ABSTRACT

Title of dissertation: ASYMPTOTIC PROBLEMS FOR
STOCHASTIC PROCESSES AND
RELATED DIFFERENTIAL EQUATIONS

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It is well known that solutions of classical initial–boundary problems for second order parabolic equations depend continuously on the coefficients if the coefficients converge to their limits in a strong enough topology.

In case of one spatial variable, we consider the question of the weakest possible topology providing convergence of the solutions. Convergence of solutions of PDE's is equivalent to weak convergence of corresponding diffusion processes. In general, continuous Markov processes corresponding to the generalized second order differential operators introduced by W. Feller can appear as limiting processes. In other words, the infinitesimal generator of limiting processes need not be a classical second order elliptic differential operator but instead can be a generalized in the sense of W. Feller [7].

Following Freidlin and Wentzell's paper [14], where processes in open intervals were considered, we study the necessary and sufficient conditions for the weak convergence of one dimensional Markov processes in closed intervals. We provide

conditions that guarantee the convergence of solutions of initial–boundary value problems for parabolic equations. Furthermore, necessary and sufficient conditions of weak convergence can be easily verified.

In a number of articles, it was proved that the solution of reaction diffusion equations with a certain nonlinearity term is close for large t to a running wave solution. However, in general, one cannot always give a simple formula for the asymptotic speed as was done in the Kolmogorov, Petrovskii and Piskunov (KPP) case in R^1. We apply our results to wave front propagation in narrow, of width $\varepsilon \ll 1$, domains. Especially, we consider the wave front propagation problem in a narrow periodic domain of width $\varepsilon \ll 1$ with the reaction term of KPP type. We focus on how a smooth or non–smooth periodic boundary of a narrow tube can affect the asymptotic speed of wave front propagation. In particular, under a non–smooth periodic boundary, the limit of solutions of initial–boundary problems is expected to satisfy a parabolic equation with a generalized second order operator, instead of a classical second order elliptic one.

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by

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Dedication

To my husband SeungHwan,

my parents and my lovely family.

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Table of Contents

Chapter 1

Introduction

1.1 The convergence of solutions of initial–boundary value problems for parabolic equations.

1.1.1 Statement of problem

For each second order elliptic differential operator

$$Lf(x) = a(x)\frac{d^2 f}{dx^2} + b(x)\frac{df}{dx}, \quad a(x) > 0 \tag{1.1}$$

with regular enough coefficients $a(x)$ and $b(x)$, there exists a diffusion process (X_t, P_x) in R^1 such that the generator of this process on twice continuously differentiable functions coincides with L. If $a(x) \in C^1(R^1)$, $a(x) > 0$ (otherwise see [10]), and $b(x) \in C^1(R^1)$, the trajectories of X_t can be constructed as the solutions of corresponding stochastic differential equations:

$$dX_t = \sqrt{2}\sigma(X_t)dW_t + b(X_t)dt, \quad X_0 = x \tag{1.2}$$

Here $\sigma(x) = (a(x))^{\frac{1}{2}}$ and W_t is the standard Wiener process in R^1. Roughly speaking, this means that locally X_t behaves like a Wiener process multiplied by a factor. In particular, X_t spends time zero at any given point $x \in R^1$; the trajectory X_t exits the interval $[x - \delta, x + \delta]$ through both ends with asymptotically equal probabilities as $\delta \downarrow 0$.

Consider a sequence of processes $X_t^{(n)}$ governed respectively by operators L_n:

$$L_n f = a_n(x)\frac{d^2 f}{dx^2} + b_n(x)\frac{df}{dx}.$$

If $\sigma_n^2(x) = a_n(x)$ and $b_n(x)$ converge uniformly to $a(x)$ and $b(x)$ and they satisfy a Lipschitz condition with the same constant K, it is simple to prove that $X_t^{(n)}$ converges in probability uniformly on any finite time interval to the process X_t corresponding to L as $n \to \infty$. This implies that the solutions of the Cauchy problem for equations $\partial f_n/\partial t = L_n f_n$ with a bounded continuous initial function converge to the solution of the corresponding problem for $\partial f/\partial t = Lf$.

But if $(a_n(x), b_n(x))$ converges to $(a(x), b(x))$ in a weaker sense, $\lim_{n\to\infty} f_n(x)$ may not exist. Even if such a limit $f(x)$ exists, $f(x)$ may not be a solution to $\partial f/\partial t = Lf$ (for example, in the case of fast oscillating periodic coefficients, see [9]). This is the situation we have studied.

In particular, we study necessary and sufficient conditions for prelimiting processes $X_t^{(n)}$ governed by a second order elliptic operator L_n with coefficients $(a_n(x), b_n(x))$ to converge weakly a limiting process X_t. The coefficients $a_n(x)$ and $b_n(x)$ are assumed to be smooth and grow no faster than linearly, and $a_n(x) > \alpha > 0$. It turns out that the limiting process X_t may not be governed by the standard second order elliptic operator $Lf = a(x)(d^2f/dx^2) + b(x)(df/dx)$. It was proven in [7] that any Markov process X_t in R^1 that is continuous with probability one, under some minimal regularity conditions, is governed by a generalized elliptic operator $D_v D_u$. Here u and v are strictly increasing functions such that $u(x)$ is continuous and $v(x)$ is right continuous, and D_u and D_v are operators of differentiation with

respect to $u(x)$ and $v(x)$ respectively, which are defined later. Thus, the limiting process can be characterized by the generalized second order differential operator introduced by W. Feller ([7]). M. Freidlin and A. Wentzell in [14] considered the problem in open intervals without boundary conditions, while boundary conditions are considered in this thesis.

1.1.2 The W. Feller generalized second order differential operator

Before we give the definition and properties of the generalized second order differential operators, we can see that every classical second order elliptic operator L of the form (1.1) can be put into a succession of two differentiations in the form $D_v D_u$ via the following recipe:

$$\begin{cases} dv(x) = (a(x))^{-1} e^{B(x)} dx, & du(x) = e^{-B(x)} dx \\ d/dv(x) = D_v, & d/du(x) = D_u \end{cases}$$

where $B(x) = \int^x b(y)(a(y))^{-1} dy$.

Then we can write

$$L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx} = a(x) e^{-B(x)} \frac{d}{dx} (e^{B(x)} \frac{d}{dx}) = D_v D_u.$$

In fact, the operator L in the form $D_v D_u$ is meaningful for an arbitrary strictly increasing function v, not necessarily continuous or bounded, and an arbitrary strictly continuous increasing function u. Moreover, the choices of the two functions u and v are not unique. We can multiply one of these functions by some positive number, and divide the other by the same number. We can also add some constant to either of them.

The infinitesimal operators for more general class of one–dimensional continuous Markov processes were calculated by W. Feller in 1954–1955. Let B be the Banach space of continuous functions on R^1 with norm $\|f\| = \sup_{x \in B} |f(x)|$. Let T_t be a semigroup and A the infinitesimal operator of the process (X_t, P_x). (More details for definitions and properties can be found in [5]). A transition function $P(t, x, \Gamma)$ of a Markov process, which is homogeneous in time, is called Fellerian if for any bounded continuous function $f(x)$, $T_t f(x) = \int_\Gamma f(y) P(t, x, dy)$ is continuous and $\lim_{t \to 0} T_t f = f$ for every $f \in B$. Here the semigroup T_t is associated with a given transition function $P(t, x, \Gamma)$ and the operator semigroup takes the space of continuous bounded functions into itself. We will call this process a Feller process. We consider here a Feller process in R^1 with continuous trajectories. It was shown in [4] that a Feller process with continuous trajectories must be a strong Markov process. A Feller semigroup is a strongly continuous positive contraction semigroup. Moreover it was proven in [7] (see also [8]) that an infinitesimal operator A of a Feller process can be characterized by two increasing functions $u(x)$ and $v(x)$, where $u(x)$ is a strictly increasing continuous function and $v(x)$ is a strictly increasing right continuous function. Here $u(x)$ is called a scale function and $v(x)$ is called a speed function. The infinitesimal operator A of a Feller process can be written in the form of a $D_v D_u$–operator. We will call the $D_v D_u$–operator the generalized second order differential operator. Every function f which belongs to the domain of the operator $D_v D_u$ ($f \in \mathcal{D}(D_v D_u)$) is continuous, but not necessary differentiable.

The formal adjoint of the classical second order elliptic operator L in (1.1) is given by $L^* g = (d^2/dx^2)(a(x)g) - (d/dx)(b(x)g)$ if the coefficients $a(x)$ and $b(x)$

are sufficiently regular. Without this highly strong condition, no formal adjoint of the form (1.1) exists. By contrast, no difficulties arise in connection with $D_v D_u$–operator. Denote by R_λ the resolvent of the generator $D_v D_u$ and $R_\lambda f = (\lambda - D_v D_u)^{-1} f$ for all $f \in B$ and $\lambda > 0$. Considered as a transformation from $\mathcal{C}$ to $\mathcal{C}$, each R_λ has an adjoint R_λ^* mapping from B into B. By using the notation $(f, d\xi) = \int f d\xi$ for inner products, the adjoint R_λ^* is defined by $(R_\lambda f, d\pi) = (f, R_\lambda^* d\pi)$. From this, $D_u D_v$ appears as the formal adjoint of the differential operator $D_v D_u$. Here $D_u D_v$ is an operator on measures.

Definition 1 *D_u and D_v are operators of differentiation with respect to $u(x)$ and $v(x)$ respectively, which are defined as follows:*

- *If the right derivative of f with respect to u exists, it is defined by*

$$D_u^+ f(x) = \lim_{h \downarrow 0} \frac{f(x+h) - f(x)}{u(x+h) - u(x)}$$

  The left derivative $D_u^- f(x)$ is defined similarly.

- *$D_u f(x)$ exists if $D_u^+ f(x) = D_u^- f(x)$*

- *If v is discontinuous at x_0, we define*

$$D_v f(x_0) = \lim_{h \downarrow 0} \frac{f(x_0 + h) - f(x_0 - h)}{v(x_0 + h) - v(x_0 - h)} = \frac{f(x_0+) - f(x_0-)}{v(x_0+) - v(x_0-)}$$

When no confusion can arise, we shall occasionally simplify the notation by inconsistently using the symbol D_u in the meaning of either D_u^+ or D_u^- even if $D_u^+ \neq D_u^-$ or if only one is defined. The value of $D_v D_u f$ is defined at each point where the derivative exists and the definition of $D_v D_u$ will be completed by specifying the

domain. In connection with the space $\mathcal{C}$ of continuous functions in R we shall require that $D_v D_u f \in \mathcal{C}$.

1.1.3 Outline and Preparation

Let us consider the Cauchy problem:

$$\begin{cases} \frac{\partial f_n(t,x)}{\partial t} = L_n f_n(t,x) & \text{for } x \in R, t > 0 \\ f_n(0,x) = g(x) & \text{for } x \in R, t = 0 \end{cases} \qquad (1.3)$$

where $L_n = a_n(x)d^2/dx^2 + b_n(x)d/dx$; the coefficients $a_n(x)$ and $b_n(x)$ are assumed to be smooth and grow not faster than linearly and $a_n(x) > 0$. Let us denote the process corresponding to L_n by $X_t^{(n)}$. Let $T_t^{(n)}$ be the semigroup in C_{0T} corresponding to $X_t^{(n)}$ governed by the operator L_n. It is known ([10], Theorem 2.5.2) that the solution $f_n(t,x)$ of problem (1.3) can be represented in the form of the expectations of appropriate functionals of the trajectories of the Markov family $(X_t^{(n),x}, P)$: $f_n(t,x) = E_x g(X_t^{(n)})$, where E_x denotes the expectation with the initial point $X_0^{(n)} = x$.

We are interested in the weak limit of $X_t^{(n)}$, $0 \le t \le T < \infty$, as $n \to \infty$. If such a limit X_t exists and is a diffusion process, its generator is the generalized second order differential operator $D_v D_u$. Consider the Cauchy problem:

$$\begin{cases} \frac{\partial f(t,x)}{\partial t} = D_v D_u f(t,x) & \text{for } x \in R, t > 0 \\ f(0,x) = g(x) & \text{for } x \in R, t = 0 \end{cases} \qquad (1.4)$$

Let X_t be the process that is governed by the $D_v D_u$–operator. Then, one can check that $f(t,x) = E_x g(X_t)$ is the unique solution to problem (1.4). Hence the

6

convergence of solutions of initial value problems follows from the weak convergence of the processes $X_t^{(n)}$ as $n \to \infty$ to a process X_t.

Now we will describe the construction of a diffusion process in the half–line R_+^1 with reflection at zero (More details can be found in [10]). Consider $Lf = a(x)(d^2f/dx^2) + b(x)(df/dx)$, which is defined for $x \in R_+^1$. It is easy to see that there exist two strictly increasing functions $u(x)$ and $v(x)$ in R_+^1 such that $Lf = D_v D_u f$:

$u(x) = u'(0) \int_0^x \exp\{-B(y)\}dy$ and $v(x) = v(0) + 1/u'(0) \int_0^x a(y)^{-1} \exp\{B(y)\}dy$,

where $B(x) = \int_0^x b(y)/a(y)dy$. Let $a(x) = \sigma^2(x)$ be positive and $\sigma(x)$ and $b(x)$ be Lipschitz continuous. Let us extend $\sigma(x)$ and $b(x)$ onto the entire space R^1 so that, for $x < 0$, $\sigma(x) = \sigma(-x)$ and $b(x) = -b(-x)$. From these extensions it can be derived that, for $x < 0$, $u(x) = -u(-x)$ and $v(x) = -v(-x)$. Consider the stochastic differential equation with extended coefficients $\sigma(x)$ and $b(x)$ in R^1:

$$dY_t = \sqrt{2}\sigma(Y_t)dW_t + b(Y_t)dt, \quad Y_0 = x.$$

Let ϕ be a symmetric mapping with respect to a point $x = 0$: $\phi(x) = -x$. Then it has the following property $P(t, x, \Gamma) = P(t, \phi(x), \phi(\Gamma))$. Here $P(t, x, \Gamma)$ is a transition function of (Y_t^x, P). Define $X_t^x = |Y_t^x|$ with measure P_x for an initial point $x \in R_+^1$. Then the process (X_t, P_x) is a Markov process in R_+^1 with instantaneous reflection on the boundary point. Furthermore, it was proven in [10] that the process (X_t, P_x) is a Feller process which also has a continuous trajectory. Therefore the process (X_t, P_x) is a strong Markov process [4]. One can verify that the infinitesimal operator A of this process (X_t, P_x) is defined for the smooth function $f(x)$ satisfying the conditions $D_u f(x) = 0$ at $x = 0$ and $Af(x) = D_v D_u f(x)$. In fact, these conditions are necessary

7

and sufficient for $f(x) \in \mathcal{D}(A)$. To prove sufficiency, one should use the symmetric property of the transition function $P(t, x, \Gamma)$ ([2], Theorem 6.10.13). Assuming that $D_u f(x) \neq 0$ at $x = 0$, it can be derived that $D_v D_u f(x)$ does not exist, so $f(x) \notin \mathcal{D}(A)$. This implies necessity.

In Chapter 2 we will give necessary and sufficient conditions for weak convergence of processes which have not only a reflection but also some delay at the boundary point.

1.2 Reaction diffusion equations in narrow domains

1.2.1 Reaction diffusion equations in narrow tubes and asymptotic speed of wave front propagation

Let G^δ, for $\delta > 0$, be a set in $R \times R^n$ such that $R \times \{0\} \subset G^\delta$, and for each $x \in R$, let the set $G_x^\delta = \{y \in R^n : (x, y) \in G^\delta\}$ be a bounded connected domain. Assume that the boundary ∂G^δ of G^δ is smooth enough and that the inward unit normal $\gamma^\delta(x, y)$ to ∂G^δ at any point $(x, y) \in \partial G^\delta$ is not parallel to R. Let $G^{\varepsilon,\delta} = \{(x, y) \in R^{1+n} : (x, y\varepsilon^{-1}) \in G^\delta\}$ for $0 < \varepsilon \ll 1$. Denote by $V^\delta(x)$ the volume of G_x^δ in R^n.

Consider the problem:

$$
\begin{cases}
\frac{\partial w^{\varepsilon,\delta}(t,x,y)}{\partial t} = \frac{1}{2}\triangle w^{\varepsilon,\delta} + f(x, y, w^{\varepsilon,\delta}), & \text{if } t > 0, (x, y) \in G^{\varepsilon,\delta} \\
\frac{\partial w^{\varepsilon,\delta}(t,x,y)}{\partial \gamma^{\varepsilon,\delta}}\Big|_{t>0,(x,y)\in\partial G^{\varepsilon,\delta}} = 0, & w^{\varepsilon,\delta}(0, x, y) = g(x),
\end{cases}
\tag{1.5}
$$

where $\gamma^{\varepsilon,\delta}$ is the inward unit normal vector to $\partial G^{\varepsilon,\delta}$. The functions f and g are sufficiently regular and bounded. From the weak convergence of the first component

$X_t^{\varepsilon,\delta}$ in $G^{\varepsilon,\delta}$ to X_t, whose generator is in the form of the W. Feller generalized second order differential operator (see Theorem 7 in Section 2.2), one can "guess" that the solution $w^{\varepsilon,\delta}$ converges as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ to w, which is the solution to the non–standard reaction diffusion equation:

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = D_v D_u w + f(x,0,w), & \text{if } t > 0, x \in R \\ w(0,x) = g(x), \end{cases} \tag{1.6}$$

where $u(x) = \lim_{\delta \downarrow 0} \int_0^x (1/V^\delta(y))dy$ and $v(x) = \lim_{\delta \downarrow 0} \int_0^x 2(V^\delta(y))dy$.

It is well known that a classical solution to (1.5) exists and is unique under some minimal regularity conditions [15]. On the other hand, the solution to (1.6) may not be differentiable in x. In Section 3.1, we will introduce the generalized solution of problem (1.6) and provide a uniqueness and an existence of solution to the problem (1.6) with a KPP–type nonlinearity term, which will be introduced in next subsection.

Consider the Wiener process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ in $G^{\varepsilon,\delta}$ with normal reflection on $\partial G^{\varepsilon,\delta}$. Its trajectories can be described by the stochastic differential equations:

$$\begin{aligned} dX_t^{\varepsilon,\delta} &= dW_t^1 + \gamma_1^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta}, \\ dY_t^{\varepsilon,\delta} &= dW_t^2 + \gamma_2^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta}. \end{aligned} \tag{1.7}$$

where W_t^1 and W_t^2 are independent Wiener processes in R and R^n respectively; $\gamma_1^{\varepsilon,\delta}(x,y)$ and $\gamma_2^{\varepsilon,\delta}(x,y)$ are projections of the unit inward normal vector to $\partial G^{\varepsilon,\delta}$ on R and R^n respectively. Moreover $L_t^{\varepsilon,\delta}$ is the local time for the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ on $\partial G^{\varepsilon,\delta}$. In case of smooth enough boundary $\partial G^{\varepsilon,\delta}$ for fixed $\delta > 0$, one can expect, as $\varepsilon \downarrow 0$, to which process the component $X_t^{\varepsilon,\delta}$ converges due to the following Theorem.

