ABSTRACT

Title of dissertation: WEAKLY O-MINIMAL STRUCTURES AND SKOLEM FUNCTIONS

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The monotonicity theorem is the first step in proving that o-minimal structures satisfy cellular decomposition, which gives a comprehensive picture of the definable subsets in an o-minimal structure. This leads to the fact that any o-minimal structure has an o-minimal theory. We first investigate the possible analogues for monotonicity in a weakly o-minimal structure, and find that having definable Skolem functions and uniform elimination of imaginaries is sufficient to guarantee that a weakly o-minimal theory satisfies one of these, the Finitary Monotonicity Property.

In much of the work on weakly o-minimal structures, it is shown that nonvaluational weakly o-minimal structures are most "like" the o-minimal case. To that end, there is a monotonicity theorem and a strong cellular decomposition for nonvaluational weakly o-minimal expansions of a group. In contrast to these results, we show that nonvaluational weakly o-minimal expansions of an o-minimal group do not have definable Skolem functions. As a partial converse, we show that certain valuational expansions of an o-minimal group, called T-immune, do have definable Skolem functions, and we calculate them explicitly via quantifier elimination.

WEAKLY O-MINIMAL STRUCTURES AND SKOLEM FUNCTIONS

by

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Dedication

For Jamie, who is missed, but who is spared the duty of reading this document.

And for Kristen, who is not.

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Chapter 0

Background

0.1 Introduction

Any o-minimal structure with a new convex subset named gives rise to a weakly o-minimal structure with a weakly o-minimal theory. Often, definable sets in models defined this way are very simple, and can be naturally described using the same techniques as in the study of o-minimal structures. In particular, the class of non-valuational weakly o-minimal expansions of groups has a cellular decomposition property that is as close as possible to the o-minimal case. On the other extreme, a valuational weakly o-minimal structure may have definable sets that are much more complicated. The flash-point for such a difference can usually be described by unary functions definable in the model.

Much of the research on o-minimal structures points to the idea that a nonvaluational structure is the most "like" an o-minimal structure. It is perhaps surprising, then, that while o-minimal theories enjoy uniformly definable Skolem functions, we find that certain nonvaluational weakly o-minimal theories fail to have Skolem functions in a strong way. Conversely, we find a large class of valuational weakly o-minimal structures which do have definable Skolem functions. We collect here some observations about monotonicity in weakly o-minimal structures, as well as selected criteria for weakly o-minimal theory to have definable Skolem functions.

0.2 Notation and conventions

We use $\mathcal{M}$ or $\mathcal{N}$ for a model, and shall generally use the the Gothic letter $\mathfrak{L}$ to define the language of a model. This frees up the Latin L to be used as a relation, which becomes useful in describing sets which we consider Less or Lower or to the Left. Similarly, we use $\mathbb{R}$ for the reals; the undecorated R will then be free to be used as a relation. Unless otherwise specified, all models are totally ordered, and all languages contain the binary relation symbol $<$ which defines this order. In general, variables will be given the letters $\{u, \ldots, z\}$; early-alphabet letters $\{a, \ldots, e\}$ and their Greek counterparts will refer to elements of the universe of a model; $\mathfrak{L}$-formulas will be given late-alphabet Greek names like $\varphi, \psi, \sigma, \tau$. And for both variables and constants, a naked letter a or x refers to a singleton; $\bar{a}$ or $\bar{x}$ refers to a tuple of finite length. Unless it becomes unclear in context, we shall denote $\lg(\bar{y})$ by n without explicitly defining it as such.

It is conventional in o-minimal model theory to work inside a large ("monster") model $\mathfrak{C}$, which is assumed to be "sufficiently saturated" in the following sense: it is immediate from the Löwenheim-Skolem Theorem that for every $\mathfrak{L}$-structure $\mathcal{M}$, there is an $\mathfrak{L}$-structure $\mathfrak{C} \succeq \mathcal{M}$ which is $|M|$-saturated. Informally, for a given theory T, we shall assume there is a "saturated enough" model $\mathfrak{C}$ of T so that every model $\mathcal{M}$ that we use is elementary in $\mathfrak{C}$ and $\mathfrak{C}$ is $|M|$-saturated; every element is an element of $\mathfrak{C}$; and a set A of elements is a subset of $\mathfrak{C}$ such that $\mathfrak{C}$ is $|A|$-saturated. When working in a monster model, it is also convention to abbreviate $\mathfrak{C} \models \theta$ by $\models \theta$.

In the same vein, unless otherwise specified, we say "definable set" or "definable function" to mean a set or function definable *with parameters from the ambient model*. In cases where the ambient model is not clear from context, we shall use the term $\mathcal{M}$-definable to mean "definable by an $\mathfrak{L}(\mathcal{M})$-formula."

We pay special attention to definability in the context of algebraic closure: an element a is *algebraic over the set* B if there is a formula $\varphi(x, \bar{y})$ over the empty set, and a set of parameters $\bar{b}$ from B with $\lg(\bar{b}) = \lg(\bar{y})$, such that $\varphi(\mathfrak{C}, \bar{b})$ is finite, and $a \in \varphi(\mathfrak{C}, \bar{b})$. In this case, $\varphi(x, \bar{b})$ is also called an *algebraic formula*. If $\varphi(x, \bar{b})$ is realized by precisely one element, namely a, then a is said to be *definable over* $\bar{b}$, or $\bar{b}$-*definable*. The *algebraic closure of* B, $\operatorname{acl}(B)$ is the set of elements which are algebraic over B; the *definable closure of* B, $\operatorname{dcl}(B)$, is the set of elements definable over B.

Note that in an ordered structure, if an element a an element is algebraic over $\bar{b}$, then a is in fact *definable* over $\bar{b}$ by a first order formula which describes the number of elements to the left and right of a in the finite set $\varphi(\mathcal{M}, \bar{b})$. Since all of our models will be ordered, we shall in general use $\operatorname{cl}(B)$ to refer to $\operatorname{acl}(B)$ or $\operatorname{dcl}(B)$.

In later chapters, we describe definable functions with many independent variables; in order to keep the pages as uncluttered as possible, we abuse notation as follows: Given a function $F : \mathcal{M}^n \to M$, we write $\operatorname{dom}(F(a_1, \ldots, x_i, \ldots, a_n))$ to mean $\{x_i \in M : F(a_1, \ldots, x_i, \ldots a_n) \text{ is defined}\}$. Similarly, we write $F(a_1, \ldots, x_i, \ldots, a_n)$ or $F_{a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_n}(x_i)$ to mean $F \restriction \{\bar{x} : x_1 = a_1, \ldots, x_{i-1} = a_{i-1}, x_{i+1} = a_{i+1}, \ldots, x_n = a_n\}$.

Another abuse of notation which is commonplace throughout mathematics but nevertheless bears warning here: we often use relations and functions in a setwise manner. We write $A < B$ for "every element of A is strictly less than every element of B." Similarly $a + B := \{a + b : b \in B\}$. These situations will be clear from context.

0.3 O-minimality

The class of o-minimal structures was given its name in the 1980s, and came about as a generalization of the nice behavior of the real field. It should be noted that the initial investigation of the real field, including quantifier elimination and a characterization of definable sets, was first undertaken by Tarski and Seidenberg in the 1930s. The seminal work on o-minimal structures and in particular, o-minimal structures on the reals, is done in [16], [13], and [10]. The definitions and results from this chapter are considered canonical and will be used elsewhere in this paper, usually without citation or proof.

For all of our work, we shall consider intervals to have endpoints in $\mathcal{M} \cup \{\pm\infty\}$.

Definition 0.3.1. For a first-order language $\mathfrak{L}$ containing a binary relation symbol $<$, an $\mathfrak{L}$-structure $(\mathcal{M}, <)$ is called *o-minimal* if $<_{\mathcal{M}}$ is dense, and every definable subset of $\mathcal{M}$ (with parameters from M allowed) is a finite union of points and open intervals. A theory is called o-minimal if all of its models are o-minimal.

The original definition of o-minimal allowed for structures that are not dense, but as all of the motivating examples are dense, it has become standard practice to assume the ordering is dense. It is proved in [13] (shown here as Corollary 0.3.11 that every o-minimal structure has an o-minimal theory; therefore we shall usually refer to o-minimal theories.

Example 0.3.2. Any model of DLO, the axioms for dense linear order without endpoints, in the language $\mathfrak{L} = \{<\}$, is o-minimal.

This is readily seen given that DLO eliminates quantifiers: any definable quantifier-free subset can be written in disjunctive normal form, and given that the only relations in $\mathfrak{L}$ are $<$ and $=$, the definable subsets are just finite Boolean combinations of points and intervals.

In most cases, we shall be concerned with that have more algebraic structure; for the most part in the paper we shall focus on expansions of ordered groups. We shall see eventually that any o-minimal or weakly o-minimal group is divisible and abelian; as such, we use $+$ for the group action. The language will always contain a constant 0. In this language, the axioms for an ordered group are as follows:

1. $\forall x(x + 0 = x)$
2. $\forall x \exists y(x + y = 0)$
3. $\forall x \forall y \forall z((x + y) + z = x + (y + z))$
4. $\forall x \forall y \forall z(x < y \rightarrow x + z < y + z)$

Axioms (1)-(3) are the group axioms; (4) gives the ordered group. Note that if we add the binary function '$-$' to the language, then axiom (2) becomes $\forall x(x+(0-x) =$

0), which is universal. For our purposes, it will not matter, but this fact is used in [7] to give the theory of ordered groups a universal axiomatization.

Example 0.3.3. $(\mathbb{R}, +, \cdot, <, 0, 1)$ is o-minimal.

This is essentially a consequence of the Tarski-Seidenberg theorem that projections of semialgebraic sets are semialgebraic. From this it is clear that the group structure $(\mathbb{R}, +, <, 0, 1)$ is also o-minimal, since any set definable in the group is definable in the field.

Example 0.3.4. $(\mathbb{Q}, +, \cdot, <, 0, 1)$ is not o-minimal.

This is a consequence of Julia Robinson's result that the integers are definable in the field structure over $\mathbb{Q}$.

0.3.1 Monotonicity and cellular decomposition

The key tool in the study of o-minimal structures is the Monotonicity Theorem. In order to clear up any possible discrepancies with conventions from other fields, we set the following definitions for the duration of the paper:

Definition 0.3.5. Let $(\mathcal{M}, <, \ldots)$ be an ordered structure. If $f : \mathcal{M} \to \mathcal{M}$ is a definable partial function and $U \subseteq \mathcal{M}$ a definable convex subset of the domain of f, then we say f is *strictly increasing on* U if $\mathcal{M} \models (\forall x, y \in U)(x < y \to f(x) < f(y))$. Similarly, f is *strictly decreasing on* U if $\mathcal{M} \models (\forall x, y \in U)(x < y \to f(x) > f(y))$. Then we say f is *strictly monotone on* U if f is strictly increasing, strictly decreasing, or constant on all of U.

Definition 0.3.6. With $\mathcal{M}$ as above, $f : \mathcal{M} \to \mathcal{M}$ a definable partial function, and U a definable convex subset of $\mathrm{dom}(f)$, we say f is *locally strictly increasing on* U if for any $a \in U$, there exists a convex open (in the order topology) set $V \subseteq U$ with $a \in V$, such that f is strictly increasing on V. Similarly, f is *locally strictly decreasing* if for any $a \in U$, there exists a convex open set $V \subseteq U$ with $a \in V$ such that f is strictly decreasing on V, and *locally constant* if for any $a \in U$, there is a convex open $V \subseteq U$ with $a \in U$ such that f is constant on V. We say f is *locally strictly monotone on* U if f is locally strictly increasing, locally strictly decreasing, or locally constant on U.

Definition 0.3.5 is used below; we shall not have need of Definition 0.3.6 until §1.1.

Theorem 0.3.7 (Monotonicity for o-minimal structures)**.** *Let* $(\mathcal{M}, <, \ldots)$ *be o-minimal, and* $F : \mathcal{M} \to \mathcal{M}$ *definable (with parameters from* $\mathcal{M}$*) function. Then there is a finite set* S *and open intervals* $I_1, \ldots, I_n$ *such that:*

- $S \cup I_1 \cup \ldots \cup I_n = dom(F)$*, and*
- *for every* $i \leq n$*,* $F \restriction I_i$ *is continuous and strictly monotone.*

In fact, much more is true. In the o-minimal case, monotonicity is the first step in the inductive proof of a complete cellular decomposition for definable sets. First, a notational convention:

Definition 0.3.8. Let $f, g : M^n \to M$ be definable functions, and $X \subseteq M^n$ a convex subset of $\mathrm{dom}(f) \cap \mathrm{dom}(g)$, such that $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in X$. Then $(f, g)_X := \{(\bar{x}, y) : f(\bar{x}) < y < g(\bar{x})\}$.

Definition 0.3.9. Let $(\mathcal{M}, <)$ be o-minimal. Cells are defined inductively as follows:

- A (0)-*cell* is a point in $\mathcal{M}$.
- A (1)-*cell* is an open interval of $\mathcal{M}$.
- If C is a $(i_0, \ldots, i_k)$-cell and $f : C \to M$ is a continuous definable function, then Γ_f, the graph of f on C, is a $(i_0, \ldots, i_k, 0)$-cell.
- If C is a $(i_0, \ldots, i_k)$-cell and $f, g : M^n \to M$ are continuous definable functions such that $f(\bar{x}) < g(\bar{x})$ for all $\bar{x} \in C$, then $(f, g)_C$ is a $(i_0, \ldots, i_k, 1)$-cell.
- There are no other cells.

We say C is a $k-$cell if C is a $(i_0, \ldots, i_k)$-cell, for some $\bar{i} \in 2^k$. It is convention to define the unique $()$-cell to be $\emptyset$.

It is clear from the definition that a k-cell of $\mathcal{M}$ is a definable subset of $\mathcal{M}^k$.

Theorem 0.3.10. *Let $\mathcal{M}$ be o-minimal, and fix $k \in \omega$. Let $f : \mathcal{M}^k \to \mathcal{M}$ be definable in $\mathcal{M}$, and $X_1, \ldots, X_n$ be definable subsets of $\mathcal{M}^k$. Then there is a partition $\mathcal{P}$ of $\mathcal{M}$ into finitely many k-cells such that:*

(i) Each X_i is a union of cells in $\mathcal{P}$.

(ii) $f \restriction X_i$ is continuous for each $i \leq k$.

Corollary 0.3.11. *If $\mathcal{M}$ is o-minimal and $\mathcal{M} \equiv \mathcal{N}$, then $\mathcal{N}$ is o-minimal.*

In reality, the 'corollary' is a consequence of a technique used in the proof of cellular decomposition. But it does follow easily from the theorem as stated: if $\mathcal{M}$ is o-minimal, and $\varphi(x, \bar{y})$ an $\mathfrak{L}$-formula (without parameters from outside the language), then $X = \varphi(\mathcal{M}^n)$ is a finite union of cells (say N-many), and:

$$\mathcal{M} \models \forall y \left((neg\exists x_1 < x_1' < \ldots < x_N < x_N' < x_{N+1}) \left(\bigwedge_i \varphi(x_i, \bar{y}) \wedge \bigwedge_j \neg\varphi(x_j', \bar{y}) \right) \right)$$

This gives a uniform bound on the number of connected components of $\varphi(x, \bar{a})$ for any $\mathcal{N} \equiv \mathcal{M}$ and $\bar{a} \in N^k$. (This property is called *uniform finiteness.*) Thus we may say equivalently that either $\mathcal{M}$ or its theory are o-minimal.

In the general o-minimal case, definable functions share some nice properties with functions definable on the real field.

Definition 0.3.12. A definable set $A \subseteq \mathcal{M}$ is *definably connected* if there is no partition of A into open definable subsets B and C such that $B \cap C = \emptyset$ and $B \cup C = A$.

Lemma 0.3.13 (van den Dries)**.** *If $\mathcal{M}$ is o-minimal, then every cell of $\mathcal{M}$ is definably connected.*

The property of definable connectedness is often also called the *intermediate value property.* It is shown in [10] that continuous definable functions in an o-minimal structure satisfy the intermediate value theorem. We shall also use the fact, shown in [11], that a continuous definable function on a closed bounded set attains a maximum value.

Often, the o-minimal context is looked to in comparison with stability theory: in practice, o-minimal theories are the ordered structures whose definable sets are most like those of a stable structure. To that end, some of the nice properties of stable theories have been shown to hold in the o-minimal case.

0.3.2 Definability results

Definition 0.3.14. An $\mathfrak{L}$-structure $\mathcal{M}$ satisfies the *exchange principle for algebraic closure* (we shall usually just say *exchange*) if, for any $a, b \in M$ and $C \subseteq M$, if $a \in \mathrm{cl}(Cb)$, then either $a \in \mathrm{cl}(C)$, or $b \in \mathrm{cl}(Ca)$.

Lemma 0.3.15. *Let $(\mathcal{M}, <, \ldots)$ be o-minimal. Then $\mathcal{M}$ satisfies exchange.*

We give a proof here, as the style of argument is often repeated in o-minimal structures. The lemma and the original proof are from [16].

