it is a first step to finding an invariant which is independent of these choices.

Definition 6.11. Let U_1, U_2, U_3, U_4 be 4×2 matrices of rank 2 spanning pairwise transverse Lagrangian planes. This means that $Gr(U_i, U_j)$ is nonsingular if $i \neq j$ and $Gr(U_i, U_i) = 0_2$. Define the *cross ratio* as

$$\begin{aligned} CR[U_1, U_2, U_3, U_4] :&= Gr(U_1, U_3)Gr^{-1}(U_4, U_3)Gr(U_4, U_2)Gr^{-1}(U_1, U_2) \\ &= [U_1^T \Omega U_3][U_4^T \Omega U_3]^{-1}[U_4^T \Omega U_2][U_1^T \Omega U_2]^{-1} \end{aligned}$$

The first trivial observation is the following:

Lemma 6.12. *The Cross Ratio is invariant under $Sp(4, \mathbb{C})$.*

Proof. Follows immediately from the fact that the Gram matrix is invariant under $\mathrm{Sp}(4, \mathbb{C})$. □

The following lemma establishes that the cross ratio for transverse quadruples of Lagrangians is well defined up to conjugacy. This means that the **spectral**

properties of the cross ratio are independent of choices of basis for each plane, and we can use the spectrum of the cross ratio to produce well defined invariants for quadruples.

Lemma 6.13. *The cross ratio for a quadruple of transverse Lagrangians is well defined up to conjugacy. More precisely, if* U_1,U_2, U_3 *and* U_4 *are Lagrangian frames spanning pairwise transverse Lagrangian planes, and* $g \in GL(2, \mathbb{C})$ *then the following cross ratios are all* $GL(2, \mathbb{C})$ *conjugate:*

- $CR[U_1, U_2, U_3, U_4]$
- $CR[U_1 g, U_2, U_3, U_4]$
- $CR[U_1, U_2 g, U_3, U_4]$
- $CR[U_1, U_2, U_3 g, U_4]$
- $CR[U_1, U_2, U_3, U_4 g]$

Proof. Consider how changing basis for each of the planes affects the cross ratio:

1. Change basis for U_1:

$$\begin{aligned} CR[U_1 g, U_2, U_3, U_4] &= [(U_1 g)^T \Omega U_3][U_4^T \Omega U_3]^{-1}[U_4^T \Omega U_2][(U_1 g)^T \Omega U_2]^{-1} \\ &= g^T [U_1^T \Omega U_3][U_4^T \Omega U_3]^{-1}[U_4^T \Omega U_2][g^T U_1^T \Omega U_2]^{-1} \\ &= g^T [U_1^T \Omega U_3][U_4^T \Omega U_3]^{-1}[U_4^T \Omega U_2][U_1^T \Omega U_2]^{-1}(g^T)^{-1} \\ &= g^T CR[U_1, U_2, U_3, U_4](g^T)^{-1} \end{aligned}$$

2. Change basis for U_2:

$$
\begin{aligned}
CR[U_1, U_2g, U_3, U_4] &= [U_1^T\Omega U_3][U_4^T\Omega U_3]^{-1}[U_4^T\Omega U_2g][U_1^T\Omega U_2g]^{-1} \\
&= [U_1^T\Omega U_3][U_4^T\Omega U_3]^{-1}[U_4^T\Omega U_2]gg^{-1}[U_1^T\Omega U_2]^{-1} \\
&= CR[U_1, U_2, U_3, U_4]
\end{aligned}
$$

3. Change basis for U_3:

$$
\begin{aligned}
CR[U_1, U_2, U_3g, U_4] &= [U_1^T\Omega U_3g][U_4^T\Omega U_3g]^{-1}[U_4^T\Omega U_2][U_1^T\Omega U_2]^{-1} \\
&= [U_1^T\Omega U_3]gg^{-1}[U_4^T\Omega U_3]^{-1}[U_4^T\Omega U_2][U_1^T\Omega U_2]^{-1} \\
&= CR[U_1, U_2, U_3, U_4]
\end{aligned}
$$

4. Change basis for U_4:

$$
\begin{aligned}
CR[U_1, U_2, U_3, U_4g] &= [U_1^T\Omega U_3][(U_4g)^T\Omega U_3]^{-1}[(U_4g)^T\Omega U_2][U_1^T\Omega U_2]^{-1} \\
&= [U_1^T\Omega U_3][g^TU_4^T\Omega U_3]^{-1}[g^TU_4^T\Omega U_2][U_1^T\Omega U_2]^{-1} \\
&= [U_1^T\Omega U_3][U_4^T\Omega U_3]^{-1}(g^T)^{-1}g^T[U_4^T\Omega U_2][U_1^T\Omega U_2]^{-1} \\
&= CR[U_1, U_2, U_3, U_4]
\end{aligned}
$$

□

There is a cool observation to be made from the above proof, namely that changing basis for U_2, U_3 or U_4 actually preserves the cross ratio; only changing basis for U_1 conjugates the cross ratio. This can simplify things if we are able to fix a basis for U_1.

The definition of cross ratio is the natural extension of Siegel's cross ratio defined on $\mathfrak{S}_2$ to $\mathrm{Lag}(\mathbb{C}^4)$, as shown in the following

Lemma 6.14. *For points $Z, Z_1 \in \mathfrak{S}_2$, the cross ratio $R(Z, Z_1)$ discussed in §3.2 satisfies:*

$$R(Z, Z_1) = CR\left[\begin{bmatrix} Z_1 \\ I_2 \end{bmatrix}, \begin{bmatrix} \overline{Z} \\ I_2 \end{bmatrix}, \begin{bmatrix} Z \\ I_2 \end{bmatrix}, \begin{bmatrix} \overline{Z_1} \\ I_2 \end{bmatrix}\right]$$

As with the classical cross ratio for quadruples of points in $\mathbb{CP}^1$, there is a natural action of the symmetric group S_4 which permute the U_i. The choice made in definition 6.11 is such that for a nonsingular 2×2 symmetric matrix l

$$CR[L_0, L_1, L, L_\infty] = CR\left[\begin{bmatrix} 0_2 \\ I_2 \end{bmatrix}, \begin{bmatrix} I_2 \\ I_2 \end{bmatrix}, \begin{bmatrix} l \\ I_2 \end{bmatrix}, \begin{bmatrix} I_2 \\ 0_2 \end{bmatrix}\right] = l$$

In light of lemma 6.13 we make the following extension to the definition of the cross ratio:

Definition 6.15. Let CR_{Eig} be the mapping defined on transverse quadruples of transverse Lagrangians which yields the (unordered) eigenvalues of the cross ratio of the quadruple. By lemma 6.13, this mapping is well defined.

The following is the direct generalization of lemma 6.2, and extends theorem 3.1 from $\mathfrak{S}_2$ to $\mathrm{Lag}(\mathbb{C}^4)$.

Theorem 6.16. *Let* $L = \begin{bmatrix} l \\ I_2 \end{bmatrix}$ *where* l *is a nonsingular* 2×2 *symmetric matrix and let* (N_0, N_1, N, N_∞) *be transverse Lagrangian planes. Then there is a symplectic transformation* $A \in Sp(4, \mathbb{C})$ *such that* $A(N_0) = L_0$, $A(N_1) = L_1$, $A(N) = L$, *and* $A(N_\infty) = L_\infty$ *if and only if their cross ratios are* $GL(2, \mathbb{C})$ *conjugate, equivalently*

$$CR_{Eig}[L_0, L_1, L, L_\infty] = CR_{Eig}[N_0, N_1, N, N_\infty]$$

Proof. ($\Rightarrow$) Supposing there is a symplectic mapping between these quadruples and applying lemma 6.12 we have that

$$CR[N_0, N_1, N, N_\infty] = CR[AN_0, AN_1, AN, AN_\infty]$$

Although the Largangians are equal, $AN_0 = L_0$, we may need to change basis for this plane to write L_0 in the appropriate coordinates, which would conjugate the cross ratio (lemma 6.13) and hence preserve the eigenvalues.

($\Leftarrow$) Applying lemma 6.6 we know that any triple of transverse Lagrangians can be mapped to L_0, L_1 and L_∞. By lemma 6.12 this mapping preserves the cross ratio. So we may assume without loss of generality that the Lagrangian planes are $N_0 = L_0$, $N_1 = L_1$ and $N_\infty = L_\infty$. We may need to change basis for these to make the Siegel homogeneous coordinates the same, which would (at worst) conjugate the cross ratio (lemma 6.13). This now implies that we can choose a basis for N of the form

$$N = \begin{bmatrix} n \\ \\ I_2 \end{bmatrix}$$

where n is a nonsingular symmetric matrix. The cross ratios are now

$$CR[L_0, L_1, L, L_\infty] = CR[N_0, N_1, N, N_\infty]$$

$$l = n$$

as desired. $\square$

The above theorem is not quite applicable to real Lagrangians. Since transverse triples are partitioned into three orbits, the cross ratio can only be used to distinguish quadruples after choosing an orbit for the triple. The corresponding theorem for the cross ratio is:

Theorem 6.17. *Let (N_0, N_i, N, N_∞) be an ordered quadruple of transverse Lagrangians and*

$$L = \begin{bmatrix} l \\ \\ I_2 \end{bmatrix}$$

where l is a 2×2 nonsingular symmetric matrix.

1. *$CR_{Eig}[N_0, N_i, N, N_\infty] = Eigenvalues(l)$ if and only if there is $A \in Sp(4, \mathbb{R})$ which maps this quadruple to one of the following:*

   - (L_0, L_1, L, L_∞)
   - $(L_0, L_{-1}, L, L_\infty)$
   - $(L_0, L_{-1,1}, L, L_\infty)$

2. *$CR_{Eig}[N_0, N_i, N, N_\infty] = Eigenvalues(l)$ if and only if there is $A \in Sp_\pm(4, \mathbb{R})$ mapping this quadruple to either*

   - (L_0, L_1, L, L_∞)
   - $(L_0, L_{-1,1}, L, L_\infty)$

We can visualize these orbits in the pictures below as follows. The point p_1 corresponding to L_1 is shown, and by theorem 6.8 the stabilizer of triple is $\approx O(2)$. Thinking of $CR_{Eig}[L_0, L_1, *, L_\infty]$ as a function on $\mathrm{Ein}^{2,1}$, the level sets of this function are shown in red. There are two types of orbits:

- Each point along the z-axis is in its own orbit. These points are stabilized by the stabilizer of the triple, and correspond to Lagrangians in the image of $\phi_{diag}(\partial\mathbb{H}^2)$ (see §5.5). In Siegel coordinates they are given by:

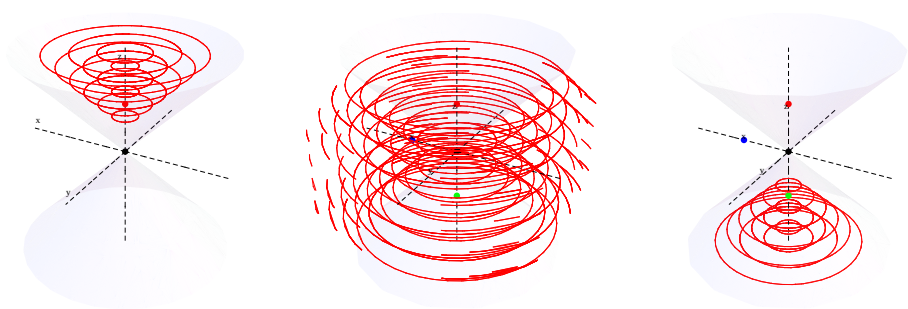
$$\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \\ 1 & 0 \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$$

For such Lagrangians, $CR_{Eig}[L_0, L_1, L, L_\infty] = (\lambda, \lambda)$.

- Circles centered along the z-axis. Each circle intersects the xz-plane in exactly 2 points, and by ordering the eigenvalues of the cross ratio we obtain a unique representative from each orbit of the form

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\lambda_1 < \lambda_2$ and $CR_{Eig}[L_0, L_1, L, L_\infty] = (\lambda_1, \lambda_2)$.

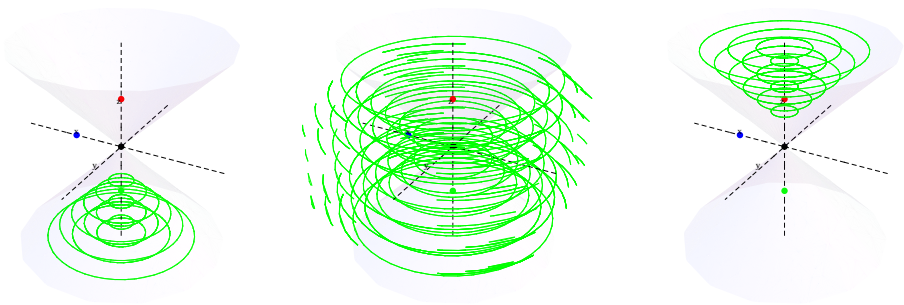


$CR[L_0, L_1, L, L_\infty] > 0$, $CR[L_0, L_1, L, L_\infty]$ indefinite, $CR[L_0, L_1, L, L_\infty] < 0$

The Lagrangians L such that $CR[L_0, L_1, L, L_\infty]$ is positive definite correspond to points in the same component of $\mathbb{E}^{2,1} - \text{Light}(p_0)$ as p_1. Those with indefinite

cross ratio correspond to points outside the light cone Light(p_0), and those with negative definite cross ratio are in the same component as $-p_1$.

We obtain a similar picture for the triple (L_0, L_{-1}, L_∞). Shown below in green is the point p_{-1} corresponding to L_{-1}. The isotropy group is the same and the orbits admit a similar description to the above. The triples (L_0, L_1, L_∞) and (L_0, L_{-1}, L_∞) are equivalent by a Lagrangian involution.



The point $p_{-1,1}$ corresponding to $L_{-1,1}$ and the level sets of $CR_{eig}[L_0, L_{-1,1}, *, L_\infty]$ are shown below. The stabilizer of the triple is $\approx O(1,1)$ (theorem 6.8) and once again there are 2 types of orbits.

- Each point along the x-axis is in its own orbit. These correspond to the image of the boundary of the anti-diagonal in the bidisk (§5.5) and are given in Siegel coordinates by:

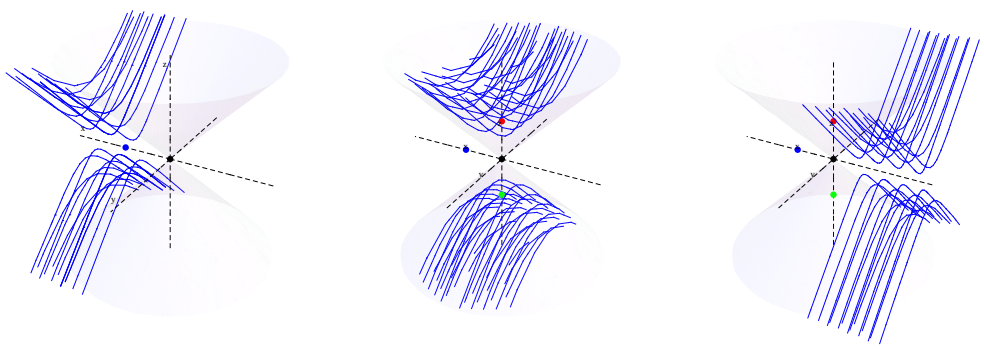
$$\left\{ \begin{bmatrix} \lambda & 0 \\ 0 & -\lambda \\ 1 & 0 \\ 0 & 1 \end{bmatrix} : \lambda \in \mathbb{R} \right\}$$

and $CR_{Eig}[L_0, L_{-1,1}, L, L_\infty] = (\lambda, -\lambda)$

- Hyperbolas in planes parallel to the yz-plane. Each such hyperbola intersects the xz-plane twice and so there is a unique representative from each orbit of the form

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

where $\lambda_1 < \lambda_2$ and $CR_{Eig}[L_0, L_{-1,1}, L, L_\infty] = (\lambda_1, \lambda_2)$.



All of these hyperbolas pass through the same ideal point which lies on the ideal circle. This point corresponds to a Lagrangian which is not transverse to either L_0 or L_∞. The cross ratio is undefined here, so the level sets necessarily have two components, one for each branch of the hyperbola. The components are equivalent via a Lagrangian involution.

Chapter 7

Dihedral Groups

In this section we will study faithful representations of dihedral groups into $\mathrm{Sp}_{\pm}(4,\mathbb{R})$. Our focus will be representations constructed by building a representation into $\mathrm{SL}_{\pm}(2,\mathbb{R})$ and then composing with one of the embeddings described in §4. Each involution generating the dihedral group will have Lagrangian eigenspaces. We will make use of the cross ratio to describe which configurations of Lagrangians correspond to dihedral groups. The first section considers dihedral groups in $\mathrm{SL}_{\pm}(2,\mathbb{R})$.

7.1 Dihedral groups in $\mathrm{SL}_{\pm}(2,\mathbb{R})$

Suppose $R_1, R_2 \in \mathrm{SL}_{\pm}(2,\mathbb{R})$ are reflections generating a dihedral group of order $2p$. For the reflection R_i, let

$$F_{R_i} = \text{Fixed Vector for } R_i = +1 \text{ eigenspace for } R_i$$

$$N_{R_i} = \text{Negated Vector for } R_i = -1 \text{ eigenspace for } R_i$$

Using the fact that $\mathrm{SL}_{\pm}(2,\mathbb{R})$ acts triply transitively on $\mathbb{RP}^1$ we may assume without loss of generality that these are:

$$F_{R_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad F_{R_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$N_{R_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \qquad N_{R_2} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

We may abuse language slightly by referring to R_i as both the reflection and its matrix representative in $\mathrm{SL}_{\pm}(2,\mathbb{R})$. With the above choices the reflections are:

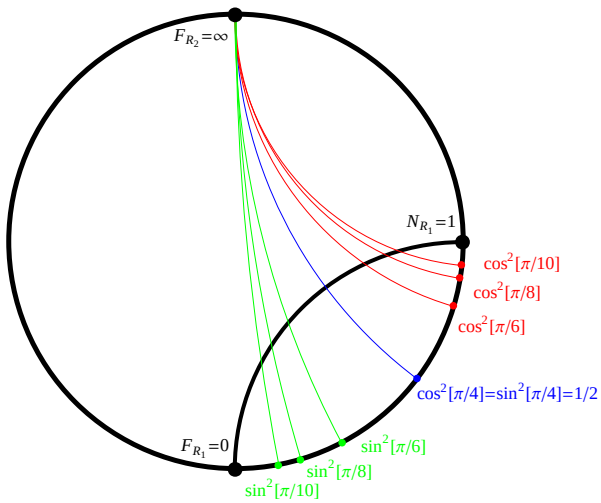
$$R_1 = \begin{pmatrix} -1 & 0 \\ -2 & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & -2\lambda \\ 0 & -1 \end{pmatrix} \qquad R_1R_2 = \begin{pmatrix} -1 & 2\lambda \\ -2 & 4\lambda - 1 \end{pmatrix}$$

Let τ be an eigenvalue for R_1R_2. The condition that $(R_1R_2)^p = I_2 \in \mathrm{PSL}_{\pm}(2,\mathbb{R})$ means that the matrix representatives satisfy $(R_1R_2)^p = \pm I_2$, hence $\tau^p = \pm 1$. So $\tau = \pm e^{\frac{k\pi i}{p}}$ for some integer $k < 2p$. Since $\det(R_1) = \det(R_2) = -1$, we know $\det(R_1R_2) = 1$ hence τ^{-1} is the other eigenvalue for R_1R_2. We can compute that the trace is:

$$\begin{aligned} Tr(R_1R_2) &= \tau + \tau^{-1} \\ &= \pm e^{\frac{k\pi i}{p}} \pm e^{-\frac{k\pi i}{p}} \\ &= \pm 2\cosh\left(\frac{k\pi i}{p}\right) \\ &= \pm 2\cos\left(\frac{k\pi}{p}\right) \\ 4\lambda - 2 &= \pm 2\cos\left(\frac{k\pi}{p}\right) \\ \lambda &= \frac{1 \pm \cos\left(\frac{k\pi}{p}\right)}{2} \\ \lambda = \cos^2\left(\frac{k\pi}{2p}\right) &\text{ or } \lambda = \sin^2\left(\frac{k\pi}{2p}\right) \end{aligned}$$

We will further restrict our attention to the situation where $k = 1$ to avoid a cone angle and obtain a representation which is geometric. This yields 2 geometric representations of the dihedral group of order $2p$ into $\mathrm{SL}_{\pm}(2,\mathbb{R})$. While these representations are not conjugate in $\mathrm{SL}(2,\mathbb{R})$ they are conjugate in $\mathrm{SL}_{\pm}(2,\mathbb{R})$. Interchanging the roles of F_{R_1} and N_{R_1} determines the same involution in $\mathrm{PSL}_{\pm}(2,\mathbb{R})$.

This can be easily seen by looking at the configuration of geodesics fixed by R_1 and R_2 in $\mathbb{H}^2$. All of our computations above can be interpreted in the upper half plane, and the picture below is the corresponding picture in the Poincare Disk under the Cayley transform (see §2.3). The geodesic between 0 and 1 is fixed by R_1, and the fixed geodesics for R_2 are shown for $p = 2, 3, 4, 5$. It is clear that the two fixed geodesics are conjugate by reflection M in the geodesic from $\frac{1}{2}$ to ∞. This reflection interchanges F_{R_1} and N_{R_1}.