Theorem 1 *(M. Freidlin, [11] (see also [12])) The processes $X_t^{\varepsilon,\delta}$, $0 \le t \le T$, converge weakly in the space of continuous functions on R as $\varepsilon \downarrow 0$ to the diffusion process X_t^δ governed by the operator*

$$\frac{1}{2}\frac{d^2}{dx^2} + \frac{1}{2}\frac{d}{dx}(\ln V^\delta(x)) \cdot \frac{d}{dx}. \tag{1.8}$$

One can expect that, under certain assumptions on the non–linear term $f(x,y,w^{\varepsilon,\delta})$ in (1.5), the solution $w^{\varepsilon,\delta}(t,x,y)$ can be approximated by a running–wave–type solution. Corresponding results on the standard reaction diffusion equations allow one to describe the asymptotic wave front propagation motion for $w^{\varepsilon,\delta}(t,x,y)$. In particular, we will consider KPP-type nonlinearity in periodic tubes in Section 3. In this case, we will study the behavior of the asymptotic speed of the wave front propagation that corresponds to (1.5) as $\varepsilon \downarrow 0$ for fixed $\delta > 0$. We will prove in Proposition 4 in Section 3.3 that it will converge to the asymptotic speed of the wave front propagation for the equation:

$$\frac{\partial w^\delta}{\partial t} = \frac{1}{2}\frac{\partial^2 w^\delta}{\partial x^2} + \frac{1}{2}\frac{\partial}{\partial x}(\ln V^\delta(x)) \cdot \frac{\partial w^\delta}{\partial x} + f(x,0,w^\delta)$$
$$w^\delta(0,x) = g(x) \tag{1.9}$$

This can be done by considering the eigenvalues for corresponding problems. We will prove that, for fixed $\delta > 0$, corresponding principal eigenvalues $\lambda^{\varepsilon,\delta}$ of the problem (1.5) in periodic domains converge as $\varepsilon \downarrow 0$ to a principal eigenvalue λ^δ of the problem (1.9). This will guarantee the convergence of the asymptotic speed of the wave front propagation as $\varepsilon \downarrow 0$. In the case that the boundary ∂G^δ converges as $\delta \downarrow 0$ to a non–smooth boundary, the asymptotic behavior of the wave front propagation is still open and is left for future work.

In next subsection we introduce KPP–type reaction diffusion equations and the notion of asymptotic speed of the wave front propagation. We will also prove that the solutions $w^{\varepsilon,\delta}$ of problem (1.5) of nonlinearity term KPP–type (defined below) converge as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ to the solution w of problem (1.6). We conclude Section 1.2 with the action functional for finite dimensional random vectors. We will use these results to find the asymptotic speed of $w^{\varepsilon,\delta}(t,x,y)$ and $w^{\delta}(t,x)$ in Section 3.

1.2.2   KPP–type reaction diffusion equations

The quasi–linear reaction diffusion equations and wavefront propagation have been studied extensively since the 1930s. Diffusion equations with non–linear terms are used for describing certain physical, chemical, or biological processes. In 1937 A.N.Kolmogorov, I.G.Petrovskii and N.S.Piskunov considered the following problem:

$$
\begin{aligned}
\frac{\partial u(t,x)}{\partial t} &= \frac{D}{2}\frac{\partial^2 u}{\partial x^2} + f(u), \quad t > 0, x \in R^1 \\
u(0,x) &= \chi_{x<0}(x) = \begin{cases} 1, & \text{for} \quad x < 0 \\ 0, & \text{for} \quad x \geq 0. \end{cases}
\end{aligned}
\tag{1.10}
$$

Here $D > 0$ and the nonlinear term $f(u) = c(u) \cdot u$, where the function $c(u)$ is supposed to be Lipschitz continuous, positive for $u < 1$ and negative for $u > 1$, and such that $c = c(0) = \max_{0 \leq u \leq 1} c(u)$. Let us denote the class of such functions $f(u)$ by $\mathcal{F}_1$ and call such a nonlinear term $f(u)$ the KPP–type.

It was proved in [18] that the solution $u(t,x)$ of (1.10) for large t will be close to a running wave type solution $v(x - \alpha t)$. The speed of the wave is $\alpha = \sqrt{2cD}$,

and the shape $v(z)$ is a solution of the problem

$$(D/2)v''(z) + \alpha v(z) + f(v(z)) = 0, \quad -\infty < z < \infty$$

$$v(-\infty) = 1, \quad v(\infty) = 0.$$

(1.11)

Problem (1.11) is solvable for $\alpha \geq \sqrt{2cD}$, and the solution is unique. So the limiting

behavior of the solution of problem (1.10) can be characterized by the speed α and

by the shape $v(z)$ of the running wave.

Using the Feynman–Kac formula, one can write the equation

$$u(t, x) = E\chi_{x<0}(x + \sqrt{D}W_t) \exp\left\{\int_0^t c(u(t - s, x + \sqrt{D}W_s))ds\right\}, \quad (1.12)$$

where W_t is the one–dimensional Wiener process starting at zero. One can introduce

the asymptotic speed independently of the shape. The number α^* is called the

asymptotic speed as $t \to \infty$ for the problem (1.10) if for any $h > 0$

$$\lim_{t\to\infty} \sup_{x>(\alpha^*+h)t} u(t, x) = 0, \quad \lim_{t\to\infty} \inf_{x<(\alpha^*-h)t} u(t, x) = 1. \quad (1.13)$$

It follows from [18] that such α^* exists and is equal to $\sqrt{2cD}$. The notion of asymp-

totic speed can be introduced in a more general situation, and the large deviation

approach allows us to calculate it. However, we cannot expect that some asymp-

totic speed will be established in the case of arbitrary diffusion coefficients and a

non–linear term $f(x, u)$ without a certain hypotheses. So we will consider the peri-

odic media case in Chapter 3; the diffusion coefficients and the non–linear term are

functions periodic in the space variables. The generalized results on the KPP–type

reaction diffusion equations in a more general situation were given by M. Freidlin

[11].

1.2.3 Action functional

When we study wave propagation for quasi–linear equations with small diffu-sion, we usually use large deviations theorems in the space of continuous functions. To calculate the asymptotic wave propagation velocity for the reaction diffusion equations to be considered in Chapter 3, we will apply large deviations theorems for some families of finite–dimensional random vectors, rather than for diffusion processes with small parameter. Let us formulate the corresponding results to be used in Chapter 3.

Let $(\Omega_\theta^t, \mathcal{F}_\theta^t, P_\theta^t)$ be a family of probability spaces, where $t \in (0, \infty)$ and the parameter θ varies over an arbitrary non–empty set Θ. Consider a family of n–dimensional random vectors η_θ^t defined on the corresponding measurable spaces $(\Omega_\theta^t, \mathcal{F}_\theta^t)$, $t \in (0, \infty)$ and $\theta \in \Theta$. Suppose that for some positive function $\varepsilon(t)$, such that $\varepsilon(t) \to 0$ as $t \to \infty$, and for all $z \in R^n$,

$$G(z) = \lim_{t \to \infty} \varepsilon(t) \ln \left(E_\theta^t \exp \left\{ \frac{1}{\varepsilon(t)} < z, \eta_\theta^t > \right\} \right). \qquad (1.14)$$

Here the limit (1.14) exists independently of $\theta \in \Theta$, uniformly in the parameter θ and the limit $G(z)$ may be the value $+\infty$, not identically equal to $+\infty$, assuming its values in $(-\infty, \infty]$. The function $G(z)$ is convex and lower semicontinuous. The notation $< \cdot, \cdot >$ indicates a scalar product in the space R^n and E_θ^t denotes the expectation with respect to the probability measure P_θ^t.

Let us introduce the action function $S(y) : R^n \to [0, \infty]$ as the Legendre transform of $G(z)$, defined as

$$S(y) = \sup_{z \in R^n} [< y, z > -G(z)], \qquad y \in R^n. \qquad (1.15)$$

It is easy to prove that the function $S(y)$ is again a function of the same class as $G(z)$. That is, it is convex, lower semicontinuous , assumes values in $(-\infty, \infty]$ and is not identically equal to $+\infty$. Let

$$D(G) = \{\nabla G(z) : \text{ the function } G(z) \text{ is finite and differentiable, } z \in R^n\},$$

$$\overline{D}(G) = \{y \in R^n : \exists \{y_k\} \subset D(G) \text{ such that } y_k \to y \text{ and } S(y_k) \to S(y)\}$$

(1.16)

For any $s \geq 0$, define

$$\Phi(s) = \{y \in R^n : S(y) \leq s\}. \tag{1.17}$$

The sets $\Phi(x)$ are closed and convex. If $G(0)$ is finite, it is easy to see that $\Phi(s)$, $s \geq 0$, are bounded. Denote by $\rho(\cdot, \cdot)$ the Euclidean metric in the space R^n. To calculate the asymptotic wave front propagation velocity in Chapter 3, the following two theorems (for the proofs of these theorems, see [13]) will be essential.

Theorem 2 *(M. Freidlin and A. Wentzell, [13]) Suppose that, for some $s \geq 0$, the set $\Phi(s)$ is nonempty and bounded. Then for any $\delta > 0$, $h > 0$, we can choose $t_0 > 0$ such that the bound*

$$P_\theta^t\{\rho(\eta_\theta^t, \Phi(s)) > \delta\} \leq \exp\left\{-\frac{1}{\varepsilon(t)}(s - h)\right\} \tag{1.18}$$

holds for $t > t_0$ and all $\theta \in \Theta$.

Theorem 3 *(M.Freidlin and A.Wentzell, [13]) For any $\delta > 0$, $h > 0$, and for all $y \in \overline{D}(G)$, a $t_0 > 0$ exists such that*

$$P_\theta^t\{\rho(\eta_\theta^t, y) < \delta\} \geq \exp\left\{-\frac{1}{\varepsilon(t)}(S(y) + h)\right\} \tag{1.19}$$

for $t > t_0$ and all $\theta \in \Theta$.

Chapter 2

Necessary and sufficient conditions for weak convergence of one

dimensional Markov processes in closed intervals

2.1 Weak convergence of a $D_v D_u$–process

As was shown in [14], for each continuous Markov process (X_t, P_x) on $[r_0, \infty)$,

there exist a monotone increasing continuous function $u(x)$, a monotone increasing

right continuous function $v(x)$, and constants α and β, not both equal to zero, such

that the generator of (X_t, P_x) is equal to a $D_v D_u$–operator, and the domain of this

generator is

$$
\left\{
\begin{array}{rl}
f(x) & : \quad D_v D_u f(x) \text{ is well defined, continuous, and} \\
& \Phi(f) = \alpha D_u f(r_0) + \beta D_v D_u f(r_0) = 0.
\end{array}
\right\}.
$$

Here P_x is a measure in the space C_{0T} corresponding to X_t. We call such a process

(X_t, P_x) a $D_v D_u$–process on $[r_0, \infty)$. If $\beta = 0$, then the $D_v D_u$–process has an

instantaneous reflection at a boundary point r_0. If $\alpha = 0$, the $D_v D_u$–process is

trapped at r_0. In other words, it will stay forever at a boundary point r_0 once it

gets there. If $\alpha \neq 0$ and $\beta \neq 0$, then the $D_v D_u$–process spends positive time at a

boundary point r_0 and also reflects there.

Theorem 4 *Let $u(x)$ be a strictly increasing continuous function and let $v(x)$ be a*

strictly increasing right continuous function on $[r_0, \infty)$ corresponding to a $D_v D_u$–

process (X_t, P_x). Let P_x be the measure on C_{0T} corresponding to the process X_t. Assume that all functions in the domain of the generator of (X_t, P_x) satisfy the following boundary condition :

$$\Phi(F) = \alpha D_u F(r_0) + \beta D_v D_u F(r_0) = 0, \ \ for \ all \ F \in \mathcal{D}(D_v D_u).$$

Let $u_n(x)$ and $v_n(x)$ be a sequence of increasing functions on $[r_0, \infty)$. Let $(X_t^{(n)}, P_x^{(n)})$ be a $D_{v_n} D_{u_n}$–process for each n. Let $P_x^{(n)}$ be the measure on C_{0T} corresponding to the process $X_t^{(n)}$. Assume that all functions in the domain of the generator of $(X_t^{(n)}, P_x^{(n)})$ satisfy the boundary condition

$$\Phi^{(n)}(F_n) = \alpha_n D_{u_n} F_n(r_0) + \beta_n D_{v_n} D_{u_n} F_n(r_0) = 0, \ \ for \ all \ F_n \in \mathcal{D}(D_{v_n} D_{u_n}).$$

For the weak convergence of measure $P_x^{(n)}$ to P_x as $n \to \infty$ for all $x \in [r_0, \infty)$, it is necessary after an appropriate choice of $u_n(x)$ and $v_n(x)$, and sufficient that

$$u_n(x) \to u(x) \quad for \ all \ x \in [r_0, \infty), \tag{2.1}$$

$$v_n(x) \to v(x) \quad for \ all \ x \in [r_0, \infty) \ that \ are \ continuity \ points \ of \ v, \tag{2.2}$$

$$\alpha_n \to \alpha, \ \ \beta_n \to \beta. \tag{2.3}$$

Proof. (Necessity) Without loss of generality, we will assume that $r_0 = 0$. Assume that $P_x^{(n)}$ converges weakly to P_x as $n \to \infty$ for all $x \in [0, \infty)$. It was proven by Freidlin and Wentzell [14] that (2.1) and (2.2) hold on open intervals. Here the proof of (2.1) and (2.2) can be carried out in the same way, so we will sketch the proof for these two conditions. For more details one may refer to [14].

16

To prove $u_n(x) \to u(x)$ as $n \to \infty$, consider the functional $I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}})$, where $I_{\{x_2\}}$ is the indicator function of the one–point set and $\tau_{(x_1,x_2)} = \min\{t : X_t \notin (x_1, x_2)\}$ for $t \in [0, \infty)$. By assumption of weak convergence of $P_x^{(n)}$ to P_x,

$$E_x^{(n)}[I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}}^{(n)})] = P_x^{(n)}\{(X_{\tau_{(x_1,x_2)}}^{(n)} = x_2)\}$$
$$\longrightarrow E_x[I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}})] = P_x\{(X_{\tau_{(x_1,x_2)}} = x_2)\}$$

(2.4)

Moreover, these probabilities can be expressed in terms of functions $u_n(x)$, $u(x)$ as follows:

$$P_x^{(n)}\{X_{\tau_{(x_1,x_2)}}^{(n)} = x_2\} = (u_n(x) - u_n(x_1))/(u_n(x_2) - u_n(x_1)),$$
$$P_x\{X_{\tau_{(x_1,x_2)}} = x_2\} = (u(x) - u(x_1))/(u(x_2) - u(x_1)).$$

(2.5)

From the two relations (2.4) and (2.5), the convergence $u_n(x) \to u(x)$ can be proved.

For proving the condition (2.2), consider the functional $I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}})e^{-\lambda\tau_{(x_1,x_2)}}$, where λ is a positive number. This functional is bounded and continuous almost everywhere with respect to P_x. By the weak convergence of $P_x^{(n)}$ to P_x, we can see that

$$F_n(x) = E_x^{(n)}[I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}}^{(n)}) \exp\{-\lambda\tau_{(x_1,x_2)}\}]$$
$$\longrightarrow F(x) = E_x[I_{\{x_2\}}(X_{\tau_{(x_1,x_2)}}) \exp\{-\lambda\tau_{(x_1,x_2)}\}]$$

(2.6)

Here the functions $F_n(x)$, $F(x)$ are solutions of

$$D_{v_n}D_{u_n}F_n(x) = \lambda F_n(x), \quad x \in (x_1, x_2), \quad F_n(x_1) = 0, \quad F_n(x_2) = 1,$$
$$D_v D_u F(x) = \lambda F(x), \quad x \in (x_1, x_2), \quad F(x_1) = 0, \quad F(x_2) = 1.$$

(2.7)

From the two relations (2.6) and (2.7), we can deduce the following:

$$\int_{x_1}^{x_2} g(x)dv_n(x) \to \int_{x_1}^{x_2} g(x)dv(x),$$

(2.8)

for every bounded continuous function $g(x)$ on the interval (x_1, x_2). Therefore the condition $v_n(x) \to v(x)$ as $n \to \infty$ follows.

17

The condition (2.3) is provided in the following.

First, if $\alpha = 0$, there is nothing further to prove because the condition $D_v D_u F(0) = 0$ implies $P(t, 0, \{0\}) = 1$ for $t \geq 0$. Thus let us assume that $\alpha \neq 0$ and $\alpha_n \neq 0$. Then we can rewrite the boundary conditions as follows :

$$D_u F(0) = \gamma D_v D_u F(0)$$

$$D_{u_n} F_n(0) = \gamma_n D_{v_n} D_{u_n} F_n(0).$$

Now, it is enough to prove that $\gamma_n \to \gamma$. Let X_t be a $D_v D_u$–process on $(-\infty, \infty)$ where $u(x)$ and $v(x)$ are antisymmetric with respect to $x = 0$. (One can put $u(0) = 0$ so that the extended function $u(x)$ is continuous on R^1.) Here X_t is a one–dimensional continuous strong Markov Process which is regular on the interval $[c_1, c_2]$ [4]. This means that every interior point is accessible. It is well known [3] that the infinitesimal operator A is equal to the restriction of the characteristic operator $\mathfrak{U}$, which is defined by

$$\mathfrak{U} f(x) = \lim_{a_1 \uparrow x, a_2 \downarrow x} \frac{E_x f[X_{\tau(a_1, a_2)}] - f(x)}{E_x \tau(a_1, a_2)}, \qquad \text{for } c_1 \leq a_1 \leq x \leq a_2 \leq c_2, \qquad (2.9)$$

where $\tau(a_1, a_2) = \inf\{t : X_t \notin (a_1, a_2)\}$.

For $c_1 \leq a_1 \leq x \leq a_2 \leq c_2$, the function $m(x) = E_x \tau(c_1, c_2)$ is the solution of the equation $D_v D_u m(x) = -1$ with boundary conditions $m(c_1) = m(c_2) = 0$, where $\tau(c_1, c_2) = \inf\{t : X_t \notin (c_1, c_2)\}$. Let $\phi_i(x, a_1, a_2) = P_x(X_{\tau(a_1, a_2)} = a_i)$ and $m(x, a, b) = E_x \tau(a, b)$, where $x \in (a, b)$. Then

$$m(x) = m(x, a_1, a_2) + \phi_1(x, a_1, a_2) m(a_1) + \phi_2(x, a_1, a_2) m(a_2).$$

According to equation (2.9),

$$
\begin{aligned}
D_v D_u F(0) &= \lim_{a_1 \uparrow 0, a_2 \downarrow 0} \frac{E_{x=0} F(X_{\tau(a_1,a_2)}) - F(0)}{E_{x=0}\tau(a_1,a_2)} \\
&= \lim_{a_1 \uparrow 0, a_2 \downarrow 0} \frac{P(X_\tau = a_1)F(a_1) + P(X_\tau = a_2)F(a_2) - F(0)}{m(0,a_1,a_2)} \quad (2.10) \\
&= \lim_{a_1 \uparrow 0, a_2 \downarrow 0} \frac{P(X_\tau = a_1)F(a_1) + P(X_\tau = a_2)F(a_2) - F(0)}{m(0) - P(X_\tau = a_1)m(a_1) - P(X_\tau = a_2)m(a_2)}.
\end{aligned}
$$

The numerator of (2.10) is

$$
\begin{aligned}
&P(X_\tau = a_1)F(a_1) + P(X_\tau = a_2)F(a_2) - F(0) \\
&= \frac{u(a_2)-u(0)}{u(a_2)-u(a_1)}F(a_1) + \frac{u(0)-u(a_1)}{u(a_2)-u(a_1)}F(a_2) - F(0) \\
&= \frac{1}{u(a_2)-u(a_1)}\{(u(a_2)-u(0))F(a_1) + (u(0)-u(a_1))F(a_2) \\
&\qquad\qquad\qquad\qquad -(u(a_2)-u(a_1))F(0) + F(0)u(0) - F(0)u(0)\} \\
&= \frac{1}{u(a_2)-u(a_1)}\{(u(a_2)-u(0))(F(a_1)-F(0)) + (u(0)-u(a_1))(F(a_2)-F(0))\}.
\end{aligned}
$$

Similarly, the denominator of (2.10) is

$$
\begin{aligned}
m(0,a_1,a_2) &= m(0) - P(X_\tau = a_1)m(a_1) - P(X_\tau = a_2)m(a_2) \\
&= m(0) - \frac{u(a_2)-u(0)}{u(a_2)-u(a_1)}m(a_1) - \frac{u(0)-u(a_1)}{u(a_2)-u(a_1)}m(a_2) \\
&= \frac{1}{u(a_2)-u(a_1)}\{-(u(a_2)-u(0))(m(a_1)-m(0)) \\
&\qquad\qquad\qquad\qquad +(u(0)-u(a_1))(m(0)-m(a_2))\}
\end{aligned}
$$

So, by dividing both parts by $(u(a_2)-u(0))(u(0)-u(a_1))$, and since $D_v D_u m(x) = -1$, we have

$$
(D_u^+ F(0) - D_u^- F(0)) = (v(0) - v(0-))D_v D_u F(0).
$$

From the fact that $u(x)$ and $v(x)$ are antisymmetric with respect to the point 0, it is easy to check $D_u^+ F(0) = -D_u^- F(0)$. So the boundary condition is

$$
D_u^+ F(0) - \frac{v(0) - v(0-)}{2} D_v D_u F(0) = 0.
$$

19

Therefore γ is equal to $(v(0) - v(0-))/2$. Similarly, for each n, $\gamma_n = (v_n(0) - v_n(0-))/2$, so $\gamma_n \to \gamma$ as $n \to \infty$. The proof of necessity is complete.

