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Covering Analysis of Flooding Algorithms on a 1-D Continuum Model

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1 Continuum Models on a line

The analysis assumes the nodes are deployed according a *Poisson point process* on a infinite line. Let $\rho \geq 0$ be the range of any node. We assume that the transmission and reception range are identical for all the nodes. Thus the connection model for this geometric random graph is the so called *Boolean connection model*. Let $\mathcal{X} = \{X_i \in \mathbb{R}, i \geq 0\}$ denote the random sequence of points of the Poisson point process of density λ. Let us denote the recurrence interval by $I_i = (X_{i-1}, X_i]$. For any domain $D \in \mathbb{R}$ let $l(D)$ and $\mathbb{N}(D)$ denote the *length* and the number of points in D respectively. Then the Poisson point process has the following properties.

1. For mutually disjoint domains $D_1, D_2, \cdots, D_k \in \mathbb{R}$ the random variables $\mathbb{N}(D_1), \mathbb{N}(D_2), \cdots, \mathbb{N}(D_k)$ are mutually independent.

2. For any bounded domain $D \in \mathbb{R}$ we have for every $k \geq 0$

$$\mathbb{P}(\mathbb{N}(D) = k) = e^{-\lambda l(D)} \frac{(\lambda l(D))^k}{k!}$$

The point process $\mathcal{X}$ induces a geometric random graph $G(V, E)$. If points $X_i, X_j \in \mathcal{X}$ satisfy $|X_i - X_j| \leq \rho$ then $i, j \in V$ are adjacent in G.

2 MPR Algorithm

To construct flooding networks for wireless ad-hoc networks, several algorithms which use local neighbourhood information have been proposed in the literature. One such popular algorithm which is used in *Optimized Link State Algorithm* (OLSR) is the *Multipoint Relaying Algorithm.* The basic idea of this algorithm is to select a subset of neighbours called *Multipoint relays* (MPRs), which would rebroadcast any flooding traffic. The selection heuristic is to *select a minimal set of one hop neighbours which cover all the two hop neighbours.* For the analysis on a line, this selection heuristic reduces to the following algorithm.

Algorithm Every host $h \in V$ selects its farthest neighbour in either sides as its MPRs, if they cover some two hop neighbours.

Suppose every node in the graph carries out the above algorithm we obtain a flooding subgraph G^f whose vertices are the MPRs selected by some host. We attempt to characterise the stochastic distribution of this flooding subgraph. As a first step, we show the computation of a thinning probability of the original graph G.

2.1 Thinning probability

In this subsection we present the calculation of the marginal thinning probability. The marginal thinning probability is the probability of removing a node in the original graph. This would corresponds to the nodes that are not chosen as MPRs by any of the nodes in the graph G. Let us consider a realization of the point process $\mathcal{X}$ as shown in figure 1.

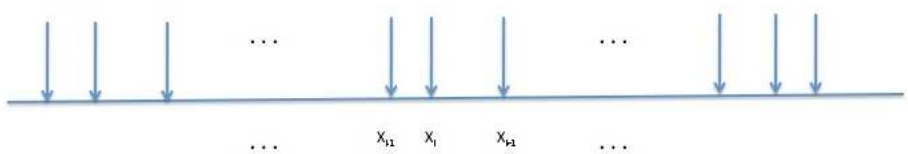


Figure 1: Realization of the Point Process $\mathcal{X}$

We next show by a series of arguments that to calculate the probability that any node $X_i \in \mathcal{X}$ is retained in the flooding graph G^f, is dependent only on the two points $X_{i-1}, X_{i+1} \in \mathcal{X}$ which are to the either side of it. The initial observation is that if either $|X_i - X_{i-1}| > \rho$ or $|X_{i+1} - X_i| > \rho$, then X_i is not chosen as an MPR. This is because in both of the above cases, X_i is not a neighbour to any host or does not cover any two hop neighbours. Thus it voids at least one of the selection criteria of the algorithm. The feasible configuration for the points for i likely to be chosen as an MPR is shown in figure 2.