Proof. Suppose $a \in \mathrm{cl}(Cb)$. Then a is definable over Cb, thus there is a function $f : \mathcal{M} \to \mathcal{M}$, definable with parameters from C, such that $f(b) = a$. If $b \notin \mathrm{cl}(Ca)$, then the set $\{\beta \in \mathcal{M} : f(\beta) = a\}$ is infinite, thus contains a maximal open interval I on which f is constantly a. By monotonicity, there are finitely many maximal open intervals on which f is constant; thus by the ordering, the endpoints of I are definable over C. Then a is C-definable as the image of I, so $a \in \mathrm{cl}(C)$. □

Definition 0.3.16. Given a theory T and set A, we denote by $S_n(A)$ (called the *Stone space of* A) the set of consistent complete n-types with parameters from A.

For each n, there is a natural topology on $S_n(A)$, with an open basis consisting of the collection of the sets $[\varphi(\bar{x})]$ with $\lg(\bar{x}) = n$ and $\varphi(\bar{x})$ a formula with parameters from A, where $[\varphi(\bar{x})] = \{p \in S_n(A) : \varphi(\bar{x}) \in p\}$. Since $S_n(A) \setminus [\varphi(\bar{x})] = [\neg\varphi(\bar{x})]$, then each $[\varphi(\bar{x})]$ is in fact clopen.

Definition 0.3.17. Let T be a theory, and $\mathfrak{C}$ a monster model of T. Given a set $A \subseteq \mathfrak{C}$, a model $\mathcal{M}$ is *prime over* A if $\mathcal{M} \models T$, $A \subseteq M$, and for every $\mathcal{N} \models T$, if $A \subseteq N$, then $\mathcal{M}$ is elementarily embeddable in $\mathcal{N}$.

Fact 0.3.18. It is shown in [13] that for an o-minimal T, for every set A, there is $\mathcal{M} \models T$ which is prime over A, and this $\mathcal{M}$ is unique up to A-isomorphism. We denote a prime model over the set A by $pr(A)$. The proof of existence of $pr(A)$ is straightforward, but we postpone it until the next section, as it is a consequence of the definability of Skolem functions.

Definition 0.3.19. Given an ordered structure $(\mathcal{M}, <)$, a *cut* of $\mathcal{M}$ is a pair $\langle C, D\rangle$ for such that $C, D \subseteq M$, and are nonempty, $C \cup D = M$, and $C < D$. The cut $\langle C, D\rangle$ is said to be definable if both C and D are.

While strictly speaking the term *cut* refers to the pair of subsets whose union is $\mathcal{M}$, we shall often refer to a complete 1-type as a cut, where an element which *realizes the cut* $\langle C, D\rangle$ over $\mathcal{M}$ is an element $a \in \mathfrak{C}$ such that $\mathfrak{C} \models c < a < d$ for every $c \in C$, $d \in D$. Given the lemma, we shall interchangeably refer to the type of an element and its cut over the model. The next lemma, originally proven in [16], says that a complete 1-type is determined by the cut that it describes.

Lemma 0.3.20. *Suppose T is o-minimal, $A \subseteq \mathfrak{C}$, and $p \in S_1(A)$. Then p is generated by:*

$$\{x < a : a \in cl(A) \text{ and } p \models x < a\} \cup \{x > a : a \in cl(A) \text{ and } p \models x > a\}$$
$$\cup \{x = a : a \in cl(A) \text{ and } p \models x = a\}$$

Proof. Fix T and A as above, and let $\mathcal{M}$ be a model of T containing A. Suppose $p \in S_1(A)$ describes a cut $\langle C, D\rangle$ of M, and let $\varphi(x)$ be an $\mathfrak{L}(A)-$formula. Then by o-minimality, both $\varphi(\mathcal{M})$ and $\neg\varphi(\mathcal{M})$ are each finite unions of intervals. Since the endpoints of the intervals are each in $\mathcal{M}$, then exactly one of φ, $\neg\varphi$ is consistent with the cut C, thus is implied by p. $\square$

The remaining results in this subsection are due to Marker and appear in [15].

Definition 0.3.21. Given an o-minimal structure $\mathcal{M} \preceq \mathfrak{C}$ and a type $p \in S_1(\mathcal{M})$, p is an *irrational cut* if p is nonalgebraic and for every $a \in M$, if $x > a \in p$, then there is $a' > a$ such that $x > a' \in p$, and for every $b \in M$, if $x < b \in p$, then there is $b' < b$ such that $x < b' \in p$. A complete 1-type p is a *noncut* if p is nonalgebraic and not an irrational cut.

Those familiar with [15] will recognize the addition of the word "irrational." This is done to eliminate possible confusion with the more general meaning of "cut," above. "Irrational" is also an apt adjective, since the type of any irrational real over the rationals is an irrational cut, while the type of an infinitesimal or infinite element is a noncut. Also note that in any T expanding a divisible ordered abelian group, any model realizing a noncut must realize it infinitely many times. The following theorem is a central point in [15]:

Theorem 0.3.22. *Let $\mathcal{M}$ be o-minimal, $p \in S(\mathcal{M})$ an irrational cut, and $q \in S(\mathcal{M})$ a noncut. Further, let $a, b \in \mathfrak{C}$ such that a realizes p and b realizes q. Then $pr(\mathcal{M} \cup \{a\})$ omits q, and $pr(\mathcal{M} \cup \{b\})$ omits p.*

Colloquially, Theorem 0.3.22 says that realizing an irrational cut does not force us to realize a noncut, and vice versa. Finally, we note an important distinction highlighted in [15]:

Definition 0.3.23. Let T be o-minimal, $\mathcal{M} \models T$, and $p \in S_1(\mathcal{M})$ be an irrational cut. Then p is *uniquely realizable* if there is an elementary extension $\mathcal{M}'$ of $\mathcal{M}$ which realizes p with precisely one element. Equivalently, given $b \in \mathfrak{C}$ which realizes p, p is realized uniquely by b in $pr(\mathcal{M} \cup \{b\})$.

0.3.3 Skolem functions and elimination of imaginaries

Definition 0.3.24. A theory T in a language L has *definable Skolem functions* if for any formula $\varphi(x, \bar{y})$ of L with $\lg(\bar{y}) = n$, there is a 0-definable n-ary function F_φ such that $T \models \forall \bar{y}\, (\exists x \varphi(x, \bar{y}) \to \varphi(F_\varphi(\bar{y}), \bar{y}))$.

Please note that due to the idiosyncracies of the author, we shall unconventionally use $\bar{y}$ as the domain set in our formulas and Skolem functions. Thus, for the remainder of this paper, a *Skolem function for* $\varphi(x, \bar{y})$ shall refer to a function $F_\varphi : M^m \to M$, where m is the length of the tuple $\bar{y}$, such that $\mathcal{M} \models \forall y\, (\exists x \varphi(x, \bar{y}) \to \varphi(F_\varphi(\bar{y}), \bar{y}))$.

Definition 0.3.25. A theory T in language L has *uniform elimination of imaginaries* if for any model $\mathcal{M} \models T$ and any 0-definable equivalence relation E on $\mathcal{M}^n$, there is a 0-definable function $f_E : \mathcal{M}^n \to \mathcal{M}$, such that for any $\bar{a}$, $\bar{b} \in \mathcal{M}^n$, $\mathcal{M} \models E(\bar{a}, \bar{b})$ if and only if $f_E(\bar{a}) = f_E(\bar{b})$.

It is routine to show that an o-minimal expansion of an ordered group has definable Skolem functions and uniform elimination of imaginaries. We repeat the argument used by van den Dries in [10], in order to build on it later in the paper.

First, we define a function δ_n from the definable subsets of $\mathcal{M}^n$ to $\mathcal{M}$. Let $(\mathcal{M}, +, <, \ldots)$ be o-minimal, and $X \subseteq M$ be nonempty and definable. Since $\mathcal{M}$ is o-minimal, then X is a finite union of intervals. Let Y be the leftmost interval of X. If Y has a least element $a \in M$, let $\delta_1(X) = a$. If not, then let Y' be the interior of Y (note $Y' = Y$ except in the case where Y contains its supremum). Write $Y' = (a, b)$ for $a, b \in M \cup \{\pm\infty\}$. Then define:

$$\delta_1(Y') = \begin{cases} \frac{a+b}{2} & a, b \in M \\ a+1 & a \in M,\, b = +\infty \\ b-1 & b \in M,\, a = -\infty \\ 0 & a = -\infty,\, b = +\infty \end{cases}$$

Inductively, let $X \subseteq \mathcal{M}^n$ be definable and nonempty with $n > 1$, and $\Pi : M^n \to M^{n-1}$ be projection onto the first $n-1$ coordinates. Define:

$$\delta_n(X) := (\delta_{n-1}(\Pi X), \delta_{n-1}(X_{\delta_{n-1}(\Pi X)}))$$

It is routine to check that for each n, the function δ is 0-definable.

Proposition 0.3.26. *Let $(\mathcal{M}, +, <, \ldots)$ be o-minimal. Then:*

(i) $\mathcal{M}$ has definable Skolem functions.

(ii) $\mathcal{M}$ has uniform elimination of imaginaries.

Proof. For (i), consider a formula $\varphi(x, \bar{y})$ of $\mathfrak{L}$ with $lg(\bar{y}) = n$. We would like a definable function $F_\varphi : \mathcal{M}^n \to \mathcal{M}$ such that $\mathcal{M} \models \forall \bar{y} \left(\exists x \varphi(x, \bar{y}) \to \varphi\left(F_\varphi(\bar{y}), \bar{y}\right)\right)$.

Define:

$$F_\varphi(\bar{y}) = \begin{cases} \delta_n(\{x \in \mathcal{M} : \mathcal{M} \models \varphi(x, \bar{y})\}) & \text{if } \mathcal{M} \models \exists x \varphi(x, \bar{y}) \\ 0 & \text{otherwise} \end{cases}$$

For (ii), let E be a 0-definable equivalence relation on $\mathcal{M}^n$, and define $X = \{(\bar{a}, \bar{b}) \in \mathcal{M}^n \times \mathcal{M}^n : E(\bar{a}, \bar{b})\}$. Then consider $f := \delta_{2n} \restriction (\{(\bar{x}, \bar{y}) : E(\bar{x}, \bar{y})\})$. Then for every $\bar{b} \in \mathcal{M}^n$, we have $f(\bar{x}, \bar{b}) = f(\bar{y}, \bar{b})$ if and only if $E(\bar{x}, \bar{y})$. $\square$

In the context of the real numbers, conditions (i) and (ii) are also known as the principle of *definable choice.*

Given the definability of Skolem functions, we are now in a position to give the proof hinted at in the statement of Fact 0.3.18:

Proof of Fact 0.3.18. We would like to show that given an o-minimal T and a set A, there is a prime model $pr(A)$ of T containing A. It suffices to show that the isolated 1-types are dense in $S_1(A)$. To show this, since the collection of sets $[\varphi(x)]$ forms a basis, it suffices to show that:

1. For every consistent $\varphi(x)$ with parameters from A, $[\varphi(x)]$ contains an isolated type.

2. For any $\varphi(x)$ and $\psi(x)$, if $[\varphi(x)] \cap [\psi(x)]$ is nonempty, then it contains an isolated type.

For (2), note that $[\varphi(x)] \cap [\psi(x)] = [(\varphi \wedge \psi)(x)]$, so it suffices to show (1). Fix a formula $\varphi(x)$ with parameters from A. If φ isolates a type over A, there is nothing to prove. Otherwise, there is an $\mathfrak{L}(\emptyset)$-formula $\psi(x, \bar{y})$ such that $\varphi(x) = \psi(x, \bar{a})$.

By definability of Skolem functions, let F be an n-ary definable function such that $T \models \forall \bar{y}(\exists x(\psi(x, \bar{y})) \rightarrow \psi(F(\bar{y}), \bar{y}))$. Then the formula $x = F(\bar{a})$ isolates a type containing $\varphi(x)$, namely $tp_{\mathfrak{C}}(F(\bar{a})/A)$. $\square$

In fact, more is true. Using definable Skolem functions, we can show that the universe of $pr(A)$ is precisely $cl_{\mathfrak{C}}(A)$. To see this, first it is clear that $cl_{\mathfrak{C}}(A)$ is a submodel of any model of T containing A. To show that $cl_{\mathfrak{C}}(A)$ is elementary in any such model, it suffices to check the Tarski-Vaught criterion: let $M = cl_{\mathfrak{C}}(A) \subseteq \mathcal{N}$; let $\varphi(x, \bar{y})$ be an $\mathfrak{L}$-formula without parameters, and let $\bar{b} \in \mathcal{M}^n$ such that $\mathcal{N} \models \exists x \varphi(x, \bar{b})$. Then there is a Skolem function $F_\varphi : \mathcal{N}^n \rightarrow \mathcal{N}$ such that $\mathcal{N} \models \varphi(F_\varphi(\bar{b}), \bar{b})$. But F_φ is definable (without parameters), so $F_\varphi(\bar{b}) \in cl_{\mathfrak{C}}(\bar{b}) \subseteq cl_{\mathfrak{C}}(A) = \mathcal{M}$.

0.4 Weak o-minimality

The study of weakly o-minimal structures arose with the work of Cherlin-Dickmann in [5] on real closed fields with convex valuation rings. Again, the key property that led to understanding the definable subsets is quantifier elimination. The characterization of weakly o-minimal structures, including most of the definitions that appear in this paper, and many of the results that are mentioned here without proof come from [1], [2], [3], and [14].

Definition 0.4.1. For a first-order language $\mathfrak{L}$ containing $<$, an $\mathfrak{L}$-structure $(\mathcal{M}, <, \ldots)$ is called *weakly o-minimal* if every definable subset of $\mathcal{M}$ (with parameters from M allowed) is a finite union of convex sets.

The convex sets allowed in Definition 0.4.1 include intervals with endpoints in $\mathcal{M}$, as well as convex sets whose supremum and infimum are not contained in the universe. An immediate fact is that any o-minimal structure is therefore weakly o-minimal. As such, we say $\mathcal{M}$ is *properly weakly o-minimal* to mean that $\mathcal{M}$ is weakly o-minimal and not o-minimal. Similarly, a set $U \subseteq \mathcal{M}$ is called *properly convex* if it is convex but not an interval in $\mathcal{M}$.

Remark 0.4.2. Note that any convex subset U of a Dedekind complete structure $\mathcal{M}$ has endpoints in $\mathcal{M} \cup \{\pm\infty\}$; thus any weakly o-minimal structure with a Dedekind complete universe is o-minimal. In particular, this means that there is no properly weakly o-minimal structure with universe $\mathbb{R}$.

Thematically, the genesis of our research is the comparison between o-minimal structures and weakly o-minimal structures. In general, the class of weakly o-minimal structures is not as tractable as the class of o-minimal structures:

1. Weakly o-minimal structures do not, in general, satisfy monotonicity or cellular decomposition.
2. A weakly o-minimal structure may fail to satisfy exchange.
3. There are structures $\mathcal{M} \preceq \mathcal{N}$ such that $\mathcal{M}$ is weakly o-minimal and $\mathcal{N}$ is not.
4. A weakly o-minimal expansion of a group may fail to have definable Skolem functions.
5. A weakly o-minimal expansion of a group may fail to have uniform elimination of imaginaries.

For some of the above, there are known compromises and fixes. The purpose of this paper is to examine each of these areas and improve upon the existing results. We summarize here the result of our efforts (the enumeration corresponds to the above, and not necessarily to the chapter structure of the paper):

1. There is an analogous monotonicity and cellular decomposition theorem for weakly o-minimal structures, but it is significantly weaker. We examine weakly o-minimal ordered groups and give a criterion for monotonicity.

2. A criterion is given in [14] for exchange in weakly o-minimal expansions of rings. We partially extend the result to expansions of groups.

3. It is shown in [14] that weakly o-minimal rings with the property of being *nonvaluational* have weakly o-minimal theories. Our monotonicity results are headed toward showing this fact for groups, which was proved independently by R. Wencel in [20]. We exhibit our progress toward that end, but reference R. Wencel for the remainder of the work.

4. The bulk of the work in the paper is dedicated to finding criteria for Skolem functions. It is immediately clear that the technique used in Proposition 0.3.26 will not work for bounded convex sets without endpoints. In fact we show that nonvaluational weakly o-minimal structures obtained by adding a convex predicate to an o-minimal group do not have definable Skolem functions. Conversely, it is shown nonconstructively in [9] that expansions of rings which are T-*convex* do have Skolem functions. We show that certain expansions of

groups which we call T-*immune* do have Skolem functions, and give a constructive proof.

5. We find, as a corollary to the existence of definable Skolem functions, a negative elimination of imaginaries result.

Chapter 1

Monotonicity

The work in this chapter was inspired by work from [14] on weakly o-minimal ordered fields. We extend some of these results to expansions of weakly o-minimal ordered groups. Many of the results in the remainder of the chapter were proved independently and in increased generality by R. Wencel in [20]. Unless otherwise marked, the proofs which appear in this section are ours. Results which are stated without proof may be assumed to have been proven definitively in [20] or in [14].

1.1 Classical results

The most general analogue of monotonicity for weakly o-minimal structures is the following, which is first proved for weakly o-minimal structures in [14], and later by R. Arefiev for all weakly o-minimal structures, in [1]:

Theorem 1.1.1. Let $(\mathcal{M}, <, \ldots)$ be weakly o-minimal and $f : M \to M$ a definable function. Then there exists a finite collection of convex sets $\{U_i : i \leq N\}$ such that:

(i) $\bigcup_{i \leq N} U_i = \mathrm{dom}(f)$

(ii) For each $i \leq N$, $f \restriction U_i$ is locally strictly monotone.