Let us prove the sufficiency of (2.1)–(2.3).

Assume that (2.1), (2.2), and (2.3) are satisfied. Let $T_t^{(n)}$ and T_t be semigroups associated with $X_t^{(n)}$ and X_t respectively. One can verify that $T_t^{(n)}$ (also T_t) is a strongly continuous contraction semigroup through the Hille–Yosida theorem ([5], Theorem 1.2.6). The weak convergence of the processes $X_t^{(n)}$ as $n \to \infty$ to a continuous Markov process X_t with the Feller property is equivalent to

$$\lim_{n \to \infty} \mid T_t^{(n)} f(x) - T_t f(x) \mid = 0, \text{ uniformly in } t, t \geq 0. \tag{2.11}$$

for all bounded continuous functions $f(x) \in B$. According to the "Trotter–Kato" theorem, the following result implies (2.12) :

$$\lim_{n \to \infty} \mid (\lambda - D_{v_n} D_{u_n})^{-1} f(x) - (\lambda - D_v D_u)^{-1} f(x) \mid = 0, \tag{2.12}$$

for every bounded continuous function $f(x) \in B$ vanishing at infinity and for all $\lambda > 0$. Therefore we will prove (2.12), instead of (2.11).

Consider the following homogeneous and inhomogeneous equations.

$$\lambda G_n(x) - D_{v_n} D_{u_n} G_n(x) = 0, \qquad \lambda F_n(x) - D_{v_n} D_{u_n} F_n(x) = f(x) \tag{2.13}$$

$$\lambda G(x) - D_v D_u G(x) = 0, \qquad \lambda F(x) - D_v D_u F(x) = f(x) \tag{2.14}$$

where $f(x)$ is a bounded continuous function and λ is a positive constant.

Under our assumptions, for each n, there exists the unique bounded solution $F_n(x)$ of the inhomogeneous differential equation in (2.13) given by

$$F_n(x) = \int G_\lambda^{(n)}(x, y) f(y) dv_n(y) + c_n \widehat{G}_{1n}(x) + d_n \widehat{G}_{2n}(x) \tag{2.15}$$

where $G_\lambda^{(n)}(x, y)$ is the Green function associated with the differential operator $D_{v_n}D_{u_n}$ ([7]). Here each $\widehat{G}_{1n}(x)$ and $\widehat{G}_{2n}(x)$ are a decreasing and an increasing solutions to the homogeneous equation in (2.14), and c_n and d_n are constants. Put $\widehat{F}_n(x) = \int G_\lambda^{(n)}(x, y) f(y) dv_n(y)$. Since the point 0 is the only accessible and regular boundary point, d_n is equal to zero and the general solution of the inhomogeneous equation (2.13) is of the form

$$F_n(x) = \widehat{F}_n(x) + c_n \hat{G}_{1n}(x), \tag{2.16}$$

where c_n is a constant, $\lambda\widehat{F}_n(x) - D_{v_n}D_{u_n}\widehat{F}_n(x) = f(x)$, and $\lambda\widehat{G}_{1n}(x) - D_{v_n}D_{u_n}\widehat{G}_{1n}(x) = 0$. Here the inhomogeneous equation (2.13) will have a unique solution with boundary condition $\Phi^{(n)}(F_n) = 0$. That is, the constant c_n will be uniquely determined by the boundary condition $D_{u_n}F_n(0) - \gamma_n D_{v_n}D_{u_n}F_n(0) = 0$. It is known ([19]) that for $\lambda > 0$, $\widehat{G}_{1n}(x)$ is positive and strictly decreasing. Therefore $D_{u_n}\widehat{G}_{1n}(0) - \gamma_n D_{v_n}D_{u_n}\widehat{G}_{1n}(0) < 0$ and c_n can be uniquely determined

$$c_n = -\frac{D_{u_n}\widehat{F}_n(0) - \gamma_n D_{v_n}D_{u_n}\widehat{F}_n(0)}{D_{u_n}\widehat{G}_{1n}(0) - \gamma_n D_{v_n}D_{u_n}\widehat{G}_{1n}(0)}. \tag{2.17}$$

Similarly the general solution of the inhomogeneous equation (2.14) is of the form $F(x) = \widehat{F}(x) + c\widehat{G}_1(x)$, where $\lambda\widehat{F}(x) - D_vD_u\widehat{F}(x) = f(x)$ and $\lambda\widehat{G}_1(x) - D_vD_u\widehat{G}_1(x) = 0$. Also, for $\lambda > 0$, $\widehat{G}_1(x)$ is positive and strictly decreasing. The constant c can be found as before to be

$$c = -\frac{D_u\hat{F}(0) - \gamma D_vD_u\hat{F}(0)}{D_u\hat{G}(0) - \gamma D_vD_u\hat{G}(0)}. \tag{2.18}$$

Assuming (2.1) and (2.2), it was shown ([14]) that $\widehat{G}_{1n}(x)$ and $\widehat{F}_n(x)$ converge uniformly to $\widehat{G}_1(x)$ and $\widehat{F}(x)$ respectively. Furthermore, the uniform convergence of

21

$\widehat{F}_n(x)$ to $\widehat{F}(x)$ and of $\widehat{G}_{1n}(x)$ to $\widehat{G}_1(x)$ as $n \to \infty$, and assumptions (2.1), (2.2), and (2.3) imply that c_n converges uniformly to c as $n \to \infty$. Therefore, the general solution $F_n(x) = \widehat{F}_n(x) + c_n \widehat{G}_{1n}(x)$ converges uniformly to $F(x) = \widehat{F}(x) + c\widehat{G}_1(x)$. That is, $\lim_{n\to\infty} F_n(x) = F(x)$. So, from the inhomogeneous equations of (2.13) and (2.14), $\lim_{n\to\infty} D_{v_n} D_{u_n} F_n(x) = D_v D_u F(x)$. According to [5], Theorem 1.6.1, the convergence of the infinitesimal operator is equivalent to the convergence of the strongly continuous contraction semigroup $T_t^{(n)}$ to the strongly continuous contraction semigroup T_t corresponding to the $D_v D_u$-operator of the limiting process X_t. It is also known ([5], Theorem 4.2.5) that the convergence of the semigroup $T_t^{(n)} f(x)$ to the semigroup $T_t f(x)$ implies the weak convergence of $P_x^{(n)}$ to P_x. $\square$

In the following theorem, the necessary and sufficient conditions for weak convergence in the closed interval $[r_0, r_1]$ will be shown, following the methodology of Theorem 1.

Theorem 5 *Let $u(x)$ be a strictly increasing continuous function and let $v(x)$ be a strictly increasing right continuous function on $[r_0, r_1]$ with the corresponding process (X_t, P_x). Let P_x be the measure on C_{0T} corresponding to X_t. Assume that all functions $F(x)$ in the domain of the generator of (X_t, P_x) satisfy the following boundary conditions :*

$$\Phi_0(F) = D_u F(r_0) - \eta^1 D_v D_u F(r_0) = 0,$$
$$\Phi_1(F) = D_u F(r_1) + \eta^2 D_v D_u F(r_1) = 0,$$

(2.19)

where η^1 and η^2 are nonnegative.

22

Let $u_n(x)$ and $v_n(x)$ be a sequence of increasing functions on $[r_0, r_1]$. Let $(X_t^{(n)}, P_x^{(n)})$, for each n, be a $D_{v_n}D_{u_n}$–process on $[r_0, r_1]$. Let $P_x^{(n)}$ be the measure on C_{0T} corresponding to X_t. Assume that all functions $F_n(x)$ in the domain of the generator of $(X_t^{(n)}, P_x^{(n)})$ satisfy the following boundary conditions at the boundary points :

$$\Phi_0^{(n)}(F_n) = D_{u_n} F_n(r_0) - \eta_n^1 D_{v_n} D_{u_n} F_n(r_0) = 0,$$
$$\Phi_1^{(n)}(F_n) = D_{u_n} F_n(r_1) + \eta_n^2 D_{v_n} D_{u_n} F_n(r_1) = 0,$$
(2.20)

where η_n^1 and η_n^2 are nonnegative.

For the weak convergence of probability $P_x^{(n)}$ to P_x as $n \to \infty$ for all $x \in [r_0, r_1]$, it is necessary after an appropriate choice of $u_n(x)$ and $v_n(x)$, and sufficient that

$$u_n(x) \to u(x) \qquad \textit{for all } x \in [r_0, r_1], \tag{2.21}$$

$$v_n(x) \to v(x) \qquad \textit{for all } x \in [r_0, r_1] \textit{ that are continuity points of } v, \tag{2.22}$$

$$\eta_n^1 \to \eta^1, \qquad \eta_n^2 \to \eta^2. \tag{2.23}$$

Proof. Most of the proof is similar to the proof of Theorem 1. Therefore we will only outline the proof of sufficiency which differs from the proof of Theorem 1. Assume that (2.21), (2.22), and (2.23) are true. Let

$$(\lambda - D_{v_n} D_{u_n})F_n(x) = f(x), \tag{2.24}$$

$$(\lambda - D_v D_u)F(x) = f(x), \tag{2.25}$$

where $f(x)$ is a bounded continuous function. Then, for each n, the general solution of (2.24) will be

$$F_n(x) = \hat{F}_n(x) + c_{1n}\hat{G}_{1n}(x) + c_{2n}\hat{G}_{2n}(x),$$

where $(\lambda - D_{v_n}D_{u_n})\hat{F}_n(x) = f(x)$, and both $\hat{G}_{1n}(x)$ and $\hat{G}_{2n}(x)$ are positive solutions of the homogeneous equation $(\lambda - D_{v_n}D_{u_n})\hat{G}_{in}(x) = 0$ for $i = 1, 2$. Here $\hat{G}_{1n}(x)$ is strictly decreasing and $\hat{G}_{2n}(x)$ is strictly increasing, and $-D_{u_n}\hat{G}_{1n}(r_0) > D_{u_n}\hat{G}_{2n}(r_0)$ and $D_{u_n}\hat{G}_{2n}(r_1) > -D_{u_n}\hat{G}_{1n}(r_1)$. (For details one may refer to [19]). Then one can check $| \Phi_0^{(n)}(\hat{G}_{1n})\Phi_1^{(n)}(\hat{G}_{2n}) - \Phi_0^{(n)}(\hat{G}_{2n})\Phi_1^{(n)}(\hat{G}_{1n}) | > 0$. Similarly we can find the general solution of (2.25) to be

$$F(x) = \hat{F}(x) + c_1\hat{G}_1(x) + c_2\hat{G}_2(x)$$

where $(\lambda - D_vD_u)\hat{F}(x) = f(x)$, $(\lambda - D_vD_u)\hat{G}_i(x) = 0$ for $i = 1, 2$. Here $\hat{G}_1(x)$ is positive and strictly decreasing, and $\hat{G}_2(x)$ is positive and strictly increasing.

In the general solution $F_n(x)$ and $F(x)$, the constants c_{1n}, c_{2n} and c_1, c_2 are uniquely determined by the boundary conditions (2.19) and (2.20) respectively ([19]). Then it is not difficult to prove $c_{1n} \to c_1$ and $c_{2n} \to c_2$ as $n \to \infty$. Therefore, the general solution $F_n(x)$ converges to $F(x)$ as $n \to \infty$. So, the convergence of $D_{v_n}D_{u_n}F_n(x)$ to $D_vD_uF(x)$ is complete which implies weak convergence. $\square$

Consider

$$L_nf = a_n(x)\frac{d^2f}{dx^2} + b_n(x)\frac{df}{dx} \quad \text{for } f \in C_o^2(R^1), \tag{2.26}$$

where $a_n(x)$ and $b_n(x)$ are continuous functions in R^1.

Define the limits of $a_n(x)$ and $b_n(x)$ as

$$\lim_{n\to\infty} a_n(x) = a(x) = \begin{cases} a_+(x) & x > 0, \\ a_-(x) & x < 0, \end{cases}$$

$$\lim_{n\to\infty} b_n(x) = b(x) = \begin{cases} b_+(x) & x > 0, \\ b_-(x) & x < 0, \end{cases}$$

where $a(x)$ and $b(x)$ may be undefined or may be discontinuous at $x = 0$. For simplicity we assume that there exists only one point $x = 0$ that is a singular point in R^1.

Define

$$P_r = \lim_{\delta\downarrow 0} \lim_{n\to\infty} P_x(X_\tau^{(n)} = \delta), \qquad P_l = \lim_{\delta\downarrow 0} \lim_{n\to\infty} P_x(X_\tau^{(n)} = -\delta)$$

$$Q = \lim_{\delta\downarrow 0} \lim_{n\to\infty} \tfrac{1}{\delta} E_x \tau^{(n)}(-\delta, \delta)$$

where $\tau^{(n)}(-\delta, \delta) = \inf\{t : X_t^{(n)} \notin (-\delta, \delta)\}$ and $x \in (-\delta, \delta)$.

Define $u(x)$ and $v(x)$ as follows :

$$u(x) = \begin{cases} (1/P_r) \int_0^x \exp\{-\int_0^y (b(z)/a(z))dz\}dy, & x \geq 0 \\ (1/P_l) \int_0^x \exp\{-\int_0^y (b(z)/a(z))dz\}dy, & x < 0, \end{cases}$$

$$v(x) = \begin{cases} Q + P_r \int_0^x a_+(y)^{-1} \exp\{\int_0^y (b(z)/a(z))dz\}dy, & x \geq 0 \\ P_l \int_0^x a_-(y)^{-1} \exp\{\int_0^y (b(z)/a(z))dz\}dy, & x < 0. \end{cases} \qquad (2.27)$$

Then $D_v D_u f(x)$ is defined by

$$D_v D_u f(x) = \begin{cases} a_+(x)(d^2 f/dx^2) + b_+(x)(df/dx), & x > 0 \\ a_-(x)(d^2 f/dx^2) + b_-(x)(df/dx), & x < 0 \end{cases}$$

$$Q D_v D_u f(x) = P_r f'_+(x) - P_l f'_-(x), \qquad x = 0$$

where $D_v D_u f(0) = \lim_{x\to 0+}(a_+(x)\frac{d^2 f}{dx^2} + b_+(x)\frac{df}{dx}) = \lim_{x\to 0-}(a_-(x)\frac{d^2 f}{dx^2} + b_-(x)\frac{df}{dx})$.

- Assumption (A1): limits P_r, P_l, and Q exist.

- Assumption (A2): $\lim_{n\to\infty} \Lambda_n(x) = \lim_{n\to\infty} \exp\{-\int_0^x (b_n(z)/a_n(z))dz\}$ exists for $x \in R^1$.

However, we would like to point out that $\lim_{n\to\infty} \Lambda_n(x)$ is not always equal to $\Lambda(x) = \exp\{-\int_0^x (b(z)/a(z))dz\}$ since there exists a singular point at $x = 0$. That is, $a_n(x)$ and $b_n(x)$ do not converge uniformly to $a(x)$ and $b(x)$ respectively. For $\epsilon > 0$, define $u'_+(0)$ and $u'_-(0)$ respectively as

$$u'_+(0) = \lim_{\epsilon\to 0} \lim_{n\to\infty} u'_n(\epsilon), \qquad u'_-(0) = \lim_{\epsilon\to 0} \lim_{n\to\infty} u'_n(-\epsilon),$$

where $u_n(x)$ and $v_n(x)$ are sequences of increasing functions such that $L_n f = D_{v_n} D_{u_n} f$. If $\lim_{n\to\infty} \Lambda_n(x)$ exists, then $u'_+(0)$ and $u'_-(0)$ exist even though $u'_+(0)$ may not be equal to $u'_-(0)$. If $\lim_{n\to\infty} \Lambda_n(x)$ does not exist, then the problem is more challenging.

Theorem 6 *With assumptions (A1) and (A2), let $P_x^{(n)}$ be the measure in the space C_{0T} corresponding to $X_t^{(n)}$ governed by $A_n f = L_n f$, for $f \in \mathcal{D}(A_n)$. Let P_x be the measure on the space C_{0T} corresponding to a process X_t governed by the $D_v D_u-$ operator where $v(x)$ and $u(x)$ are defined by (2.27). Then $P_x^{(n)}$ converges weakly in C_{0T} to P_x as $n \to \infty$.*

Proof. From $D_{v_n} D_{u_n} f = a_n(x)(d^2 f/dx^2) + b_n(x)(df/dx)$ for $f \in C_o^2(R)$, $u_n(x)$ and $v_n(x)$ are written as

$$
\begin{aligned}
u_n(x) &= \int_0^x \exp\{-\int_0^y (b_n(z)/a_n(z))dz\}dy, \\
v_n(x) &= \int_0^x a_n(y)^{-1} \exp\{\int_0^y (b_n(z)/a_n(z))dz\}dy.
\end{aligned}
$$

According to [14], it is enough to show that $u_n(x)$ and $v_n(x)$ converge uniformly to $u(x)$ and $v(x)$ respectively. From the definition of the $D_v D_u$-operator, we have

$$\lim_{\delta \to 0}(v(\delta) - v(-\delta))D_v D_u f(0) = D_u^+ f(0) - D_u^- f(0).$$

For $f \in \mathcal{D}(D_v D_u)$, the gluing condition is given at $x = 0$ as

$$QD_v D_u f(0) = P_r f_+'(0) - P_l f_-'(0).$$

Since $\frac{P_l}{P_r} = \frac{u_+'(0)}{u_-'(0)}$, we have the following ratio

$$\frac{\lim_{\delta \downarrow 0}\{v(\delta) - v(-\delta)\}}{Q} = \frac{1}{P_r u_+'(0)} = \frac{1}{P_l u_-'(0)} = K,$$

where K is a positive constant. As mentioned before, even though we multiply one of functions $u(x)$ and $v(x)$ by some positive number, and divide the other by the same number, the operator $D_v D_u$ is still same. So, we multiply $v(x)$ by K and divide $u(x)$ by K. Therefore $u(x)$ and $v(x)$ are rewritten as follows

$$u(x) = \begin{cases} u_+'(0) \int_0^x \exp\{-\int_0^y (b(z)/a(z))dz\}dy, & x \geq 0 \\ u_-'(0) \int_0^x \exp\{-\int_0^y (b(z)/a(z))dz\}dy, & x < 0, \end{cases}$$

$$v(x) = \begin{cases} v(0) + (1/u_+'(0)) \int_0^x (a_+(y))^{-1} \exp\{\int_0^y (b(z)/a(z))dz\}dy, & x \geq 0 \\ (1/u_-'(0)) \int_0^x (a_-(y))^{-1} \exp\{\int_0^y (b(z)/a(z))dz\}dy, & x < 0. \end{cases}$$

Then, for $x > 0$ and $\delta > 0$

$$\lim_{n \to \infty} | u_n(x) - u(x) |$$

$$= \lim_{n \to \infty} | \int_0^x e^{-\int_0^y \frac{b_n(z)}{a_n(z)}dz}dy - u_+'(0) \int_0^x e^{-\int_0^y \frac{b(z)}{a(z)}dz}dy |$$

$$= \lim_{n \to \infty} | e^{-\int_0^\delta \frac{b_n(z)}{a_n(z)}} \int_0^x e^{-\int_\delta^y \frac{b_n(z)}{a_n(z)}dz}dy - u_+'(0) \int_0^x e^{-\int_0^y \frac{b(z)}{a(z)}dz}dy |$$

$$= \lim_{\delta \to 0} \lim_{n \to \infty} | u_n'(\delta) \int_0^x e^{-\int_\delta^y \frac{b_n(z)}{a_n(z)}dz}dy - u_+'(0) \int_0^x e^{-\int_0^y \frac{b(z)}{a(z)}dz}dy |= 0.$$

The last equality is true because $a_n(x)$ and $b_n(x)$ converge uniformly to $a(x)$ and $b(x)$ respectively for $x \in (-\infty, -\delta]$ and $x \in [\delta, \infty)$ for $\delta > 0$. Hence $u_n(x)$ converges uniformly to $u(x)$. Similarly it can be proven that $\lim_{n \to \infty} \mid v_n(x) - v(x) \mid = 0$. $\square$

It is well known that the uniform convergence of $a_n(x)$ and $b_n(x)$ immediately implies the uniform convergence of $u_n(x)$ and $v_n(x)$ at continuous points of $v(x)$. Then, from the main theorem in [14], the weak convergence of $P_x^{(n)}$ to P_x follows. We want to emphasize that this theorem is meaningful even though $a_n(x)$ and $b_n(x)$ do not converge uniformly to $a(x)$ and $b(x)$ respectively under our assumptions.

