Abstract

Title of thesis: PRECISE ESTIMATES FOR WEIGHT FUNCTIONS SATISFYING A WEIGHTED FOURIER TRANSFORM INEQUALITY

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This paper shows that for a given weighted Fourier transform inequality, certain weight functions will satisfy it. The work done in my paper is a continuation of similar ideas found in Yuki Yayama's thesis. She proved that a nonessentially increasing weight function w with a finite number of zeros can satisfy a given weighted Fourier transform inequality. Her proof includes estimations of distribution functions, the sine and the arcsine functions both near zero. My paper provides another proof by using precise values of distribution functions certain approximations used only when necessary.

PRECISE ESTIMATES FOR WEIGHT FUNCTIONS SATISFYING A WEIGHTED FOURIER TRANSFORM INEQUALITY

by

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1 Introduction

1.1 Discussion of theorem for essentially increasing, even weight functions w

The weighted Fourier transform inequality studied in this paper is:

$$\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 w\Big(\frac{1}{x}\Big)\,dx \le C\int_{-\infty}^{\infty} |f(x)|^2 w(x)\,dx, \tag{1}$$

where w is a non-negative weight function, C is a constant, and $\widehat{f}$ the Fourier transform defined by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} e^{-\imath xy} f(y)\,dy.$$

The conditions of weight functions that satisfy (1) have been discussed in [1] and [2]. It was shown in [2] that if $w(x)$ is an even weight function, nondecreasing on $(0,\infty)$, we have (1) if and only if $w(x) \in A_2$, where A_2 is a Muckenhoupt weight class and consists of all non-negative locally integrable functions $w(x)$, such that for all intervals (a,b),

$$\Big(\int_a^b w(x)\,dx\Big)^{1/2}\Big(\int_a^b w(x)^{-1}\,dx\Big)^{1/2} \le C(b-a) \tag{2}$$

holds. This theorem is still valid if one replaces $w(x)$ by even weight functions which are essentially increasing on $(0,\infty)$ and have no zeros. This was shown in Yayama [1]. A weight $w(x)$ is an essentially increasing function on $(0,\infty)$ if there exists an increasing function $U(x)$ and positive constants C_1, C_2, such that $C_1 \le \frac{w(x)}{U(x)} \le C_2$. In her thesis, Yayama proved that a weight function $w(x)$ with n zeros which is not essentially increasing can satisfy (1). She proved this by using estimations of the distribution function and also using the fact that $\sin t \sim t$ and $\arcsin t \sim t$ for $t \sim 0$. This paper will discuss the computation of the precise value of the distribution function and make the approximations only when necessary, thereby providing another proof of the result.

1.2 Essentially increasing weight functions w with a finite number of zeros

We show that if w is essentially increasing and there exists at least one z such that $w(z) = 0$, the weight w cannot satisfy (1). We suppose that $w(x)$ is an essentially increasing function with $w(z) = 0$. Then there is an increasing function $U(x)$ and positive constants C_1, C_2, such that $C_1 \le \frac{w(x)}{U(x)} \le C_2$.

Then $U(x) = 0$ for $x \le z$ and so $w(x) = 0$ for $x \le z$. If we also suppose that w belongs to A_2, then the inequality (1) must hold. If we choose

$$f(x) = \begin{cases} 1 & |x| \le \frac{z}{2} \\ \\ 0 & |x| > \frac{z}{2}, \end{cases}$$

then

$$\int_{-\infty}^{\infty} |f(x)|^2 w(x)\, dx = 0$$

Since $|\widehat{f}(x)|^2 > 0$ on $\frac{1}{2z} \le x \le \frac{1}{z}$, we have

$$\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 w\left(\frac{1}{x}\right) dx > 0.$$

Hence, $w(x)$ cannot satisfy (1) [1].

2 Definitions

Consider the following definitions from [2] and [3].

Definition 2.1
Let E be any subset of R. If a function f belongs to $L_p(E)$, the distribution function D_f of a function f is defined by $D_f(s) = m\{x \in E : |f(x)| > s\}$, where m is the Lebesgue measure.

Definition 2.2
The nonincreasing rearrangement f^* of a measurable function f is defined on a measure space by $f^*(t)$=inf$\{s : D_f(s) \le t\}$, where D_f is the distribution function defined above.

Definition 2.3
The weight class $F^*_{2,2}$, is the collection of all pairs of non-negative, locally integrable functions (u, v) on R such that

$$\sup_{s>0} \left(\int_0^{1/s} u^*(t)\, dt \right)^{1/2} \left(\int_0^s \left(\frac{1}{v}\right)^*(t)\, dt \right)^{1/2} < \infty. \qquad (3)$$

We write $(u, v) \in F^*_{2,2}$. Note that if $(u, v) \in F^*_{2,2}$, then u satisfies inequality (1). Here, we interpret $\infty \cdot 0$ to be 0.

It is shown in [3] that, if (u,v)$\in F^*_{2,2}$, then

$$\int_{-\infty}^{\infty} |\widehat{f}(x)|^2 u(x)\, dx \le C \int_{-\infty}^{\infty} |f(x)|^2 v(x)\, dx \tag{4}$$

holds for some constant C, for every $f(x)$ such that the right hand side is finite.

Remarks
1. Both D_f and f^* are nonincreasing functions on the positive real axis.
2. Clearly, if f is a nonincreasing function, $f = D_f$.

3 Main Problem

We will examine even, nonessentially increasing weight functions with a finite number of zeros by giving concrete examples, and then determine if they can satisfy (1).

