# Rationalizing Momentum Interactions

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# Rationalizing Momentum Interactions

Momentum profitability concentrates in high information uncertainty and high credit risk firms and is virtually nonexistent otherwise. This paper rationalizes such momentum interactions in equilibrium asset pricing. In our paradigm, dividend growth is mean reverting, expected dividend growth is persistent, the representative agent is endowed with stochastic differential utility of Duffie and Epstein (1992), and leverage, which proxies for credit risk, is modeled based on the Abel's (1999) formulation. Using reasonable risk aversion levels we are able to produce the observational momentum effects. In particular, momentum profitability is especially large in the interaction between high levered and risky cash flow firms. It rapidly deteriorates and ultimately disappears as leverage or cash flow risk diminishes.

# 1 Introduction

Momentum effects in stock returns are robust. Fama and French (1996) show that momentum is the only deviation from the CAPM unexplained by the Fama and French (1993) model. Schwert (2003) demonstrates that profit opportunities, such as the size and value effects as well as equity premium predictability, typically disappear, reverse, or attenuate following their discovery. Momentum is an exception. Specifically, Jegadeesh and Titman (2001, 2002) document momentum profitability in the period after its discovery in Jegadeesh and Titman (1993). Korajczyk and Sadka (2004) find that momentum survives trading costs, whereas Avramov, Chordia, and Goyal (2006) show that the profitability of the other past-return anomaly, namely reversal, disappears in the presence of trading costs. Fama and French (2007) argue that momentum is among the few robust anomalies. Momentum robustness has generated a plethora of behavioral and rational explanations.[1]

Empirical work also uncovers momentum interactions. In particular, Zhang (2006) finds that momentum concentrates in high information uncertainty stocks, i.e., stocks with high return volatility, high cash flow volatility, small market capitalization, or high analysts' earnings forecast dispersion, and points to behavioral interpretations. Avramov, Chordia, Jostova, and Philipov (2007) document that momentum prevails only among high credit risk stocks and is nonexistent otherwise and, moreover, the credit risk effect dominates the information uncertainty effect. Indeed, the momentum-credit risk relation could point to

[1]Behavioral: Barberis, Shleifer, and Vishny (1998), Daniel, Hirshleifer, and Subrahmanyam (1998), and Hon, Lim, and Stein (2000). Rational: Berk, Green, and Naik (1999) and Johnson (2002). Moskowitz and Grinblatt (1999) argue that industry momentum explains momentum in individual stocks. Avramov and Chordia (2006) show that while momentum is unexplained by risk based asset pricing models, removing business cycle effects from model mispricing completely eliminates momentum effects. These last two studies indicate that momentum could be attributable to missing factors in rational asset pricing models.

rational interpretations of momentum. However, this is an empirical finding that has not been formalized in an equilibrium model.[2] Indeed, thus far there has not been any attempt, to our knowledge, to theoretically rationalize or even behavioralize momentum interactions.

This paper studies momentum interactions from an equilibrium perspective and ultimately shows that the concentration of momentum in high information uncertainty as well as high credit risk stocks is perfectly consistent with rational asset pricing. In particular, we develop a representative agent general equilibrium paradigm extending the Lucas (1978) economy. Here, dividend growth is mean reverting, expected dividend growth is persistent, and the representative agent is endowed with the recursive utility form of Duffie and Epstein (1992), which is the continuous time analog of the Epstein and Zin (1989) and Weil (1989) preferences. Moreover, based on the novel formulation of Abel (1999), our equilibrium clearing condition requires that the stream of dividends serve for both consumption and debt repayment. The leverage role in asset pricing goes back at least to Merton (1974).

Our model is fairly general from the perspective of both preferences and dynamics. The utility specification breaks the tight association between the elasticity of intertemporal substitution and the risk aversion measure, which are reciprocals of each other under power preferences. Moreover, our consumption dynamics closely follows the high economic growth along with small consumption growth documented in the US. Collectively, our model allows one to match key asset pricing regularities based on reasonable risk aversion measures.

In essence, we extend the theoretical work of Johnson (2002) along two important di-

[2]Another interaction that we do not explore here is the volume effect on momentum profitability documented by Lee and Swaminathan (2000).

mensions. First, whereas Johnson does not explicitly formulate investor preferences, here the dynamics of the underlying economic fundamentals is tied down to stochastic differential utility of the representative agent. Moreover, Johnson focuses on understanding momentum effects, whereas we attempt to rationalize the concentration of momentum profitability among stocks with some particular styles. That is, the question of interest is how momentum payoffs vary in equilibrium with financial leverage, which proxies for credit risk, as well as information uncertainty measures such as stock return volatility and cash flow volatility.[3]

Based on simulations, we are able to generate high enough equity premium using reasonable risk aversion measures, consumption dynamics, and leverage. More importantly, we are able to rationalize momentum interactions, as we generate strong momentum effects for the interaction between high leverage and risky cash flow firms. We demonstrate that momentum effects deteriorate and ultimately disappear as leverage or cash flow risk diminishes. More specifically, the correlation between observed and expected return is positive and *monotonically* increases with leverage. The monotonic relation holds at high, medium, and low levels of expected growth rate volatility, at high and low autocorrelation of expected growth rate, and at the entire range of risk aversion measures (5 to 10) analyzed.

Next, we show that the expected return spread between the highest and lowest past year cumulative return portfolios increases with leverage. For example, when operating cash flows are highly volatile, the expected return spread is 11.49% for high leverage and only 0.27% for low leverage stocks. On the other hand, the overall spreads are small when

[3]The leverage proxy for credit risk is sound. For one, there is a strong correlation (the time series mean of the cross sectional Spearman Rank Correlation is 0.34) between S&P credit rating and leverage. Moreover, leverage is a crucial determinant in modeling default risk.

either the volatility of expected growth in cash flows is small or the expected growth in cash flows is not highly persistent. Indeed, while leverage is crucial, risk and persistence of cash flow growth are also important determinants of momentum effects in the cross section of returns. On the other hand, momentum is only mildly related to investor's risk aversion.

To summarize, it has been documented that momentum interacts with firm-level information uncertainty measures and credit conditions. Our collective evidence shows that equilibrium momentum indeed concentrates in the interaction between risky cash flows and high credit risk firms. In the presence of recursive preferences, financial leverage, and persistent dividend growth, one can match equity premia, riskfree rate, return predictability, and especially the observational momentum profitability at reasonable risk-aversion levels.

The paper proceeds as follows. Section 2 describes the economic setup including investor preferences and the dynamics of the underlying economic fundamentals. Section 3 derives the interaction of momentum with credit risk and information uncertainty. Section 4 reports the simulation results coming up from the theoretical formulations. Section 5 concludes and provides suggestions for future work. Technical details are in the appendix.

# 2 The economic setup

## 2.1 Preferences and Dynamics

In formulating investor preferences, we depart from the regular power utility specification. Theoretically, power preferences put a heavy restriction on elasticity of inter-temporal substitution (EIS) and risk aversion – they are reciprocals of each other – even when risk aversion and EIS are distinct economic quantities. EIS is about a deterministic consumption path as it measures the willingness to exchange consumption today for consumption tomorrow conditioned on a current riskfree interest rate, whereas risk aversion is about preference over a random quantity. Empirically, the power utility restriction gives rise to the equity premium and riskfree rate puzzles as well as the failure of the consumption based asset pricing model to explain the cross section dispersion in average stock returns.

Instead, we employ stochastic differential utility (SDU) of Duffie and Epstein (1992). The SDU is the continuous time analog of the Epstein-Zin (1989) and Weil's (1990) recursive preferences, which break the tight association between risk aversion and EIS. The SDU is identified by a pair of functions $(f^*, A(J))$, called an aggregator, where $A(J)$ is local risk-aversion and $f^*$ represents the relative preference between immediate consumption and the certainty equivalent of utility derived from future consumption. An ordinally equivalent representation of the SDU is given by the normalized aggregator $(f, 0)$, which is formulated as

$$f(C, J) = \frac{\beta(1-\gamma)J}{1-\frac{1}{\psi}} \left[ C^{1-\frac{1}{\psi}} \left((1-\gamma)J\right)^{\frac{\frac{1}{\psi}-1}{1-\gamma}} - 1 \right]. \tag{1}$$