Lastly, let us consider the following Cauchy problems.

$$
\begin{cases}
\frac{\partial f_n(t,x)}{\partial t} & = \quad L_n f_n + c_n(x) f_n \\
\\
f_n(0,x) & = \quad g(x),
\end{cases}
\tag{2.28}
$$

where $D_{v_n} D_{u_n} f_n = L_n f_n = \frac{1}{2} a_n(x) \frac{\partial^2 f_n}{\partial x^2} + b_n(x) \frac{\partial f_n}{\partial x}$, and $c_n(x)$ is a bounded continuous function on R. For a bounded continuous function $g(x)$, the solution of the problem (2.28) can be written in the form

$$
f_n(t,x) = E_x \left(g(X_t^{(n)}) \exp \left\{ \int_0^t c_n(X_s^{(n)}) ds \right\} \right).
\tag{2.29}
$$

If $c_n(x)$ converges uniformly to $c(x)$, and $a_n(x)$ and $b_n(x)$ satisfy assumptions of Theorem 6, then it is easy to see that the limit of $f_n(t,x)$ converges to the solution of the following problem:

$$
\begin{cases}
\frac{\partial f(t,x)}{\partial t} & = \quad D_v D_u f + c(x) f \\
\\
f(0,x) & = \quad g(x),
\end{cases}
\tag{2.30}
$$

where $u(x) = \lim_{n \to \infty} u_n(x)$ and $v(x) = \lim_{n \to \infty} v_n(x)$. However, if $c_n(x)$ converges to $c(x)$ in a weaker sense, it cannot be guarantee that solutions $f_n(t,x)$ converge to $f(t,x)$. So, we will finish this subsection with a counterexample for this case. For simplicity, assume that diffusion coefficients $a_n(x)$ and $a(x)$ are equal to 1 and a drift coefficient $b(x)$ is given by the following picture.
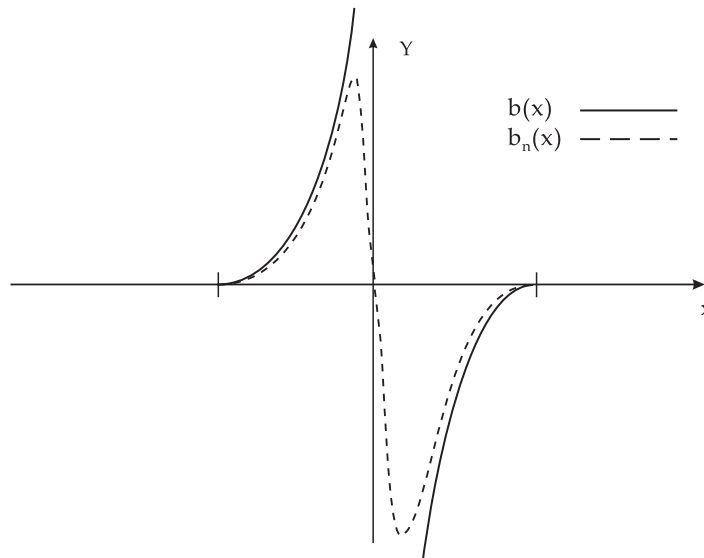
Figure 2.1: Functions $b(x)$ and $b_n(x)$.

Then, for each n, we can construct continuous functions $b_n(x)$ throughout the interval $[-1, 1]$, which converge pointwise to a function $b(x)$ as $n \to \infty$. From (2.28), the first derivative of the function $v_n(x)$ is $v'_n(x) = 2 \exp \left\{ \int_{-1}^{x} b_n(z) dz \right\}$. Thus we can see that the first derivative of function $v_n(x)$ at $x = 0$ with a function $b_n(x)$ converges to a delta function as $n \to \infty$. That is, the processes $X_t^{(n)}$ have some delay proportional to the difference of the change of the function $v_n(x)$ near the point $x = 0$. Let us choose functions $c_n(x)$ which converge pointwise to a function

29

$c(x)$ as follows :

$$c_n(x) = (1- \mid x \mid)^n, \quad \text{for each } n,$$

$$c(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{2.31}$$

Then $\int_0^t c_n(X_s^{(n)})ds = \tau_0^n$ is a strictly positive value, where τ_0^n indicates the exit time of the neighborhood of a point $x = 0$ and $\int_0^t c_n(X_s^{(n)})ds$ does not converge to $\int_0^t c(X_s)ds = 0$ as $n \to \infty$. So, even thought we have the weak convergence of $P_x^{(n)}$ to P_x, the solutions $f_n(t,x)$ of problem (2.28) does not converge to a solution $f(t,x)$ of problem (2.30) unless $c_n(x)$ converges uniformly to $c(x)$.

2.2 Some applications of weak convergence of $D_v D_u$–processes.

Let $h^\delta(x)$ and $k^\delta(x)$, for any $\delta > 0$, be smooth positive functions in R^1 as shown in Figure 1 and define the strip $G^{\varepsilon,\delta}$ in R^2 as follows

$$G^{\varepsilon,\delta} = \{(x,y) \in R^2 : -\varepsilon k^\delta(x) \leq y \leq \varepsilon h^\delta(x)\}, \text{ for } \varepsilon, \delta > 0.$$
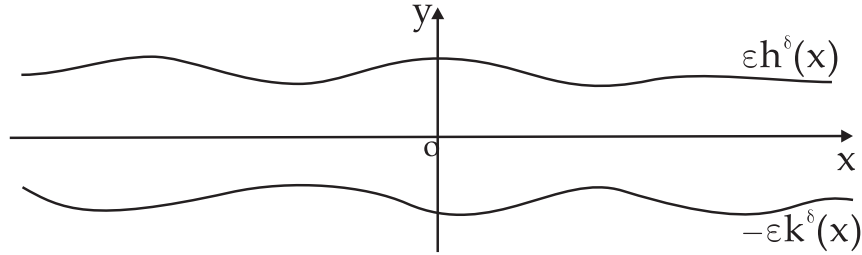

Figure 2.2: The strip $G^{\varepsilon,\delta}$.

Let $\varepsilon l^\delta(x)$ be the width of the cross–section of the strip $G^{\varepsilon,\delta}$ at x, that is,

$$l^\delta(x) = h^\delta(x) + k^\delta(x).$$

Let $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ be the Wiener process in $G^{\varepsilon,\delta}$ with a normal reflection on the boundary. The process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ is a solution to the stochastic differential equations

$$\begin{aligned}
dX_t^{\varepsilon,\delta} &= dW_t^1 + \gamma_1^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta}, \\
dY_t^{\varepsilon,\delta} &= dW_t^2 + \gamma_2^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta},
\end{aligned} \tag{2.32}$$

where W_t^1 and W_t^2 are independent Wiener processes in R^1, and $\gamma^{\varepsilon,\delta} = (\gamma_1^{\varepsilon,\delta}(x,y), \gamma_2^{\varepsilon,\delta}(x,y))$ is the unit inward normal to $\partial G^{\varepsilon,\delta}$. That is, $\gamma_1^{\varepsilon,\delta}$ and $\gamma_2^{\varepsilon,\delta}$ are projections of the unit inward normal vector to $\partial G^{\varepsilon,\delta}$ on corresponding axes. Moreover $L_t^{\varepsilon,\delta}$ is the local time for the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ on $\partial G^{\varepsilon,\delta}$. The local time is a continuous, non–decreasing process which increases only when the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ touches the boundary of $G^{\varepsilon,\delta}$ ([16]). As $\varepsilon \downarrow 0$, the component $X_t^{\varepsilon,\delta}$ becomes a slow motion and the component $Y_t^{\varepsilon,\delta}$ becomes a fast motion in $G^{\varepsilon,\delta}$.

Theorem 7 *Assume that strictly increasing functions $u(x)$ and $v(x)$ exist, $u(x)$ is continuous and $v(x)$ is right continuous such that $u^\delta(x) = \int_0^x (1/l^\delta(y))dy$ converges to $u(x)$ as $\delta \downarrow 0$ for each x, and $v^\delta(x) = \int_0^x 2(l^\delta(y))dy$ converges to $v(x)$ as $\delta \downarrow 0$ at each continuity point of $v(x)$. Let X_t be the process governed by the $D_v D_u$–operator. Then the component $X_t^{\varepsilon,\delta}$ converges weakly to X_t as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.*

Proof. The process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ is a solution to the stochastic differential equation (2.32) with

$$\gamma^{\varepsilon,\delta}(x,y) = \begin{cases} \frac{1}{\sqrt{1+(\varepsilon h'^\delta(x))^2}}(\varepsilon h'^\delta(x), -1), & \text{for } y = \varepsilon h^\delta(x) \\[2mm] \frac{1}{\sqrt{1+(\varepsilon k'^\delta(x))^2}}(\varepsilon k'^\delta(x), 1), & \text{for } y = -\varepsilon k^\delta(x). \end{cases}$$

31

It was proven ([11]) (see also [12]) that, as $\varepsilon \downarrow 0$, the component $X_t^{\varepsilon,\delta}$ converges to the diffusion process X_t^δ which satisfies the following stochastic differential equation

$$dX_t^\delta = dW_t^1 + b^\delta(X_t^\delta)dt, \qquad b^\delta(x) = \frac{1}{2}\frac{d}{dx}\ln(l^\delta(x)). \qquad (2.33)$$

Here the generator of X_t^δ is equal to a $D_{v^\delta}D_{u^\delta}$–operator, where $u^\delta(x) = \int_0^x(1/l^\delta(y))dy$ and $v^\delta(x) = \int_0^x 2(l^\delta(y))dy$. According to [14], since $\lim_{\delta \downarrow 0} \mid v^\delta(x) - v(x) \mid = 0$ at each continuity point of $v(x)$ and $\lim_{\delta \downarrow 0} \mid u^\delta(x) - u(x) \mid = 0$ at each point x, X_t^δ converges weakly to X_t, whose generator is a $D_v D_u$–operator, as $\delta \downarrow 0$ and the proof of Theorem 4 is complete. $\square$

Lastly we will show some examples of Theorem 4. In these following examples three different strips $G^{\varepsilon,\delta}$ will be considered (in the second and third case, we have one more small parameter ϵ_1. So in these cases we denote the strip by $G^{\varepsilon,\delta,\epsilon_1}$ instead of $G^{\varepsilon,\delta}$).

Examples (1). Let $G^{\varepsilon,\delta} = \{(x,y) \in R^2 : -\varepsilon k^\delta(x) \le y \le \varepsilon h^\delta(x)\}$, for $\varepsilon, \delta > 0$. Assume that, as $\delta \downarrow 0$, the smooth positive functions $h^\delta(x)$ and $k^\delta(x)$ have following properties, respectively, (see Figure 2-(1))

$$(h^\delta(x)/\delta) \longrightarrow \begin{cases} K_1, & x < 0 \\ K_3, & x > 0, \end{cases} \qquad (k^\delta(x)/\delta) \longrightarrow \begin{cases} K_2, & x < 0 \\ K_4, & x > 0, \end{cases}$$

where K_1, K_2, K_3, K_4 are positive constants. Let $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ be the Wiener process in $G^{\varepsilon,\delta}$ with normal reflection on the boundary.

Let X_t be governed by the infinitesimal operator $Af = (1/2)f''$, for $f \in \mathcal{D}(A)$

32

with

$$\mathcal{D}(A) = \left\{ \begin{array}{c} f \in C_o(R): \quad f' \text{ and } f'' \text{ exist and are continuous except at zero }, \\[2mm] \lim_{x\to 0+}(K_3 + K_4)f'_+(x) = \lim_{x\to 0-}(K_1 + K_2)f'_-(x), \\[2mm] \text{and } \lim_{x\to 0+} f''_+(x) = \lim_{x\to 0-} f''_-(x) \end{array} \right\}.$$

Then $X_t^{\varepsilon,\delta}$ converges weakly to X_t as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.

Figure 2.3: $(h^\delta(x)/\delta)$, $(k^\delta(x)/\delta)$, $(h^{\delta,\epsilon_1}(x)/\delta)$, and $(k^{\delta,\epsilon_1}(x)/\delta)$ as $\delta \downarrow 0$.

(**2**). Let $G^{\varepsilon,\delta,\epsilon_1} = \{(x,y) \in R_+ \times R : 0 \leq y \leq \varepsilon h^{\delta,\epsilon_1}(x)\}$, where, as $\delta \downarrow 0$, the

smooth positive function $h^{\delta,\epsilon_1}(x)$ has the following property (see Figure 2-(2)) :

$$(h^{\delta,\epsilon_1}(x)/\delta) \longrightarrow \left\{ \begin{array}{ll} (K_1/\epsilon_1), & 0 \leq x \leq \epsilon_1 \\[2mm] K_2, & x > \epsilon_1, \end{array} \right.$$

33

where K_1 and K_2 are positive constants and $\epsilon_1 > 0$. The process $(X_t^{\varepsilon,\delta,\epsilon_1}, Y_t^{\varepsilon,\delta,\epsilon_1})$ is the Wiener process in $G^{\varepsilon,\delta,\epsilon_1}$ with normal reflection on the boundary.

Let X_t be governed by the infinitesimal operator $Af = (1/2)f''$, for $f \in \mathcal{D}(A)$ with

$$\mathcal{D}(A) = \left\{ \begin{array}{c} f \in C_o(R): \quad f' \text{ and } f'' \text{ exist and are continuous except at zero }, \\ \lim_{x \to 0+} K_2 f'_+(x) = \lim_{x \to 0+} K_1 f''_+(x) \\ \text{and } \lim_{x \to 0+} f''_+(x) = \lim_{x \to 0-} f''_-(x) \end{array} \right\}.$$

Then $X_t^{\varepsilon,\delta,\epsilon_1}$ converges weakly to X_t as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, $\epsilon_1 \downarrow 0$.
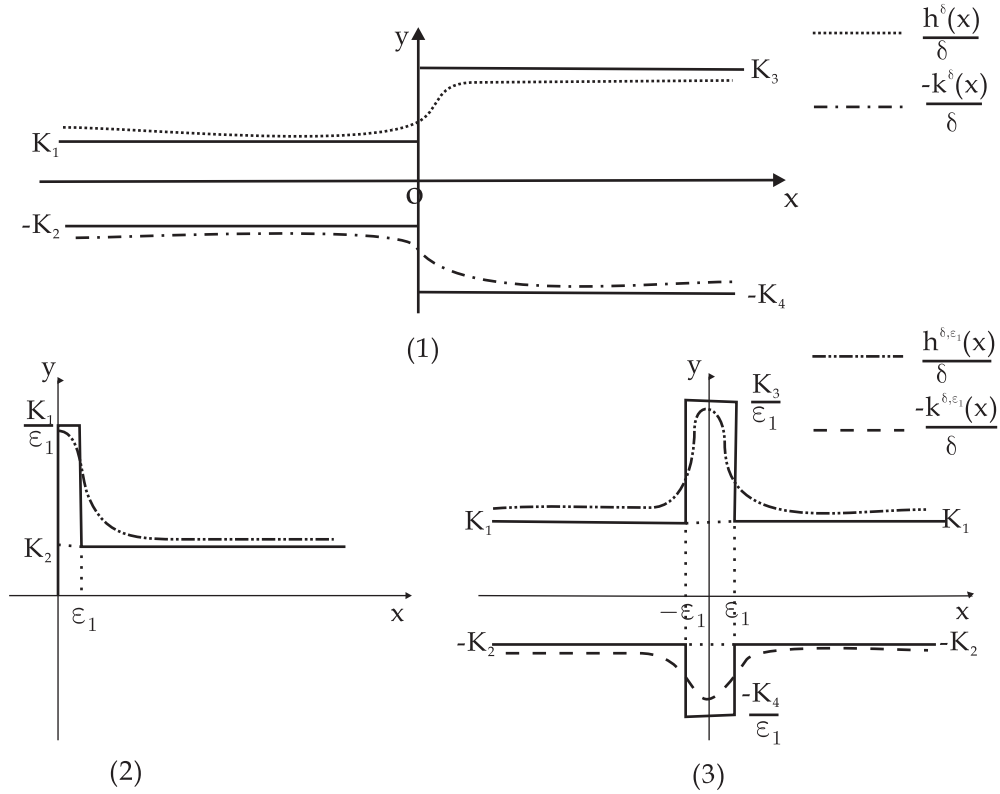
(**3**). Let $G^{\varepsilon,\delta,\epsilon_1} = \{(x,y) \in R^2 : -\varepsilon k^{\delta,\epsilon_1}(x) \leq y \leq \varepsilon h^{\delta,\epsilon_1}(x)\}$, where, as $\delta \downarrow 0$, smooth positive functions $h^{\delta,\epsilon_1}(x)$ and $k^{\delta,\epsilon_1}(x)$ have following properties (see Figure 2-(3)) :

$$(h^{\delta,\epsilon_1}(x)/\delta) \longrightarrow \left\{ \begin{array}{ll} K_1, & |x| > \epsilon_1 \\ (K_3/\epsilon_1), & |x| \leq \epsilon_1, \end{array} \right. \qquad (k^{\delta,\epsilon_1}(x)/\delta) \longrightarrow \left\{ \begin{array}{ll} K_2, & |x| > \epsilon_1 \\ (K_4/\epsilon_1), & |x| \leq \epsilon_1, \end{array} \right.$$

where K_1, K_2, K_3, and K_4 are positive constants. The process $(X_t^{\varepsilon,\delta,\epsilon_1}, Y_t^{\varepsilon,\delta,\epsilon_1})$ is the Wiener process in $G^{\varepsilon,\delta,\epsilon_1}$ with normal reflection on the boundary.

