ABSTRACT

Title of thesis: Jointly optimal placement
and power allocation of nodes
of a wireless network

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In this thesis, we investigate the optimal design of wireless networks. We consider wireless networks that have fixed and movable nodes, and we assume that all nodes feature adjustable transmission power. Hence, we aim at maximizing network centric objectives, by optimizing over admissible choices of the positions of the movable nodes as well as the transmission power at all the nodes. We adopt exponential path loss, which is a realistic assumption in urban and sub sea environments, and we propose ways of using this assumption to obtain a tractable optimization problem. Our formulation allows for the optimization of typical network centric objectives, such as power and throughput. It also allows signal-to-interference based constraints, such as rate-regions and outage probabilities, under the high signal to interference regime. We show that our optimization paradigm is convex and that it can be solved up to an arbitrary degree of accuracy via geometric programming techniques. By using a primal-dual decomposition, we also provide a case-study that illustrates how certain instances of our optimization paradigm can be solved via distributed iterative algorithms. We show that such a solution method

also leads to a convenient layering in the primal step, whereby the power allocation and the node placement become two independent sub-problems.

Jointly optimal placement and power allocation of nodes of a wireless network

by

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Chapter 1

Introduction

Optimal node placement in wireless networks has received significant attention in the networking, robotics and computer science research communities. Examples of optimal placement paradigms are the maximization of the coverage of a sensor network [1], or the design of a wireless network so as to minimize the number of relays [3, 2], under a combination of power, longevity and rate-region constraints. The resulting optimization paradigm depends on the model of the wireless medium, the cost function and the constraints. Some of the existing paradigms are inherently combinatorial, while others rely on suboptimal strategies. In this thesis, we observe that, by adopting an exponential path loss model, we can integrate optimal node placement in existing convex programming techniques, which so far have been used for optimal power allocation in wireless networks. Hence, we obtain an optimization paradigm, for wireless network design, that is jointly parameterized by the power allocation of all nodes and by the placement of nodes that can be moved. Notice that exponential path loss is characteristic of high absorption media, such as radio frequency communication in urban [9] and in sub sea environments [5].

Our formulation is general enough to model the effects of interference and to include constraints on the rate region and on the outage probability at pre-defined routing paths, in the high signal to interference regime. In addition, we show that our paradigm

is convex and that it can be solved with arbitrary accuracy using geometric programming techniques, which are highly desirable due to their guaranteed polynomial time properties [11, 16]. We present a case study, where we exemplify how certain instances of our paradigm can be solved via a primal-dual iterative scheme. An attractive characteristic of such a scheme is that, in the primal step, the power allocation and the positions of the nodes can be optimized independently, which can be viewed as layering. In addition, our case-study illustrates how the primal-dual algorithm might be implemented in a distributed way.

This thesis is organized as follows: in the remaining of this chapter, we mention some preliminary definitions and assumptions regarding our communication model while in chapter 2, we give a precise description of our design problem and some examples that comply with that formulation. In chapter 3, we give an approximate solution to the placement problem by utilizing Geometric Programming. Then in chapter 4 we focus on a particular placement optimization problem and we propose a layering approach together with an efficient primal-dual algorithm that leads to a decentralized solution to that problem. Chapter 5, is dedicated to some simulations to picture the performance of the proposed algorithms and finally in chapter 6, we give the possible extensions to our optimization framework.

1.1 Preliminary definitions and assumptions

Before we give a description of the model of the wireless network adopted in this work, we introduce the following basic notation:

- Design parameters that are integers are represented using large caps Greek letters, such as Ω, while scalar or finite vectors of real numbers are represented using small caps Greek letters, such as ϕ.

- Optimization variables are indicated using boldface fonts, such as $\mathbf{P}$ and $\mathbf{x}$.

- The letters i and j are reserved for use as subscripts for integer indexing, with respect to the nodes of the wireless network. The letters k and l are also set aside for integer indexing.

- Functions are represented in calligraphic font, such as $\mathcal{U}$.

1.2 Basic description of the nodes in the network

Consider a wireless network consisting of a collection of nodes placed in a Cartesian plane. A non-empty sub-collection of these nodes is fixed, i.e., their positions in the Cartesian plane are pre-selected, while the locations of the remaining nodes are optimization parameters. Denote by Δ the number of fixed nodes and by Ω the number of remaining (movable) nodes in the network. The nodes are uniquely identified by an integer index in the set $\{1, \ldots, \Delta + \Omega\}$. We adopt the convention of allocating the first Δ indexes for the fixed nodes and the last Ω indexes for the movable nodes. We indicate the positions of the fixed nodes using ordered pairs (χ_1, γ_1) through $(\chi_\Delta, \gamma_\Delta)$, while the locations of the remaining nodes are specified by $(\mathbf{x}_{\Delta+1}, \mathbf{y}_{\Delta+1})$ through $(\mathbf{x}_{\Delta+\Omega}, \mathbf{y}_{\Delta+\Omega})$.

1.3 Wireless medium sharing assumptions

Each node of the wireless network has a communication module comprising a receiver and a transmitter. In addition, we assume that each node has a distinct reception channel assigned to it. As such, any given node will *tune* into and receive information transmitted through its ascribed channel. In addition,we assume that inter-channel interference is negligible. However, in our formulation, we allow multiplexing at each channel, so as to allow more than one source node to send information to any given destination node. In practice, multiple sources can send their messages through the same channel via multiplexing techniques, such as CDMA (asynchronous code division multiplexing) [4] among many other possibilities [12]. In order to quantify the impact of channel multiplexing, we adopt a formulation that is suitable for performance metrics and constraints that are based on the signal to interference ratio (see Section 2).

1.4 Power allocation and propagation loss model

For each node index i, in the set $\{1, \ldots, \Delta + \Omega\}$, we adopt the following model for quantifying the total power $\mathbf{P}_i^{total}$ used by node i, in $dBmW$ power units:

$$\mathbf{P}_i^{total} = 10 \log_{10} \left[\sum_{k \in \{1,\ldots,\Delta+\Omega\}-\{i\}} \phi 10^{0.1\mathbf{P}_{i\to k}+\alpha \mathbf{e}_{i,k}} \right] \tag{1.1}$$

where $\mathbf{e}_{i,k}$ is the Euclidean distance that separates nodes i and k, given by:

$$\mathbf{e}_{i,k} = \begin{cases} \sqrt{(\mathbf{x}_i - \mathbf{x}_k)^2 + (\mathbf{y}_i - \mathbf{y}_k)^2} & \text{if } i,k \geq \Delta + 1 \\ \sqrt{(\mathbf{x}_i - \chi_k)^2 + (\mathbf{y}_i - \gamma_k)^2} & \text{if } i \geq \Delta + 1, k \leq \Delta \\ \sqrt{(\mathbf{x}_k - \chi_i)^2 + (\mathbf{y}_k - \gamma_i)^2} & \text{if } k \geq \Delta + 1, i \leq \Delta \\ \sqrt{(\chi_i - \chi_k)^2 + (\gamma_i - \gamma_k)^2} & \text{if } i \leq \Delta, k \leq \Delta \end{cases} \tag{1.2}$$

In addition, the constants ϕ and α in (1.1) are positive real parameters that depend on the characteristics of the wireless medium and $\mathbf{P}_{i \to k}$ represents the received power in $dBmW$, as measured at the destination node k, of the signal transmitted by node i. We express power in $dBmW$ not only because it is a standard and convenient option for wireless communication [12], but also because commercial radio frequency amplifiers often feature controllable amplification gains that have uniform (linear) resolution in the $dBmW$ scale. Similarly, commercially available radio frequency power meters usually provide readings with a resolution that is uniform in the $dBmW$ scale. Implicit in (1.1) is the simplifying assumption that the transmitted signal between any two nodes, say i and k, is attenuated in dB according to an affine law of the distance, given by $-\log(\phi) - \alpha \mathbf{e}_{i,k}$. This assumption will be discussed in more detail in the following Section.

1.4.1 Validity of our propagation loss model for urban and sub sea environments

It has been shown, both analytically and empirically, that in a (logarithmic) dB scale (consistent with $dBmW$), the path loss attenuation of radio frequency waves in urban [9] and in sub sea environments [5] is well approximated by an affine function of the

Euclidean distance between the source and the receiver. More specifically, the authors of [5] suggest that attenuation, in $dBmW$, is essentially an affine law for distances of four meters or above. For distances below two meters our model may become conservative, which is not an issue since most applications will not require placement of wireless nodes that close. Commercial underwater radio frequency modems operate over distances as large as fifty meters [7], which indicates that our propagation loss model is accurate for underwater communications in the four to fifty meters range. Similarly, in urban environments, the authors of [9] have shown that an affine law is very accurate for distances of fifty meters or above. In the setting of [9], transmitter and receiver can communicate over distances of at least three hundred and fifty meters, which indicates that an affine law is an accurate model for propagation loss in urban environments, for distances that range from fifty to three hundred and fifty meters.

1.5 Comment on radio frequency communication underwater

In contrast to what was believed until recently, underwater radio modems are viable with loop antennas of one meter radius or less and modem housing of thirty centimeters (see [6] for an example). Underwater radio can be used for communication over ranges up to (typically) fifty meters, where it is far superior to acoustic based communication both in terms of delay and immunity to turbulence and noise. These features make underwater radio communications very suitable for mobile applications or (and) when tight clock synchronization is required for extended periods of time, such as in monitoring

operations[1].

[1]Note that clock synchronization underwater is critical for packet stamping and that it cannot be performed using global positioning systems. The work by [8] explains the difficulties of clock synchronization over networks, including sensitivity to communication delay

Chapter 2

Problem formulation

In this chapter, we formulate the central optimization paradigm of this thesis. We start by specifying the following class of functions, which we use to express the cost function as well as the constraints of our optimization paradigm. In Section 2.2, we provide network design examples and we show how they can be cast using the framework put forward in this Section. Such examples are intended to illustrate the wide applicability of our formulation.

Definition 2.0.1. *Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given a non-negative integer Ξ representing the number of auxiliary optimization variables, we define $\mathbf{F}_{\Delta,\Omega,\Xi}$ as the set of all functions $\mathcal{F}$ that can be written in the following form:*

$$\mathcal{F}\left(\mathbf{Q}\right) = \sum_{k=1}^{\Gamma} \varsigma_k 10^{\sum_{l=1}^{\Xi} \xi_{l,k}\mathbf{z}_l + \sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega} \beta_{i,j,k}\mathbf{P}_{i\rightarrow j} + \tau_{i,j,k}\mathbf{e}_{i,j}} \tag{2.1}$$

where Γ is a positive integer, ς_k and $\tau_{i,j,k}$ are non-negative real constants, while $\beta_{i,j,k}$ and $\xi_{l,k}$ are real constants. Moreover, $\mathbf{z}_1$ through $\mathbf{z}_\Xi$ are non-negative real auxiliary optimization variables. In addition, $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables given by $\{\mathbf{P}_{i\rightarrow j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$.

Notice that Definition 2.0.1 might be viewed as an extension of the class of posynomial functions [11], so as to include the Euclidean distance between any pair of nodes.

Indeed, if we select $\tau_{i,j,k} = 0$ in (2.1) then the resulting function is a posynomial in $\tilde{\mathbf{P}}_{i\to j}$ and $\tilde{\mathbf{z}}_l$, where $\tilde{\mathbf{P}}_{i\to j} \stackrel{def}{=} 10^{\mathbf{P}_{i\to j}}$ and $\tilde{\mathbf{z}}_l \stackrel{def}{=} 10^{\mathbf{z}_l}$.

Definition 2.0.2. *(**Main constraints**) Let Δ and Ω be positive integers representing the number of fixed and movable nodes in a wireless network. Given a non negative constant Ξ, a positive integer Φ and functions $\mathcal{F}_1$ through $\mathcal{F}_\Phi$ in the set $\mathbb{F}_{\Delta,\omega,\Xi}$, we consider constraints expressed by the following inequalities:*

$$\mathcal{F}_k(\mathbf{Q}) \leq 1,\ k \in \{1,\ldots,\Phi\} \tag{2.2}$$

An immediate and central example of application of Definition 2.0.2 is imposing constraints on the total power at every node, which could be expressed as $\mathbf{P}_i^{total} \leq \Psi$, where $\mathbf{P}_i^{total}$ is given by (2.21) and Ψ quantifies the total power available at each node. Similarly, we can adopt cost functions such as $10^{\sum_{i=1}^{\Delta+\Omega}\lambda_i\mathbf{P}_i^{total}}$, where λ_1 through $\lambda_{\Delta+\Omega}$ are nonnegative weights.