In equation (1), $C$ denotes the current consumption, $J$ is the continuation utility (or the value function) attributable to future consumption streams, $\psi$ is the EIS, $\beta$ is the discount rate standing for the time preference, and $\gamma$ is the relative risk-aversion parameter. Under this convention, the time $t$ value function of an agent can be written as

$$J_t = E_t \int_t^{\infty} f(c_s, J_s) ds. \tag{2}$$

We derive below an explicit solution for $J_t$ assuming that $\psi = 1$. This assumption is innocuous. In particular, Bansal and Yaron (2004) and Ai (2007) show that $\psi = 1$ is the point of equivalence between wealth and substitution effects, but preferences over risk are still determined by risk-aversion. That is, if $\psi < 1$ ($\psi > 1$), the income (substitution) effect dominates. In other words, high growth rate can lead an agent to consume more (income effect) or invest more (substitution effect). Which course of action is undertaken depends upon the $\psi$ parameter. Indeed, EIS primarily deals with determining the riskfree rate, which is not at the core of our study. The normalized aggregator based on $\psi = 1$ is the limit of (1) taking the form

$$f(C, J) = \beta(1-\gamma)J \left[\log C - \frac{\log(1-\gamma)J}{1-\gamma}\right]. \tag{3}$$

Next, as in Johnson (2002), we assume that the dividend growth follows a geometric path with stochastic expected growth rate. Specifically, the joint system of the observed dividend growth and the unobserved expected dividend growth are formulated as

$$\frac{dD_t}{D_t} = X_t dt + \sigma_D dW_1, \tag{4}$$

$$dX_t = \kappa(\bar{X} - X_t)dt + \sigma_x dW_2, \tag{5}$$

where $X_t$ is the expected dividend growth, $\kappa$ stands for the speed of mean reversion, $\bar{X}$ is the long run mean of $X_t$, $\sigma_D$ is the volatility of dividend growth, $\sigma_x$ is the volatility of expected dividend growth, and the correlation between the two Brownian motions is $\rho$.

To account for leverage, we build on the novel formulation of Abel (1999). Abel is able to generate, in a fairly simple setting, low variability of the riskfree rate along with a large equity premium, both of which are patterns documented in the US economy in the post-war period. In particular, we assume that the equilibrium consumption is a portion of dividend

$$C = D^{\lambda}, \tag{6}$$

while the remainder of the dividend stream is distributed as adjustment cost or debt payment in the economy. The no-leverage case $\lambda = 1$ depicts an economy with no adjustment cost wherein the agent consumes the full dividend streams. In the $\lambda < 1$ case, which is at the core of our analysis, the rest of dividends go towards debt payment. Then stocks in this economy are residual claims on the consumption stream net debt payments.

In the presence of leverage, the consumption growth dynamics takes the form

$$\frac{dC}{C} = \mu_C(X_t)dt + \sigma_C dW_1, \tag{7}$$

where

$$\mu_C(X_t) = \lambda \left[ X_t + \frac{1}{2}(\lambda - 1)\sigma_D^2 \right] \tag{8}$$

$$\sigma_C = \lambda \sigma_D, \tag{9}$$

suggesting that expected consumption growth is slower than expected dividend growth as long as $\lambda < 1$. This is consistent with the US economy which demonstrates slow consumption growth along with relatively fast economic growth. Notice also that in a levered economy, the volatility of consumption $\lambda\sigma_D$ is smaller than the volatility of economic growth.

The utility process $J$ satisfies the Bellman equation with respect to equilibrium consumption

$$\mathcal{D}\mathcal{J}(C, X, t) + f(C, J) = 0 \tag{10}$$

where $\mathcal{D}\mathcal{J}$ is the differential operator applied to $J$ with respect to $\{C, X, t\}$ with the boundary condition $J(C, x, T) = 0$. In the analysis that follows, we are interested in the equilibrium as $T \to \infty$. Thus, we drop the explicit time dependence assuming that the agent is infinitely long-lived and has reached the equilibrium over time. We can formulate an exact solution to the value function which is stated in the following proposition.

**Proposition 1.** *An exact solution to the differential equation in (10) is given by*

$$J(C_t, X_t) = \frac{C_t^{1-\gamma}}{1-\gamma} \exp\left(u_1 X_t + u_2\right) \tag{11}$$

*where*

$$u_1 = \frac{(1-\gamma)\lambda}{\kappa + \beta} \tag{12}$$

$$u_2 = \frac{(1-\gamma)\lambda}{\beta}\left[\frac{(\lambda - 1 - \lambda\gamma)\sigma_D^2}{2} + \frac{\kappa\bar{X}}{\kappa+\beta} + \frac{(1-\gamma)\lambda}{\kappa+\beta}\left[\frac{\sigma_x^2}{2(\kappa+\beta)} + \sigma_D\sigma_x\rho\right]\right]. \tag{13}$$

*Proof: see the appendix.*

Unlike in power preferences, SDU incorporates the agent's value function in the current utility. Thus, expected growth rate enters into the agent's utility through the value function $J$. Indeed, expected growth rate has important implications for future consumption streams. Higher expected growth rate indicates higher expected future consumption and hence higher future utility which is reflected through higher current value function. Thus, $J_X$ is positive. On the other hand, $J_{XX}$ is negative suggesting that $J$ is concave in $X$ or $J_X$ increases in $X$ in diminishing rates. We also run some simulations to learn the impact of the value function on the current utility. The current utility is increasing (decreasing) in growth rates under low (high) leverage and low (high) uncertainty of future growth rates.

Moreover, the autocorrelation of $X$ is important for understanding the expected growth rate effect on future utility. If expected growth rate is highly persistent (low $\kappa$) then high $X_t$ implies high future expected growth rate, which, in turn, implies that the agent expects to consume more in the future. Thus $J_X$ increases with expected growth rate persistence. Finally, $J_X$ decreases in $\beta$. In particular, if the agent discounts the future more (higher $\beta$) then the impact of expected growth rate on the value function (future utility) diminishes.

The next section derives the pricing kernel dynamics, the asset return dynamics, the correlation between observed realized returns and expected returns, and especially the link between leverage, information uncertainty, and stock return momentum.

# 3 Asset Pricing

Duffie and Epstein (1992) show that the pricing kernel for SDU is given by $\Lambda_t = \exp(\int_0^t f_J ds) f_C$, where $f_J$ and $f_C$ are the derivatives of $f(C, J)$ in (3) with respect to $J$ and $C$. Hence we can find the explicit pricing kernel dynamics as stated in the proposition below.

**Proposition 2.** *The pricing kernel dynamics is given by*

$$\frac{d\Lambda}{\Lambda} = -r_t^f dt - \lambda\gamma\sigma_D dW_1 + u_1\sigma_x dW_2 \tag{14}$$

*where*

$$\begin{aligned} r_t^f &= \lambda X_t + u_1\lambda\gamma\sigma_D\sigma_x\rho + \beta(u_2+1) - u_1\kappa\bar{X} - \frac{1}{2}\lambda\gamma\sigma_D^2(\lambda\gamma+1) - \frac{1}{2}\sigma_x^2 u_1^2 \\ &= \mu_C(X_t) + \beta - \sigma_C^2\gamma + \frac{(1-\gamma)\lambda}{\kappa+\beta}\sigma_C\sigma_x\rho. \end{aligned} \tag{15}$$

*Proof: see the appendix.*

Observe from equation (14) that leverage is an important determinant of both the drift and diffusion of the pricing kernel dynamics. Moreover, our equilibrium riskfree rate formulated in (15) has two attractive features compared to its power utility counterpart. For comparison, the corresponding riskfree rate for power utility is $r_t^f = \gamma\mu_C(X_t) + \beta - \frac{\gamma(\gamma+1)\sigma_C^2}{2}$.

First, the riskfree rate based on the power utility is highly sensitive to expected consumption growth. In particular, a one-percent increase in the growth rate is followed by $\gamma$-percent increase in the riskfree rate - a nuisance at the heart of the riskfree rate puzzle. Indeed, the riskfree rate determines the intertemporal substitution effect between deterministic consumption streams, whereas risk-aversion determines the agent's preference over risky bets. Thus, risk aversion should not exert such a considerable influence on the consumption growth riskfree rate sensitivity. In the SDU case, there is a much more realistic one-to-one relationship between expected consumption growth and riskfree rate.

Second, for realistic values of $\mu_C$ and $\sigma_C$, the power utility riskfree rate is increasing under believable values of risk-aversion - which again misconstrues the nature of the riskfree rate. In other words, higher risk-aversion is required to match the equity-premia, it also correspondingly increases the riskfree rate. In our case, as in Duffie and Epstein (1992), the riskfree rate is strictly decreasing in risk-aversion. Thus, if indeed higher risk-aversion is needed to match the equity premia, it does not pose any challenge to match low riskfree rate. It should also be noted that the riskfree rate does vary through time with $X_t$.

We next establish the equilibrium dividend price ratio and the return dynamics.