Note that the insertion of 'locally' allows for the existence of certain pathological examples. The below example is described in full detail in [14]:

Example 1.1.2. Define $\mathcal{M} = (M, L, R, <, f)$ as follows: M is a copy of $\mathbb{Q}+(\mathbb{Q}\times\mathbb{Q})$ (where $(\mathbb{Q} \times \mathbb{Q})$ is ordered lexicographically); $L^{\mathcal{M}}$ is the copy of $\mathbb{Q}$ on the left, and $R^{\mathcal{M}}$ is $\mathcal{M} \setminus L^{\mathcal{M}}$. Define $f^{\mathcal{M}} : R \to L$ such that $f(n, m) = n$. Then $\mathcal{M}$ with f interpreted as above is weakly o-minimal. But while f is clearly locally constant, in fact $f \restriction \mathbb{Q} \times \{m\}$ is constant for each m, it is also easily seen that f is *not* a constant function, and no finite partition of the domain will yield the analogous monotonicity.

Note that in this example, since f is the only function in the language, there is no way of defining other functions using f; in particular, there is no function from $L^{\mathcal{M}}$ to $R^{\mathcal{M}}$ at all. To circumvent this issue, we consider weakly o-minimal models which contain a group operation and whose theories contain the axioms ordered groups. Note that among the many nice properties enjoyed by o-minimal groups, such structures have no definable nontrivial proper subgroups. This is used to prove that any o-minimal group must be particularly nice, namely divisible abelian (which we abbreviate *DOAG*, for 'divisible ordered abelian group'). The situation is not quite as good in the case of a weakly o-minimal structure, in that there may be definable subgroups, but these must be convex, which still turns out to be enough to prove that any *weakly* o-minimal ordered group be a model of *DOAG*. Lemma 1.1.5 is proved in [14], but we present a slightly more general proof here which uses a different technique and may be of interest. Recall from Definition 0.3.1 that we assume all structures to be densely ordered.

Lemma 1.1.3. *Let $\mathcal{G} = (G, <, +, 0, \ldots)$ be a weakly o-minimal ordered group, and*

$H \subseteq G$ a definable subgroup of G. Then H is convex in G.

Proof. Given $\mathcal{G}$ and H as above, suppose H is not convex. Then there are $0 < a < b < c$ with $a, c \in H$ and $b \notin H$. By the group properties, for any $n \in \omega$, $na + b \notin H$ and $na + c \in H$, while $b < c < a + b < a + c < \cdots < na + b < na + C$. Thus H is not a finite union of convex sets, contradicting weak o-minimality. $\square$

For divisibility of $\mathcal{G}$, fix a positive integer n, and let $G' = \{ng : g \in G\}$, a definable subgroup. Then G' is convex; thus if $G' \neq G$, there is $g > G'$, in which case $ng > G'$ by the ordered group properties, contradicting $ng \in G'$ by definition of G'.

For the abelian property, we use the following more general lemma:

Lemma 1.1.4. *Let $(\mathcal{G}, <, +, 0, \ldots)$ be a weakly o-minimal ordered group, and let $U(x, y)$ be a definable relation which is reflexive and symmetric on G, such that for each $g \in G$, $U(\mathcal{G}, g)$ is a subgroup of G. Then $U^{\mathcal{G}} = G \times G$.*

Proof. Fix $g \in G$, and let $X_g = U(\mathcal{G}, g)$. X_g is a subgroup of G, therefore convex. Suppose $X_g \neq G$. Then there is $g' > X_g$ (thus $0 < g < g'$). But $X_{g'}$ is also a convex subgroup, so in particular contains 0. Thus by convexity, $g \in X_{g'}$, in which case, $\mathcal{G} \models U(g, g')$, and so $g' \in X_g$. $\square$

Lemma 1.1.5. *Let $\mathcal{M}$ be a weakly o-minimal expansion of an ordered group. Then $\mathcal{M} \models DOAG$.*

Note that the relation $U(x, y) \leftrightarrow x + y = y + x$ is reflexive and symmetric, yielding the result immediately.

1.2 Finitary monotonicity and strong monotonicity

1.2.1 A naïve notion of monotonicity

We now turn our attention to the study of weakly o-minimal models of $DOAG$. In keeping with the pedagogy of o-minimal structures, we would like to find a more powerful monotonicity that will give rise to an analogue for cellular decomposition. A naïve attempt is the following:

Definition 1.2.1. Let $(\mathcal{M}, <, \ldots)$ be a weakly o-minimal structure. Then $\mathcal{M}$ has the *finitary monotonicity property (FMP)* if for any definable $f : \mathcal{M} \to \mathcal{M}$, there exists a finite partition of $\mathrm{dom}(f)$ into convex sets $\{U_i : i \leq n\}$ such that, for each $i \leq n$, $f \restriction_{U_i}$ is strictly monotone (*i.e.* strictly increasing, strictly decreasing, or constant), and continuous.

The question of when a structure has the FMP is interesting in its own right, and we investigate it later on in this chapter. However, the FMP is not quite the right notion. To see this, suppose $\mathcal{M} = (\mathbb{Q}, +, P, <, 0, 1)$, with $P^{\mathcal{M}} = \{q \in \mathbb{Q} : -\pi < q < \pi\}$. We would like to follow the work done in the o-minimal case, and use the monotonicity theorem as a stepping stone to building a full cellular decomposition. But a simple definable subset of $\mathcal{M}^2$ is the open box $X = I \times P^{\mathcal{M}}$ where I is the interval $(0, 1)$. X is definable as $\{(q, r) \in \mathbb{Q}^2 : 0 < q < 1 \wedge P(r)\}$. Though X is a convex open box, it cannot be written as the area between two boundary functions with image in $\mathcal{M}$. It is easy to see that any properly weakly o-minimal structure will have a definable set with this property.

Remark 1.2.2. The above example will be used often in the paper, and is a paradigm of a *nonvaluational* weakly o-minimal structure (defined below) with a weakly o-minimal theory. Weak o-minimality is a consequence of the general theorem in [2]; it can also be shown more directly by quantifier elimination of $(\mathbb{Q}, +, -, P, <, 0, 1, \{\lambda_q\}_{q\in\mathbb{Q}})$, where λ_q is interpreted as multiplication by q. The quantifier elimination is exhibited in [6].

1.2.2 Functions to the Dedekind completion

To reach the right notion of monotonicity, we follow §1.2 of [14] and consider functions $\mathcal{M} \to \overline{\mathcal{M}}$, where $\overline{\mathcal{M}}$ is the Dedekind completion of $\mathcal{M}$, in the following sense:

Let $Y \subseteq \mathcal{M}^{n+1}$ be 0-definable, and let $\Pi : \mathcal{M}^{n+1} \to \mathcal{M}^n$ be projection onto the first n coordinates, and $Z := \Pi(Y)$. For each $\bar{a} \in Z$, define $Y_{\bar{a}} := \{y : (\bar{a}, y) \in Y\}$. Suppose for all $\bar{a} \in Z$, the set $Y_{\bar{a}}$ is properly convex; we may assume $Y_{\bar{a}}$ is bounded above, but has no supremum in M. Define an equivalence relation E as follows:

$$
\begin{aligned}
E(\bar{a}, \bar{b}) \leftrightarrow \quad & (\bar{a}, \bar{b} \in M^n \setminus Z), \\
& \text{or } \sup Y_{\bar{a}} = \sup Y_{\bar{b}}, \text{ for all } \bar{a}, \bar{b} \in Z
\end{aligned}
$$

Let $\overline{Z} := Z/E$, and for each $\bar{a} \in Z$, denote the E-class of $\bar{a}$ by $\|\bar{a}\|$. There is a natural 0-definable total order on $M \cup \overline{Z}$ defined as follows:

Let $\bar{a} \in Z$ and $c \in M$. Then $\|\bar{a}\| < c$ if and only if $w < c$ for all $w \in Y_{\bar{a}}$. If $\neg E(\bar{a}, \bar{b})$, then there is some $x \in M$ such that $\|\bar{a}\| < x < \|\bar{b}\|$ or $\|\bar{b}\| < x < \|\bar{a}\|$, so $<$ is total on $M \cup \overline{Z}$. The authors of [14] call such a set $\overline{Z}$ a *sort* in $\overline{\mathcal{M}}$, and can be seen as naturally embedded in $\overline{\mathcal{M}}$. Note that a sort depends strongly on the set Y

itself. In this sense, we may talk about a definable function $F : \mathcal{M}^n \to \overline{\mathcal{M}}$, when we really mean a definable function $\mathcal{M}^n \to \overline{Z}$, where $\overline{Z}$ is a sort which depends on the function F.

As an example, we consider $\mathcal{M}$ as in the examples above. Then over $\overline{\mathcal{M}}$, the function $F(x) = y \leftrightarrow \forall z_1 \forall z_2 (z_1 < y - x < z_2 \leftrightarrow P(z_1) \wedge \neg P(z_2) \wedge z_2 > 0$ is an $\mathcal{M}$-definable function $\mathcal{M} \to \overline{\mathcal{M}}$ which has the same graph as the $\overline{\mathcal{M}}$-definable function $y = x + \pi$, where π is understood to be the name for the equivalence class of elements realizing the cut defined by P.

Definition 1.2.3. Let $(\mathcal{M}, <, \ldots)$ be a weakly o-minimal structure. Then $\mathcal{M}$ has the *strong monotonicity property (SMP)* iff for any definable $f : \mathcal{M} \to \overline{\mathcal{M}}$, there exists a finite partition of $\mathrm{dom}(f)$ into convex sets $\{U_i : i \leq n\}$ such that, for each $i \leq n$, $f \upharpoonright U_i$ is strictly monotone and continuous.

Note that in any $\mathfrak{L}$-structure $\mathcal{M}$, a witness to the failure of the FMP will also witness the failure of the SMP; thus if $\mathcal{M}$ has the SMP, then $\mathcal{M}$ has the FMP. The monotonicity properties are closely connected with the additive group structure.

Definition 1.2.4. Given an ordered group $(\mathcal{M}, +, <, \ldots)$, a cut $\langle C, D \rangle$ on $\mathcal{M}$ is called is *valuational* if there is an element $\varepsilon > 0$ in M such that $C + \varepsilon = C$. An ordered group $(\mathcal{M}, +, <, \ldots)$ is *valuational* if there is a definable valuational cut $\langle C, D \rangle$ on $\mathcal{M}$. Otherwise, $(\mathcal{M}, +, <, \ldots)$ is *nonvaluational.*

For our purposes, the definability of a valuational cut is necessary.

Definition 1.2.5. An ordered group $(\mathcal{M}, +, <, \ldots)$ is archimedean if for any $a \in M$ and any $b > 0$ with $b \in M$, there is $n \in \omega$ such that $nb > a$.

It is clear that a valuational group is nonarchimedean, but the converse is not true in general: consider a nonarchimedean group $(\mathcal{M}, +, <)$ with no additional structure, and let $0 < a < b$ be in M such that $na < b$ for all $n \in \omega$. Then while there is a valuational cut of $\mathcal{M}$, namely $\langle C, D\rangle$, where C is the downward closure of $\{na : n \in \omega\}$ and $D = M \setminus C$, there is no way to define it in $\mathfrak{L}$ without any additional structure.

Lemma 1.2.6. *Let $(\mathcal{M}, +, <, \ldots)$ be weakly o-minimal. The following are equivalent:*

(i) $\mathcal{M}$ is valuational.

(ii) $\mathcal{M}$ has a nontrivial proper definable convex subgroup.

Proof. (i)⇒(ii) Let $\langle C, D\rangle$ be a definable valuational cut on $\mathcal{M}$. Let $\varphi(x) = \forall y(C(y) \to C(x + y))$, and let $H = \varphi^{\mathcal{M}}$. Then H is nontrivial by the definition of valuational. To see that H is a group: clearly $0 \in H$. And if $\delta, \varepsilon \in H$ with $\delta, \varepsilon > 0$, and $c \in C$, then $\delta \in H$ gives $c + \delta \in C$; similarly $\varepsilon \in H$ gives $c + \delta + \varepsilon \in C$, so $\delta + \varepsilon \in H$. To show H is proper, note that C and D are nonempty, and let $c \in C$, $d \in D$. Then $d - c \notin H$. And since H is a group, it is convex by Lemma 1.1.3.

(ii)⇒(i) Let $H < \mathcal{M}$ be a proper definable subgroup. Define $C(x) \leftrightarrow H(x) \vee x < H$, and $D(x) \leftrightarrow \neg C(x)$. Then $\langle C, D\rangle$ is a definable valuational cut of $\mathcal{M}$. □

Remark 1.2.7. In addition, note that if $\mathcal{M}$ has a proper definable subgroup H, there is a definable equivalence relation E on $\mathcal{M}$ with infinitely many infinite convex equivalence classes: define $E(x,y) \leftrightarrow H(y-x)$. H contains 0 and is abelian, associative, proper, and convex, which gives that E is reflexive, resp. symmetric, transitive, convex, and contains infinitely many classes.

Proposition 1.2.8. *Let $(\mathcal{M},+,<,\ldots)$ be nonvaluational, such that $Th(\mathcal{M})$ is weakly o-minimal. Then $\mathcal{M}$ satisfies the FMP.*

Proof. Suppose the proposition fails, witnessed by a definable function $f : \mathcal{M} \to \mathcal{M}$, such that there is no finite partition of $\mathrm{dom}(f)$ into convex sets on which f is strictly monotone. By Theorem 1.1.1, there is a partition of $\mathrm{dom}(f)$ into finitely many convex sets on which f is locally strictly monotone. Since the FMP fails, there is a convex set U such that $f \restriction U$ is *locally* strictly monotone but not strictly monotone on all of U. Without loss of generality, suppose f is locally strictly decreasing but not strictly decreasing on all of U. Without loss of generality, assume that $\mathrm{dom}(f) = U$.

Define an equivalence relation on $U^{\mathcal{M}}$ as follows:

$$E(x,y) \leftrightarrow x < y \wedge \forall z_1 \forall z_2 (x < z_1 < z_2 < y \rightarrow f(x) > f(z_1) > f(z_2) > f(y)) \vee$$

$$x > y \wedge \forall z_1 \forall z_2 (x > z_1 > z_2 > y \rightarrow f(x) < f(z_1) < f(z_2) < f(y)) \vee$$

$$x = y$$

Then the E-classes are each maximal convex sets on which f is strictly decreasing. For any $x \in M$, define $U_x := E(x,\mathcal{M})$, the E-class of x. By throwing away finitely many points, we may assume that each U_x is nondegenerate. Failure of the FMP for f implies there are infinitely many U_x's.

Now let:

$$\varphi(x,\varepsilon) := \forall y \forall z\, ((x-\varepsilon < y < x < z < x+\varepsilon) \rightarrow (y \in U_x \wedge z \in U_x))$$

Note that $\varphi(x,\varepsilon)$ holds precisely when the ε-ball surrounding x is contained in U_x. Since $\mathcal{M}$ is nonvaluational, we get that for $\varepsilon > 0$ 'small enough', the set $\varphi(\mathcal{M},\varepsilon)$ contains a union of disjoint convex sets with a measurable distance between them. Thus, given any ε, the set of all classes U_x with $U_x \cap \varphi(x,\varepsilon) \neq \emptyset$ is finite.

Since there are infinitely many sets U_x, each nondegenerate and convex, we must have:

$$\text{For all } \varepsilon > 0 \text{, every } a \in U \text{, there is } 0 < \varepsilon' \leq \varepsilon \text{ such that } \mathcal{M} \models \varphi(a,\varepsilon'). \tag{1.1}$$

Finally, define:

$$\psi_n(x,y) := \varphi(x,y) \wedge (\exists x_0 < y_0 < \cdots < x_{n-1} < y_{n-1} < x_n) \left(\bigwedge_i \varphi(x_i,y) \wedge \neg\varphi(y_i,y) \right)$$

By weak o-minimality of $Th(\mathcal{M})$, we know that the type $\Psi(x,y) = \{\psi_n : n \in \omega\}$ is not realized on any model of $Th(\mathcal{M})$, thus by compactness there is $N \in \omega$ such that $\mathcal{M} \models \forall x \forall y \neg\psi_N(x,y)$, contradicting 1.1 above.

The cases for locally increasing and locally constant can be handled analogously. $\square$

Note that this proof required us to assume $Th(\mathcal{M})$ to be weakly o-minimal, and essentially uses uniform finiteness. R. Wencel proves in [20] the stronger fact that weakly o-minimal expansions of ordered groups in fact have the SMP, and this in turn leads to a proof of a strong cellular decomposition and the following results, which we shall refer to later:

Lemma 1.2.9 (Wencel)**.** *Let* $(\mathcal{M}, +, <, \dots)$ *be weakly o-minimal. Then the following are equivalent:*

(i) $\mathcal{M}$ *has the* SMP*.*

(ii) There is no definable equivalence relation on $\mathcal{M}$ *with infinitely many infinite convex classes.*

Theorem 1.2.10 (Wencel)**.** *If* $(\mathcal{M}, +, <, \dots)$ *is weakly o-minimal and nonvaluational, then* $\mathcal{M}$ *has the* SMP*.*

Theorem 1.2.11 (Wencel)**.** *If* $(\mathcal{M}, +, <, \dots)$ *is weakly o-minimal and nonvaluational, then* $Th(\mathcal{M})$ *is weakly o-minimal.*

Wencel also introduces the notion of a *canonical o-minimal extension* for nonvaluational structures, which, simply stated, says that a weakly o-minimal nonvaluational structure is essentially an o-minimal structure with a certain collection of points removed. These results echo the sentiment expressed in [14] and [20], that a nonvaluational weakly o-minimal expansion of an ordered group, is "very close" to an o-minimal structure.