Further examples of constraints, with network-centric significance, can be expressed using the following definition of signal-to-interference ratio:

Definition 2.0.3. *(**Signal to interference ratio**) Let Δ and Ω quantify the number of fixed and movable nodes of a wireless network, respectively. Let i and j be distinct integers in the set $\{1,\ldots,\Delta+\Omega\}$ representing nodes of a wireless network. The signal to interference ratio for the logical link from node i to node j is defined as:*

$$\mathbf{S}_{i\to j} \stackrel{def}{=} \frac{\eta_{i,i}10^{0.1\mathbf{P}_{i\to j}}}{\sum_{k\in\{1,\ldots,\Delta+\Omega\}-\{i,j\}}\eta_{k,j}10^{0.1\mathbf{P}_{k\to j}}+\sigma_N^2} \tag{2.3}$$

In (2.3) $\eta_{k,i}$ are positive real coding gains that quantify the fact that, for instance, multiplexing codes are not perfectly orthogonal and $\mathbf{P}_{k\to j}$ represents the received power, as measured at the destination node j, of the signal transmitted by node k.

Clearly, constraints of the type $\mathbf{S}_{i,j} \geq \lambda_{i,j}$ can be cast as in Definition 2.0.2. In Section 2.2 we provide more examples of constraints and cost functions that can be expressed as in Definition 2.0.2.

Using the following class of constraints, we can impose that the movable nodes are placed inside pre-specified polyhedral convex subsets of the Cartesian plane.

Definition 2.0.4. *(**Polyhedral convex set placement constraints**) Let Δ and Ω be positive integers representing the number of fixed and movable nodes in a wireless network, respectively. Given a subset of $\{\Delta+1,\ldots,\Delta+\Omega\}$ denoted by $\mathbb{S}$, an integer Γ, real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$ with (i,k) in the set $\mathbb{S}\times\{1,\ldots,\Gamma\}$, we consider the following class of constraints:*

$$\zeta_{i,k}\mathbf{x}_i + \vartheta_{i,k}\mathbf{y}_i \leq 1,\ (i,k) \in \mathbb{S}\times\{1,\ldots,\Gamma\} \tag{2.4}$$

The following is the description of the main paradigm addressed in this thesis.

Problem 2.0.1. *(**Jointly optimal placement and power allocation**) Consider that Δ and Ω are positive integers quantifying the number of fixed and movable nodes in a wireless network, respectively. Let a non negative integer Ξ, positive integers Φ and Γ, functions $\mathcal{F}_1$ through $\mathcal{F}_\Phi$ in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$, a subset of $\{1,\ldots,\Delta+\Omega\}$ denoted by $\mathbb{S}$ and real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$, with (i,k) in the set $\mathbb{S}\times\{1,\ldots,\Gamma\}$, be given design parameters. Given a cost function $\mathcal{U}$ in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$, we want to find the solution to the following optimization paradigm:*

$$\mathbf{Q}^* = arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \tag{2.5}$$

subject to constraints (2.2)-(2.4), where $\mathbf{Q}$ represents the entire collection of optimization variables given by $\{\mathbf{P}_{i\rightarrow j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i,\mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$. In (2.5) $\mathbf{Q}^*$ is used to

indicate the collection of optimization variables at an optimum.

2.1 Basic optimality properties of Problem 2.0.1

In Section 2.1.1, we show that Problem 2.0.1 is convex. This is very useful attribute because it guarantees that an optimum can be found via standard constrained optimization techniques.

We also show, in Section 3, that Problem 2.0.1 can be arbitrarily well approximated by a geometric program. This is quite desirable, since geometric programs can be solved via polynomial time algorithms, which are available in existing software packages. In addition, in a like manner to linear programs, geometric programming solvers provide a certificate of infeasibility, in case the problem is not feasible.

Since constrained optimization algorithms may feature slow converge, we propose the use of geometric programs for obtaining a first approximate solution, which can be used as an initial condition in any method that is adopted for solving the exact problem.

2.1.1 Proof that Problem 2.0.1 is convex.

In order to prove that Problem 2.0.1 is convex, it suffices to prove that the class of functions specified in Definition 2.0.1 is convex. The main argument is given in the following Proposition:

Proposition 2.1.1. *Given positive integers Δ, Ω and Ξ, along with nonnegative real constants $\{\tau_{i,j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$ and real constants $\{\beta_{i,j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$ and $\{\xi_l\}_{l=1}^{\Xi}$, consider the follow-*

ing function:

$$\mathcal{G}\left(\mathbf{Q}\right) = 10^{\sum_{l=1}^{\Xi} \xi_l \mathbf{z}_l + \sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega} \beta_{i,j} \mathbf{P}_{i\to j} + \tau_{i,j} \mathbf{e}_{i,j}} \tag{2.6}$$

where $\mathbf{e}_{i,j}$ *is the Euclidean distance (1.2), while* $\mathbf{Q}$ *is a shorthand notation for representing the entire collection of optimization variables, given by* $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ *and* $\{\mathbf{z}_l\}_{l=1}^{\Xi}$. *The function* $\mathcal{G}$ *is convex.*

Proof. We start by defining the following functions:

$$\mathcal{G}_2(\mathbf{Q}) \stackrel{def}{=} \sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega} \tau_{i,j} \mathbf{e}_{i,j} \tag{2.7}$$

$$\mathcal{G}_1(\mathbf{Q}) \stackrel{def}{=} \sum_{l=1}^{\Xi} \xi_l \mathbf{z}_l + \sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega} \beta_{i,j} \mathbf{P}_{i\to j} + \mathcal{G}_2(\mathbf{Q}) \tag{2.8}$$

Now notice that $\tau_{i,j}$ are nonnegative and the Euclidean distance is itself a convex function of its parameters. Hence, $\mathcal{G}_2$ is convex function of $\{(\mathbf{x}_i, \mathbf{y}_i)\}_{i=\Delta+1}^{\Delta+\Omega}$ because it is a sum of convex functions. This also implies that $\mathcal{G}_1$ is convex and since exponentiation is a convex function too [1], we can conclude that $\mathcal{G}\left(\mathbf{Q}\right) = 10^{\mathcal{G}_1(\mathbf{Q})}$ is convex. □

Recall that the coefficients ς_k, in Definition 2.0.1, are non-negative. Therefore, it follows from Proposition 2.1.1 that functions in the set $\mathbb{F}_{\Delta,\Omega,\Xi}$ are convex because they are the sum of convex functions. This implies that the cost $\mathcal{U}$ of Problem 2.0.1 and all of its constraints are convex, which includes the ones specified in Definitions 2.0.2 and 2.0.4. This fact leads to the conclusion that Problem 2.0.1 is convex.

[1] Recall that the composition of a convex function with another increasing convex function is convex.

2.2 Examples of design problems that comply with the problem formulation of Section 2

In this Section, we give design examples that we can cast in the framework of Section 2. By way of these examples, we expect to illustrate the pertinence of our framework for the design of wireless networks, with respect to jointly optimal power allocation and node placement.

2.2.1 Optimal relay placement, power allocation and routing, under the high signal to interference ratio assumption

In this Subsection, we delineate a design example which involves the maximization of the bit-rate between a pre-specified collection of fixed nodes. Throughout, we will describe how our design example can be cast in the formulation of Section 2. An interesting attribute of such an exercise is that it also shows, for the present example, how we can integrate optimization of routing, in addition to power allocation and placement of the movable nodes. For simplicity of notation, we describe our example for two fixed and two movable nodes, but our approach can be used for any number of nodes at the expense of a potentially large number of auxiliary variables. In addition, this example admits routing with at most two hops, but the number of hops can be increased at the expense of using more auxiliary variables.

Example 2.2.1. *Consider a wireless network with two fixed and two movable nodes. The movable nodes, indexed by* 3 *and* 4*, are intended to act as relays so as to as to maximize*

the rate of communication between the fixed nodes, which are identified by indices 1 *and* 2*. In order to precisely state our design example, assume that we are given the following design parameters:*

- *We are given the parameters* ϕ *and* α *that are needed in the total power formula (1.1).*
- *We pre-specify a positive real constant* Ψ *representing the maximal power available at each node.*
- *We are given the positions of the fixed nodes* (χ_1, γ_1) *and* (χ_2, γ_2)*.*

In addition, we adopt the following cost function:

$$\mathcal{U}^{example\ 2.2.1}(\mathbf{Q}) = -\min\{\mathbf{R}_{1\to 2}^{total}, \mathbf{R}_{2\to 1}^{total}\} \tag{2.9}$$

where $\mathbf{R}_{1\to 2}^{total}$ *and* $\mathbf{R}_{2\to 1}^{total}$ *represent the total aggregated rates that stream from node* 1 *to node* 2 *and from node* 2 *to node* 1*, respectively. These aggregated rates can be computed by the following formulae:*

$$\mathbf{R}_{1\to 2}^{total} = \mathbf{R}_{1\to 2} + \underbrace{\min\{\mathbf{R}_{1\to 3}, \mathbf{R}_{3\to 2}\}}_{(A)} + \underbrace{\min\{\mathbf{R}_{1\to 4}, \mathbf{R}_{4\to 2}\}}_{(B)} \tag{2.10}$$

$$\mathbf{R}_{2\to 1}^{total} = \mathbf{R}_{2\to 1} + \underbrace{\min\{\mathbf{R}_{2\to 3}, \mathbf{R}_{3\to 1}\}}_{(C)} + \underbrace{\min\{\mathbf{R}_{2\to 4}, \mathbf{R}_{4\to 1}\}}_{(D)} \tag{2.11}$$

Here we use $\mathbf{R}_{i\to j}$ *to represent the average rate through the direct communication link (point-to-point, i.e., no relaying) from node* i *to node* j*, in bits per time unit. An interpretation for the flux constraints (2.10)-(2.11) is that the information that is transferred between nodes* 1 *and* 2 *can flow through three different routes. It can flow directly (point-to-point) between nodes* 1 *and* 2 *and it can be routed through nodes* 3 *and* 4*.*

Hereon, we use the following formula for relating the point-to-point rate $\mathbf{R}_{i\to j}$ with the signal to interference ratio [11, page 68], which is valid in the high signal to interference regime:

$$\mathbf{R}_{i\to j} = \frac{1}{\Upsilon}\log_{10}\left(\kappa \mathbf{S}_{i\to j}\right) \tag{2.12}$$

where Υ and κ are positive real constants. We can now precisely state the optimization paradigm of our example, which consists of finding $\{\mathbf{P}_{i,j}\}_{i=1,j=1}^{i=4,j=4}$ and $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=3}^{4}$ so as to minimize $\mathcal{U}^{example\ 2.2.1}(\mathbf{Q})$, subject to the following constraints:

$$\mathbf{P}_i^{total} \leq \Psi,\ i \in \{1,\ldots,4\} \tag{2.13}$$

Now we show that (2.9)-(2.13), which constitute the specification of Example 2.2.1, can be put in the framework of Section 2. We start by noticing that the following optimization paradigm is equivalent to Example 2.2.1:

$$\min 10^{-\mathbf{z}_1} \tag{2.14}$$

subject to the following constraints:

$$\mathbf{z_2} \geq \mathbf{z}_1 \text{ and } \mathbf{z_3} \geq \mathbf{z}_1 \tag{2.15}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 2}}} \geq 10^{\mathbf{z}_2-\mathbf{z}_4-\mathbf{z}_5} \text{ and } \underbrace{\kappa \mathbf{S}_{2\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 1}}} \geq 10^{\mathbf{z}_3-\mathbf{z}_6-\mathbf{z}_7} \tag{2.16}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 3}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 3}}} \geq 10^{\mathbf{z_4}} \text{ and } \underbrace{\kappa \mathbf{S}_{3\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{3\to 2}}} \geq 10^{\mathbf{z_4}} \tag{2.17}$$

$$\underbrace{\kappa \mathbf{S}_{1\to 4}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{1\to 4}}} \geq 10^{\mathbf{z_5}} \text{ and } \underbrace{\kappa \mathbf{S}_{4\to 2}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{4\to 2}}} \geq 10^{\mathbf{z_5}} \tag{2.18}$$

$$\underbrace{\kappa \mathbf{S}_{2\to 3}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 3}}} \geq 10^{\mathbf{z_6}} \text{ and } \underbrace{\kappa \mathbf{S}_{3\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{3\to 1}}} \geq 10^{\mathbf{z_6}} \tag{2.19}$$

$$\underbrace{\kappa \mathbf{S}_{2\to 4}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{2\to 4}}} \geq 10^{\mathbf{z_7}} \text{ and } \underbrace{\kappa \mathbf{S}_{4\to 1}^{\frac{1}{\Upsilon}}}_{10^{\mathbf{R}_{4\to 1}}} \geq 10^{\mathbf{z_7}} \tag{2.20}$$

$$10^{\mathbf{P}_i^{total}-\Psi} \leq 1,\ i \in \{1,\ldots,4\} \tag{2.21}$$

In the inequality constraints above, the auxiliary variables $\mathbf{z}_1$ through $\mathbf{z}_7$ are introduced to construct the cost function (2.9). In particular, (2.15) implements the minimum in (2.9), while the auxiliary variables $\mathbf{z}_2$ and $\mathbf{z}_3$ represent $\mathbf{R}_{1\to 2}^{total}$ and $\mathbf{R}_{2\to 1}^{total}$, respectively. Similarly, the auxiliary variables $\mathbf{z}_4$ and $\mathbf{z}_5$ implement the two terms, denoted by (A) and (B), at the right hand side of (2.10), while $\mathbf{z}_6$ and $\mathbf{z}_7$ implement the two terms (C) and (D) at the right hand side of (2.11). The power constraint (2.13) is also re-written in the form (2.21).