Let X_t be governed by the infinitesimal operator $Af = (1/2)f''$, for $f \in \mathcal{D}(A)$ with

$$\mathcal{D}(A) = \left\{ \begin{array}{c} f \in C_o(R): \quad f' \text{ and } f'' \text{ exist and are continuous except at zero }, \\ \lim_{x \to 0+}(K_1 + K_2)f'_+(x) - \lim_{x \to 0-}(K_1 + K_2)f'_-(x) \\ = \lim_{x \to 0} 2(K_3 + K_4)f''(x), \\ \text{and } \lim_{x \to 0+} f''_+(x) = \lim_{x \to 0-} f''_-(x) \end{array} \right\}.$$

Then $X_t^{\varepsilon,\delta,\epsilon_1}$ converges weakly to X_t as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, $\epsilon_1 \downarrow 0$.

Proof (1). From the infinitesimal operator A and the gluing condition, $v(x)$ and $u(x)$ are determined as follows

$$v(x) = \begin{cases} 2(K_3 + K_4)x, & x \geq 0 \\ 2(K_1 + K_2)x, & x < 0, \end{cases} \qquad u(x) = \begin{cases} \frac{1}{K_3+K_4}x, & x \geq 0 \\ \frac{1}{K_1+K_2}x, & x < 0. \end{cases}$$

For $x < 0$,

$$\lim_{\delta \downarrow 0} \mid v^\delta(x) - v(x) \mid$$

$$= \lim_{\delta \downarrow 0} \mid \delta^{-1} \int_0^x 2(h^\delta(y) + k^\delta(y))dy - 2(K_1 + K_2)x \mid = 0.$$

and it is easy to check that $\lim_{\delta \downarrow 0} \mid u^\delta(x) - u(x) \mid = 0$. Similarly, for $x \geq 0$, it can be proven that $\lim_{\delta \downarrow 0} \mid v^\delta(x) - v(x) \mid = 0$ and $\lim_{\delta \downarrow 0} \mid u^\delta(x) - u(x) \mid = 0$. Therefore these results imply that $X_t^{\varepsilon,\delta}$ converges weakly to X_t.

(2). From the infinitesimal operator A and the gluing condition, $v(x)$ and $u(x)$ are determined as follows

$$v(x) = 2K_1 + 2K_2 x, \qquad u(x) = x/K_2 \quad \text{for } x > 0.$$

Let $(\widetilde{X}_t^{\varepsilon,\delta,\epsilon_1}, \widetilde{Y}_t^{\varepsilon,\delta,\epsilon_1})$ be a solution to the stochastic differential equation (2.32) with

$$\gamma^{\varepsilon,\delta,\epsilon_1}(x,y) = \begin{cases} \frac{1}{\sqrt{1+\varepsilon^2\widetilde{h}'^{\delta,\epsilon_1}(x)^2}}(\varepsilon\widetilde{h}'^{\delta,\epsilon_1}(x), -1), & \text{for } y = \varepsilon\widetilde{h}^{\delta,\epsilon_1}(x) \\ (0,1), & \text{for } y = 0 \end{cases}$$

where $\widetilde{h}^{\delta,\epsilon_1}(x) = h^{\delta,\epsilon_1}(x)$, for $x \geq 0$ and $\widetilde{h}^{\delta,\epsilon_1}(x) = h^{\delta,\epsilon_1}(-x)$, for $x < 0$. For the component $X_t^{\varepsilon,\delta,\epsilon_1}$ defined on the half–line $R_+ = \{x \geq 0\}$ with instantaneous reflection at zero, it is known (see [10], Ch.1.6) that $X_t^{\varepsilon,\delta,\epsilon_1}$ is $\mid \widetilde{X}_t^{\varepsilon,\delta,\epsilon_1} \mid$ in law.

Let $\widetilde{X}_t$ be governed by the $D_{\widetilde{v}}D_{\widetilde{u}}$–operator where for $x < 0$, $\widetilde{v}(x) = -v(-x)$ and $\widetilde{u}(x) = -u(-x)$, and for $x \geq 0$, $\widetilde{v}(x) = v(x)$ and $\widetilde{u}(x) = u(x)$. Then X_t is also $|\widetilde{X}_t|$ in law.

Therefore this problem is equivalent to proving that $\widetilde{X}_t^{\varepsilon,\delta,\epsilon_1}$ converges weakly to $\widetilde{X}_t$ as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$ and $\epsilon_1 \downarrow 0$.

As $\varepsilon \downarrow 0$, the component $\widetilde{X}_t^{\varepsilon,\delta,\epsilon_1}$ converges to the diffusion process $\widetilde{X}_t^{\delta,\epsilon_1}$ governed by the $D_{\widetilde{v}^{\delta,\epsilon_1}} D_{\widetilde{u}^{\delta,\epsilon_1}}$–operator, where $D_{\widetilde{v}^{\delta,\epsilon_1}} D_{\widetilde{u}^{\delta,\epsilon_1}} f = \frac{1}{2}f''(x) + \frac{1}{2}(d\ln(\widetilde{h}^{\delta,\epsilon_1}(x))/dx) \times f'(x)$ for $\delta > 0$ and $\epsilon_1 > 0$. Here $\widetilde{v}^{\delta,\epsilon_1}(x) = \int_0^x (2(\widetilde{h}^{\delta,\epsilon_1}(y))/\delta)dy$ for $x \geq 0$ and $\widetilde{v}^{\delta,\epsilon_1}(x) = \int_0^x (2(\widetilde{h}^{\delta,\epsilon_1}(y))/\delta)dy$ for $x < 0$, and $\widetilde{u}^{\delta,\epsilon_1}(x) = \int_0^x (\delta/\widetilde{h}^{\delta,\epsilon_1}(y))dy$ for $x \in R^1$. One can check that $\widetilde{v}^{\delta,\epsilon_1}(x)$ and $\widetilde{u}^{\delta,\epsilon_1}(x)$ are antisymmetric with respect to zero. Lastly it is not difficult to show that $\lim_{\epsilon_1 \downarrow 0} \lim_{\delta \downarrow 0} |\widetilde{v}^{\delta,\epsilon_1}(x) - \widetilde{v}(x)| = 0$ and $\lim_{\epsilon_1 \downarrow 0} \lim_{\delta \downarrow 0} |\widetilde{u}^{\delta,\epsilon_1}(x) - \widetilde{u}(x)| = 0$. Therefore $\widetilde{X}_t^{\delta,\epsilon_1}$ converges weakly to $\widetilde{X}_t$ as $\delta \downarrow 0$ and then $\epsilon_1 \downarrow 0$. The proof is completed in this case.

(3). From the infinitesimal operator A and the gluing condition, $v(x)$ and $u(x)$ are determined as follows

$$v(x) = \begin{cases} 4(K_3 + K_4) + 2(K_1 + K_2)x, & x \geq 0 \\ 2(K_1 + K_2)x, & x < 0, \end{cases} \qquad u(x) = \frac{1}{K_1 + K_2}x.$$

The proof can be complete in the same way as (1) and (2). $\square$

Chapter 3

KPP-Type Reaction Diffusion Equations in Narrow Domains

3.1   Generalized solution of Cauchy problem

For each $x \in R^1$ and $\delta > 0$, suppose that G_x^δ is a bounded domain in R^n with a smooth boundary ∂G_x^δ. Consider the domain $G^\delta = \{(x, y) : x \in R^1, y \in G_x^\delta\} \subset R \times R^n$. Assume that the boundary ∂G^δ of G^δ is smooth enough and denote by $\gamma^\delta(x, y)$ the inward unit normal to ∂G^δ. Consider a narrow tube $G^{\varepsilon,\delta} = \{(x, y) : x \in R^1, \varepsilon y \in G_x^\delta\}$ for $0 < \varepsilon \ll 1$ and the following problem in $G^{\varepsilon,\delta}$:

$$
\begin{cases}
\frac{\partial w^{\varepsilon,\delta}(t,x,y)}{\partial t} = \frac{1}{2}\triangle w^{\varepsilon,\delta} + f(x, y, w^{\varepsilon,\delta}), & \text{if } t > 0, (x, y) \in G^{\varepsilon,\delta} \\
\left.\frac{\partial w^{\varepsilon,\delta}(t,x,y)}{\partial \gamma^{\varepsilon,\delta}}\right|_{t>0,(x,y)\in\partial G^{\varepsilon,\delta}} = 0, & w^{\varepsilon,\delta}(0, x, y) = g(x),
\end{cases}
\tag{3.1}
$$

where $\gamma^{\varepsilon,\delta}$ is the inward unit normal to $\partial G^{\varepsilon,\delta}$ and $\triangle$ is the Laplacian in x and y. Here, the nonlinear term $f(x, y, w^{\varepsilon,\delta})$ is assumed to be bounded and nonnegative. We consider the nonlinearity of Kolmogorov, Petrovskii and Piskunov (KPP) type, that is, $f(x, y, 0) = f(x, y, 1) = 0$, $f(x, y, w^{\varepsilon,\delta}) > 0$ for $w^{\varepsilon,\delta} \in (0, 1)$, and $f(x, y, w^{\varepsilon,\delta}) < 0$ for $w^{\varepsilon,\delta} \notin [0, 1]$. Let $f(x, y, w^{\varepsilon,\delta}) = c(x, y, w^{\varepsilon,\delta})w^{\varepsilon,\delta}$ for $w^{\varepsilon,\delta} > 0$ and $c(x, y) = c(x, y, 0) = \max_{0 \le w^{\varepsilon,\delta} \le 1} c(x, y, w^{\varepsilon,\delta})$. Assume that the function $c(x, y, w^{\varepsilon,\delta})$ for $w^{\varepsilon,\delta} \in [0, \infty)$ is continuous and satisfies a Lipschitz condition in $w^{\varepsilon,\delta}$.

Consider the Markov process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ in $G^{\varepsilon,\delta}$ with a normal reflection on the boundary. The process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ is a solution to the stochastic differential

equation:

$$dX_t^{\varepsilon,\delta} = dW_t^1 + \gamma_1^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta},$$

$$dY_t^{\varepsilon,\delta} = dW_t^2 + \gamma_2^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})dL_t^{\varepsilon,\delta},$$

(3.2)

where W_t^1 and W_t^2 are independent Wiener processes in R^1 and R^n, respectively, and $\gamma^{\varepsilon,\delta} = (\gamma_1^{\varepsilon,\delta}(x,y), \gamma_2^{\varepsilon,\delta}(x,y))$ is the unit inward normal to $\partial G^{\varepsilon,\delta}$. That is, $\gamma_1^{\varepsilon,\delta}$ and $\gamma_2^{\varepsilon,\delta}$ are projections of the unit inward normal vector to $\partial G^{\varepsilon,\delta}$ on the corresponding axes. Moreover, $L_t^{\varepsilon,\delta}$ is the local time for the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ on $\partial G^{\varepsilon,\delta}$. The local time is a continuous, non–decreasing process which increases only when the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ touches the boundary of $G^{\varepsilon,\delta}$. Then, using the Feynman–Kac formula, $w^{\varepsilon,\delta}(t,x,y)$ is the unique bounded solution of problem (3.1):

$$w^{\varepsilon,\delta}(t,x,y) = E_{x,y}^{\varepsilon,\delta}\left[g(X_t^{\varepsilon,\delta})\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t-s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}))ds\right\}\right].$$

(3.3)

As has been shown in [11] and [12], as $\varepsilon \downarrow 0$ and for fixed $\delta > 0$, the component $X_t^{\varepsilon,\delta}$, $0 \le t \le T$, converges weakly in the space of continuous functions on R to the one–dimensional diffusion process X_t^δ which satisfies the following stochastic differential equation:

$$dX_t^\delta = dW_t^1 + b(X_t^\delta)dt, \quad b(x) = (1/2)(d\ln(V^\delta(x))/dx),$$

(3.4)

where $V^\delta(x)$ is the volume of G_x^δ in R^n. Furthermore, this implies that, for fixed $\delta > 0$, $w^{\varepsilon,\delta}(t,x,y)$ converges as $\varepsilon \downarrow 0$ to the solution $w^\delta(t,x)$ of the problem:

$$\begin{cases} \frac{\partial w^\delta(t,x)}{\partial t} = \frac{1}{2}w_{xx}^\delta + \frac{1}{2}(d\ln(V^\delta(x))/dx)w_x^\delta + c(x,0,w^\delta)w^\delta, & \text{if } t > 0, x \in R \\ w^\delta(0,x) = g(x), & \text{if } x \in R. \end{cases}$$

(3.5)

Using the Feynman–Kac Formula, we can write the solution $w^\delta(t,x)$ of (3.5) in the form of the expectation of a functional of the trajectories of the corresponding

process:

$$w^\delta(t, x) = E_x^\delta \left[g(X_t^\delta) \exp \left\{ \int_0^t c(X_s^\delta, 0, w^\delta(t - s, X_s^\delta)) ds \right\} \right]. \qquad (3.6)$$

It is worthy of note that the classical solution of problem (3.5), if it exists, satisfies the system of equations (3.4) and (3.6). The function $w^\delta(t, x)$ is called the generalized solution of problem (3.5) provided it satisfies the system of equations (3.4) and (3.6). According to Theorem 7, we expect that the solution w^δ converges as $\delta \downarrow 0$ to w, which is the solution to the non–standard reaction diffusion equation:

$$\begin{cases} \frac{\partial w(t,x)}{\partial t} = D_v D_u w + f(x, 0, w), & \text{if } t > 0, x \in R \\ w(0, x) = g(x), \end{cases} \qquad (3.7)$$

where $u(x) = \lim_{\delta \downarrow 0} \int_0^x (1/V^\delta(y)) dy$ and $v(x) = \lim_{\delta \downarrow 0} \int_0^x 2(V^\delta(y)) dy$. Here we will consider the generalized solution of problem (3.7). For the operator $\widetilde{A}$ given by

$$\widetilde{A}w = -\frac{dw}{dt} + Aw + c(x, 0, w)w = -\frac{dw}{dt} + D_v D_u w + c(x, 0, w)w, \qquad (3.8)$$

there exists a corresponding Markov family and a corresponding process $Y_s = (t - s, X_s)$ which is homogeneous in time in the state space $(-\infty, T] \times R$, $T > 0$. Define $w(t, x) = g(x)$ for $t \leq 0$. Using the Feynman–Kac formula, the solution of problem (3.7) may be written in the form

$$w(t, x) = E_x g(X_t) \exp \left\{ \int_0^t c(X_s, 0, w(t - s, X_s)) ds \right\}. \qquad (3.9)$$

where the process X_t is governed by the generator $D_v D_u$. Since $c(x, 0, w)$ is Lipschitz continuous in w, we can derive from (3.9) the existence and uniqueness of the generalized solution of problem (3.7) via the method of successive approximation.

Theorem 8 *There exists a unique generalized solution for the problem (3.7).*

Proof. First, let us prove the existence of the generalized solution to (3.7).

Let us put

$$
\begin{cases}
w_{n+1}(t,x) = E_x g(X_t) \exp\left\{ \int_0^t c(X_s, 0, w_n(t-s, X_s)) ds \right\}, \\
w_1(t,x) = g(x),
\end{cases}
\tag{3.10}
$$

where the process X_t is governed by the generator $D_v D_u$.

Set $\delta_n(t) = \sup_x \mid w_n(t,x) - w_{n-1}(t,x) \mid$ for $0 \le t \le T$. Then,

$$
|w_{n+1}(t,x) - w_n(t,x)|
$$

$$
= \left| E_x g(X_t) \left(\exp\left\{ \int_0^t c(X_s, 0, w_n(t-s, X_s)) ds \right\} \right. \right.
$$

$$
\left. \left. - \exp\left\{ \int_0^t c(X_s, 0, w_{n-1}(t-s, X_s)) ds \right\} \right) \right|
$$

$$
\le E_x \left| g(X_t) \exp\left\{ \int_0^t c(X_s) ds \right\} \left(\int_0^t c(X_s, 0, w_n(t-s, X_s)) \right. \right.
$$

$$
\left. \left. - c(X_s, 0, w_{n-1}(t-s, X_s)) ds \right) \right|
$$

$$
\le E_x |g(X_t)| \exp\left\{ \int_0^t c(X_s) ds \right\} \int_0^t K_c |w_n(t-s, X_s) - w_{n-1}(t-s, X_s)| \, ds
$$

$$
\le K_c E_x \left| g(X_t) \exp\left\{ \int_0^t c(X_s) ds \right\} \right| \int_0^t \delta_n(u) du
$$

$$
\le K \int_0^t \delta_n(u) du,
$$

$$
\tag{3.11}
$$

where K_c is a Lipschitz constant of the function $c(x, 0, w)$ in w. Here a constant K can be properly chosen since the initial function $g(x)$ and the function $c(x)$ are bounded. Moreover, K is independent of t and n. That is, there exists a constant K such that $\delta_{n+1}(t) \le K \int_0^t \delta_n(u) du$. Thus we can induce

$$
\delta_{n+1}(t) \le \frac{(KT)^n}{n!} \parallel g \parallel \qquad \text{for each } n
\tag{3.12}
$$

where $\parallel g \parallel = \sup_x \mid g(x) \mid$. Since $w_n(t,x) \le \parallel g \parallel \sum_{k=1}^{n-1} \frac{(KT)^k}{k!}$, the limit $w_n(t,x)$ exists as $n \to \infty$ and this convergence is uniform on the set $\{x \in R, 0 \le t \le T\}$.

40

Let us prove the uniqueness by using successive approximation again.

Set $l(t) = \max_{0 \leq s \leq t, x \in R} | u(s, x) - v(s, x) |$ for $0 \leq t \leq T$. Then,

$| u(s, x) - v(s, x) |$

$$
\begin{aligned}
= \quad & \left| E_x g(X_s) \left(\exp \left\{ \int_0^s c(X_r, 0, u(s-r, X_r)) dr \right\} \right. \right. \\
& \qquad\qquad \left. \left. - \exp \left\{ \int_0^s c(X_r, 0, v(s-r, X_r)) dr \right\} \right) \right| \\
\leq \quad & E_x \left| g(X_s) \exp \left\{ \int_0^s c(X_r) dr \right\} \left(\int_0^s c(X_r, 0, u(s-r, X_r)) \right. \right. \\
& \qquad\qquad\qquad\qquad \left. \left. - c(X_r, 0, v(s-r, X_r)) dr \right) \right| \\
\leq \quad & K_c E_x \left| g(X_s) \exp \left\{ \int_0^s c(X_r) dr \right\} \right| \int_0^s l(z) dz.
\end{aligned}
\tag{3.13}
$$

where K_c is a Lipschitz constant of the function $c(x, 0, w)$ in w. Since the initial function $g(x)$ and the function $c(x)$ are bounded, and for $0 \leq z \leq t$, $l(z) \leq l(t)$, by the definition, there exists a constant C such that

$$
l(t) \leq (Ct) l(t), \qquad \text{for } 0 \leq t \leq T. \tag{3.14}
$$

If $t < 1/C$, then $l(t)$ is equal to zero. Since the constant C does not depend on t, one can conclude that $u(t, x) = v(t, x)$ for $0 \leq t \leq T$. $\square$

Proposition 1 *If $w(t, x)$ is Lipschitz continuous in x and $0 < t \leq T$, then $w(t, x)$ is Hölder continuous in t with exponent 0.5.*

Proof. We want to prove that there exists a constant C such that, for $h > 0$, $| w(t+h, x) - w(t, x) | \leq C\sqrt{h}$. Assume that $| w(t, x) - w(t, y) | \leq C_1 | x - y |$, where C_1 is a Lipschitz constant. From (3.9) and the Markov property of the process X_t, it follows that, for $h > 0$,

$$
w(t+h, x) = E \left[w(t, X_h) \exp \left\{ \int_0^h c(X_s, 0, w(t+h-s, X_s)) ds \right\} \right]. \tag{3.15}
$$

41

Then, we can deduce that

$$
\begin{aligned}
| \, w(t+h, x) - w(t, x) \, | & \\
\leq \quad & E \, |w(t, X_h) - w(t, x)| \exp \left\{ \int_0^h c(X_s, 0, w(t+h-s, X_s)) ds \right\} \\
& + \| \, w(t, x) \, \| \, E \left| \exp \left\{ \int_0^h c(X_s, 0, w(t+h-s, X_s)) ds \right\} - 1 \right| \\
\leq \quad & K_1 E \, |X_h - x| + K_2 E \left| \int_0^h c(X_s, 0, w(t+h-s, X_s)) ds \right|
\end{aligned}
\tag{3.16}
$$

where $\| \, w(t, x) \, \| = \sup_{t, x} | \, w(t, x) \, |$, and K_1 and K_2 are constants. Here if we can

show that $E \, | \, X_h - x \, |^2 \leq K_3 h$ with a constant K_3, then our proof can be done. We

are going to use a successive approximation method to prove this. Set

$$
\begin{cases}
\Psi_n(x) & = \quad \int_c^x \int_c^y \Psi_{n-1}(z) du(z) du(y) \\
\Psi_0(x) & = \quad 1,
\end{cases}
\tag{3.17}
$$

where c is a constant. Then, it is easy to see that $D_v D_u \Psi_n(x) = \Psi_{n-1}(x)$ for all n.