1.3 Monotonicity in valuational structures

We turn to valuational structures. As pointed out earlier, there is no hope for a valuational structure to satisfy the Strong Monotonicity Property, since by Remark 1.2.7, a valuational structure will have a convex equivalence relation, and thus by 1.2.9, will fail to have the SMP.

1.3.1 Adding Skolem functions and imaginaries

The original intent of the material in this section is to establish a minimum set of conditions for a structure (possibly valuational) to satisfy the finitary monotonicity property. It turns out that Skolem functions and elimination of imaginaries are indeed sufficient to guarantee that a weakly o-minimal theory satisfies the FMP; however as we shall see, there cannot be a valuational structure with Skolem functions and elimination of imaginaries. We use the following lemma:

Lemma 1.3.1. *Let $\mathcal{M}$ be a model of a weakly o-minimal theory T which has definable Skolem functions and uniform elimination of imaginaries. Then there is no equivalence relation E definable on $\mathcal{M}$ with infinitely many convex equivalence classes of nonzero length.*

Proof. Let E be a 0-definable equivalence relation on M such that E has infinitely many convex equivalence classes, each containing an open set. Thus by uniform elimination of imaginaries, there is a definable $g : \mathcal{M} \longrightarrow \mathcal{M}$ such that each E-class is mapped to a single and unique element. Then since there are infinitely many classes, the range of g is infinite, hence contains a convex set I (by weak o-minimality).

Now let $\varphi(x, y) \leftrightarrow g(x) = y$ ("y is the name for the E-class containing x"). Then since T has definable Skolem functions, there is a function $h = h_{\varphi(x,y)} : \mathcal{M} \longrightarrow \mathcal{M}$, such that if $\exists x\varphi(x, y)$ ("y is in the range of g"), then $\varphi(h(y), y)$ ("y is in the class named by $g(y)$").

Since the elements of I are imaginary elements for distinct classes, $h \restriction I$ is one-to-one; in particular, $h(I)$ is infinite. But each element of $h(I)$ is an element of a distinct class, hence the $h(I)$ is discrete as well, which contradicts weak o-minimality. $\square$

Corollary 1.3.2. *Let T be a weakly o-minimal theory with uniform elimination of imaginaries and definable Skolem functions, and $\mathcal{M} \models T$. Then $\mathcal{M}$ is nonvaluational.*

Proof. Suppose $\mathcal{M}$ has a definable valuational cut $\langle C, D \rangle$, and let $H \leq M$ be the definable subgroup of $\mathcal{M}$ given by $\{x \in M : x + C = C\}$. Define the equivalence relation $E_H(x, y) \leftrightarrow y - x \in H$. Then it is clear that E_H has infinitely many convex equivalence classes, a contradiction. $\square$

We shall also see in the next chapter by a different method that certain non-valuational structures fail to have definable Skolem functions.

Corollary 1.3.3. *Let T be a weakly o-minimal theory with uniform elimination of imaginaries and definable Skolem functions, and $\mathcal{M} \models T$. Then $\mathcal{M}$ has the FMP.*

Proof. Let $\mathcal{M}$ and T be as above, and suppose the FMP fails. Then we may assume there is a definable function $f : \mathcal{M} \to \mathcal{M}$ which is locally strictly monotone everywhere on its domain, but for which there is no partition of $dom(f)$ into finitely many definable convex sets such that the restriction of f to each of these is strictly monotone. Assume without loss of generality that f is locally constant.

For any $a \in \mathrm{dom}(f)$, define $U_a(x)$ if and only if there are $c, d \in dom(f)$ with $c < a < d$ satisfying:

$$\forall y_1 \forall y_2 (c < y_1 < y_2 < a \rightarrow f(y_1) = f(y_2)) \wedge$$

$$\forall y_1 \forall y_2 (a < y_1 < y_2 < d \rightarrow f(y_1) = f(y_2)) \wedge$$

$$c < x < d$$

In words, x is inside the maximal convex set containing a on which f is truly constant. Note that $U_a(x)$ is definable over a. Now define:

$$\begin{aligned} E_f(x, y) \leftrightarrow \quad & \exists z \in \mathrm{dom}(f)(x \in U_z \wedge y \in U_z) \vee \\ & \neg \exists z \in \mathrm{dom}(f)(x \in U_z) \wedge \neg \exists z \in \mathrm{dom}(f)(y \in U_z) \end{aligned}$$

Then E_f is an equivalence relation on all of $\mathcal{M}$, definable over $\emptyset$, and each class is $U_a(\mathcal{M})$ for some $a \in \mathcal{M}$. Thus by Lemma 1.3.1 there are only finitely many convex equivalence classes, contradicting the failure of finitary monotonicity for f. $\square$

Chapter 2

Skolem functions in nonvaluational structures

2.1 Introduction

The very property of having definable Skolem functions is of interest, as theories with built-in Skolem functions are model complete. In our case we are especially interested in establishing conditions for Skolem functions, having noted their relationship to monotonicity. As such, Skolem functions are the focus of this chapter and the following. In both cases, we simplify matters by looking at weakly o-minimal structures obtained by adding a convex predicate to an o-minimal structure. We know from [2] that such structures have weakly o-minimal theories as well, thus the results from the previous chapter apply. Given that nonvaluational structures are generally understood to have definable sets that are most like those in an o-minimal structure, it is perhaps surprising, then, that our research shows that nonvaluational expansions of groups fail to have Skolem functions.

The test for Skolem functions by van den Dries (described in [7]) is a natural place to start, but it is apparently not as useful in the case of nonvaluational structures. The van den Dries test requires an examination of substructures, but a basic example shows that this is not a tractable process in the nonvaluational case. Note that the direct translation of the definition of 'nonvaluational' is an $\exists\forall$-sentence: Considered in $\mathcal{M}$, the cut $\langle C, D\rangle$ is nonvaluational if

$\mathcal{M} \models (\exists x > 0)\,(\forall y(C(y) \to C(y+x)))$. A universally axiomatizable theory T has the property that if $\mathcal{M} \models T$ and $\mathcal{N} \subseteq \mathcal{M}$ is a substructure of $\mathcal{M}$, then $\mathcal{M} \models T$ as well. The following example demonstrates the complications:

Example 2.1.1. Let $\mathcal{M} = (\mathbb{Q}^3, +, <, U)$, where $<^{\mathcal{M}}$ is lexicographic, $+^{\mathcal{M}}$ is componentwise addition in $\mathbb{Q}$, and $U^{\mathcal{M}} = \{\bar{x} \in \mathbb{Q}^3 : \bar{x} < (1,1,\pi)\} = \{(n,p,q) : n \leq 1,\, p \leq 1,\, q < \pi\}$. It is easy to see that $\mathcal{M}$ is weakly o-minimal, since $(\mathbb{Q}^3, +, <) \models DOAG$ and is hence o-minimal, and $U^{\mathcal{M}}$ is convex. And $\mathcal{M}$ is valuational since every 'small' element is of the form $(0,0,q)$ for some $q \in \mathbb{Q}$, and $\mathbb{Q}$ is itself archimedean.

But notice that $\mathcal{M}$ has substructures $\mathcal{M}'$ and $\mathcal{M}''$ such that $\mathcal{M}'$ is valuational, and $\mathcal{M}''$ is o-minimal! Let $\mathcal{M}'$ be the substructure generated by $+^{\mathcal{M}}$ with universe $\{(n,p,q) : p = 0\}$. Then $U^{\mathcal{M}'} = \{(n,0,p) : n \leq 1\}$, which is valuational in $\mathcal{M}'$, witnessed by $\varepsilon = (0,0,1)$. And let $\mathcal{M}''$ be the substructure with universe $\{(n,p,q) : q = 0\}$. Then $U^{\mathcal{M}''} = \{(n,p,0) : (n,p,0) \leq (1,1,0)\}$, which is an interval with endpoints in $\mathcal{M}''$. Thus $\mathcal{M}''$ is o-minimal.

As such, our work in these chapters will be based on a direct calculation of Skolem functions.

2.2 A negative answer

We first state the main theorem of this section:

Theorem 2.2.1. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group in the language $\mathfrak{L}$ (note this means $\mathcal{M} \models DOAG$). Let U be a new unary predicate symbol, $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$, and $\mathcal{M}' = (\mathcal{M}, U)$, where $U^{\mathcal{M}'}$ is a downward-closed convex set which* *defines a properly convex nonvaluational cut. Then $\mathcal{M}'$ does not have definable Skolem functions in $\mathfrak{L}'$.*

Note that the hypotheses of the theorem imply that $Th(\mathcal{M}')$ is properly weakly o-minimal.

The proof of Theorem 2.2.1 relies heavily on the analysis by L. van den Dries in [8] on definable functions in dense pairs of o-minimal structures. To that end, we would like to be able to be able to talk about the definable subsets of our weakly o-minimal structure in this context. We shall need to use the fact that o-minimal theories have prime models over sets, as well as the classical results of Marker on irrational cuts and unique realizability cited in §0.3.2.

We next clarify the relationship between uniquely realized cuts and convex sets in weakly o-minimal structures. To that end we establish the following convention:

Definition 2.2.2. Let $(\mathcal{M}, <, \ldots)$ be an ordered structure, and $U \subseteq M$ be a convex set. Then $tp_{\mathfrak{C}}(\sup U/M) := \{x \geq a : a \in U^{\mathcal{M}}\} \cup \{x \leq b : b > U^{\mathcal{M}}\}$.

Note that if U has a supremum $a \in M$, then a realizes $tp_{\mathfrak{C}}(\sup U/M)$. Also if U is definable in $\mathcal{M}$, then $tp_{\mathfrak{C}}(\sup U/M)$ is isolated by a single formula.

Lemma 2.2.3. *Let $\mathcal{M}$ and U be as in the statement of Theorem 2.2.1 above. Let $p = tp_{\mathfrak{C}}(\sup U/M)$. If p is a noncut, then $\mathcal{M}'$ is o-minimal.*

Proof. If p is the type of an element infinitesimally close on the right (resp., left) to an element $a \in M$, then $U^{\mathcal{M}'} = (-\infty, a]$ (resp., $(-\infty, a)$), which is already definable in $\mathcal{M}$, thus U adds no new definable sets. Similarly, if p is the type of an infinite large (resp., small) element, then $U^{\mathcal{M}'} = M$ (resp., $\emptyset$). $\square$

Lemma 2.2.4. *Let $\mathcal{M}$ and U and p be as above, and p an irrational cut.*

The following are equivalent:

(i) For any $b \in \mathfrak{C}$ realizing p, $pr(\mathcal{M} \cup \{b\})$ contains a unique realization of p.

(ii) $(\mathcal{M}, U)$ is nonvaluational. (In particular, U defines a nonvaluational cut on M.)

Proof. (i)⇒(ii) Suppose $\mathcal{M}'$ is valuational. Then let $\varepsilon \in M$ with $0 < \varepsilon$ be such that for any $a \in U^{\mathcal{M}'}$, $a + \varepsilon \in U^{\mathcal{M}'}$. Note that this means for any $d \in M$, if $d > U^{\mathcal{M}'}$, then $d - \varepsilon > U^{\mathcal{M}'}$ (otherwise $(d - \varepsilon) \in U^{\mathcal{M}'}$ and $(d - \varepsilon) + \varepsilon \notin U^{\mathcal{M}'}$).

Claim: $b + \varepsilon \models p$. To see this, suppose not: then there is $d \in M$ such that $d > U^{\mathcal{M}'}$ and $b + \varepsilon > d$, so clearly $b > d - \varepsilon$. But by the above comment, $d - \varepsilon > U^{\mathcal{M}'}$, a contradiction. So p is realized in $pr(\mathcal{M} \cup \{b\})$ by b and $b + \varepsilon$, and in particular b is not the unique realization.

(ii)⇒(i) Suppose $a, b \in \mathfrak{C}$, with $a < b$ and both a and b realizing p in $pr(\mathcal{M} \cup \{b\})$. By Theorem 0.3.22, there is $\varepsilon \in M$ with $0 < \varepsilon < b - a$. Let $c \in U^{\mathcal{M}}$. Then $c + \varepsilon < a + \varepsilon < b$. Since $c, \varepsilon \in M$, then $c + \varepsilon \in M$; and $tp_{\mathcal{M}}(b)$ is generated by $tp_{\mathfrak{C}}(\sup U/M)$ which implies that $c + \varepsilon \in U^{\mathcal{M}'}$. Thus $\mathcal{M}'$ is valuational. □

This equivalence will be later expanded to include a more generalized notion of valuational cut.

Corollary 2.2.5. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and U a predicate for a downward-closed convex set such that $\mathcal{M}' = (\mathcal{M}, U)$ is nonvaluational. Let $p = tp_{\mathfrak{C}}(\sup U/M)$, $b \in \mathfrak{C}$ a realization of p, and $\mathcal{N} = pr(\mathcal{M} \cup \{b\})$. Then $\mathcal{M}$ is dense in $\mathcal{N}$.*

Proof. By Lemma 2.2.4, p is uniquely realized in $\mathcal{N}$ by b. Thus *any* irrational cut over M which is realized in $\mathcal{N}$ is *uniquely* realized in $\mathcal{N}$ (else there are $d < d'$ such that $tp_{\mathfrak{C}}(d/M) = tp_{\mathfrak{C}}(d'/M)$, whence $d + (b - d)$ and $d' + (b - d)$ both realize p). Thus for any $b < b' \in \mathfrak{C}$, b and b' must realize different cuts over M; as such, there is $a \in M$ such that $b < a < b'$. $\square$

This gives us a clear context in which we may use the van den Dries analysis of definable functions in dense pairs. Formally, a *dense pair* is a pair $(\mathcal{N}, \mathcal{M})$ such that $\mathcal{M}$ and $\mathcal{N}$ are o-minimal ordered abelian groups with $\mathcal{M} \preceq \mathcal{N}$, and such that $\mathcal{M}$ is dense in $\mathcal{N}$. We make use of Corollary 3.6 (also stated as Theorem 3, part (3) in [8]):

Theorem 2.2.6. *Let $(\mathcal{N}, \mathcal{M})$ be a dense pair, and let $f : M^n \to M$ be definable in $(\mathcal{N}, \mathcal{M})$. Then there are $f_1, \ldots, f_k : M^n \to M$ definable in $\mathcal{M}$ such that for each $m \in M^n$, we have $f(m) = f_i(m)$ for some $i \in \{1, \ldots, k\}$.*

Working in a weakly o-minimal structure obtained by adding a nonvaluational convex predicate to an o-minimal structure is essentially the same as working in a dense pair:

Lemma 2.2.7. *Let $\mathcal{M}$ be o-minimal with language $\mathfrak{L}$; let $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$, and $\mathcal{M}' = (\mathcal{M}, U)$ with $U^{\mathcal{M}'}$ a downward-closed nonvaluational convex set, and $\mathcal{N} = pr(\mathcal{M} \cup \{b\})$, where b realizes $tp_{\mathfrak{C}}(\sup U/M)$. Then for any $X \subseteq M$ definable in $\mathcal{M}'$, there is an $\mathfrak{L}$-formula $\varphi_X(\bar{x}, y)$ such that $X = \varphi_X(\mathcal{N}^n, b) \cap \mathcal{M}^n$.*

Proof. Given the formula $\psi(x)$ which defines X in $\mathcal{M}'$, replace all instances of $U(x)$ in ψ with $x < y$. $\square$

We now move to the proof of the main result, via a single counterexample which will apply to all structures satisfying the hypotheses of Theorem 2.2.1.

Proposition 2.2.8. *Let $(\mathcal{M}, +, <, \ldots)$ be an o-minimal expansion of an ordered group in the language $\mathfrak{L}$, and $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$ for U a unary predicate. Assume $\mathcal{M}' = (\mathcal{M}, U)$ is a nonvaluational weakly o-minimal structure. Then there is no $\mathcal{M}'$-definable function $f : M \to M$ such that for every $a \in U^{\mathcal{M}'}$, we have $a < f(a)$ and $f(a) \in U$.*

Note that the proposition implies Theorem 2.2.1: If $\varphi(x, y) = U(x) \wedge U(y) \wedge y < x$, then f would be a Skolem function corresponding to $F_{\exists x \varphi(x,y)}$.