In order to show that (2.14)-(2.21) comply with the formulation of Section 2, it suffices to notice the following facts:

- By a direct substitution of (2.3), we can write inequalities (2.17)-(2.20) as in Definition 2.0.2.
- Similarly, by using (1.1), (2.21) can be re-written so as to comply with Definition 2.0.2.

2.2.2 Further examples of optimization constraints that comply with the formulation of Section 2

The following Example illustrates how linear inequalities on the rates among distinct nodes can be expressed as in Definition 2.0.2.

Example 2.2.2. *(**Linear inequalities on rates**) Let Λ be a given integer and λ_1 through λ_Λ be given positive real constants. In addition, consider real parameters given by $\varrho_{i,j,k}$, with $(i,j) \in \{1,\ldots,\Delta+\Omega\}^2$ and $k \in \{1,\ldots,\Lambda\}$. We consider the following collection of constraints*

$$\sum_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} \varrho_{i,j,k}\mathbf{R}_{i\to j} \geq \lambda_k,\ k \in \{1,\ldots,\Lambda\} \tag{2.22}$$

where $\mathbf{R}_{i\to j}$ represents the rate of transmission (in bits per channel use) from node i to node j. Using (2.12), we can re-write (2.22) as:

$$\prod_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} (\kappa\mathbf{S}_{i\to j})^{\frac{\varrho_{i,j,k}}{\Upsilon}} \geq 2^{\lambda_k},\ k \in \{1,\ldots,\Lambda\} \tag{2.23}$$

which clearly complies with Definition 2.0.2.

Inequalities of the form (2.22) can be used to specify any convex polyhedral rate region among any subcollection of source and destination nodes. In addition, necessary and sufficient conditions for multi-terminal omniscience, in the presence of an overlay node [14], can be cast as in (2.22). This class of inequalities can also be used to specify the rate of certain distributed secret key generation mechanisms [13].

Yet another example is the specification of constraints on the outage probabilities over a path, which can be cast as in Definition 2.0.2, under the assumption of no single dominant interferer (see [11, Page 68] for more details).

Chapter 3

Approximate Solutions to Problem 2.0.1 via Geometric Programming

Geometric programming is an optimization paradigm that has been widely studied for more than thirty years. Due to the increase in computational power verified in recent years, efficient solvers for geometric programs are now easily accessible. Hence, the recent significant interest in casting the optimal design of communication systems as geometric programs [11]. In a way that is similar to linear programs, solvers for geometric programs can efficiently handle hundreds of variables and constrains, which is very relevant for the method proposed in this Section, where a large number of auxiliary variables might be needed. Even when the original problem cannot be cast as a geometric program, in many cases a satisfactory solution can be found via an approximate geometric program [11, 15]. In this section, we follow such an approach, by proposing an approximate solution to Problem 2.0.1 via geometric programming. In order to accomplish this goal, we follow a strategy where it suffices to approximate the Euclidean distance by a geometric program. No other approximations are needed. A comprehensive account of the uses of geometric programming in various other fields can be found in [15], where a rich portfolio of examples is also provided.

Below we give a brief description of the standard geometric programming paradigm. We start with the definition of the class of posynomial functions.

Definition 3.0.1. *Given a finite collection of non-negative real variables denoted by*

$\mathbf{W} \stackrel{def}{=} \{\mathbf{w}_i\}_{i=1}^{\Xi}$. *The class of posynomials over* $\mathbf{W}$ *is formed by functions* $\mathcal{P}$ *with the following structure:*

$$\mathcal{P}(\mathbf{W}) = \sum_{k=1}^{\Gamma} \varsigma_k \prod_{l=1}^{\Xi} \mathbf{w}_l^{\xi_{l,k}} \tag{3.1}$$

where ς_k *are positive real and* $\xi_{l,k}$ *are real (possibly negative) constants.*

The following is the definition of the general form of a geometric program in standard form.

Definition 3.0.2. *Let a finite collection of non-negative real variables denoted by* $\mathbf{W} \stackrel{def}{=} \{\mathbf{w}_i\}_{i=1}^{\Xi}$, *and posynomials over* $\mathbf{W}$ *denoted by* $\mathcal{U}$ *and* $\mathcal{P}_1$ *through* $\mathcal{P}_\Phi$ *be given. The following optimization paradigm is a geometric program:*

$$\mathbf{W}^* = arg \min_{\mathbf{W}} \mathcal{U}(\mathbf{W}) \tag{3.2}$$

subject to the following inequality constraints:

$$\mathcal{P}_i(\mathbf{W}) \leq 1, \ i \in \{1, \ldots, \Phi\} \tag{3.3}$$

3.1 Specification of a geometric program that approximates Problem 2.0.1

In what follows, we specify a geometric program that approximates Problem 2.0.1, to an arbitrary degree of accuracy. However, we should note that higher accuracy is attained at the expense of a larger number of optimization variables. Nonetheless, as we explain in Section 2.1, obtaining an approximate solution is important because it can be used as an initial condition on any constrained optimization algorithm that is applied to

the exact problem. Another reason for adopting this procedure is that solvers provide a certificate if a geometric program is unfeasible.

Notice that Problem 2.0.1 is not a geometric program because the constraints involve the Euclidean distance. The main idea, in what follows, is to approximate the Euclidean distance in Problem 2.0.1 with an appropriate function so as to obtain a geometric program. No other approximations are required. The following is the class of distance functions that we will use in our approximate geometric program.

Definition 3.1.1. *(**Convex polygonal distances**) Let $\mathcal{D}$ be a distance function in the Cartesian plane that satisfies the properties of a norm. We qualify $\mathcal{D}$ as a convex polygonal distance if and only if the unit ball, according to $\mathcal{D}$, is a convex polygon[10, Chapter 19]. We define the class of convex polygonal distances as $\mathbb{D}$.*

The following remark states the main reason why the class $\mathbb{D}$ is a suitable choice for approximating the Euclidean distance.

Remark 3.1.1. *Given any two positive real constants $\wp_1$ and $\wp_2$ satisfying $\wp_1 < 1$ and $\wp_2 > 1$, we can always find a distance function $\mathcal{D}$ in the set $\mathbb{D}$ such that the following holds:*

$$\wp_1 \mathcal{D}(u_1, u_2) < \|u_1 - u_2\|_2 < \wp_2 \mathcal{D}(u_1, u_2),\ u_1, u_2 \in \mathbb{R}^2 \tag{3.4}$$

where $\|u_1 - u_2\|_2$ is the Euclidean distance between u_1 and u_2.

In order to prove Remark 3.1.1, one only needs to realize that the unit ball, associated with the Euclidean distance, can be approximated arbitrarily well by a convex polygon. Hence, the proof follows by selecting the convex polygonal distance that corre-

sponds to the approximating convex polygon. The following remark provides a systematic method for obtaining such a distance function, which will also be useful later in this subsection.

Remark 3.1.2. *If $\mathcal{D}$ is a convex polygonal distance then there an integer m and a finite collection of m vectors $(\nu_{1,1}, \nu_{2,1})$ through $(\nu_{1,m}, \nu_{2,m})$ in the Cartesian plane, for which the following holds [10, pp. 173]:*

$$\mathcal{D}((x_i, y_i), (x_j, y_j)) = arg \min \mathbf{h} \tag{3.5}$$

subject to:

$$\nu_{1,k}(x_j - x_i) + \nu_{2,k}(y_j - y_i) \leq \mathbf{h}, k \in \{1, \ldots, m\} \tag{3.6}$$

for any vectors (x_i, y_i) and (x_j, y_j) in the Cartesian plane.

Example 3.1.1. *Now we consider an example of application of Remark 3.1.2. Consider that we wish to specify a distance whose unit ball is the unit square, centered at the origin. The corresponding distance can be obtained from Remark 3.1.2 by selecting the four vectors $(\nu_{1,1}, \nu_{2,1}) = (1, 0)$, $(\nu_{1,2}, \nu_{2,2}) = (-1, 0)$, $(\nu_{1,3}, \nu_{2,3}) = (0, 1)$ and $(\nu_{1,4}, \nu_{2,4}) = (0, -1)$. Likewise, if the unit ball is an hexagon then the distance would be specified by $(\nu_{1,i}, \nu_{2,i}) = (\cos(i\frac{\pi}{3}), \sin(i\frac{\pi}{3}))$, with $i \in \{1, \ldots, 6\}$.*

The class of functions specified in Definition 2.0.1 plays a central role in the statement of Problem 2.0.1. By replacing the Euclidean distance with an approximating convex polygonal distance we obtain the class of functions defined below. From Remark 3.1.1, we conclude that functions in such a class may be used as an approximation to the functions given in Definition 2.0.1.

Definition 3.1.2. *Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given a convex polyhedral distance $\mathcal{D}$ and a non-negative integer Ξ representing the number of auxiliary optimization variables, we define $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ as the set of all functions $\mathcal{H}$ that can be written in the following form:*

$$\mathcal{H}\left(\mathbf{Q}\right) = \sum_{k=1}^{\Gamma} \varsigma_k 10^{\sum_{l=1}^{\Xi} \xi_{l,k}\mathbf{z}_l + \sum_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega} \beta_{i,j,k}\mathbf{P}_{i\rightarrow j} + \tau_{i,j,k}\mathbf{d}_{i,j}} \tag{3.7}$$

where $\mathbf{d}_{i,j} = \mathcal{D}\left((\mathbf{x}_i,\mathbf{y}_i),(\mathbf{x}_j,\mathbf{y}_j)\right)$, Γ is a positive integer, ς_k, $\xi_{l,k}$ and $\tau_{i,j,k}$ are non-negative real constants, while $\beta_{i,j,k}$ are real constants. Moreover, $\mathbf{z}_1$ through $\mathbf{z}_\Xi$ are non-negative real auxiliary optimization variables. In addition, $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables given by $\{\mathbf{P}_{i\rightarrow j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i,\mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$.

The following Problem 3.1.1 is a modified version of Problem 2.0.1, where the class of functions $\mathbb{F}$ is replaced with $\mathbb{H}$. Since functions in $\mathbb{F}$ can be approximated by functions in $\mathbb{H}$, we can view Problem 3.1.1 as an approximate version of Problem 2.0.1. In addition, as we show in Theorem 3.1.1, Problem 3.1.1 can be cast as a geometric program.