**Proposition 3.** *The equilibrium price-dividend ratio* $\frac{P_t}{D_t}$*, denoted by* $G(X_t)$*, is*

$$G(X_t) = \int_0^\infty \exp(P_1(\tau)X_t + P_2(\tau))d\tau \tag{16}$$

*where* $P_1$ *and* $P_2$ *are the solutions of a system of ODEs given in the appendix. Furthermore, the excess return dynamics is given by*

$$\begin{aligned} dR_t &= \mu_t^R dt + \sigma_D dW_1 + \frac{G_X}{G}\sigma_x dW_2 & (17) \\ d\mu_t^R &= (\cdot)dt + \left(\frac{G_X}{G}\right)_X (\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)\sigma_x dW_2. & (18) \end{aligned}$$

It immediately follows that the instantaneous covariance between realized and expected return is given by

$$E_t\left[\left(dR_t - \mu_t^R dt\right)\left(d\mu_t^R - (\cdot)dt\right)\right] = \left(\frac{G_X}{G}\right)_X \left(\sigma_D\sigma_x\rho + \frac{G_X}{G}\sigma_X^2\right)(\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2). \tag{19}$$

We can now go further and formulate the SDEs of realized cumulative excess returns and expected excess returns for an investment horizon of $l$ periods. In particular,

$$R_{t,t+l} = R_t + \int_t^{t+l} \mu_s^R ds + \int_t^{t+l} \sigma_D dW_1 + \int_t^{t+l} \frac{G_X}{G} \sigma_x dW_2 \tag{20}$$

$$\mu_{t,t+l}^R = \mu_t^R + \int_t^{t+l} (\cdot) ds + \int_t^{t+l} \left(\frac{G_X}{G}\right)_X (\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)\sigma_x dW_2 \tag{21}$$

As a first pass to understand momentum effects in stock returns we compute the correlation between $R_{t,t+l}$ and $\mu_{t,t+l}^R$. The covariance between realized and expected return is

$$\begin{aligned} Cov_t(R_{t,t+l}, \mu_{t,t+l}^R) &= E_t\left[(R_{t,t+l} - E_t R_{t,t+l})(\mu_{t,t+l}^R - E_t\mu_{t,t+l}^R)\right] \\ &= \sigma_x(\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2) \int_t^{t+l} \left(\frac{G_X}{G}\right)_X \left(\sigma_D\rho + \frac{G_X}{G}\sigma_x\right) ds. \end{aligned} \tag{22}$$

Then, the variances are computed as

$$V_t(\mu_{t+l}^R) = (\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)^2\sigma_x^2 \int_t^{t+l} \left(\left(\frac{G_X}{G}\right)_X\right)^2 ds \tag{23}$$

$$V_t(R_{t+l}) = \int_t^{t+l} \left(\sigma_D^2 + \left(\frac{G_X}{G}\sigma_x\right)^2 + 2\frac{G_X}{G}\sigma_D\sigma_x\rho\right) ds. \tag{24}$$

Hence, the correlation is

$$\Gamma(l) = \frac{\int_t^{t+l} \left(\frac{G_X}{G}\right)_X (\sigma_D\rho + \frac{G_X}{G}\sigma_x) ds}{\sqrt{\int_t^{t+l} \left(\left(\frac{G_X}{G}\right)_X\right)^2 ds \cdot \int_t^{t+l} \left(\sigma_D^2 + \left(\frac{G_X}{G}\sigma_x\right)^2 + 2\frac{G_X}{G}\sigma_D\sigma_x\rho\right) ds}}. \tag{25}$$

The $\Gamma(l)$ function depicts the correlation between expected return and realized return up to horizon $l$. As $l \to 0$, $\Gamma(l)$ is the instantaneous correlation between realized return up until now (time 0) and expected return. As $l$ increases, it becomes the correlation between realized return over the period $0 \to l$ with the expected return at the end of that period.

The leading term on the numerator of $\Gamma(l)$ can be written as

$$\frac{GG_{XX} - (G_X)^2}{G^2} = \frac{1}{G^2}\left[\int_0^\infty \exp(\cdot)d\tau \int_0^\infty \exp(\cdot)P_1^2(\tau)d\tau - \left(\int_0^\infty \exp(\cdot)P_1(\tau)d\tau\right)^2\right] \quad (26)$$

which is positive from direct application of Cauchy-Schwartz inequality to functions $P_1(\tau)\sqrt{\exp(\cdot)}$ and $\sqrt{\exp(\cdot)}$, both of which are integrable in the domain. As we show below, the term $\frac{G_X}{G}$ is always positive for $\lambda < 1$. The autocorrelation is positive unless $\rho < 0$ *and* $|\sigma_D \rho| > \frac{G_X}{G}\sigma_x$.

## 3.1 Interpretation and implications

### 3.1.1 Consumption Growth

The equilibrium condition produces consumption growth which follows the dynamics in (7). If there is no leverage ($\lambda = 1$), then consumption growth is the same as the dividend growth and we are back to a Lucas (1978) economy setting. In the case of a slave economy ($\lambda = 0$), there is no consumption growth and consumption at every point in time is nonrandom. Of course, one can also attribute the $\lambda = 0$ case to a riskfree bond. At intermediate level of leverage ($0 < \lambda < 1$), which is the focus of our analysis below, we observe key features of post-war US data. First, the volatility of consumption growth is much smaller than the volatility of dividend growth. Furthermore, in the presence of high growth rate of dividends the consumption growth rate can still be low especially if the economy is highly levered.

### 3.1.2 Price-Dividend ratio

The business cycle effect on the $P/D$ ratio is instantly observable as

$$G_X = \frac{1-\lambda}{\kappa} \int_0^\infty \exp(P_1(\tau)X_t + P_2(\tau)) \left(1 - e^{-\kappa\tau}\right) d\tau. \tag{27}$$

Thus the $P/D$ ratio is increasing in the growth rate $X_t$ as long as $\lambda < 1$. The effect is most pronounced for low $\lambda$ (high leverage), it deteriorates as $\lambda$ grows (lower leverage), and ultimately vanishes as $\lambda$ approaches one (no leverage). It should be noted that in the power utility case $G_X$ is positive only if $\lambda\gamma < 1$. Essentially, this restriction puts an undue burden on $\gamma$ to be less than $\frac{1}{\lambda}$. Here, the condition that the $P/D$ ratio increases in the business cycle variable does not depend on risk-aversion. This appealing feature is due to the use of the stochastic differential utility. Notice also that the $\lambda < 1$ case points to a wealth effect in the economy in the presence of increasing economic growth rate. That is, if $X_t$ increases, the stock price rises relative to the dividend - and even more so relative to consumption.

The equilibrium $D/P$ ratio is shown in Figure 1 for plausible values of the parameters underlying the stock return dynamics. The effect of risk-aversion is straightforward. Higher risk-aversion increases expected return thus reducing the current stock price or increasing the $D/P$ ratio. The effect of leverage on the $D/P$ ratio, however, is not straightforward.

Figure 1 shows that leverage and the $D/P$ ratio are nonlinearly related. The $D/P$ ratio first increases as $\lambda$ grows and then decreases as $\lambda$ approaches one. This pattern is attributed to the interaction of leverage with the growth rate $X_t$. In the no leverage ($\lambda = 1$) case the $P_1(\tau)$ in (16) is 0, suggesting that leverage and growth rate do not interact. At high growth rates with small leverage (high $\lambda$), agents encounter high consumption growth and thus low marginal utility of consumption. Agents would therefore invest more and consequently the stock price would rise, triggering low $D/P$ ratio. It is intuitive to think that the reverse would take place at high leverage (small $\lambda$). However, at high leverage, $P_1(\tau)$ is high and it interacts strongly with the growth rate. Thus, the stock price rises with high leverage and high growth rate and the $D/P$ ratio again falls. In essence, when leverage is high, consumption reacts slowly in response to high growth rate and the agent invests more in anticipation of good times in the future, thus raising prices and lowering the $D/P$ ratio.

#### 3.1.3 Expected return and leverage

The expected excess stock return is given by

$$\mu_t^R = (\lambda\gamma\sigma_D^2 - u_1\sigma_D\sigma_x\rho) + G_X(\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)\frac{D}{P}. \tag{28}$$

Thus, $G_X$ – the response of the $P/D$ ratio to the economic growth rate – guides the expected return dynamics. Notice that for stocks with high $\lambda$ (low leverage) the ability of the dividend yield to forecast future returns diminishes. In particular, for $\lambda = 1$, $G_X = 0$ suggesting that all time series effects on expected stock return vanish completely. Likewise, at $\lambda = 0$, $\mu_t^R = 0$ which implies that in a fully levered economy, there is no consumption growth and the only security available is a bond with a constant stream of consumption. As such, the risk-premia vanishes and the bond yields a rate of return equal to the discount rate $\beta$. Thus, return predictability is pronounced for stocks with intermediate values of $\lambda$.

