ABSTRACT

Title of Thesis: SYNGE'S TRICK REVISITED

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We review Synge's trick and present some of its applications to Riemannian Geometry. We use it to prove Frankel, Weinstein-Synge and Wilking's theorems, which concern manifolds of positive sectional curvature. First, we give the classical proofs of these theorems and then we present a reformulation of Synge's trick in terms of a lower bound for the index of a special kind of geodesics.

SYNGE'S TRICK REVISITED

by

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A la memoria de mi padre.

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Chapter 1

Introduction

In the present work, we will study some theorems about manifolds of positive curvature, all of which use a common idea introduced by Synge in [10]. As in [11], we call this idea Synge's trick. Among these theorems are Frankel's theorem, Weinstein's theorem and Wilking's theorem. This work was inspired by some ideas of my advisor, Prof. Grove, and also by [9].

First, we see the classical proofs of Frankel and Weinstein-Synge's theorems, which are proved by contradiction. Then, we present the same theorems but now being reformulated as a computation of a lower bound for the index of a special kinds of geodesics, called N-geodesics. Then, we give a direct proof of Frankel and Weinstein's theorems. We also present Wilking's theorem, which also uses Synge's trick to find a lower bound for these special kind of geodesics. At the end, we formulate and prove an optimal general index theorem which encompass the theorems mentioned above.

In the first part of the thesis, we present the first and the second variational formulas of the energy function [1] of paths. We also state and give proofs of the

[1] The energy function we use in this work, it is known in Physics as the action functional.

theorems mentioned in the above paragraph and illustrate how Synge's trick in conjunction with the second variational formula for the energy function of paths is used in each of them.

In the second part of the thesis, we explain how these theorems have really to do with the index of geodesics in a manifold M.

Chapter 2

Synge's Trick and Some Theorems About Positive Sectional Curvature

2.1 First and Second Variational Formulas of the Energy Function

In this section, we compute the first and second variational formulas of the energy function, we proceed similarly as in [8] and [2].

Let $(M, <>)$ be a Riemannian manifold and let $p, q \in M$ be two points of M. Let $\gamma : [0,1] \to M$ be a curve that joins p with q; i.e. $\gamma(0) = p$ and $\gamma(1) = q$. Consider a differentiable variation $f(s,t)$ of the curve γ. That is, f is a differentiable mapping

$$\begin{aligned} f : (-\varepsilon, \varepsilon) \times [0,1] &\to M \\ (s,t) \qquad &\mapsto f(s,t) \end{aligned}$$

such that $f(0,t) = \gamma(t)$.

Remark 1. *For t =constant, the curves $f_t(s) := f(s,t) : (-\varepsilon, \varepsilon) \to M$ are called the transversal curves of the variation.*

Remark 2. *Note that the vector field* $\frac{\partial f}{\partial t}(0,t)$ *is the velocity vector field* $\gamma'(t)$ *along the curve* $\gamma(t)$.

Remark 3. *The vector field* $W(t) := \frac{\partial f}{\partial s}(0,t)$ *is called the variational vector field along* $\gamma(t)$.

Next, we will compute the first variational formula of the energy function

$$E(0) := E(f_t(0)) := \frac{1}{2}\int_0^1 \|\gamma'(t)\|^2 dt,$$

where $\|\gamma'(t)\|^2 =< \gamma'(t), \gamma'(t) >$.

Theorem 2.1.1 (First Variational Formula of the Energy Function)**.** *Let* $f(s,t)$ *be a differentiable variation of the curve* γ*, then*

$$E'(0) = \frac{d}{ds}E(f_t(0)) = \big\langle W(1), \gamma'(1)\big\rangle - \big\langle W(0), \gamma'(0)\big\rangle - \int_0^1 \big\langle W(t), \frac{D}{dt}\gamma'(t)\big\rangle dt.$$

Proof. By the properties of the Levi-Civita connection, we have that the following identities are satisfied for any parametrized surface f in a Riemannian manifold

$$\frac{d}{ds}\big\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle = 2\big\langle \frac{D}{ds}\frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle$$
$$\frac{D}{ds}\frac{\partial f}{\partial t} = \frac{D}{dt}\frac{\partial f}{\partial s}$$

where we have used the compatibility of the metric in the first equation and the symmetry of the metric in the second equation. Then

$$\begin{aligned} E'(s) = \frac{d}{ds}E(f_t(s)) =& \frac{1}{2}\int_0^1 \frac{d}{ds}\big\langle \frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle dt \\ =& \frac{1}{2}\int_0^1 2\big\langle \frac{D}{ds}\frac{\partial f}{\partial t}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle dt \\ =& \int_0^1 \big\langle \frac{D}{dt}\frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle dt. \end{aligned}$$

Using again the compatibility of the connection with the metric we have that

$$\frac{d}{dt}\big\langle \frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle = \big\langle \frac{D}{dt}\frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle + \big\langle \frac{\partial f}{\partial s}(s,t), \frac{D}{dt}\frac{\partial f}{\partial t}(s,t)\big\rangle.$$

Then

$$\begin{aligned}
E'(s) =& \int_0^1 \frac{d}{dt}\big\langle \frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle dt - \int_0^1 \big\langle \frac{\partial f}{\partial s}(s,t), \frac{D}{dt}\frac{\partial f}{\partial t}(s,t)\big\rangle dt \\
=& \big\langle \frac{\partial f}{\partial s}(s,t), \frac{\partial f}{\partial t}(s,t)\big\rangle\Big|_{t=0}^{t=1} - \int_0^1 \big\langle \frac{\partial f}{\partial s}(s,t), \frac{D}{dt}\frac{\partial f}{\partial t}(s,t)\big\rangle dt \\
=& \big\langle \frac{\partial f}{\partial s}(s,1), \frac{\partial f}{\partial t}(s,1)\big\rangle - \big\langle \frac{\partial f}{\partial s}(s,0), \frac{\partial f}{\partial t}(s,0)\big\rangle \\
&- \int_0^1 \big\langle \frac{\partial f}{\partial s}(s,t), \frac{D}{dt}\frac{\partial f}{\partial t}(s,t)\big\rangle dt.
\end{aligned}$$

So, at $s = 0$ we have that

$$\begin{aligned}
E'(0) =& \big\langle \frac{\partial f}{\partial s}(0,1), \frac{\partial f}{\partial t}(0,1)\big\rangle - \big\langle \frac{\partial f}{\partial s}(0,0), \frac{\partial f}{\partial t}(0,0)\big\rangle - \int_0^1 \big\langle \frac{\partial f}{\partial s}(0,t), \frac{D}{dt}\frac{\partial f}{\partial t}(0,t)\big\rangle dt \\
=& \big\langle W(1), \gamma'(1)\big\rangle - \big\langle W(0), \gamma'(0)\big\rangle - \int_0^1 \big\langle W(t), \frac{D}{dt}\gamma'(t)\big\rangle dt.
\end{aligned}$$

□

Now, let us compute the second variational formula of the energy function.

Let $\gamma : [0,1] \to M$ be a geodesic that joins the points p and q of M(i.e. a critical point of the energy function $E : \Omega(M; p, q) \to \mathbf{R}$.)