To more easily generalize the above argument, we can reformulate the above condition on the trace in terms of cross ratios. The cross ratio $[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \lambda$, thus we could have found the above configuration of geodesics by solving the equations:

$$[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \cos^2\left(\frac{\pi}{2p}\right) \text{ or } [F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \sin^2\left(\frac{\pi}{2p}\right).$$

It is easy to see that the reflection M interchanges these solutions. Since M preserves the cross ratio

$$[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = [M(F_{R_1}), M(N_{R_1}), M(N_{R_2}), M(F_{R_2})]$$

$$[0, 1, \lambda, \infty] = [M(0), M(1), M(\lambda), M(\infty)]$$

$$\lambda = [1, 0, M(\lambda), \infty]$$

$$\lambda = 1 - M(\lambda)$$

$$\cos^2\left(\frac{\pi}{2p}\right) = 1 - \sin^2\left(\frac{\pi}{2p}\right)$$

Remark 7.1. A key note here is that neither $\mathrm{tr}(R_1R_2)$ nor the cross ratio can be used to distinguish between the $+1$ and -1 eigenspaces. Although we will continue to use the notation F_{R_i} and N_{R_i}, it really makes no difference which eigenspace is which. Considering the action on $\mathbb{H}^2$, interchanging these ideal fixed points still determines the same reflection. Interchanging the eigenspaces of both reflections preserves the cross ratio, interchanging just one pair of eigenspaces negates the cross ratio and adds 1.

$$[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = [N_{R_1}, F_{R_1}, F_{R_2}, N_{R_2}]$$

$$1 - [N_{R_1}, F_{R_1}, N_{R_2}, F_{R_2}] = 1 - [F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]$$

We conclude that

Theorem 7.2. *There is a unique faithful geometric representation of the dihedral group of order $2p$ in $SL_{\pm}(2, \mathbb{R})$ up to conjugacy.*

Remark 7.3. If $k \neq 1$ we can still build dihedral representations where the cross ratio will be $\sin^2\left(\frac{k\pi}{2p}\right)$. Such representations will give rise to structures with a cone point. The fixed geodesics of the generators will intersect in an angle which is $\frac{k\pi}{p}$ and the "wedge" in between these geodesics will not be a fundamental domain.

7.2 Dihedral Groups in $\mathrm{Sp}_{\pm}(4,\mathbb{R})$

Let $R_1, R_2 \in \mathrm{Sp}_{\pm}(4,\mathbb{R})$ be Lagrangian involutions generating a dihedral group of order $2p$. Similar to the $\mathrm{SL}_{\pm}(2,\mathbb{R})$ case, let

$$F_{R_i} = \text{Fixed Lagrangian for } R_i = +1 \text{ eigenspace for } R_i$$

$$N_{R_i} = \text{Negated Largangian for } R_i = -1 \text{ eigenspace for } R_i$$

$\mathrm{Sp}(4,\mathbb{R})$ is acts doubly transitively on $\mathrm{Lag}(\mathbb{R}^4)$ (see §6.2), so without any loss of generality we may assume that $F_{R_1} = L_0$ and $F_{R_2} = L_\infty$. For a third Lagrangian we must make a choice of a triple. We may choose N_{R_1} to be either

$$L_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ or } L_{1,-1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

In either case, we may then apply an element which stabilizes the triple (theorem 6.8) to assume that

$$N_{R_2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

thus all elements of the quadruple have a diagonal top block. Such Lagrangians are in the image of the bidisk (see §5.5). So without any loss of generality we may assume that a dihedral group representation factors through the bidisk.

7.3 The definite triple

First consider choosing $N_{R_1} = L_1$. It is now straightforward to compute that

$$R_1 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & 0 & -2\lambda 1 & 0 \\ 0 & 1 & 0 & -2\lambda 2 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

We can now express $\mathrm{tr}(R_1R_2)$ and $\mathrm{tr}((R_1R_2)^2)$ as polynomial functions of λ_1, λ_2

$$\mathrm{tr}(R_1R_2) = 4(\lambda_1 + \lambda_2) - 4$$

$$\mathrm{tr}((R_1R_2)^2) = 4(2(\lambda_1 + \lambda_2) - 1)^2 - 32\lambda_1\lambda_2$$

Since we know the cross ratio is

$$CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

the above equations can be reformulated as polynomial functions of the spectrum of the cross ratio. Let

$$T_p := \mathrm{tr}(CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]) = \lambda_1 + \lambda_2$$

$$D_p := \det(CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]) = \lambda_1\lambda_2$$

so we can rewrite the above equations as

$$\mathrm{tr}(R_1R_2) = 4T_p - 4$$

$$\mathrm{tr}((R_1R_2)^2) = 4(2T_p - 1)^2 - 32D_p$$

Let τ be an eigenvalue for R_1R_2. As in the case of $\mathrm{SL}_\pm(2,\mathbb{R})$, $\tau = \pm e^{\frac{k\pi i}{p}}$, and we will primarily focus on the case $k = 1$. Since $R_1R_2 \in \mathrm{Sp}(4,\mathbb{R})$, lemma 1.7 implies that eigenvalues come in inverse pairs. There are now 2 different cases to consider depending upon whether or not R_1R_2 has a repeated eigenvalue.

7.3.1 The diagonal case: Repeated eigenvalues

If the dihedral representation factors through the diagonal embedding of $\mathrm{SL}_\pm(2,\mathbb{R})$ then R_1R_2 has repeated eigenvalues. We will show that the converse is also true: if R_1R_2 has repeated eigenvalues then it is conjugate to a representation factoring through Φ_{diag}. The eigenvalues are $\pm\tau, \pm\tau, \pm\tau^{-1}, \pm\tau^{-1}$ and

$$\begin{aligned}
\mathrm{tr}(R_1R_2) &= \pm 2(\tau + \tau^{-1}) & \mathrm{tr}((R_1R_2)^2) &= 2(\tau^2 + \tau^{-2}) \\
&= \pm 2(e^{\frac{\pi i}{p}} + e^{-\frac{\pi i}{p}}) & &= 2(e^{\frac{2\pi i}{p}} + e^{-\frac{2\pi i}{p}}) \\
&= \pm 4\cosh\left(\frac{\pi i}{p}\right) & &= 4\cosh\left(\frac{2\pi i}{p}\right) \\
&= \pm 4\cos\left(\frac{\pi}{p}\right) & &= 4\cos\left(\frac{2\pi}{p}\right)
\end{aligned}$$

This yields a system of 2 equations in the unknowns λ_1 and λ_2, or equivalently in the unknowns T_p and D_p. Solving these systems yields

$$\lambda_1 = \lambda_2 = \cos^2\left(\tfrac{\pi}{2p}\right)$$

or

$$\lambda_1 = \lambda_2 = \sin^2\left(\tfrac{\pi}{2p}\right)$$

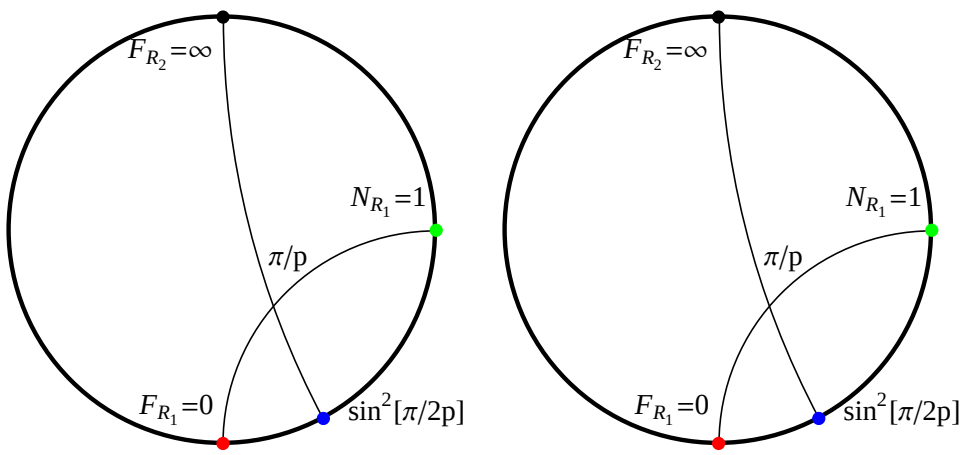
equivalently

$$T_p = 2\cos^2\left(\tfrac{\pi}{2p}\right) \text{ and } D_p = \cos^4\left(\tfrac{\pi}{2p}\right)$$

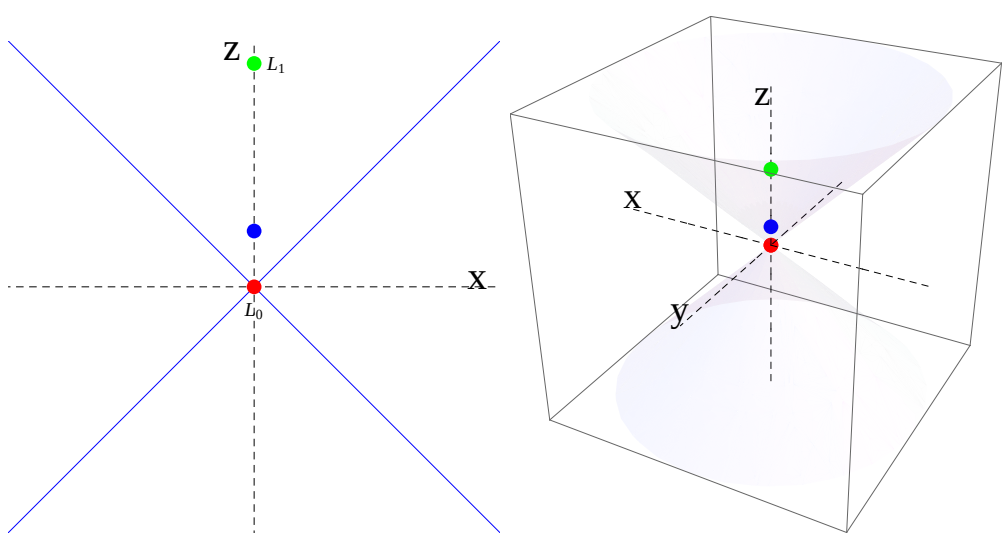
or

$$T_p = 2\sin^2\left(\frac{\pi}{2p}\right) \text{ and } D_p = \sin^4\left(\frac{\pi}{2p}\right)$$

These are precisely the solutions obtained above for $\mathrm{SL}_{\pm}(2,\mathbb{R})$. Such a representation can be constructed by taking a representation in $\mathrm{SL}_{\pm}(2,\mathbb{R})$ whose generators fix the geodesics $(0,1)$ and $\left(\sin^2\left(\frac{\pi}{2p}\right),\infty\right)$ and composing with Φ_{diag}. The picture below illustrates this configuration in the bidisk:



The image of the fixed points can be seen in $\mathrm{Ein}^{2,1}$ below. Recall from §5.5 that $\phi_{diag}(\partial\mathbb{H}^2)$ identifies with the z-axis in the standard Minkowski patch.



The points $F_{R_1} = L_0$ and $N_{R_1} = L_1$ correspond to p_0 and p_1 respectively. $F_{R_2} = L_\infty$ corresponds to the improper point p_∞ and N_{R_2} is the point in the Minkowski patch $\left(0, 0, \sin^2\left(\frac{\pi}{2p}\right)\right)$. The stabilizer of the triple (L_0, L_1, L_∞) also stabilizes N_{R_2}, hence no deformations of this solution is possible. We conclude that

Theorem 7.4. *The (unique) geometric faithful representation of the dihedral group or order $2p$ which factors through Φ_{bidisk} is locally rigid, i.e. there are no $Sp_\pm(4, \mathbb{R})$ deformations of this representation.*

7.3.2 Non-diagonal: Distinct eigenvalues

In this case the eigenvalues of R_1R_2 are $\tau, -\tau, \tau^{-1}, -\tau^{-1}$ and we can compute

$$\begin{aligned} \operatorname{tr}(R_1R_2) &= \tau - \tau + \tau^{-1} - \tau^{-1} & \operatorname{tr}((R_1R_2)^2) &= 2(\tau^2 + \tau^{-2}) \\ &= 0 & &= \pm 4\cos\left(\frac{2\pi}{p}\right) \end{aligned}$$

Solving the system now yields the solutions

$$\lambda_1 = \cos^2\left(\tfrac{\pi}{2p}\right) \text{ and } \lambda_2 = \sin^2\left(\tfrac{\pi}{2p}\right)$$

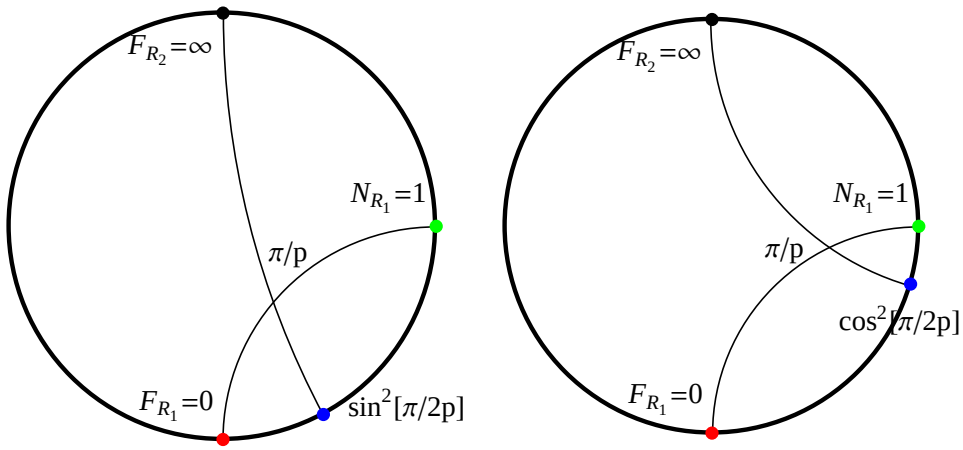
or

$$\lambda_1 = \sin^2\left(\tfrac{\pi}{2p}\right) \text{ and } \lambda_2 = \cos^2\left(\tfrac{\pi}{2p}\right)$$

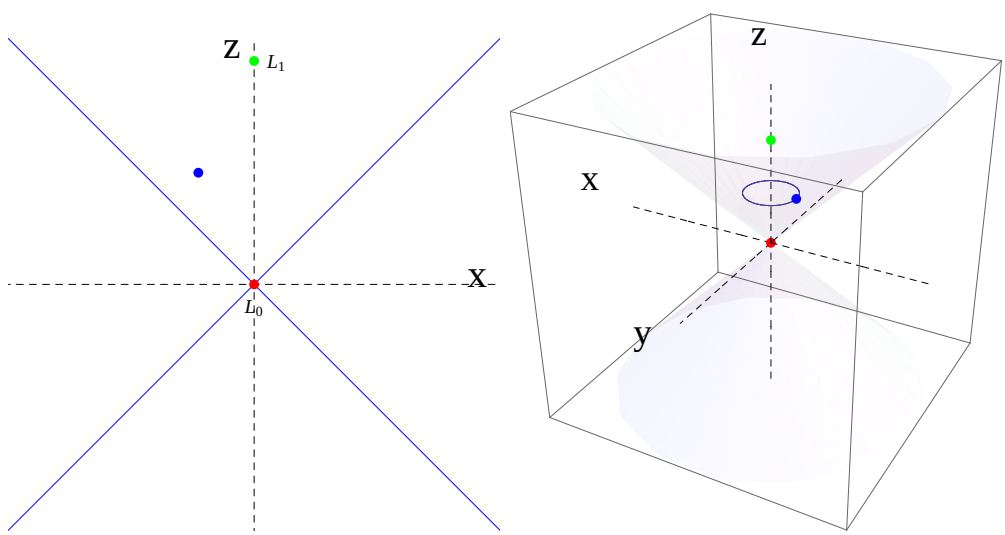
equivalently

$$T_p = 1 \text{ and } D_p = \tfrac{1}{4}\sin^2\left(\tfrac{\pi}{p}\right)$$

Such a representation can be constructed as follows: Begin with a dihedral representation in each factor of the bidisk where the cross ratio for the first quadruple is $\sin^2\left(\frac{\pi}{2p}\right)$ and for the second factor it is $\cos^2\left(\frac{\pi}{2p}\right)$. In the bidisk these configurations look like:



The image of these fixed points under ϕ_{bidisk} yield the desired configuration in $\mathrm{Lag}(\mathbb{R}^4)$. We can visualize this configuration in $\mathrm{Ein}^{2,1}$ in a manner analogous to above. F_{R_1}, N_{R_1} and F_{R_2} are the same and N_{R_2} corresponds to the point $\left(-\frac{1}{2}\cos\left(\frac{\pi}{p}\right), 0, \frac{1}{2}\right)$.



The stabilizer of the triple $(F_{R_1}, N_{R_1}, F_{R_2})$ is $O(2)$ (see thm 6.8) and if $p \neq 2$ then N_{R_2} is not fixed by this subgroup. There is a 1 parameter family of solutions, all of whom have the same cross ratio. This is shown in the picture above as a circle centered at the point corresponding to the solution for $p = 2$.

Remark 7.5. The eigenvalues of the cross ratio for these representations are related by the trigonometric identity

$$\sin^2\left(\frac{\pi}{2p}\right) = \cos^2\left(\frac{(p+1)\pi}{2p}\right)$$

We could in a similar manner construct representations where the cross ratio is conjugate to

$$\begin{bmatrix} \sin^2\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \sin^2\left(\frac{k\pi}{2p}\right) \end{bmatrix}$$

For each k there is a one parameter family of such representations, all of which are conjugate to one factoring through the bidisk.

Theorem 7.6. *For $p \neq 2$ and all $k < 2p$ there is a one parameter family of faithful representations of the dihedral group of order $2p$ where the cross ratio is conjugate to*

$$\begin{bmatrix} \sin^2\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \sin^2\left(\frac{k\pi}{2p}\right) \end{bmatrix}$$

In particular these representations are conjugate to one which factors through Φ_{bidisk} but not through Φ_{diag}.

7.4 The indefinite triple

Now we suppose that $N_{R_1} = L_{1,-1}$. This is surprisingly similar to the case using the definite triple. The eigenspaces for the generators are:

$$F_{R_1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad F_{R_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \quad N_{R_1} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N_{R_2} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The cross ratio is

$$CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix}$$

and it is straightforward to compute that the equations are the same

$$T_p := \mathrm{tr}(CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]) = \lambda_1 - \lambda_2$$

$$D_p := \det(CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]) = -\lambda_1\lambda_2$$

$$\mathrm{tr}(R_1R_2) = 4T_p - 4$$

$$\mathrm{tr}((R_1R_2)^2) = 4(2T_p - 1)^2 - 32D_p$$

Once again we have two cases to consider depending upon whether or not R_1R_2 has a repeated eigenvalue.

7.4.1 The anti-diagonal case: Repeated eigenvalues

As above, if R_1R_2 has a repeated eigenvalue then

$$\mathrm{tr}(R_1R_2) = \pm 4\cos\left(\tfrac{\pi}{p}\right) \qquad \mathrm{tr}((R_1R_2)^2) = 4\cos\left(\tfrac{2\pi}{p}\right)$$

The solving for λ_1 and λ_2 yields

$$\lambda_1 = \cos^2\left(\tfrac{\pi}{2p}\right) \text{ and } \lambda_2 = -\cos^2\left(\tfrac{\pi}{2p}\right)$$

or

$$\lambda_1 = \sin^2\left(\tfrac{\pi}{2p}\right) \text{ and } \lambda_2 = -\sin^2\left(\tfrac{\pi}{2p}\right)$$

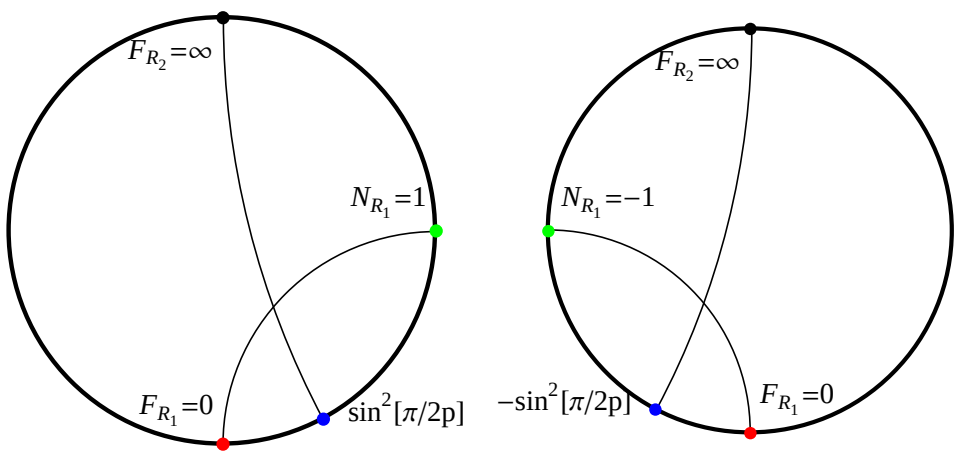
Since the eigenvalues of the cross ratio are λ_1 and $-\lambda_2$ we equivalently have

$$T_p = 2\cos^2\left(\tfrac{\pi}{2p}\right) \text{ and } D_p = \cos^4\left(\tfrac{\pi}{2p}\right)$$

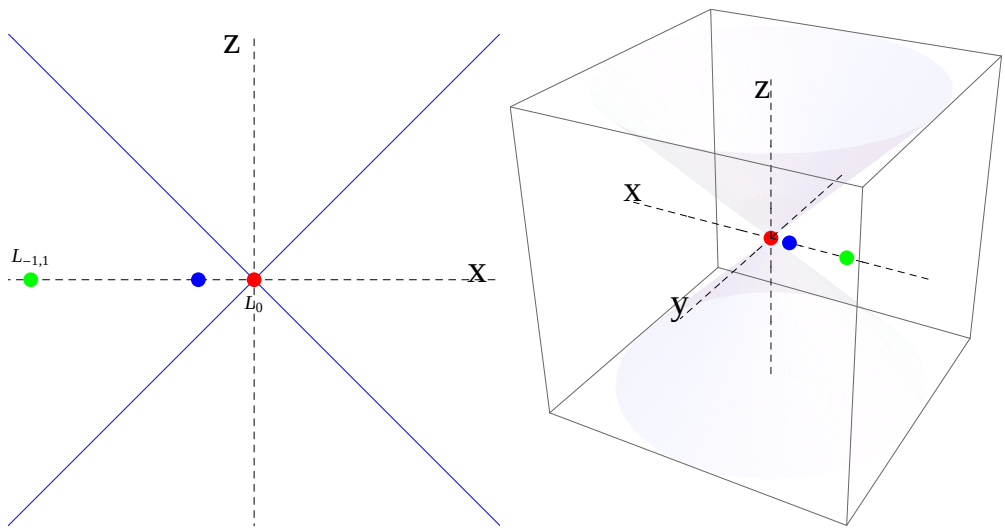
or

$$T_p = 2\sin^2\left(\tfrac{\pi}{2p}\right) \text{ and } D_p = \sin^4\left(\tfrac{\pi}{2p}\right)$$

We can construct such a representation by taking a dihedral representation into $\mathrm{SL}_{\pm}(2,\mathbb{R})$ in one factor, and a conjugate of that group by reflection over the geodesic $(0,\infty)$ in the other factor. The configuration of fixed geodesics in the bidisk is shown below.