Let us define a function $\Phi_n(t)$:

$$
\Phi_n(t) = \Psi_n(X_t) - \int_0^t \Psi_{n-1}(X_s) ds
\tag{3.18}
$$

Since $f(X_t) - \int_0^t D_v D_u f(X_s) ds$ is a martingale, for any $f \in \mathcal{D}(D_v D_u)$ ([5], Ch.4),

$\Phi_n(t)$ is a martingale. It follows that $E_{x=c}(\Phi_n(t)) = 0$ and $\Phi_1(t) = \Psi_1(X_t) - t$. Then

$$
E_{x=c}(\Psi_1(X_t)) = t, \quad E_{x=c}(\Psi_2(X_t)) = \tfrac{1}{2} t^2.
\tag{3.19}
$$

Since both $u(x)$ and $v(x)$ are strictly increasing functions, there exist positive con-

stants a_1 and a_2 such that $du(x) \geq a_1 dx$ and $dv(x) \geq a_2 dx$ for all x. Choose

$b = \min(a_1, a_2)$. Then, for $b > 0$,

$$
\begin{aligned}
\Psi_1(x) & = \quad \int_c^x \int_c^y \Psi_0(z) dv(z) du(y) \quad \geq \quad \int_c^x \int_c^y b^2 dz dy \quad = \quad \tfrac{b^2}{2} (x-c)^2, \\
\Psi_2(x) & = \quad \int_c^x \int_c^y \Psi_1(z) dv(z) du(y) \quad \geq \quad \tfrac{b^4}{4!} (x-c)^4.
\end{aligned}
\tag{3.20}
$$

42

So $E((X_t - X_0)^4 b^4/4!) \leq E(\Psi_2(X_t)) = 1/2t^2$. We have $E((X_t - X_0)^4) \leq 12/b^4 t^2$.

Finally, from equation (3.16), using Lyapunov's Inequality, we can conclude that

$$
\begin{aligned}
\mid w(t+h,x) - w(t,x) \mid \; &\leq \; K_1 E(\mid X_h - x \mid^2)^{1/2} + K_2 \parallel c \parallel h \\
&\leq \; K_1 \sqrt{K_3 h} + K_2 \parallel c \parallel h \leq K_4 \sqrt{h}
\end{aligned}
\tag{3.21}
$$

where $K_3 = \sqrt{12}/b^2$ and K_4 can be chosen as a positive constant. Therefore, the proof is complete. $\square$

Lastly, we can see that the generalized solutions $w^{\varepsilon,\delta}$ of the system (3.2) and (3.3) in narrow domains converge, as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$, to the generalized solution w of the problem (3.9) as follows.

Proposition 2 *The solutions $w^{\varepsilon,\delta}$ in (3.1) converge to the solution w in (3.7) as first $\varepsilon \downarrow 0$ and then $\delta \downarrow 0$.*

Proof. From the bounded continuous function $w^{\varepsilon,\delta}$ in (3.3) and the Ascoli–Arzela theorem, there exists a subsequence of $\{w^{\varepsilon,\delta}\}$ for fixed $\delta > 0$ and w^δ such that

$$
w^{\varepsilon,\delta} \longrightarrow w^\delta, \quad \text{as } \varepsilon \to 0, \text{ uniformly in compacts.}
$$

For convenience, we denote a subsequence of $\{w^{\varepsilon,\delta}\}$ by $\{w^{\varepsilon,\delta}\}$ again. Then, using the weak convergence of processes $X_t^{\varepsilon,\delta}$ to a process X_t^δ as $\varepsilon \downarrow 0$,

$$
\sup_{0 \leq t \leq T} E \left| \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}) - c(X_s^\delta, 0, w^\delta) ds \right| \to 0, \text{ as } \varepsilon \downarrow 0.
\tag{3.22}
$$

Therefore, we can see that w^δ actually satisfies (3.6) which implies that w^δ satisfies the problem (3.5) as follows:

$$
\begin{aligned}
&\left| w^{\varepsilon,\delta}(t,x,y) - E_x g(X_t^\delta) \exp \left\{ \int_0^t c(X_s^\delta, 0, w^\delta) ds \right\} \right| \\
&\leq \; K_1 E_x \left| g(X_t^{\varepsilon,\delta}) - g(X_t^\delta) \right| + K_2 E_x \int_0^t \left| c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}) - c(X_s^\delta, 0, w^\delta) \right| ds
\end{aligned}
\tag{3.23}
$$

43

where K_1 is a constant determined by the bounded function $c(x, y)$, and K_2 is also a constant determined by the bounded functions $g(x)$ and $c(x, y)$. Thus, by the definition of weak convergence and (3.22), all terms on the right side in (3.23) go to zero as $\varepsilon \downarrow 0$. Therefore, the convergence of $w^{\varepsilon, \delta}(t, x, y)$ to $w^\delta(t, x)$ is completed. In the same way, using Theorem 8, it is easy to prove the convergence of $w^\delta(t, x)$ to $w(t, x)$ in (3.9) as $\delta \downarrow 0$. $\square$

3.2   Wave front propagation in periodic media

Assume that the boundary ∂G^δ of G^δ is of period one in x. We will use equations of the type of (3.3) and (3.6) in periodic media for studying asymptotic properties of the solutions as $t \to \infty$.
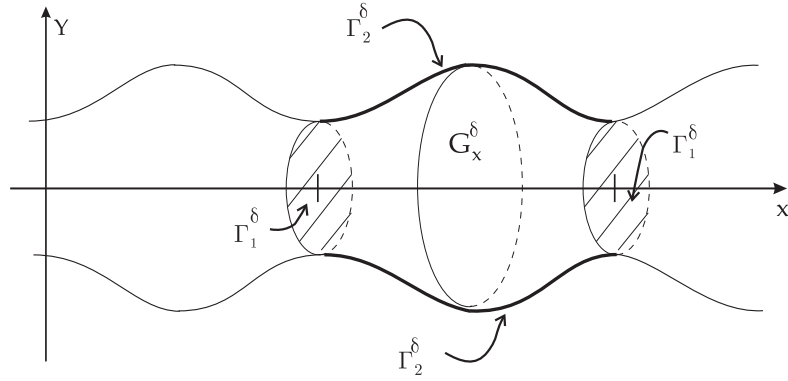

Figure 3.1: The periodic domain $G^{\varepsilon, \delta}$.

Since the medium $G^{\varepsilon, \delta}$ is periodic in the component x with period one, it is worthy of note that, for each $x \in R$ and an integer z_1, the distribution in the path space of the process $(X_t^{\varepsilon, \delta}, Y_t^{\varepsilon, \delta}, P_{x,y}^{\varepsilon, \delta})$ with respect to the probability measure $P_{(x+z_1, y)}^{\varepsilon, \delta}$ coincides with the distribution of the process $(X_t^{\varepsilon, \delta} + z_1, Y_t^{\varepsilon, \delta})$ with respect

to $P_{x,y}^{\varepsilon,\delta}$. Let $C_\pi^{\varepsilon,\delta}$ be the Banach space of all functions continuous in $R \times R^n$ and periodic in $x \in R$ with period one with the uniform norm.
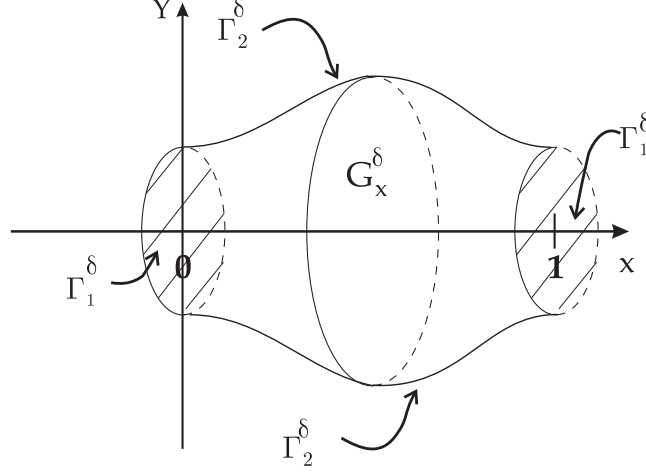

Figure 3.2: One period $G_1^{\varepsilon,\delta}$ of the narrow domain $G^{\varepsilon,\delta}$.

Define the function $H^{\varepsilon,\delta}(x,y)$ by

$$H^{\varepsilon,\delta}(x,y) = \sup_{z=(z_1,\mathbf{0})\in R\times R^n} \left[< (x,y), z > -\lambda^{\varepsilon,\delta}(z) \right], \qquad (x,y) \in G^{\varepsilon,\delta}, \qquad (3.24)$$

where $\lambda^{\varepsilon,\delta}(z)$ is a simple eigenvalue for the differential operator $L^{z,\varepsilon,\delta}$,

$$L^{z,\varepsilon,\delta} = \frac{1}{2} \triangle - < z, \bigtriangledown > + c(x,y) + \frac{1}{2} < z, z >, \qquad (3.25)$$

in the space $C_\pi^{\varepsilon,\delta}$ corresponding to a positive eigenfunction, which also satisfies the boundary conditions of problem (3.1). Here $< \cdot, \cdot >$ denotes an inner product.

First, we formulate the main theorem in the periodic narrow domain with smooth boundary.

Theorem 9 *(i) For any closed set $F \subset \left\{ (x,y) \in G^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x,y) > 0 \right\}$,*

$$\lim_{t\to\infty,(tx,y)\in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t,tx,y) = 0, \qquad uniformly \ in \ (x,y) \in F. \qquad (3.26)$$

45

(ii) For any compact set $K \subset \left\{ (x, y) \in G^{\varepsilon, \delta} : H^{\varepsilon, \delta}(x, y) < 0 \right\}$,

$$\lim_{t \to \infty, (tx, y) \in G^{\varepsilon, \delta}} w^{\varepsilon, \delta}(t, tx, y) = 1, \qquad \text{unifromly in } (x, y) \in K. \tag{3.27}$$

and the set $t \cdot \left\{ (x, y) \in G^{\varepsilon, \delta} : H^{\varepsilon, \delta}(x, y) = 0 \right\}$ may be interpreted as the wave front propagation.

Let C_{π}^{δ} be the Banach space of all periodic continuous functions of period one in R with the uniform norm. Similarly, for the problem (3.5), define the function $H^{\delta}(x)$ as given by

$$H^{\delta}(x) = \sup_{z \in R} \left[xz - \lambda^{\delta}(z) \right], \qquad x \in R, \tag{3.28}$$

where $\lambda^{\delta}(z)$ is a simple eigenvalue for the differential operator $L^{z, \delta}$,

$$L^{z, \delta} = \frac{1}{2} \frac{d^2}{dx^2} + \left(\frac{1}{2} \frac{d}{dx} (\ln(V^{\delta}(x))) - z \right) \frac{d}{dx} + \left(c(x) - \frac{z}{2} \frac{d}{dx} (\ln(V^{\delta}(x))) + \frac{z^2}{2} \right),$$

$$\tag{3.29}$$

in the space C_{π}^{δ} corresponding to a positive eigenfunction, which also satisfies the boundary conditions of problem (3.5). We formulate the following Theorem for a solution of the problem (3.5).

Theorem 10 *(i) For any closed set $F \subset \left\{ x \in R : H^{\delta}(x) > 0 \right\}$,*

$$\lim_{t \to \infty, tx \in R} w^{\delta}(t, tx) = 0, \qquad \text{uniformly in } x \in F. \tag{3.30}$$

(ii) For any compact set $K \subset \left\{ y \in R : H^{\delta}(x) < 0 \right\}$,

$$\lim_{t \to \infty, tx \in R} w^{\delta}(t, tx) = 1, \qquad \text{uniformly in } x \in K. \tag{3.31}$$

and the set $t \cdot \left\{ x \in R : H^{\delta}(x) = 0 \right\}$ may be interpreted as the wave front propagation.

In the following section, we will calculate the large deviations for the family of random vectors

$$\eta_{x,y}^{\varepsilon,\delta,t} = \frac{1}{t}(x - X_t^{\varepsilon,\delta} + \int_0^t \gamma_1(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})dL_s^{\varepsilon,\delta}, t(y - Y_t^{\varepsilon,\delta} + \int_0^t \gamma_2(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})dL_s^{\varepsilon,\delta}))$$

(3.32)

defined on the probability spaces $(\Omega^{\varepsilon,\delta}, \mathcal{F}^{\varepsilon,\delta}, P_{x,y}^{\varepsilon,\delta,t})$, $t > 0$, $x \in R$ and $y \in R^n$. Here the family of probability measures $P_{x,y}^{\varepsilon,\delta,t}$ is defined by

$$P_{x,y}^{\varepsilon,\delta,t}(A) = \frac{E_{x,y}^{\varepsilon,\delta}\left[\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\} \chi_A(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})\right]}{E_{x,y}^{\varepsilon,\delta}\left[\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\}\right]},$$

(3.33)

where χ_A is an indicator function of A. The exponential bounds of Theorem 2 and Theorem 3 are valid for all $s \geq 0$, all $(x, y) \in G^{\varepsilon,\delta}$ with $\varepsilon(t) = 1/t$. The sets $\Phi(s)$ are compact and the action function has the form: $S^{\varepsilon,\delta}(x, y) = H^{\varepsilon,\delta}(x, y) + \lambda^{\varepsilon,\delta}(0)$, where $H^{\varepsilon,\delta}(x, y)$ is defined in (3.24) as the Legendre transform of $\lambda^{\varepsilon,\delta}(z)$. In the same way, we will use the corresponding notations without "ε" for the problem (3.5) unless there is no confusion. This approach follows Mark Freidlin's work in [10].

3.3   Calculation of the action functional and the proof of Theorem 9.

Lemma 1 *For any $z = (z_1, 0) \in R \times R^n$ and an integer z_1,*

$$\lim_{t \to \infty} \frac{1}{t} \ln E_{x,y}^{\varepsilon,\delta,t} \exp\left\{t < z, \eta_{x,y}^{\varepsilon,\delta,t} >\right\} = \lambda^{\varepsilon,\delta}(z) - \lambda^{\varepsilon,\delta}(0)$$

(3.34)

exists uniformly in $(x, y) \in G^{\varepsilon,\delta}$. Here $\lambda^{\varepsilon,\delta}(z)$ is a simple eigenvalue of the operator $L^{z,\varepsilon,\delta}$ in (3.25) corresponding to a positive eigenfunction meeting the mixed boundary condition in (3.1). The function $\lambda^{\varepsilon,\delta}(z)$ is differentiable.

47

Proof. Consider a Markov process $(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z), P_{x,y}^{z,\varepsilon,\delta})$ in $G^{\varepsilon,\delta} \subset R \times R^n$ which is a solution of the following stochastic differential equations:

$$
\begin{aligned}
dX_t^{\varepsilon,\delta}(z) &= dW_t^1 - z_1 dt + \gamma_1^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z)) dL_t^{\varepsilon,\delta}, & X_0^{\varepsilon,\delta}(z) = x, \\
dY_t^{\varepsilon,\delta}(z) &= dW_t^2 + \gamma_2^{\varepsilon,\delta}(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z)) dL_t^{\varepsilon,\delta}, & Y_0^{\varepsilon,\delta}(z) = y,
\end{aligned}
\tag{3.35}
$$

where W_t^1 and W_t^2 are independent Wiener processes in R and R^n respectively, (x, y) is a point inside $G^{\varepsilon,\delta}$, and $\gamma^{\varepsilon,\delta} = (\gamma_1^{\varepsilon,\delta}(x,y), \gamma_2^{\varepsilon,\delta}(x,y))$ is the unit inward normal to $\partial G^{\varepsilon,\delta}$. The operator $(Q_t^{z,\varepsilon,\delta}\psi)(x,y)$, which is given by

$$
\begin{aligned}
(Q_t^{z,\varepsilon,\delta}\psi)(x,y) &= \\
E_{x,y}^{z,\varepsilon,\delta} &\left[\psi(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z)) \exp\left\{ \int_0^t c(X_s^{\varepsilon,\delta}(z), Y_s^{\varepsilon,\delta}(z)) + \tfrac{1}{2} < z, z > ds \right\} \right],
\end{aligned}
\tag{3.36}
$$

defines a continuous semigroup of linear bounded operators depending on the parameters z, ε and δ in $C_\pi^{\varepsilon,\delta}$. Denote by $\bar{c}(x,y)$ the restriction of the periodic function $c(x,y)$ (periodic in x) to the set $G_1^{\varepsilon,\delta}$ with finite Lebesgue measure. The process $(\bar{X}_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ indicates a diffusion process over $G_1^{\varepsilon,\delta}$ with reflection at the boundary Γ_2^δ and zero boundary condition at the boundary Γ_1^δ. We denote by $(\bar{Q}_t^{z,\varepsilon,\delta}\bar{\psi})(x,y)$ the semigroup of linear bounded operators:

$$
\begin{aligned}
(\bar{Q}_t^{z,\varepsilon,\delta}\bar{\psi})(x,y) &= \\
E_{x,y}^{z,\varepsilon,\delta} &\left[\bar{\psi}(\bar{X}_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z)) \exp\left\{ \int_0^t \bar{c}(\bar{X}_s^{\varepsilon,\delta}(z), Y_s^{\varepsilon,\delta}(z)) + \tfrac{1}{2} < z, z > ds \right\} \right],
\end{aligned}
\tag{3.37}
$$

where $(x,y) \in G_1^{\varepsilon,\delta}$ and $\bar{\psi} \in C_{G_1^{\varepsilon,\delta}}$. Here $C_{G_1^{\varepsilon,\delta}}$ denotes a space of continuous functions in $G_1^{\varepsilon,\delta}$. Since the Doeblin condition is satisfied for a non–degenerate diffusion process on any compact smooth manifold, it was proven in [20] that there exists a simple eigenvalue $\exp\{t\lambda^{\varepsilon,\delta}(z)\}$ of the operator $\bar{Q}_t^{z,\varepsilon,\delta}$ with a positive eigenfunction $\bar{u}^{z,\varepsilon,\delta}$ in-

dependent of t. Then $\exp\{t\lambda^{\varepsilon,\delta}(z)\}$ is also an eigenvalue of the operator $Q_t^{z,\varepsilon,\delta}$ with a strictly positive eigenfunction $u^{z,\varepsilon,\delta} \in C_\pi^{\varepsilon,\delta}$, which corresponds to $\bar{u}^{z,\varepsilon,\delta} \in C_{G_1^{\varepsilon,\delta}}$. By the definition of the infinitesimal operator of the corresponding semigroup, it is easy to see that $\lambda^{\varepsilon,\delta}(z)$ is the eigenvalue of the operator $L^{z,\varepsilon,\delta}$ in $C_\pi^{\varepsilon,\delta}$ corresponding to a strictly positive eigenfunction $u^{z,\varepsilon,\delta}$.