Problem 3.1.1. *(**Jointly optimal placement and power allocation with polyhedral distances**) Consider that Δ and Ω are positive integers quantifying the number of fixed and movable nodes in a wireless network, respectively. Let a polyhedral convex distance $\mathcal{D}$, a positive integer Φ, functions $\mathcal{H}_1$ through $\mathcal{H}_\Phi$ in the set $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$, a subset of $\{1,\ldots,\Delta+\Omega\}$ denoted by $\mathbb{S}$ and real constants $\zeta_{i,k}$ and $\vartheta_{i,k}$, with (i,k) in the set $\mathbb{S}\times\{1,\ldots,\Gamma\}$, be given design parameters. Given a cost $\mathcal{U}$ in the set $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$, we want to find the solution to the following optimization paradigm:*

$$\mathbf{Q}^* = arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \tag{3.8}$$

$$\mathcal{H}_k\left(\mathbf{Q}\right) \leq 1,\ k \in \{1, \ldots, \Phi\} \tag{3.9}$$

subject also to the placement constraints (2.4). Here $\mathbf{Q}^*$ *is used to indicate the entire collection of optimization variables* $\{\mathbf{P}_{i\rightarrow j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ *and* $\{\mathbf{z}_l\}_{l=1}^{\Xi}$ *at an optimum.*

Theorem 3.1.1. *Problem 3.1.1 can be cast as a standard geometric program.*

Proof. Let the parameters of Problem 3.1.1 be given. In particular, consider the class of functions $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$, along with the associated optimization variables $\{\mathbf{P}_{i\rightarrow j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$ and $\{\mathbf{z}_l\}_{l=1}^{\Xi}$. We start the proof by defining the following supplemental variables $\tilde{\mathbf{P}}_{i\rightarrow j} \stackrel{def}{=} 10^{\mathbf{P}_{i\rightarrow j}}$, $\tilde{\mathbf{x}}_i \stackrel{def}{=} 10^{\mathbf{x}_i}$, $\tilde{\mathbf{y}}_j \stackrel{def}{=} 10^{\mathbf{x}_j}$, $\tilde{\mathbf{z}}_l \stackrel{def}{=} 10^{\mathbf{z}_l}$ and $\tilde{\mathbf{d}}_{i,j} \stackrel{def}{=} 10^{\mathbf{d}_{i,j}}$, where $\mathbf{d}_{i,j} = \mathcal{D}\left((\mathbf{x}_i, \mathbf{y}_i), (\mathbf{x}_j, \mathbf{y}_j)\right)$. The proof follows as a conclusion based on the following two facts: **(Fact 1)** By inspecting (3.7), we conclude that any function in class $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ can be re-written as a posynomial in terms of the supplemental variables $\tilde{\mathbf{P}}_{i\rightarrow j}$,$\tilde{\mathbf{x}}_i$, $\tilde{\mathbf{y}}_j$, $\tilde{\mathbf{z}}_l$ and $\tilde{\mathbf{d}}_{i,j}$. In fact, the cost function in (3.8) and the left hand side of the inequality constraints in (2.4) and (3.9) can be re-cast as posynomial functions. As a result, we infer that Problem 3.1.1 can be written as a Geometric program with respect to the supplemental variables. However, according to the statement of Problem 3.1.1, $\tilde{\mathbf{d}}_{i,j}$ is not one of the desired optimization variables, i.e., we want to obtain an answer in terms of the positions and not the distances. In addition, if we optimize with respect to $\tilde{\mathbf{d}}_{i,j}$ directly, and without further constraints, then the optimum may be such that the resulting distances are not consistent with the positions. In order to address this problem, we make use of the following fact: **(Fact 2)** From Remark 3.1.2, we conclude that there exists a positive integer m and vectors $(\nu_{1,1}, \nu_{2,1})$ through $(\nu_{1,m}, \nu_{2,m})$ such that $\tilde{\mathbf{d}}_{i,j}$ can be expressed as a function of $\tilde{\mathbf{x}}_i$

and $\tilde{\mathbf{y}}_j$ via the following geometric program:

$$\tilde{\mathbf{d}}_{i,j} = \arg\min \tilde{\mathbf{h}}_{i,j} \tag{3.10}$$

subject to:

$$(\tilde{\mathbf{x}}_j \tilde{\mathbf{x}}_i^{-1})^{\nu_{1,l}} (\tilde{\mathbf{y}}_j \tilde{\mathbf{y}}_i^{-1})^{\nu_{2,l}} \tilde{\mathbf{h}}_{i,j}^{-1} \leq 1,\ l \in \{1, \ldots, m\} \tag{3.11}$$

Now notice that in the definition of $\mathbb{H}_{\Delta,\Omega,\Xi,\mathcal{D}}$ (see (3.7)), the coefficients of the distances $\mathbf{d}_{i,j}$, by definition, satisfy $\tau_{i,j,k} \geq 0$. This means that if any given $\mathbf{d}_{i,j}$ is present in the cost $\mathcal{U}$ or in an active constraint, say $\mathcal{H}_k \leq 1$, then $\tilde{\mathbf{d}}_{i,j}$ is implicitly minimized. This observation, together with Fact 1 and Fact 2 show that Problem 3.1.1 can be written as a geometric program by re-expressing the left hand side of (2.4), (3.9) and $\mathcal{U}$ as posynomials and by adding the following constraints:

$$(\tilde{\mathbf{x}}_j \tilde{\mathbf{x}}_i^{-1})^{\nu_{1,l}} (\tilde{\mathbf{y}}_j \tilde{\mathbf{y}}_i^{-1})^{\nu_{2,l}} \tilde{\mathbf{d}}_{i,j}^{-1} \leq 1,\ l \in \{1, \ldots, m\} \tag{3.12}$$

□

Chapter 4

Layering and distributed implementation

In Section 2, we defined Problem 2.0.1 which constitutes the main paradigm in this paper. In addition, in Section 2.1.1, we proved that such a problem can be cast as a convex problem and in Section 3 we provided a method for obtaining approximate solutions via geometric programming.

In this Section, we illustrate how particular instances of Problem 2.0.1 can be solved via iterative algorithms based on the primal-dual principle. As we illustrate here, the primal step consists of an optimization problem that can be decomposed into smaller subproblems (Layering) which can be solved independently, while the dual step is, typically, a simple price update rule. Our iterative solution and associated layering decomposition has the following advantages: (1) The primal step decomposes into two simpler and independent optimization subproblems, namely, the placement and the power allocation get decoupled. The dimension of such subproblems is smaller than the original problem and they can be solved in parallel. In addition, at each iteration, these subproblems can be solved using the updated prices and variables that are mostly local at each agent. This means that, in our algorithm, the coupling among agents is implemented via price exchange plus a reduced number of variables. (2) If the cost function is additive and constraints are imposed within a neighborhood of each node then it suffices to implement price and variable exchanges within those same neighborhoods. Such an implementation,

where prices and a few variables are exchanged within neighborhoods, is what we qualify as distributed solution.

Hereafter, we analyze the following case-study, which is a particular case of Problem 2.0.1:

Problem 4.0.2. ***(total power minimization)*** *Let Δ and Ω be positive integers representing the number of fixed and movable nodes of a wireless network, respectively. Given non-negative real constants φ_1 through $\varphi_{\Delta+\Omega}$, define the cost function $\mathcal{U}$ to be a weighted sum of consumed powers in each node:*

$$\mathcal{U}(\mathbf{Q}) = \sum_{i=1}^{\Delta+\Omega} \varphi_i 10^{0.1\mathbf{P}_i^{total}} \tag{4.1}$$

where $\mathbf{P}_i^{total}$ is given by (1.1) and $\mathbf{Q}$ is a shorthand notation for representing the entire collection of optimization variables given by $\{\mathbf{P}_{i\to j}\}_{i=1,j=1}^{\Delta+\Omega,\Delta+\Omega}$, $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$. Consider also that a collection $\{\mathbb{O}(i)\}_{i=1}^{\Delta+\Omega}$ of subsets of $\{1, \ldots, \Delta+\Omega\}$ is given, representing the neighborhoods or destination nodes in the outgoing links from each node i. Let $\{\varrho_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ and $\{\Psi_i\}_{i=1}^{\Delta+\Omega}$ be two sets of design parameters, representing the minimum required rates in the links and the maximum power available in the transmitters, respectively. We want to find the solution of the following optimization paradigm:

$$\mathbf{Q}^* = arg\min_{\mathbf{Q}} \mathcal{U}(\mathbf{Q}) \tag{4.2}$$

Subject to:

$$\mathcal{R}(\mathbf{S}_{i\to j}) \geq \varrho_{i\to j}, i \in \{1, \ldots, \Delta+\Omega\}, j \in \mathbb{O}(i) \tag{4.3}$$

$$\mathbf{P}_i^{total} \leq \Psi_i,\ i \in \{1, \ldots, \Delta+\Omega\} \tag{4.4}$$

where $\mathbf{S}_{i\to j}$ is the signal to interference ratio of the transmission from link i to j and $\mathcal{R}$ is any positive and increasing function that satisfies $\lim_{\gamma\to\infty}\mathcal{R}(\gamma)=\infty$. We use $\mathcal{R}$ to quantify the rate of data transmission from node i to node j. Notice that not only the approximation in (2.12), i.e., $\mathcal{R}(\mathbf{S}_{i\to j})=\frac{1}{\Upsilon}\log_{10}(\kappa\mathbf{S}_{i\to j})$, but also the exact formula $\mathcal{R}(\mathbf{S}_{i\to j})=\frac{1}{\Upsilon}\log_{10}(1+\kappa\mathbf{S}_{i\to j})$ are valid choices here.

Remark 4.0.3. *Since $\mathcal{R}$ is an increasing function that is invertible in the positive reals, the constraints in (4.3) can be replaced by the following signal to interference ratio constraints:*

$$\mathbf{S}_{i\to j}\geq\mathcal{R}^{-1}(\varrho_{i\to j}),i\in\{1,\ldots,\Delta+\Omega\},j\in\mathbb{O}(i) \tag{4.5}$$

where $\mathcal{R}^{-1}$ represents the inverse function of $\mathcal{R}$.

In what follows, we describe how we obtain a distributed algorithm that converges to the optimal solution of Problem 4.0.2. In particular, in Section 4.1 we show how Problem 4.0.2 can be decomposed using a primal-dual approach. We also provide a layering of the primal step, where the optimal placement and the optimal power allocation subproblems become decoupled. In Section 4.3, we introduce efficient distributed solutions for each of these subproblems.

4.1 Primal-dual decomposition and layering of the primal subproblem

Utilizing a dual decomposition approach, we break down Problem 4.0.2 into smaller subproblems which can be solved efficiently in a distributed fashion. One of these subproblems, which we denote as *node placement subproblem*, only involves the location of the mobile nodes, i.e., $(\mathbf{x}_{\Delta+1},\mathbf{y}_{\Delta+1})$ through $(\mathbf{x}_{\Delta+\Omega},\mathbf{y}_{\Delta+\Omega})$. The second subproblem op-

timizes with respect to the received powers $\mathbf{P}_{i\to k}$, and we label it as *power allocation subproblem*. The correct coupling between these two subproblems, via message exchange, leads to an algorithm that converges to the optimal solution of Problem 4.0.2.

The main idea for breaking down our overall optimization problem is applying Lagrange relaxation to power constraints in (4.4). These power constraints involve both sets of variables $\{\mathbf{x}_{\Delta+i}, \mathbf{y}_{\Delta+i}\}_{i=1}^{\Omega}$ and $\{\mathbf{P}_{i\to j}\}$ in a coupled fashion. In order to start the decoupling process, we need to replace these power constraints for each transmitter $i \in \{1, 2, ..., \Delta + \Omega\}$, with the following set of inequalities:

$$\phi 10^{0.1\mathbf{P}_{i\to j}+\alpha\mathbf{e}_{i,j}} \leq 10^{0.1\mathbf{v}_{i\to j}},\ j \in \mathbb{O}(i) \tag{4.6}$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i} \tag{4.7}$$

Where $\mathbf{v}_{i\to j}$ represents an auxiliary variable that can be interpreted as the maximum power that is allocated to the link from the transmitter i to the receiver j, at the transmitter side (in $dBmW$). We can also re-write the utility function (4.1) in terms of these supplemental variables. As such, Problem 4.0.2 can be re-formulated as:

Problem 4.1.1. *Let all parameters needed in the definition of Problem 4.0.2 be given. Adopting the same parameters, consider the following augmented optimization paradigm:*

$$Q^* = arg\min_{\mathbf{Q}} \left[\min_{\mathbf{V}} \tilde{\mathcal{U}}(\mathbf{V})\right] \tag{4.8}$$

$$\tilde{\mathcal{U}}(\mathbf{V}) \stackrel{def}{=} \sum_{i=1}^{\Delta+\Omega} \sum_{j\in\mathbb{O}(i)} \varphi_i 10^{0.1\mathbf{v}_{i\to j}} \tag{4.9}$$

$$\mathbf{V} \stackrel{def}{=} \{\mathbf{v}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega} \tag{4.10}$$

subject to constraints (4.5)-(4.7).