Let $W_1, W_2 \in T_\gamma\Omega$ be two vector fields along the curve γ. Let f be a 2-parameter differentiable variation of γ; that is, a differentiable function

$$f : U \times [0,1] \to \mathbf{R}$$

$$(s_1, s_2, t) \mapsto f(s_1, s_2, t)$$

where U is a neighborhood of $(0,0) \in \mathbf{R}^2$ in such a way that

$$f(0,0,t) = \gamma(t), \quad \frac{\partial f}{\partial s_1}(0,0,t) = W_1(t) \quad \text{and} \quad \frac{\partial f}{\partial s_2}(0,0,t) = W_2(t).$$

Then as in [8], the Hessian $E_{**}(W_1, W_2)$ is defined by

$$E_{**}(W_1, W_2) := \frac{\partial^2}{\partial s_2 \partial s_1} E(f_t(s_1, s_2))\big|_{(0,0)}.$$

Theorem 2.1.2 (Second Variational Formula of the Energy Function)**.**

Let $\gamma : [0,1] \to M$ be a geodesic that joins the points p and q of M. Let f be a 2-parameter variation of γ as above. Then

$$\begin{aligned} E_{**}(W_1, W_2) = \frac{\partial^2 E}{\partial s_2 \partial s_1}(0,0) =& \big\langle \frac{D}{ds_2} W_1(1), \gamma'(1)\big\rangle + \big\langle W_1(1), \frac{D}{dt} W_2(1)\big\rangle \\ & - \big\langle \frac{D}{ds_2} W_1(0), \gamma'(0)\big\rangle - \big\langle W_1(0), \frac{D}{dt} W_2(0)\big\rangle \\ & - \int_0^1 \big\langle W_1(t), \frac{D^2}{dt^2} W_2(t) + R\big(\gamma'(t), W_2(t)\big)\gamma'(t)\big\rangle. \end{aligned}$$

Proof. By the first variational formula of the energy, we have that

$$\begin{aligned} \frac{\partial}{\partial s_1} E(f_t(s_1, s_2)) =& \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 1), \frac{\partial f}{\partial t}(s_1, s_2, 1)\big\rangle - \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 0), \frac{\partial f}{\partial t}(s_1, s_2, 0)\big\rangle \\ & - \int_0^1 \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, t), \frac{D}{dt}\frac{\partial f}{\partial t}(s_1, s_2, t)\big\rangle dt. \end{aligned}$$

Then

$$\begin{aligned} \frac{\partial E}{\partial s_2 \partial s_1}(s_1, s_2) =& \frac{\partial}{\partial s_2}\Big\{\big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 1), \frac{\partial f}{\partial t}(s_1, s_2, 1)\big\rangle - \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 0), \frac{\partial f}{\partial t}(s_1, s_2, 0)\big\rangle \\ & - \int_0^1 \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, t), \frac{D}{dt}\frac{\partial f}{\partial t}(s_1, s_2, t)\big\rangle dt\Big\} \\ =& \big\langle \frac{D}{ds_2}\frac{\partial f}{\partial s_1}(s_1, s_2, 1), \frac{\partial f}{\partial t}(s_1, s_2, 1)\big\rangle + \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 1), \frac{D}{ds_2}\frac{\partial f}{\partial t}(s_1, s_2, 1)\big\rangle \\ & - \big\langle \frac{D}{ds_2}\frac{\partial f}{\partial s_1}(s_1, s_2, 0), \frac{\partial f}{\partial t}(s_1, s_2, 0)\big\rangle - \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, 0), \frac{D}{ds_2}\frac{\partial f}{\partial t}(s_1, s_2, 0)\big\rangle \\ & - \int_0^1 \big\langle \frac{D}{ds_2}\frac{\partial f}{\partial s_1}(s_1, s_2, t), \frac{D}{dt}\frac{\partial f}{\partial t}(s_1, s_2, t)\big\rangle dt \\ & - \int_0^1 \big\langle \frac{\partial f}{\partial s_1}(s_1, s_2, t), \frac{D}{ds_2}\frac{D}{dt}\frac{\partial f}{\partial t}(s_1, s_2, t)\big\rangle dt. \end{aligned}$$

Now, evaluate the above expression at $(s_1, s_2) = (0,0)$. Since $\gamma(t) = f(0,0,t)$ is a geodesic (i.e. $\frac{D}{dt}\frac{\partial f}{\partial t}(0,0,t) = 0$) then we have that the fifth term of the above equality is zero. Also, if we interchange the order of the derivatives of the second and fourth terms and using the fact that for any parametrized surface f

$$R\Big(\frac{\partial f}{\partial t}(0,0,t), \frac{\partial f}{\partial s_2}(0,0,t)\Big)\frac{\partial f}{\partial t}(0,0,t) = \frac{D}{ds_2}\frac{D}{dt}\frac{\partial f}{\partial t}(0,0,t) - \frac{D}{dt}\frac{D}{ds_2}\frac{\partial f}{\partial t}(0,0,t)$$

in the last term of the above equality, we get precisely the second varational formula of the energy function.

□

2.2 Synge's Trick

In the study of manifolds with positive sectional curvature there are several important results that make use of Synge's trick. Among these results are: Frankel's theorem, Weinstein-Synge's theorem and Wilking's theorem.

Although these theorems have different hypothesis, Synge's trick can be used once we have a basic setting: given a complete connected Riemannian manifold M with positive sectional curvature and two distinct points p and q in M, suppose we have a minimal geodesic $\gamma(t)$ that joins p and q; then choose a unit parallel vector field $W_1(t)$ along $\gamma(t)$, which is orthogonal to the tangent field of the curve $\gamma(t)$; then make a 1-parameter variation $f(s,t) = f(s_1 + s_2, t) = f(s_1, s_2, t)$ of γ with variational field $\frac{\partial f}{\partial s}(s,t)|_{s=0} = \frac{\partial f}{\partial s_i}(0,0,t) = W_1(t)$ for $i = 1, 2$ (in which the transversal curves $\alpha(s) := f(s,0)$ and $\beta(s) := f(s,1)$ to γ are geodesics); then by the second variational formula of the energy function of curves we get that

$$
\begin{aligned}
E_{**}(W_1, W_1) =& \langle \frac{D}{ds} W_1(1), \gamma'(1) \rangle + \langle W_1(1), \frac{D}{dt} W_1(1) \rangle \\
& - \langle \frac{D}{ds} W_1(0), \gamma'(0) \rangle - \langle W_1(0), \frac{D}{dt} W_1(0) \rangle \\
& - \int_0^1 \langle W_1(t), \frac{D^2}{dt^2} W_1(t) + R\big(\gamma'(t), W_1(t)\big)\gamma'(t) \rangle dt.
\end{aligned}
$$

Since the vector field $W_1(t)$ is parallel, then $\frac{D}{dt} W_1(t) = 0$, so the second and fourth terms of the above expression are zero. Also, since the curves $\alpha(s)$ and $\beta(s)$ are geodesics then $\frac{D}{ds}\alpha'(s) = 0 = \frac{D}{ds}\beta'(s)$, so in particular,

$$
\frac{D}{ds} W_1(0) = \frac{D}{ds}\alpha'(0) = 0 \quad \text{and} \quad \frac{D}{ds} W_1(1) = \frac{D}{ds}\beta'(0) = 0,
$$

so the first and third terms of the second variational formula of the energy are also zero. Since the manifold M has positive sectional curvature, the second variational formula of the energy becomes

$$
\begin{aligned}
E_{**}(W_1, W_1) &= -\int_0^1 \langle W_1(t), R\big(\gamma'(t), W_1(t)\big)\gamma'(t) \rangle dt \\
&= -\int_0^1 sec(\gamma'(t), W_1(t)) dt < 0,
\end{aligned}
$$

then one gets shorter curves $f_s(t)$ (that join the point $\alpha(s)$ to $\beta(s)$) close to the original geodesic $\gamma(t)$. In different situations, this is going to give us a contradiction, as we will see in the following theorems.

2.3 Classical Theorems: Frankel and Weinstein-Synge's theorems

Let us see how Synge's trick is used in the theorems mentioned above. The following theorem was proved by Frankel in 1960 in [3].