Since the process $(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z), P_{x,y}^{z,\varepsilon,\delta})$ differs from the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta}, P_{x,y}^{\varepsilon,\delta})$ only by a drift in the first component with a non–degenerate diffusion coefficient, by the Girsanov theorem, the law of the process $(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z))$ is absolutely continuous with respect to the law of the original process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$; that is, the density of the measure $P_{x,y}^{z,\varepsilon,\delta}$ with respect to the density of the original measure $P_{x,y}^{\varepsilon,\delta}$ is written as follows:

$$\frac{dP_{x,y}^{z,\varepsilon,\delta}}{dP_{x,y}^{\varepsilon,\delta}}(X_\cdot^{\varepsilon,\delta}) = \exp\left\{-z_1 W_t^1 - \frac{1}{2}z_1^2 t\right\}, \tag{3.38}$$

where $W_t = (W_t^1, W_t^2)$ is a Wiener process in $R \times R^n$. So, the operator $(Q_t^{z,\varepsilon,\delta}\psi)(x,y)$ can be rewritten as follows:

$$
\begin{aligned}
&(Q_t^{z,\varepsilon,\delta}\psi)(x,y)\\
&= E_{x,y}^{z,\varepsilon,\delta}\left[\psi(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z))\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}(z), Y_s^{\varepsilon,\delta}(z)) + \frac{1}{2}z_1^2 ds\right\}\right]\\
&= \int \psi(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds+ <t\eta_{x,y}^{\varepsilon,\delta,t}, z>\right\}dP_{x,y}^{\varepsilon,\delta}\\
&= E_{x,y}^{\varepsilon,\delta}\left[\psi(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds+ <t\eta_{x,y}^{\varepsilon,\delta,t}, z>\right\}\right]
\end{aligned}
\tag{3.39}
$$

Since the eigenfunction $u^{z,\varepsilon,\delta}$ corresponding to the eigenvalue $\lambda^{\varepsilon,\delta}(z)$ is strictly positive, we can deduce the following:

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{1}{t}\ln\left(E_{x,y}^{\varepsilon,\delta}\exp\left\{\int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds+ <t\eta_{x,y}^{\varepsilon,\delta,t}, z>\right\}\mathbf{1}\right)\\
&= \lim_{t\to\infty}\frac{1}{t}\ln(Q_t^{z,\varepsilon,\delta}\mathbf{1})(x,y) = \lambda^{\varepsilon,\delta}(z)
\end{aligned}
\tag{3.40}
$$

This convergence is uniform in $(x, y) \in R \times R^n$. From the definition of $P_{x,y}^{\varepsilon,\delta,t}(A)$, the first part can be completed. Lastly, from the fact that the operator function $z \to Q_t^{z,\varepsilon,\delta}$ is differentiable, this implies the differentiability of the eigenvalue $\lambda^{\varepsilon,\delta}(z)$ due to the perturbation theory of linear operator in [17]. $\square$

Lemma 2 *For any closed set $F \subset \left\{ (x, y) \in G^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x, y) > 0 \right\}$,*

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \ln \left(\sup_{(x,y) \in F, (tx,y) \in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t, tx, y) \right) \leq - \min_{(x,y) \in F} H^{\varepsilon,\delta}(x, y). \tag{3.41}$$

Proof. For $0 < s < \min_{(x,y) \in F} H^{\varepsilon,\delta}(x, y)$, there exists a positive number δ_1 such that $2\delta_1 = d(F, \Psi(s))$, where $d(\cdot, \cdot)$ is the distance between two sets and $\Psi(s) = \{(x, y) \in G^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x, y) \leq s\}$, because a closed set F does not intersect the compact set $\Psi(s)$. For t sufficiently large, the support of the initial function g is contained in $U_{\delta_1 t}(0)$, a $\delta_1 t$–neighborhood of the point 0. Then the solution $w^{\varepsilon,\delta}(t, tx, y)$ of problem (3.1) implies the following bound by using definitions $\eta_{x,y}^{\varepsilon,\delta,t}$ and $P_{x,y}^{\varepsilon,\delta,t}$:

$\sup_{(x,y) \in F} w^{\varepsilon,\delta}(t, tx, y)$

$$\leq \| g \| E_{tx,y}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\} \chi_{U_{(\delta_1 t, \delta_1)}(0)}(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$$

$$\leq \| g \| \sup_{(x,y) \in F} P_{tx,y}^{\varepsilon,\delta,t} \{ \rho(\eta_{tx,y}^{\varepsilon,\delta,t}, (x, y)) < \delta_1 \} E_{tx,y}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\}$$

$$\leq \| g \| \sup_{(x,y) \in G^{\varepsilon,\delta}} P_{x,y}^{\varepsilon,\delta,t} \{ \rho(\eta_{x,y}^{\varepsilon,\delta,t}, \Psi(s)) > \delta_1 \}$$

$$\times \sup_{(x,y) \in G^{\varepsilon,\delta}} E_{x,y}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\}.$$

where $\| g \| = \sup_{(x,y) \in G^{\varepsilon,\delta}} | g(x) |$, $U_{(\delta_1 t, \delta_1)}(0)$ is a $\delta_1 t$–neighborhood of the first component 0 and a δ_1–neighborhood of remaining components 0.

Using Theorem 2 and the relation (3.40), since $\Psi(s) = \Phi(s + \lambda^{\varepsilon,\delta}(0))$, we can deduce

the following for an arbitrary value $h > 0$:

$$\overline{\lim_{t \to \infty}} \frac{1}{t} \ln \left(\sup_{(x,y) \in F} w^{\varepsilon,\delta}(t, tx, y) \right) \le -(s + \lambda^{\varepsilon,\delta}(0)) + h + \lambda^{\varepsilon,\delta}(0) \le -s. \qquad (3.42)$$

Here s can be chosen arbitrarily close to $\min_{(x,y) \in F} H^{\varepsilon,\delta}(x,y)$. So the proof is complete. $\square$

Lemma 3 *For all $(x, y) \in G^{\varepsilon,\delta}$ with $H^{\varepsilon,\delta}(x, y) > 0$,*

$$\lim_{t \to \infty} \frac{1}{t} \ln \left(\inf_{(\tilde{x},\tilde{y}) \in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}(t, t\tilde{x}, \tilde{y}) \right) \qquad (3.43)$$

$$\ge \lim_{t \to \infty} \frac{1}{t} \ln \left(\inf_{(p,q) \in G_1^{\varepsilon,\delta}, (\tilde{x},\tilde{y}) \in U_{2\delta_1}(x,y)} E_{x,y}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\} \right.$$

$$\left. \times \chi_{(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta}) \in U_{\delta_1}(p - t\tilde{x}, q - \tilde{y})} \right)$$

provided δ_1 is small enough. Here $U_{\delta_1}(x, y)$ is the δ_1–neighborhood of a point (x, y) in $G^{\varepsilon,\delta}$.

    Proof. Let

$$l = \lim_{t \to \infty} \frac{1}{t} \ln \left(\inf_{(\tilde{x},\tilde{y}) \in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}(t, t\tilde{x}, \tilde{y}) \right). \qquad (3.44)$$

First, we will prove that l is bounded below, that is, $l > -\infty$, by using the Markov property of the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$. For any $(\tilde{x}, \tilde{y}) \in G^{\varepsilon,\delta}$,

$$E_{t\tilde{x},\tilde{y}}^{\varepsilon,\delta} g(X_t^{\varepsilon,\delta}) \ge \prod_{k=1}^{[t]-1} \inf_{(x,y) \in U_{\delta_1}((t-k+1)\tilde{x},\tilde{y})} P_{x,y}^{\varepsilon,\delta} \{ (X_1^{\varepsilon,\delta}, Y_1^{\varepsilon,\delta}) \in U_{\delta_1}((t-k)\tilde{x}, \tilde{y}) \}$$

$$\times \inf_{(x,y) \in U_{\delta_1}((t-[t]+1)\tilde{x},\tilde{y})} E_{x,y}^{\varepsilon,\delta} g(X_{t-[t]+1}^{\varepsilon,\delta}),$$

$$(3.45)$$

where $[t]$ is the integer part of t. Since the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ is in $G^{\varepsilon,\delta}$, which is

periodic in x,

$$\inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} E_{t\tilde{x},\tilde{y}}^{\varepsilon,\delta} g(X_t^{\varepsilon,\delta})$$

$$\geq \left(\inf_{(p,q)\in G_1^{\varepsilon,\delta},(\tilde{x},\tilde{y})\in U_{2\delta_1}(x,y)} P_{p,q}^{\varepsilon,\delta}\{(X_1^\varepsilon, Y_1^\varepsilon)\in U_\delta(p-\tilde{x}, q-\tilde{y})\}\right)^{[t]-1} \qquad (3.46)$$

$$\times \inf_{\{(p,q)\in U_{2|(x,y)|+3\delta_1}(0),1\leq s\leq 2\}} E_{p,q}^{\varepsilon,\delta} g(X_s^{\varepsilon,\delta}) > 0.$$

Thus we can see the lower bound for l :

$$l = \varliminf_{t\to\infty} \frac{1}{t} \ln \inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}(t, t\tilde{x}, \tilde{y})$$

$$\geq \varliminf_{t\to\infty} \left\{\frac{1}{t} \ln \inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} E_{t\tilde{x},\tilde{y}}^{\varepsilon,\delta} g(X_t^{\varepsilon,\delta}) + \max_{0\leq s\leq t} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})\right\} \qquad (3.47)$$

$$> -\infty.$$

For each $(a,b) \in G^{\varepsilon,\delta}$, $\eta > 0$, and t, let us define the Markov times:

$$\sigma_\eta(t) = \min\{s \geq 0 :| (X_s^{\varepsilon,\delta} - (t-s)x, Y_s^{\varepsilon,\delta} - y) |\geq \eta t\},$$
$$\qquad (3.48)$$
$$\tau_{(a,b),\eta}(t) = \min\{s \geq 0 :| (X_s^{\varepsilon,\delta} - a + sx, Y_s^{\varepsilon,\delta} - b) |\geq \eta t\}.$$

Choose $\epsilon > 0$ so that $U_{2\epsilon}(x,y) \subset \{(x,y) : H^{\varepsilon,\delta}(x,y) > 0\}$, $\delta_1 \in (0, \epsilon/3)$, and $h \in (0,1)$. By using Lemma 2, if $\sigma_\epsilon(t) > t$, $w^{\varepsilon,\delta}(t-s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) \leq h$ from some time on because $H^{\varepsilon,\delta}(x,y)$ is strictly positive. Let $c_h(x,y) = \inf_{w^{\varepsilon,\delta}\in[0,h]} c(x,y,w^{\varepsilon,\delta})$. Then it is easy to see that $c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t-s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})) \geq c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})$ for $s \in [0, t/2]$. Using the Markov property again, we can see that, for any $\kappa \in (0, 1/2)$,

$$\inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}(t, t\tilde{x}, \tilde{y})$$

$$\geq \inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} E_{t\tilde{x},\tilde{y}}^{\varepsilon,\delta} \left(\exp\left\{\int_0^{\kappa t} c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\} \chi_A\right)$$

$$\times \inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}((1-\kappa)t, (1-\kappa)t\tilde{x}, (1-\kappa)\tilde{y}) \qquad (3.49)$$

$$\geq \inf_{(p,q)\in G_1^{\varepsilon,\delta},(\hat{x},\hat{y})\in U_{2\delta_1}(x,y)} E_{p,q}^{\varepsilon,\delta} \left(\exp\left\{\int_0^{\kappa t} c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\} \chi_B\right)$$

$$\times \inf_{(\tilde{x},\tilde{y})\in U_{\delta_1}(x,y)} w^{\varepsilon,\delta}((1-\kappa)t, (1-\kappa)t\tilde{x}, (1-\kappa)\tilde{y}).$$

where χ_A and χ_B are indicator functions:

$$\chi_A = \chi_{\{\sigma_\epsilon(t)>\kappa t,(X_{\kappa t},Y_{\kappa t})\in U_{(1-\kappa)\delta_1}(t((1-\kappa)tx),(1-\kappa)y))\}},$$

$$\chi_B = \chi_{\{\tau_{(p,q),\delta_1/\kappa}(\kappa t)>\kappa t,(X^{\varepsilon,\delta}_{\kappa t},Y^{\varepsilon,\delta}_{\kappa t})\in U_{\kappa\delta_1}(t(p-\kappa t\hat{x}),q-\kappa\hat{y})\}}.$$

The second inequality comes from the periodicity of the process $(X^{\varepsilon,\delta}_t,Y^{\varepsilon,\delta}_t)$ and the

following relations:

$$U_\epsilon(t(p-sx-t(\tilde{x}-x)),q-y-(\tilde{y}-y)) \supset U_{\delta_1}(t(p-sx),q-y),$$

$$U_{(1-\kappa)\delta_1}(t(p-t\tilde{x}+(1-\kappa)tx),q-\tilde{y}+(1-\kappa)y) \supset U_{\kappa\delta_1}(t(p-\kappa t\hat{x}),q-\kappa\hat{y}) \tag{3.50}$$

where p and q are the fractional parts of $t\tilde{x}$ and $\tilde{y}$, respectively, and $(\hat{x},\hat{y}) = (x,y) + 2(\tilde{x}-x,\tilde{y}-y)$. Since l is bounded below, we see the following inequality from (3.49):

$$l \geq \varliminf_{t\to\infty}\tfrac{1}{t}\ln\inf_{(p,q)\in G^{\varepsilon,\delta}_1,(\tilde{x},\tilde{y})\in U_{2\delta_1}((x,y))} E^{\varepsilon,\delta}_{p,q}\exp\left\{\int_0^t c_h(X^{\varepsilon,\delta}_s,Y^{\varepsilon,\delta}_s)ds\right\} \tag{3.51}$$

$$\times\chi_{\tau_{(p,q),\eta}(t)>t,(X^{\varepsilon,\delta}_t,Y^{\varepsilon,\delta}_t)\in U_{\delta_1}(t(p-t\tilde{x}),q-\tilde{y})}.$$

If we denote the right side in (3.51) by $\varliminf_{t\to\infty}(p_{h,\delta_1\kappa^{-1}}(t))/t$, the function $p_{h,\eta}(t)$ is

semi–additive, i.e, $p_{h,\eta}(s+t) \geq p_{h,\eta}(s)+p_{h,\eta}(t)$, for $s,t > 0$ due to the Markov prop-

erty and the periodicity of the process $(X^{\varepsilon,\delta}_t,Y^{\varepsilon,\delta}_t)$, and $p_{h,\eta}(t) \leq t\max_{(x,y)}c(x,y)$

for $t > 0$. Thus, $\lim_{t\to\infty}p_{h,\eta}(t)/t$ exists and it is equal to $\sup_{t>0}p_{h,\eta}(t)/t$ due to the

semi–additive and the upper bound. Then, from (3.51), $l\geq \sup_{t>0}p_{h,\delta_1\kappa^{-1}}(t)/t$. Let

us define a function $p(t)$ as follows:

$$p(t) = \ln\inf_{(p,q)\in G^{\varepsilon,\delta}_1,(\tilde{x},\tilde{y})\in U_{2\delta_1}(x,y)} E^{\varepsilon,\delta}_{p,q}\left(\exp\left\{\int_0^t c(X^{\varepsilon,\delta}_s,Y^{\varepsilon,\delta}_s)ds\right\}\right. \tag{3.52}$$

$$\left.\times\chi_{(X^{\varepsilon,\delta}_t,Y^{\varepsilon,\delta}_t)\in U_{\delta_1}(t(p-t\tilde{x}),q-t\tilde{y})}\right).$$

Since $c_h(x,y) \uparrow c(x,y)$ as $h\downarrow 0$ and, by Fatou's lemma, $p_{h,\eta}(t) \uparrow p(t)$ as $h\downarrow 0$ and

$\eta\uparrow\infty$, we have

$$l \geq \sup_{t>0}(p(t)/t), \quad \text{as } h\downarrow 0,\kappa\downarrow 0. \tag{3.53}$$

53

Therefore, the proof is complete.□

Lemma 4 *For any compact* $K \subset \left\{ (x,y) \in G^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x,y) < 0 \right\}$,

$$\lim_{t \to \infty} \frac{1}{t} \ln \left(\inf_{(x,y) \in K, (tx,y) \in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t,tx,y) \right) \geq - \max_{(x,y) \in K} H^{\varepsilon,\delta}(x,y) \qquad (3.54)$$

Proof. For any compact set K, it is sufficient to prove that, for any $(x,y) \in G^{\varepsilon,\delta}$, $H^{\varepsilon,\delta}(x,y) > 0$, and $\epsilon > 0$, there exists a $\delta_1 > 0$ such that

$$\liminf_{t \to \infty} \tfrac{1}{t} \ln \left(\inf_{(\tilde{x},\tilde{y}) \in U_{\delta_1}(x,y),(t\tilde{x},\tilde{y}) \in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t,t\tilde{x},\tilde{y}) \right)$$

$$\geq \quad \liminf_{t \to \infty} \tfrac{1}{t} \ln \inf_{(p,q) \in G_1^{\varepsilon,\delta},(\tilde{x},\tilde{y}) \in U_{2\delta_1}(x,y)} E_{p,q}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\} \chi_A$$

$$\geq \quad - \sup_{(\tilde{x},\tilde{y}) \in U_{2\delta_1}(x,y)} H^{\varepsilon,\delta}(\tilde{x},\tilde{y}).$$

$$(3.55)$$

where $\chi_A = \chi_{(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta}) \in U_{\delta_1}(t(p-t\tilde{x}),q-\tilde{y})}$. The first inequality above was proved in Lemma 3. To show the second inequality in (3.55), due to the definitions of the measures $P_{x,y}^{\varepsilon,\delta,t}$ and random vectors $\eta_{x,y}^{\varepsilon,\delta,t}$,

$$\inf_{(p,q) \in G_1^{\varepsilon,\delta},(\tilde{x},\tilde{y}) \in U_{2\delta_1}(x,y)} E_{p,q}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\} \chi_A$$

$$\geq \quad \inf_{(\tilde{x},\tilde{y}) \in U_{2\delta_1}(x,y)} \inf_{(p,q) \in G_1^{\varepsilon,\delta}} P_{p,q}^{\varepsilon,\delta,t} \{ \eta_{p,q}^{\varepsilon,\delta,t} \in U_{\delta_1}(\tilde{x},\tilde{y}) \} \qquad (3.56)$$

$$\times \inf_{(p,q) \in G_1^{\varepsilon,\delta}} E_{p,q}^{\varepsilon,\delta} \exp \left\{ \int_0^t c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) ds \right\}.$$

From Theorem 3, we can see that, for any $h > 0$,

$$\liminf_{t \to \infty} \tfrac{1}{t} \ln \left(\inf_{(\tilde{x},\tilde{y}) \in U_{\delta_1}(x,y),(t\tilde{x},\tilde{y}) \in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t,t\tilde{x},\tilde{y}) \right)$$

$$\qquad \qquad (3.57)$$

$$\geq \quad -(S^{\varepsilon,\delta}(x,y) + h) + \lambda^{\varepsilon,\delta}(0) = -H^{\varepsilon,\delta}(x,y) - h.$$

Because of the compactness of K, the proof is complete.□

Now we will prove Theorem 9 in Section 3.2.