Proposition 4.1.1. *Let all parameters needed in the definition of Problem 4.0.2 be given. The following Lagrangian based min-max optimization paradigm is equivalent to Problems 4.0.2 and 4.1.1:*

$$(\mathbf{Q}^{\star}, \mathbf{V}^{\star}, \mathbf{H}^{\star}) = arg \left[\max_{\mathbf{H}} \underbrace{\min_{(\mathbf{Q},\mathbf{V})} \mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H})}_{primal\ problem} \right] \tag{4.11}$$

$$\mathbf{H} \overset{def}{=} \{\mathbf{h}_{i,j}\}_{i=1, j\in\mathbb{O}(i)}^{\Delta+\Omega} \tag{4.12}$$

$$\mathbf{h}_{i,j} \geq 0,\ (i,j) \in \bigcup_{i=1}^{\Delta+\Omega} \{i\} \times \mathbb{O}(i) \tag{4.13}$$

subject to constraints (4.5) and (4.7). Here we use Lagrange multipliers $\mathbf{H}$ *to impose constraint (4.6), while* $\mathcal{L}$ *is the following Lagragean:*

$$\mathcal{L}(\mathbf{Q}, \mathbf{V}, \mathbf{H}) \overset{def}{=} \underbrace{\mathcal{L}_1\left(\{\mathbf{P}_{i\to j}\}_{i=1, j\in\mathbb{O}(i)}^{\Delta+\Omega}, \mathbf{V}, \mathbf{H}\right)}_{power\ allocation} + \underbrace{\mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}, \mathbf{H}\right)}_{node\ placement} \tag{4.14}$$

where

$$\mathcal{L}_1\left(\{\mathbf{P}_{i\to j}\}_{i=1, j\in\mathbb{O}(i)}^{i=\Delta+\Omega}, \mathbf{V}, \mathbf{H}\right) \overset{def}{=} \tilde{\mathcal{U}}(\mathbf{V}) + \sum_{i=1}^{\Delta+\Omega} \sum_{j\in\mathbb{O}(i)} \mathbf{h}_{i,j} \left[\mathbf{P}_{i\to j} - \mathbf{v}_{i\to j} + 10\log\varphi\right] \tag{4.15}$$

$$\mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta}, \mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}, \mathbf{H}\right) \overset{def}{=} \sum_{i=1}^{\Delta+\Omega} \sum_{j\in\mathbb{O}(i)} \mathbf{h}_{i,j} 10\alpha \mathbf{e}_{i,j} \tag{4.16}$$

By inspection, we find that the primal component in (4.11) can be decomposed into two independent subproblems, as indicated in the following Remark:

Remark 4.1.1. ***(Layering)*** *The Lagrangian (4.14) comprises the two additive terms given by* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*. As such, given any choice of* $\mathbf{H}$*, the primal problem in (4.11) can be recast*

as follows:

$$\min_{(\mathbf{Q},\mathbf{V})} \mathcal{L}\left(\mathbf{Q},\mathbf{V},\mathbf{H}\right) \quad = \quad \begin{array}{l} \textit{Subject to: (4.5) and (4.7)} \\ \underbrace{\begin{array}{l} \min_{\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega},\mathbf{V}} \mathcal{L}_1\left(\{\mathbf{P}_{i\to j}\}_{i=1,j\in\mathbb{O}(i)}^{i=\Delta+\Omega},\mathbf{V},\mathbf{H}\right) \\ \textit{Subject to (4.5) and (4.7)} \end{array}}_{\textit{Power allocation subproblem}} + \\ \underbrace{\min_{\{\mathbf{x}_{i+\Delta},\mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega}} \mathcal{L}_2\left(\{\mathbf{x}_{i+\Delta},\mathbf{y}_{i+\Delta}\}_{i=1}^{\Omega},\mathbf{H}\right)}_{\textit{Node placement subproblem}} \end{array} \quad (4.17)$$

Notice that the first term in the right hand side of (4.17) involves only the power allocation and the supplemental variables, while the second term addresses node placement. We denote these independent subproblems as power allocation and node placement, respectively.

Proof of Proposition 4.1.1 Since Problems 4.0.2 and 4.1.1 are equivalent, we only need to prove that Problem 4.1.1 can be re-cast as in the statement of the Proposition. We start by noticing that the inequalities in (4.6) can be written in linear form as $10\log\varphi + \mathbf{P}_{i\to k} + 10\alpha\mathbf{e}_{i,k} - \mathbf{v}_{i\to k} \leq 0$. Relaxing these linear constraints by introducing the multipliers $\mathbf{H}$, leads to the following Lagrangian for Problem 4.1.1, which is identical to the Lagrangian in (4.14):

$$\mathcal{L}(\mathbf{Q},\mathbf{V},\mathbf{H}) \;=\; \tilde{\mathcal{U}}(\mathbf{Q}) \;+\; \sum_{i=1}^{\Delta+\Omega}\sum_{j\in\mathbb{O}(i)} \mathbf{h}_{i,j}\left[\mathbf{P}_{i\to j} - \mathbf{v}_{i\to j} + 10\log\varphi + 10\alpha\mathbf{e}_{i,j}\right] \quad (4.18)$$

Notice that the sole role of the Lagrange multipliers $\mathbf{H}$ is to impose (4.6) and that constraints (4.5) and (4.7) still need to be observed. Given these facts, the statement in the Proposition follows from standard use of Lagrange multiplier theory [16]. □

4.2 A primal-dual iterative solution to Problem 4.1.1

To complete the primal-dual solution framework in this part, we can apply a simple subgradient method to solve the above Lagrange dual problem. The outcome is an iterative algorithm that solves optimization problem 4.0.2, globally. It is important to note that the key requirement that allows to solve our optimization problem efficiently via its dual is the fact that strong duality holds that is a consequence of underlying convexity of problem. More specifically, we now propose the following primal-dual algorithm that solves 4.0.2 iteratively.

Primal-Dual iteration : Consider the following coupled iterative equations, where k is a non negative integer counter:

(Primal step)

$$\left(\mathcal{Q}^{k+1}, \mathcal{V}^{k+1}\right) = \begin{cases} \arg\min_{(\mathbf{Q},\mathbf{V})} \mathcal{L}\left(\mathbf{Q}, \mathbf{V}, \mathcal{H}(k)\right) \\ \text{subject to (4.5) and (4.7)} \end{cases} \tag{4.19}$$

Where $\mathcal{Q}^{k+1} = \{\ \{P_{i\to j}^{k+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}\ ,\ \{e_{i\to j}^{k+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}\ \}$,

$\mathcal{V}^{k+1} = \{v_{i\to j}^{k+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$

(Price update)

Update the elements of $\mathcal{H}(k+1) = \{h_{ij}^{k+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ by:

$$h_{ij}^{k+1} = [h_{ij}^{k} + \varepsilon^{k}(10\log\varphi + P_{i\to j}^{k} + 10\alpha {e_{i,j}}^{k} - v_{i\to j}^{k})]^{+} \tag{4.20}$$

Where $[.]^{+}$ denotes $max(.,0)$.

Proposition 4.2.1. *The above primal-dual algorithm converges to the global optimum of problem 4.0.2, provided that the feasible set has a non-empty interior (i.e., there exists a strictly feasible solution in the constraint set).*

Proof. We outline the proof here, which is inspired from [19]. The convexity of the overall optimization problem 4.0.2 was shown in section 2.1.1 for a more general case. More, we know that the Slater's condition holds when there exists a strictly feasible point in the constraint set. Since the strong duality holds for the class of convex optimization problems under the Slater's condition, finding the optimal solution of 4.0.2 is equivalent to solving the dual maximization. The proposed iterative algorithm simply solves this dual maximization problem with a simple subgradient method. Note that (4.20) simply describes the update of the dual variables in the gradient direction. The convergence of this subgradient method is guaranteed, if the step sizes ε^{τ} are chosen following a diminishing step size rule[21](e.g., any square summable but not summable sequence). Hence, the above algorithm converges to the global dual optimum that coincides with the optimum solution in the primal domain. □

In decoupling of the joint placement and power allocation problem, the dual variables(shadow prices) $\mathbf{h}_{ij}$ play a key role in coordinating the optimal placement of the nodes $demand$ for satisfying rate constraints in one hand and the physical layer power $supply$ on the other hand. In particular, $\mathbf{h}_{ij}$ can be interpreted as the rate cost in the link between transmitter i and receiver j, i.e. the higher value of the $\mathbf{h}_{ij}$ signals to the power allocation subproblem that more power should be allocated to this link. At the same time, it signals to the node placement subproblem that increasing the distance in this link is expensive.

This primal-dual optimization framework provides a layered approach to the total power minimization problem 4.0.2. This kind of layering breaks the overall optimiza-

tion problem into independent modules by utilizing a dual decomposition approach. This modularity is one of the main features of the primal-dual algorithm that will be used later in this section for proposing a decentralized solution for problem 4.0.2. On the other hand, these independent subproblems give a nice interpretation of the solution of the original problem. In particular, the node placement part in (4.17), is nothing but a minimization of a weighted sum of the distances between the nodes that communicate with each other. These weights are lagrangian multipliers that are controlled by the minimum rate constraints that we need on the links. This implies that our optimization problem 4.0.2 in this section, which is an instance of problem 2.0.1 with the objective of minimizing the total power consumption, is somehow equivalent to minimizing a weighted sum of the distances between the nodes.

4.3 Distributed implementation

After breaking the problem into smaller independent subproblems with exchanging prices, it is the time to solve each of these subproblems with an efficient distributed algorithm. After addressing a distributed solution for each of the subproblem modules, we return to complete the overall decentralized algorithm by applying them into the primal-dual algorithm.

4.3.1 Distributed power allocation

Substituting the cost function from (4.1) in the Lagrangian (4.15), we can see that the power allocation subproblem can be broken further into disjoint smaller parts that

can be solved locally at the transmitters or the receivers. One part is the set of disjoint optimization problems of the form:

$$\text{minimize} \ : \ \sum_{j\in\mathbb{O}(i)} \varphi_i 10^{0.1\mathbf{v}_{i\to j}} - \mathbf{h}_{ij}\mathbf{v}_{i\to j}$$

$$s.t. \ : \ \sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i} \tag{4.21}$$

That can be solved independently at each transmitter node $i \in \{1, ..., \Delta + \Omega\}$. Note that the prices used in these subproblems are locally available in each transmitter. These optimization problems actually shows that how the power supply at each transmitter should be distributed between the outgoing links according to the prices of the links at each iteration. The analytic solution of these simple constrained convex optimization problems is derived in appendix A and the results can be readily used in this part.

The other part of the power allocation subproblem is the following set of disjoint problems that can be solved at each receiver, independently:

$$\min \ \sum_{i\in\mathbb{I}(j)} \mathbf{h}_{ij}\mathbf{P}_{i\to j}$$

$$s.t. \ : \ \mathbf{S}_{i\to j} \geq \mathcal{R}^{-1}(\varrho_{i\to j}) \ \ \forall i \in \mathbb{I}(j) \tag{4.22}$$

Where $\mathbb{I}(j)$ denotes the set of transmitter nodes that have incoming links to destination node j. The fact that we can solve these problems in each receiver j independently, is a direct consequence of our wireless medium sharing assumptions in section 1.3. Recall that in our model, we assumed that each receiver node has a distinct reception assigned channel, that can be used for sharing between different incoming links that want to send

information to it. More, we specified that we do not consider inter-channel interference in our problem formulation. This assumption means that we only consider interference for the links that send information to a common destination node by sharing the channel in time or frequency domain, imperfectly. This simplifying assumption, leads to the fact that the power allocation can be done in each receiver node for all incoming links, independently.

For the above reasons, the power allocation subproblem in our case-study can be decomposed into a set of disjoint local optimization problems (4.21) and (4.22) that contain only local variables and prices. Therefore, it can have an efficient distributed implementation without extra message passing inside the network.