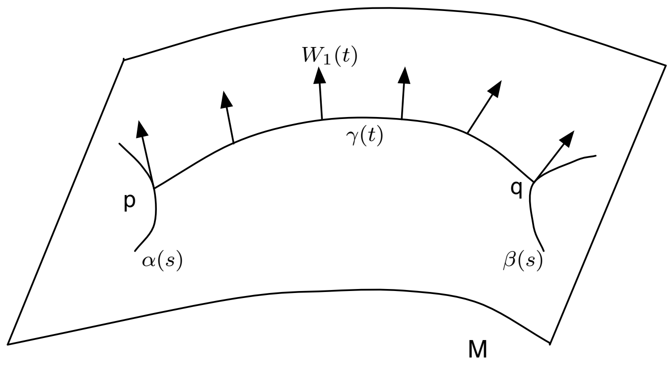

Figure 2.1: Synge's trick

Theorem 2.3.1 (Frankel)**.** *Let M^m be a complete connected Riemannian manifold with positive sectional curvature and let U^r and W^s be compact totally geodesic submanifolds such that $r+s \geq m$, then U^r and W^s have a non-empty intersection.*

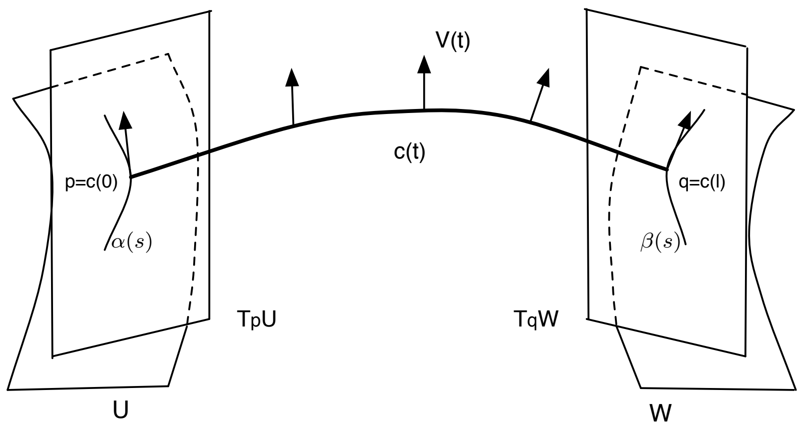

Figure 2.2: Frankel's theorem

Proof. Let U and W be two totally geodesic submanifolds of M. Suppose that U and W does not intersect. Then, there exists a minimal geodesic $c(t)$, with length

l, that joins U and W say at the points $p = c(0) \in U$, $q = c(l) \in W$. Since $c(t)$ is a minimal geodesic joining U and W, then $c(t)$ is orthogonal to U and W at $c(0)$ and $c(l)$, respectively.

Let $\widetilde{W} \subseteq T_qM$ be the linear subspace obtained by parallel transporting T_pU along $c(t)$ at the point q.

Since T_pU is orthogonal to c at p, then $\widetilde{W}$ is also orthogonal to $c(t)$ at q.

Note that $T_qW, \widetilde{W} \subseteq T_qM$ are linear subspaces of T_qM. Then

$$\dim(T_qW \cap \widetilde{W}) \geq \dim T_qW + \dim \widetilde{W} - (\dim T_qM - 1) = s + r - m + 1 \geq 1.$$

So, $T_qW \cap \widetilde{W} \neq \emptyset$. Then, there exists a parallel vector field $V(t)$ along $c(t)$ such that $V(0) \in T_pU$ and $V(l) \in T_qW$.

Now, let $f(s,t)$, $s \in [-\varepsilon, \varepsilon]$, $t \in [0,l]$ be a variation along the curve $c(t)$, in which the transversal curves are geodesics and which has variational field $V(t)$.

Note that since U and W are totally geodesic submanifolds of M then the geodesics

$$\alpha(s) = f(s,0), \quad \beta(s) = f(s,l)$$

lie entirely in U and W, respectively. Then

$$\frac{D}{ds}V(0) = \frac{D}{ds}\left(\frac{\partial f}{\partial s}(s,0)\Big|_{s=0}\right) = \frac{D}{ds}\alpha'(s)|_{s=0} = 0$$

and

$$\frac{D}{ds}V(l) = \frac{D}{ds}\left(\frac{\partial f}{\partial s}(s,l)|_{s=0}\right) = \frac{D}{ds}\beta'(s)|_{s=0} = 0.$$

Therefore, by the second variation formula of energy we get that

$$\begin{aligned}\frac{d^2E}{ds^2}(0) &= \left\langle \frac{D}{ds}V(l), c'(l) \right\rangle + \left\langle V(l), \frac{D}{dt}V(l) \right\rangle \\ &\quad - \left\langle \frac{D}{ds}V(0), c'(0) \right\rangle - \left\langle V(0), \frac{D}{dt}V(0) \right\rangle \\ &\quad - \int_0^l \left\langle V(t), R\left(\frac{dc}{dt}, V\right)\frac{dc}{dt} \right\rangle dt \\ &= -\int_0^l sec\left(\frac{dc}{dt}, V\right) dt < 0,\end{aligned}$$

so we get shorter curves $f_s(t)$ than $c(t)$ (that join U with W), which contradicts that $c(t)$ was minimal.

□

Also Weinstein used Synge's trick in [11], the following theorem is a corollary of a more general result mentioned in this paper (Weinstein proved his statement for conformal diffeomorphisms not just isometries.)

Theorem 2.3.2 (Weinstein). *Let $f : M^m \to M^m$ be an isometry of a compact, oriented Riemannian manifold M^m with $secM > 0$. If f is orientation-preserving if m is even and orientation-reversing if m is odd, then $Fix(f) = \{x \in M : f(x) = x\} \neq \emptyset$.*

Proof. Suppose that $q \neq f(q)$ for all $q \in M$. Let $d(q, f(q))$ be the distance between $q \in M$ and $f(q) \in M$. Then $d(q, f(q)) > 0$ for all $q \in M$.

Since M is compact and $d : M \longrightarrow \mathbf{R}^+$ is a continuous function given by $q \longmapsto d(q, f(q))$, then there exists $p \in M$ such that $d(p, f(p)) > 0$ is a minimum value for d.

Since M is complete, then there exists a minimal geodesic $\gamma : [0, l] \longrightarrow M$ that joins p and $f(p)$ (i.e. $\gamma(0) = p$ and $\gamma(l) = f(p)$) and such that $\|\gamma'(t)\| = 1$ for all

$t \in [0, l]$.

Let N and N' be the orthogonal complement of γ' at p and $f(p)$, respectively. Note that N is a subspace of T_pM and N' is a subspace of $T_{f(p)}M$.

We claim that $df_p(N) = N'$, or equivalently that

$$df_p(\gamma'(0)) = \gamma'(l).$$

In fact, consider the curve $f \circ \gamma : [0, l] \longrightarrow M$. Note that $f \circ \gamma$ is a geodesic.

Let $p' = \gamma'(t')$, where $t' \neq 0, l$. Then, by the triangle inequality and since γ is a minimal geodesic, we have that

$$d(p', f(p')) \leq d(p', f(p)) + d(f(p), f(p')) = d(p', f(p)) + d(p, p') = d(p, f(p)),$$

since p is a minimum value for d, the above inequality becomes equality, so the curve $\gamma * (f \circ \gamma)$, formed by juxtaposition of γ and $f \circ \gamma$, is a geodesic, so

$$\gamma'(l) = \frac{d}{dt}(f \circ \gamma)(t)\Big|_{t=0} = df_p(\gamma'(0)),$$

as we claimed.

Now, consider the linear automorphism

$$P_{f(p),p} \circ df_p : T_pM \longrightarrow T_pM,$$

where $P_{f(p),p} : T_{f(p)}M \longrightarrow T_pM$ is the parallel transport from $\gamma(l) = f(p)$ to $\gamma(0) = p$ along $\gamma(t)$. Note that $P_{f(p),p} \circ df_p$ leaves N invariant, since

$$P_{f(p),p} \circ df_p(\gamma'(0)) = P_{f(p),p}(\gamma'(l)) = \gamma'(0).$$

Let $A : N \longrightarrow N$ be the restriction of $P_{f(p),p} \circ df_p$ to N. Since $P_{f(p),p}$ is orientation-preserving and $det(df_p) = (-1)^m$, we have that

$$det(A) = det(P_{f(p),p} \circ df_p) = det(P_{f(p),p})det(df_p) = (-1)^m.$$

Since $A : \mathbf{R^{m-1}} \to \mathbf{R^{m-1}}$ is an orthogonal transformation with $detA = (-1)^m$, we have that A has a fixed point $v \in N$.