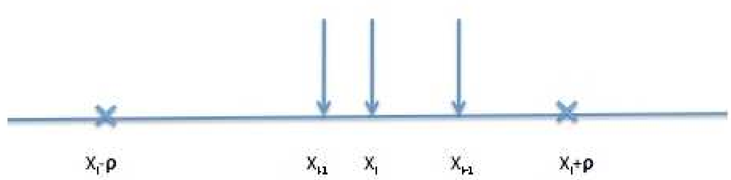


Figure 2: Feasible configuration

Suppose i is chosen as an MPR of any host $h \in V$, it should chosen as a *right* or *left* MPR of that host h. Those hosts which choose i as a right MPR lie to the left of i within a distance ρ from X_i. Next, i is chosen as an MPR only if it is the farthest neighbour of any host. Thus hosts to the left can lie only in the $D_i^r = [X_i - \rho, X_{i+1} - \rho)$. By a similar argument the host to the right (those which select i as their left MPR) must lie in $D_i^l = (X_{i-1} - \rho, X_i]$. Thus the event that i is chosen as an MPR is dependent on the number of nodes in D_i^r and D_i^l. That is,

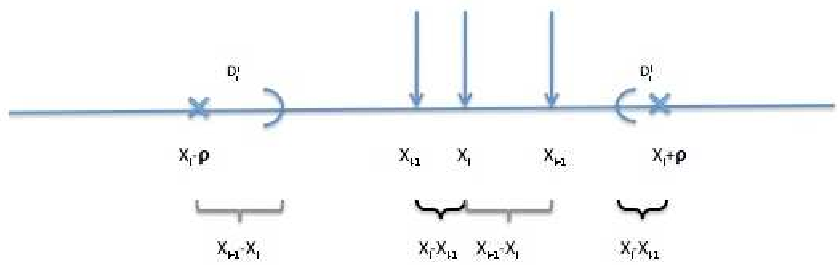
$$\{i \text{ is chosen as a MPR}\} = \{\mathbb{N}(D_i^l) + \mathbb{N}(D_i^r) > 0\} \cap \{l(I_i) \leq \rho\} \cap \{l(I_{i+1}) \leq \rho\} \tag{1}$$

The next observation characterises the relative positions of X_{i-1} and X_{i+1}. Consider the two cases shown in figure 3. In the first case shown in sub-figure 3a, the positions of the immediate neighbours do not affect the events in D_i^l and D_i^r. In the other case shown in sub-figure 3b, immediate neighbours lie within D_i^l and D_i^r making the event in equation 1 is true *surely*. As seen in the figures, the two cases are characterized by the relative positions of X_{i-1} and X_{i+1}. Case 1 corresponds to $|X_{i+1} - X_{i-1}| \leq \rho$ and case 2 corresponds $|X_{i+1} - X_{i-1}| > \rho$. We observe all the events are characterized by the relative positions of X_{i-1}, X_i and X_{i+1}. In other words, the events can be characterized by $l(I_{i+1})$ and $l(I_i)$ distributions of which are well defined. The various conditions are illustrated figure 4.

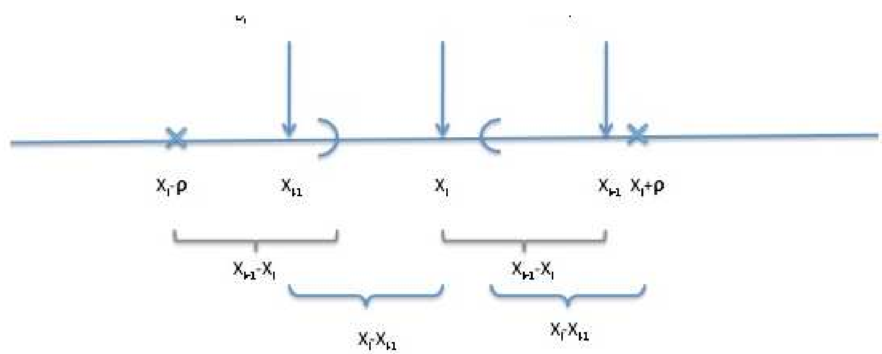
We now proceed to show the computation of the *marginal non-thinning* probability of a node i in the relay selection algorithm. The joint density of $(l(I_i), l(I_{i+1}))$ is given by

$$f_{l(I_i),l(I_{i+1})}(\tau_i, \tau_{i+1}) = \lambda^2 \exp(-\lambda(\tau_i + \tau_{i+1}))$$