Interestingly, observe from Figure 2 that even in the absence of time series effects on expected stock return, our model can still deliver the relatively high equity premium observed in the US over the past century. In particular, the second term in the first parenthesis in (28), $u_1\sigma_D\sigma_x\rho = \frac{(1-\gamma)\lambda\sigma_D\sigma_x\rho}{\kappa+\beta}$, which is a contribution of the stochastic differential utility, provides the explanation. Small values of $\kappa$ and $\beta$ could greatly magnify this expected return component, thus producing expected return terms that match the high equity premium at a relatively low risk-aversion.

Figure 2 also displays the non-linear dependence of expected return on leverage. Clearly, at $\lambda = 0$, $\mu_t^R = 0$ and at $\lambda = 1$, excess return is constant. The time-series dependence and its interaction with leverage is provided at intermediate levels of $\lambda$. In such intermediate levels, whenever we observe high $D/P$ ratio in Figure 1, we also observe high expected return level since $G_X > 0$. The effect is magnified as risk-aversion increases, as expected. This positive relationship is empirically shown through the positive coefficients in predictive regressions as in Cochrane (2007).

# 4 Understanding momentum interactions

As noted earlier for $\lambda = 1$ or $\lambda = 0$ the excess return is zero or constant, respectively, and the momentum effect is nonexistent. We focus on the more realistic $0 < \lambda < 1$ domain where the full dynamics of the model can be interacted with leverage. Our goal is to calibrate the correlation between observed return and expected return as well as the overall momentum profitability for 42 distinct specifications of parameters underlying the return dynamics with each specification standing for one particular class of firms. Thus, our model could ultimately deliver prominent predictions on the cross section of such firms, and it does, as we show below.

The parameter settings are described in Table 1. In the simulation exercises the seven settings described in Table 1 will be interacted with six leverage levels - thus overall we consider 42 distinct cases. For all the seven [A-G] settings, we assume that the discount rate for our infinitely lived agent is small ($\beta = 0.01$) and the expected dividend growth is highly autoregressive ($\kappa = 0.05$) with the long run mean growth rate ($\bar{X}$) at 5%. Based on the relationship in (25) and (27), the claim going forward is that high variance in expected dividend growth rate generates the momentum effect that is more pronounced for high levered firms. High variance could emerge due to both high volatility in expected growth rate innovation (high $\sigma_x$) and highly persistent expected dividend growth (low $\kappa$).

In Table 1, settings A and B display high variance of dividend growth (high $\sigma_D$ and $\sigma_x$), C and D display low, and E exhibits medium. The distinction between A and B (C and D) is higher risk-aversion parameter for B (D). Both E and F exhibit the same volatility of dividend growth ($\sigma_D$) but F displays lower expected growth rate volatility ($\sigma_x$). Setting G takes higher $\kappa$, which means less persistent expected dividend growth rate, and at the same time holds $\sigma_D$ and $\sigma_x$ at a moderate level.

The next section describes the simulations made to assess the impact of leverage, volatility of dividend growth, volatility of expected dividend growth, risk aversion, and persistence of expected dividend growth on momentum effects.

## 4.1 Correlation between realized and expected returns

We first explore the correlation between realized and expected returns. The system of realized and expected returns is simulated forward (starting from $X_0 = \bar{X}$) for investment horizons of 3, 6, 9, and 12 months and then the correlation between observed investment return and expected return is computed. Table 2 shows the correlation for all eight settings, each of which is interacted with six levels of leverage ranging between 0.15 (high leverage) and 0.90 (low leverage). The figures represent the correlation formulated in (25) for investment horizons of 3-12 months. The unchanging correlation pattern represents the stability of correlation over time. Johnson (2002) depicts similar correlation structure.

The correlation between expected excess return and cumulative excess return *monotonically* increases with leverage. The monotonic relationship holds at high, medium, and low levels of growth rate volatility, at high and low autocorrelation of expected growth rate, and at the entire range of risk aversion measures considered here. While all cases examined exhibit positive correlation between observed and expected returns, we show below that momentum profitability prevails only in a few of the settings and is nonexistent in others.

## 4.2 Instantaneous expected return spreads

Table 3 exhibits expected excess return over a one-year period in which past cumulative returns have been classified into ten deciles, with column 1 (10) pertaining to the lowest (highest) observed returns. There are several insights emerging about the role that leverage, dividend growth volatility, expected dividend growth volatility, risk aversion, and persistence play in generating momentum effects. Here, momentum profitability is defined as the expected return spread between the highest and lowest observed return portfolios.

**Momentum and leverage.** Momentum profitability monotonically increases with leverage regardless of the case considered, with settings A, B, and E displaying large and economically meaningful expected return spreads. Focusing on A, the expected return spread is 11.49% [26.51%-15.02%] per year for $\lambda = 0.15$ while it is only 0.27% for $\lambda = 0.9$. Moving to B, the expected return spread is 13.16% for $\lambda = 0.15$ and is 1.71% for $\lambda = 0.9$. The corresponding figures for E are 9.87% and 0.26%. On the other hand, the overall spreads are small for the C, D, and G settings. This indicates that while leverage is crucial there are some other important determinants of momentum effects.

Before we move on, it should be noted that the expected returns reported in Table 3 might seem quite large, especially the ones indicating strong momentum effects. However, notice that the time discount parameter $\beta$ is small (0.01) at this stage. We reexamine some of the sub-cases using higher time-discount parameter. The evidence is reported in Table 4. Indeed, increasing $\beta$ brings down expected return at every decile to more realistic levels. The interesting evidence, however, is that the expected return spread between the highest and lowest deciles is preserved. To illustrate, for case A and $\lambda = 0.25$ the spread is 7.56%, for B and $\lambda = 0.15$ is 9.96%, for B and $\lambda = 0.9$ is 0.57%, and for E and $\lambda = 0.2$ is 5.87%.

**Momentum and information uncertainty.** Settings A, B, and E display the highest expected return spreads across the ten deciles, as noted earlier. For the other settings the spreads are much lower and are often even negligible. To illustrate, the highest spread for C is 2.95% [15.01%-12.06%], 4.86% for D, 1.05% for F, and only 0.29% for G. Cases A, B, and E are all characterized by high (0.07) to moderate (0.05) expected dividend growth volatility, which, from the firm perspective, amounts to relatively high volatility cash flows.

To this point we are able to rationalize previously documented momentum interactions. In particular Zhang (2006) finds that momentum concentrates in high information uncertainty stocks and points to behavioral interpretations. Avramov, Chordia, Jostova, and Philipov (2007) document that momentum prevails only among high credit risk stocks. Whereas such momentum-credit risk interaction could point to rational interpretations, this is purely an empirical finding thus far that has not been formalized in an equilibrium model. The collective evidence here shows that equilibrium momentum effects should concentrate in the interaction of risky cash flows and highly levered firms. Interestingly, neither leverage alone nor cash flow volatility alone are sufficient to generate momentum effects.

Focusing on information uncertainty measures, a valid point to make is that the volatility of expected dividend growth ($\sigma_x$) is the primary force of momentum effects, whereas the volatility of the unexpected dividend growth ($\sigma_D$) plays a marginal role. Note in particular that cases E and F are virtually identical with the only exception being $\sigma_x = 0.05$ in E versus $\sigma_x = 0.02$ in F. Nevertheless, the expected return spreads in E are considerably higher for all leverage levels.

**Momentum and risk aversion.** Setting G features the highest risk aversion but nevertheless yields the lowest expected return spreads between the ten portfolios, ranging between 0.02% and 0.29%. The immediate takeout is that the risk aversion measure is not a key parameter in generating momentum effects. Let us also compare A versus B as well as C versus D. For cases B and D, the higher risk-aversion simply increases expected return at every level of leverage and still preserves expected return spread across the deciles. Indeed, the momentum profitability in B (D) is slightly higher than than of A (C), suggesting that risk aversion has some effect, albeit relatively small, in explaining the return spread.

**Momentum and expected growth rate persistence.** Case G is different from the previous settings in that it features the lowest autocorrelation of expected growth rate, which reduces the total variance of expected growth rate. In case G, the shocks to the system are exactly the same as in case E ($\sigma_D = 0.06$ and $\sigma_x = 0.05$). Therefore, with higher risk-aversion (10 versus 5) we can only expect the momentum effect of case E to be exacerbated, just like case B compared to case A. However, with lower $\kappa$ the expected return spread across the high and low performing portfolios is minuscule, even at high levels of leverage. Therefore, high $\kappa$, which reduces the variance of expected growth rates, exhibits no expected return differential that characterizes the momentum effect in the data. Persistence is indeed crucial in generating momentum effects and it overwhelms risk aversion.