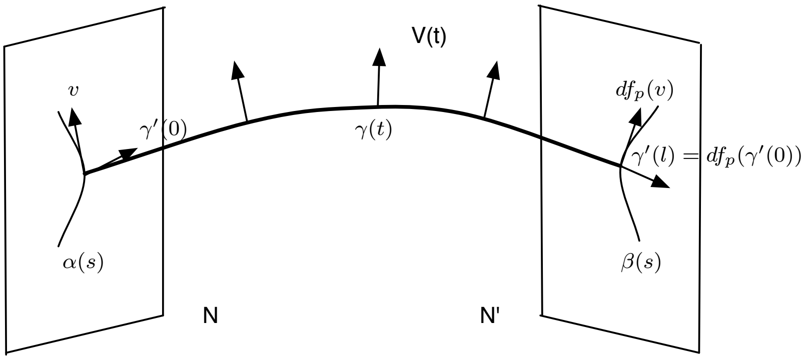

Figure 2.3: Weinstein-Synge's theorem

Now, we have all the necessary ingredients to use Synge's trick.

Let $V(t)$ be a parallel extension of the vector $v \in N$ along the curve $\gamma(t)$ (i.e. $V(0) = v \in N$, $V(l) = df_p(v) \in N'$ and $\frac{DV}{dt}(t) = 0$ for all $t \in [0, l]$.)

Now, let $h(s,t)$, $s \in [-\varepsilon, \varepsilon]$, $t \in [0, l]$, be a variation along the curve $\gamma(t)$ in which the transversal curves are geodesics and which has variational field $V(t)$. For example, $h(s,t)$ can be constructed in the following way: let $\alpha(s)$, $s \in [-\varepsilon, \varepsilon]$, be a curve such that $\alpha(0) = p$ and $\alpha'(0) = v \in N$. Then, the curve $\beta(s) := f \circ \alpha(s)$ is such that $\beta(0) = f(\alpha(0)) = f(p)$ and $\beta'(0) = df_p(\alpha'(0)) = df_p(v) \in N'$. Let

$$h(s,t) = exp_{\gamma(t)}(sV(t)), \qquad s \in [-\varepsilon, \varepsilon], \quad t \in [0, l].$$

Then

$$h(s,0) = exp_{\gamma(0)}(sV(0)) = exp_p(sv) = \alpha(s),$$
$$h(s,l) = exp_{\gamma(l)}(sV(l)) = exp_{f(p)}(sdf_p(v)) = \beta(s)$$

and

$$\frac{\partial}{\partial s}h(s,t)\big|_{s=0} = exp_{\gamma(t)*0}\left(V(t)\right) = Id_{T_{\gamma(t)}M}(V(t)) = V(t).$$

So $V(t)$ is the variational field of $h(s,t)$ with $\frac{DV}{dt} = 0$. Also, since α and β are geodesics, then

$$\begin{aligned}\frac{D}{ds}V(0) &= \frac{D}{ds}\left(\frac{\partial h}{\partial s}(s,0)|_{s=0}\right) = \frac{D}{ds}\alpha'(0) = 0 \quad \text{and}\\ \frac{D}{ds}V(l) &= \frac{D}{ds}\left(\frac{\partial h}{\partial s}(s,l)|_{s=0}\right) = \frac{D}{ds}\beta'(0) = 0.\end{aligned}$$

Then, by the second variation formula of the energy and since M has positive sectional curvature we get that

$$\begin{aligned}\frac{d^2E}{ds^2}(0) = &\left\langle \frac{D}{ds}V(l), \gamma'(l)\right\rangle + \left\langle V(l), \frac{D}{dt}V(l)\right\rangle \\ &- \left\langle \frac{D}{ds}V(0), \gamma'(0)\right\rangle - \left\langle V(0), \frac{D}{dt}V(0)\right\rangle \\ &- \int_0^l \left\langle V(t), \frac{D^2V}{dt^2} + R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle dt \\ = &-\int_0^l \left\langle V(t), R\left(\frac{d\gamma}{dt}, V\right)\frac{d\gamma}{dt}\right\rangle dt \\ = &-\int_0^l sec\left(\frac{d\gamma}{dt}, V\right) dt < 0,\end{aligned}$$

so we get shorter curves $h_s(t)$ (which join points $q \in M$ to $f(q)$ where $q \neq p$) than $\gamma(t)$, which contradicts that $d(p, f(p))$ was a minimum.

$\square$

The hypothesis about $f : M^{2m} \to M^{2m}$ being orientation preserving can not be relaxed, as it is shown in the following

Example 1. *Let $M = S^2$ and $f = -id$ be the antipodal map of S^2*

$$-id : S^2 \longrightarrow S^2$$
$$x \longmapsto -x.$$

Then f is orientation-reversing since $degf = -1$, also note that

$$-id_{*_x} : T_xS^2 \longrightarrow T_xS^2$$
$$v \longmapsto -v$$

and clearly $-id$ does not have fixed points.

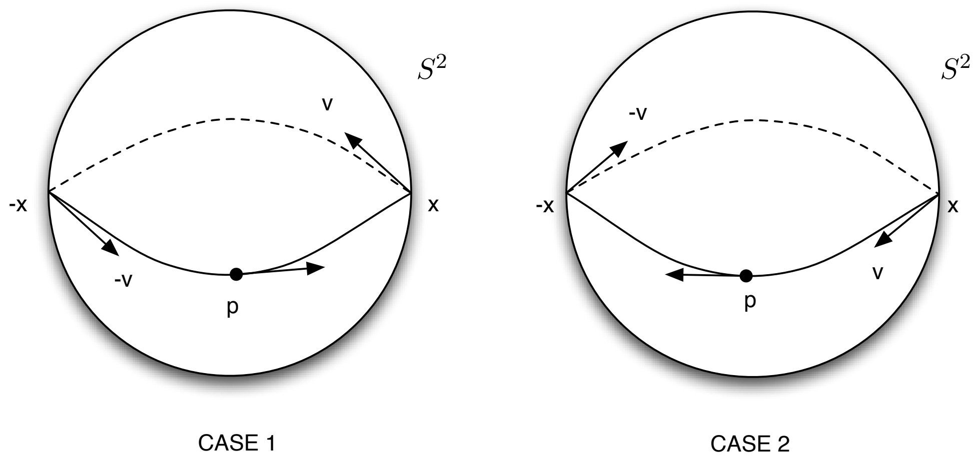

Figure 2.4: Antipodal map on S^2

Similarly, the hypothesis about $f : M^{2m+1} \to M^{2m+1}$ being orientation-reversing can not be relaxed as it shows the following

Example 2. *Let $M = S^3$ and $f = -id$ be the antipodal map of S^3. Then f is orientation-preserving, since $degf = 1$, and f does not have fix points.*

As a corollary of Weinstein's theorem we have a weaker version proved earlier by Synge. I borrowed the main ideas of the proof from [1], [11] and [7].

Corollary 2.3.1 (Synge)**.** *Let M^m be a compact, connected, Riemannian manifold with positive sectional curvature. Then,*

a) If m is even and

(1) M^m is orientable then $\pi_1(M) = 1$.

(2) M^m is not orientable then $\pi_1(M) = \mathbf{Z}_2$.

b) If m is odd, then M^m is orientable.

Proof.

(a1) Suppose m is even and M^m is orientable. Let $p : \widetilde{M} \longrightarrow M$ be the universal cover of M. Let $\widetilde{M}$ have the covering metric (i.e. the pull-back metric of M) and let $\widetilde{M}$ be oriented in such a way that p preserves the orientation. Since $\widetilde{M}$ satisfies the same curvarture conditions as M, then $\widetilde{M}$ has positive sectional curvature. Then, by Myers' theorem, $\widetilde{M}$ is compact.