4.3.2 Distributed placement

Solving the node placement subproblem in (4.17) which is just a minimization of the weighted sum of the distances in the network is a simple unconstraint convex optimization problem. This convexity is an immediate consequence of the fact that in each iteration the prices $\mathbf{h}_{ij}$ are merely positive constant coefficients and the distances $\mathbf{e}_{i,j}$ are Euclidean norms that are convex functions of the location of the mobile nodes, i.e $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$. The only problem that we encounter is that the Euclidean norms that are the summands of our objective function (4.16) are not differentiable at origin. In other words, this problem is in the class of non-smooth convex optimization problems. Therefore, instead of using a simple gradient method we apply a subgradient descent method [20, section 3.2.3] to find the optimum positions in an iterative way with the following

updates for the position of the mobile nodes in each step :

$$(\mathbf{x}_i, \mathbf{y}_i)^{t+1} = (\mathbf{x}_i, \mathbf{y}_i)^t + \varepsilon^t \sum_{j \in \{\mathbb{O}(i) \bigcup \mathbb{I}(i)\}} \mathbf{h}_{ij} \overrightarrow{\theta_{ij}} \tag{4.23}$$

Where $\{\varepsilon^t\}$ is an appropriate[1] step size sequence and $\overrightarrow{\theta_{ij}}$ is defined to be:

$$\overrightarrow{\theta_{ij}} = \begin{cases} 1/\mathbf{e}_{i,j}(\mathbf{x}_j - \mathbf{x}_i, \mathbf{y}_j - \mathbf{y}_i) & \text{if } \mathbf{e}_{i,j} \neq 0 \\ (0,0) & Otherwise \end{cases} \tag{4.24}$$

Which is actually a unit size vector with direction from node i to j unless $\mathbf{e}_{i,j} = 0$.[2] Notice that $\{\mathbb{O}(i) \bigcup \mathbb{I}(i)\}$ describes the union of the all wireless nodes that receive data from node i *or* send something to it.

The key idea that makes it possible to implement this subgradient method in a distributed fashion is that the updated positions of the mobile nodes in (4.23) can be computed locally in each step, without extra message-passing inside the network. In other word, the updates in (4.23) implies that each mobile node just needs the prices and the direction of its own outgoing links and these pieces of information is available locally.

Now after proposing distributed solutions for placement subproblem and power allocation part, we are ready to introduce our decentralized solution that solves problem 4.0.2 efficiently, which is the main algorithm of this section. The idea is to plug these subproblem modules into our previous primal-dual algorithm as follows:

[1] e.g. any square summable but not summable sequence

[2] See appendix B for a detailed proof of the convergence of the proposed method

---

Initialize $\mathbf{H}$

(Primal step)

- Solve the node placement subproblem in a distributed manner as discussed in 4.3.2 and let $\{e_{i\rightarrow j}^{\tau+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ be the optimal distances
- Solve the power allocation subproblem in a distributed manner as discussed in 4.3.1 and let $\{P_{i\rightarrow j}^{\tau+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ and $\{v_{i\rightarrow j}^{\tau+1}\}_{i=1,j\in\mathbb{O}(i)}^{\Delta+\Omega}$ be the optimal solution

(Price update)

- Update the dual variables locally at each link:

$$h_{ij}^{\tau+1} = [h_{ij}^{\tau} + \varepsilon^{\tau}(10log\varphi + P_{i\rightarrow j}^{\tau} + 10\alpha e_{i,j}^{\tau} - v_{i\rightarrow j}^{\tau})]^{+} \tag{4.25}$$

Notice that in each iteration of this algorithm corresponding to counter τ, we have to run the distributed placement module in primal step, which is an iterative algorithm itself that may need infinitely many steps to converge to it's optimum solution. What we will do later in our simulations is to replace the node placement module with a finite time iterative algorithm that leads to a sub-optimal solution to the whole optimization problem. This sub-optimality of the final solution is the penalty for coming up with a decentralized solution. Nonetheless, we can get closer to the optimum solution by increasing the iterations in the node placement module.

What we have done so far in this section, was a distributed solution for the total power minimization problem 4.0.2 as an instance of our general jointly optimal placement and power allocation problem 2.0.1. The main concern of this case study was minimizing the total power at the physical layer while we assumed fixed rate constraints on the links. As another example, we can generalize this framework to a cross-layer optimization problem with the objective of maximizing the throughput of the wireless networks(see [17] for details). Chapter 6 also provides other possible extensions to this framework, but the main goal here was to illustrate how the primal-dual algorithm might be implemented in a decentralized fashion for the class of network optimization problems that contain placement optimization.

Chapter 5

Simulations

We simulate two examples in this section to illustrate our decentralized optimization framework, proposed in chapter 4. The first example would be a simple case where we only have one mobile node to picture some basic results. Then we increase the number of the mobile nodes to exemplify its performance in more demanding situations.

Example 5.0.1. *Consider a simple scenario when we have four fixed nodes ($\Delta = 4$) that want to send information to a single mobile node ($\Omega = 1$). Each of these four links will try to satisfy it's minimum bit rate constraint which is equal for all and the objective would be to minimize the total power consumption of the network. We also consider two different cases for channel interference: a low interference case where the the average ratio of the desired channel gain to the sum of interference coefficients is 15dB and a high interference case where the the average ratio of the desired channel gain to the sum of interference coefficients is 5dB. In both cases the channel gain and interference coefficient are generated according to log-normal fading. We use our proposed distributed algorithm in section 4 to find the optimal power of the transmitters and the position of the mobile nodes.*

Fig. 5.1 illustrates the optimal position of the mobile node in the case of high interference. It also shows the trajectory of the mobile node during the algorithm until it converges to the optimum position. Fig. 5.2 depicts the power allocated in the transmitter

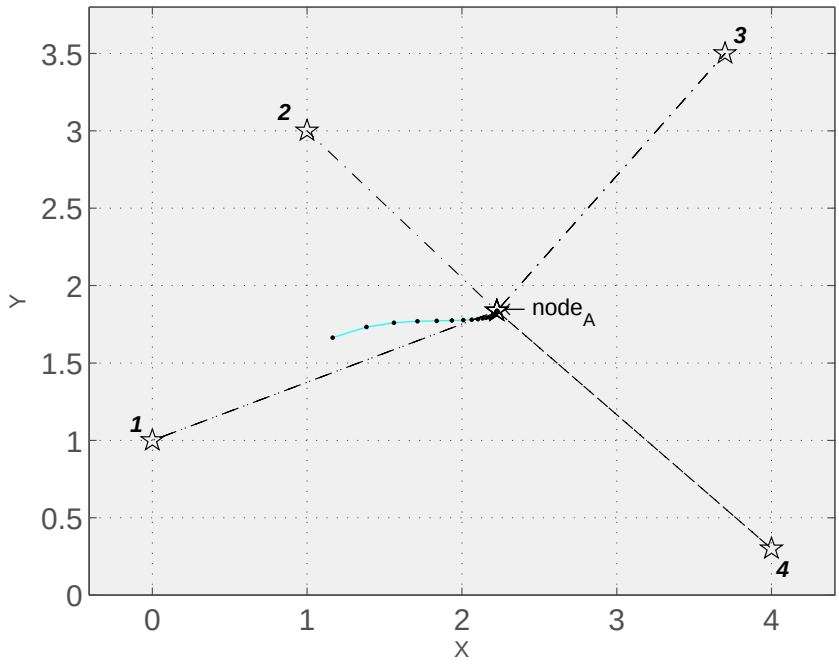

Figure 5.1: Optimal position of the mobile node ($node_A$) in the case of high-interference, Example 5.0.1

side of each link during the iterations.

In order to compare the speed of convergence in high-interference and low-interference scenarios, we have plotted the dual variables(shadow prices) in both cases in Fig. 5.3 As we can see, with the same step sizes, convergence is faster in low-interference case(i.e. 30 iterations) than in high-interference case(i.e. 80 iterations).

As we discussed we have a maximum power constraint, Ψ_i, in the transmitters. We have reduced this quantity so that the signal to interference constraints cannot be satisfied. Fig. 5.4 illustrates the dual variables in this case where the total optimization problem is infeasible. As we can see, the prices do not converge. Refereing to the interpretation of the prices, in this case and in each iteration each link increases its price in

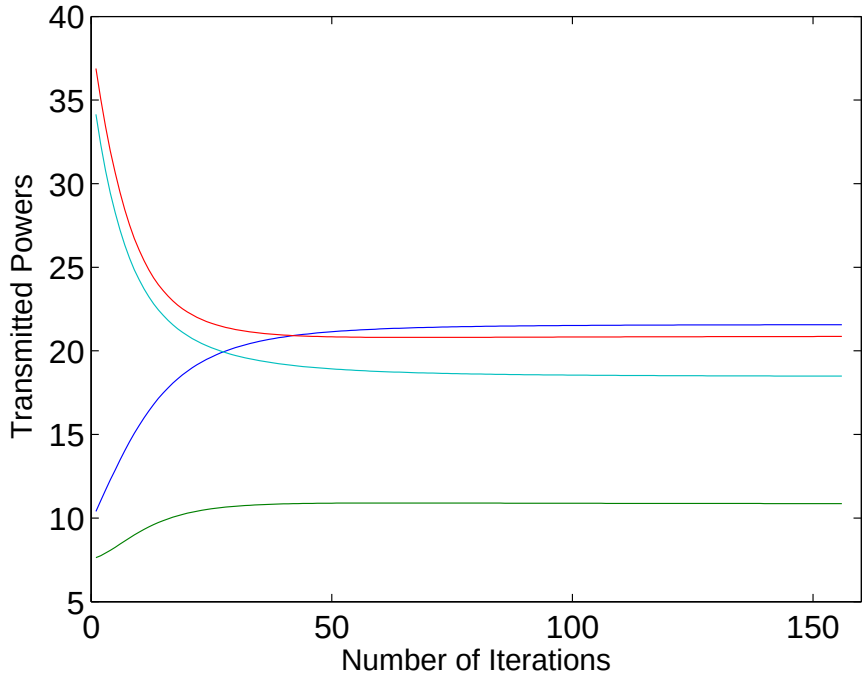

Figure 5.2: Power allocation in the transmitter sides While we have high-interference, Example 5.0.1

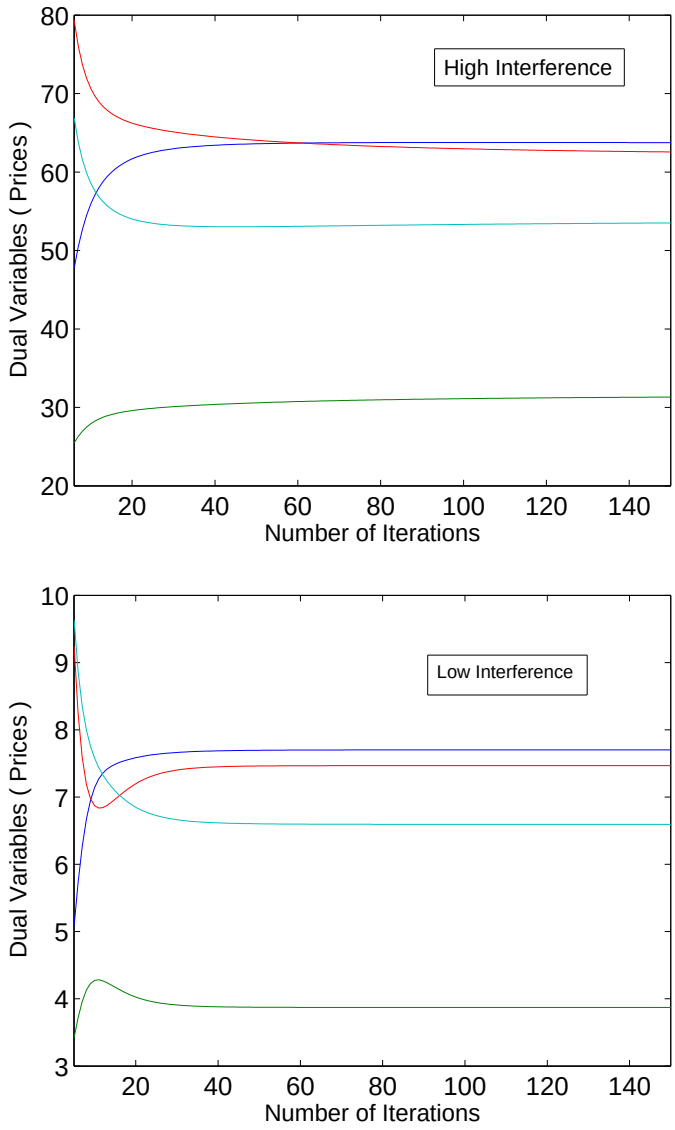

Figure 5.3: Shadow Prices (High-Interference v.s Low-Interference), Example 5.0.1

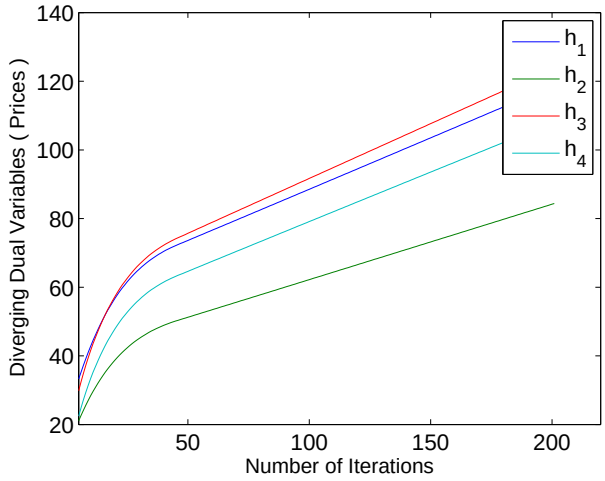

Figure 5.4: Dual variables in the case that the whole optimization problem is infeasible, Example 5.0.1

order to get more power, but since there is no more power available in the transmitter side there is noway for convergence of the algorithm. The feasibility of the whole optimization problem can be checked in our distributed algorithm by the convergence of the dual variables.