The evidence emerging from Table 3 also suggests that even when there is strong correlation between realized and expected return at high levels of leverage in high, medium, and low levels of volatility (see Table 2), low cash flow volatility does not produce economically significant expected return spreads across the different deciles even when leverage is high.

## 4.3 Holding period return spreads

What makes the momentum effect a conundrum is the holding period profit. The strategy of buying winners and selling short losers produces 8-12% ex post payoffs according to Jegadeesh and Titman (1993). We next simulate holding period returns based on one year formation period and conventional holding periods of 3-12 months. Table 5 reports momentum profitability which is the return spread between the top and bottom past return deciles.

Consistent with the evidence reported thus far it follows that the high volatility case A, which produces high ex-ante expected excess returns, also generates high ex post holding period returns. Focusing on the one year holding period, momentum profitability is 10.35% for $\lambda = 0.15$ and is only 0.37% for $\lambda = 0.9$. The low volatility case D fails to generate high momentum profitability. For the one year holding period the momentum payoff is 3.2% for high leverage and 0.13% at low leverage. Furthermore, the moderate volatility case E generates moderate levels of investment returns ranging between 0.26% and 8.98% for the one year holding period. The evidence in Case G with the lowest autocorrelation confirms the earlier observation that holding period returns based on low autocorrelation and high volatility are small and similar in magnitude to the high autocorrelation low volatility case D (recall $\sigma_x = 0.05$ in case G while $\sigma_x = 0.03$ in case D). In summary, we confirm that momentum effects concentrate in firms with high leverage as well as highly volatile and persistent expected cash flows growth.

# 5 Conclusion

Previous work shows that momentum effects in stock returns are robust, thus invoking a plethora of behavioral and rational explanations. Previous work also uncovers momentum interactions. In particular, momentum concentrates in stocks with high return volatility, high cash flow volatility, small market capitalization, high analysts' earnings forecast dispersion, as well as high credit risk. Thus far, there has not been any attempt, to our knowledge, to theoretically rationalize or even behavioralize such momentum interactions.

This paper embraces this task. In particular, it analyzes momentum interactions from a rational equilibrium perspective and ultimately shows that the concentration of momentum in high information uncertainty as well as high credit risk stocks is perfectly consistent with rational asset pricing. Our economic setup is fairly general from the perspectives of both preferences and dynamics. The stochastic differential utility of Duffie and Epstein (1992) employed here breaks the tight association between the elasticity of inter-temporal substitution and the risk aversion measure. Moreover, our consumption dynamics, which is based on the novel formulation of Abel (1999), closely follows the high economic growth along with small consumption growth documented in the US post-war data. Collectively, our model allows one to match key regularities in asset pricing using reasonable risk aversion measures.

We use simulations to find out that our paradigm indeed predicts strong equilibrium momentum effects for the interaction between high leverage and risky cash flow firms. Momentum profitability deteriorates and ultimately disappears as either leverage or cash flow risk diminishes. More specifically, the correlation between observed and expected returns is positive and *monotonically* increasing with leverage. The monotonic relationship holds at high, medium, and low levels of expected growth rate volatility, at high and low autocorrelation of expected growth rate, and at the entire range of risk aversion measures analyzed.

Moreover, the expected return spread between the highest and lowest past year cumulative return portfolios increases with leverage. For example, when operating cash flows are highly volatile, the expected return spread is 11.49% for high leverage and only 0.27% for low leverage. On the other hand, the overall spreads are small when either the volatility of expected cash flows growth is small, or the expected growth in cash flows in not highly persistent. This indicates that while leverage is crucial, risk and persistence in cash flows growth are both important determinants of momentum effects. The collective evidence thus shows that equilibrium momentum profitability concentrates in the interaction between risky cash flows and high levered firms which is perfectly consistent with data.

Looking forward, there are several suggestions for future work. First, currently leverage is exogenous. We ask: given a particular leverage level - what is the overall momentum effect? It would be quite appealing to endogenize leverage and make it a firm-decision variable. Next, our focus here has been on firm level interactions. It has also been shown that momentum displays strong business cycle effects. Our setting can readily be extended to analyze possible rational business cycle effects in momentum profitability. Finally, from an empirical perspective, one could analyze the joint effect of leverage, expected dividend growth risk, and dividend growth persistence on the cross section of average returns.

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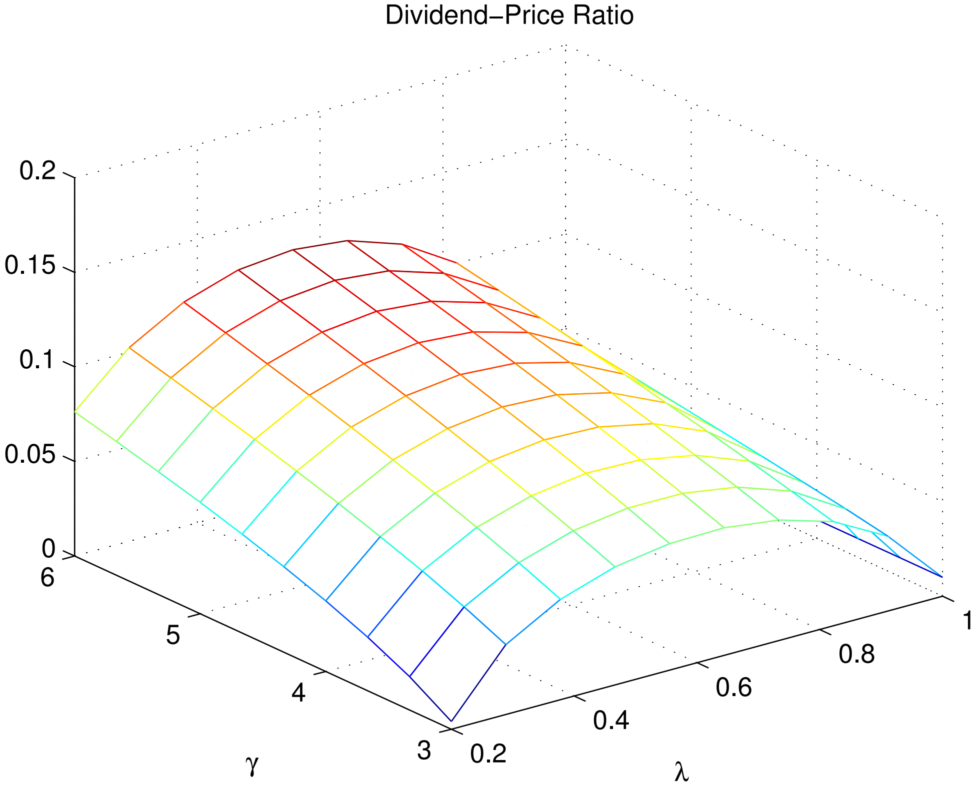


Figure 1: *Dividend-Price ratios implied by the model for* $\beta = 0.01$, $\kappa = 0.1$, $\bar{X} = 0.05$, $\sigma_D = .05$, $\sigma_x = .035$, $\rho = .35$. *The state is set to* $\bar{X}$, *i.e.*$X_0 = \bar{X}$
.