Let $D : \widetilde{M} \longrightarrow \widetilde{M}$ be a Deck transformation of M. Then, by the way we oriented $\widetilde{M}$, D is an orientation-preserving isometry. Since m is even, by Weinstein's theorem, we have that D has a fixed point, and therefore D is the identity map of $\widetilde{M}$. Since $\pi_1(\widetilde{M}) = 1$, then the group of Deck transformations G can be identified with $\pi_1(M)$. Then $\pi_1(M) \cong G = \{1\}$.

(a2) Suppose m is even and M^m is not orientable. Let $\widetilde{M}$ be the orientable double cover of M and let $\widetilde{M}$ have the covering metric. As in part (a1), since $\widetilde{M}$

satisfies the same curvature conditions as M, then $\widetilde{M}$ has positive sectional curvature. Then, by Myers' theorem, $\widetilde{M}$ is compact.

Since m is even and $\widetilde{M}$ is orientable, by (a1) we have that $\pi_1(\widetilde{M}) = 1$. Then, since $\widetilde{M}$ is a double cover of M we have that $|\pi_1(M)| = 2$, then $\pi_1(M) \cong \mathbf{Z}_2$.

(b) Suppose m is odd and M is not orientable. Let $\widetilde{M}$ be the orientable double cover of M. Let $\widetilde{M}$ have the covering metric. As in part (a) $\widetilde{M}$ is also compact (by Myers' theorem or being the double cover of a compact manifold.)

Since $\widetilde{M}$ is a double cover of M, we have that $[\pi_1(M) : p_\#(\pi_1(\widetilde{M}))] = 2$, so $p_\#(\pi_1(\widetilde{M})) \trianglelefteq \pi_1(M)$. So the covering $p : \widetilde{M} \longrightarrow M$ is regular. Then, the group of Deck transformations $G \cong \pi_1(M)/p_\#(\pi_1(\widetilde{M})) \cong \mathbf{Z}_2$.

Let $D \in G$ and such that $D \neq id$. Then, D is an orientation-reversing isometry of $\widetilde{M}$. Since m is odd, by Weinstein's theorem, D has a fixed point, but this contradicts that $D \neq id$.

$\square$

Chapter 3

Index and Connectivity

3.1 Lower Bound for the Index of non-trivial N-geodesics

All the previous theorems can be studied considering the index of N-geodesics in M introduced in [5]. That is, let M^m be a complete, connected Riemannian manifold with positive sectional curvature. Let $N^n \hookrightarrow M^m \times M^m$ be a closed totally geodesic submanifold. Let $M_N^I = \{\gamma \in C^0(I, M) : (\gamma(0), \gamma(1)) \in N\}$; i.e. M_N^I is the space of curves $\gamma(t) : [0, 1] \longrightarrow M$ such that $(\gamma(0), \gamma(1)) \in N$.

As in [5] we say that the geodesic γ in M is an N-geodesic if $\gamma \in M_N^I$ and

$$(\gamma'(0), -\gamma'(1)) \quad \text{is normal to } N, \tag{3.1}$$

where $\gamma'(t)$ denotes the velocity vector of γ at t and $M \times M$ has the product metric.

Observe that

$$T_\gamma M_N^I = \{X_\gamma \in T_\gamma C^0(I, M) : (X_\gamma(0), X_\gamma(1)) \in T_{(\gamma(0),\gamma(1))}N\}.$$

From [4], we get the following remarks.

Remark 4. *When $N = \triangle \subset M \times M$ is the diagonal submanifold of $M \times M$, we have that N-geodesics correspond just to closed geodesics in M.*

Remark 5. *When $N = U \times W$, where U and W are submanifolds of M, an N-geodesic γ in M is just a geodesic γ in M that starts orthogonal to U and ends orthogonal to W. Then*

$$\begin{aligned} T_\gamma M_N^I &= \{X_\gamma \in T_\gamma C^0(I, M) : (X_\gamma(0), X_\gamma(1)) \in T_{(\gamma(0),\gamma(1))}N\} \\ &\cong \{X_\gamma \in T_\gamma C^0(I, M) : (X_\gamma(0), X_\gamma(1)) \in T_{\gamma(0)}U \times T_{\gamma(1)}W\}. \end{aligned}$$

Remark 6. *If $f : M \to M$ is an isometry on M and $N = Graph(f)$, then an N-geodesic γ corresponds to an f-invariant geodesic; i.e. a geodesic γ in M with the property $df_{\gamma(0)}(\gamma'(0)) = \gamma'(1)$. Also, since*

$$T_{(\gamma(0),\gamma(1))}Graph(f) = \{(v, df_{\gamma(0)}(v)) : v \in T_{\gamma(0)}M)\},$$

we have that

$$\begin{aligned} T_\gamma M_N^I &= \{X_\gamma \in C^0(I, M) : (X_\gamma(0), X_\gamma(1)) \in T_{(\gamma(0),\gamma(1))}Graph(f)\} \\ &= \{X_\gamma \in C^0(I, M) : (X_\gamma(0), df_{\gamma(0)}(X_\gamma(0))) \in T_{\gamma(0)}M \times df_{\gamma(0)}(T_{\gamma(0)}M)\}. \end{aligned}$$

So, in particular $\gamma' \in T_\gamma M_N^I$.

3.1.1 Reformulation of the Classical Theorems as an Index of N-geodesics

Now, let us reformulate the idea of finding shorter curves in the case of Frankel and Weinstein theorems as an index of non-trivial N-geodesics.

Theorem 3.1.1 (Frankel). *Let M^m be a complete connected Riemannian manifold with positive sectional curvature. Let $N = U^r \times W^s \subset M \times M$ where U^r and W^s are compact totally geodesic submanifolds of M such that $r+s \geq m$, then the index of non-trivial N-geodesics is $\geq s+r-m+1$.*

Proof. Let $\gamma(t) : [0,1] \to M$ be a non-trivial N-geodesic. Then by the above remark, we have that γ is a geodesic that joins U to W starting orthogonal to U and ending orthogonal to W. Using a similar procedure as in the original proof of Frankel's theorem, we can find a parallel vector field $V(t)$ along the curve $\gamma(t)$, which is orthogonal to this curve, so $V(0) \in T_{\gamma(0)}U$ and $V(1) \in T_{\gamma(1)}W$; i.e $V \in T_\gamma M_N^I$.

Then, performing a variation of γ that has variational field $V(t)$, in which the transversal curves are geodesics, using the second varational formula $E_{**} : T_\gamma M_N^I \times T_\gamma M_N^I \to \mathbf{R}$ of the energy function and the fact that M has positive sectional curvature we obtain that

$$E_{**}(V,V) = -\int_0^1 sec(\gamma', V)dt < 0,$$

so the index of γ is $\geq s+r-m+1$. □

Theorem 3.1.2 (Weinstein). *Let $f : M^m \to M^m$ be an isometry of a compact, oriented Riemannian manifold M^m with $secM > 0$. Let $N = Graph(f)$. If f is orientation-preserving if m is even and orientation-reversing if m is odd, then the index of non-trivial N-geodesics is ≥ 1.*

Proof. Let $\gamma(t) : [0,1] \to M$ be a non-trivial N-geodesic. Then, by the above remark γ is an f-invariant geodesic. Then, using an analogous procedure as in the original proof of Weinstein's theorem, we find a parallel vector field $V(t)$ along $\gamma(t)$

which is orthogonal to this curve such that $V(1) = df_p(V(0))$ (note that $V \neq \gamma'$). Then, by the above remark we have that $V \in T_\gamma M_N^I$.