Example 5.0.2. *In this part, we increase the number of mobile nodes in our simulation to illustrate the performance of our decentralized algorithm in a more challenging setting. We consider three mobile nodes ($\Omega = 3$) that are labeled A, B and C. The connectivity graph of the nodes is illustrated in Fig. 5.5. The double arrow between mobile nodes A and B means that both A and B want to communicate with each other. We have the same bit rate constraints in all of the links and the objective is still minimizing the total power consumption in the network. Fig. 5.6 illustrates the optimal position of the mobile nodes, the power allocations in the links and the dual variables all in the case of high-interference.*

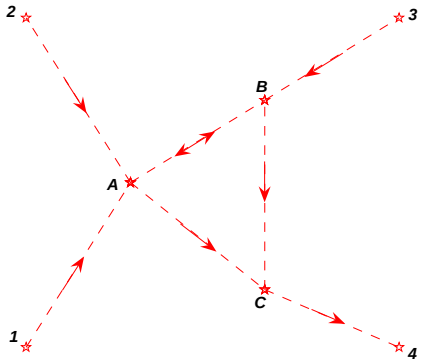

Figure 5.5: The Network Topology for Example 5.0.2

As we discussed in section (5) we have to solve the placement optimization subproblem in each iteration via a gradient descent method. In order to reduce the complexity, we can find the suboptimal solution of each these placement optimization problems. This is possible by reducing the number of steps in the gradient method that is discussed in 4.3.2. We have reduced the number of these steps in each iteration to one and illustrated the converging solution in Fig. 5.7. It can be seen that the optimum solution is still the same. This figure also shows the trajectory of the mobile nodes during the algorithm until it converges to the optimum positions.

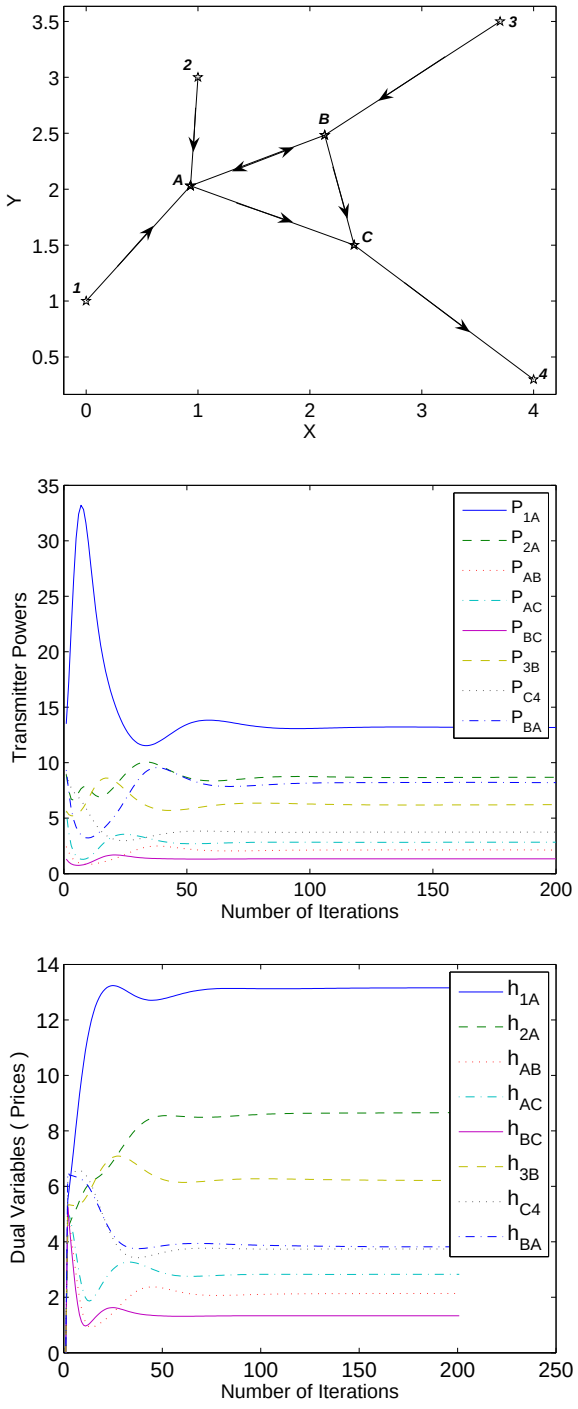

Figure 5.6: (a) Optimal positions of the mobile nodes $A, B\ and\ C$; (b) power allocations; (c) Dual variables

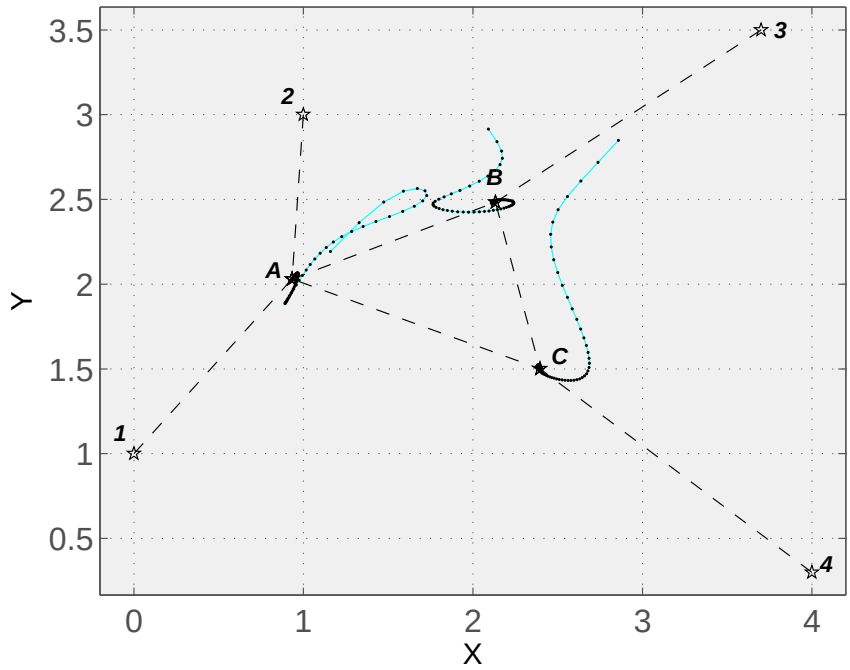

Figure 5.7: Trajectory of the mobile nodes during the iterations

Chapter 6

Extensions

In this chapter, we will point out two possible extensions of our framework.

6.1 Generalized Propagation Loss Model

The complexity of signal propagation makes it difficult to obtain a single model for characterizing propagation loss across a range of different environments. Therefore, there are several models that have been developed over the years to predict path-loss in different wireless environments. What we discussed here in this thesis was the case where

the pass-loss function[1] can be expressed as an exponentially decreasing function of the distance between source and receiver, i.e $P_L = \varphi 10^{\alpha\ \mathbf{d}}$. In chapter (1.4.1) we mentioned some motivations for such path-loss function, specially in urban and sub sea environment. Though, in this extension, we want to emphasize on general path-loss function $P_L(d)$ that is modeling the variation of received signal power over distance.

According to what we showed in section 2.1.1, under the assumption of exponential path-loss function, the general placement optimization problem is a convex optimization problem. The main difficulty of general path-loss functions is that the convexity of the problem doesn't hold anymore for arbitrary models, necessarily. In other words, the main watershed between convexity and non-convexity in placement optimization of the wireless nodes is the model that we use for propagation loss. Our main goal in this subsection is to mitigate the disadvantages of non-convexity by introducing an iterative method that is a convex approximation of the original problem in each step.

Based on what we have done earlier in this thesis, we know that our placement optimization problem is in a nice convex form if we adopt the exponential path-loss function. This underlying convexity, leaded us to cast the problem as standard GP in chapter 3 and later was the substructure of our decentralized algorithm in chapter4. The method that we want to propose in this subsection for solving the placement optimization problem

[1] What we mean by *path-loss function*(P_L) for the communication channel is the ratio of the transmitted power to received power(Note that in this ratio the powers are expressed in W). Generally, path-loss is a function of transmitter-receiver distance, transmission frequency, transmitter and receiver antenna gain and also the environment (all are assumed to be fixed, except the distance that includes in our optimization variables.)

with general path-loss function, starts by solving a problem in which we approximate the path-loss functions for each link with an exponential function. After that, in each iteration, we try to find the constants φ and α, for each link, in a way that the exponential function $\varphi 10^{\alpha\ \mathbf{d}}$, becomes the first order approximation of the actual path-loss function around the optimal distances that we have found in the previous iteration. Note that, since we are using exponential path-loss function in each iteration, we have to solve a convex optimization problem in each step which is possible with fast algorithms such as interior point methods or primal dual algorithms[18]. It is not hard to see that the converging optimum solution that we would find by this iterative method is also a local optimum for the original problem with arbitrary path-loss function. Meanwhile, there is no guarantee for the convergence of the algorithm or finding the global optimum, since the overall problem is not convex anymore.

The idea of solving the problem for generalized path-loss models iteratively, can be generalized to our distributed algorithm that is proposed in chapter 4. More specifically, each link can update it's path loss model parameters, φ and α, during the iterations. This update can be done by measuring the separation between the transmitter and the receiver and trying to fit the exponential path-loss model, $\varphi 10^{\alpha\ \mathbf{d}}$, to the first order approximation of the actual path-loss function around the measured distance.

In order to check the efficiency of the proposed method, we repeat our simulation in the second example of chapter 5 with a new path-loss function of the following form [12]:

$$P_L(d) = K(d/d_0)^{\gamma} \tag{6.1}$$

where d_0 is a reference distance for the antenna far-field, γ is the path loss fall off exponent which is typically a number between 2 and 6 and K is a constant that encompass the transmission frequency and the gain of the antennas. We use $d_0 = 0.5$, $\gamma = 4$ and $K = 1$ for our simulation. Also, we assume the same bit rate constraints in all of the links and the objective is still minimizing the total power consumption in the network with a decentralized method. Fig. 6.1 illustrates the optimal position of the mobile nodes, and parameters φ and α for different links during the iterations. This figure also shows the trajectory of the mobile nodes during the algorithm until it converges to the optimum position. Note that in this case the convergence happens after approximately two hundred iterations which is roughly two times slower in comparison with the previous case where we assumed pure exponential path-loss model(see Fig. 5.6).

What we proposed in this extension was a simple approach for solving the placement optimization problems where we have general path-loss model. Although we show the convergence of the algorithm for a specific path-loss model by simulation, there is still a lack of proof for convergence and even global optimality of the solution for general case and needs further studies.

