

CONVERGENCE OF SPECTRAL METHODS FOR HYPERBOLIC INITIAL-BOUNDARY VALUE SYSTEMS*

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Abstract. We present here a convergence proof for spectral approximations for hyperbolic systems with initial and boundary conditions. We treat in detail Chebyshev collocation, but the final result is readily applicable to other spectral methods, such as Legendre collocation or tau-methods.

Key words. spectral methods, hyperbolic systems, convergence, stability

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Introduction. In the paper [1], we derived stability results for spectral methods applied to initial-boundary value problems for hyperbolic systems. It is shown there that one can bound certain weighted L_2 spatial norms of the solution in terms of norms of the boundary data (homogenous initial conditions are assumed). The bounds also contain powers of N , which is the degree of the approximating polynomials.

Here we show that the approximations discussed above actually converge to the exact solution, at least when this solution is smooth. We bound the error in the numerical method by a power of N multiplying a term which depends only on the exact solution—more precisely, this is the interpolation error of the initial value and boundary derivatives. For sufficiently differentiable functions, this interpolation error will decay fast enough to drive the full approximation error to zero. We have not attempted to derive the sharpest bound of this type, but merely to show that such a bound exists.

The method of proof here is similar to the one in [1], [2], where basic results are first deduced for a scalar equation, and then extended to the full system. Accordingly, the paper is divided into two sections, the first dealing with the scalar case, and the second with the system. By the means of Gauss-Lobatto quadrature formulas we first bound the error at outflow for a single scalar equation. Then we use this estimate, together with the basic stability result of [1] to bound the overall error for a system.

1. The scalar case—Chebyshev collocation. Given the equation

$$(1.1) \quad \begin{aligned} u_t &= u_x, & |x| < 1, & t > 0, \\ u(x, 0) &= f(x), & u(1, t) &= g(t), \end{aligned}$$

we consider the pseudospectral method based on collocating at the extrema of T_{N+1} , where T_m is the Chebyshev polynomial of degree m

$$T_m(x) = \cos(m \cos^{-1}(x)).$$

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It is shown in [3], [4] that the pseudospectral approximation $v = v_N(x, t)$ satisfies

$$(1.2) \quad \begin{aligned} \frac{\partial v}{\partial t} &= \frac{\partial v}{\partial x} + \tau(t) T'_{N+1}(x), \\ v(x, 0) &= P_N f, \quad v(1, t) = g(t), \end{aligned}$$

with the projection operator P_N defined by

$$(1.3) \quad \begin{aligned} P_N f &\text{ is a polynomial of degree } \leq N, \\ (P_N f)(x) &= f(x) \quad \text{at } x = x_j = \cos \frac{\pi j}{N+1}, \quad 0 \leq j \leq N. \end{aligned}$$

Define now $\delta_N(x, t)$ as the difference between v_N and the projection of u

$$\delta_N(x, t) = v - P_N u.$$

This function satisfies the equation

$$(1.4) \quad \begin{aligned} \frac{\partial}{\partial t} \delta_N(x, t) &= \frac{\partial}{\partial x} \delta_N(x, t) - \left[P_N \frac{\partial u}{\partial x} - \frac{\partial}{\partial x} P_N u \right] + \tau T'_{N+1}, \\ \delta_N(x, 0) &= 0, \quad \delta_N(1, t) = 0. \end{aligned}$$

We note at this stage that the polynomial $(1+x)\delta_N T'_{N+1}$ is of degree $2N+1$ and therefore may be integrated exactly by the Gauss-Lobatto quadrature rule. This results in the following:

$$(1.5) \quad \int_{-1}^1 \frac{(1+x)\delta_N T'_{N+1}}{\sqrt{1-x^2}} dx = \frac{\pi}{N} \sum_{j=0}^{N+1} \frac{1}{c_j} (1+x_j)\delta_N(x_j) T'_{N+1}(x_j) = 0,$$

$c_0 = c_{N+1} = 2, c_j = 1 \quad \text{for } 0 < j \leq N,$

since $1+x_{N+1} = 0, \delta_N(x_0) = 0$ and $T'_{N+1}(x_j) = 0$ for $j = 1, \dots, N$. In fact, because of the term $(1+x)$ and the boundary condition, the indices in the quadrature sums may run only from 1 to N (instead of 0 to $N+1$).