Then, making a variation of γ that has variational vector field $V(t)$, using the second variational formula $E_{**} : T_\gamma M_N^I \times T_\gamma M_N^I \to \mathbf{R}$ of the energy function and the fact that M has positive sectional curvature we obtain that

$$E_{**}(V, V) = -\int_0^1 sec(\gamma', V)dt < 0,$$

so the index of γ is at least 1. □

The hypothesis of $f : M^m \to M^m$ about being orientation-preserving (reversing) according to m is even (odd) for the computations of the index of N-geodesics can not be relaxed. The same examples given in the first section illustrate this fact.

Example 3. *Let $M = S^2$ and $f = -id$ be the antipodal map of S^2. Then f is an orientation-reversing isometry. Let $x \in S^2$ and consider a minimal geodesic γ in S^2 that joins x to $-x$. Then γ is a non-trivial $Graph(f)$-geodesic; however, it has index 0.*

Example 4. *Let $M = S^3$ and $f = -id$ be the antipodal map of S^3. Then f is an orientation-preserving isometry. Let $x \in S^3$ and consider a minimal geodesic γ in S^3 that joins x to $-x$. Then γ is a non-trivial $Graph(f)$-geodesic; however, it has index 0.*

3.1.2 Direct proofs of the Classical Theorems

Now, let us focus our attention to the energy function for paths restricted to the space M_N^I. As in [4], the critical points of the energy function restricted to M_N^I

correspond to N-geodesics. This is true, since if γ is a critical point for the energy function restricted to M_N^I, then $E'(0) = 0$. The fact that γ is a geodesic is a consequence of proposition 1.5 in [4]. Then, by the first variational formula of the energy function, we have that

$$\begin{aligned} 0 = E'(0) &= \langle W(1), \gamma'(1)\rangle - \langle W(0), \gamma'(0)\rangle \\ &= \langle (W(1), W(0)), (\gamma'(0), -\gamma'(1))\rangle_{M\times M} \end{aligned}$$

for any $W(t) \in T_\gamma M_N^I$. The above equality is precisely condition (1) of the definition of an N-geodesic; that is,

$$(\gamma'(0), -\gamma'(1)) \in (T_{(\gamma(0),\gamma(1))}N)^{\perp}.$$

That an N-geodesic is a critical point of the energy function restricted to M_N^I is clear if we use the facts that γ is a geodesic and satisfies condition (3.1) in the first variational formula of the energy function.

Remark 7. *Grove and Halperin proved in [6] that the energy function $E : M_N^I \to \mathbf{R}$ satisfies condition (C)of Palais and Smale (a necessary condition for making critical point theory, like Morse theory on infinite dimensional manifolds) iff the function $dist \circ e : N \to \mathbf{R}$ is proper, where $dist : M \times M \to \mathbf{R}$ is the distance function in M and $e : N \hookrightarrow M \times M$ is the inclusion map.*

With the computations of a lower bound for the index of non-trivial N-geodesics in the cases discussed above, we can now give a direct proof to Frankel's and Weinstein's theorems.

Theorem 3.1.3 (Frankel). *Let M^m be a complete connected Riemannian manifold with positive sectional curvature and let U^r and W^s be compact totally geodesic submanifolds such that $r + s \geq m$, then U^r and W^s have a non-empty intersection.*

Proof. Let $N = U \times W$. Let γ be a non-trivial N-geodesic. So, γ is a critical point of the energy function (restricted to M_N^I). By our index estimates, we have that the index of $\gamma \geq s + r - m + 1$. Then γ is not a minimum for the energy function, otherwise it would have index 0. Then, the minimum is reached[1] at trivial N-geodesics, i.e. points of $N = U \times W$. This means that U must intersect W. $\square$

Theorem 3.1.4 (Weinstein). *Let $f : M^m \to M^m$ be an isometry of a compact, oriented Riemannian manifold M^m with $secM > 0$. If f is orientation-preserving if m is even and orientation-reversing if m is odd, then $Fix(f) = \{x \in M : f(x) = x\} \neq \emptyset$.*

Proof. Let $N = Graph(f)$. Let γ be a non-trivial N-geodesic. So, γ is a critical point of the energy function (restricted to M_N^I). By our index estimates, we have that the index of $\gamma \geq 1$. Then γ is not a minimum for the energy function, otherwise it would have index 0. Then, the minimum is reached[2] at trivial N-geodesics, i.e. points of $N = Graph(f) = \{(p, f(p)) : p \in M\}$. This means that there exists a point $x \in M$ such that $x = f(x)$. $\square$

3.2 Wilking's theorem: The Connectedness Principle

The next theorem, which uses Synge's trick and Morse Theory, was proved by Wilking in [12]. It is in some sense, a special case of Frankel's theorem where

[1]see remark 7

[2]see remark 7

$N = U \times U \subset M \times M$ and U is a compact totally geodesic submanifold of M. It uses Synge's trick similarly as in the case of Frankel's theorem to compute a lower bound for the index of N-geodesics.

Theorem 3.2.1 (Connectedness Principle)**.** *Let M^m be a compact manifold with positive sectional curvature, suppose that $U^{m-k} \subset M^m$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $U^{m-k} \hookrightarrow M^m$ is (m-2k+1)-connected.*

Proof. Let $N = U \times U \subset M \times M$. Let $M_N^I = M_{U\times U}^I$ be the space of smooth curves $\gamma(t) : [0,1] \longrightarrow M$ such that $(\gamma(0), \gamma(1)) \in U \times U$; i.e. the space of smooth curves in M that start and end in U. Consider the energy function defined on $M_N^I = M_{U\times U}^I$ by

$$E(\gamma) = \frac{1}{2}\int_0^1 \|\gamma'(t)\|^2 dt.$$

Since U^{m-k} can be embedded in $M_{U\times U}^I$ as the set of point paths, we have that $E^{-1}(0) = U^{n-k}$. We will use Morse Theory to show that the space $M_{U\times U}^I$ has the homotopy type of a CW-complex which consists of cells of dimension $\geq m-2k+1$ attached to the 0−skeleton U^{m-k}.

By the first variational formula for the energy of paths, we have that the critical points of E are precisely the geodesics of M that start and end perpendicular to U; i.e. $(U \times U)$-geodesics. Let $\gamma : [0,1] \longrightarrow M$ be one of such geodesics. We will use the same idea as in the proof of Frankel's theorem, to compute a lower bound for the index of $\gamma(t)$. Let $W_1 = T_{\gamma(0)}U \subset T_{\gamma(0)}M$ be the tangent space to U at the point $\gamma(0)$ and let $W_2 \subset T_{\gamma(0)}M$ be the linear subspace obtained by parallel transporting $T_{\gamma(1)}U$ along $\gamma(t)$ at the point $\gamma(0)$. Since $T_{\gamma(1)}U$ is orthogonal to γ

at $\gamma(1)$, then W_2 is also orthogonal to γ at $\gamma(0)$. Then

$$\dim(W_1 \cap W_2) \geq \dim W_1 + \dim W_2 - (\dim M - 1) = 2(m-k) - (m-1) = m - 2k + 1.$$

Then, there exist $m-2k+1$ parallel fields $V(t)$ along $\gamma(t)$ such that $V(0) \in T_{\gamma(0)}U$ and $V(1) \in T_{\gamma(1)}U$. Similarly as in the proof of Frankel's theorem, for every parallel field $V(t)$ we can construct a variation of γ with variational field $V(t)$ in which the transversal curves are geodesics, so using the second variational formula for the energy and the fact that $secM > 0$, we have that $\frac{d^2E}{ds^2}(0) < 0$. Then, the index of γ is $\geq m - 2k + 1$.

Now using Morse Theory, see [8] section 22, we have that there is a Morse function E' that is C^∞ close to E, such that $E' = E$ on a neighborhood of $U = E^{-1}(0)$, and has index $\geq m - 2k + 1$ on any geodesic $\gamma \in M^I_{U \times U} \backslash U$. Then $\pi_r(M^I_{U\times U}, U) = 0$ for $0 < r \leq m - 2k$; then by the short exact homotopy sequence of a pair, it follows that the inclusion map $U \hookrightarrow M^I_{U\times U}$ is $(m-2k)$-connected.