Proof of Theorem 9. Part(i) follows from Lemma 2. So we will prove part

(ii). Define the following sets:

$$\Upsilon(s) = \{(x, y) \in G_1^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x, y) = s\}$$
$$\widehat{\Upsilon}(s) = \{(x, y) \in G_1^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x, y) \leq s\}$$

(3.58)

Also, for any $\delta_1 > 0$ and $T > 1$,

$$\Gamma_T = [\{1\} \times \widehat{\Upsilon}(\delta_1)] \cup [\cup_{1 \leq t \leq T}\{t\} \times (t\Upsilon(\delta_1))].$$

(3.59)

From equation (3.3) and Lemma 4, it is easy to see that $w^{\varepsilon,\delta}(1, x, y) > 0$ for all

$(x, y) \in G^{\varepsilon,\delta}$ and, for sufficiently large t and all $(s, x, y) \in \Gamma_T$, $w^{\varepsilon,\delta}(s, x, y) \geq e^{-2\delta_1 t}$.

We introduce the Markov times, for $t > 0$, $\varsigma > 0$, and $h \in (0, 1)$:

$$\sigma_\Gamma^{\varepsilon,\delta}(t) = \min\{s \geq 0 : (t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) \in \Gamma_t\},$$
$$\sigma_h^{\varepsilon,\delta}(t) = \min\{s \geq 0 : w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) \geq h\},$$
$$\tau_\varsigma^{\varepsilon,\delta}(t) = \min\{s \geq 0 :\mid X_s^{\varepsilon,\delta} - x \mid> \varsigma t\}.$$

(3.60)

If $w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) < h$ for all $s \in [0, t]$, then we will take $\sigma_h^{\varepsilon,\delta}(t) = +\infty$. In

general, using the Ito's equation, $w(t, x, y) = P_{x,y}\{\tau_D > t\}$ satisfies the problem:

$$\frac{\partial w}{\partial t} = Lw, \qquad t > 0, (x, y) \in D$$
$$w(0, x, y) = 1, \quad w(t, x, y)|_{(x,y) \in \partial D} = 0.$$

(3.61)

where $\tau_D = \min\{t : (X_t, Y_t) \notin \partial D\}$. From this, one can prove that for any $\varsigma > 0$,

as $t \to \infty$,

$$\sup_{(x,y) \in G^{\varepsilon,\delta}} P_{x,y}^{\varepsilon,\delta}\{\tau_\varsigma^{\varepsilon,\delta}(t) \leq t\} \to 0.$$

(3.62)

Choose ς so that the distance between the ς–neighborhood of the set K and the set

$\{(x, y) \in G^{\varepsilon,\delta} : H^{\varepsilon,\delta}(x, y) > 0\}$ is positive. Then there exists a number $\kappa \in (0, 1)$

such that for all $(x, y) \in K$,

$$\kappa t < \sigma_\Gamma^{\varepsilon,\delta}(t) \leq t - 1, \qquad \text{if } \tau_\varsigma^{\varepsilon,\delta}(t) > t. \tag{3.63}$$

Here, if $\tau_\varsigma^{\varepsilon,\delta}(t) > t$, then the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ is still in the ς–neighborhood of the set K. By the strong Markov property of the process $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$, for any Markov time $\tau^{\varepsilon,\delta}$ and $(x, y) \in G^{\varepsilon,\delta}$,

$$
\begin{aligned}
w^{\varepsilon,\delta}(t, x, y) &= E_{x,y}^{\varepsilon,\delta} \exp\left\{ \int_0^{\tau^{\varepsilon,\delta} \wedge t} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})) ds \right\} \\
&\quad \times w^{\varepsilon,\delta}(t - (\tau^{\varepsilon,\delta} \wedge t), X_{\tau^{\varepsilon,\delta} \wedge t}^{\varepsilon,\delta}, Y_{\tau^{\varepsilon,\delta} \wedge t}^{\varepsilon,\delta}).
\end{aligned}
\tag{3.64}
$$

If we put $\tau^{\varepsilon,\delta} = \sigma_h^{\varepsilon,\delta}$, since $0 < w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}) < h$ for $0 \leq s \leq \sigma_h^{\varepsilon,\delta}$, we can have the following lower bound:

$$
\begin{aligned}
w^{\varepsilon,\delta}(t, x, y) &= E_{x,y}^{\varepsilon,\delta} \exp\left\{ \int_0^{\sigma_h^{\varepsilon,\delta}} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})) ds \right\} \\
&\quad \times w^{\varepsilon,\delta}(t - \sigma_h^{\varepsilon,\delta}, X_{\sigma_h^{\varepsilon,\delta}}^{\varepsilon,\delta}, Y_{\sigma_h^{\varepsilon,\delta}}^{\varepsilon,\delta}) \cdot \chi_{\{\sigma_h^{\varepsilon,\delta} \leq t\}} \\
&\geq E_{x,y}^{\varepsilon,\delta} \left(\exp\left\{ \int_0^{\sigma_h^{\varepsilon,\delta}} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})) ds \right\} \cdot h \cdot \chi_{\{\sigma_h^{\varepsilon,\delta} \leq t\}} \right) \\
&\geq h P_{x,y}^{\varepsilon,\delta}\{\sigma_h^{\varepsilon,\delta} \leq t\}.
\end{aligned}
\tag{3.65}
$$

From the definition of the Markov time $\sigma_h^{\varepsilon,\delta}$, the second inequality can be true and the last inequality follows from $c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}) > 0$.

Since $\overline{\lim}_{t \to \infty} \sup_{(x,y) \in G^{\varepsilon,\delta}} w^{\varepsilon,\delta}(t, x, y) \leq 1$ and (3.65), if we can prove that for any $h \in (0.1)$,

$$P_{tx,y}^{\varepsilon,\delta}\{\sigma_h^{\varepsilon,\delta}(t) \geq t, \tau_\varsigma^{\varepsilon,\delta}(t) \geq t\} \to 0 \quad \text{uniformly in } (x, y) \in K \text{ as } t \to \infty \tag{3.66}$$

then our proof for the part (ii) of Theorem 9 can be completed.

$$P_{tx,y}^{\varepsilon,\delta}\{\sigma_h^{\varepsilon,\delta}(t) > t, \tau_\varsigma^{\varepsilon,\delta}(t) > t\} \leq P_{tx,y}^{\varepsilon,\delta}\{\kappa t < \sigma_\Gamma^{\varepsilon,\delta}(t) \leq \sigma_h^{\varepsilon,\delta}(t)\}$$

$$\leq \ E_{tx,y}^{\varepsilon,\delta} \exp\left\{\delta_1 t + \tfrac{1}{2}\int_0^{\sigma_\Gamma^{\varepsilon,\delta}(t)} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}))ds\right\}$$

$$\times (w^{\varepsilon,\delta}(t - \sigma_\Gamma^{\varepsilon,\delta}(t), X_{\sigma_\Gamma^{\varepsilon,\delta}(t)}^{\varepsilon,\delta}, Y_{\sigma_\Gamma^{\varepsilon,\delta}(t)}^{\varepsilon,\delta}))^{1/2}$$

$$\times \exp\left\{-\tfrac{1}{2}\int_0^{\sigma_\Gamma^{\varepsilon,\delta}(t)} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}))ds\right\} \times \chi_{\{\kappa t < \sigma_\Gamma^{\varepsilon,\delta}(t) \leq \sigma_h^{\varepsilon,\delta}(t)\}}$$

$$\leq \ E_{tx,y}^{\varepsilon,\delta} \exp\left\{\delta_1 t + \tfrac{1}{2}\int_0^{\sigma_\Gamma^{\varepsilon,\delta}(t)} c(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}, w^{\varepsilon,\delta}(t - s, X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta}))ds\right\}$$

$$(w^{\varepsilon,\delta}(t - \sigma_\Gamma^{\varepsilon,\delta}(t), X_{\sigma_\Gamma^{\varepsilon,\delta}(t)}^{\varepsilon,\delta}, Y_{\sigma_\Gamma^{\varepsilon,\delta}(t)}^{\varepsilon,\delta}))^{1/2} \times \exp\left\{-\tfrac{1}{2}\int_0^{\kappa t} c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\},$$

$$(3.67)$$

where $c_h(x, y) = \inf_{w^{\varepsilon,\delta} \in [0,h]} c(x, y, w^{\varepsilon,\delta})$. Here second inequality follows from the Hölder inequality and all $(s, x, y) \in \Gamma_T$, $w^{\varepsilon,\delta}(s, x, y) \geq e^{-2\delta_1 t}$. With $\tau^{\varepsilon,\delta} = \sigma_\Gamma^{\varepsilon,\delta}(t)$ in (3.64), we can induce that

$$P_{tx,y}^{\varepsilon,\delta}\{\sigma_h^{\varepsilon,\delta}(t) > t, \tau_\varsigma^{\varepsilon,\delta}(t) > t\}$$

$$\leq \ e^{\delta_1 t}(w^{\varepsilon,\delta}(t, tx, y))^{1/2}\left(E_{tx,y}^{\varepsilon,\delta} \exp\left\{-\int_0^{\kappa t} c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\}\right)^{1/2}$$

$$(3.68)$$

Since the function $c_h(x, y)$ is non–negative and not identically equal to zero,

$$\lim_{t\to\infty} \frac{1}{t} \ln \sup_{(x,y)\in G^{\varepsilon,\delta}} E_{x,y}^{\varepsilon,\delta} \exp\left\{-\int_0^t c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\} = \lambda_h^{\varepsilon,\delta} < 0. \qquad (3.69)$$

From (3.40), $\lambda_h^{\varepsilon,\delta}$ is the eigenvalue of the operator $L^{\varepsilon,\delta} - c_h(x, y)$ in $G_1^{\varepsilon,\delta}$ such that the corresponding eigenfunction is positive. If we choose $\delta_1 < (\kappa \mid \lambda_h^{\varepsilon,\delta} \mid)/2$, then

$$P_{tx,y}^{\varepsilon,\delta}\{\sigma_h^{\varepsilon,\delta}(t) > t, \tau_\varsigma^{\varepsilon,\delta}(t) > t\}$$

$$\leq \ e^{\delta_1 t}(w^{\varepsilon,\delta}(t, tx, y))^{1/2}\left(E_{tx,y}^{\varepsilon,\delta} \exp\left\{-\tfrac{1}{2}\int_0^{\kappa t} c_h(X_s^{\varepsilon,\delta}, Y_s^{\varepsilon,\delta})ds\right\}\right)^{1/2} \qquad (3.70)$$

$$\leq \ e^{(\delta_1 - \kappa|\lambda_h^{\varepsilon,\delta}|/2)t}(w^{\varepsilon,\delta}(t, tx, y))^{1/2} \to 0 \qquad \text{as } t \to \infty,$$

where $w^{\varepsilon,\delta}(t, tx, y)$ is bounded. This implies (3.66), so the proof is completed. $\square$

Since we can prove the Theorem 10 in the same way, we will not repeat the proof here. In the following section, we will prove the convergence of the asymptotic wave propagation velocity as $\varepsilon \downarrow 0$.

3.4   Convergence of asymptotic speed of wave front propagation

According to Theorem 9, the wave front propagation for $w^{\varepsilon,\delta}$ in periodic domain $G^{\varepsilon,\delta}$ is associated with a simple eigenvalue corresponding to the differential operator $L^{z,\varepsilon,\delta}$ in (3.25). In this section, we will investigate the convergence of the asymptotic speed of the wave front propagation as $\varepsilon \downarrow 0$ for fixed $\delta > 0$.

First, consider the eigenvalue problem for a nonsymmetric elliptic operator $L + c$:

$$(L + c)w = -\sum_{i,j=1}^{n} a_{ij} w_{x_i x_j} + \sum_{i=1}^{n} b_i w_{x_i} + cw = \lambda w, \quad \text{in a domain } G \qquad (3.71)$$

with w subject to the mixed boundary conditions

$$\begin{cases} w = 0 & \text{on } \Gamma_1, \\ \frac{\partial w}{\partial \nu} = 0 & \text{on } \Gamma_2. \end{cases} \qquad (3.72)$$

Here G is bounded, $\Gamma_1 \cup \Gamma_2 = \partial G$, and $\partial/\partial \nu$ is a directional derivative in a direction outward from G. Let us assume that the matrix a_{ij} is symmetric and positive definite, and the coefficients a_{ij} and b_i are uniformly bounded. Let λ_1 be the principal eigenvalue of the nonsymmetric elliptic operator $L + c$ in (3.71). Since the operator $L + c$ is not equal to its formal adjoint, in general, $L + c$ will have complex eigenvalues and eigenfunctions. However it was proven in [22] that at least the principal eigenvalue λ_1 for the nonsymmetric elliptic operator $L + c$, taken with the mixed

58

boundary condition, in (3.71) is real and corresponding to a positive eigenfunction w_1 within G. Furthermore, if λ is any other eigenvalue, we have $\mathrm{Re}\lambda \geq \lambda_1$. The principal eigenvalue λ_1 is simple, that is, if u is another corresponding eigenfunction for λ_1, then u is a multiple of w_1. If the boundary condition in (3.72) is simply a zero boundary, the same conclusion holds for the principal eigenvalue for the non-symmetric elliptic operator $L + c$.

Thus, the principal eigenvalue $\lambda_1^{\varepsilon,\delta}$ for the nonsymmetric elliptic operator $L^{z,\varepsilon,\delta}$, taken with the mixed boundary conditions, in (3.25) is real, simple and corresponding to a positive eigenfunction. It is explicitly given by

$$\lambda_1^{\varepsilon,\delta}(z) = \sup_{w} \inf_{(x,y)} \left[\frac{\frac{1}{2}\triangle w(x,y) - (z, \triangledown w(x,y)) + (c(x,y) + \frac{1}{2}(z,z))w(x,y)}{w(x,y)} \right] \quad (3.73)$$

where the "sup" taken over functions $w \in C^\infty(\bar{G}^{\varepsilon,\delta})$ with $w > 0$ in $G^{\varepsilon,\delta}$, taken the mixed conditions (3.1) on $\partial G^{\varepsilon,\delta}$, and the "inf" taken over points $(x,y) \in G^{\varepsilon,\delta}$.

Similarly, the principal eigenvalue $\lambda_1^{\delta}(z)$ of the operator $L^{z,\delta}$ in (3.29) is given by

$$\lambda_1^{\delta}(z) = \sup_{w} \inf_{x} \left[\frac{\frac{1}{2}\frac{d^2 w}{dx^2} + \left(\frac{1}{2}\frac{d(\ln(V^\delta(x)))}{dx} - z\right)\frac{dw}{dx} + \left(c(x) - \frac{z}{2}\frac{d(\ln(V^\delta(x)))}{dx} + \frac{z^2}{2}\right)w}{w} \right]$$
$$(3.74)$$

where the "sup" taken over functions $w \in C^\infty(\bar{G}^\delta)$ with $w > 0$ in G^δ, taken the initial condition (3.5) on ∂G^δ, and the "inf" taken over points $x \in G^\delta$.

Proposition 3 *For any $1 + n$–dimensional vector $z = (z_1, 0) \in R \times R^n$ with an integer z_1, the principal eigenvalue $\lambda_1^{\varepsilon,\delta}(z)$ for the operator $L^{z,\varepsilon,\delta}$ in (3.25) converges as $\varepsilon \downarrow 0$ to $\lambda_1^{\delta}(z_1)$, which is the principal eigenvalue for the operator $L^{z,\delta}$ with a corresponding positive eigenfunction meeting the boundary condition in (3.5).*

Proof. The processes $(X_t^{\varepsilon,\delta}(z), Y_t^{\varepsilon,\delta}(z))$ in (3.35) differ from the processes $(X_t^{\varepsilon,\delta}, Y_t^{\varepsilon,\delta})$ in (3.2) only by a drift $z = (z_1, 0)$. Since the processes $X_t^{\varepsilon,\delta}$, $0 \le t \le T$, converge weakly in the space of continuous functions on R as $\varepsilon \downarrow 0$ for fixed $\delta > 0$ to the diffusion process X_t^δ by Theorem 1, this implies the weak convergence of the processes $X_t^{\varepsilon,\delta}(z)$ to the process $X_t^\delta(z_1)$, which is the solution to the stochastic differential equation, for $\delta > 0$,

$$dX_t^\delta(z_1) = dW_t^1 + \left(b(X_t^\delta(z_1)) - z_1\right) dt, \quad b(x) = \frac{1}{2}\frac{d}{dx}\ln(V^\delta(x)), \tag{3.75}$$

where $V^\delta(x)$ is the volume of G_x^δ in R^n. According to the paper [1], the principal eigenvalue $\lambda_1^{\varepsilon,\delta}(z)$ for the operator $L^{z,\varepsilon,\delta}$, taken with the mixed boundary condition in (3.1) is given by

$$\lambda_1^{\varepsilon,\delta}(z) = \lim_{t\to\infty} \frac{1}{t}\log \sup_{x\in G_1^{\varepsilon,\delta}} \int_{G_1^{\varepsilon,\delta}} P^{\varepsilon,\delta}(t, x, dy), \tag{3.76}$$

where $x = (x_1, x_Y) \in R \times R^n$ and $L^{z,\varepsilon,\delta}$ is the infinitesimal generator of a semigroup $T_t^{\varepsilon,\delta}$ given by a measure $P^{\varepsilon,\delta}(t, x, dy)$. Similarly the principal $\lambda_1^\delta(z)$ for the operator $L^{z,\delta}$ meeting with the boundary condition in (3.5) can be expressed in terms of a corresponding measure $P^\delta(t, x, dy)$. Therefore, due to the weak convergence of $P^{\varepsilon,\delta}$ to P^δ, this proof can be completed. $\square$

If $\min_{z=(z_1,z_Y)\in R\times R^n} \lambda^{\varepsilon,\delta}(z) > 0$, due to the definition of the function $H^{\varepsilon,\delta}$, $H^{\varepsilon,\delta}(0,0)$ is negative and the equation $H^{\varepsilon,\delta}(\nu^{\varepsilon,\delta}e) = 0$ has a unique positive solution $\nu^{\varepsilon,\delta} = \nu^{\varepsilon,\delta}(e)$ given by:

$$\nu^{\varepsilon,\delta}(e) = \inf_{z:(e,z)>0} \frac{\lambda_1^{\varepsilon,\delta}(z)}{(e, z)} \tag{3.77}$$

Here, $\nu^{\varepsilon,\delta}(e)$ is the asymptotic wave propagation velocity in the direction of the vector e, which is a unit vector in $R \times R^n$.

Similarly, the asymptotic wave propagation velocity ν^δ for the solution $w^\delta(t, tx)$ of the problem (3.5) is given by

$$\nu^\delta = \inf_{z_1} \frac{\lambda_1^\delta(z_1)}{z_1}, \tag{3.78}$$

where the infimum is taken over all positive integer $z_1 \in R^+$.

Proposition 4 *The asymptotic wave propagation velocity $\nu^{\varepsilon,\delta}(e_1)$ in the direction of the vector $e_1 = (1,0) \in R \times R^n$ converges as $\varepsilon \downarrow 0$ to ν^δ.*

The proof for Proposition 4 follows from Proposition 3 by the definition of the asymptotice wave propagation velocity.

As a future work, we will investigate all these procedures for the problem (3.7) to examine the asymptotic behavior of the function w as $t \to \infty$. According to the paper [1], since the limiting process is still a Feller process, we expect that the corresponding principal eigenvalue for the problem (3.7) in periodic domains could be expressed with the corresponding measure P of a limiting process X_t. Then, as we have been seen the weak convergence of P_x^δ to P_x in Theorem 7, it will not be difficult to see the convergence of asymptotic speed of wave propagation of ν^δ to ν as $\delta \downarrow 0$, where ν is the asymptotic wave propagation speed for the solution $w(t, tx)$ of the problem (3.7).

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