The image of the fixed points in $\mathrm{Ein}^{2,1}$ are shown below. These all lie on the x-axis in the standard Minkowski patch. As discussed in §5.5, this is precisely the image of anti-diagonally embedded $\partial\mathbb{H}^2$.



The stabilizer of the triple $(F_{R_1}, N_{R_1}, F_{R_2})$ is isomorphic to $O(1,1)$ (see thm 6.8) and this fixes N_{R_2} as well.

7.4.2 Not anti-diagonal: Distinct eigenvalues

If R_1R_2 has distinct eigenvalues then

$$\mathrm{tr}(R_1R_2) = 0 \qquad \mathrm{tr}((R_1R_2)^2) = \pm 4\cos\left(\tfrac{2\pi}{p}\right)$$

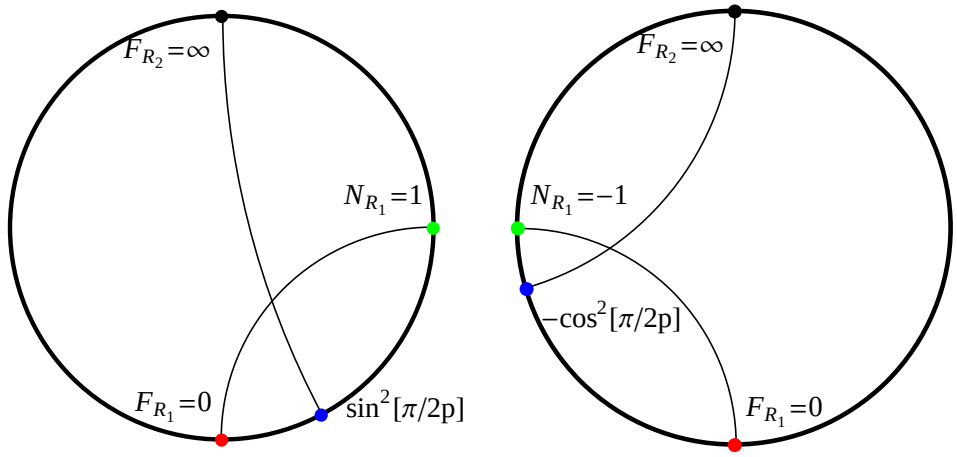
The cross ratio is then conjugate to

$$\begin{pmatrix} \cos^2\left(\frac{\pi}{2p}\right) & 0 \\ 0 & \sin^2\left(\frac{\pi}{2p}\right) \end{pmatrix}$$

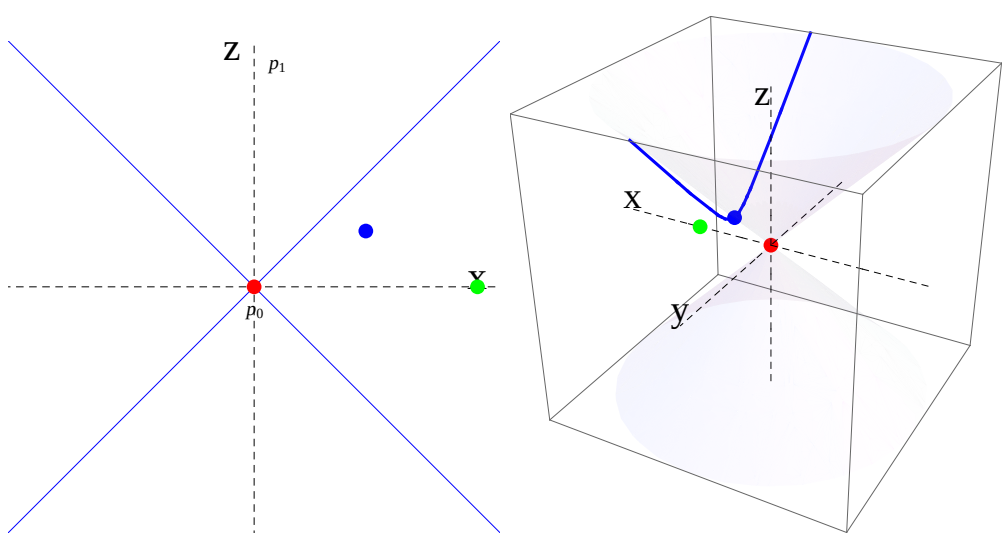
thus the trace and determinant of the cross ratio is

$$T_p = 1 \text{ and } D_p = \tfrac{1}{4}\sin^2\left(\tfrac{\pi}{p}\right)$$

Once again such a configuration can be constructed from two dihedral groups in $\mathrm{SL}_{\pm}(2,\mathbb{R})$ which are conjugate by orientation reversing transformations and have distinct cross ratios. Such a configuration is shown here in the bidisk



The corresponding configuration of fixed Lagrangians is shown here in $\mathrm{Ein}^{2,1}$. The stabilizer of the triple $(F_{R_1}, N_{R_1}, F_{R_2})$ is isomorphic to $O(1,1)$ and N_{R_2} is not fixed by this subgroup. The result is a one parameter family of such configurations all which are conjugate.



7.5 Key Lemma on the characteristic polynomial of the cross ratio

The following trivial lemma regarding trace and determinant of the cross ratio will be essential to studying triangle groups. Let

$$CR = CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]$$

$$T_p = \mathrm{tr}(CR)$$

$$D_p = \det(CR)$$

Lemma 7.7. *The discriminant of the characteristic polynomial for CR is $T_p^2 - 4D_p$. This is simply square of the difference of the eigenvalues of CR. In particular:*

- *If CR has a repeated eigenvalue then $T_p^2 - 4D_p = 0$*
- *If CR has distinct eigenvalues then $T_p^2 - 4D_p = \cos^2\left(\frac{\pi}{p}\right)$*

Proof. If CR has a repeated eigenvalue λ, we saw above that either $\lambda = \cos^2\left(\frac{\pi}{2p}\right)$ or $\lambda = \sin^2\left(\frac{\pi}{2p}\right)$.

$$T_p^2 - 4D_p = (2\lambda)^2 - 4\lambda^2 = 0$$

If CR has distinct eigenvalues then we saw in above that $T_p = 1$ and $D_p = \frac{1}{4}\sin^2\left(\frac{\pi}{p}\right)$. Then

$$T_p^2 - 4D_p = 1 - \sin^2\left(\frac{\pi}{p}\right) = \cos^2\left(\frac{\pi}{p}\right)$$

□

Chapter 8

Triangle Groups

In this section we consider deformations of hyperbolic triangle groups. Our approach will be to construct representations of these groups into $\mathrm{SL}_{\pm}(2,\mathbb{R})$, where we will show they are rigid. Then composing with an embedding arising from the bidisk (§4.1) will yield a representation into $\mathrm{Sp}_{\pm}(4,\mathbb{R})$. We will then look for deformations of such representations inside $\mathrm{Sp}_{\pm}(4,\mathbb{R})$

Definition 8.1. Let p, q, r be positive integers. A pqr-triangle group is a group that has a presentation

$$\Delta_{pqr} := \langle R_1, R_2, R_3 : R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^r = 1\rangle$$

The group is called *hyperbolic* if $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$, and in this case the generators can be realized as reflections in the sides of a triangle in the hyperbolic plane. Any 2 generators generate a dihedral group; hence the study in the previous section will be essential.

The fact that representations of triangle groups into the isometry group of the hyperbolic plane are rigid is perhaps easier to see geometrically by the well known fact that any two similar triangles in $\mathbb{H}^2$ are congruent. We will reprove this local rigidity in a geometric fashion by considering configurations of 6-tuples of distinct points on $\partial\mathbb{H}^2$. This technique will generalize nicely to the symplectic case. In §8.3.1 we will prove theorem 1.1, that representations factoring through Φ_{diag} are locally

rigid. In §8.4.1 we will present evidence that representations factoring through the anti-diagonal have non-trivial deformations.

Lastly in §8.5 we will illustrate some more exotic constructions factoring through the bi-disk which are neither diagonal nor anti-diagonal. These representations are still quite mysterious. We will give some evidence that such representations may not be rigid.

8.1 Hyperbolic Triangle Groups in $\mathrm{SL}_{\pm}(2,\mathbb{R})$

Let $R_1, R_2, R_3 \in \mathrm{SL}_{\pm}(2,\mathbb{R})$ be reflections generating a pqr-triangle group. These determine 3 fixed vectors and 3 negated vectors which must be distinct. Using the fact that $\mathrm{SL}_{\pm}(2,\mathbb{R})$ acts triply transitively on $\mathbb{RP}^1$, we may assume without loss of generality that

$$F_{R_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad F_{R_2} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad F_{R_3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

and the negated vectors are

$$N_{R_1} = \begin{bmatrix} 1 \\ m \end{bmatrix} \qquad N_{R_2} = \begin{bmatrix} n \\ 1 \end{bmatrix} \qquad N_{R_3} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix}$$

where m, n, λ are not zero or one. Imposing the relations of a triangle group yields equations which we can solve for m, n, λ. We first sketch a purely algebraic approach to this problem. We will find values of m, n, λ which yield a triangle group and prove that such groups have no deformations.

8.1.1 Algebraic Setup using Traces

It is straightforward to compute with the above choices that

$$R_1 = \begin{pmatrix} -1 & 0 \\ -2m & 1 \end{pmatrix} \qquad R_2 = \begin{pmatrix} 1 & -2n \\ 0 & -1 \end{pmatrix} \qquad R_3 = \begin{pmatrix} \frac{\lambda+1}{1-\lambda} & \frac{2\lambda}{\lambda-1} \\ -\frac{2}{\lambda-1} & \frac{\lambda+1}{\lambda-1} \end{pmatrix}$$

Any two of these reflections generates a dihedral group, so imposing the triangle group relations determines 3 equations that can expressed in terms of traces. Computing the traces of the pairwise products we obtain

$$\begin{aligned} \mathrm{tr}(R_1R_2) &= -2 + 4mn \\ \mathrm{tr}(R_2R_3) &= -\frac{2(-2n+\lambda+1)}{\lambda-1} \\ \mathrm{tr}(R_3R_1) &= \frac{(2-4m)\lambda+2}{\lambda-1} \end{aligned}$$

Let x, y and z denote these traces:

$$x = \mathrm{tr}(R_1R_2) \qquad y = \mathrm{tr}(R_2R_3) \qquad z = \mathrm{tr}(R_3R_1).$$

The triple (x, y, z) is called the *trace coordinates* for the representation (see [13]). Solving this system for m, n, λ we obtain:

$$\begin{aligned} m &= \frac{-x+y+z-2 \pm \sqrt{x^2+y^2+z^2-xyz-4}}{2y-4} \\ n &= \frac{-x+y+z-2 \mp \sqrt{x^2+y^2+z^2-xyz-4}}{2z-4} \\ \lambda &= \frac{-2x+yz \mp 2\sqrt{x^2+y^2+z^2-xyz-4}}{(y+2)\,(z-2)} \end{aligned}$$

It is not clear that there even are trace coordinates yielding real solutions. It is also unclear whether or not possible trace coordinates yielding real solutions are conjugate. We make a change of variables to the above system which yields equivalent solutions and make these a bit clearer.

8.1.2 Geometric Solution using Cross Ratios

The cross ratios are given by

$$CR_{12} := [F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = nm$$

$$CR_{23} := [F_{R_2}, N_{R_2}, N_{R_3}, F_{R_3}] = \frac{n-1}{\lambda - 1}$$

$$CR_{31} := [F_{R_3}, N_{R_3}, N_{R_1}, F_{R_1}] = -\frac{(m-1)\lambda}{\lambda - 1}$$

Let C_p, C_q, C_r be the values of the cross ratios determined by the relations for the group. By §7.1 we know that $C_p = \cos^2\left(\frac{\pi}{2p}\right)$ or $C_p = \sin^2\left(\frac{\pi}{2p}\right)$, etc. The relationship between the trace coordinates and the cross ratios is

$$(x, y, z) = (4C_p - 2, 4C_q - 2, 4C_r - 2)$$

The equivalent system of equations is

$$CR_{12} = C_p \qquad CR_{23} = C_q \qquad CR_{31} = C_r \tag{8.1}$$

Solving this system for m, n, λ we obtain

$$\begin{aligned}
m &= \frac{-1 - C_p + C_q + C_r \pm \sqrt{-4C_qC_rC_p + C_p^2 + C_q^2 + C_r^2 - 2\left(C_p + C_q + C_r\right) + 2\left(C_pC_q + C_rC_q + C_pC_r\right) + 1}}{2\left(C_q - 1\right)} \\
n &= \frac{-1 - C_p + C_q + C_r \mp \sqrt{-4C_qC_rC_p + C_p^2 + C_q^2 + C_r^2 - 2\left(C_p + C_q + C_r\right) + 2\left(C_pC_q + C_rC_q + C_pC_r\right) + 1}}{2\left(C_r - 1\right)} \\
\lambda &= \frac{1 - C_p - C_q + 2C_qC_r - C_r \mp \sqrt{-4C_qC_rC_p + C_p^2 + C_q^2 + C_r^2 - 2\left(C_p + C_q + C_r\right) + 2\left(C_pC_q + C_rC_q + C_pC_r\right) + 1}}{2C_q\left(C_r - 1\right)}
\end{aligned} \tag{8.2}$$

The radicand in each of the above expressions is the same, so a real solution to this system exists provided we can choose (C_p, C_q, C_r) such that the radicand is nonnegative. We will consider possible tuples yielding real solutions in a moment. First we claim that if the radicand is positive, both solutions above are conjugate by an orientation preserving transformation. The first solution is obtained from

the second by interchanging the roles of the $+1$ and -1 eigenspaces for all three reflections.

Proposition 8.2. *Let (m, n, λ) be the first solution to (8.1) and $(\tilde{m}, \tilde{n}, \tilde{\lambda})$ be the second. There exists $T \in SL(2, \mathbb{R})$ conjugating these groups. More precisely, if $(\tilde{m}, \tilde{n}, \tilde{\lambda}) \neq (m, n, \lambda)$ then*

$$T(0) = \frac{1}{m} \qquad T(\infty) = \frac{1}{n} \qquad T(1) = \lambda$$
$$T\left(\frac{1}{\tilde{m}}\right) = 0 \qquad T(\tilde{n}) = \infty \qquad T(\tilde{\lambda}) = 1$$

Proof. Since both (m, n, λ) and $(\tilde{m}, \tilde{n}, \tilde{\lambda})$ are solutions to the same system of equations, the relevant cross ratios yield 3 equations

$$C_p = [0, \tfrac{1}{m}, n, \infty] = [0, \tfrac{1}{\tilde{m}}, \tilde{n}, \infty]$$
$$C_q = [\infty, n, \lambda, 1] = [\infty, \tilde{n}, \tilde{\lambda}, 1]$$
$$C_q = [1, \lambda, \tfrac{1}{m}, 0] = [1, \tilde{\lambda}, \tfrac{1}{\tilde{m}}, 0]$$

which allows us to write $(\tilde{m}, \tilde{n}, \tilde{\lambda})$ in terms of m, n, λ. We obtain

$$(\tilde{m}, \tilde{n}, \tilde{\lambda}) = (m, n, \lambda) \qquad \text{or} \qquad (\tilde{m}, \tilde{n}, \tilde{\lambda}) = \left(\tfrac{n - mn\lambda}{n - \lambda}, \tfrac{m(\lambda - n)}{m\lambda - 1}, \tfrac{(m-1)(n-\lambda)}{(n-1)(m\lambda - 1)}\right)$$

In the first case we can take T to be the identity map. This occurs if and only if the radicand above is zero which occurs only for Euclidean triangle groups.

In the second case, let T be the $\mathrm{SL}(2, \mathbb{R})$ transformation taking $(0, 1, \infty) \to (\frac{1}{m}, \lambda, n)$. As a matrix in $\mathrm{PGL}(2, \mathbb{R})$ this is

$$T = \begin{pmatrix} n\left(\frac{1}{m} - \lambda\right) & \frac{\lambda - n}{m} \\ \frac{1}{m} - \lambda & \lambda - n \end{pmatrix}$$

Straightforward computations now yield

$$T\left(\tfrac{1}{\tilde{m}}\right) = 0 \qquad T(\tilde{n}) = \infty \qquad T(\tilde{\lambda}) = 1$$

and $\det(T) > 0$ as desired. $\square$

Now let us consider how to obtain real solutions to (8.1). Let

$$\kappa(x, y, z) := x^2 + y^2 + z^2 - xyz - 2$$

be the polynomial defined in appendix B. The radicands in (8.2) are all

$$\tfrac{1}{16}\left(\kappa(4C_p - 2, 4C_q - 2, 4C_r - 2) - 2\right)$$

so to obtain real solutions we need this value to be positive. Lemma B.3 says that

$$\kappa\left(-2\cos\left(\tfrac{\pi}{p}\right), -2\cos\left(\tfrac{\pi}{q}\right), -2\cos\left(\tfrac{\pi}{r}\right)\right) > 2$$

Applying relevant trigonometric identities yields

$$\begin{aligned}
\kappa\left(4\sin^2\left(\frac{\pi}{2p}\right) - 2, 4\sin^2\left(\frac{\pi}{2q}\right) - 2, 4\sin^2\left(\frac{\pi}{2r}\right) - 2\right) &> 2 \\
\kappa\left(4\sin^2\left(\frac{\pi}{2p}\right) - 2, 4\sin^2\left(\frac{\pi}{2q}\right) - 2, 4\sin^2\left(\frac{\pi}{2r}\right) - 2\right) - 2 &> 0 \\
\frac{1}{16}\left(\kappa\left(4\sin^2\left(\frac{\pi}{2p}\right) - 2, 4\sin^2\left(\frac{\pi}{2q}\right) - 2, 4\sin^2\left(\frac{\pi}{2r}\right) - 2\right) - 2\right) &> 0
\end{aligned}$$

thus we obtain real solutions when

$$(C_p, C_q, C_r) = \left(\sin^2\left(\tfrac{\pi}{2p}\right), \sin^2\left(\tfrac{\pi}{2q}\right), \sin^2\left(\tfrac{\pi}{2r}\right)\right)$$

Lemma B.4 implies that other possible solutions are

- $(C_p, C_q, C_r) = \left(\cos^2\left(\tfrac{\pi}{2p}\right), \cos^2\left(\tfrac{\pi}{2q}\right), \sin^2\left(\tfrac{\pi}{2r}\right)\right)$
- $(C_p, C_q, C_r) = \left(\cos^2\left(\tfrac{\pi}{2p}\right), \sin^2\left(\tfrac{\pi}{2q}\right), \cos^2\left(\tfrac{\pi}{2r}\right)\right)$

- $(C_p, C_q, C_r) = \left(\sin^2\left(\frac{\pi}{2p}\right), \cos^2\left(\frac{\pi}{2q}\right), \cos^2\left(\frac{\pi}{2r}\right)\right)$

We now claim that the three solutions to (8.1) obtained using each of these are conjugate by an orientation reversing transformation to the one obtained using $(C_p, C_q, C_r) = \left(\sin^2\left(\frac{\pi}{2p}\right), \sin^2\left(\frac{\pi}{2q}\right), \sin^2\left(\frac{\pi}{2r}\right)\right)$. The conjugating map is obtained by interchanging the eigenspaces for one or two of the reflections but not all three.

Proposition 8.3. *Fix p, q, r satisfing the relations for a hyperbolic triangle group. Let $(m_{sss}, n_{sss}, \lambda_{sss})$ be the solution to (8.2) obtained using*

$$(C_p, C_q, C_r) = \left(\sin^2\left(\frac{\pi}{2p}\right), \sin^2\left(\frac{\pi}{2q}\right), \sin^2\left(\frac{\pi}{2r}\right)\right)$$

and $(m_{ccs}, n_{ccs}, \lambda_{ccs})$ be the solution obtained using

$$(C_p, C_q, C_r) = \left(\cos^2\left(\frac{\pi}{2p}\right), \cos^2\left(\frac{\pi}{2q}\right), \sin^2\left(\frac{\pi}{2r}\right)\right).$$

Then there is $T \in SL_{\pm}(2, \mathbb{R})$ conjugating these groups.

Proof. The cross ratios are related by

$$[0, \frac{1}{m_{sss}}, n_{sss}, \infty] = 1 - [0, \frac{1}{m_{ccs}}, n_{ccs}, \infty]$$

$$[\infty, n_{sss}, \lambda_{sss}, 1] = 1 - [\infty, n_{ccs}, \lambda_{ccs}, 1]$$

$$[1, \lambda_{sss}, \frac{1}{m_{sss}}, 0] = [1, \lambda_{ccs}, \frac{1}{m_{ccs}}, 0]$$

We can solve for $(m_{ccs}, n_{ccs}, \lambda_{ccs})$ in terms of $(m_{sss}, n_{sss}, \lambda_{sss})$ to obtain

$$(m_{ccs}, n_{ccs}, \lambda_{ccs}) = \left(1 - m_{sss}\lambda_{sss}, \frac{m_{sss}n_{sss}-1}{m_{sss}\lambda_{sss}-1}, \frac{m_{sss}-1}{m_{sss}\lambda_{sss}-1}\right)$$

or

$$(m_{ccs}, n_{ccs}, \lambda_{ccs}) = \left(\frac{m_{sss}n_{sss}-1}{n_{sss}-1}, 1 - n_{sss}, \frac{(n_{sss}-1)\lambda_{sss}}{n_{sss}-\lambda_{sss}}\right)$$

In the first case, let T be the transformation mapping $(0, 1, \infty) \to (0, 1, n_{sss})$. Direct computation shows that

$$(0, 1, \infty, \tfrac{1}{m_{ccs}}, n_{ccs}, \lambda_{ccs}) \xrightarrow{T} (0, 1, n_{sss}, \tfrac{1}{m_{sss}}, \infty, \lambda_{sss})$$

Hence T conjugates the groups.