$$\begin{aligned}
& \mathbb{P}(\{i \text{ is chosen as an MPR}\}) \\
= & \int_{\tau_{i+1}=0}^{\infty} \int_{\tau_i=0}^{\infty} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
= & \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
= & \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho-\tau_{i+1}} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
& + \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=\rho-\tau_{i+1}}^{\rho} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
= & \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho-\tau_{i+1}} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
& + \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=\rho-\tau_{i+1}}^{\rho} 1 \times f d\tau_i d\tau_{i+1} \\
= & \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho-\tau_{i+1}} \mathbb{P}(\{\mathbb{N}(\tau_i) + \mathbb{N}(\tau_{i+1}) > 0\}/l(I_i) = \tau_i, l(I_{i+1}) = \tau_{i+1}) f d\tau_i d\tau_{i+1} \\
& + \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=\rho-\tau_{i+1}}^{\rho} 1 \times f d\tau_i d\tau_{i+1}
\end{aligned}$$



(a) Case 1



(b) Case 2

Figure 3: Host regions

$$
\begin{aligned}
&= \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho-\tau_{i+1}} \mathbb{P}(\{\mathbb{N}(\tau_i) + \mathbb{N}(\tau_{i+1}) > 0\}) f d\tau_i d\tau_{i+1} \\
&\quad + \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=\rho-\tau_{i+1}}^{\rho} 1 \times f d\tau_i d\tau_{i+1} \\
&= \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho-\tau_{i+1}} (1 - exp(-\lambda(\tau_i + \tau_{i+1}))) f d\tau_i d\tau_{i+1} \\
&\quad + \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=\rho-\tau_{i+1}}^{\rho} 1 \times f d\tau_i d\tau_{i+1} \\
&= \frac{3}{4} + (\frac{5}{4} + \frac{\lambda\rho}{2}) \exp(-2\lambda\rho) - 2\exp(-\lambda\rho)
\end{aligned}
$$

3 2-Vajra Algorithm

The 2-vajra algorithm tries to provide robustness to the flooding network by selecting two relay nodes to reach every two hop neighbour whenever possible. Again the selection algorithm on a line reduces to

Algorithm Every host $h \in V$ selects two of its farthest neighbour in either sides as its relay whenever possible, if they cover some two hop neighbours.

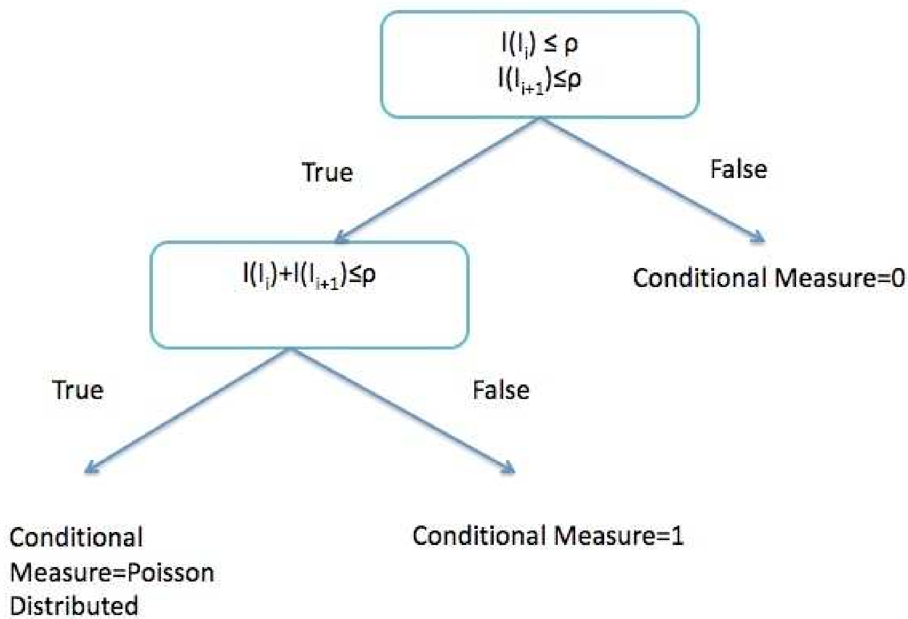


Figure 4: Conditional Measures for each of the conditions

Suppose every node in the graph carries out the above algorithm we obtain a flooding subgraph G^f whose vertices are the vajra relays selected by some host. As with the previous analysis, we compute the thinning probability of the original graph G.