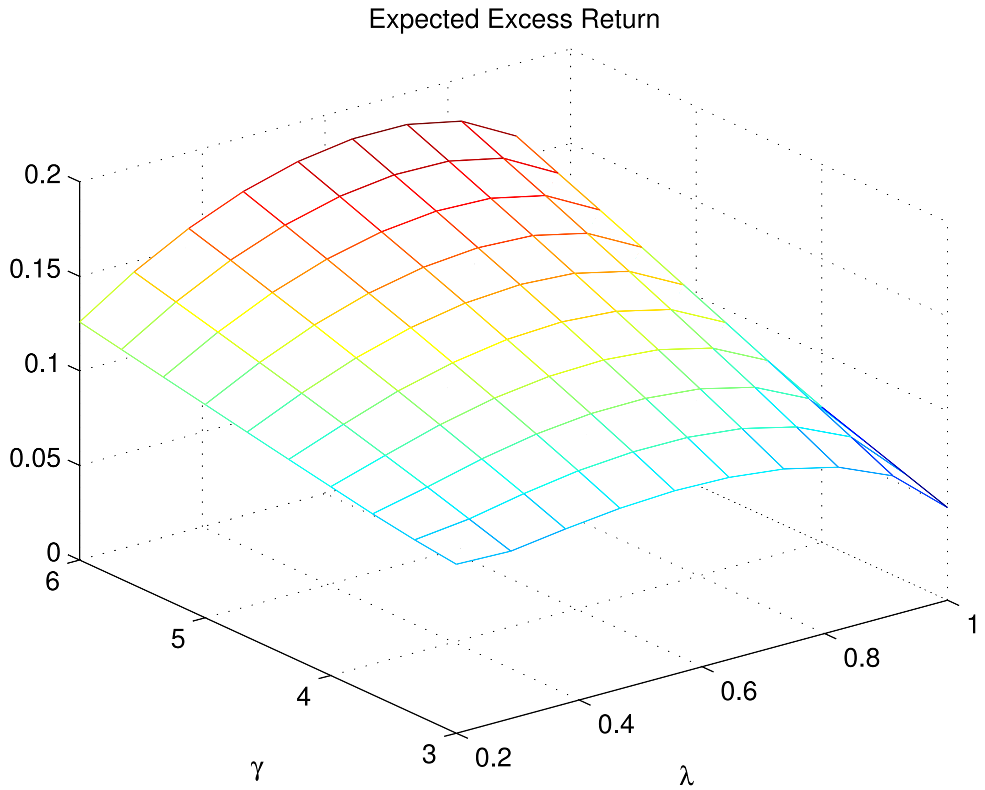


Figure 2: *Expected excess return implied by the model for* $\beta = 0.01$, $\kappa = 0.1$, $\bar{X} = 0.05$, $\sigma_D = .05$, $\sigma_x = .035$, $\rho = .35$. *The state is set to* $\bar{X}$, *i.e.* $X_0 = \bar{X}$
.

**Table 1:** This table lists the set of parameters we are considering. A and B are cases with high volatility of dividend growth and expected dividend growth. Case B is the same as A with higher risk-aversion. Cases C and D are with low volatilities with D having higher risk-aversion than C. E and F have moderate volatilities and higher correlation where F has lower volatility of expected dividend growth. G is similar to E, except with lower autocorrelation.

| | $\beta$ | $\gamma$ | $\kappa$ | $\bar{X}$ | $\sigma_D$ | $\sigma_x$ | $\rho$ |
|---|---|---|---|---|---|---|---|
| $A$ | 0.01 | 5.0 | 0.05 | 0.05 | 0.08 | 0.07 | 0.2 |
| $B$ | 0.01 | 8.0 | 0.05 | 0.05 | 0.08 | 0.07 | 0.2 |
| $C$ | 0.01 | 5.0 | 0.05 | 0.05 | 0.04 | 0.03 | 0.1 |
| $D$ | 0.01 | 8.0 | 0.05 | 0.05 | 0.04 | 0.03 | 0.1 |
| $E$ | 0.01 | 5.0 | 0.05 | 0.05 | 0.06 | 0.05 | 0.2 |
| $F$ | 0.01 | 5.0 | 0.05 | 0.05 | 0.06 | 0.02 | 0.2 |
| $G$ | 0.01 | 10.0 | 0.20 | 0.05 | 0.06 | 0.05 | 0.2 |

**Table 2:** This table shows the correlation between expected excess return and cumulative excess return based on investment horizons of 3,6,9 and 12 months. The system is simulated forward for 5000 paths using Monte Carlo integration subsequent to which we compute the $\Gamma(l)$ function. Then the average of the $\Gamma(l)$ function is reported for each $l$. In all cases, $X_0 = \bar{X}$.

| | $\lambda$ | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $A$ | 0.15 | 0.97 | 0.97 | 0.97 | 0.97 |
| | 0.25 | 0.95 | 0.95 | 0.95 | 0.95 |
| | 0.30 | 0.91 | 0.91 | 0.91 | 0.91 |
| | 0.40 | 0.87 | 0.87 | 0.87 | 0.87 |
| | 0.70 | 0.71 | 0.71 | 0.71 | 0.71 |
| | 0.90 | 0.51 | 0.51 | 0.51 | 0.51 |
| $D$ | 0.15 | 0.98 | 0.98 | 0.98 | 0.98 |
| | 0.20 | 0.97 | 0.97 | 0.97 | 0.97 |
| | 0.30 | 0.95 | 0.95 | 0.95 | 0.95 |
| | 0.40 | 0.92 | 0.92 | 0.92 | 0.92 |
| | 0.70 | 0.77 | 0.77 | 0.77 | 0.77 |
| | 0.90 | 0.52 | 0.52 | 0.52 | 0.52 |
| $E$ | 0.15 | 0.99 | 0.99 | 0.99 | 0.99 |
| | 0.20 | 0.97 | 0.97 | 0.97 | 0.97 |
| | 0.30 | 0.94 | 0.94 | 0.94 | 0.94 |
| | 0.40 | 0.91 | 0.91 | 0.91 | 0.91 |
| | 0.70 | 0.75 | 0.75 | 0.75 | 0.75 |
| | 0.90 | 0.57 | 0.57 | 0.57 | 0.57 |
| $G$ | 0.15 | 0.96 | 0.96 | 0.96 | 0.96 |
| | 0.20 | 0.95 | 0.95 | 0.95 | 0.95 |
| | 0.30 | 0.92 | 0.92 | 0.92 | 0.92 |
| | 0.40 | 0.89 | 0.89 | 0.89 | 0.89 |
| | 0.70 | 0.73 | 0.73 | 0.73 | 0.73 |
| | 0.90 | 0.47 | 0.47 | 0.47 | 0.47 |

**Table 3:** This table shows the average instantaneous expected excess return (annualized percentage) under the different parameter settings subsequent to one year in which the cumulative return has fallen into 1 of the 10 deciles labeled 1-10. Column 1 is the expected excess return for the lowest decile and Column 10 is for the highest. The system is simulated forward for 5000 different paths for one year, and each path here depicts one security over the year. For each path, we compute the observed return in (20) and the corresponding expected return. At the end of one year, we sort the 5000 paths based on observed return and assign them into the ten portfolios. The average of the expected return for each portfolio (equally-weighted) is then reported. In all cases, $X_0 = \bar{X}$.

| | $\lambda$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A$ | 0.15 | 15.02 | 17.13 | 18.25 | 18.83 | 19.48 | 20.04 | 20.67 | 21.39 | 22.99 | 26.51 |
| | 0.25 | 18.75 | 20.32 | 21.08 | 21.50 | 21.91 | 22.26 | 22.68 | 23.12 | 24.09 | 26.13 |
| | 0.30 | 20.37 | 21.79 | 22.43 | 22.85 | 23.22 | 23.52 | 23.86 | 24.28 | 25.11 | 26.88 |
| | 0.40 | 22.94 | 24.11 | 24.65 | 24.98 | 25.26 | 25.50 | 25.76 | 26.13 | 26.73 | 28.12 |
| | 0.70 | 25.77 | 26.21 | 26.43 | 26.55 | 26.65 | 26.74 | 26.83 | 26.97 | 27.22 | 27.71 |
| | 0.90 | 21.61 | 21.68 | 21.74 | 21.76 | 21.78 | 21.79 | 21.81 | 21.84 | 21.90 | 21.98 |
| $B$ | 0.15 | 20.13 | 21.91 | 22.81 | 23.28 | 23.76 | 24.15 | 24.66 | 25.18 | 26.32 | 33.29 |
| | 0.20 | 23.45 | 24.98 | 25.73 | 26.21 | 26.65 | 26.99 | 27.37 | 27.88 | 28.83 | 30.89 |
| | 0.30 | 28.56 | 29.92 | 30.56 | 30.97 | 31.25 | 31.54 | 31.88 | 32.26 | 33.00 | 34.64 |
| | 0.40 | 32.43 | 33.55 | 34.05 | 34.43 | 34.61 | 34.81 | 35.14 | 35.41 | 35.99 | 37.27 |
| | 0.70 | 34.25 | 34.78 | 35.12 | 35.77 | 36.12 | 36.88 | 37.22 | 37.45 | 37.89 | 38.14 |
| | 0.90 | 37.30 | 37.66 | 37.89 | 37.98 | 38.08 | 38.15 | 38.24 | 38.36 | 38.62 | 39.01 |
| $C$ | 0.16 | 12.06 | 13.07 | 13.53 | 13.71 | 13.87 | 14.01 | 14.16 | 14.30 | 14.58 | 15.01 |
| | 0.25 | 7.88 | 8.51 | 8.85 | 9.00 | 9.14 | 9.30 | 9.44 | 9.61 | 9.95 | 10.62 |
| | 0.30 | 8.17 | 8.72 | 9.01 | 9.14 | 9.27 | 9.40 | 9.53 | 9.69 | 9.97 | 10.57 |
| | 0.40 | 8.71 | 9.15 | 9.38 | 9.48 | 9.58 | 9.69 | 9.79 | 9.90 | 10.14 | 10.62 |
| | 0.70 | 8.57 | 8.76 | 8.84 | 8.88 | 8.93 | 8.96 | 8.99 | 9.04 | 9.13 | 9.30 |
| | 0.90 | 5.96 | 5.98 | 6.00 | 6.01 | 6.02 | 6.02 | 6.02 | 6.03 | 6.05 | 6.07 |
| $D$ | 0.15 | 10.56 | 11.94 | 12.63 | 12.93 | 13.20 | 13.46 | 13.72 | 13.99 | 14.52 | 15.42 |
| | 0.20 | 9.69 | 10.32 | 10.67 | 10.81 | 10.97 | 11.11 | 11.27 | 11.43 | 11.79 | 12.52 |
| | 0.30 | 10.83 | 11.38 | 11.66 | 11.79 | 11.92 | 12.06 | 12.19 | 12.33 | 12.63 | 13.24 |
| | 0.40 | 11.83 | 12.31 | 12.52 | 12.64 | 12.76 | 12.85 | 12.96 | 13.07 | 13.32 | 13.83 |
| | 0.70 | 12.05 | 12.47 | 12.33 | 12.39 | 12.42 | 12.46 | 12.50 | 12.55 | 12.64 | 12.82 |
| | 0.90 | 8.77 | 8.79 | 8.81 | 8.82 | 8.82 | 8.83 | 8.83 | 8.84 | 8.86 | 8.88 |