Defining

$$Q_N = \frac{\partial}{\partial x} (P_N u) - P_N \frac{\partial u}{\partial x},$$

multiply (1.4) by $(1+x)\delta_N$ and integrate to get

$$(1.6) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-1}^1 \frac{(1+x)\delta_N^2(x, t)}{\sqrt{1-x^2}} dx \\ = \int_{-1}^1 \frac{(1+x)\delta_N(x, t)}{\sqrt{1-x^2}} \frac{\partial}{\partial x} \delta_N(x, t) dx + \int_{-1}^1 \frac{(1+x)\delta_N(x, t) Q_N}{\sqrt{1-x^2}} dx. \end{aligned}$$

An integration by parts on the right-hand side produces

$$(1.7) \quad \begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{-1}^1 \frac{(1+x)\delta_N^2(x, t)}{\sqrt{1-x^2}} dx \\ = -\frac{1}{2} \int_{-1}^1 \frac{\delta_N^2(x, t)}{(1-x)\sqrt{1-x^2}} dx + \int_{-1}^1 \frac{(1+x)\delta_N(x, t) Q_N(x, t)}{\sqrt{1-x^2}} dx. \end{aligned}$$

We use again the Gauss-Lobatto formula, and reach

$$\begin{aligned}
 & \frac{1}{2} \frac{d}{dt} \int_{-1}^1 \frac{(1+x) \delta_N^2(x, t)}{\sqrt{1-x^2}} dx \\
 (1.8) \quad &= -\frac{\pi}{2N} \sum_{j=1}^N \frac{\delta_N^2(x_j, t)}{1-x_j} - \frac{\pi}{8N} \delta_N^2(-1, t) + \frac{\pi}{N} \sum_{j=1}^N (1+x_j) \delta_N(x_j, t) Q_N(x_j, t) \\
 &\leq -\frac{\pi}{2N} \sum_{j=1}^N \frac{\delta_N^2(x_j, t)}{1-x_j} - \frac{\pi}{8N} \delta_N^2(-1, t) + \frac{\pi}{2N} \sum_{j=1}^N \frac{\delta_N^2(x_j, t)}{1-x_j} \\
 &\quad + \frac{\pi}{2N} \sum_{j=1}^N (1+x_j)^2 (1-x_j) Q_N^2(x_j, t).
 \end{aligned}$$

Equation (1.8) yields immediately, since $\delta_N(x, 0) = 0$,

$$(1.9) \quad \int_{-1}^1 \frac{(1+x)}{\sqrt{1-x^2}} \delta_N^2(x, t) dx + \frac{\pi}{4N} \int_0^t \delta_N^2(-1, t) dt \leq \int_0^t \|Q_N(x, t)\|^2 dt$$

where

$$(1.10) \quad \|Q(x, t)\|^2 = \frac{\pi}{N} \sum_{j=1}^N (1+x_j)^2 (1-x_j) Q_N^2(x_j, t).$$

In the next section we will need a different version of (1.9), in which the time integral is weighted by $e^{-2\eta t}$, $\eta > 0$:

$$\begin{aligned}
 (1.11) \quad & \int_0^\infty e^{-2\eta t} \int_{-1}^1 \frac{(1+x)}{\sqrt{1-x^2}} \delta_N^2(x, t) dx dt + \frac{\pi}{8N\eta} \int_0^\infty e^{-2\eta t} \delta_N^2(-1, t) dt \\
 & \leq \frac{1}{2\eta} \int_0^\infty e^{-2\eta t} \|Q_N(x, t)\|^2 dt.
 \end{aligned}$$

This form matches the Laplace-Fourier transforms which are used in the basic stability estimates.

It should be noted that the bound on $\delta_N^2(-1, t)$ obtained in (1.11),

$$\begin{aligned}
 (1.12) \quad & \int_0^\infty e^{-2\eta t} \delta_N^2(-1, t) dt \leq \alpha(N) \int_0^\infty e^{-2\eta t} \|Q_N(x, t)\|^2 dt, \\
 & \alpha(N) = \frac{4N}{\pi},
 \end{aligned}$$

is very crude. A better estimate can be found for the boundary error $u(-1, t) - v(-1, t)$ by taking the Laplace transform of (1.1) and (1.2) and analyzing the difference. This approach was used by Dubiner [5], who obtained $(u-v)(-1, t) \approx 1/N$. The same bound holds for $P_N u - v$, since, for smooth functions, $P_N u$ approaches u faster than any power of $1/N$.