Proof of Proposition 2.2.8. Let $f : M \to M$ be such a function, and assume f is $\mathcal{M}'$-definable. Consider $\Gamma(f) \subseteq M^2$, the graph of f on M. Let $b \in \mathfrak{C}$ be a realization of $tp_{\mathfrak{C}}(sup(U)/\mathcal{M}')$, and $\mathcal{N} = pr(\mathcal{M} \cup \{b\})$. Then by Lemma 2.2.7, there are $\varphi(x, y, \bar{z})$ and $\bar{c}$ from N such that $\{(x, y) : \mathcal{N} \models \varphi(x, y, \bar{c})\} \cap M^2 = \Gamma(f)$. Note that the solution set of $\varphi(x, y, \bar{c})$ in $\mathcal{N}$ need not be the graph of a function, but the formula defines $\Gamma(f)$ in the pair $(\mathcal{N}, \mathcal{M})$. Hence, by Theorem 2.2.6, there are $\mathcal{M}$-definable functions $f_1, \ldots, f_k$ such that for each $a \in U$, $f(a) = f_i(a)$ for some $i \leq k$. Each of these functions f_i is also definable in $\mathcal{M}'$, so by weak o-minimality, there is a convex set I, which is unbounded in U, and $i \leq k$ such that $f \upharpoonright I = f_i$. Restricting the domain of f, we may assume $f = f_i$.

The main point of the dense pair argument was to get this function f to be definable *in* $\mathcal{M}$. Thus by monotonicity for o-minimal structures, we may assume I is an open *interval*. Since I is unbounded in $U^{\mathcal{M}'}$ and $tp_{\mathfrak{C}}(\sup U/M)$ is omitted in

$\mathcal{M}$, then I must be (c, d) for some $c, d \in M$, $c \in U^{\mathcal{M}'}$ and $d > U^{\mathcal{M}'}$.

By the hypotheses for f in the statement of the Proposition, we may shrink the domain further to assume f is strictly increasing on I, and for all $a \in U$, $a < f(a)$ and $a \in U$. Let $g(x) := f(x) - x$. Then we may also assume by shrinking that g is strictly decreasing and positive on I. Again by the hypotheses on f, we can say in a certain sense that $\lim_{x \to \sup U} g(x) = 0$:

Since $\mathcal{M}'$ is nonvaluational, for any $\varepsilon > 0$, there is $a \in U^{\mathcal{M}'}$ and $b > U^{\mathcal{M}'}$ such that $b - a \leq \varepsilon$. Then $a < f(a) < b$ implies that $f(a) - a < b - a$, and thus $g(a) = f(a) - a < \varepsilon$.

But since I is an interval with right endpoint $d \in M$, $d > U^{\mathcal{M}'}$, there is $d' \in I \cap U^{\mathcal{M}'}$. Let $\varepsilon_0 = f(d') - d' = g(d')$. Then by the above, there is $a' \in U^{\mathcal{M}'}$ such that $g(a') < \varepsilon_0$. But g was strictly decreasing on I, so we must have $g(d') < \varepsilon_0$, an impossibility. This proves the Proposition, and thus Theorem 2.2.1. $\square$

Corollary 2.2.9. *Let $(\mathcal{M}, +, <, \ldots)$ be o-minimal, U a new convex unary predicate symbol, and $\mathcal{M}' = (M, +, <, U, \ldots)$ be the weakly o-minimal structure obtained by adding U to the language. Suppose $\mathcal{M}'$ has definable Skolem functions and uniform elimination of imaginaries. Then $\mathcal{M}'$ is o-minimal. Equivalently, U is an interval already definable in $\mathcal{M}$.*

Proof. Suppose $U^{\mathcal{M}'}$ is not already definable in $\mathcal{M}$. If U is a new nonvaluational cut, then by the main theorem in this section, $(\mathcal{M}, U)$ does not have definable Skolem functions. If U is valuational, then there is a definable subgroup in $(\mathcal{M}, U)$, thus by Lemma 1.3.1, $(\mathcal{M}, U)$ cannot have both definable Skolem functions and uniform elimination of imaginaries, a contradiction. $\square$

2.3 Pluslike functions

Motivated by the correspondence between nonvaluational weakly o-minimal structures and dense pairs, we introduce the following definitions:

Definition 2.3.1. Let $(\mathcal{M}, <)$ be an ordered group, and $F : M^2 \to M$ a definable function. Given a convex set $W \subseteq M$, F is *pluslike on* W if the following hold:

- F is continuous on $W \times M$;
- For every $a \in W$, $F(a, y)$ is increasing; and
- For every $b \in M$, $F(x, b)$ is increasing.

Definition 2.3.2. Let $(\mathcal{M}, +, <, \ldots)$ be a weakly o-minimal expansion of an ordered group, $\langle C, D\rangle$ a definable cut, and $F : \mathcal{M}^2 \to \mathcal{M}$ a definable function. Then $\langle C, D\rangle$ is F-*valuational* if there is there is a convex set $W \subseteq M$ such that $W \cap C$ and $W \cap D$ are each nonempty, F is pluslike on W, and there is $\varepsilon > 0$ from M such that for all $a \in C$, $F(a, \varepsilon) \in C$. A definable convex set $U \subseteq M$ is F-*valuational* if the cut $\langle U^{\mathcal{M}} \cup \{x \in M : x < U\}, \{x \in M : x > U\}\rangle$ is F-valuational.

Proposition 2.3.3. *Let $(\mathcal{M}, +, <, \ldots)$ be an o-minimal expansion of an ordered group, U a new convex predicate, and $\mathcal{M}' = (\mathcal{M}, U)$. The following are equivalent:*

(i) U is F-valuational for some definable pluslike function F.

(ii) U is F-valuational for all definable pluslike functions F.

(iii) U is valuational.

(iv) $p := tp_{\mathfrak{C}}(\sup U/\mathcal{M})$ is nonuniquely realizable.

Proof. (ii)⇒(i) is trivial, and we previously showed (iii)⇔(iv); thus it suffices to show (i)⇒(iv)⇒(ii).

For (i)⇒(iv), suppose F and ε witness the fact that U is F-valuational. First note that since F is pluslike, it is increasing in both variables; thus for a fixed $a \in \mathcal{M}$, $F(a,x)$ is a bijection, and has a definable continuous inverse F_a^{-1}. Now note that for any $d \in M$, $d > U^{\mathcal{M}}$ implies that $F_d^{-1}(\varepsilon) > U^{\mathcal{M}}$ (or else $d > U^{\mathcal{M}}$ and $F_d^{-1}(\varepsilon) \in U^{\mathcal{M}}$, so $F(F_d^{-1}(\varepsilon), \varepsilon) = d \in U^{\mathcal{M}}$, a contradiction).

Now suppose p is uniquely realized by b in $pr(\mathcal{M} \cup \{b\})$. We claim that $F(b,\varepsilon) \models p$. If not, then there is $d \in M$ such that $d > U^{\mathcal{M}}$, and $F(b,\varepsilon) > d$. Thus, $b > F_d^{-1}(\varepsilon)$. But $F_d^{-1}(\varepsilon) > U^{\mathcal{M}}$, contradicting the statement in the above paragraph. So, $b \models p$ and $F(b,\varepsilon) \models p$. And since F is definable, $F(b,\varepsilon) \in pr(\mathcal{M} \cup \{b\})$. Since F is strictly increasing, $F(b,\varepsilon) \neq b$, contradicting unique realizability of p.

To show (iv)⇒(ii), suppose b realizes p, and $a < b$ such that $a \models p$ and $a \in pr(\mathcal{M} \cup \{b\})$. Let F be pluslike. Let $\varepsilon := F_a^{-1}(b) =$, the unique ε such that $F(a,\varepsilon) = b$.

If $\varepsilon \in M$, then let $c \in U^{\mathcal{M}}$ such that $F(c,\varepsilon) < F(a,\varepsilon) = b$. Then $c \in M$ and $\varepsilon \in M$ implies $F(c,\varepsilon) \in M$, and thus $F(c,\varepsilon) \in U^{\mathcal{M}}$, so U is F-valuational. And if $\varepsilon \notin M$, then since p is an irrational cut, then again by Theorem 0.3.22, there is $\varepsilon_0 \in M$ such that $0 < \varepsilon_0 < \varepsilon$. We may then redo the above argument with ε_0. □

Corollary 2.3.4. *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group in the* *language $\mathfrak{L}$, with a definable pluslike function $F : \mathcal{M}^2 \to \mathcal{M}$. Let U be a new unary predicate symbol, $\mathfrak{L}' = \mathfrak{L} \cup \{U\}$, and $\mathcal{M}' = (\mathcal{M}, U)$, where $U^{\mathcal{M}'}$ is a convex set which defines an F-nonvaluational cut with no supremum in $\mathcal{M}$. Then $\mathcal{M}'$ does not have definable Skolem functions in $\mathfrak{L}'$.*

Proof. It only remains to check that U need not be downward-closed. But if U were not downward closed, we could define U' to be the downward closure of U, and the result follows. $\square$

Chapter 3

Skolem functions in a valuational expansion of a group

3.1 A weakly o-minimal group with a convex subgroup

Theorem 3.1.1. *Fix $\mathfrak{L} = \{+, <, 0, 1, c, U\}$. Let $\mathcal{M} = (M, +, <, 0, 1, c, U)$, where $U^{\mathcal{M}}$ is a nontrivial convex subgroup of M, $1^{\mathcal{M}}$ a positive element of $U^{\mathcal{M}}$, $c^{\mathcal{M}}$ a positive element of $\mathcal{M} \setminus U^{\mathcal{M}}$, and such that $M \restriction \mathfrak{L} \setminus \{c, U\}$ is o-minimal. Then $\mathcal{M}$ admits elimination of quantifiers and has definable Skolem functions in the language $\mathfrak{L}$.*

Proof. The weak o-minimality is a direct consequence of [3]. For Skolem functions, we begin by proving quantifier elimination for $\mathcal{M}$ in the language $\mathfrak{L}$. The quantifier elimination result may be obtained by a few different methods, perhaps more efficiently than is done below; however we use a special formulation which will be very useful in contriving a simple definition for the Skolem functions.

Note that any atomic or negated-atomic formula is equivalent to a formula of one of the following forms:

- $x \Box t(\bar{y})$, for $\Box \in \{=, \neq, <, >, \leq, \geq\}$ and t an L-term;
- $U(ax + t(\bar{y}))$, for $a \in \mathbb{Q}$ and an L-term t; or
- $\neg U(ax + t(\bar{y}))$, for $a \in \mathbb{Q}$ and an L-term t

The following lemma simplifies notation a bit. (The proof is immediate from the fact that $U^{\mathcal{M}}$ is a divisible ordered abelian group.)

Lemma 3.1.2. *For any $a \in \mathbb{Q}$, $\mathcal{M} \models \forall \bar{y} \forall x \big(U(ax + t(\bar{y})) \leftrightarrow U(x + \frac{t(\bar{y})}{a})\big)$*

Using the lemma, we may replace the above instances of U to exclude the coefficient for x.

In the usual method, we would eliminate the quantifier from an arbitrary existential formula by showing that we can eliminate $\exists$ from a primitive formula, of the form $\exists x \bigwedge_i \varphi_i(x, \bar{y})$ for each $\varphi_i(x, \bar{y})$ an atomic or negated-atomic formula. However, we want all of our conjuncts to have convex solution sets. The only two of the forms above which do do not have convex solution sets are $\neg U(ax + t(\bar{b}))$ and $x \neq t(\bar{y})$. We get around these difficulties with the following definitions.

First, for simplicity, we expand the language $\mathfrak{L}$. Let $\mathfrak{L}' = \mathfrak{L} \cup \{L, R\}$ for L and R unary relation symbols. Define $L^{\mathcal{M}}(x) := \{a \in M : \mathcal{M} \models \neg U(a) \wedge a < 0\}$, and $R^{\mathcal{M}}(x) := \{a \in M : \mathcal{M} \models \neg U(a) \wedge a > 0\}$. Note that $\mathcal{M} \models \forall x(\neg(U(x)) \leftrightarrow (L(x) \vee R(x))$, and that L and R are both quantifier-free definable in $\mathfrak{L}$, so the expansion by definitions is harmless for showing quantifier elimination and definable Skolem functions.

Definition 3.1.3. A **$*$-atomic formula** is an $\mathfrak{L}'$-formula of one of the following forms:

- $x \square t(\bar{y})$, for $\square \in \{=, <, >, \leq, \geq\}$ and t an L-term;
- $U(x + t(\bar{y}))$, for an L-term t;

- $L(x + t(\bar{y}))$, for an L-term t; or

- $R(x + t(\bar{y}))$, for an L-term t;

A **$*$-primitive formula** is any finite conjunction of $*$-atomic formulas.

Our definition for $*$-primitive differs from the standard definition of primitive in that we do require the $*$-primitive formula to be quantifier-free. Note that for any fixed parameter from $\mathcal{M}^n$, each of the $*$-atomic formulas is realized by a convex set. We removed "$\neq$" from the list for $\square$, as $x \neq y$ may be replaced by $x < y \wedge x > y$.

Lemma 3.1.4. *Let $\varphi(x, \bar{y})$ be a quantifier free formula in $\mathfrak{L}'$. Then there are $*$-primitive formulas $\psi_0(x, \bar{y}), \ldots, \psi_{n-1}(x, \bar{y})$ such that such that $\mathcal{M} \models \forall \bar{y} \forall x \Big(\varphi(x, \bar{y}) \leftrightarrow \bigvee_{i<n} \psi_i(x, \bar{y}) \Big)$. (We shall later call this the $*$-disjunctive normal form.)*

Proof. First write $\varphi(x, \bar{y})$ in disjunctive normal form and apply the equivalences above, so that φ is a Boolean combination of $*$-atomic formulas, and apply the tautology $\alpha \wedge (\beta \vee \gamma) \leftrightarrow (\alpha \wedge \beta) \vee (\alpha \wedge \gamma)$. $\square$

Thus for quantifier-elimination, it will suffice to show that we can eliminate $\exists$ from $\exists x \varphi(x, \bar{y})$ for any $*$-primitive $\varphi(x, \bar{y})$.

3.1.1 Helly's Theorem

The usefulness of the above machinery becomes clear when we consider the following purely topological lemma, which is essentially Helly's Theorem for $n = 1$, generalized to an arbitrary linear order:

Lemma 3.1.5. *Let $U_0, \ldots, U_{n-1}$ be convex subsets of a linearly ordered set. Then there are $j, k < n$ (possibly $j = k$) such that $\bigcap_{i<n} U_i = U_j \cap U_k$. In particular, $\bigcap_{i<n} U_i$ is nonempty if and only if for each $i, j < n$, $U_j \cap U_k$ is nonempty.*

Proof. For convex sets U and V, define the orderings $\prec_L$ and $\prec_R$ as follows:

- $U \prec_L V$ if and only if $U \subseteq V$ or there is $a \in V$ such that $a < U$.

- $U \prec_R V$ if and only if $U \subseteq V$ or there is $b \in V$ such that $b > U$.

Then $\prec_L$ and $\prec_R$ are each total orderings on the set $\{U_i\}_{i<n}$, and thus there are $j, k < n$ such that U_j and U_k are minimal with respect to $\prec_L$ and $\prec_R$, respectively. And by definition of the orderings, we get $\bigcap_{i<n} U_i = U_j \cap U_k$. □

The general form of the lemma will be used in the definition of our Skolem functions; for QE, we shall only require the second statement.

3.1.2 Quantifier elimination

Using lemma 3.1.4, it suffices to eliminate $\exists$ from an existential $*$-primitive formula. Let $\varphi(x, \bar{y})$ be $*$-primitive, so: $\varphi(x, \bar{y}) = \bigwedge_{i<n} \varphi_i(x, \bar{y})$ for some finite collection of $*$-atomic formulas $\varphi_0(x, \bar{y}), \ldots, \varphi_{n-1}(x, \bar{y})$. By lemma 3.1.5:

$$\mathcal{M} \models \forall \bar{y} \left[\Big(\exists x \bigwedge_{i<n} \varphi_i(x, \bar{y}) \Big) \leftrightarrow \Big(\bigwedge_{j,k<n} \exists x (\varphi_j(x, \bar{y}) \wedge \varphi_k(x, \bar{y})) \Big) \right]$$

Thus, it suffices to eliminate the existential quantifier from any pair of $*$-atomic formulas. We do this by cases below. While it will perhaps be pedantic to do each of the cases in detail, we shall refer back to this section in order to define Skolem functions. For the sequel, $s(\bar{y})$, $t(\bar{y})$, $u(\bar{y})$, $v(\bar{y})$ will refer to $\mathfrak{L}'$-terms. Most of the cases are solved by algebraic calculations in $\mathcal{M}$.

Case 1: $\exists x(x \Box s(\bar{y}) \wedge x \Box t(\bar{y}))$ for $\Box \in \{=, <, >, \leq, \geq\}$

This case is handled by the classic quantifier-elimination result for DOAG.