**Table 3 (Continued):**

| | $\lambda$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $E$ | 0.15 | 11.54 | 13.39 | 14.47 | 14.98 | 15.50 | 16.03 | 16.57 | 17.21 | 18.57 | 21.41 |
| | 0.20 | 13.15 | 14.28 | 14.84 | 15.13 | 15.43 | 15.68 | 15.97 | 16.27 | 16.96 | 18.38 |
| | 0.30 | 14.11 | 15.13 | 15.64 | 15.89 | 16.15 | 16.37 | 16.62 | 16.90 | 17.48 | 18.72 |
| | 0.40 | 15.68 | 16.51 | 16.89 | 17.11 | 17.33 | 17.49 | 17.69 | 17.91 | 18.37 | 19.30 |
| | 0.70 | 16.94 | 17.27 | 17.42 | 17.53 | 17.58 | 17.65 | 17.74 | 17.82 | 17.99 | 18.37 |
| | 0.90 | 13.49 | 13.55 | 13.59 | 13.60 | 13.61 | 13.62 | 13.64 | 13.66 | 13.69 | 13.75 |
| $F$ | 0.20 | 7.16 | 7.47 | 7.62 | 7.68 | 7.74 | 7.79 | 7.85 | 7.90 | 8.01 | 8.21 |
| | 0.25 | 6.30 | 6.64 | 6.79 | 6.86 | 6.94 | 7.00 | 7.07 | 7.14 | 7.28 | 7.57 |
| | 0.30 | 6.34 | 6.63 | 6.78 | 6.84 | 6.91 | 6.96 | 7.03 | 7.09 | 7.23 | 7.50 |
| | 0.40 | 6.72 | 6.95 | 7.05 | 7.11 | 7.17 | 7.21 | 7.25 | 7.31 | 7.42 | 7.63 |
| | 0.70 | 7.09 | 7.17 | 7.20 | 7.22 | 7.24 | 7.25 | 7.27 | 7.28 | 7.32 | 7.40 |
| | 0.90 | 5.74 | 5.75 | 5.75 | 5.76 | 5.76 | 5.77 | 5.77 | 5.78 | 5.78 | 5.79 |
| $G$ | 0.15 | 7.25 | 7.34 | 7.38 | 7.39 | 7.41 | 7.42 | 7.44 | 7.45 | 7.48 | 7.54 |
| | 0.20 | 8.33 | 8.48 | 8.54 | 8.57 | 8.60 | 8.63 | 8.66 | 8.68 | 8.74 | 8.85 |
| | 0.30 | 10.02 | 10.21 | 10.30 | 10.34 | 10.34 | 10.41 | 10.45 | 10.49 | 10.57 | 10.71 |
| | 0.40 | 11.22 | 11.34 | 11.48 | 11.52 | 11.56 | 11.59 | 11.62 | 11.69 | 11.74 | 11.90 |
| | 0.70 | 12.07 | 12.14 | 12.17 | 12.19 | 12.20 | 12.21 | 12.22 | 12.24 | 12.27 | 12.33 |
| | 0.90 | 9.54 | 9.54 | 9.55 | 9.55 | 9.55 | 9.55 | 9.55 | 9.55 | 9.55 | 9.56 |

**Table 4:** This table repeats the exercise in Table 3 with different time preference parameter $\beta$. Certain sub-cases are taken and they are replicated with a higher $\beta$. The first line for each sub-case is copied from the corresponding line on Table 3, and the following line is the repeat of the same simulation with higher beta, such that the transversality condition is still satisfied. This table shows that increasing $\beta$ lowers expected return at every decile but still maintains a healthy difference between the highest and lowest deciles.

| | $\beta$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $A(\lambda = 0.25)$ | 0.01 | 18.75 | 20.32 | 21.08 | 21.50 | 21.91 | 22.26 | 22.68 | 23.12 | 24.09 | 26.13 |
| | 0.08 | 10.88 | 12.23 | 12.96 | 13.34 | 13.76 | 14.11 | 14.53 | 14.99 | 16.05 | 18.42 |
| $B(\lambda = 0.15)$ | 0.01 | 20.13 | 21.91 | 22.81 | 23.28 | 23.76 | 24.15 | 24.66 | 25.18 | 26.32 | 33.29 |
| | 0.08 | 11.80 | 13.44 | 14.33 | 14.82 | 15.37 | 15.81 | 16.36 | 16.98 | 18.40 | 21.76 |
| $B(\lambda = 0.90)$ | 0.01 | 37.30 | 37.66 | 37.89 | 37.98 | 38.08 | 38.15 | 38.24 | 38.36 | 38.62 | 39.01 |
| | 0.08 | 18.86 | 18.91 | 18.96 | 18.97 | 18.99 | 18.99 | 19.01 | 19.02 | 19.07 | 19.13 |
| $E(\lambda = 0.20)$ | 0.01 | 13.15 | 14.28 | 14.84 | 15.13 | 15.43 | 15.68 | 15.97 | 16.27 | 16.96 | 18.38 |
| | 0.055 | 7.64 | 8.69 | 9.29 | 9.59 | 9.89 | 10.20 | 10.54 | 10.89 | 11.71 | 13.51 |

**Table 5:** This table shows the holding period return differential based on investment horizons of 3-12 months. The formation period is one year. We use 5000 different paths where each path denotes one stock. At the end we sort the observed return into ten equally weighted portfolios. The average difference between the top and bottom decile is reported in this table. In all cases, $X_0 = \bar{X}$.

| | $\lambda$ | 3 | 6 | 9 | 12 |
|---|---|---|---|---|---|
| $A$ | 0.15 | 2.78 | 5.52 | 7.87 | 10.35 |
| | 0.25 | 1.78 | 3.49 | 5.05 | 6.65 |
| | 0.30 | 1.58 | 3.11 | 4.46 | 5.87 |
| | 0.40 | 1.24 | 2.46 | 3.54 | 4.66 |
| | 0.70 | 0.46 | 0.89 | 1.29 | 1.82 |
| | 0.90 | 0.09 | 0.18 | 0.26 | 0.37 |
| $D$ | 0.15 | 3.21 | 3.25 | 3.28 | 3.20 |
| | 0.20 | 2.63 | 2.65 | 2.71 | 2.63 |
| | 0.30 | 2.25 | 2.27 | 2.32 | 2.23 |
| | 0.40 | 1.85 | 1.85 | 1.91 | 1.85 |
| | 0.70 | 0.72 | 0.67 | 0.73 | 0.72 |
| | 0.90 | 0.11 | 0.12 | 0.14 | 0.13 |
| $E$ | 0.15 | 2.38 | 4.75 | 6.75 | 8.98 |
| | 0.20 | 1.48 | 2.96 | 4.23 | 5.62 |
| | 0.30 | 1.10 | 2.18 | 3.13 | 4.14 |
| | 0.40 | 0.87 | 1.72 | 2.48 | 3.27 |
| | 0.70 | 0.33 | 0.65 | 0.95 | 1.30 |
| | 0.90 | 0.06 | 0.13 | 0.18 | 0.26 |
| $G$ | 0.15 | 1.41 | 2.78 | 4.00 | 3.96 |
| | 0.20 | 1.26 | 2.47 | 3.57 | 3.49 |
| | 0.30 | 1.04 | 2.03 | 2.97 | 2.83 |
| | 0.40 | 0.83 | 1.63 | 2.36 | 2.27 |
| | 0.70 | 0.31 | 0.61 | 0.86 | 0.87 |
| | 0.90 | 0.06 | 0.12 | 0.19 | 0.18 |