2. Convergence proof for systems. Consider the first-order hyperbolic system of partial differential equations

$$(2.1) \quad \frac{\partial u}{\partial t} = A \frac{\partial u}{\partial x}, \quad |x| < 1, \quad t > 0$$

where

$$u = u(x, t) = (u^{(1)}(x, t), u^{(2)}(x, t), \dots, u^{(n)}(x, t))'$$

is the vector of unknowns and A is a fixed $n \times n$ coefficient matrix. Since by hyperbolicity A is similar to a real diagonal matrix we may, without loss of generality, take it diagonal:

$$(2.2) \quad A = \begin{pmatrix} A^I & 0 \\ 0 & A^{II} \end{pmatrix},$$

$$A^I = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_l \end{pmatrix} < 0, \quad A^{II} = \begin{pmatrix} a_{l+1} & & \\ & \ddots & \\ & & a_n \end{pmatrix} > 0.$$

The solution of this system is uniquely determined if we specify initial conditions

$$(2.3) \quad u(x, 0) = f(x) = (f^I(x), f^{II}(x))$$

and boundary conditions

$$(2.4) \quad u^I(-1, t) = Lu^{II}(-1, t) + g^I(t), \quad u^{II}(1, t) = Ru^I(1, t) + g^{II}(t).$$

In these formulas, f and $g = g(t) = (g^I(t), g^{II}(t))$ are prescribed n -vectors, and

$$(2.5) \quad (u^I, u^{II}), \quad (f^I, f^{II}), \quad (g^I, g^{II})$$

is the partition of these vectors into inflow and outflow components—corresponding to the partition of A in (2.2). L and R are constant reflection matrices of order $l \times (n - l)$ and $(n - l) \times l$, respectively.

We shall discuss only problems whose solutions decay in time, and therefore postulate:

Assumption I. There exists a constant $\gamma > 0$, such that

$$|R||L| \leq 1 - \gamma < 1.$$

($|A|$ and $|v|$ denote the Euclidean norm of a matrix A or vector v , respectively.)

In a pseudospectral Chebyshev approximation to (2.1), one seeks a vector $v = v_N$ of N -degree polynomials such that

$$(2.6a) \quad \frac{\partial v}{\partial t} = A \frac{\partial v}{\partial x} + \tau(t) T'_{N+1}(x).$$

The multiplier $\tau = (\tau^I, \tau^{II})$ is determined by the boundary conditions

$$(2.6b) \quad v^I(-1, t) = Lv^{II}(-1, t) + g^I(t), \quad v^{II}(1, t) = Rv^I(1, t) + g^{II}(t).$$

Initially, v is defined by collocation:

$$(2.6c) \quad \begin{aligned} v^I(x, 0) &= f^I(x) & \text{at } x = x_j, \quad j = 1, 2, \dots, N + 1, \\ v^{II}(x, 0) &= f^{II}(x) & \text{at } x = x_j, \quad j = 0, 1, \dots, N. \end{aligned}$$

In [1] the stability of the approximation (2.6) has been established, under zero initial conditions. Here we shall prove the convergence of $v_N(x, t)$ to $u(x, t)$ as N tends to infinity.

We define a pair of projection operators $P = (P^I, P^{II})$ by requiring that for any function F , $P^I F$ and $P^{II} F$ be polynomials of degree N at most, satisfying

$$(2.7) \quad \begin{aligned} (P^I F)(x_j) &= F(x_j), & j = 1, 2, \dots, N + 1, \\ (P^{II} F)(x_j) &= F(x_j), & j = 0, 1, \dots, N. \end{aligned}$$

Note that two distinct sets of nodes are used for collocation—cf. (2.6c).

We can state now the main convergence result:

THEOREM. *Let $\varepsilon = \varepsilon_N(x, t) = v - Pu$ be the error in the pseudospectral approximation (2.6) to the hyperbolic system (1.1). Let $Q = (Q^I, Q^{II})$ be the approximation error*

$$(2.8) \quad Q^I = A^I \left(P^I \frac{\partial u^I}{\partial x} - \frac{\partial}{\partial x} P^I u^I \right), \quad Q^{II} = A^{II} \left(P^{II} \frac{\partial u^{II}}{\partial x} - \frac{\partial}{\partial x} P^{II} u^{II} \right).$$

Then

$$(2.9) \quad \int_0^\infty e^{-2\eta t} \|\varepsilon(x, t)\|^2 dt \leq \frac{KN\alpha(N)}{2\eta} \int_0^\infty e^{-2\eta t} \|Q(x, t)\|^2 dt$$

with K independent of N , $\alpha(N)$ as defined in (1.12) and

$$(2.10) \quad \begin{aligned} \|\varepsilon(x, t)\|^2 &= \int_{-1}^1 (1-x) |\varepsilon^I(x, t)|^2 \frac{dx}{\sqrt{1-x^2}} + \int_{-1}^1 (1+x) |\varepsilon^{II}(x, t)|^2 \frac{dx}{\sqrt{1-x^2}}, \\ \|Q(x, t)\| &= \frac{\pi}{N} \sum_{j=1}^N (1-x_j)^2 (1+x_j) |Q^I(x_j, t)|^2 \\ &\quad + \frac{\pi}{N} \sum_{j=1}^N (1+x_j)^2 (1-x_j) |Q^{II}(x_j, t)|^2. \end{aligned}$$

(Here we generalize the seminorm defined in (1.11).)