Case 2: $\mathcal{M} \models \Big[\exists x(U(x + s(\bar{y})) \wedge U(x + t(\bar{y})))\Big] \leftrightarrow U(t(\bar{y}) - s(\bar{y}))$

Case 3: $\mathcal{M} \models \Big[\exists x(U(x + s(\bar{y})) \wedge L(x + t(\bar{y})))\Big] \leftrightarrow L(t(\bar{y}) - s(\bar{y}))$

Case 4: $\mathcal{M} \models \Big[\exists x(U(x + s(\bar{y})) \wedge R(x + t(\bar{y})))\Big] \leftrightarrow R(t(\bar{y}) - s(\bar{y}))$

Case 5: Clearly $L^{\mathcal{M}}$ and $R^{\mathcal{M}}$ are mutually exclusive, hence:

$\mathcal{M} \models \Big[\exists x(L(x + s(\bar{y})) \wedge R(x + t(\bar{y})))\Big] \leftrightarrow \bar{y} \neq \bar{y}$

Case 6: Conversely, $\inf L^{\mathcal{M}} = -\infty$, so $\mathcal{M} \models \Big[\exists x(L(x+s(\bar{y})) \wedge L(x+t(\bar{y})))\Big] \leftrightarrow \bar{y} = \bar{y}$

Case 7: Similarly, $\sup R^{\mathcal{M}} = +\infty$, so $\mathcal{M} \models \Big[\exists x(R(x+s(\bar{y})) \wedge R(x+t(\bar{y})))\Big] \leftrightarrow \bar{y} = \bar{y}$

Case 8: $\exists x(x \Box s(\bar{y}) \wedge U(x + t(\bar{y})))$ We cover the case where $\Box$ is $<$; other cases are similar. Note that for any $\bar{y}$, the solution set for the formula may be written $(-\infty, s(\bar{y})) \cap U^{\mathcal{M}}(x + t(\bar{y}))$, which is nonempty if and only if $s(\bar{y})$ falls inside the set defined by $U^{\mathcal{M}}(x + t(\bar{y}))$, so if and only if $\models U(-t(\bar{y}) - s(\bar{y}))$.

Case 9 - 10: As a representative case, we consider $\exists x(x < s(\bar{y}) \wedge L(x + t(\bar{y})))$. Here the solution set $(-\infty, s(\bar{y})) \cap L^{\mathcal{M}}(x + t(\bar{y}))$ is nonempty if and only if $s(\bar{y})$ falls to the left of the set defined by $U^{\mathcal{M}}(x + t(\bar{y}))$, so if and only if $\models L(-t(\bar{y}) - s(\bar{y}))$.

$\Box$

3.1.3 Skolem functions

In order to establish Skolem functions, we show that it suffices to definably choose an element from each pair of $*$-atomic formulas. Suppose we have an $\mathfrak{L}'$-formula $\varphi(x, \bar{y})$. By the quantifier elimination above, we may assume φ is quantifier-free. Thus we may also assume φ is in $*$-disjunctive normal form, i.e. $\varphi = \bigvee_{i<n} \varphi_i$ for φ_i's each $*$-primitive. Suppose we have defined a Skolem function $F_i = F_{\varphi_i} : \mathcal{M}^m \to M$ for each $i < n$. Then define a function $F : \mathcal{M}^m \to \mathcal{M}$ as follows:

$$F(\bar{a}) := \begin{cases} 0, \text{ if } \mathcal{M} \models \neg\exists x \varphi(x, \bar{a}) \\ \text{least } x_i \text{ s.t. } \mathcal{M} \models \exists x \varphi_i(x, \bar{a}) \text{ and } F_i(\bar{a}) = x_i, \text{ otherwise} \end{cases}$$

It is clear that F is a definable Skolem function for $\varphi(x, \bar{y})$. Thus, we may assume $\varphi(x, \bar{y})$ is $*$-primitive, written as $\varphi(x, \bar{y}) = \bigwedge_{i<n} \varphi_i(x, \bar{y})$, for each of the φ_i's $*$-atomic. As such, we suppose that for each i, j there is a definable 'pairwise' Skolem function $F_{ij} : \mathcal{M}^m \to M$ such that for all $\bar{a} \in M^m$, $\mathcal{M} \models (\exists x(\varphi_i(x, \bar{a}) \wedge \varphi_j(x, \bar{a})) \to \varphi_i(F_{ij}(\bar{a}), \bar{a}) \wedge \varphi_j(F_{ij}(\bar{a}), \bar{a}))$.

Using the stronger form of Lemma 3.1.5, we may define:

$$F(\bar{a}) := \begin{cases} 0 & \text{if } \mathcal{M} \models \neg\exists x \varphi(x, \bar{a}) \\ \text{least } x_{ij} \text{ such that:} & \text{otherwise} \\ \Big(\mathcal{M} \models \forall x \left(\varphi_i(x, \bar{a}) \wedge \varphi_j(x, \bar{a}) \leftrightarrow \varphi(x, \bar{y})\right) & \\ \text{ and } F_{ij}(\bar{a}) = x_{ij}\Big) & \end{cases}$$

Then this will be a definable Skolem function for $\varphi(x, \bar{y})$. By this analysis, it suffices to show the following:

Lemma 3.1.6. *For any two $*$-atomic formulas $\varphi_0(x, \bar{y})$ and $\varphi_1(x, \bar{y})$, there is a* *definable function $F_\varphi : M^m \to M$ such that $\mathcal{M} \models \forall y(\exists x(\varphi_0(x, \bar{y}) \wedge \varphi_1(x, \bar{y})) \to \varphi(F_\varphi(\bar{y}), \bar{y}))$.*

Proof. We return to the case analysis from the proof of quantifier elimination above. As usual, we shall always define $F_\varphi(\bar{a})$ to be 0 if the solution set $\varphi(\mathcal{M}, \bar{a}) = \emptyset$. As such, except where mentioned otherwise, the value calculated in the list below is implicitly the value provided that the formulas have a nonempty intersection for the given parameter. Following conventions from above, $s(\bar{a})$, $t(\bar{a})$ are $\mathfrak{L}'$-terms.

Case 1: $\varphi(x, \bar{a}) = x \square s(\bar{a}) \wedge x \square t(\bar{a})$ for $\square \in \{=, <, >, \leq, \geq\}$

This case is handled by the proof of definable choice in [10].

Case 2: $\varphi(x, \bar{a}) = U(x + s(\bar{a})) \wedge U(x + t(\bar{a}))$

Let $F_\varphi(\bar{a}) = -s(\bar{a})$.

Case 3: $\varphi(x, \bar{a}) = U(x + s(\bar{a})) \wedge L(x + t(\bar{a}))$

Let $F_\varphi(\bar{a}) = -s(\bar{a})$.

Case 4: $\varphi(x, \bar{a}) = U(x + s(\bar{a})) \wedge R(x + t(\bar{a}))$

Let $F_\varphi(\bar{a}) = -s(\bar{a})$.

Case 5: $\varphi(x, \bar{a}) = L(x + s(\bar{a})) \wedge R(x + t(\bar{a}))$

As this intersection is empty, we define $F_\varphi(\bar{a}) = 0$.

Case 6: $\varphi(x, \bar{a}) = L(x + s(\bar{a})) \wedge L(x + t(\bar{a}))$

In order to find an element to the left of both of the convex sets described by the formulas in the conjunction, we shall need to use the constant c. Define $F_\varphi(\bar{a}) = \min\{-|t(\bar{a})|, -|s(\bar{a})|\} - c^{\mathcal{M}}$.

Case 7: $\varphi(x, \bar{a}) = R(x + s(\bar{a})) \wedge R(x + t(\bar{a}))$

Similarly, to find an element to the right of both convex sets, define: $F_\varphi(\bar{a}) = \max\{|t(\bar{a})|, |s(\bar{a})|\} + c^{\mathcal{M}}$.

Case 8: $\varphi(x, \bar{a}) = x \Box s(\bar{a}) \wedge U(x + t(\bar{a}))$

Again, we cover the case where $\Box$ is $<$; other cases are similar: $\varphi(x, \bar{a}) = x < s(\bar{a}) \wedge U(x + t(\bar{a}))$. Note from the quantifier elimination that if $\mathcal{M} \models \exists x(x < s(\bar{a}) \wedge U(x + t(\bar{a})))$, then $\mathcal{M} \models U(t(\bar{a}) + s(\bar{a}))$. Now since $1^{\mathcal{M}}$ is interpreted by an element of U, we define $F_\varphi(\bar{a}) = s(\bar{a}) - 1$.

Case 9 - 10: As a representative case, we consider $x < s(\bar{y}) \wedge L(x + t(\bar{y}))$.

As in Case 8, because $L(\mathcal{M})$ is closed under addition by elements of $U(\mathcal{M})$, we define $F_\varphi(\bar{a}) = s(\bar{a}) - 1$. $\Box$

Corollary 3.1.7. *$\mathcal{M}$ has Skolem functions in the language $\mathfrak{L}$.* $\Box$

This sheds some light on one of the most intuitive examples of a nonvaluational weakly o-minimal structure.

Corollary 3.1.8. *$\mathbb{R}^*$ be a nonstandard extension of the reals realizing the $\mathfrak{L}$-type $\Phi(x) = \{x > n : n \in \omega\}$, and let $\mathcal{M} = \{\mathbb{R}^*, +, <, 0, 1, c, U\}$, where $U^{\mathcal{M}} = \{r \in \mathbb{R}^* : \exists n < m \in \mathbb{Z}$ s.t. $n < r < m\}$, the convex hull of the reals in $\mathbb{R}^*$, and $c^{\mathcal{M}}$ is interpreted by some large nonstandard element $c > 0$, $c \in \mathbb{R}^* \setminus U^M$. Then $\mathcal{M}$ has definable Skolem functions in the language $\mathfrak{L}$.* $\Box$

Chapter 4

T-immunity

4.1 T-convexity

In [11] and [9], the authors define and investigate the property of T-convexity in ordered fields. Given an o-minimal field $\mathcal{M}$ and its complete theory T, a pair $(\mathcal{M}, V)$ is T-convex (we say $(\mathcal{M}, V) \models T_{convex}$) if V is a proper convex subring of $\mathcal{M}$, and is closed under every 0-definable continuous function with domain M. T-convex fields are shown to have some very desirable properties. The main results that concern us are as follows:

Theorem 4.1.1. *(van den Dries - Lewenberg) If T admits quantifier elimination and is universally axiomatizable, then T_{convex} admits quantifier elimination and is complete.*

Theorem 4.1.2. *(van den Dries) Given $(\mathcal{M}, V) \models T_{convex}$ and $c \in M \setminus V^{\mathcal{M}}$, the $\mathfrak{L}(c)$-theory of $(\mathcal{M}, V)$ (namely $T_{convex,c}$) has definable Skolem functions.*

We define the strictly stronger notion of T-immunity, and give an alternate proof that T-immune theories have definable Skolem functions, echoing the construction in the previous chapter via quantifier elimination.

4.2 Definition; Statement of the main theorem

Definition 4.2.1. Let $\mathcal{M}$ be an o-minimal expansion of an ordered group, and $V \subseteq M$ be a convex set. We say that the pair $(\mathcal{M}, V)$ is *T-immune* if $V \subsetneq \mathcal{M}$ and for any 0-definable function $F : M \to M$ and any open convex set $I \subseteq V^{\mathcal{M}}$, if $F \restriction I$ is continuous, then $F(V) \subseteq V$.

For the rest of this section, we assume that $(\mathcal{M}, V)$ is T-immune.

In order to clear up notation, we forego model-theoretic formality, and shall write simply V when referring to $V^{\mathcal{M}} = \{a \in M : \mathcal{M} \models V(m)\}$.

It is easy to see that if $\mathcal{M}$ is a group with no additional structure and V a convex subgroup, then $(\mathcal{M}, V)$ is both T-convex and T-immune. In many cases, T-immunity is a strictly stronger property than T-convexity:

Example 4.2.2. Let $\mathcal{R} = (R, +, \cdot, 0, 1, <) \models RCOF$, and $V \subseteq M$ a proper convex subring of $\mathcal{R}$. Then $(\mathcal{R}, V) \models T_{convex}$ but $(\mathcal{R}, V)$ is not T-immune. T-convexity of this structure is shown in [11]. To see that T-immunity fails, let $b^* > V$, so $\dfrac{1}{b^*} \in V$, and let $0 < \delta < \dfrac{1}{b^*}$. Then it is clear that the function $F(x) = \dfrac{1}{x}$ is continuous on the interval (δ, ∞), but $F\left(\dfrac{1}{b^*}\right) = b^* \notin V$.

The key difference in the above example is that the function $F(x) = \dfrac{1}{x}$ is continuous *on its domain*, namely the nonzero elements, but there is no extension of the function to 0 which preserves continuity.

Theorem 4.2.3. *Let $(\mathcal{M}, +, <, 0, \varepsilon, \ldots)$ be an o-minimal expansion of a group with named positive element ε in a language $\mathfrak{L}$ which admits elimination of quantifiers,* *and $V \subseteq M$ such that (M, V) is T-immune. (Note that since $\varepsilon \in \mathfrak{L}$, then $\varepsilon^{\mathcal{M}} \in V$.) Let c be a new constant symbol and $c^{\mathcal{M}} > 0$ an element of $M \setminus V$. Then (M, V, c) has definable Skolem functions in the language $\mathfrak{L} \cup \{V, c\}$.*

Our proof will take the form of an explicit quantifier elimination. Assume $\mathcal{M}$ is as above, and eliminates quantifiers in the language $\mathfrak{L}$. To show that $(\mathcal{M}, V)$ eliminates quantifiers in $\mathfrak{L} \cup \{V\}$, it will suffice to show that we can eliminate quantifiers from a very special type of formula. The proof relies essentially on Lemmas 4.2.5 and 4.2.6, which we prove before giving the above a technical treatment.

Lemma 4.2.4. *Let $F : \mathcal{M}^n \to \mathcal{M}$ be a 0-definable continuous function with convex domain. Then $F(V^n) \subseteq V$.*

Proof. Fix $\bar{a}$ in the domain of F. Define:

$$F'(x) := \begin{cases} |F(\bar{a})| & \text{if } x < 0 \\ \max\left\{ |F(\bar{y})| : \bigwedge_i |y_i - a_i| \leq x \right\} & \text{if } x \geq 0 \end{cases}$$

The function is well-defined, since any continuous function defined in an o-minimal structure on a closed bounded set attains a maximum value. F' is continuous on its domain, since F is. Finally, V is closed under additive inverses, and thus $|F(V^n)| \subseteq F'(V) \subseteq V$ gives the desired result. □

Lemma 4.2.5. *Let $F(x, \bar{y})$ be a 0-definable function which is continuous with convex domain, monotone in each variable, and nonconstant in x for any $\bar{b}$. Fix $\bar{b} \in M^n$. Without loss of generality, assume that $F(x, \bar{b})$ is increasing. Then there is no triple $a_1 < a^* < a_2$ from M such that:*

$$\mathcal{M} \models \forall x_1, x_2 \left(a_1 < x_1 < a^* \to V(F(x, \bar{b})) \wedge a^* < x_2 < a_2 \to \neg V(F(x, \bar{b})) \right)$$

Proof. Such a point a^* would be a 'break point' in M at which the value of F jumps from V to a point outside of V. The proof relies on T-immunity in an essential way:

Suppose we have such a triple. We claim $F(a^*, \bar{b}) \notin V$: If $F(a^*, \bar{b}) \in V$, then since $F(x, \bar{b})$ is increasing on (a_1, a_2), there is $\gamma > 0 \in V$ such that $a^* + \gamma < a_2$ and $F(a^* + \gamma, \bar{b}) > F(a*, \bar{b})$. Let $\delta = F(a^* + \gamma, \bar{b}) - F(a^*, \bar{b})$. If $\delta \in V$, then $F(a^*, \bar{b}) + \delta \in V$, which is impossible. And if $\delta \notin V$, there is $0 < \delta' < \delta$ such that $\delta' \in V$. Then since F is continuous, by the Intermediate Value Theorem for $\mathcal{M}$, there is $c^* > a^*$ such that $F(c^*, \bar{b}) = F(a, \bar{b}) + \delta'$. By T-immunity, we get that $F(c^*, \bar{b}) \in V$, which is also impossible. Thus $F(a^*, \bar{b}) \notin V$. We may then apply the above argument to the left side of the interval (a_1, a_2) to get another contradiction. □

Thus we have no 'break point' in M for F. As in the above lemma, we would like to be able to characterize precisely when the function F may 'jump' between V and $\mathcal{M} \setminus V$.

Lemma 4.2.6. *Let $F(x, \bar{y})$ be a 0-definable function which is continuous, monotone in each variable, and has a convex domain. Then for every $\bar{b}$ from M, either $F(V, \bar{b}) \subseteq V$ or $F(V, \bar{b}) \subseteq M \setminus V$.*

To ease notation in the case where $\bar{b}$ contains elements from V and from outside of V, we introduce the following definition.

Definition 4.2.7. $\bar{b} \in M^n$ is *V-dependent* if there are $b \in M$, $\bar{c} \in M^{n-1}$ such that $\bar{b} = b\bar{c}$, and an $\mathfrak{L}(V)$-definable function $g : \mathcal{M}^{n-1} \to \mathcal{M}$ such that $g(\bar{c}) = b$. Otherwise, we say $\bar{b}$ is *V-independent*.