In the second case, let T be the transformation mapping $(0, 1, \infty) \to (\frac{1}{m_{sss}}, \lambda_{sss}, \infty)$. Direct computation shows that

$$(0, 1, \infty, \tfrac{1}{m_{ccs}}, n_{ccs}, \lambda_{ccs}) \xrightarrow{T} (\tfrac{1}{m_{sss}}, \lambda_{sss}, \infty, 0, n_{sss}, 1)$$

hence T conjugates the groups. □

The same argument can be used to show that the other possible triples for (C_p, C_q, C_r) also yield conjugate groups. Since all these solutions yield conjugate groups we have proven:

Theorem 8.4 (Rigidity of triangle groups in $\mathrm{SL}_{\pm}(2, \mathbb{R})$)**.** *Let Δ_{pqr} be a hyperbolic triangle group. There is a unique faithful representation $\Delta_{pqr} \to SL_{\pm}(2, \mathbb{R})$ up to conjugacy.*

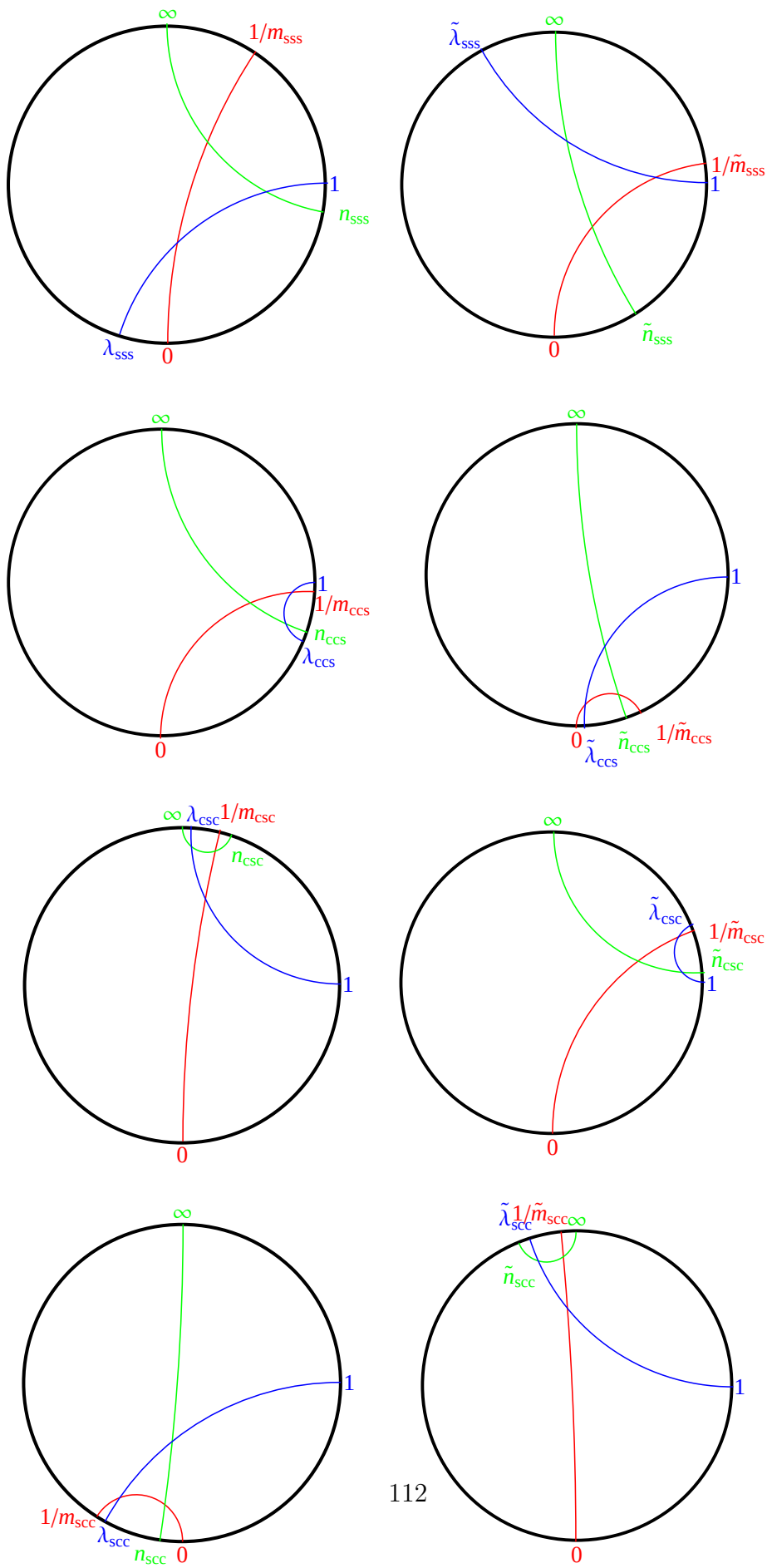
Although all of these groups are $\mathrm{SL}_{\pm}(2, \mathbb{R})$ conjugate, there are eight possible pqr-triangle groups with $0, 1, \infty$ as eigenvectors for one reflection: two solutions in (8.2) and four possible values for the cross ratios (C_p, C_q, C_r). To add to the confusion, we establish some notation.

Notation 8.5. Fix a hyperbolic triangle group Δ_{pqr}. Denote each of the eight representations ρ and the corresponding values of (m, n, λ) as follows:

$\underline{\rho}$	$\underline{(m,n,\lambda)}$	$\underline{\text{(8.2) Solution}}$	$\underline{(C_p, C_q, C_r)}$
ρ_{sss}	$(m_{sss}, n_{sss}, \lambda_{sss})$	First	$\left(\sin^2\left(\frac{\pi}{2p}\right), \sin^2\left(\frac{\pi}{2q}\right), \sin^2\left(\frac{\pi}{2r}\right)\right)$
$\widetilde{\rho_{sss}}$	$(\widetilde{m_{sss}}, \widetilde{n_{sss}}, \widetilde{\lambda_{sss}})$	Second	
ρ_{ccs}	$(m_{ccs}, n_{ccs}, \lambda_{ccs})$	First	$\left(\cos^2\left(\frac{\pi}{2p}\right), \cos^2\left(\frac{\pi}{2q}\right), \sin^2\left(\frac{\pi}{2r}\right)\right)$
$\widetilde{\rho_{ccs}}$	$(\widetilde{m_{ccs}}, \widetilde{n_{ccs}}, \widetilde{\lambda_{ccs}})$	Second	
ρ_{csc}	$(m_{csc}, n_{csc}, \lambda_{csc})$	First	$\left(\cos^2\left(\frac{\pi}{2p}\right), \sin^2\left(\frac{\pi}{2q}\right), \cos^2\left(\frac{\pi}{2r}\right)\right)$
$\widetilde{\rho_{csc}}$	$(\widetilde{m_{csc}}, \widetilde{n_{csc}}, \widetilde{\lambda_{csc}})$	Second	
ρ_{scc}	$(m_{scc}, n_{scc}, \lambda_{scc})$	First	$\left(\sin^2\left(\frac{\pi}{2p}\right), \cos^2\left(\frac{\pi}{2q}\right), \cos^2\left(\frac{\pi}{2r}\right)\right)$
$\widetilde{\rho_{scc}}$	$(\widetilde{m_{scc}}, \widetilde{n_{scc}}, \widetilde{\lambda_{scc}})$	Second	

The picture below illustrates this notation with the example of Δ_{345}. When the choice of cross ratios is irrelevant we will write ρ_*, (m_*, n_*, λ_*) for the first solution to (8.2) and $\widetilde{\rho_*}$, $(\widetilde{m_*}, \widetilde{n_*}, \widetilde{\lambda_*})$ for the second.

Theorem 8.4 says that all these representations are conjugate, but the way in which they are conjugate will be important. ρ_* and $\widetilde{\rho_*}$ are conjugate by an orientation preserving transformation. ρ_{sss} is conjugate to ρ_{ccs}, ρ_{csc} and ρ_{scc} by orientation reversing transformations. The rest can be determined by composing these. The picture below also helps make these conjugacies apparent.


∞
1/m_sss
1
n_sss
λ_sss
0
λ̃_sss
∞
1/m̃_sss
1
ñ_sss
0
∞
1
1/m_ccs
n_ccs
λ_ccs
0
∞
1
0
λ̃_ccs
ñ_ccs
1/m̃_ccs
∞
λ_csc
1/m_csc
n_csc
1
0
∞
λ̃_csc
1/m̃_csc
ñ_csc
1
0
∞
1
1/m_scc
λ_scc
n_scc
0
λ̃_scc
1/m̃_scc
∞
ñ_scc
1
0

8.2 Triangle Groups in $\mathrm{Sp}_{\pm}(4,\mathbb{R})$

Let $R_1, R_2, R_3 \in \mathrm{Sp}_{\pm}(4,\mathbb{R})$ be Lagrangian involutions generating a pqr triangle group. The eigenspaces for these forms a 6-tuple of Lagrangians which we will assume is transverse. Again we will denote the $+1$ eigenspaces as $F_{R_1}, F_{R_2}, F_{R_3}$ and the -1 eigenspaces as $N_{R_1}, N_{R_2}, N_{R_3}$.

The $+1$ eigenspaces form a triple and thus we have two possibilities for which orbit this triple is in. We may assume without loss of generality that the fixed eigenspaces have Siegel homogeneous coordinates

$$F_{R_1} = L_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad F_{R_2} = L_\infty = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \qquad F_{R_3} = \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Assuming that the -1 eigenspaces are transverse to F_{R_1} and F_{R_2} implies that the Siegel homogeneous coordinates for these spaces have top and bottom blocks which are nonsingular (see §6.3). Thus we may choose coordinates such that one of the blocks is the identity matrix and the other is a nonsingular symmetric matrix. Applying elements in the isotropy subgroup of the $+1$ eigenspaces (see theorem 6.8) allows us to further assume that one of those symmetric matrices is diagonal. So we can choose coordinates of the form

$$N_{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix} \qquad N_{R_2} = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad N_{R_3} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that without any loss of generality we can assume that $F_{R_1}, F_{R_2}, F_{R_3}$ and N_{R_3} all have a top block which is a diagonal matrix. As noted in §5.5, such Lagrangians are in the image of $\phi_{bidisk}(\partial\mathbb{H}^2 \times \partial\mathbb{H}^2)$. This image intersects the Minkowski patch in the xz-plane.

We can construct a faithful $\mathrm{Sp}_\pm(4,\mathbb{R})$ representation ρ_0 by building representations into $\mathrm{SL}_\pm(2,\mathbb{R}) \times \mathrm{SL}_\pm(2,\mathbb{R})$ and then composing with Φ_{bidisk}

$$\rho_0 : \Delta_{pqr} \to \mathrm{SL}_\pm(2,\mathbb{R}) \times \mathrm{SL}_\pm(2,\mathbb{R}) \hookrightarrow \mathrm{Sp}_\pm(4,\mathbb{R})$$

Some care must be taken to ensure that the fixed points for the representation in $\mathrm{SL}_\pm(2,\mathbb{R})$ map to the Lagrangians given above. In the previous section we saw that there were 8 possible representations into $\mathrm{SL}_\pm(2,\mathbb{R})$ whose generators had fixed eigenspaces $0, \infty, 1$. Conjugating by the isometry fixing 0 and ∞ gives a representation whose fixed eigenspaces are $0, \infty - 1$. Combinations of these composed with the bidisk embedding will yield a representation factoring through the bidisk whose eigenspaces are as desired. For all of these representations the off diagonal entries $m_{12} = n_{12} = 0$.

8.2.1 Possibilities for ρ_0

We can produce representations ρ_0 having the form $\Phi_{bidisk}(\rho_1, \rho_2)$ where ρ_1,ρ_2 are representations in $\mathrm{SL}_\pm(2, \mathbb{R})$. A choice of ρ_0 amounts to choosing a component of the representation variety $\mathrm{Hom}(\Delta_{pqr}, \mathrm{Sp}_\pm(4, \mathbb{R}))/\mathrm{Sp}_\pm(4, \mathbb{R})$ to investigate.

Choosing ρ_1 to be one of the possibilities from the table in the previous section ensures that the $+1$ eigenspaces in $\mathbb{R}^2$ map to the desired Lagrangians defined above. By theorem 8.4, all possibilities for ρ_1 are conjugate to ρ_{sss}, so there is $T \in \mathrm{SL}_\pm(2, \mathbb{R})$ such that $T^{-1}\rho_{sss}T = \rho_1$. Then

$$\begin{aligned}
\rho_0 &= \Phi_{bidisk}(\rho_1, \rho_2) \\
&= \Phi_{bidisk}(T^{-1}\rho_{sss}T, \rho_2) \\
&= \Phi_{bidisk}(T^{-1}, T^{-1})\Phi_{bidisk}(\rho_{sss}, T\rho_2 T^{-1})\Phi_{bidisk}(T, T)
\end{aligned}$$

thus ρ_0 is conjugate to a representation factoring through $\Phi_{bidisk}(\rho_{sss}, *)$.

Remark 8.6. A key point here is that the above argument applies regardless of whether T is orientation preserving or reversing. We are using the result in §4.5.1 which says that Φ_{bidisk} extends not to all of $\mathrm{SL}_\pm(2, \mathbb{R}) \times \mathrm{SL}_\pm(2, \mathbb{R})$ but only to a mapping of the index two subgroup

$$\Phi_{bidisk} : \{(A, B) \in \mathrm{SL}_\pm(2, \mathbb{R}) \times \mathrm{SL}_\pm(2, R) : \det(A) = \det(B)\} \to \mathrm{Sp}_\pm(4, \mathbb{R})$$

Thus $\Phi_{bidisk}(T, T) \in \mathrm{Sp}_\pm(4, \mathbb{R})$ regardless of the determinant of T.

So without any loss of generality we may assume that $\rho_0 = \Phi_{bidisk}(\rho_{sss}, \rho_2)$. We must choose ρ_2 such that the $+1$ eigenspaces are either $(0, \infty, 1)$ or $(0, \infty, -1)$ to ensure that the F_{R_i} are the Lagrangians from above. The representations defined

on the table at the end of the previous section all have $+1$ eigenspaces $(0, \infty, 1)$. Choosing ρ_2 to be one of these we obtain representations such that $(F_{R_1}, F_{R_2}, F_{R_3})$ is the definite triple. Conjugating any of these by the involution in $\mathrm{SL}_{\pm}(2, \mathbb{R})$ fixing the geodesic $(0, \infty)$ yields a representation where $(F_{R_1}, F_{R_2}, F_{R_3})$ is the indefinite triple.

Theorem 8.4 again implies that ρ_2 is conjugate to ρ_{sss}, so there is $T \in \mathrm{SL}_{\pm}(2, \mathbb{R})$ (different T than above) such that $\rho_2 = T^{-1}\rho_{sss}T$. Thus

$$\rho_0 = \Phi_{bidisk}(\rho_{sss}, T^{-1}\rho_{sss}T)$$

There are now several cases to consider, depending upon $\det(T)$. The key point in what follows is that if $\det(T) = -1$, then $(I_2, T) \in \mathrm{SL}_{\pm}(2, \mathbb{R}) \times \mathrm{SL}_{\pm}(2, R)$ but not in the domain of Φ_{bidisk}.

1. **(Diagonal)** If T is orientation preserving then $\Phi_{bidisk}(I_2, T) \in \mathrm{Sp}_{\pm}(4, \mathbb{R})$ and

$$\begin{aligned}\rho_0 &= \Phi_{bidisk}(\rho_{sss}, T^{-1}\rho_{sss}T) \\ &= \Phi_{bidisk}(I_2, T)\Phi_{bidisk}(\rho_{sss}, \rho_{sss})\Phi_{bidisk}(I, T^{-1}) \\ &= \Phi_{bidisk}(I_2, T)\Phi_{diag}(\rho_{sss})\Phi_{bidisk}(I, T^{-1})\end{aligned}$$

   hence ρ_0 is conjugate to a representation factoring through Φ_{diag}. The triple $(F_{R_1}, F_{R_2}, F_{R_3})$ is the definite triple.

2. **(Anti-Diagonal)** If T is the involution fixing $(0, \infty)$ then $\det(T) = -1$. So $\Phi_{bidisk}(\rho_{sss}, T^{-1}\rho_{sss}T)$ is not conjugate in $\mathrm{Sp}_{\pm}(4, \mathbb{R})$ to the diagonal representation. This is conjugate to the anti-diagonal representation obtained by

composing ρ_{sss} with the conjugate of Φ_{PDS} discussed in §4.3.1. The triple $(F_{R_1}, F_{R_2}, F_{R_3})$ is the indefinite triple

3. **(Mysterious)** If T is orientation reversing but does not fix $(0, \infty)$ we obtain some more exotic representations. For instance ρ_{scc} is conjugate to ρ_{sss} by an orientation reversing transformation and the $+1$ eigenspaces map to the definite triple as desired. We will investigate

$$\rho_0 = \Phi_{bidisk}(\rho_{sss}, \rho_{scc})$$

The result is a triangle group representation where two of the three cross ratios have distinct eigenvalues.

8.3 Equations obtained using the definite triple

Suppose that F_{R_i} is the definite triple, so $F_{R_3} = L_1$. Suppose $\rho(t)$ is a continuous path of faithful representations of Δ_{pqr} and $\rho(0) = \rho_0$. We denote the eigenspaces for the generators of $\rho(t)$ by $F_{R_i}(t)$ and $N_{R_i}(t)$ for $i = 1, 2, 3$. As seen above, we assume without loss of generality that $F_{R_i}(t)$ is constant for $i = 1, 2, 3$ and that $N_{R_3}(t)$ has a top block which is diagonal. Thinking of each of the variables as functions of t, the cross ratios for the eigenspaces for $\rho(t)$ are:

$$CR_{12} := CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}] = \begin{pmatrix} n_{11} & n_{12} \\ n_{12} & n_{22} \end{pmatrix} \begin{pmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{pmatrix}$$

$$= \begin{pmatrix} m_{11}n_{11} + m_{12}n_{12} & m_{12}n_{11} + m_{22}n_{12} \\ m_{11}n_{12} + m_{12}n_{22} & m_{12}n_{12} + m_{22}n_{22} \end{pmatrix}$$

$$CR_{23} := CR[F_{R_2}, N_{R_2}, N_{R_3}, F_{R_3}] = \begin{pmatrix} \lambda_1 - 1 & 0 \\ 0 & \lambda_2 - 1 \end{pmatrix}^{-1} \begin{pmatrix} n_{11} - 1 & n_{12} \\ n_{12} & n_{22} - 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{n_{11}-1}{\lambda_1 - 1} & \frac{n_{12}}{\lambda_1 - 1} \\ \frac{n_{12}}{\lambda_2 - 1} & \frac{n_{22}-1}{\lambda_2 - 1} \end{pmatrix}$$

$$CR_{31} := CR[F_{R_3}, N_{R_3}, N_{R_1}, F_{R_1}] = \begin{pmatrix} m_{11} - 1 & m_{12} \\ m_{12} & m_{22} - 1 \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \begin{pmatrix} 1 - \lambda_1 & 0 \\ 0 & 1 - \lambda_2 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} -\frac{(m_{11}-1)\lambda_1}{\lambda_1 - 1} & \frac{m_{12}\lambda_2}{1-\lambda_2} \\ \frac{m_{12}\lambda_1}{1-\lambda_1} & -\frac{(m_{22}-1)\lambda_2}{\lambda_2 - 1} \end{pmatrix}$$

Let T_p, T_q, T_r denote the traces of the cross ratios and D_p, D_q, D_r denote the determinants. Imposing the triangle group relations determines a system of six equations in these 8 variables:

$$\begin{aligned} T_p &= \operatorname{tr}(CR_{12}) & D_p &= \det(CR_{12}) \\ &= m_{11}n_{11} + 2m_{12}n_{12} + m_{22}n_{22} & &= \left(m_{12}^2 - m_{11}m_{22}\right)\left(n_{12}^2 - n_{11}n_{22}\right) \end{aligned} \tag{8.3}$$

$$\begin{aligned} T_q &= \operatorname{tr}(CR_{23}) & D_q &= \det(CR_{23}) \\ &= \frac{n_{11} - 1}{\lambda_1 - 1} + \frac{n_{22} - 1}{\lambda_2 - 1} & &= -\frac{n_{12}^2 + n_{11} - n_{11}n_{22} + n_{22} - 1}{(\lambda_1 - 1)(\lambda_2 - 1)} \end{aligned} \tag{8.4}$$

$$T_r = \mathrm{tr}(CR_{31}) \qquad\qquad D_r = \det(CR_{31})$$

$$= -\frac{(m_{11}-1)\,\lambda_1}{\lambda_1 - 1} - \frac{(m_{22}-1)\,\lambda_2}{\lambda_2 - 1} \qquad = \frac{(-m_{12}^2 - m_{11} + m_{11}m_{22} - m_{22} + 1)\,\lambda_1\lambda_2}{(\lambda_1 - 1)\,(\lambda_2 - 1)} \tag{8.5}$$

Lemma 7.7 implies that $T_p^2 - 4D_p = 0$ if the cross ratio has a repeated eigenvalue and $\cos^2\left(\frac{\pi}{p}\right)$ if it does not. Thus the triple $(T_p^2 - 4D_p, T_q^2 - 4D_q, T_r^2 - 4D_r)$ can only take on 8 possible values. The mapping which takes a representation $\rho(t)$ to any of the corresponding cross ratios is a continuous mapping into a discrete (finite) space, hence is constant. So any local deformation $\rho(t)$ of ρ_0 must take on the same values for this triple as ρ_0.