3.1 Thinning Probability

In this subsection we show that the thinning probability for the *2-vajra* algorithm depends only on distribution of the relative positions of the X_{i-2}, X_{i-1}, X_i, X_{i+1}, X_{i+2}. As in the previous analysis $|X_i - X_{i-1}| \leq \rho$ and $|X_{i+1} - X_i| \leq \rho$ for i to be a feasible relay. The host regions of interest, for i to be a right MPR in this case would be $D_i^r = [X_i - \rho, X_{i+2} - \rho)$. This accounts for i being one of the two farthest neighbour of some host to its left. In this case however D_i^r is restricted by the presence of X_i and hence is given by $D_i^r = [X_i - \rho, \min\{X_i, X_{i+2} - \rho\})$. By similar arguments the right host region is given by $D_i^l = (\max\{X_i, X_{i-2} + \rho\}, X_i + \rho]$. These host regions are shown in figure 5. In this case however, there are many other conditions that we need to consider in the computation. Supposing if either $X_{i+2} > X_i + \rho$ or

$X_{i-2} < X_i - \rho$, then the host regions for sure contain X_{i+1} or X_{i-1}. In this case the conditional measure of the event $\{i \text{ is chosen an relay}\}$ is 1. This situation is illustrated in sub-figure 6a. Otherwise, it might happen that both X_{i-2} and X_{i+2} lie within the radio range of X_i. This will give rise to another two cases. Suppose $|X_{i+2}-X_{i-2}| \leq \rho$, then both D_i^l and D_i^r have a *Poisson* number of points independent of the $X_{i-2}, X_{i-1}, X_i, X_{i+1}$ and X_{i+2}. This is shown in sub-figure 6b. Otherwise as illustrated in sub-figure 6c atleast one host lies in the host region making the conditional measure be to 1 again. These various conditions are shown concisely in figure 7

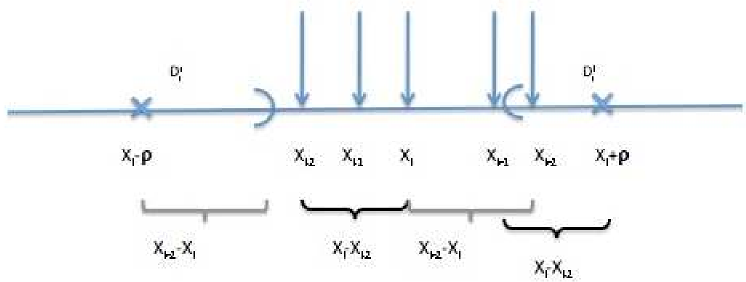


Figure 5: Host regions for 2-Vajra Construction

We now proceed to show the computation of the *marginal non-thinning* probability of a node i in the relay selection algorithm. The joint density of $(l(I_{i-1}), l(I_i), l(I_{i+1}), l(I_{i+2}))$ is given by

$$f_{l(I_{i-1}),l(I_i),l(I_{i+1}),l(I_{i+2})}(\tau_{i-1}, \tau_i, \tau_{i+1}, \tau_{i+2}) = \lambda^4 \exp(-\lambda(\tau_{i-1} + \tau_i + \tau_{i+1} + \tau_{i+2}))$$

$$\begin{aligned}
\mathcal{I} &= \mathbb{P}(\{i \text{ is chosen as an MPR}\}) \\
&= \textstyle\int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho} \int_{\tau_i+2=0}^{\infty} \int_{\tau_{i-1}=0}^{\infty} \mathbb{P}(\{i \text{ is an MPR}\}/l(I_{i-1})=\tau_{i-1}, l(I_i)=\tau_i, l(I_{i+1})=\tau_{i+1}, l(I_{i+2})=\tau_{i+2}) \\
&\qquad f_{l(I_{i-1}),l(I_i),l(I_{i+1}),l(I_{i+2})}(\tau_{i-1}, \tau_i, \tau_{i+1}, \tau_{i+2}) d\tau_{i+2} d\tau_{i+2} d\tau_i d\tau_{i-1}
\end{aligned}$$

For the sake of clarity we break down integral I into three sub-integrals $\mathcal{I}_1$,$\mathcal{I}_2$ and $\mathcal{I}_3$.

$$\mathcal{I} = \int_{\tau_{i+1}=0}^{\rho} \int_{\tau_i=0}^{\rho} (\mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3) d\tau_{i+1} d\tau_i$$