Lemma 4.2.8. *Let $F(x, \bar{y})$ be definable in $\mathfrak{L}(V)$, and fix $\bar{b} \in M^n$. Then there is an $\mathfrak{L}(V)$-definable function $F^*(x, \bar{y})$ and a V-independent $\bar{b}' \subseteq \bar{b}$ such that $\mathcal{M} \models \forall x \left(F(x, \bar{b}) = F^*(x, \bar{b}')\right)$.*

Proof. Assume $\bar{b}'$ is not V-independent and say $\bar{b} = b\bar{c}$, $g(\bar{c}) = b$, for g an $\mathfrak{L}(V)$-definable function. Define $F^*(x, \bar{c}) = w \leftrightarrow \mathcal{M} \models \exists x \, [g(\bar{c}) = y \wedge F(x, y\bar{c}) = w]$. Since $lg(\bar{b})$ is finite, repeat this step finitely many times until we get a V-independent tuple $\bar{b}'$. $\square$

Proof of Lemma 4.2.6. Let $F(x, \bar{y})$ be 0-definable, fix $\bar{b} \in M^n$, and suppose that $X := F(V, \bar{b}) \cap V$ and $Y := F(V, \bar{b}) \cap M \setminus V$ are both nonempty. By the weak o-minimality of $(\mathcal{M}, V)$, X and Y are both finite unions of convex sets. Assume without loss of generality that X and Y are convex (hence adjacent). Again without loss of generality, suppose $X < Y$. Then let $p = tp_{\mathfrak{C}}((\sup X)/M)$ (omitted in M by the proper convexity of V). Let $\mathcal{M}'$ be an elementary extension of $\mathcal{M}$ realizing p by an element a^*, and write $V' := V(\mathcal{M}')$. Observe that, since the cut between X and Y lies in V, a^* is bounded by V as well. Further observe that by definition of X and Y, there are $c_1 \in X$ and $c_2 \in Y$ such that:

$$(M', V') \models \forall x_1, x_2 \left((c_1 < x_1 < a^* < x_2 < c_2) \rightarrow V(F(x_1, \bar{b})) \wedge \neg V(F(x_2, \bar{b}))\right)$$

Then by the proof of Lemma 4.2.5 applied to a^*, we may assume that $F(a^*, \bar{b})$ realizes the type $q = tp_{\mathfrak{C}}((\sup V)/M)$. We show by induction that this is impossible.

By Lemma 4.2.4, we may assume $\bar{b} \in (\mathcal{M} \setminus V)^n$. And by Lemma 4.2.8, we may assume $\bar{b}$ is $V-independent$. We proceed inductively on $lg(\bar{b})$. The case $lg(\bar{b}) = 0$ is precisely the proof of Lemma 4.2.4. So, assume the Lemma holds for all V-definable

$F(x, \bar{y})$ with $lg(\bar{y}) \leq n$. Let $\bar{c} \in M^n$ and $b \in M$ such that $\bar{c}$ is V-independent, and suppose by way of contradiction that $a^* \in M'$ with $|a^*| < V$, and $F(a^*, \bar{c}, b) \models q$.

By assumption, $F(a^*, \bar{c}, y)$ is strictly monotone in an open interval containing b. Suppose $F(a^*, \bar{c}, y)$ is constant in this neighborhood. Then since F is definable in V, let b_1, b_2 be the endpoints of I, *i.e.* $(b_1, b_2) = \{y : F(a^*, \bar{c}, y) = F(a^*, \bar{c}, b)\}$. Note that for $i = 1, 2$, we have $b_i \in \mathrm{cl}(\bar{c}, a^*)$. And if $b_i \notin \mathrm{cl}(V, \bar{c})$, then by the exchange principle for definable closure, we have $a^* \in \mathrm{cl}(\bar{c}, a^*)$, contradicting $a^* \notin M$. Thus $b_i \in \mathrm{cl}(V, \bar{c})$. So let $h_i : \mathcal{M}^n \to \mathcal{M}$ be a V-definable function such that $h_i(c_i) = b_i$, and define a new function $G : \mathcal{M}^n \to \mathcal{M}$ as follows:

$$G(x, \bar{c}) = w \leftrightarrow \forall y \in (h_1(\bar{c}), h_2(\bar{c})) \rightarrow) \, (F(x, \bar{c}, y) = w)$$

Then $F(a^*, \bar{c}, b) = G(a^*, \bar{c})$, and thus $G(a^*, \bar{c}) \models q$, contradicting the inductive hypothesis.

Thus we may assume $F(a^*, \bar{c}, y)$ is strictly increasing or decreasing in a neighborhood of b. We write out the details for the increasing case; the decreasing case is similar. Since F is increasing near b, this is reflected in $tp_{\mathcal{M}'}(a^*/\mathcal{M})$. In particular define:

$$\varphi_l(x, y) = \exists y_1 \, (y_1 < y \wedge \forall z_1 \, (y_1 < z_1 < y \rightarrow F(x, \bar{c}, y_1) < F(x, \bar{c}, y)))$$

$$\varphi_r(x, y) = \exists y_2 \, (y < y_2 \wedge \forall z_2 \, (y < z_2 < y_2 \rightarrow F(x, \bar{c}, y) < F(x, \bar{c}, y_2)))$$

Then $\varphi_l(x, b) \wedge \varphi_r(x, b) \in tp_{\mathcal{M}'}(a^*/\mathcal{M})$, thus is consistent with $\mathcal{M}$, and realized by some $a' < a^*$. Now suppose $F(a', \bar{c}, b) = e \in V$. Then $F(a', \bar{c}, y)$ is also strictly increasing (thus a bijection) on a definable neighborhood I containing b. Thus the formula $\psi(x) := y \in I \wedge F(a', \bar{c}, y) = e$ defines b over V, contradicting the V-independence of $\bar{c}b$. $\square$

We may restate the result in Lemma 4.2.6 as follows: Given $F(x, \bar{y})$, let $C := \{x \in V : F(x, \bar{b}) \in V\}$. Then either $C = \emptyset$ or $C = V \cap \text{dom}(F(x, \bar{b}))$. Let us define $C_e := \{x \in (V + e) : F(x, \bar{b}) \in V\}$.

Corollary 4.2.9. *For any $e \in cl(\bar{b})$, either $C_e = \emptyset$ or $C_e = (V + e) \cap dom(F(x, \bar{b}))$.*

Proof. Suppose $c, d \in V$, $F(c + e, \bar{b}) \in V$, and $F(d + e, \bar{b}) \notin V$. Then $F'(x, \bar{b}) = F(x + e, \bar{b})$ is an $\mathfrak{L}(V, \bar{b})$-definable function, thus $F(V + e, \bar{b}) = F'(V, \bar{b}) \subseteq V$. $\square$

Following are the final pieces needed for the proof. The first is a routine exercise in cellular decomposition (this version is from [10]).

Theorem 4.2.10 (Regular cellular decomposition)**.** *Let $\mathcal{M}$ be an o-minimal expansion of an ordered group. Then for each definable function $F : \mathcal{M}^n \to \mathcal{M}$, there is a partition of dom(F) into definable cells (called "regular cells") such that, on each cell, F is continuous and monotone in each variable. Furthermore, if $Th(\mathcal{M})$ admits elimination of quantifiers, then each of the cells is quantifier-free definable.*

Corollary 4.2.11 (to Corollary 4.2.9)**.** *Let $F(x, \bar{y}) : \mathcal{M} \times \mathcal{M}^n \to \mathcal{M}$ be a 0-definable function, and $\bar{b} \in M^n$ fixed. Suppose $(a\bar{b})$ and $(a'\bar{b})$ are in the same regular $n+1$-cell C, and $\mathcal{M} \models V(a' - a)$. Then $\mathcal{M} \models V(F(a, \bar{b}))$ if and only if $\mathcal{M} \models V(F(a', \bar{b}))$.*

Proof. Let $\Pi : \mathcal{M}^{n+1} \to \mathcal{M}$ be projection onto the first coordinate. If $F_{\bar{b}}$ is constant on $\Pi(C)$, then the result is trivial, since a and a' will be sent to the same value.

Otherwise, suppose $F_{\bar{b}}$ is strictly increasing on $\Pi(C)$. Let $e := a' - a$, and apply Corollary 4.2.9. $\square$

Corollary 4.2.12. *Let $F(x, \bar{y})$ be 0-definable, $\bar{b}$ fixed, and $I =$dom$(F_{\bar{b}})$. Assume that $F_{\bar{b}} \restriction I$ is continuous, strictly monotone, and nonconstant, and let $a < a' \in I$ be such that $\mathcal{M} \models \neg V(a' - a)$. Then $\mathcal{M} \models \neg V(F(a, \bar{b}) - F(a', \bar{b}))$.*

Proof. Since $F_{\bar{b}}$ is continuous, strictly monotone, and nonconstant, then the inverse $F_{\bar{b}}^{-1}$ exists and is definable, continuous, and strictly monotone on $F(I, \bar{b})$. Then apply Corollary 4.2.11 substituting $F_{\bar{b}}^{-1}$ for $F_{\bar{b}}$, and substituting $F(a, \bar{b})$ for a and $F(a', \bar{b})$ for a'. $\square$

The next lemma allows us to compare the behavior of two different functions which have a shared domain.

Lemma 4.2.13. *Let $F(x, \bar{y})$ and $G(x, \bar{y})$ be 0-definable, and b fixed. Suppose that dom$(F_{\bar{b}})$ = dom$(G_{\bar{b}}) = I$, and $F_{\bar{b}}$ and $G_{\bar{b}}$ are both strictly increasing on I. Further suppose that there is $a \in I$ such that $\mathcal{M} \models V(F(a, \bar{b}))$. If $\mathcal{M} \models \neg V(G(a, \bar{b}))$, then $\mathcal{M} \models (\forall x \in I)\left(V(F(x, \bar{b})) \to \neg V(G(x, \bar{b}))\right)$.*

Proof. Suppose without loss of generality that there is $a' > a$ such that $\mathcal{M} \models V(F(a', \bar{b}))$, and $\mathcal{M} \models V(G(a', \bar{b}))$. Then since $\mathcal{M} \models \neg V(G(a, \bar{b}))$, then $\mathcal{M} \models \neg V(G(a' - a, \bar{b}))$, and thus by Corollary 4.2.11, we get that $\mathcal{M} \models \neg V(a' - a)$. Similarly, since $V(F(a', \bar{b}))$, then by Corollary 4.2.12, we get that $\mathcal{M} \models V(a' - a)$, a contradiction. $\square$

Lemma 4.2.14. *Let $(\mathcal{M}, +, <, \ldots)$ be an o-minimal expansion of an ordered group which admits elimination of quantifiers. Then there are finitely many quantifier-free formulas $\varphi_i(x, \bar{y})$ such that each $\varphi_i(\mathcal{M}, \bar{y})$ is convex for each $\bar{y} \in \mathcal{M}^n$, and such that:*

$$\mathcal{M} \models \forall \bar{y} \forall x \left(\varphi(x, \bar{y}) \leftrightarrow \bigvee_i \varphi_i(x, \bar{y}) \right)$$

Proof. Since $\mathcal{M}$ is o-minimal, there is a uniform bound N on the number of components of $\varphi(\mathcal{M}, \bar{y})$ as $\bar{y}$ varies. So define:

$$\varphi_i(x, \bar{y}) = \begin{cases} \text{“}x \text{ is in the } i^{th} \text{ component of } \varphi(\mathcal{M}, \bar{y})\text{”} & \text{if it exists} \\ x \neq x & \text{otherwise} \end{cases}$$

Then $\mathcal{M} \models \forall \bar{y} \forall x \left(\varphi(x, \bar{y}) \leftrightarrow \bigvee_{i=1}^{N} \varphi_i(x, \bar{y}) \right)$. And each of the φ_i is equivalent to a quantifier-free formula by elimination of quantifiers for $\mathcal{M}$. $\square$

4.3 Construction of Skolem functions

Proof of Theorem 4.2.3. We eliminate quantifiers and define Skolem functions simultaneously. For quantifier elimination, it suffices to show that we can eliminate the quantifier from $\exists x \varphi(x, \bar{y})$, for any primitive φ of $\mathfrak{L} \cup \{V, c\}$. And since $\varphi(x, \bar{y})$ is a primitive $\mathfrak{L} \cup \{V\}$-formula, it is equivalent to:

$$\bigwedge_i \varphi_i(x, \bar{y}) \wedge \bigwedge_j V(F_i(x, \bar{y})) \wedge \bigwedge_k \neg V(G_k(x, \bar{y}))$$

where the φ_i are finitely many $\mathfrak{L}$-formulas, and F_j, G_k are finitely many $\mathfrak{L}$-terms.

By Lemma 4.2.14, we may rewrite each φ_i as a finite disjunction $\bigvee_l \varphi_{i_l}(x, \bar{y})$ such that each φ_{i_l} is quantifier-free and with the property that $\varphi_{i_l}(\mathcal{M}, \bar{b})$ is convex for each $\bar{b} \in \mathcal{M}^n$. And by regular cellular decomposition, there are finitely many definable cells C_{j_m} and $C_{k_{m'}}$ such that $\bigcup_m C_{j_m} = \operatorname{dom}(F_j)$ and $\bigcup_{m'} C_{k_{m'}} = \operatorname{dom}(G_k)$, and $F_j \restriction C_{j_m}$ and $G_k \restriction C_{k_{m'}}$ are each monotone in each variable and continuous. Note that since $\mathcal{M}$ admits elimination of quantifiers, each of the cells C_{j_m} and $C_{k_{m'}}$ are quantifier-free definable.

It is important to note that given an $\mathfrak{L}$-term $F(x, \bar{y})$ and a quantifier-free definable regular cell $C(x, \bar{y})$, the $\mathfrak{L}$-sentence "$V(F \restriction C)$" is first-order expressible by $V(F(x, \bar{y})) \wedge C(x, \bar{y})$, which is still quantifier-free. For the sake of clarity, for the remainder of the section, we shall suppress the the notation indicating conjunction with the regular cell C.

Finally, we define $L(x) \leftrightarrow \neg V(x) \wedge x < 0$ and $R(x) \leftrightarrow \neg V(x) \wedge x > 0$.

Now, call a formula $\psi(x, \bar{y})$ $*$-*atomic* if one of the following hold:

- $\psi(x, \bar{y})$ is a quantifier-free $\mathfrak{L}$-formula such that for all $\bar{b} \in \mathcal{M}$, $\psi(\mathcal{M}, \bar{b})$ is convex.

- $\psi(x, \bar{y})$ is equivalent to $V(F(x, \bar{y}))$ for F a quantifier-free definable partial function whose domain is a cell C, such that F is continuous and monotone in each variable on C.

- $\psi(x, \bar{y})$ is equivalent to $L(F(x, \bar{y}))$ for F a quantifier-free definable partial function whose domain is a cell C, such that F is continuous and monotone in each variable on C.

- $\psi(x, \bar{y})$ is equivalent to $R(F(x, \bar{y}))$ for F a quantifier-free definable partial function whose domain is a cell C, such that F is continuous and monotone in each variable on C.

A $*$-*primitive* formula is a finite conjunction of $*$-atomic formulas.

Note that if ψ is $*$-atomic, then for every $\bar{b} \in \mathcal{M}^n$, $\psi(\mathcal{M}, \bar{b})$ is convex. Thus, by the above analysis, the formula $\varphi(x, \bar{y})$ can be written as a finite disjunction of $*$-primitive formulas ($*$-*disjunctive normal form*, or $*$*DNF*). Again as in the previous chapter, by Lemma 3.1.5, it suffices to eliminate the quantifier from $\exists x(\psi_1(x, \bar{y}) \wedge \psi_2(x, \bar{y}))$ for any two $*$-atomic formulas ψ_i, ψ_2.

Thus, we may assume $\exists x \Phi(x, \bar{y})$ takes one of the following forms:

(1) $\exists x \varphi(x, \bar{y})$, for φ an $\mathfrak{L}$-formula and $\varphi(\mathcal{M}, \bar{b})$ convex for every $\bar{b} \in \mathcal{M}^n$

(2) $\exists x\,(V(F(x, \bar{y})))$

(3) $\exists x\,(L(F(x, \bar{y})))$ or

$\exists x\,(R(F(x, \bar{y})))$

(4) $\exists x\,(\varphi(x, \bar{y}) \wedge V(F(x, \bar{y})))$ or

$\exists x\,(\varphi(x, \bar{y}) \wedge L(F(x, \bar{y})))$ or

$\exists x\,(\varphi(x, \bar{y}) \wedge R(F(x, \bar{y})))$, for φ as above

(5) $\exists x\,(V(F(x, \bar{y})) \wedge V(G(x, \bar{y})))$

(6) $\exists x\,(V(F(x, \bar{y})) \wedge L(G(x, \bar{y})))$ or

$\exists x\,(V(F(x, \bar{y})) \wedge R(G(x, \bar{y})))$ or

$\exists x\,(L(F(x, \bar{y})) \wedge L(G(x, \bar{y})))$ or

$\exists x\,(L(F(x, \bar{y})) \wedge R(G(x, \bar{y})))$ or

$\exists x\,(R(F(x, \bar{y})) \wedge R(G(x, \bar{y})))$

Case (1) is taken care of by quantifier elimination and Skolem functions for $\mathcal{M}$.

For (2)-(6), recall that by the $*$DNF, and regular cellular decomposition, we may assume in each case that F and G are continuous and monotone in each variable.

Case (2). The equivalent formula for general $\bar{y}$ is quite long and perhaps not as illustrative as each of the special cases for $\bar{y}$. Since F is defined on a regular cell, then $F_{\bar{b}}(x)$ is strictly monotone for each $\bar{b}$; for a given $\bar{b}$, whether $F_{\bar{b}}$ is strictly increasing, strictly decreasing, or constant, is definable, so if we are able to define a function for each kind of behavior, then our equivalent quantifier-free formula and definable Skolem function will be a definition by cases. Thus as a special case, we fix $\bar{b}$ with $\mathrm{dom}(F_{\bar{b}}) = (\alpha, \beta)$ where $\alpha, \beta \in \mathcal{M}$. For the equivalence, we shall make use of the fact that ε is a constant symbol in the language $\mathfrak{L}$, and $\varepsilon^{\mathcal{M}} \in V$; therefore by T-immunity, for any $a \in \mathcal{M}$, we have $a \in V \leftrightarrow a + \varepsilon \in V$.