Consider the last two equations (8.5) induced by the relation $(R_3R_1)^r = Id$. Solving these for m_{11} and m_{22} yields

$$m_{11} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_1 + 2\lambda_1\lambda_2 \pm \sqrt{(T_r^2 - 4D_r)\,(\lambda_1 - 1)^2\,(\lambda_2 - 1)^2 - 4m_{12}^2\,(\lambda_1 - 1)\,\lambda_1\,(\lambda_2 - 1)\,\lambda_2}}{2\lambda_1\,(\lambda_2 - 1)}$$

$$m_{22} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_2 + 2\lambda_1\lambda_2 \mp \sqrt{(T_r^2 - 4D_r)\,(\lambda_1 - 1)^2\,(\lambda_2 - 1)^2 - 4m_{12}^2\,(\lambda_1 - 1)\,\lambda_1\,(\lambda_2 - 1)\,\lambda_2}}{2\,(\lambda_1 - 1)\,\lambda_2} \tag{8.6}$$

Similarly equations (8.4) are induced by the relation $(R_2R_3)^q = Id$ and we can solve these for n_{11} and n_{22} yielding

$$n_{11} = \frac{-2 - \lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q + T_q + 2\lambda_2 \pm \sqrt{(T_q^2 - 4D_q)\,(\lambda_1 - 1)^2\,(\lambda_2 - 1)^2 - 4n_{12}^2\,(\lambda_1 - 1)\,(\lambda_2 - 1)}}{2\,(\lambda_2 - 1)}$$

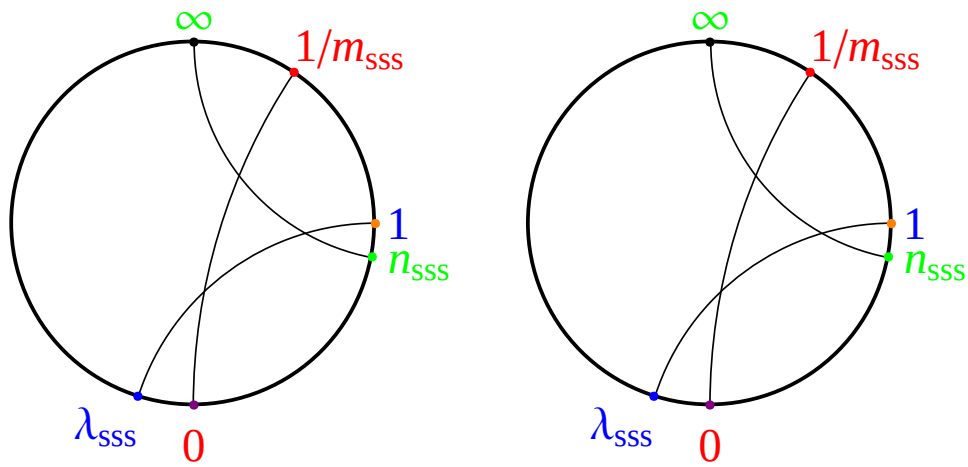
$$n_{22} = \frac{-2 - \lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q + T_q + 2\lambda_1 \mp \sqrt{(T_q^2 - 4D_q)\,(\lambda_1 - 1)^2\,(\lambda_2 - 1)^2 - 4n_{12}^2\,(\lambda_1 - 1)\,(\lambda_2 - 1)}}{2\,(\lambda_1 - 1)} \tag{8.7}$$

Each of the radicands in the above solutions contains a factor of $(T_r^2 - 4D_r)$ or $(T_q^2 - 4D_q)$ in one term.

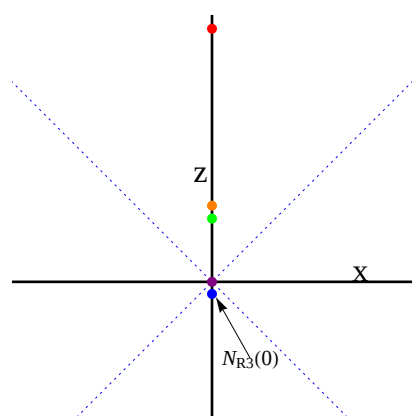
8.3.1 ρ_0 factors through Φ_{diag}

As noted in §8.2.1 we may assume without loss of generality that $\rho_0 = \Phi_{diag}(\rho_{sss})$. The projections of the fixed geodesics onto each factor of the bidisk yield the con-

figuration shown here:



The images of these fixed ideal points under ϕ_{diag} yield a configuration of 6 Lagrangians which, except for the improper point, all like on the z-axis in the Minkowski patch:



In this case, all cross ratios have a repeated eigenvalue. Lemma 7.7 implies that $T_p^2 - 4D_p = T_q^2 - 4D_q = T_r^2 - 4D_r = 0$. The above solutions (8.6) and (8.7) reduce to

$$m_{11} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_1 + 2\lambda_1\lambda_2 \pm \sqrt{-4m_{12}^2\left(\lambda_1 - 1\right)\lambda_1\left(\lambda_2 - 1\right)\lambda_2}}{2\lambda_1\left(\lambda_2 - 1\right)}$$
$$m_{22} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_2 + 2\lambda_1\lambda_2 \mp \sqrt{-4m_{12}^2\left(\lambda_1 - 1\right)\lambda_1\left(\lambda_2 - 1\right)\lambda_2}}{2\left(\lambda_1 - 1\right)\lambda_2} \tag{8.8}$$

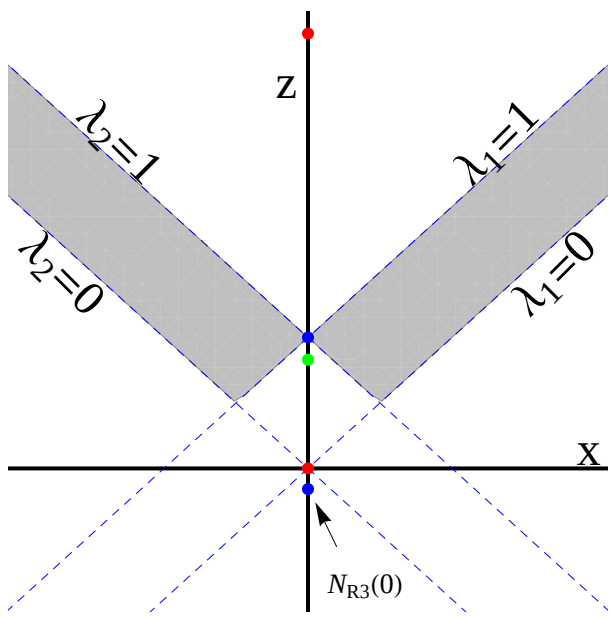
$$
\begin{aligned}
n_{11} &= \frac{-2-\lambda_1 T_q+\lambda_1\lambda_2 T_q-\lambda_2 T_q+T_q+2\lambda_2 \pm \sqrt{-4n_{12}^2\left(\lambda_1-1\right)\left(\lambda_2-1\right)}}{2\left(\lambda_2-1\right)} \\
n_{22} &= \frac{-2-\lambda_1 T_q+\lambda_1\lambda_2 T_q-\lambda_2 T_q+T_q+2\lambda_1 \mp \sqrt{-4n_{12}^2\left(\lambda_1-1\right)\left(\lambda_2-1\right)}}{2\left(\lambda_1-1\right)}
\end{aligned}
\tag{8.9}
$$

Consider the off diagonal entries: If $m_{12} \neq 0$ or $n_{12} \neq 0$ then to obtain a real solution we need the radicands

$$-4m_{12}^2\left(\lambda_1-1\right)\lambda_1\left(\lambda_2-1\right)\lambda_2 \geq 0$$

$$-4n_{12}^2\left(\lambda_1-1\right)\left(\lambda_2-1\right) \geq 0$$

The solutions to the above inequalities are shaded below in the xz-plane. The fixed points for ρ_0 are shown and of course all lie on the z-axis.



There is a neighborhood of $p_{N_{R_3}(0)} \in \mathrm{Ein}^{2,1}$ which is disjoint from the shaded region, i.e. there exists a neighborhood of the Lagrangian $N_{R_3}(0)$ in $\mathrm{Lag}(\mathbb{R}^4)$ which contains no values for λ_1 and λ_2 making the radicands positive.

Thus any deformation of ρ_0 must then have $m_{12} = n_{12} = 0$, hence it must factor through Φ_{bidisk}. Then $N_{R_1}(t)$ and $N_{R_2}(t)$ are also in the image of $\phi_{bidisk}(\partial\mathbb{H}^2 \times \partial\mathbb{H}^2)$ and correspond to points in the xz-plane. The solutions (8.8) and (8.9) reduce to

$$m_{11} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_1 + 2\lambda_1\lambda_2}{2\lambda_1(\lambda_2 - 1)}$$

$$m_{22} = \frac{\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r - T_r - 2\lambda_2 + 2\lambda_1\lambda_2}{2(\lambda_1 - 1)\lambda_2}$$

$$n_{11} = \frac{-2 - \lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q + T_q + 2\lambda_2}{2(\lambda_2 - 1)}$$

$$n_{22} = \frac{-2 - \lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q + T_q + 2\lambda_1}{2(\lambda_1 - 1)}$$

Since $m_{12} = n_{12} = 0$ and $T_p^2 - 4D_p = 0$, the equations (8.3) reduce to

$$T_p = m_{11}n_{11} + m_{22}n_{22} \qquad \frac{T_p^2}{4} = m_{11}m_{22}n_{11}n_{22}$$

Now substituting the above reductions into these equations and solving for λ_1 and λ_2 we obtain 4 possible solutions

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{-T_p - T_q + T_q T_r - T_r + 2 \pm \sqrt{T_p^2 + 2(-T_r T_q + T_q + T_r - 2)T_p + (T_q + T_r - 2)^2}}{T_q(T_r - 2)} \\ \frac{-T_p - T_q + T_q T_r - T_r + 2 \pm \sqrt{T_p^2 + 2(-T_r T_q + T_q + T_r - 2)T_p + (T_q + T_r - 2)^2}}{T_q(T_r - 2)} \end{bmatrix}$$

or

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \frac{-T_p - T_q + T_q T_r - T_r + 2 \pm \sqrt{T_p^2 + 2(-T_r T_q + T_q + T_r - 2)T_p + (T_q + T_r - 2)^2}}{T_q(T_r - 2)} \\ \frac{-T_p - T_q + T_q T_r - T_r + 2 \mp \sqrt{T_p^2 + 2(-T_r T_q + T_q + T_r - 2)T_p + (T_q + T_r - 2)^2}}{T_q(T_r - 2)} \end{bmatrix}$$

The first two of these solutions correspond to the images of $\phi_{diag}(\lambda_*)$ and $\phi_{diag}(\widetilde{\lambda}_*)$. The second two correspond to the images of $\phi_{bidisk}(\lambda_*, \widetilde{\lambda}_*)$ and $\phi_{bidisk}(\widetilde{\lambda}_*, \lambda_*)$. Thus $N_{R_3}(t)$ is constant and hence so are N_{R_1} and N_{R_2}. Thus $\rho(t)$ is constant and we have proven

Theorem 8.7. *Let $\rho_0 : \Delta_{pqr} \to Sp_{\pm}(4, \mathbb{R})$ be a faithful representation conjugate to a representation factoring through Φ_{diag}. Then ρ_0 is locally rigid, i.e. there does not exist any $Sp_{\pm}(4, \mathbb{R})$ deformations.*

In particular we have

Corollary 8.8 (Rigidity of diagonally embedded triangle groups)**.** *All faithful representations of Δ_{pqr} factoring through Φ_{diag} are locally rigid.*

8.4 Equations obtained using the indefinite triple

Suppose now that $F_{R_3} = L_{1,-1}$. The equations obtained in this case are extremely similar to 8.3, 8.4 and 8.5, and we refer to the Mathematica notebook (see [15]) for the exact equations. Taking them one cross ratio at a time and solving in a manner similar to above allows us to write m_{11} and m_{22} as functions of λ_1, λ_2 and m_{12} with the parameters being T_r and D_r:

$$\begin{aligned} m_{11} &= \frac{-\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r + T_r + 2\lambda_1 + 2\lambda_1\lambda_2 \mp \sqrt{(\lambda_1 - 1)(\lambda_2 + 1)\left(4\lambda_1\lambda_2 m_{12}^2 + (T_r^2 - 4D_r)(\lambda_1 - 1)(\lambda_2 + 1)\right)}}{2\lambda_1(\lambda_2 + 1)} \\ m_{22} &= \frac{\lambda_1 T_r + \lambda_1\lambda_2 T_r - \lambda_2 T_r - T_r - 2\lambda_1\lambda_2 + 2\lambda_2 \mp \sqrt{(\lambda_1 - 1)(\lambda_2 + 1)\left(4\lambda_1\lambda_2 m_{12}^2 + (T_r^2 - 4D_r)(\lambda_1 - 1)(\lambda_2 + 1)\right)}}{2(\lambda_1 - 1)\lambda_2} \end{aligned} \tag{8.10}$$

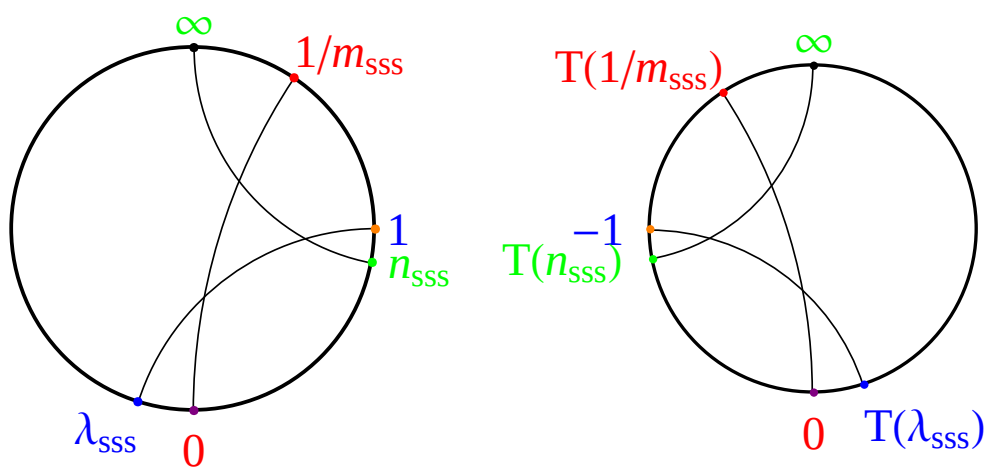
Similarly we can write n_{11} and n_{22} as functions of λ_1, λ_2 and n_{12} with the parameters being T_q and D_q:

$$\begin{aligned} n_{11} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q + 2\lambda_2 \pm \sqrt{(1 - \lambda_1)(\lambda_2 + 1)\left(4n_{12}^2 - \left(T_q^2 - 4D_q\right)(\lambda_1 - 1)(\lambda_2 + 1)\right)} + 2}{2(\lambda_2 + 1)} \\ n_{22} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q - 2\lambda_1 \pm \sqrt{(1 - \lambda_1)(\lambda_2 + 1)\left(4n_{12}^2 - \left(T_q^2 - 4D_q\right)(\lambda_1 - 1)(\lambda_2 + 1)\right)} + 2}{2(\lambda_1 - 1)} \end{aligned} \tag{8.11}$$

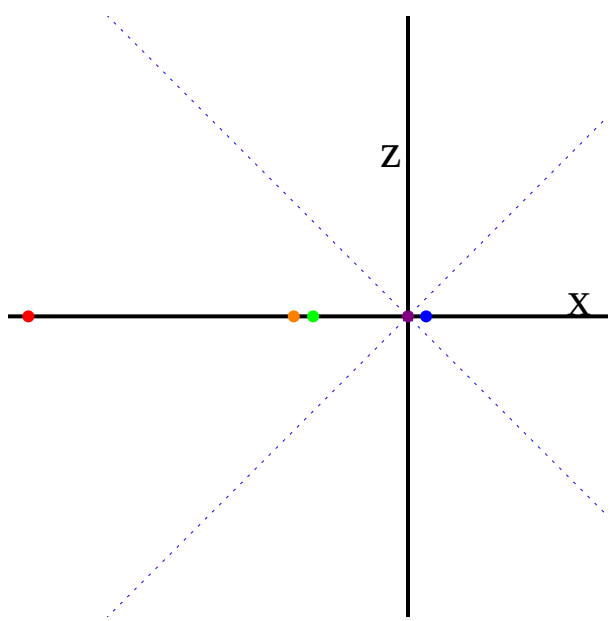
Once again observe that the radicand of these expressions contains the characteristic polynomial of the cross ratio.

8.4.1 ρ_0 factors through the anti-diagonal

By §8.2.1 we may assume that $\rho_0 = \Phi_{bidisk}(\rho_{sss}, T^{-1}\rho_{sss}T)$ where $T \in \mathrm{SL}_{\pm}(2,\mathbb{R})$ is the involution fixing $(0,\infty)$. Thus the projections of our starting configurations onto each factor of the bidisk is



The image of this configuration in the bidisk of course lies entirely on the x-axis in $\mathrm{Ein}^{2,1}$

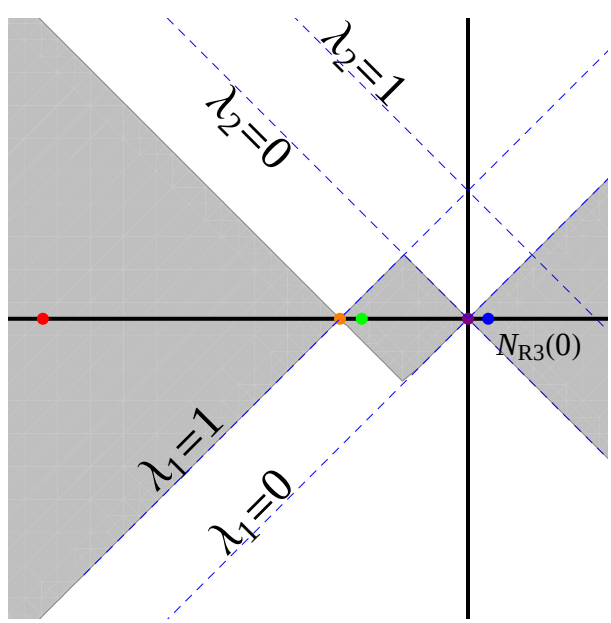


We may try to proceed analogously to the diagonal case. It is straightforward to compute that all cross ratios ave a repeated eigenvalue. Lemma 7.7 implies that $T_p^2 - 4D_p = T_q^2 - 4D_q = T_r^2 - 4D_r = 0$ and equations 8.10 and 8.11 reduce to:

$$
\begin{aligned}
m_{11} &= \frac{-\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r + T_r + 2\lambda_1 + 2\lambda_1\lambda_2 \mp \sqrt{4\lambda_1\lambda_2 m_{12}^2 (\lambda_1 - 1)(\lambda_2 + 1)}}{2\lambda_1(\lambda_2 + 1)} \\
m_{22} &= \frac{\lambda_1 T_r + \lambda_1\lambda_2 T_r - \lambda_2 T_r - T_r - 2\lambda_1\lambda_2 + 2\lambda_2 \mp \sqrt{4\lambda_1\lambda_2 m_{12}^2 (\lambda_1 - 1)(\lambda_2 + 1)}}{2(\lambda_1 - 1)\lambda_2}
\end{aligned}
\tag{8.12}
$$

$$
\begin{aligned}
n_{11} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q + 2\lambda_2 \pm \sqrt{4n_{12}^2 (1 - \lambda_1)(\lambda_2 + 1)} + 2}{2(\lambda_2 + 1)} \\
n_{22} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q - 2\lambda_1 \pm \sqrt{4n_{12}^2 (1 - \lambda_1)(\lambda_2 + 1)} + 2}{2(\lambda_1 - 1)}
\end{aligned}
\tag{8.13}
$$

We can again analyze the radicands in these expressions. These radicands are positive in the shaded region below. Unfortunately we are not so lucky as in the diagonal case. There is a neighborhood of $N_{R_3}(0)$ which is entirely contained within this region, and within this region the radicands all are non-negative.



Thus we are unable to conclude that m_{12} and n_{12} are zero. We must proceed in an another manner, which actually would have worked for the diagonal case as well. We utilize the following trivial observation.

Lemma 8.9. *A cross ratio having a repeated eigenvalue means that the quadruple lies on a common time-like or space-like circle.*

Proof. If three of the four Lagrangians are in the orbit of the definite triple, then we may assume that they are L_0, L_1, L_∞. If the cross ratio has a repeated eigenvalue then the fourth Lagrangian is fixed by the stabilizer of the definite triple. All four Lagrangians then correspond to points which lie on the (closure of) the z-axis in $Ein^{2,1}$. This is a time-like circle in $Ein^{2,1}$.

Similarly if there of the four are in the orbit of indefinite triple, all points lie on the x-axis, which is a space-like circle. □

Our approach will be to try to parameterize all deformations by N_{R_3}. Suppose we allow N_{R_3} to vary within a small neighborhood of $N_{R_3}(0)$. CR_{23} has a repeated eigenvalue, hence the quadruple $(F_{R_2}, N_{R_2}, F_{R_3}, N_{R_3})$ all lie on a common space-like circle. Since F_{R_2} is L_∞, this space-like circle intersects the Minkowski patch in a line. This line passes through $F_{R_3} = L_{1,-1}$ and N_{R_3}. Both of these points lie in the xz-plane in the Minkowski patch, thus the space-like circle is a line contained in the xz-plane. In particular this means that N_{R_2} is constrained to lie in the xz-plane, hence $n_{12} = 0$.

CR_{12} has a repeated eigenvalue hence the quadruple $(F_{R_1}, N_{R_1}, F_{R_2}, N_{R_2})$ all lie on a common space-like circle. Again since $F_{R_2} = L_\infty$ this space-like circle

intersects the Minkowski patch in a line. This line passes through $F_{R_1} = L_0$ and N_{R_2}, both of which lie in the xz-plane, hence the entire line lies in the xz-plane. This implies that F_{R_1} also lies in the xz-plane, hence $m_{12} = 0$.