6.2 Generalization of the Distributed Solution for Linear Rate Constraints

As we saw in example 2.2.2, under the assumption of high signal to interference ratio, we can re-write the linear rate constraints in the communication links, as (2.23) that contains the product of signal to interference ratios in the associated links. Here we want to extend our proposed distributed algorithm in section 4 so that we could handle

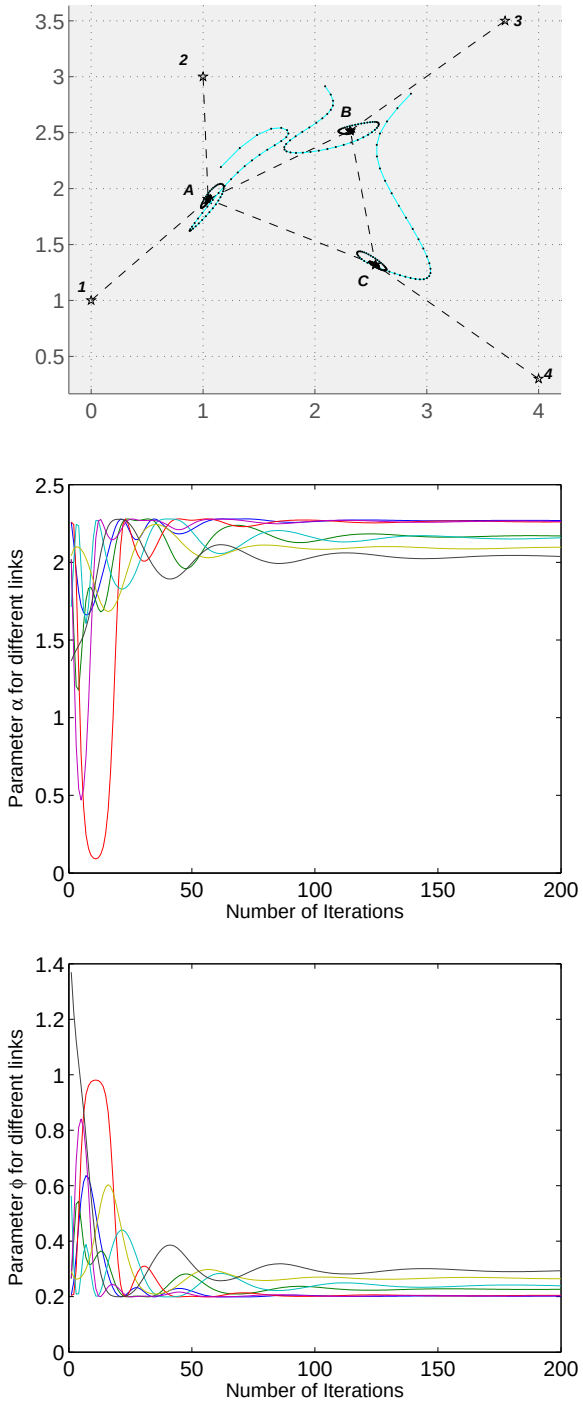

Figure 6.1: (a) Optimal positions of the mobile nodes $A, B\ and\ C$; (b) parameter α for different links ; (c) parameter φ for different links

these linear inequalities on rates. For simplicity, suppose that we only have one linear rate constraint which is re-formulated in terms of signal to interference ratio as:

$$\prod_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} \left(\kappa \mathbf{S}_{i\rightarrow j}\right)^{\frac{\varrho_{i,j}}{\Upsilon}} \geq 2^r \tag{6.2}$$

Where r is a design parameter. The procedure of governing this inequality in our proposed decentralized algorithm, is exactly the same as what we did for power constraints in (4.4). Specifically, in order to break this inequality into some local constraints, we replace it by the following set of equivalent inequalities:

$$\prod_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} 2^{\widetilde{\mathbf{R}}_{i\rightarrow j}} \geq 2^r \tag{6.3}$$

$$\left(\kappa \mathbf{S}_{i\rightarrow j}\right)^{\frac{\varrho_{i,j}}{\Upsilon}} \geq 2^{\widetilde{\mathbf{R}}_{i\rightarrow j}} \quad ,(i,j) \in \{1,\ldots,\Delta+\Omega\}^2 \tag{6.4}$$

Where $\widetilde{\mathbf{R}}_{i\rightarrow j}$ is an auxiliary variable that can be interpreted as the maximum rate constraint in the link from transmitter i to the receiver j. Following the same procedure of section 4 and by introducing some price $\widetilde{\mathbf{h}}$ into the objective, we can relax (6.3) [2]into our previous Lagrangian L in (4.18). As a result, the new Lagrange optimization problem in the primal domain would be updated to minimizing :

$$\widetilde{L} = L + \widetilde{\mathbf{h}} \left(r - \sum_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} \widetilde{\mathbf{R}}_{i\rightarrow j} \right) \tag{6.5}$$

Subject to new single rate constraints (6.4), plus our previous rate and power constraints (4.7) and (4.5). This Lagrange minimization problem can still be decoupled into two disjoint parts: the node placement subproblem and the power allocation subproblem. Note that the node placement part remains unchanged while in the power allocation part, the

[2]Note that this inequality is equivalent to the following linear constraint: $\sum_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} \widetilde{\mathbf{R}}_{i\rightarrow j} \geq r$

objective would change a bit and we should also include our new single rate constraints (6.4). Both of these subproblems can still be solved in a decentralized fashion with a similar approach of section 4.3. Therefore, we can apply our decentralized solution with identical steps, by considering the fact that the updates for the new Lagrange multiplier $\widetilde{\mathbf{h}}$ in dual update step would be:

$$\widetilde{\mathbf{h}}^{\tau+1} = \widetilde{\mathbf{h}}^{\tau} + \varepsilon^{\tau} \left(r - \sum_{(i,j)\in\{1,\ldots,\Delta+\Omega\}^2} \widetilde{\mathbf{R}}_{i\to j}^{\tau} \right) \tag{6.6}$$

Where $\widetilde{\mathbf{R}}_{i\to j}^{\tau}$ is the optimum rate obtained in the power allocation subproblem in primal step. The important message which is implied from this price update formula, is that we need the optimum value of the achieved rate in each iteration. In other words, the communication links that are involved in the linear rate constraint inequality, should be able to contact with each other during the algorithm in order to update the prices. As it is expected, this extra message passing may cause some practical problems in the distributed algorithm, specially if the communication links that are involved in the linear rate constraint be far apart from each other. Nonetheless, this a natural problem with distributed optimization problems that have coupled inequalities or objectives.

Chapter 7

Conclusions and open problems

In this thesis, we proposed a paradigm for the optimal design of wireless networks, with respect to power allocation and the placement of nodes in a Cartesian plane. We consider the optimization of network-centric figures of merit, which are functions of the transmission power and of the signal to interference ratio. In the high signal to interference regime, we can also adopt constraints and cost functions that involve rate-regions or throughput and under the ”no dominant interferer” assumption we can also include outage probabilities. Under the assumption of exponential path loss, we show that our paradigm is convex and that it admits an approximate solution via geometric programming. We also provide a case study that illustrates how certain instance of our paradigm can be optimized, with no approximation, via a primal-dual iterative algorithm. This solution also leads to a layering in the primal step, where the power allocation and the node placement can be optimized independently. The following problems require further investigation, and we believe that this thesis might be a stepping stone towards their solution:

- It is important to handle cost and the constraints that are functions of the transmission rate, while lifting the high signal to interference assumption. In the paper[**?**], this problem was solved with respect to power allocation.

- It is also important to obtain the solution to our paradigm for the case where the path loss is polynomial.

Chapter A

Analytic Solution of (4.21)

Here we want to find the optimal solution of the following convex optimization problem which is used in our distributed implementation.

$$\text{minimize} \ : \ \textstyle\sum_{j\in\mathbb{O}(i)} \varphi_i 10^{0.1\mathbf{v}_{i\to j}} - \mathbf{h}_{ij}\mathbf{v}_{i\to j}$$

$$s.t. \ : \ \sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}_{i\to j}} \leq 10^{0.1\Psi_i} \tag{A.1}$$

First note that , neglecting the constraint, the global solution that minimize the cost function is : $\mathbf{v}^*_{i\to k} = 10\log\frac{10\ \mathbf{h}_{ij}}{\ln(10)\varphi_i}$. It is clear that if these optimal solutions are satisfying the constraint, we have solved (A.1). Now consider the case that the global solution of the unconstrained problem is not feasible. In this case the level-set of the cost function at the optimal point should be tangent to the feasible set at optimal point. It is equivalent to the following equalities:

$$\frac{\varphi_i\ 10^{0.1\mathbf{v}^*_{i\to j}} - \mathbf{h}_{ij}}{10^{0.1\mathbf{v}^*_{i\to j}}} = K \quad , for\ \forall j \in \mathbb{O}(i) \tag{A.2}$$

$$\sum_{j\in\mathbb{O}(i)} 10^{0.1\mathbf{v}^*_{i\to k}} = 10^{0.1\Psi_i} \tag{A.3}$$

Where K is a constant. Equation (A.2) is from the fact that the gradient of the level-set of the cost function should have the same direction as the gradient of the boundary of the feasible set at optimal point and (A.3) is the fact that the optimum is in the boundary of

the feasible set. [1] Solving (A.2) and (A.3) simultaneously, we can conclude that $10^{0.1\mathbf{v}^*_{i\to j}}$, which is actually the optimum value of the maximum power that should be allocated in each iteration, is $\frac{\mathbf{h}_{ij}}{\sum_{j\in\mathbb{O}(i)}\mathbf{h}_{ij}}10^{0.1P_i^{max}}$. This result has also an interesting interpretation in dual domain that says the maximum power which should be allocated in each iteration to the links is proportional to the prices $\mathbf{h}_{ij}$.

[1] Note that these equalities can also be derived from the KKT conditions.

Chapter B

Subgradient Method for Node Placement Module

As we mentioned in section 4.3.2 the node placement sub-problem (4.17) is a non-smooth convex optimization problem. Here in this appendix we want to take a quick look at the proposed subgradient method in section 4.3.2 and see how does it work. In particular, we describe the update formula (4.23) which is nothing but moving in the opposite direction of the subgradient. To this end we start by the definition of the subgradient for a nonsmooth convex function and we proceed by utilizing some simple lemmas in order to compute the subgradient of our main objective (4.16).

Definition B.0.1. *Let f be a convex function. A vector g is called a* subgradient *of function f at point $z_0 \in domf$ if for any $z \in domf$ we have*

$$f(z) \geq f(z_0) + < g, z - z_0 > . \tag{B.1}$$

Where $< ., . >$ is a dot product. The set of all subgradients of f at z_0, $\partial f(z_0)$, is called subdifferential *of function f at point z_0.*

Recall that the optimization variables in this problem is the position of the mobile nodes $\{\mathbf{x}_i, \mathbf{y}_i\}_{i=\Delta+1}^{\Delta+\Omega}$, and we need to update them during the iterations according to the subgradient of the cost function (4.16). Hence we need to calculate the subgradient for a weighted sum of the Euclidean distances which is not difficult by using the following lemmas.

Lemma B.0.1. *Let $\mathbf{e}_{i,j}$ be the Euclidean distance function given by (1.1). Suppose $i \geq \Delta + 1, j \leq \Delta$, hence $\mathbf{e}_{i,j}$ is merely a function of $(\mathbf{x}_i, \mathbf{y}_i)$. For this case we have:*

$$\partial \mathbf{e}_{i,j}(\mathbf{x}_i, \mathbf{y}_i) = \begin{cases} \frac{(\mathbf{x}_i - \chi_j, \mathbf{y}_i - \gamma_j)}{\sqrt{(\mathbf{x}_i - \chi_k)^2 + (\mathbf{y}_i - \gamma_k)^2}} & (\mathbf{x}_i, \mathbf{y}_i) \neq (\chi_j, \gamma_j) \\ B_2(0,1) & Otherwise \end{cases} \tag{B.2}$$

Where $B_2(0,1) = \{(x,y) \mid \sqrt{x^2 + y^2} \leq 1\}$

Proof. see [20, example 3.1.5]. □

Note that, according to the definition of $\overrightarrow{\theta_{ij}}$ in (4.24), we have $\overrightarrow{\theta_{ij}} \in \partial \mathbf{e}_{i,j}(\mathbf{x}_i, \mathbf{y}_i)$, i.e. $\overrightarrow{\theta_{ij}}$ is a valid subgradient for Euclidean norm function $\mathbf{e}_{i,j}$. We can derive similar expressions for the cases when $\{j \geq \Delta + 1, i \leq \Delta\}$ or $\{j \geq \Delta + 1, i \geq \Delta + 1\}$.

Lemma B.0.2. *Let $f_1(z)$ and $f_2(z)$ be convex functions with same domain and $h_1 \geq 0,\ h_2 \geq 0$. For the function $f(z) = h_1 f_1(z) + h_2 f_2(z)$ we have:*

$$\partial f(z) = h_1 \partial f_1(z) + h_2 \partial f_2(z) \tag{B.3}$$

Proof. The proof is in [20, page 131]. It is also trivial to generalize this result for a weighted sum of more than two functions. □

Now we can conclude that the proposed iterative method in section 4.3.2 is a sub-gradient descent method[20, section 3.2.3], and the update formula (4.23) is just moving in the opposite direction of subgradient of the objective function.

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