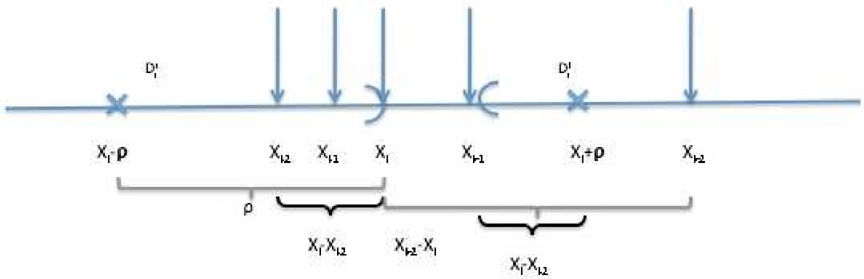
$$\begin{aligned}
\mathcal{I}_1 &= \int_{\tau i+2=0}^{\rho-\tau_{i+1}-\tau_i} \int_{\tau_{i-1}=\rho-\tau_{i+1}-\tau_i-\tau_{i+2}}^{\rho-\tau_i} 1 \times f d\tau_{i-1} d\tau_{i+2} \\
&\quad + \int_{\tau i+2=\rho-\tau_{i+1}-\tau_i}^{\rho-\tau_{i+1}} \int_{\tau_{i-1}=0}^{\rho-\tau_i} 1 \times f d\tau_{i-1} d\tau_{i+2}
\end{aligned}$$

$$\mathcal{I}_2 = \int_{\tau i+2=0}^{\rho-\tau_{i+1}-\tau_i} \int_{\tau_{i-1}=0}^{\rho-\tau_{i+1}-\tau_i-\tau_{i+2}} (1-\exp(-\lambda(\tau_{i-1}+\tau_i+\tau_{i+1}+\tau_{i+2})) \times f d\tau_{i-1} d\tau_{i+2}$$

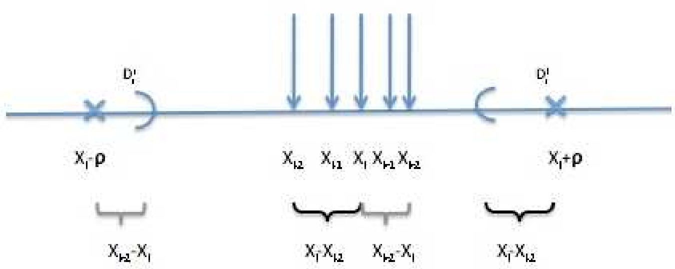
$$\begin{aligned}
\mathcal{I}_3 &= \int_{\tau i+2=0}^{\infty} \int_{\tau_{i-1}=\rho-\tau_i}^{\infty} 1 \times f d\tau_{i-1} d\tau_{i+2} \\
&\quad + \int_{\tau i+2=\rho-\tau_{i+1}}^{\infty} \int_{\tau_{i-1}=0}^{\rho-\tau_i} 1 \times f d\tau_{i-1} d\tau_{i+2}
\end{aligned}$$

These integrals clearly demark the conditions described in figure 7. On solving these integrals we obtain

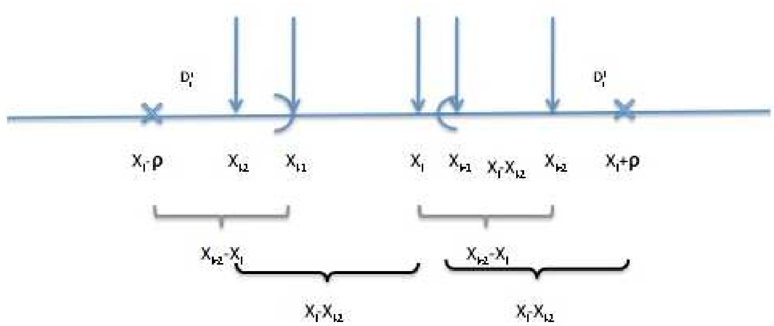
$$\mathcal{I} = \frac{1}{16}(15 - 32\exp(-\lambda\rho) + (18 + 4\lambda^2\rho^2)\exp(-2\lambda\rho) - \exp(-4\lambda\rho))$$