If $F(x, \bar{b})$ is strictly increasing on (α, β), then $\mathcal{M} \models \exists x(F(x, \bar{b}))$ if and only if:

$$\begin{aligned} \mathcal{M} \models \quad & \big(\beta - \alpha > \varepsilon \wedge \big(V(F(\alpha + \varepsilon, \bar{b})) \vee V(F(\beta - \varepsilon, \bar{b}))\big)\big) && (*) \\ \vee \quad & \big(\beta - \alpha > \varepsilon \wedge \big(L(F(\alpha + \varepsilon, \bar{b})) \wedge R(F(\beta - \varepsilon, \bar{b}))\big)\big) && (**) \\ \vee \quad & \big(\beta - \alpha \leq \varepsilon \wedge V(F(\tfrac{\alpha+\beta}{2}, \bar{b}))\big) && (***) \end{aligned}$$

If $F_{\bar{b}}$ is strictly decreasing on (α, β), then $\mathcal{M} \models \exists x(F(x, \bar{b}))$ if and only if:

$$\begin{aligned} \mathcal{M} \models \quad & \big(\beta - \alpha > \varepsilon \wedge \big(V(F(\alpha + \varepsilon, \bar{b})) \vee V(F(\beta - \varepsilon, \bar{b}))\big)\big) \\ \vee \quad & \big(\beta - \alpha > \varepsilon \wedge \big(R(F(\alpha + \varepsilon, \bar{b})) \wedge L(F(\beta - \varepsilon, \bar{b}))\big)\big) \\ \vee \quad & \big(\beta - \alpha \leq \varepsilon \wedge V(F(\tfrac{\alpha+\beta}{2}, \bar{b}))\big) \end{aligned}$$

And if $F_{\bar{b}}$ is constant on (α, β), then $\mathcal{M} \models \exists x(F(x, \bar{b}))$ if and only if $\mathcal{M} \models V\left(F\left(\dfrac{\alpha + \beta}{2}, \bar{b}\right)\right)$.

Of course $F_{\bar{b}}$ may have an unbounded domain. If $\mathrm{dom}(F_{\bar{b}}) = (-\infty, \infty) = \mathcal{M}$, then there are three subcases to consider. If $F_{\bar{b}}$ is constant on $\mathcal{M}$, then $\mathcal{M} \models \exists x(F(x,\bar{b})) \leftrightarrow V(F(0,\bar{b}))$. If $F_{\bar{b}}$ is strictly increasing or strictly decreasing on $\mathcal{M}$, then by Corollary 4.2.12, if the range of $F_{\bar{b}}$ has a nonempty intersection with V, then V is contained in the range of $F_{\bar{b}}$. Thus, $\mathcal{M} \models \exists x(F(x,\bar{b})) \leftrightarrow \exists x(F(x,\bar{b}) = 0)$. Then since $\exists x(F(x,\bar{b}) = 0)$ is a formula of $\mathfrak{L}(\bar{b})$, by the quantifier elimination for $\mathcal{M}$ there is an equivalent quantifier free formula in $\mathfrak{L}(\bar{b})$. Similarly, if $\mathrm{dom}(F_{\bar{b}})$ is (α, ∞) for α from $\mathcal{M}$, then it suffices to check the value of $F(\alpha + \varepsilon, \bar{b})$ and the behavior of $F_{\bar{b}}$ on (α, ∞).

Each of the equivalences is quantifier-free-definable, and the behavior of $F_{\bar{y}}$ (either strictly increasing, strictly decreasing, or constant) is quantifier-free definable; thus the above leads to a full quantifier elimination for any formula of type (2).

For Skolem functions, we focus on the case where $F_{\bar{b}}$ is strictly increasing. We define the following (note that the function symbol $F_{\Phi(x,\bar{y})}$ is the Skolem function for the formula Φ):

$$F_{\Phi(x,\bar{y})}(\bar{b}) := \begin{cases} \alpha + \varepsilon & \text{if } (*) \text{ and } \mathcal{M} \models V(F(\alpha + \varepsilon, \bar{b})) \\ \beta - \varepsilon & \text{if } (*) \text{ and } \mathcal{M} \models \neg V(F(\alpha + \varepsilon, \bar{b})) \wedge V(F(\beta - \varepsilon, \bar{b})) \\ F_{\bar{b}}^{-1}(0) & \text{if } (**) \\ \frac{\alpha+\beta}{2} & \text{if } (***) \\ 0 & \text{otherwise} \end{cases}$$

If $F_{\bar{b}}$ is strictly decreasing on (α, β), then the Skolem function is defined analogously.

If $F_{\bar{b}}$ is constant on (α, β) for $\alpha, \beta \in \mathcal{M}$, then $F_{\Phi(x,\bar{y})}(\bar{y}) := \dfrac{\alpha + \beta}{2}$. Again, note that the behavior of $F_{\bar{y}}$ is definable, thus each subcase for the Skolem function is definable as well.

Case (3). We choose $\exists x\,(L(F(x,\bar{y})))$ as a representative case. As above, fix $\bar{b} \in \mathcal{M}^n$, and let (α, β) be the domain of $F_{\bar{b}}$. If (α, β) is bounded, then since $F_{\bar{b}}$ is strictly monotone, we have $\mathcal{M} \models \exists x(L(x, \bar{b}))$ if and only if:

$$\begin{aligned} \mathcal{M} \models \quad & \big(\beta - \alpha > \varepsilon \wedge \big(L(F(\alpha+\varepsilon, \bar{b})) \vee L(F(\beta-\varepsilon, \bar{b}))\big)\big) && (*) \\ \vee \quad & \big(\beta - \alpha \leq \varepsilon \wedge L(F(\tfrac{\alpha+\beta}{2}, \bar{b}))\big) && (**) \end{aligned}$$

We use the above cases to define the Skolem function:

$$F_{\Phi(x,\bar{y})}(\bar{b}) := \begin{cases} \alpha+\varepsilon & \text{if } (*) \text{ and } \mathcal{M} \models L(F(\alpha+\varepsilon, \bar{b})) \\ \beta-\varepsilon & \text{if } (*) \text{ and } \mathcal{M} \models \neg L(F(\alpha+\varepsilon, \bar{b})) \wedge L(F(\beta-\varepsilon, \bar{b})) \\ \frac{\alpha+\beta}{2} & \text{if } (**) \\ 0 & \text{otherwise} \end{cases}$$

If $F_{\bar{b}}$ has an unbounded domain, then we shall need to make use of the constant c which was added to the language. Suppose $\alpha = -\infty$ and $\beta \in \mathcal{M}$. Then there are several subcases to consider. If $F_{\bar{b}}$ is constant on $\mathcal{M}$, then it suffices to check the value of $F(\beta - \varepsilon, \bar{b})$. Suppose that $F_{\bar{b}}$ is strictly increasing (the strictly decreasing case is symmetric). We define the equivalent formula and give a uniformly definable element in each case:

- If $\mathrm{ran}(F_{\bar{b}})$ is unbounded below and $\mathcal{M} \models V(F(\beta-1, \bar{b})) \vee R(F(\beta-1, \bar{b}))$, then $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow 0 = 0$. We let $F_{\Phi(x,\bar{y})}(\bar{b}) = F_{\bar{b}}^{-1}(-c)$.

- If $\mathrm{ran}(F_{\bar{b}})$ is unbounded below and $\mathcal{M} \models L(F(\beta - 1, \bar{b}))$, then $\mathrm{ran}(F_{\bar{b}})$ is bounded above. Thus $\sup(\mathrm{ran}(F_{\bar{b}})) = \gamma \in \mathcal{M}$ by o-minimality of $\mathcal{M}$. Then again, $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow 0 = 0$. We let $F_{\Phi(x,\bar{y})}(\bar{b}) = F_{\bar{b}}^{-1}(\gamma - \varepsilon)$, which exists and is well-defined by the fact that $F_{\bar{b}}$ is strictly increasing and continuous.

- If $\mathrm{ran}(F_{\bar{b}})$ is bounded below, say $\inf(\mathrm{ran}(F_{\bar{b}})) = \gamma \in \mathcal{M}$, then since $F_{\bar{b}}$ is strictly increasing, $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow V(\gamma + \varepsilon)$, and we let $F_{\Phi(x,\bar{y})}(\bar{b}) = F_{\bar{b}}^{-1}(\gamma + \varepsilon)$.

Suppose $(\alpha, \beta) = \mathcal{M}$. If $F_{\bar{b}}$ is constant on $\mathcal{M}$, then it suffices to check the value of $F(0, \bar{b})$. We suppose again for simplicity that $F_{\bar{b}}$ is strictly increasing on $\mathcal{M}$. We consider the subcases:

- If $\mathrm{ran}(F_{\bar{b}}) = \mathcal{M}$, then $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow 0 = 0$, and we let $F_{\Phi(x,\bar{y})}(\bar{b}) = F_{\bar{b}}^{-1}(-c)$.

- If $\mathrm{ran}(F_{\bar{b}})$ is unbounded below and bounded above with $\sup(\mathrm{ran}(F_{\bar{b}})) = \gamma \in \mathcal{M}$, then $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow 0 = 0$, and we let:

$$F_{\Phi(x,\bar{y})}(\bar{b}) = \begin{cases} F_{\bar{b}}^{-1}(\gamma - \varepsilon) & \text{if } L(\gamma) \\ F_{\bar{b}}^{-1}(-c) & \text{otherwise} \end{cases}$$

- If $\mathrm{ran}(F_{\bar{b}})$ is bounded below, say $\inf(\mathrm{ran}(F_{\bar{b}})) = \gamma \in \mathcal{M}$, then since $F_{\bar{b}}$ is strictly increasing, $\mathcal{M} \models \exists x L(F(x, \bar{b})) \leftrightarrow V(\gamma + \varepsilon)$, and we let $F_{\Phi(x,\bar{y})}(\bar{b}) = F_{\bar{b}}^{-1}(\gamma + \varepsilon)$.

The case for $F_{\bar{b}}$ strictly decreasing may be argued symmetrically.

Case (4). Since we assume $\varphi(x, \bar{b})$ to be convex for each $\bar{b} \in \mathcal{M}$, then there are quantifier-free definable functions $h_1, h_2 : \mathcal{M}^n \to \mathcal{M} \cup \{\pm\infty\}$ such that $\varphi(\mathcal{M}, \bar{b}) = (h_1(\bar{b}), h_2(\bar{b}))$. Then we may repeat the above arguments substituting the definable interval $(\alpha, \beta) \cap (h_1(\bar{b}), h_2(\bar{b}))$ for the domain of $F_{\bar{b}}$.

Case (5). We choose $\exists x\,(V(F(x,\bar{y}))\wedge V(G(x,\bar{y})))$ as a representative case. Again, for reasons outlined above, we fix $\bar{b}\in\mathcal{M}^n$ such that $F_{\bar{b}}$ and $G_{\bar{b}}$ are both strictly increasing. We are only interested in the x-values where $F_{\bar{b}}$ and $G_{\bar{b}}$ are both defined, so we restrict to a domain (α,β) on which both $F_{\bar{b}}$ and $G_{\bar{b}}$ are defined and strictly increasing. By regular cellular decomposition, we further restrict to a cell on which the function $(F-G)(x,\bar{b})$ is strictly monotone. Without loss of generality, assume $F(x,\bar{b})>G(x,\bar{b})$ on (α,β). Evidently:

$$\models \exists x\,(V(F(x,\bar{y}))\wedge V(G(x,\bar{y})))\rightarrow \exists x(V(F(x,\bar{y})))\wedge \exists x(V(G(x,\bar{y})))$$

Assuming that the domain of both $F_{\bar{b}}$ and $G_{\bar{b}}$ is an interval (α,β) with $\alpha,\beta\in\mathcal{M}$ and $\beta-\alpha>\varepsilon$, and using the equivalence from Case (1) applied to both F and G, we obtain the following chart of possibilities for F and G:

F-1.	$V(F(\alpha+1,\bar{b}))$	$V(G(\alpha+1,\bar{b}))$	G-1.
F-2.	$V(F(\beta-1,\bar{b}))$	$V(G(\beta-1,\bar{b}))$	G-2.
F-3.	$L(F(\alpha+1,\bar{b}))\wedge R(F(\beta-1,\bar{b}))$	$L(F(\alpha+1,\bar{b}))\wedge R(F(\beta-1,\bar{b}))$	G-3.

Abbreviate "F-i and G-j holds" by [F-i,G-j]. We show by case analysis that:

$$\mathcal{M}\models \exists x\Phi(x,\bar{b})\leftrightarrow ([\text{F-1,G-1}]\vee[\text{F-2,G-2}]\vee\big([\text{F-3,G-3}]\wedge V(G(F_{\bar{b}}^{-1}(0),\bar{b}))\big)$$

The subcases [F-1,G-1] and [F-2,G-2] each indicate that the functions F and G match on one of the endpoints (α,β), and thus in these two subcases, $\mathcal{M}\models \exists x(\Phi(x,\bar{b}))$.

In case [F-3,G-3], the functions $F_{\bar{b}}$ and $G_{\bar{b}}$ send $\alpha + \varepsilon$ below V, and send $\beta - \varepsilon$ above V. Thus by the intermediate value theorem, the range of both $F_{\bar{b}}$ and $G_{\bar{b}}$ contains all of V. In particular, 0 is in the range of $F_{\bar{b}}$. If $G(F_{\bar{b}}^{-1}(0), \bar{b}) \in V$, then of course $F_{\bar{b}}^{-1}(0)$ witnesses $\mathcal{M} \models \exists x(\Phi(x, \bar{b}))$. Otherwise, we have that $\neg V(G(F_{\bar{b}}^{-1}(0), \bar{b}))$, and thus by Lemma 4.2.13, $\mathcal{M} \models (\forall \alpha < x < \beta)(\neg V((F - G)(x, \bar{b})))$. Therefore $\mathcal{M} \models \neg\exists x(\Phi(x, \bar{b}))$.

To see that these are the only positive subcases, suppose that [F-1$\wedge\neg$F-2,$\neg$G-1$\wedge$G-2] or [F-1, G-3] hold. Specifically, $\mathcal{M} \models V(F(\alpha + \varepsilon, \bar{b})) \wedge \neg V(G(\alpha + \varepsilon, \bar{b}))$. Then by a direct application of Lemma 4.2.13, we get that $\mathcal{M} \models (\forall \alpha < x < \beta)(\neg V((F - G)(x, \bar{b})))$. Therefore $\mathcal{M} \models \neg\exists x(\Phi(x, \bar{b}))$. Analogously, this applies to subcases [$\neg$F-1$\wedge$F-2, G-1$\wedge\neg$G-2] and [F-2, G-3] with $\beta - \varepsilon$.

Finally, since we assumed $F_{\bar{b}}(x) > G_{\bar{b}}(x)$ on the domain, these are the only possibilities for F-i and G-j.

Given the equivalence above, we define the Skolem function as follows:

$$F_{\Phi(x,\bar{y})}(\bar{y}) := \begin{cases} \alpha + \varepsilon & \text{if [F-1,G-1] holds} \\ \beta - \varepsilon & \text{if [F-2,G-2] holds} \\ F_{\bar{b}}^{-1}(0) & \text{if [F-3,G-3]} \wedge V(G(F_{\bar{b}}^{-1}(0), \bar{b})) \text{ holds} \\ 0 & \text{otherwise} \end{cases}$$

Case (6) may be argued analogously to Case (5).

We have now shown how to eliminate a single existential quantifier from any pair of $*$-atomic formulas in $Th(\mathcal{M}, V)$, which, as argued above, is sufficient to show quantifier elimination for $Th(\mathcal{M}, V)$. Thus for Skolem functions, we define F_φ for an arbitrary $*$-primitive formula φ just as in §3.1.3. $\square$

Corollary 4.3.1. *Let* $(\mathcal{M}, V)$ *be* T*-immune. Then* $(\mathcal{M}, V)$ *does not have uniform elimination of imaginaries.*

Proof. Immediate, by Corollary 1.3.2. $\square$

Afterword

Much of the early research documented in the paper was halted by contemporary discoveries: in particular, it appears that the work on monotonicity and cellular decomposition properties in nonvaluational weakly o-minimal structures has been exhaustively done, in [20]. The canonical o-minimal expansion is a definitive answer to the question, "What does a nonvaluational weakly o-minimal structure look like?" As a result, the potential for future work in that area is limited.

Our work on Skolem functions is noticeably incomplete: it is known that T-convex expansions of o-minimal groups have Skolem functions, and it is known that o-minimal groups with a nonvaluational convex predicate do not. For technical reasons, we cannot say in general that an expansion which is not T-convex fails to have Skolem functions, but there should be a reasonable answer to the general question of which valuational structures retain Skolem functions. Also, while we have shown that certain valuational structures do not eliminate imaginaries, it is not known whether nonvaluational structures may.

Finally, we note that our results for nonvaluational structures are heavily dependent on the condition that the structures are obtained by adding a convex predicate to an o-minimal structure. In contrast to the nonvaluational case, there is not currently a wealth of results on arbitrary valuational structures.

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