Proof. Let r be the solution of

$$(2.11) \quad \begin{aligned} \frac{\partial r}{\partial t} &= A \frac{\partial r}{\partial x}, \\ r(x, 0) &= u(x, 0), \\ r^I(-1, t) &= u^I(-1, t), \quad r^{II}(1, t) = u^{II}(1, t) \end{aligned}$$

and let s be the pseudospectral Chebyshev approximation to (2.11), i.e.

$$(2.12) \quad \begin{aligned} \frac{\partial s}{\partial t} &= A \frac{\partial s}{\partial x} + T'_{N+1}(x) \theta(t), \\ s(x, 0) &= Pu(x, 0), \\ s^I(-1, t) &= u^I(-1, t), \quad s^{II}(1, t) = u^{II}(1, t). \end{aligned}$$

It is obvious that $r \equiv u$, but s does not satisfy the boundary conditions (2.6c), and therefore its multiplier θ is distinct from τ . In any case, for $\delta = s - Pr$, we have

$$(2.13) \quad \begin{aligned} \int_0^\infty e^{-2\eta t} \|\delta(x, t)\|^2 dt + \frac{1}{2N\eta} \int_0^\infty e^{-2\eta t} |\delta^I(1, t)|^2 dt \\ + \frac{1}{2N\eta} \int_0^\infty e^{-2\eta t} |\delta^{II}(-1, t)|^2 dt \leq \frac{1}{\eta} \int_0^\infty e^{-2\eta t} \|Q(x, t)\|^2 dt. \end{aligned}$$

This is clearly a restatement of (1.11). Next, we compare the spectral solution s with v —as defined by (2.6). We can show that

$$(2.14) \quad \int_0^\infty e^{-2\eta t} \|s - v\|^2 dt \leq \frac{c}{\eta} N \int_0^\infty e^{-2\eta t} [|\delta^I(1, t)|^2 + |\delta^{II}(-1, t)|^2] dt.$$

Indeed, $v - s$ satisfies

$$\begin{aligned}\frac{\partial}{\partial t}(v - s) &= \frac{\partial}{\partial x}(v - s) + T'_{N+1}(\tau - \theta), \\ (v - s)(x, 0) &= 0, \\ (v^I - s^I)(-1, t) &= L(v^{II} - s^{II})(-1, t) + Ls^{II}(-1, t) - u^I(-1, t) + g^I \\ &= L(v^{II} - s^{II})(-1, t) + L\delta^{II}(-1, t), \\ (v^{II} - s^{II})(-1, t) &= R(v^I - s^I)(1, t) + R\delta^I(1, t),\end{aligned}$$

and the inequality (2.14) follows from the stability estimate in [1, Thm. 5.1].

Now, since $Pr = Pu$ we have

$$\varepsilon - \delta = v - s,$$

and hence,

$$\int_0^\infty e^{-2\eta t} \|\varepsilon(x, t)\|^2 dt \leq \int_0^\infty e^{-2\eta t} \|\delta(x, t)\|^2 dt + \int_0^\infty e^{-2\eta t} \|v - s\|^2 dt.$$

The first term is majorized in (2.13) and the second in (2.14), thus establishing the theorem.

Again, we emphasize that the bound for $u - v_N$ is the same as the bound for $Pu - v_N$ —which we just have computed—for smooth functions u , which Pu approaches rapidly.

We conclude with two remarks:

(a) The stability estimate of (2.14) explicitly uses boundary values. This is why these have to be bounded beforehand, which was done in the preceding section, (1.10).

(b) Our result applies not only to Chebyshev collocation, but also to other spectral methods (which, however, have to satisfy Assumptions I, II and III of [1]). Indeed, the only quantity that varies with the spectral method employed is the coefficient weight at ± 1 for the Gauss-Lobatto quadrature. Once it is known that this weight is $\approx 1/N^k$, which is the case for Gegenbauer collocation, as shown in [1], the same proof follows through. In particular, it is sufficient to evaluate outflow errors for a scalar equation in order to estimate errors at both boundary points for a system.

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