# 6 Appendix

### Preferences

In continuous time, the recursive utility function takes the form of stochastic differential utility. The stochastic differential utility $U : \mathcal{L}^2 \to \mathcal{R}$ is a mapping from a square integrable space to the real line and is defined by two primitive functions: $(f, A)$ where $f : \mathcal{R}^+ \times \mathcal{R} \to \mathcal{R}$ and $A : \mathcal{R} \to \mathcal{R}$. For any consumption process $C \in \mathcal{L}^2$, the utility process $J$ is the unique SDE

$$dJ_t = \left[ -f(C_t, J_t) - \frac{1}{2} A(J_t) \sigma_v \sigma_v' \right] dt + \sigma_v dB_t$$

with boundary condition $J_T = 0$. The different components are - $J_t$, a continuation utility for the agent given consumption $C_t$, $f(C_t, J_t)$ is an ordinal map of date $t's$ consumption and continuation utility, and $A(J_t)$ is a measure of local risk-aversion. If given an initial consumption $C_t$ and as long as the solution of the above SDE is well-defined, the utility at time t is defined as $U(C_t) = J_t$. Under certain conditions, the above SDE is well-defined and hence the utility exists. The function $U$ is monotonic and risk-averse for $A \leq 0$. Given an $f$ and two functions $A^*$ and $A$, let $U^*$ and $U$ be the two utilities corresponding to the aggregators $(f, A^*)$ and $(f, A)$. If $A^* \leq A$, then $U^*$ is more risk-averse than $U$, i.e. any consumption stream rejected by a deterministic consumption path by one will also be rejected by another. A convenient normalization that produces an ordinally equivalent utility function is achieved by setting $A = 0$, which means the above SDE solves $E_t[dJ_t] + f(C, J) = 0$ for normalized aggregator $(f, 0)$. The normalization is useful because it produces a much simpler Bellman equation to be solved than if $A \neq 0$. Fortunately, there exists a transformation from $(\bar{f}, A)$ to $(f, 0)$ such that the utilities generated from both will be ordinally equivalent. Further discussion of the aggregators and the normalization that leads to an ordinally equivalent representation of the aggregators is given in Duffie and Epstein (1992).

**Proof of Proposition 1:** The Bellman equation in (10) can be written as

$$\begin{aligned}J_C C\lambda\left[X_t+\frac{1}{2}(\lambda-1)\sigma_D^2\right]+J_X\kappa(\bar{X}-X_t)+\\ \frac{1}{2}J_{CC}C^2\lambda^2\sigma_D^2+\frac{1}{2}J_{XX}\sigma_X^2+J_{XC}C\lambda\sigma_D\sigma_x\rho+f(C,J)=0\end{aligned}$$

The continuation utility $J$ has a solution of the form

$$(1-\gamma)J=\exp(u_0\ln C_t+u_1X_t+u_2)$$

Substituting it in and collecting terms, reduces the above equation to a system of ODEs that can be solved recursively

$$\begin{aligned}u_0 &= (1-\gamma)\\ u_1 &= \frac{(1-\gamma)\lambda}{\kappa+\beta}\\ u_2 &= \frac{(1-\gamma)\lambda}{\beta}\left[\frac{(\lambda-1-\lambda\gamma)\sigma_D^2}{2}+\frac{\kappa\bar{X}}{\kappa+\beta}+\frac{(1-\gamma)\lambda}{\kappa+\beta}\left[\frac{\sigma_x^2}{2(\kappa+\beta)}+\sigma_D\sigma_x\rho\right]\right]\end{aligned}$$

Thus, the continuation utility function reduces to $J(C_t,X_t)=\frac{C_t^{1-\gamma}}{1-\gamma}\exp(u_1X_t+u_2)$.

**Proof of Proposition 2:** The pricing kernel for stochastic differential utility can be written as

$$\frac{d\Lambda}{\Lambda} = \frac{df_C}{f_C} + f_J dt$$

Using the above utility function, define $g = f_C = \frac{\beta(1-\gamma)J}{C}$ and $f_J = -\beta(1 + u_1 X + u_2)$. Use Ito's Lemma on $g$ and (5) and (7) one can rewrite the pricing kernel as

$$\begin{aligned}
\frac{d\Lambda}{\Lambda} &= -r_t^f dt - \lambda\gamma\sigma_D dW_1 + u_1\sigma_x dW_2 \\
r_t^f &= \lambda X_t + u_1\lambda\gamma\sigma_D\sigma_x\rho + \beta(u_2+1) - u_1\kappa\bar{X} - \frac{1}{2}\lambda\gamma\sigma_D^2(\lambda\gamma+1) - \frac{1}{2}\sigma_x^2 u_1^2
\end{aligned}$$

**Proof of Proposition 3:** The firm stock price is given by

$$\begin{aligned}
P_t &= \frac{1}{\Lambda_t} E_t \int_t^\infty \Lambda_s D_s ds \\
&= \frac{1}{\Lambda_t} \int_t^\infty E_t \Lambda_s D_s ds
\end{aligned}$$

Applying Feynman-Kac, we know $E_t\left[\Lambda_s D_s\right] = f(\Lambda_t D_t, X_t, s-t)$. Applying Ito's Lemma,

$$P_t = D_t G(X_t)$$

where $G(X_t) = \int_t^\infty \exp(P_1(s-t)X_t + P_2(s-t))ds$. Making a change of variable $\tau = s - t$, $G(X_t) = \int_0^\infty \exp(P_1(\tau)X_t + P_2(\tau))d\tau$. $P_1(\tau)$ and $P_2(\tau)$ satisfy a set of ODEs that can be solved recursively with initial conditions $P_1(0) = P_2(0) = 0$

$$P_1'(\tau) = -(\lambda - 1) - \kappa P_1(\tau)$$

$$P_2'(\tau) = u_1\kappa\bar{X} + \frac{1}{2}\sigma_D^2(\lambda\gamma - 1) - \beta(u_2+1) + P_1(\tau)\left[\kappa\bar{X} + u_1\sigma_x^2 - (\lambda\gamma-1)\sigma_D\sigma_x\rho\right]$$
$$+\frac{1}{2}\left[(\lambda\gamma-1)^2\sigma_D^2 + u_1^2\sigma_x^2 - 2u_1(\lambda\gamma-1)\sigma_D\sigma_x\rho + P_1^2(\tau)\sigma_x^2\right]$$

The solution of $P_1(\tau)$ is given by $P_1(\tau) = \frac{1-\lambda}{\kappa}\left(1 - e^{-\kappa\tau}\right)$ and then that can be used to solve for $P_2(\tau)$. Plugging in $P_1(\tau)$, $P_2(\tau)$ becomes

$$P_2(\tau) = a\tau + b(e^{-\kappa\tau} - 1) + c(1 - e^{-2\kappa\tau})$$

where

$$\begin{aligned}
a &= \left(u_1 + \frac{1-\lambda}{\kappa}\right)\left(\bar{X}\kappa - (\lambda\gamma-1)\sigma_D\sigma_x\rho + \frac{1}{2}\sigma_x^2\left(u_1 + \frac{1-\lambda}{\kappa}\right)\right) + \frac{1}{2}\sigma_D^2\lambda\gamma(\lambda\gamma-1) \\
&\quad -\beta(u_2+1) \\
b &= \frac{1-\lambda}{\kappa^2}\left[\frac{\sigma_x^2(1-\lambda)}{\kappa} + \kappa\bar{X} + u_1\sigma_x^2 - (\lambda\gamma-1)\sigma_D\sigma_x\rho\right] \\
c &= \frac{\sigma_x^2}{4}\frac{(1-\lambda)^2}{\kappa^3}
\end{aligned}$$

The transversality condition holds for $a < 0$, which holds for believable parameter values. Applying Ito's lemma to $P_t = D_tG(X_t)$, we derive the process for cumulative excess return $dR_t = \frac{D_t + dP}{P_t} - r_t^f$

$$\begin{aligned}
dR_t &= \mu_t^R dt + \sigma_D dW_1 + \frac{G_X}{G}\sigma_x dW_2 \\
d\mu_t^R &= (\cdot)dt + \left(\frac{G_X}{G}\right)_X (\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)\sigma_x dW_2
\end{aligned}$$

where $\mu_t^R = (\lambda\gamma\sigma_D^2 - u_1\sigma_D\sigma_x\rho) + \frac{G_X}{G}(\lambda\gamma\sigma_D\sigma_x\rho - u_1\sigma_x^2)$. The latter is derived from the equilibrium argument that the expected excess return is given by $\mu_t^R = -Cov_t\left(\frac{d\Lambda_t}{\Lambda_t}, dP_t\right)$.