Since $m_{12} = n_{12} = 0$ the solutions 8.12 and 8.13 reduce to

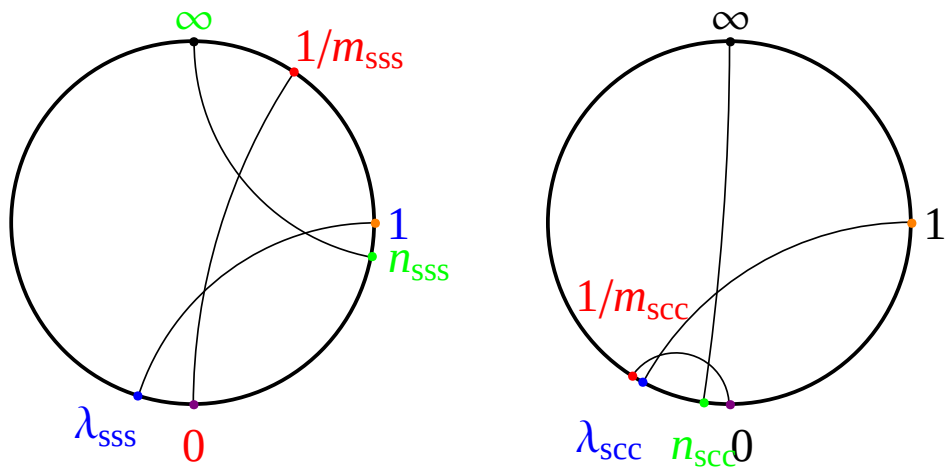
$$\begin{aligned}
m_{11} &= \frac{-\lambda_1 T_r - \lambda_1\lambda_2 T_r + \lambda_2 T_r + T_r + 2\lambda_1 + 2\lambda_1\lambda_2}{2\lambda_1(\lambda_2+1)} \\
m_{22} &= \frac{\lambda_1 T_r + \lambda_1\lambda_2 T_r - \lambda_2 T_r - T_r - 2\lambda_1\lambda_2 + 2\lambda_2}{2(\lambda_1-1)\lambda_2} \\
n_{11} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q + 2\lambda_2 + 2}{2(\lambda_2+1)} \\
n_{22} &= \frac{\lambda_1 T_q + \lambda_1\lambda_2 T_q - \lambda_2 T_q - T_q - 2\lambda_1 + 2}{2(\lambda_1-1)}
\end{aligned}$$

Substituting these into the equations obtained from $CR12$ yields 2 equations in the unknowns λ_1 and λ_2. Solving we obtain precisely four solutions for λ_1 and λ_2, all of which are images of triangle groups factoring through Φ_{bidisk}. Since the solution set is discrete we obtain

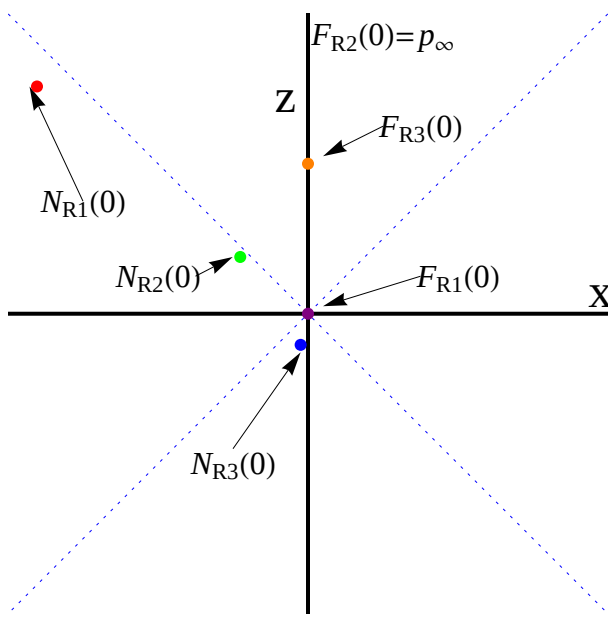
Theorem 8.10. *Let $\rho_0 : \Delta_{pqr} \to Sp_{\pm}(4,\mathbb{R})$ be a faithful representation conjugate to a representation factoring through the anti-diagonal. Then ρ_0 is locally rigid.*

8.5 More exotic constructions factoring through Φ_{bidisk}

Consider the case $\rho_0 = \Phi_{bidisk}(\rho_{sss}, \rho_{scc})$. The projection of the configuration of fixed geodesics to each factor of the bidisk is



The image of this configuration of Lagrangians in $\mathrm{Ein}^{2,1}$ is shown below.



We will attempt to parameterize representations close to ρ_0 by N_{R_3}. As above $N_{R_3}(t)$ is assumed to be in the image of the bi-disk. We will allow N_{R_3} to vary in the bi-disk in a small neighborhood of $N_{R_3}(0)$. The starting point is

$$N_{R_3}(0) = \begin{pmatrix} \lambda_{sss} & 0 \\ 0 & \lambda_{scc} \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where

$$\lambda_{sss} = \frac{-2\cos\left(\frac{\pi}{p}\right)-2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)+\sqrt{4\cos^2\left(\frac{\pi}{p}\right)+4\cos^2\left(\frac{\pi}{q}\right)+4\cos^2\left(\frac{\pi}{r}\right)+8\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)-4}}{8\sin^2\left(\frac{\pi}{2q}\right)\cos^2\left(\frac{\pi}{2r}\right)}$$
$$\lambda_{scc} = \frac{-2\cos\left(\frac{\pi}{p}\right)-2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)+\sqrt{4\cos^2\left(\frac{\pi}{p}\right)+4\cos^2\left(\frac{\pi}{q}\right)+4\cos^2\left(\frac{\pi}{r}\right)+8\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)-4}}{8\sin^2\left(\frac{\pi}{2r}\right)\cos^2\left(\frac{\pi}{2q}\right)}$$

The following lemma characterizes the coordinates for $N_{R_3}(0)$

Lemma 8.11. *λ_{sss} and λ_{scc} are both negative.*

Proof. Rather tedious but straightforward: Using the above expressions for λ_{sss} and λ_{scc} it suffices to show that

$$-2\cos\left(\frac{\pi}{p}\right) - 2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right) + \sqrt{4\cos^2\left(\frac{\pi}{p}\right) + 4\cos^2\left(\frac{\pi}{q}\right) + 4\cos^2\left(\frac{\pi}{r}\right) + 8\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right) - 4} < 0$$
$$\sqrt{4\cos^2\left(\frac{\pi}{p}\right) + 4\cos^2\left(\frac{\pi}{q}\right) + 4\cos^2\left(\frac{\pi}{r}\right) + 8\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right) - 4} < 2\cos\left(\frac{\pi}{p}\right) + 2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)$$

Squaring both sides then canceling we obtain

$$4\cos^2\left(\frac{\pi}{q}\right) + 4\cos^2\left(\frac{\pi}{r}\right) - 4 < 4\cos^2\left(\frac{\pi}{q}\right)\cos^2\left(\frac{\pi}{r}\right)$$
$$\cos^2\left(\frac{\pi}{q}\right) - \cos^2\left(\frac{\pi}{q}\right)\cos^2\left(\frac{\pi}{r}\right) + \cos^2\left(\frac{\pi}{r}\right) < 1$$
$$\cos^2\left(\frac{\pi}{q}\right)\left(1 - \cos^2\left(\frac{\pi}{r}\right)\right) + \cos^2\left(\frac{\pi}{r}\right) < 1$$
$$\cos^2\left(\frac{\pi}{q}\right)\sin^2\left(\frac{\pi}{r}\right) + \cos^2\left(\frac{\pi}{r}\right) < 1$$

□

The first cross ratio CR_{12} has a repeated eigenvalue and the remaining two have distinct eigenvalues. Specifically:

- $\operatorname{tr}(CR_{12}) = 2\sin^2\left(\frac{\pi}{2p}\right)$

  $\det(CR_{12}) = \sin^4\left(\frac{\pi}{2p}\right)$

  $T_p^2 - 4D_p = 0$

- $\operatorname{tr}(CR_{23}) = 1$

  $\det(CR_{23}) = \frac{1}{4}\sin^2\left(\frac{\pi}{q}\right)$

  $T_q^2 - 4D_q = \cos^2\left(\frac{\pi}{q}\right)$

- $\operatorname{tr}(CR_{31}) = 1$

  $\det(CR_{31}) = \frac{1}{4}\sin^2\left(\frac{\pi}{r}\right)$

  $T_r^2 - 4D_r = \cos^2\left(\frac{\pi}{r}\right)$

The solutions (8.6) and (8.7) reduce to

$$m_{11} = \frac{\lambda_2\lambda_1 - \lambda_1 + \lambda_2 \pm \sqrt{\cos^2\left(\frac{\pi}{r}\right)(\lambda_1 - 1)^2(\lambda_2 - 1)^2 - 4m_{12}^2(\lambda_1 - 1)\lambda_1(\lambda_2 - 1)\lambda_2} - 1}{2\lambda_1(\lambda_2 - 1)}$$

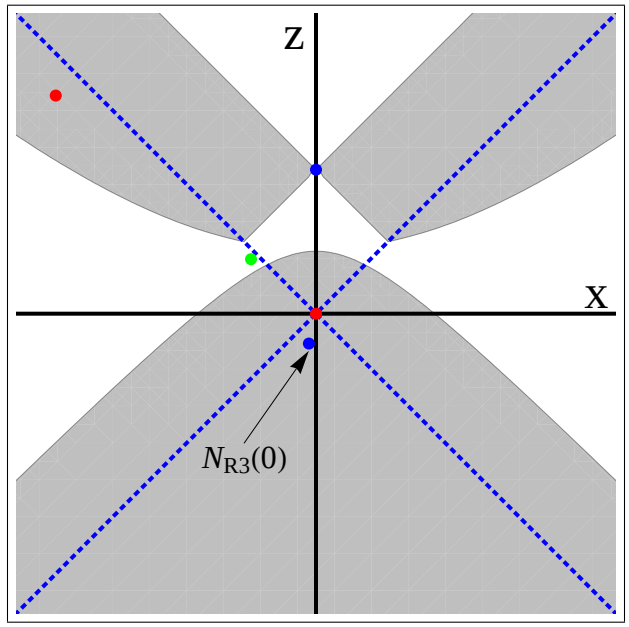
$$m_{22} = \frac{\lambda_2\lambda_1 + \lambda_1 - \lambda_2 \mp \sqrt{\cos^2\left(\frac{\pi}{r}\right)(\lambda_1 - 1)^2(\lambda_2 - 1)^2 - 4m_{12}^2(\lambda_1 - 1)\lambda_1(\lambda_2 - 1)\lambda_2} - 1}{2(\lambda_1 - 1)\lambda_2} \tag{8.14}$$

$$n_{11} = \frac{\lambda_2\lambda_1 - \lambda_1 + \lambda_2 \pm \sqrt{\cos^2\left(\frac{\pi}{q}\right)(\lambda_1 - 1)^2(\lambda_2 - 1)^2 - 4n_{12}^2(\lambda_1 - 1)(\lambda_2 - 1)} - 1}{2(\lambda_2 - 1)}$$

$$n_{22} = \frac{\lambda_2\lambda_1 + \lambda_1 - \lambda_2 \mp \sqrt{\cos^2\left(\frac{\pi}{q}\right)(\lambda_1 - 1)^2(\lambda_2 - 1)^2 - 4n_{12}^2(\lambda_1 - 1)(\lambda_2 - 1)} - 1}{2(\lambda_1 - 1)} \tag{8.15}$$

Again we may examine the radicands, but this time we are unable to conclude anything about m_{12} and n_{12}. In particular if both are zero, λ_1 and λ_2 can take any value. The picture below shows the subset of the xz-plane (equivalently the

$\lambda_1\lambda_2$-plane) where the radicands are both positive for some small values of m_{12} and n_{12}. The blue point shown is $N_{R_3}(0)$ and lemma 8.11 ensures that this point lies in the interior of the region where both radicands are positive.



So we must take an alternative approach in this case, attempting to satisfy each cross ratio one at a time. N_{R_3} will be our parameter and try to write N_{R_1} and N_{R_2} in terms of this.

8.5.1 Satisfying CR_{31}

Equations 8.14 yields m_{11} and m_{22} as functions of λ_1, λ_2 and m_{12}, and the key obstruction is that we can't conclude $m_{12} = 0$. If however it is zero we obtain a solution within the bidisk. Equations 8.14 reduce further and we effectively have $N_{R_1}(0)$ as a function of $N_{R_3}(0)$:

$$N_{R_1}(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\cos\left(\frac{\pi}{r}\right)(\lambda_1-1)+\lambda_1+1}{2\lambda_1} & 0 \\ 0 & \frac{-\cos\left(\frac{\pi}{r}\right)(\lambda_2-1)+\lambda_2+1}{2\lambda_2} \end{pmatrix}$$

We can now find all quadruples which satisfy CR_{31} by applying the stabilizer of the triple $(F_{R_1}, F_{R_3}, N_{R_3})$ to this solution. By theorem 6.8 this stabilizer is isomorphic to $O(2)$. We obtain a whole circle's worth of possibilities for N_{R_1}, given here parameterized by θ:

$$N_{R_1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{\cos\left(\frac{\pi}{r}\right)\cos(2\theta)(\lambda_1-1)+\lambda_1+1}{2\lambda_1} & -\frac{1}{2}\cos\left(\frac{\pi}{r}\right)\sin(2\theta)\sqrt{\frac{(\lambda_1-1)(\lambda_2-1)}{\lambda_1\lambda_2}} \\ -\frac{1}{2}\cos\left(\frac{\pi}{r}\right)\sin(2\theta)\sqrt{\frac{(\lambda_1-1)(\lambda_2-1)}{\lambda_1\lambda_2}} & \frac{-\cos\left(\frac{\pi}{r}\right)\cos(2\theta)(\lambda_2-1)+\lambda_2+1}{2\lambda_2} \end{pmatrix}$$

8.5.2 Satisfying CR_{23}

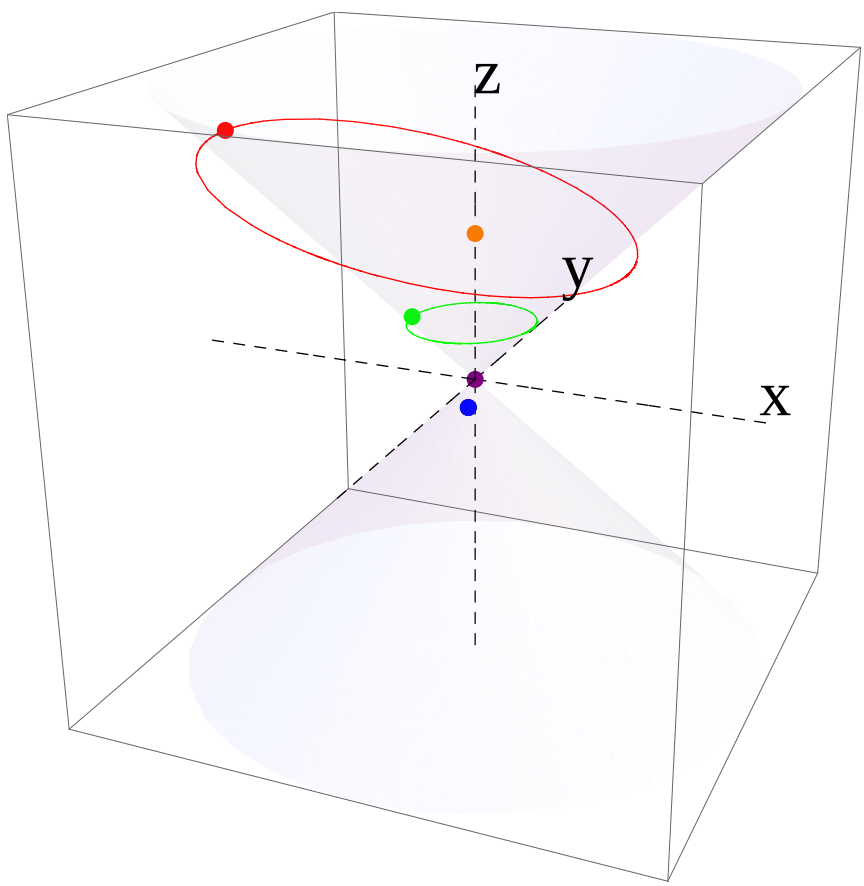
Similarly equations 8.15 give n_{11} and n_{22} as functions of λ_1, λ_2 and n_{12}. Supposing that $n_{12} = 0$ yields a N_{R_2} as a function of N_{R_3}:

$$N_{R_2}(0) = \begin{pmatrix} \frac{1}{2}\left(-\cos\left(\frac{\pi}{q}\right)(\lambda_1-1)+\lambda_1+1\right) & 0 \\ 0 & \frac{1}{2}\left(\cos\left(\frac{\pi}{q}\right)(\lambda_2-1)+\lambda_2+1\right) \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

We obtain all quadruples satisfying CR_{23} by computing the stabilizer of $(F_{R_2}, F_{R_3}, N_{R_3})$ and applying it to this solution. Again the stabilizer is isomorphic to $O(2)$ and we obtain a whole circle's worth of possibilities parameterized by ϕ:

$$N_{R_2} = \begin{pmatrix} \frac{1}{2}\left(-\cos\left(\frac{\pi}{q}\right)\cos(2\phi)\left(\lambda_1 - 1\right) + \lambda_1 + 1\right) & -\frac{1}{2}\cos\left(\frac{\pi}{q}\right)\sin(2\phi)\sqrt{\left(\lambda_1 - 1\right)\left(\lambda_2 - 1\right)} \\ -\frac{1}{2}\cos\left(\frac{\pi}{q}\right)\sin(2\phi)\sqrt{\left(\lambda_1 - 1\right)\left(\lambda_2 - 1\right)} & \frac{1}{2}\left(\cos\left(\frac{\pi}{q}\right)\cos(2\phi)\left(\lambda_2 - 1\right) + \lambda_2 + 1\right) \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The picture below shows these Lagrangians for Δ_{345}. The red points are the eigenspaces for $R_1(0)$, the green are for $R_2(0)$ (one of which is p_∞ and can't be seen) and the blue are for $R_3(0)$. The circle in red is all possibilities for N_{R_1} which make CR_{31} is as desired and the green circle is all possibilities for N_{R_2} so that CR_{23} is as desired.



As N_{R_3} varies within a neighborhood of the initial $N_{R_3}(0)$, these circles also vary slightly. We would like to find a smooth path on these circles corresponding

to N_{R_1} and N_{R_2} such that the third cross ratio is as desired. The product of these circles is a torus T and we want a smooth path on T which is causes CR_{12} to have a repeated eigenvalue of $\sin^2\left(\frac{\pi}{2p}\right)$.

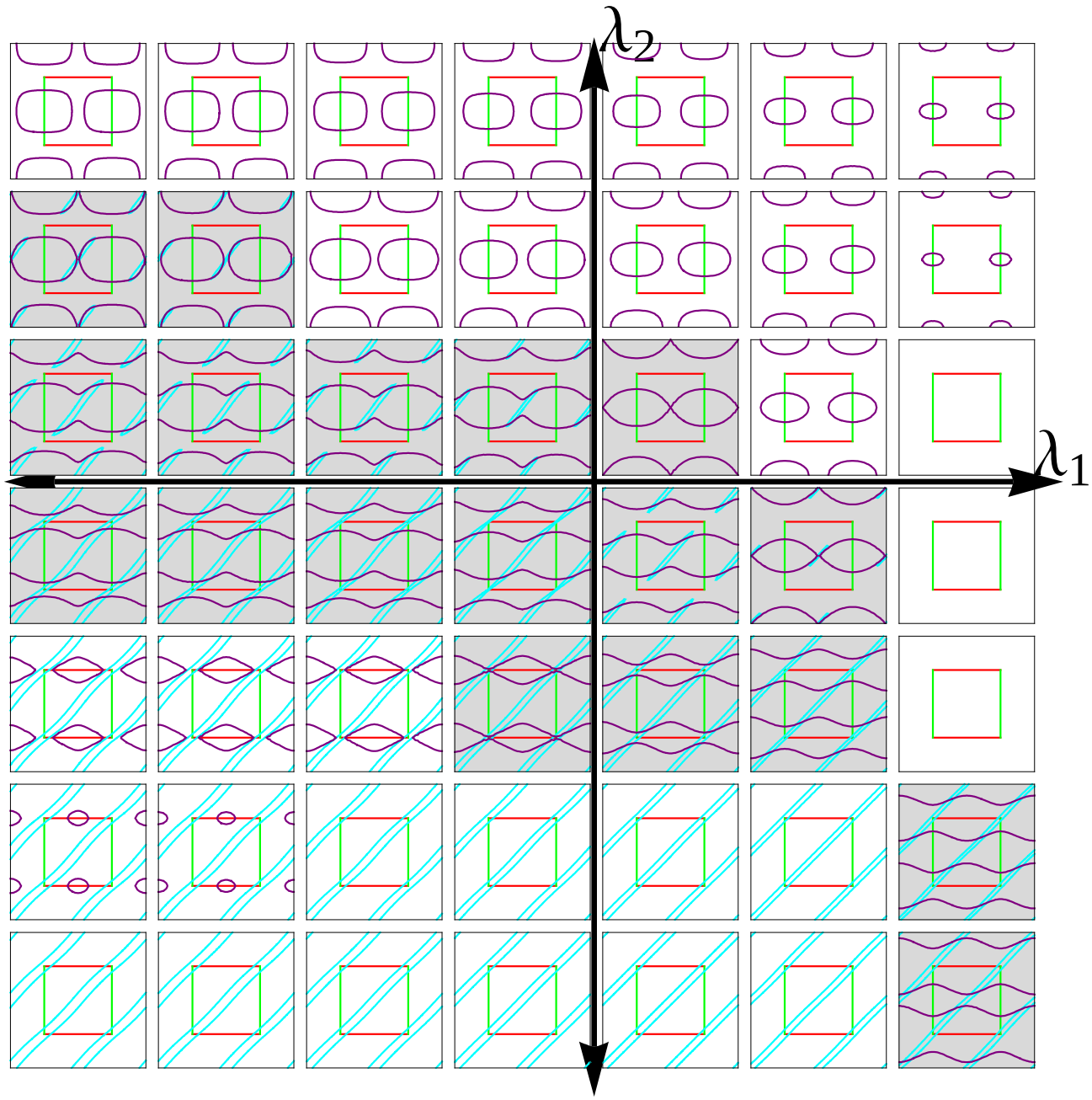
8.5.3 Satisfying CR_{12}

$CR_{12} = CR[F_{R_1}, N_{R_1}, N_{R_2}, F_{R_2}]$ can now be restricted to T. Using the expressions for N_{R_1} and N_{R_2} above we can write CR_{12} as a rather complicated looking function of λ_1,λ_2, θ and ϕ. The trace and determinant of the cross ratio can be thought of as real valued functions on the torus T. Explicitly these functions are:

$$\mathrm{tr}(CR_{12}) = \frac{1}{4}\left(2\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)\sin(2\theta)\sqrt{\frac{(\lambda_1-1)^2(\lambda_2-1)^2}{\lambda_1\lambda_2}}\sin(2\phi) + \frac{\left(\cos\left(\frac{\pi}{r}\right)\cos(2\theta)(\lambda_1-1)+\lambda_1+1\right)\left(-\cos\right.}{\lambda_1}\right.$$

$$\det(CR_{12}) = \frac{\left(4\cos\left(\frac{\pi}{q}\right)\cos(2\phi)(\lambda_1-\lambda_2)+\cos\left(\frac{2\pi}{q}\right)(\lambda_1-1)(\lambda_2-1)-3\lambda_2-\lambda_1(\lambda_2+3)-1\right)\left(\cos\left(\frac{2\pi}{r}\right)(\lambda_1-1)(\lambda_2-1)-3\lambda_2-\lambda_1(\lambda_2\right.}{64\lambda_1\lambda_2}$$

Each frame in the picture below shows the universal cover of the torus T and the square in the middle of each frame is a fundamental domain for T. The example in the picture is for Δ_{345}. Shown in light blue is the curve on T where $\mathrm{tr}(CR_{12}) = 2\sin^2\left(\frac{\pi}{2p}\right)$, and shown in purple is the curve on T where $\det = \sin^4\left(\frac{\pi}{2p}\right)$. The intersection of these curves then represents a point on the torus where CR_{12} is as desired. $N_{R_3}(0)$ is the frame where the λ_1 and λ_2 axes intersect. The frames where these curves actually intersect are shaded. Generically it seems these curves intersect in four points.