Let D^i be the closed disk with boundary ∂D^i. Since any map $(D^i, \partial D^i) \longrightarrow (M, U)$ induces a map $(D^{i-1}, \partial D^{i-1}) \longrightarrow (M^I_{U\times U}, U)$, and viceversa, we have that $\pi_i(M, U) \cong \pi_{i-1}(M^I_{U\times U}, U)$. So, $\pi_i(M, U) = 0$ for $i = 1, ..., m - 2k + 1$. Using the short exact sequence for a pair, we have that the inclusion map $U \hookrightarrow M$ is $(m - 2k + 1)$-connected.

$\square$

3.3 Optimal General Index Theorem

In this section, we are going to formulate and prove an optimal general index theorem for a lower bound of the index of non-trivial N-geodesics. This theorem is going to put together Frankel, Wilking and Weinstein-Synge's theorems.

Observe that we can identify M_N^I, the space of curves γ in M that start in the first coordinate of N and end in the second coordinate of N (i.e. $(\gamma(0), \gamma(1)) \in N$), with $(M \times M)_{N\times\triangle}^{I'}$, where $I' = [0, \frac{1}{2}]$, the space of curves in $M \times M$ that start in N and end in $\triangle$, in the following way:

$$\begin{aligned} M_N^I \quad &\longrightarrow (M \times M)_{N\times\triangle}^{I'} \\ \gamma(t), \quad t \in I &\longmapsto \hat{\gamma}(t) = (\gamma_1(t), \gamma_2(t)), \quad t \in I', \end{aligned}$$

where $\gamma_1(t) = \gamma(t)$ and $\gamma_2(t) = \gamma(1-t)$, $t \in I'$. This is well-defined, since $\hat{\gamma}(0) = (\gamma_1(0), \gamma_2(0)) = (\gamma(0), \gamma(1)) \in N$ and if we denote $\gamma(1/2) = p$, we have that $\gamma_1(\frac{1}{2}) = p = \gamma_2(\frac{1}{2})$, so $\hat{\gamma}(\frac{1}{2}) = (\gamma_1(\frac{1}{2}), \gamma_2(\frac{1}{2})) = (p, p) \in \triangle$.

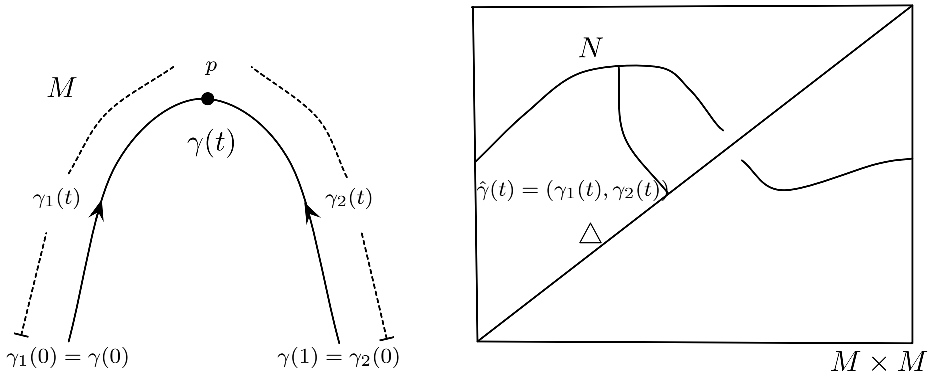

Figure 3.1: Identification $M_N^I \longrightarrow (M \times M)_{N\times\triangle}^{I'}$

The identification in the other direction is given by

$$\begin{aligned} (M \times M)_{N\times\triangle}^{I'} \quad &\longrightarrow M_N^I \\ \hat{\gamma}(t) = (\gamma_1(t), \gamma_2(t)) \quad t \in I' &\longmapsto \gamma(t), \quad t \in I, \end{aligned}$$

where $\gamma(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, \frac{1}{2}], \\ \gamma_2(1-t) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases}$

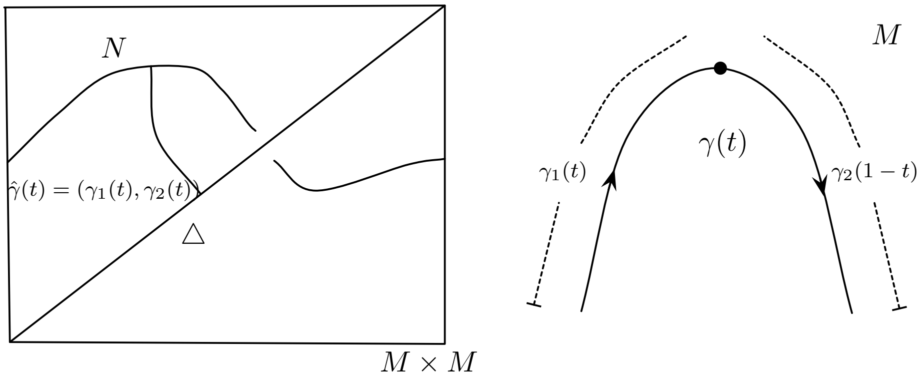

Figure 3.2: Identification $M_N^I \longrightarrow (M \times M)_{N \times \triangle}^{I'}$

Remark 8. *N-geodesics are in 1-1 correspondence with $(N \times \triangle)$-geodesics under the identification given above. This can be verified either by checking that $E|'_{M_N^I}(0) = E|'_{(M \times M)_{N\times\triangle}^I}(0)$ or by using directly the definition of a curve being an N-geodesic and $(N \times \triangle)$-geodesic, respectively and proving inclusion of sets.*

Let $\gamma \in M_N^I$ be a non-trivial N-geodesic. Let $\hat{\gamma} \in (M \times M)_{N \times \triangle}^{I'}$ be the corresponding $(N \times \triangle)$-geodesic. That is,

$$\hat{\gamma}(t) = (\gamma_1(t), \gamma_2(t)) = (\gamma(t), \gamma(1-t)) \quad t \in I'.$$

Then since $\hat{\gamma}$ is an $(N \times \triangle)$-geodesic, we have by the remark (5) that γ is a geodesic that starts orthogonal to N and ends orthogonal to $\triangle$.

Let $\hat{P}(t)$ be a vector field along $\hat{\gamma}$ such that $\hat{P}(0) \in T_{\hat{\gamma}(0)}N$ and $\hat{P}(1) \in T_{\hat{\gamma}(1)}\triangle$; that is, $\hat{P} \in T_{\hat{\gamma}}(M \times M)_{N \times \triangle}^{I'}$. Then $\hat{P}$ corresponds to a vector field P along γ given by

$$P(t) = \begin{cases} P_1(t) & \text{if } t \in [0, \frac{1}{2}], \\ P_2(1-t) & \text{if } t \in [\frac{1}{2}, 1] \end{cases}$$

such that $(P(0), P(1)) \in T_{(\gamma(0),\gamma(1))}N$; i.e. $P \in T_\gamma M_N^I$.

Indeed, $P(t)$ is well-defined since the condition

$$\hat{P}(1/2) = (P_1(1/2), P_2(1/2)) \in T_{\hat{\gamma}(\frac{1}{2})}\triangle = T_{(\gamma(\frac{1}{2}),\gamma(\frac{1}{2}))}\triangle$$

means that $P_1(\frac{1}{2}) = P_2(\frac{1}{2})$.

Since $\hat{P}(0) = (P_1(0), P_2(0)) = (P(0), P(1))$ and $\hat{\gamma}(0) = (\gamma_1(0), \gamma_2(0)) = (\gamma(0), \gamma(1))$, then the condition $\hat{P}(0) \in T_{\hat{\gamma}(0)}N$ is a different way to write $(P(0), P(1)) \in T_{(\gamma(0),\gamma(1))}N$.