(a) Case 1



(b) Case 2



(c) Case 3

Figure 6: Host regions

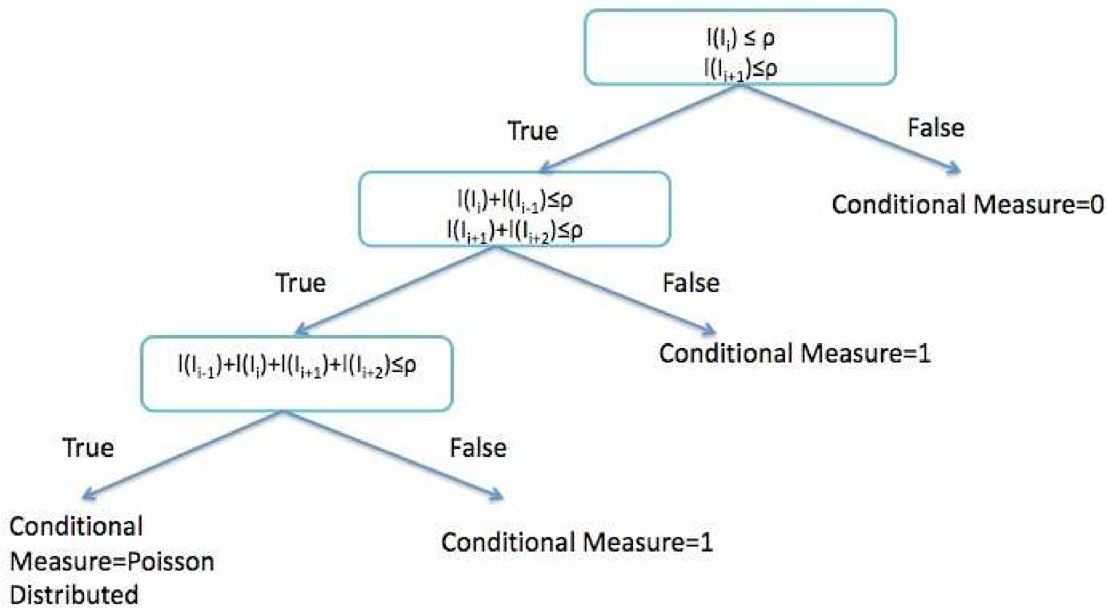


Figure 7: Conditional Measures for the 2-vajra scheme

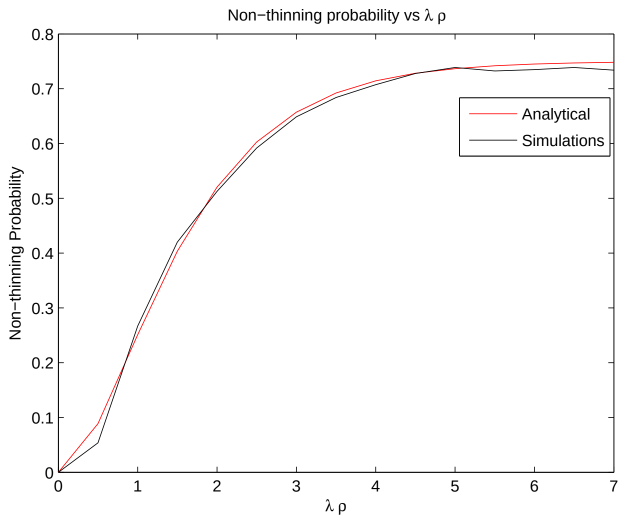


Figure 8: Validation of analysis for MPR thinning

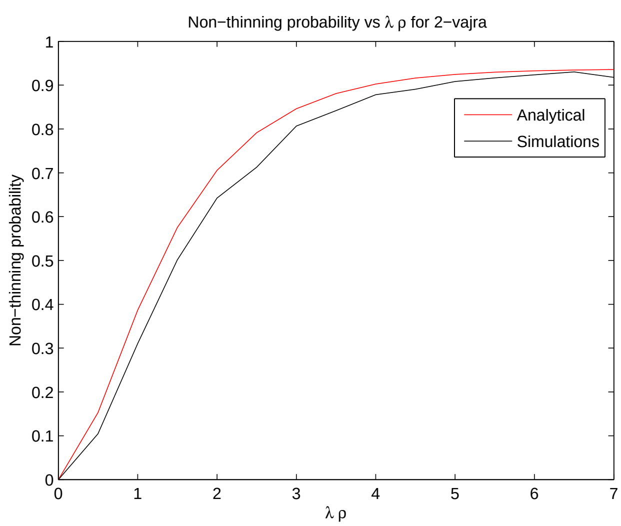


Figure 9: Validation of analysis for 2-vajra thinning