Some interactive Mathematica programs showing how T and these level sets vary are available at [15]. The experimental evidence suggests

Conjecture 8.12. *There are nontrivial deformations of the* $\rho_0 = \Phi_{bidisk}(\rho_{sss}, \rho_{scc})$. *The component of* $Hom(\Delta_{pqr}, Sp_{\pm}(4,\mathbb{R}))/Sp_{\pm}(4,\mathbb{R})$ *containing* ρ_0 *consists of 4 copies*

of a conical region parameterized by λ_1,λ_2 (the shaded frames above) glued together at ρ_0.

The conjecture could be confirmed if we could explicitly compute these intersections, i.e. find θ and ϕ in terms of λ_1 and λ_2. I have so far been unsuccessful at doing this explicitly.

Appendices

Appendix A

The Tangent Space and $\mathfrak{k}^{\perp}$

In §1.4 we saw that $\mathfrak{S}_2$ is a homogeneous space diffeomorphic to $\mathrm{Sp}(4,\mathbb{R})/\mathrm{U}(2)$. This diffeomorphism induces an identification of the tangent space $T_{iI_2}\mathfrak{S}_2$ with the a subalgebra of the Lie algebra $\mathfrak{sp}(4,\mathbb{R})$. In this appendix we explicitly compute this identification for $\mathfrak{S}_2$. While necessary for defining an invariant metric on $\mathfrak{S}_2$, these computations are a bit lengthy and tedious and are hence relegated to this appendix.

An essential tool in this analysis will be the *Trace Form*. For a Lie algebra of matrices $\mathfrak{g}$, the trace form is a symmetric non-degenerate $\mathrm{Aut}(\mathfrak{g})$-invariant bilinear form defined for $X \in \mathfrak{g}$ by $\mathrm{tr}(X^2)$. For any simple Lie algebra, such as $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sp}(4,\mathbb{R})$, all non-degenerate $\mathrm{Aut}(\mathfrak{g})$-invariant bilinear forms are scalar multiples of the *Killing form* (see [10],§14.2). Although it is perhaps more thorough to use the Killing form in the following analysis, the trace form tends to be easier to compute and will suffice for our purposes.

A.1 The Tangent Space at i to $\mathbb{H}^2$

Consider first the situation in hyperbolic geometry: the upper half plane is obtained as the quotient space for the mapping $ev : \mathrm{SL}(2,\mathbb{R}) \to \mathbb{H}^2$ given by evaluation at the basepoint i. We can obtain an identification of the tangent space $T_i\mathbb{H}^2$

with the image of the differential of this map.

The Lie algebra $\mathfrak{sl}(2,\mathbb{R})$ is

$$\mathfrak{sl}(2,\mathbb{R}) = \left\{ \begin{bmatrix} a & b \\ c & -a \end{bmatrix} : a,b,c \in \mathbb{R} \right\}$$

The stabilizer of the base point i is the subgroup $K = \mathrm{SO}(2)$, and its Lie algebra $\mathfrak{k} = \mathfrak{so}(2)$ is the 1 dimensional subalgebra generated by $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. The trace form is given in the basis above as:

$$\mathrm{tr}(X,X) = \mathrm{tr}(X^2) = 2(a^2 + bc)$$

It is easy to compute using this form that the orthogonal complement of $\mathfrak{k}$, denoted $\mathfrak{p} := \mathfrak{so}(2)^{\perp}$, consists of symmetric traceless matrices. The decomposition of the Lie algebra as the direct sum

$$\mathfrak{sl}(2,\mathbb{R}) = \mathfrak{k} \bigoplus \mathfrak{p}$$

is called the *Cartan Decomposition.*

We wish to compute explicitly the differential of the evaluation mapping ev_*. It is convenient to first consider $\mathrm{SL}(2,\mathbb{R})$ as a subset of $\mathrm{GL}(2,\mathbb{R})$ and the upper half plane $\mathbb{H}^2$ as a subset of the complex projective line $\mathbb{CP}^1$ as described in §2.1. In this way we can avoid having to use coordinate charts on $\mathrm{SL}(2,\mathbb{R})$ which has dimension 3, and use the canonical single chart on $\mathrm{GL}(2,\mathbb{R})$ which has dimension 4.

Using homogeneous coordinates on $\mathbb{CP}^1$, the upper half plane maps into $\mathbb{CP}^1$ via the inclusion $z \overset{j}{\hookrightarrow} \begin{bmatrix} z \\ 1 \end{bmatrix}$. Let $b := \begin{bmatrix} i \\ 1 \end{bmatrix} \in \mathbb{CP}^1$ be the image of the basepoint

i under this inclusion, and let $\pi : \mathbb{C}^2 \to \mathbb{CP}^1$ be the natural projection map. Define $f : \mathrm{GL}(2,\mathbb{R}) \to \mathbb{C}^2$ to be evaluation at b, that is $f(M) = Mb$. Consider the commutative diagrams:

$$\begin{tikzcd}
\mathrm{GL}(2,\mathbb{R}) \arrow[r, "f"] & \mathbb{C}^2 \arrow[d, "="] \\
\mathrm{SL}(2,\mathbb{R}) \arrow[u, hook] \arrow[r, "f"] \arrow[dr, "ev"'] & \mathbb{C}^2 \arrow[d, "\pi"] \\
& \mathbb{C}P^1 \\
& \mathbb{H}^2 \arrow[u, hook, "j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{gl}(2,\mathbb{R}) \arrow[r, "f_*"] & T_b\mathbb{C}^2 \approx \mathbb{C}^2 \arrow[d, "="] \\
\mathfrak{sl}(2,\mathbb{R}) \arrow[u, hook] \arrow[r, "f_*"] \arrow[dr, "ev_*"'] & T_b\mathbb{C}^2 \approx \mathbb{C}^2 \arrow[d, "\pi_*"] \\
& T_{\pi(b)}\mathbb{CP}^1 \\
& T_i\mathbb{H}^2 \arrow[u, hook, "j_*"']
\end{tikzcd}$$

The inverse of the inclusion j^{-1} (restricted to its image of course!) is an affine chart on $\mathbb{CP}^1$ and in this chart

$$(j^{-1} \circ \pi) \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \to \tfrac{z_1}{z_2}.$$

Further j_* is simply the identity map so $T_{\pi(b)}\mathbb{CP}^1 = T_i\mathbb{H}^2$ and π_* is the linear map given by multiplication by the row vector $\begin{bmatrix} \frac{1}{z_2} & \frac{-z_1}{z_2^2} \end{bmatrix}$, so at the basepoint b we have $(\pi_*)_b = [1\ , -i]$.

Note now that the restriction to $\mathrm{SL}(2,\mathbb{R})$ of the compositions is precisely the evaluation map from above:

$$j^{-1} \circ \pi \circ f = ev$$

Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathfrak{gl}(2,\mathbb{R})$ and compute the differential of this composition:

$$
\begin{aligned}
ev_*(X) &= (j^{-1} \circ \pi \circ f)_*(X) \\
&= (\pi_*)_b (f_*)_I (X) \\
&= [1 \ , -i] \begin{bmatrix} ai + b \\ ci + d \end{bmatrix} \\
&= (a - d)i + (b + c)
\end{aligned}
$$

Restricting this to $\mathfrak{sl}(2, \mathbb{R})$ simply means that $d = -a$. So

$$ev_*(X) = 2ai + (b + c).$$

Using the Cartan decomposition of $\mathfrak{sl}(2, \mathbb{R}) = \mathfrak{k} \bigoplus \mathfrak{p}$, we can easily see that $\mathfrak{k}$ is the kernel of ev_*, and $ev_*(\mathfrak{p})$ is surjective. For $X \in \mathfrak{p}$, ev_* induces the identification:

$$\mathfrak{p} \leftrightarrow T_i \mathbb{H}^2$$

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} \leftrightarrow 2(ai + b) \tag{A.1}$$

This identification allows us to choose a multiple of the trace form making this identification an isometry with the usual Euclidean metric on $T_i \mathbb{H}^2$. We choose the bilinear form on $\mathfrak{sl}(2, \mathbb{R})$ given by

$$
\begin{aligned}
B(X, X) &= 2\mathrm{tr}(X^2) \\
&= 4(a^2 + bc)
\end{aligned}
$$

For a tangent vector $ai + b \in T_i \mathbb{H}^2$, which identifies via the above correspondence with the matrix $X = \begin{bmatrix} \frac{a}{2} & \frac{b}{2} \\ \frac{b}{2} & -\frac{a}{2} \end{bmatrix}$, we have

$$B(X,X) = 4\left(\left(\tfrac{a}{2}\right)^2 + \left(\tfrac{b}{2}\right)^2\right) = a^2 + b^2.$$

As desired the restriction of this form to $\mathfrak{p}$ is positive definite (and equal to the Euclidean metric). This form can be used to induce an invariant Riemannian metric on $\mathbb{H}^2$.

A.2 The Tangent Space at iI_2 to $\mathfrak{S}_2$

The tangent space $T_{iI_2}\mathfrak{S}_2$ identifies with an open set of all 2×2 complex symmetric matrices. Since $\mathfrak{S}_2$ can be obtained via a quotient of the evaluation map $ev : \mathrm{Sp}(4,\mathbb{R}) \to \mathfrak{S}_2$, the tangent space $T_{iI_2}\mathfrak{S}_2$ can be seen as the image of the differential of this map. This differential is a linear map $\mathfrak{sp}(4,\mathbb{R}) \to \mathrm{Mat}(2,\mathbb{C})$ whose kernel is $\mathfrak{u}(2)$. The image then identifies with the orthogonal complement of $\mathfrak{k} = \mathfrak{u}(2)$.

First let us explicitly compute $\mathfrak{p}$. In §1.3 we saw that $X \in \mathfrak{sp}(4,\mathbb{R})$ has the form

$$X = \begin{bmatrix} a & a_{12} & b_{11} & b_{12} \\ a_{21} & b & b_{12} & b_{22} \\ c_{11} & c_{12} & -a & -a_{21} \\ c_{12} & c_{22} & -a_{12} & -b \end{bmatrix}$$

The trace form on $\mathfrak{sp}(4,\mathbb{R})$ is given by:

$$\mathrm{tr}(X,X) := \mathrm{tr}(X^2) = 2\left(a^2 + b^2 + 2a_{12}a_{21} + b_{11}c_{11} + 2b_{12}c_{12} + b_{22}c_{22}\right)$$

The orthogonal complement $\mathfrak{p} = \mathfrak{u}^\perp$ can then be found by pairing the arbitrary X with each of the basis vectors for $\mathfrak{u}(2)$ found in §1.3 to obtain:

$$\mathrm{Tr}(X, B_1) = 2(a_{21} - a_{12})$$

$$\mathrm{Tr}(X, B_2) = c_{11} - b_{11}$$

$$\mathrm{Tr}(X, B_3) = 2(c_{12} - b_{12})$$

$$\mathrm{Tr}(X, B_4) = c_{22} - b_{22}$$

Setting these equations equal to zero immediately implies that the 2×2 blocks are symmetric:

$$\mathfrak{p} := \mathfrak{k}^{\perp} = \left\{ \begin{bmatrix} A & B \\ B & -A^T \end{bmatrix} : A = A^T \ \& \ B = B^T \right\} \subset \mathfrak{sp}(4, \mathbb{R}).$$

Let us now compute ev_* to obtain a natural identification of $\mathfrak{p}$ with $T_{iI}\mathfrak{S}_2$. It will be convenient to consider $\mathrm{Sp}(4, \mathbb{R}) \subset \mathrm{Sp}(4, \mathbb{C})$ and $\mathfrak{S}_2 \subset \mathrm{Lag}(\mathbb{C}^4)$ as described in §2.2. To avoid messy charts on $\mathrm{Sp}(4, *)$ we will in fact go one step further and consider $\mathrm{Sp}(4, \mathbb{C}) \subset \mathrm{GL}(4, \mathbb{C})$.

Let Fr_2 denote the space of 2-frames in $\mathbb{C}^4$, and Fr_2^{Lag} denote the subspace of Lagrangian 2-frames in $\mathbb{C}^4$. Let π be the projection map $Fr_2^{Lag} \to \mathrm{Lag}(\mathbb{C}^4)$, and j be the natural embedding of $\mathfrak{S}_2$ given by $Z \xrightarrow{j} \begin{bmatrix} Z \\ I_2 \end{bmatrix}$. Let $b = j(iI) \in Fr_2^{Lag} \subset Fr_2$ be the basepoint and let $f : \mathrm{GL}(4, \mathbb{C}) \to Fr_2$ be evaluation at b. This of course restricts to $\mathrm{Sp}(4, \mathbb{R})$ and we have the following commutative diagrams:

$$
\begin{tikzcd}
\mathrm{GL}(4,\mathbb{C}) \arrow[r, "f"] & Fr_2 \\
\mathrm{Sp}(4,\mathbb{C}) \arrow[u, hook] \arrow[r, "f"] & Fr_2^{Lag} \arrow[u, hook] \arrow[d, "="] \\
\mathrm{Sp}(4,\mathbb{R}) \arrow[u, hook] \arrow[r, "f"] \arrow[dr, "ev"'] & Fr_2^{Lag} \arrow[d, "\pi"] \\
 & Lag(\mathbb{C}^4_\omega) \\
 & \mathfrak{S}_2 \arrow[u, hook, "j"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathfrak{gl}(4,\mathbb{C}) \arrow[r, "f_*"] & T_bFr_2 \\
\mathfrak{sp}(4,\mathbb{C}) \arrow[u, hook] \arrow[r, "f_*"] & T_bFr_2^{Lag} \arrow[u, hook] \arrow[d, "="] \\
\mathfrak{sp}(4,\mathbb{R}) \arrow[u, hook] \arrow[r, "f_*"] \arrow[dr, "ev_*"'] & T_bFr_2^{Lag} \arrow[d, "\pi_*"] \\
 & T_{\pi(b)}Lag(\mathbb{C}^4_\omega) \\
 & T_i\mathfrak{S}_2 \arrow[u, hook, "j_*"']
\end{tikzcd}
$$

Note now that in the chart where Z_2 is nonsingular, $(\jmath^{-1} \circ \pi) \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = Z_1 Z_2^{-1}$ and the differential $\jmath_*^{-1} \circ \pi_*$ is left multiplication by $\left[Z_2^{-1}\ ,\ -Z_1 Z_2^{-2}\right]$. So in particular at the base point b, the differential is $[I_2\ ,\ -iI_2]$. So for a given $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \in \mathfrak{gl}(4,\mathbb{C})$, the differential can now be computed as

$$
\begin{aligned}
(j^{-1} \circ \pi \circ f)_*(X) &= ((j^{-1} \circ \pi)_*)_b (f_*)_I(X)) \\
&= [I_2, -iI_2] \begin{bmatrix} Ai + B \\ Ci + D \end{bmatrix}. \\
&= (A - D)i + (B + C)
\end{aligned}
$$

Restricting our attention to $\mathfrak{sp}(4,\mathbb{R}) \subset \mathfrak{gl}(4,\mathbb{C})$ which consists of matrices where $D = -A^T$ and B, C are symmetric, we see $ev_*(X) = (A + A^T)i + (B + C)$. Note that $ev_*(X)$ is clearly symmetric. The Cartan Decomposition is once again:

$$\mathfrak{sp}(4,\mathbb{R}) = \mathfrak{k} \bigoplus \mathfrak{p}$$

Recall now from section §1.3 that $\mathfrak{k} = \mathfrak{u}(2)$ consisted of matrices where $A = -A^T$ and $B = -C$, hence $\mathfrak{k} \subset \ker ev_*$. Further we noted above that $\mathfrak{p}$ consists of

matrices where $A = A^T$ and $B = C$, so for $X \in \mathfrak{p}$, $ev_*(X) = 2(Ai + B)$. So the tangent space $T_{iI}\mathfrak{S}_2$ is the image of $\mathfrak{p}$ under ev_*. The identification is

Lemma A.1. *The tangent space to $\mathfrak{S}_2$ at the base point iI_2 identifies with a subspace of the Lie algebra $\mathfrak{sp}(4, \mathbb{R})$ by*

$$\begin{array}{c} \mathfrak{p} \leftrightarrow T_{iI}\mathfrak{S}_2 \\ \begin{bmatrix} A & B \\ B & -A^T \end{bmatrix} \leftrightarrow 2(Ai + B) \end{array}$$

Since $T_{iI}\mathfrak{S}_2$ is simply the space of all symmetric 2×2 complex matrices, a reasonably canonical inner product on this space is given by the trace form. We can now choose a multiple of the trace form on $\mathfrak{sp}(4, \mathbb{R})$ in such a way that the above identification is an isometry. For $X \in \mathfrak{sp}(4, \mathbb{R})$ let

$$\begin{aligned} B(X, X) &= 2\mathrm{tr}(X^2) \\ &= 4\left(a^2 + b^2 + 2a_{12}a_{21} + b_{11}c_{11} + 2b_{12}c_{12} + b_{22}c_{22}\right) \\ &= \mathrm{tr}[(ev_*X)\overline{(ev_*X)}] \end{aligned}$$

The restriction of this bilinear form to $\mathfrak{p}$ is positive definite, and this will be the quadratic form we will use on $\mathfrak{sp}(4, \mathbb{R})$. It is used in §3.1 to construct a Riemannian metric on $\mathfrak{S}_2$.

A.3 The Action of K on Tangent Vectors

For any homogeneous space G/K, the group G acts by left multiplication G/K. Let e denote the identity in G. The subgroup K fixes the base point eK of G/K, hence the differential yields a linear action of K on the tangent space at eK.

It would be desirable to lift this action of K on $T_{eK}(G/K)$ to an automorphism of $T_eG = \mathfrak{g}$ which commutes with the identification developed in the previous section of $\mathfrak{p}$ with $T_{eK}(G/K)$. One might naturally be tempted to lift the action of K on G/K by left multiplication to the action of K on G by left multiplication. This is of course incorrect since the action of K on G by left multiplication does not preserve the base point e. Instead note that the action of K on G/K by left multiplication is exactly the same as the conjugation action of K on G/K. Specifically, if $k \in K$ and $g \in G$, then

$$k(gK)k^{-1} = kgKk^{-1} = (kg)K$$

The conjugation action of K on G does fix e, hence the differential yields the *Adjoint action* of K on $T_eG = \mathfrak{g}$. Further this action commutes with the identification in the previous section as desired. For each $k \in K$ we have the following commutative diagram:

$$\begin{tikzcd}
G \arrow[r, "Inn(k)"] & G \arrow[r, "ev"] & G/K \\
\mathfrak{g} \arrow[u, "exp"] \arrow[r, "Ad(k)"] & \mathfrak{g} \arrow[u, "exp"] \arrow[r, "ev_*"] & T_{eK}(G/K) \arrow[u, "exp"]
\end{tikzcd}$$

In what follows we will analyze the Adjoint action of K on the upper half plane model for $\mathbb{H}^2$ and $\mathfrak{S}_2$. This is perhaps the first time we will see a significant difference between these 2 spaces. $K = SO(2)$ acts transitively on the unit sphere in $T_i\mathbb{H}^2$, but $K = \mathfrak{u}(2)$ does not act transitively on the unit sphere in $T_{iI}\mathfrak{S}_2$. Instead we will show that any tangent vector is $Ad(K)$ equivalent to one a 2 dimensional subspace of $T_{iI}\mathfrak{S}_2$ that is the image of a Cartan subalgebra.

A.3.1 The Action of $SO(2)$ on $T_i\mathbb{H}^2$

In the hyperbolic plane, $G = \mathrm{SL}(2,\mathbb{R})$, K=SO(2), $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$, $\mathfrak{k} = \mathfrak{so}(2)$ and $\mathfrak{p} = \mathfrak{so}(2)^{\perp}$ are as defined previously. The tangent space $T_i\mathbb{H}^2$ can be identified either with the complex plane $\mathbb{C}$ or with elements in $\mathfrak{p}$. The next lemma asserts that K acts transitively on the unit tangent vectors:

Lemma A.2. *Let $v = x + iy \in T_i^1\mathbb{H}^2$ be a unit tangent vector. Then there is $M \in SO(2)$ such that $M(v) = i$.*

Proof. Using the identification (A.1) of $T_i\mathbb{H}^2$ with $\mathfrak{p}$, the tangent vector v identifies with the matrix $V = \begin{bmatrix} \frac{y}{2} & \frac{x}{2} \\ \frac{x}{2} & -\frac{y}{2} \end{bmatrix} \in \mathfrak{p}$ and the tangent vector i identifies with $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$. The lemma is then equivalent to showing that there is $M \in \mathrm{SO}(2)$ such that $Ad(M)(V) = MVM^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$.