Therefore, $\hat{P} \in T_{\hat{\gamma}}(M \times M)^{I'}_{N\times\triangle}$ if and only if $P \in T_\gamma M_N^I$.

Moreover, it is clear that $\hat{P} \in T_{\hat{\gamma}}(M \times M)^{I'}_{N\times\triangle}$ is a parallel vector field along $\hat{\gamma}$ if and only if $P \in T_\gamma M_N^I$ is a parallel vector field along γ.

Now, let us compute a lower bound for the index of a non-trivial $N - \mathit{geodesic}$ (which is the same that the index of the corresponding non-trivial $(N \times \triangle)$-geodesic.)

Using the same idea as in the classical proof of Frankel's theorem the number of parallel fields $\hat{P}$ along a non-trivial $(N \times \triangle)$-geodesic $\hat{\gamma}$ which are orthogonal to the velocity vector field of the curve $\hat{\gamma}$ is at least

$$\dim N + \dim \triangle - (\dim(M \times M) - 1) = n + m - (2m - 1) = n - m + 1.$$

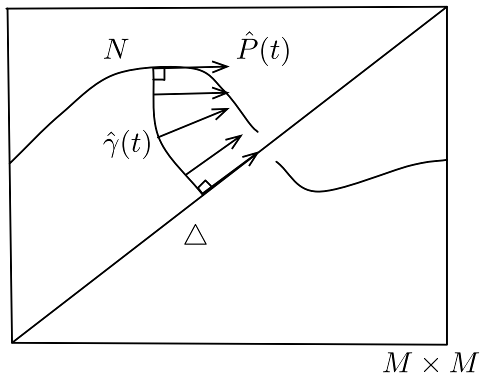

Figure 3.3: Parallel vector fields $\hat{P}$ along $\hat{\gamma}$
such that $\hat{P} \perp \hat{\gamma}'$

Note that the condition $\hat{P} \perp \hat{\gamma}'$ just means for P and γ' that

$$
\begin{aligned}
0 &= \langle (P_1(t), P_2(t)), (\gamma_1'(t), \gamma_2'(t)) \rangle_{M\times M} \quad \forall\, t \in I' \\
&= \langle (P(t), P(1-t)), (\gamma'(t), -\gamma'(1-t)) \rangle_{M\times M} \quad \forall\, t \in I' \\
&= \langle (P(t), \gamma'(t)) \rangle_M - \langle P(1-t), \gamma'(1-t) \rangle_M \quad \forall t \in I'
\end{aligned}
$$

which happens if and only if

$$\langle (P(t), \gamma'(t)) \rangle_M = \langle P(1-t), \gamma'(1-t) \rangle_M \quad \forall t \in I'. \tag{3.2}$$

So, there are at least $n - m + 1$ candidates $P(t)$ in which the Hessian of the energy function $E|_{M_N^I}$ is going to be negative-definite.

Let $\mathcal{P} = \{P \in T_\gamma M_N^I : P$ is parallel along γ and satisfies (3.2)$\}$. Then

$$\dim \mathcal{P} \geq n - m + 1.$$

So we get the following

Theorem 3.3.1 (Lower bound for the index of non-trivial N-geodesics)**.** *Let M^m be a complete, connected Riemannian manifold that has positive sectional curvature. Let $N^n \subset M^m \times M^m$ be a closed totally geodesic submanifold of $M \times M$. Then, the index of a non-trivial N-geodesic γ is at least*

(1) $n - m$ if $\gamma' \in \mathcal{P}$.

(2) $n - m + 1$ if $\gamma' \notin \mathcal{P}$.

Proof. Suppose $W \in \mathcal{P}$ and $W \notin \operatorname{span}(\gamma')$, then by the second variational formula of energy we get

$$\begin{aligned} E_{**}(W, W) = & \left\langle \frac{D}{ds} W(1), \gamma'(1) \right\rangle + \left\langle W(1), \frac{D}{dt} W(1) \right\rangle \\ & - \left\langle \frac{D}{ds} W(0), \gamma'(0) \right\rangle - \left\langle W(0), \frac{D}{dt} W(0) \right\rangle \\ & - \int_0^1 \left\langle W(t), \frac{D^2}{dt^2} W(t) + R(\gamma'(t), W(t))\gamma'(t) \right\rangle dt \\ = & - \int_0^1 \langle W(t), R(\gamma'(t), W(t))\gamma'(t) \rangle \, dt \\ = & - \int_0^1 sec(M) dt < 0, \end{aligned}$$

where we have used in the second equality the fact that $W(t)$ is parallel along γ (i.e. $\frac{D}{dt} W(t) = 0$ for all $t \in I$) and also we have performed a variation $h(s,t)$ with variational vector field $W(t)$ in which the transversal curves $h(s, 0)$ and $h(s, 1)$ are geodesics, so $\frac{D}{ds} W(1) = 0 = \frac{D}{ds} W(0)$ (this last step can be done since $N \subset M \times M$ is totally geodesic).

Now, suppose $\gamma' \in \mathcal{P}$. Then, by the same facts as in the above formula, the first four terms in the second variational formula of energy are zero, then we have that

$$E_{**}(\gamma', \gamma') = - \int_0^1 \langle \gamma'(t), R(\gamma'(t), \gamma'(t))\gamma'(t) \rangle \, dt = 0,$$

since $R(\gamma'(t), \gamma'(t))\gamma'(t) = 0$ for all $t \in I$.

Therefore, the index of a non-trivial $N - geodesic$ γ is at least $n - m$ if $\gamma' \in \mathcal{P}$ and if $\gamma' \notin \mathcal{P}$ then the index is at least $n - m + 1$. $\square$

Remark 9. *Note that since γ is a geodesic, then γ' is parallel and satisfies trivially the condition (3.2) above. Then $\gamma' \in \mathcal{P}$ whether or not*

$$\gamma' \in T_\gamma M_N^I = \{X_\gamma \in T_\gamma C^0(I, M) : (X_\gamma(0), X_\gamma(1)) \in T_{(\gamma(0),\gamma(1))}N\},$$

that is, whether or not $(\gamma'(0), \gamma'(1)) \in T_{(\gamma(0),\gamma(1))}N$.

So, in the case $N = U^r \times V^s$ with $U^r, V^s \subset M$ compact totally geodesic submanifolds of M^m, we obtain using the theorem above that the index of non-trivial N-geodesics is at least $r + s - m + 1$, since in this case $(\gamma'(0), \gamma'(1)) \notin T_{(\gamma(0),\gamma(1))}N$ (since $(\gamma'(0), \gamma'(1)) \in (T_{(\gamma(0),\gamma(1))}N)^\perp$). So, as in Frankel's theorem, when $r + s \geq m$ then the index of γ is at least 1.

However, in the case $N = Graph(f)$, where $f : M \to M$ is an isometry, we get using the above theorem that the index of a non-trivial N-geodesic is at least $n - m = m - m = 0$, since in this case $\gamma' \in \mathcal{P}$ (since $(\gamma'(0), f_*(\gamma'(0))) = (\gamma'(0), \gamma'(1)) \in T_{(\gamma(0),\gamma(1))}N$). So, in this case we do not get anything we already knew. So, if we assume that M^m is compact, the lower bound for a non-trivial N-geodesic we obtain using Weinstein-Synge's theorem (i.e. at least 1) is stronger than the lower bound we obtain using the above theorem, the reason is that the theorem for the lower bound for non-trivial N-geodesics neglects the hypothesis about f being orientation-preserving (reversing) in case m is even (odd). Precisely, examples (3) and (4) illustrates this fact.

Notice that the bounds in this theorem are sharp: examples (3) and (4) satisfy part 1 of the theorem whereas Frankel's theorem satisfies part 2.

It is surprising that although Synge's theorem was the first one to be proved, it is actually more difficult to obtain than the other theorems (Frankel, Wilking) using the computation of the index on non-trivial N-geodesics.

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