The isotropy subalgebra $\mathfrak{k}$ is one dimensional with $X = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ as a basis element, so elements in K are rotations of the form

$$R_\theta = \exp(\theta X) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Then the action of K is:

$$R_\theta(v) = R_\theta V R_\theta^{-1} = \frac{1}{2} \begin{bmatrix} y\cos(2\theta) + x\sin(2\theta) & x\cos(2\theta) - y\sin(2\theta) \\ x\cos(2\theta) - y\sin(2\theta) & -y\cos(2\theta) - x\sin(2\theta) \end{bmatrix}$$

Setting $x\cos(2\theta) - y\sin(2\theta) = 0$ we obtain the solution

$$\theta_0 = \begin{cases} \frac{1}{2}\arctan\left(\frac{x}{y}\right) & \text{if } y \neq 0 \\ \\ \frac{\pi}{4} & \text{if } y = 0 \end{cases}$$

Using this solution it is straightforward to check that

$$\begin{aligned} R_{\theta_0} V R_{\theta_0}^{-1} &= \frac{1}{2}\begin{bmatrix} \sqrt{x^2+y^2} & 0 \\ 0 & -\sqrt{x^2+y^2} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad \text{(since } v \text{ is a unit vector)} \end{aligned}$$

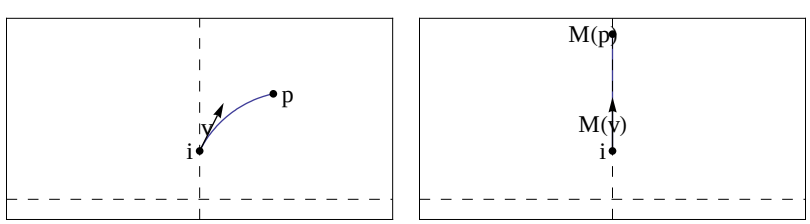
which identifies with the tangent vector i as desired.

□

An immediate corollary of this lemma is:

Corollary A.3. *Given any point $p = x + iy \in \mathbb{H}^2$ there is a transformation $M \in SO(2)$ such that $M(p) = ie^{\lambda}$ where λ is the hyperbolic distance from p to i.*

The Action of SO(2)



Proof. Let γ be the geodesic segment from i to p, parameterized by arc length. Then $\gamma'(0) \in T_i^1\mathbb{H}^2$ and by the lemma there is $M \in \mathrm{SO}(2)$ such that $M(\gamma'(0)) = i$. The image of γ is then a geodesic segment starting at i in the direction of the positive y-axis. This is the geodesic

$$M(\gamma(t)) = exp\left(t\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right)(i) = ie^t$$

i.e. $M(\gamma)$ is the y-axis. So the image of p is on the y-axis. Further since M is an isometry, the distance from i to $M(p)$ is λ, so $M(p) = ie^{\lambda}$. $\square$

A.3.2 The Action of $U(2)$ on $T_{iI}\mathfrak{S}_2$

We now obtain analogous results for the Siegel space. The action on unit tangent vectors is not transitive; rather, every unit tangent vector is equivalent to one in a Cartan subalgebra.

For the Siegel space, we have seen that $G = \mathrm{Sp}(4,\mathbb{R})$, $K \approx \mathrm{U}(2)$, $\mathfrak{g} = \mathfrak{sp}(4,\mathbb{R})$, $\mathfrak{k} \approx \mathfrak{u}(2)$, and $\mathfrak{p} = \mathfrak{k}^{\perp}$. A Cartan Subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is 2 dimensional and consists of diagonal matrices of the form

$$H_{\frac{a}{2},\frac{b}{2}} = \begin{bmatrix} \frac{a}{2} & 0 & 0 & 0 \\ 0 & \frac{b}{2} & 0 & 0 \\ 0 & 0 & -\frac{a}{2} & 0 \\ 0 & 0 & 0 & -\frac{b}{2} \end{bmatrix}$$

Notice that in fact $\mathfrak{h} \subset \mathfrak{p}$, so using the identification (A.1) from §A.2, the image of $H_{\frac{a}{2},\frac{b}{2}}$ under ev_* identifies with a tangent vector in $T_{iI}\mathfrak{S}_2$, namely

$$ev_*\left(H_{\frac{a}{2},\frac{b}{2}}\right) = i\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \in T_{iI}\mathfrak{S}_2$$

$ev_*(\mathfrak{h})$ is a 2 dimensional subspace of the tangent space and contains a circle's worth of unit tangent vectors (with respect to the bilinear form defined at the end of §A.2).

The next lemma says that every unit tangent vector is equivalent by an element of K to a tangent vector on this circle.

Lemma A.4. *Let $v \in T^1_{iI}\mathfrak{S}_2$. Then there is $M \in K$ such that $M(v) = i \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $a^2 + b^2 = 1$.*

Proof. This proof is basically like playing with a Rubik's cube! We will take an arbitrary unit tangent vector $v \in T^1_{iI}\mathfrak{S}_2$, and apply certain rotation matrices in K in the correct order until we have a vector in $ev_*(\mathfrak{h})$. Some rotations will mess up what previous ones have done (like with a Rubik's cube), but if done in the correct order we obtain the desired result. The details are a bit hairy but it is worked out explicitly in the *Mathematica* notebook found in [15].

Using the identification (A.1) in §A.2, an arbitrary unit tangent vector

$$v = \begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} + i \begin{bmatrix} a & a_{12} \\ a_{12} & b \end{bmatrix} \in T_{iI}\mathfrak{S}_2$$

identifies with the matrix

$$P = \frac{1}{2} \begin{bmatrix} a & a_{12} & b_{11} & b_{12} \\ a_{12} & b & b_{12} & b_{22} \\ b_{11} & b_{12} & -a & -a_{12} \\ b_{12} & b_{22} & -a_{12} & -b \end{bmatrix} \in \mathfrak{p}$$

so $ev_*(P) = v$. Since P is a unit tangent vector we have that

$$B(P, P) = 2Tr(P^2) = 1.$$

$\mathfrak{k} = \mathfrak{u}(2)$ is 4 dimensional and we will use the basis $\{B_1, B_2, B_3, B_4\}$ defined in §1.3. K is generated by the images of these under the exponential map. These skew symmetric matrices all exponentiate to rotation matrices, so K is generated by matrices of the form:

$$M_1(\theta) := \exp(\theta B_1) = \begin{pmatrix} \cos(\theta) & \sin(\theta) & 0 & 0 \\ -\sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & \cos(\theta) & \sin(\theta) \\ 0 & 0 & -\sin(\theta) & \cos(\theta) \end{pmatrix}$$

$$M_2(\theta) := \exp(\theta B_2) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) & 0 \\ 0 & 1 & 0 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$M_3(\theta) := \exp(\theta B_3) = \begin{pmatrix} \cos(\theta) & 0 & 0 & \sin(\theta) \\ 0 & \cos(\theta) & \sin(\theta) & 0 \\ 0 & -\sin(\theta) & \cos(\theta) & 0 \\ -\sin(\theta) & 0 & 0 & \cos(\theta) \end{pmatrix}$$

$$M_4(\theta) := \exp(\theta B_4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos(\theta) & 0 & \sin(\theta) \\ 0 & 0 & 1 & 0 \\ 0 & -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}$$

Refer to [15] for full details of the following direct and messy computations. In what follows, the $*$'s indicate a messy unimportant expression which you can

find exactly in [15]. What is important is that certain rotations zero out some coordinates. Should any of the denominators in the fractions below equal zero, simply extend the definition of $\tan^{-1}(\pm\infty) = \frac{\pi}{2}$. Apply to P the adjoint action of the following 6 matrices in K:

1. $P_1 := Ad\{M_2\left(\frac{1}{2}\tan^{-1}\left(\frac{b_{11}}{a}\right)\right)\}P = \begin{bmatrix} * & * & 0 & * \\ * & * & * & * \\ 0 & * & * & * \\ * & * & * & * \end{bmatrix}$

2. $P_2 := Ad\{M_4\left(\frac{1}{2}\tan^{-1}\left(\frac{b_{22}}{b}\right)\right)\}P_1 = \begin{bmatrix} * & * & 0 & * \\ * & * & * & 0 \\ 0 & * & * & * \\ * & 0 & * & * \end{bmatrix}$

   So without any loss of generality we may assume that the unit tangent vector has the form:

$$P_2 = \begin{bmatrix} a & a_{12} & 0 & b_{12} \\ a_{12} & b & b_{12} & 0 \\ 0 & b_{12} & -a & -a_{12} \\ b_{12} & 0 & -a_{12} & -b \end{bmatrix}$$

3. Continuing to apply matrices in K, the adjoint action of M_1 does not leave the above zero entries alone, rather rotates them in a very nice controlled way:

$$P_3 := Ad\{M_1\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{2a_{12}}{a-b}\right)\right)\}P_2 = \begin{bmatrix} * & 0 & \bigstar & * \\ 0 & * & * & -\bigstar \\ \bigstar & * & * & 0 \\ * & -\bigstar & 0 & * \end{bmatrix}$$

So without loss of generality we may assume that the tangent vector has the form:

$$P_3 = \begin{bmatrix} a & 0 & b_{11} & b_{12} \\ 0 & b & b_{12} & -b_{11} \\ b_{11} & b_{12} & -a & 0 \\ b_{12} & -b_{11} & 0 & -b \end{bmatrix}$$

4. The adjoint action of M_3 now fixes the above zeros and b_{11} coordinates, and we obtain:

$$P_4 := Ad\{M_3\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{2b_{12}}{a+b}\right)\right)\}P_3 = \begin{bmatrix} * & 0 & b_{11} & 0 \\ 0 & * & 0 & -b_{11} \\ b_{11} & 0 & * & 0 \\ 0 & -b_{11} & 0 & * \end{bmatrix}$$

So now without loss of generality we assume the tangent vector has the form

$$P_4 = \begin{bmatrix} a & 0 & b_{11} & 0 \\ 0 & b & 0 & -b_{11} \\ b_{11} & 0 & -a & 0 \\ 0 & -b_{11} & 0 & -b \end{bmatrix}$$

5. Applying M_2 zeros out one of the remaining coordinates:

$$P_5 := Ad\{M_2\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{b_{11}}{a}\right)\right)\}P_4 = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & b & 0 & -b_{11} \\ 0 & 0 & * & 0 \\ 0 & -b_{11} & 0 & -b \end{bmatrix}$$

So without loss of generality we the tangent vector has the form:

$$P_5 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & -b_{11} \\ 0 & 0 & -a & 0 \\ 0 & -b_{11} & 0 & -b \end{bmatrix}$$

6. Finally applying M_4 zeros out the remaining coordinate:

$$P_6 := Ad\{M_4\left(\tfrac{1}{2}\tan^{-1}\left(\tfrac{b_{22}}{b}\right)\right)\}P_5 = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

The resulting matrix is contained in the Cartan subalgebra $\mathfrak{h}$. Since $ad(K)$ acts preserving the trace form (it is simply conjugation), $B(P, P)$ is preserved by each of these transformations, and hence the image of P is a unit vector in $\mathfrak{h}$. $\square$

The immediate consequence of the above lemma is the following corollary, whose proof is identical to the corresponding statement in the hyperbolic plane above.

Corollary A.5. *Given any point $Z \in \mathfrak{S}_2$ there is $M \in K$ such that*

$$M(Z) = i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}.$$

The image of the Cartan subalgebra $\mathfrak{h}$ under the exponential map is the Cartan subgroup of diagonal matrices in $\mathrm{Sp}(4, \mathbb{R})$. The image of this subgroup in $\mathfrak{S}_2$ under the evaluation map will be a 2 dimensional *Flat*, that is a 2 dimensional subspace isometric to the Euclidean plane. We will refer to this as the *Standard Flat*. The above corollary says that every point in $\mathfrak{S}_2$ is K equivalent to a point in the standard flat. In the next section we will improve on this corollary a bit by realizing that the Weyl Group of $\mathrm{Sp}(4, \mathbb{R})$ is in fact a subgroup of K which leaves this flat invariant.

A.4 The Weyl Group as a subgroup of K

The Weyl group for $\mathrm{Sp}(4, \mathbb{R})$ is isomorphic to the dihedral group D_8 and can be canonically identified with a subgroup of K. It acts on the Cartan subalgebra $\mathfrak{h}$ consisting of diagonal matrices of the form:

$$H_{a,b} = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -b \end{bmatrix}$$

The seven nontrivial elements in the group act on $\mathfrak{h}$ in the following way:

1. $Ad\{M_1(\frac{\pi}{2})\}H_{a,b} = H_{b,a}$

2. $Ad\{M_2(\frac{\pi}{2})\}H_{a,b} = H_{-a,b}$

3. $Ad\{M_3(\frac{\pi}{2})\}H_{a,b} = H_{-b,-a}$

4. $Ad\{M_4(\frac{\pi}{2})\}H_{a,b} = H_{a,-b}$

5. $Ad\{M_1(\frac{\pi}{2})M_2(\frac{\pi}{2})\}H_{a,b} = H_{b,-a}$

6. $Ad\{\big(M_1(\frac{\pi}{2})M_2(\frac{\pi}{2})\big)^2\}H_{a,b} = H_{-a,-b}$

7. $Ad\{\big(M_1(\frac{\pi}{2})M_2(\frac{\pi}{2})\big)^3\}H_{a,b} = H_{-b,a}$

The action on $\mathfrak{h}$ induces an action on the Cartan subgroup, and hence an action on the homogeneous space $\mathfrak{S}_2$ preserving the standard flat. Corollary A.5 says that any point in $\mathfrak{S}_2$ is equivalent to a point in the standard flat. Any point in this flat is then equivalent by an element of the Weyl Group to a point within a specified Weyl Chamber, so we obtain a strengthening of the previous corollary:

Corollary A.6. *Any point $Z \in \mathfrak{S}_2$ is K-equivalent to a point $i \begin{bmatrix} y_1 & 0 \\ 0 & y_2 \end{bmatrix}$ where $1 \leq y_1 \leq y_2$.*

Appendix B

The κ Polynomial

Definition B.1. Let $\kappa(x, y, z) = x^2 + y^2 + z^2 - xyz - 2$.

This polynomial has been studied extensively (see [12] and [13]) as it relates heavily to character varieties for rank 2 surface groups. In this appendix we will only collect a few facts for use elsewhere in this paper.

Recall that for positive integers p, q, r, a (p, q, r) *triangle group* is a group generated by three involutions and has a presentation:

$$\Delta_{pqr} := \langle R_1, R_2, R_3 : R_1^2 = R_2^2 = R_3^2 = (R_1R_2)^p = (R_2R_3)^q = (R_3R_1)^r = 1\rangle$$

The group is called

- *Spherical* if $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} > \pi$
- *Euclidean* if $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} = \pi$
- *Hyperbolic* if $\frac{\pi}{p} + \frac{\pi}{q} + \frac{\pi}{r} < \pi$

Such a group can be realized as a group of reflections in the 3 sides of a triangle the appropriate space.

Lemma B.2. *If Δ_{pqr} is a Euclidean triangle group then*

$$\kappa\left(-2\cos\left(\frac{\pi}{p}\right), -2\cos\left(\frac{\pi}{q}\right), -2\cos\left(\frac{\pi}{r}\right)\right) = 2$$

Proof. Supposing $\frac{\pi}{p}+\frac{\pi}{q}+\frac{\pi}{r}=\pi$, then fun with trigonometric identities yields:

$$\begin{aligned}
\frac{\pi}{p}+\frac{\pi}{q} &= \pi-\frac{\pi}{r}\\
\cos\left(\frac{\pi}{p}+\frac{\pi}{q}\right) &= \cos\left(\pi-\frac{\pi}{r}\right)\\
\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)-\sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right) &= -\cos\left(\frac{\pi}{r}\right)\\
\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)+\cos\left(\frac{\pi}{r}\right) &= \sin\left(\frac{\pi}{p}\right)\sin\left(\frac{\pi}{q}\right)\\
\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)+\cos\left(\frac{\pi}{r}\right) &= \sqrt{\left(1-\cos^2\left(\frac{\pi}{p}\right)\right)\left(1-\cos^2\left(\frac{\pi}{q}\right)\right)}
\end{aligned}$$

Squaring both sides we get:

$$\begin{aligned}
\text{LHS} &= \cos^2\left(\frac{\pi}{p}\right)\cos^2\left(\frac{\pi}{q}\right)+2\cos\left(\frac{\pi}{p}\right)\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)+\cos^2\left(\frac{\pi}{r}\right)\\
\text{RHS} &= 1-\cos^2\left(\frac{\pi}{p}\right)-\cos^2\left(\frac{\pi}{q}\right)+\cos^2\left(\frac{\pi}{p}\right)\cos^2\left(\frac{\pi}{q}\right)
\end{aligned}$$

Canceling and rearranging we obtain

$$\cos^2\left(\tfrac{\pi}{p}\right)+\cos^2\left(\tfrac{\pi}{q}\right)+\cos^2\left(\tfrac{\pi}{r}\right)+2\cos\left(\tfrac{\pi}{p}\right)\cos\left(\tfrac{\pi}{q}\right)\cos\left(\tfrac{\pi}{r}\right)-1=0$$

Then multiplying both sides through by 4 we obtain

$$\begin{aligned}
&\left(2\cos\left(\tfrac{\pi}{p}\right)\right)^2+\left(2\cos\left(\tfrac{\pi}{q}\right)\right)^2+\left(2\cos\left(\tfrac{\pi}{r}\right)\right)^2+\\
&\left(2\cos\left(\tfrac{\pi}{p}\right)\right)\left(2\cos\left(\tfrac{\pi}{q}\right)\right)\left(2\cos\left(\tfrac{\pi}{r}\right)\right)-4=0
\end{aligned}$$

Observe now that this is precisely:

$$\kappa\left(-2\cos\left(\tfrac{\pi}{p}\right),-2\cos\left(\tfrac{\pi}{q}\right),-2\cos\left(\tfrac{\pi}{r}\right)\right)=2$$

□

Lemma B.3. *If Δ_{pqr} is a hyperbolic triangle group then*

$$\kappa\left(-2\cos\left(\tfrac{\pi}{p}\right),-2\cos\left(\tfrac{\pi}{q}\right),-2\cos\left(\tfrac{\pi}{r}\right)\right)>2$$

Proof. Let $\Delta_{p_0q_0r_0}$ be a Euclidean triangle group. We will show that all directional derivatives of κ are increasing at this point, and nonzero elsewhere. Hence this is a nondecreasing function of p, q, r and a hyperbolic triangle group can be obtained by increasing any one of the variables. Any hyperbolic triangle group can be obtained in this manner from some Euclidean triangle group. Treating q and r as constants let

$$f(p) = \kappa\left(-2\cos\left(\tfrac{\pi}{p}\right), -2\cos\left(\tfrac{\pi}{q_0}\right), -2\cos\left(\tfrac{\pi}{r_0}\right)\right)$$

The directional derivative $\frac{\partial\kappa}{\partial p}$ evaluated at the Euclidean triangle group is $f'(p_0)$.

$$\begin{aligned}
f'(p) &= \frac{\partial\kappa}{\partial x}\frac{\partial x}{\partial p} \\
&= (2x - yz)\left(-\frac{2\pi}{p^2}\sin\left(\frac{\pi}{p}\right)\right) \\
&= \left(-4\cos\left(\frac{\pi}{p}\right) - 4\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)\right)\left(-\frac{2\pi}{p^2}\sin\left(\frac{\pi}{p}\right)\right) \\
&= \frac{8\pi}{p^2}\sin\left(\frac{\pi}{p}\right)\left(\cos\left(\frac{\pi}{p}\right) + \cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)\right)
\end{aligned}$$

At p_0 we know that $\frac{\pi}{p_0} = \pi - \frac{\pi}{q_0} - \frac{\pi}{r_0}$ we get

$$\begin{aligned}
f'(p_0) &= \frac{\partial\kappa}{\partial p}\Big|_{\text{Euclidean Case}} \\
&= \frac{8\pi}{p_0^2}\sin\left(\pi - \frac{\pi}{q_0} - \frac{\pi}{r_0}\right)\left(\cos\left(\pi - \frac{\pi}{q_0} - \frac{\pi}{r_0}\right) + \cos\left(\frac{\pi}{q_0}\right)\cos\left(\frac{\pi}{r_0}\right)\right) \\
&= \frac{8\pi}{p_0^2}\sin\left(\frac{\pi}{q_0} + \frac{\pi}{r_0}\right)\left(-\cos\left(\frac{\pi}{q_0} + \frac{\pi}{r_0}\right) + \cos\left(\frac{\pi}{q_0}\right)\cos\left(\frac{\pi}{r_0}\right)\right) \\
&= \frac{8\pi}{p_0^2}\sin\left(\frac{\pi}{q_0} + \frac{\pi}{r_0}\right)\left(\sin\left(\frac{\pi}{q_0}\right)\sin\left(\frac{\pi}{r_0}\right)\right)
\end{aligned}$$

Since $0 < \frac{\pi}{q_0}, \frac{\pi}{r_0} \leq \frac{\pi}{2}$, we know all factors in the above expression are positive, hence $f'(p_0) > 0$ and f is increasing.

Finally, $f'(p) = \frac{8\pi}{p^2} \sin\left(\frac{\pi}{p}\right)\left(\cos\left(\frac{\pi}{p}\right) + \cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)\right)$ could only be zero if

$$\cos\left(\frac{\pi}{p}\right) + \cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right) = 0$$
$$\cos\left(\frac{\pi}{p}\right) = -\cos\left(\frac{\pi}{q}\right)\cos\left(\frac{\pi}{r}\right)$$

which can not occur since all angles are acute. Hence κ is increasing at a Euclidean triangle group and has no critical points elsewhere. We know by lemma B.2 that $f(p_0) = 2$ so for a hyperbolic triangle group $\kappa > 2$. □

Lemma B.4. *κ is invariant under "sign change automorphisms" (teminology from [12]), more precisely:*

$$\kappa(x, y, z) = \kappa(-x, -y, z) = \kappa(-x, y, -z) = \kappa(x, -y, -z)$$

Proof. Straightforward computation, the key point being that only the cubic term xyz is affected by any sign changes. □

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