3.1 The case for a weight function w_1 with one zero for $0 < x < \infty$
First, consider the case for $n = 1$:

$$w_1(x) = \begin{cases} |\sin x|^a & |x| \le \frac{3\pi}{2} \\ 1 & |x| > \frac{3\pi}{2} \end{cases} ,$$

where w_1 has two zeros, at $x = 0$ and $x = \pi$. Then,

$$\tilde{w}_1(x) = w_1\left(\frac{1}{x}\right) = \begin{cases} |\sin \frac{1}{x}|^a & |x| \ge \frac{2}{3\pi} \\ 1 & |x| < \frac{2}{3\pi} \end{cases} ,$$

and

$$\frac{1}{w_1}(x) = \begin{cases} \frac{1}{|\sin x|^a} & |x| \le \frac{3\pi}{2} \\ 1 & |x| > \frac{3\pi}{2} \end{cases} .$$

The graphs of $w_1(x)$, $\tilde{w}_1(x)$ and $\frac{1}{w_1}(x)$ are labeled Figure 1, Figure 2 and Figure 3 respectively in the List of Figures. Because w_1, $\frac{1}{w_1}$ and $\tilde{w}_1$ are all even functions, then only consider the positive real axis for each of these functions. If $A(x) \le B(x)$, then the following is true:

$$D_A(s) \le D_B(s) \text{ and } A^*(t) \le B^*(t). \tag{5}$$

The distribution function of $\frac{1}{w_1}(x)$ is

$$D_{\frac{1}{w_1}}(s) = 2\Big[m\Big(0 \le x \le \frac{3\pi}{2} : \frac{1}{|\sin x|^a} > s\Big) + m\Big(x > \frac{3\pi}{2} : 1 > s\Big)\Big] \le p_1(s),$$

where

$$p_1(s) = \begin{cases} 6\arcsin\left(\frac{1}{s^{1/a}}\right), & s \ge 1 \\ \infty & 0 \le s < 1. \end{cases}$$

By definition of the nonincreasing rearrangement function,

$$\left(\frac{1}{w_1}\right)^*(t) = \inf\left\{s : D_{\frac{1}{w_1}}(s) \le t\right\}.$$

Since $D_{\frac{1}{w_1}}(s) \le p_1(s)$, then by (5)

$$\inf\left\{s : D_{\frac{1}{w_1}}(s) \le t\right\} \le \inf\{s : p_1(s) \le t\}. \tag{6}$$

Therefore,

$$\left(\frac{1}{w_1}\right)^*(t) \le \inf\{s : p_1(s) \le t\}.$$

Recall that if $s \to 1$, then $s^{1/a} \to 1$ and $\frac{1}{s^{1/a}} \to 1$ also. This implies that $\arcsin\left(\frac{1}{s^{1/a}}\right)$ approaches $\arcsin(1) = \frac{\pi}{2}$. Moreover, as $s \to 1$, then $p_1(s) = 6\arcsin\left(\frac{1}{s^{1/a}}\right) \longrightarrow 6\Big(\frac{\pi}{2}\Big) = 3\pi$. Consider

$$\begin{aligned} 6\arcsin\left(\frac{1}{s^{1/a}}\right) &= t \\ \arcsin\left(\frac{1}{s^{1/a}}\right) &= \frac{t}{6} \\ \frac{1}{s^{1/a}} &= \sin\left(\frac{t}{6}\right) \\ s^{1/a} &= \csc\left(\frac{t}{6}\right) \\ s &= \left(\csc\frac{t}{6}\right)^a. \end{aligned}$$

Hence,

$$\left(\frac{1}{w_1}\right)^*(t) \le P_1^*(t)$$

where

$$P_1^*(t) = \begin{cases} \left(\csc \frac{t}{6}\right)^a, & 0 \le t < 3\pi \\ 1, & t \ge 3\pi \end{cases}$$

The distribution function of $\tilde{w}_1(x)$ is

$$D_{\tilde{w}_1}(s) = m\{x : \tilde{w}_1(x) > s\}.$$

To simplify $D_{\tilde{w}_1}(s)$, consider the following background calculations. Let $|\sin \frac{1}{x}|^a = s$. For $x > 0$, then we have that

$$\left(\sin \frac{1}{x}\right)^a = s$$

$$\sin \frac{1}{x} = s^{1/a}.$$

The points x_1, x_2, and x_3 where $\left(\sin \frac{1}{x}\right)^a = s$ occur when

$$\frac{2}{3\pi} \le \frac{1}{x_1} \le \frac{1}{\pi}, \quad \frac{1}{\pi} \le \frac{1}{x_2} \le \frac{2}{\pi}, \quad \frac{2}{\pi} \le \frac{1}{x_3} \le 0.$$

That is, when

$$\pi \le x_1 \le \frac{3\pi}{2}, \quad \frac{\pi}{2} \le x_2 \le \pi, \quad 0 \le x_3 \le \frac{\pi}{2}.$$

After further calculations,

$$x_1 = \frac{1}{\pi + \arcsin s^{1/a}}, \quad x_2 = \frac{1}{\pi - \arcsin s^{1/a}}, \quad x_3 = \frac{1}{\arcsin s^{1/a}}.$$

For ease of calculations, let $T = \arcsin s^{1/a}$. From these values, we get that

$$\begin{aligned} D_{\tilde{w}_1}(s) &= 2\Big((x_1 - 0) + (x_3 - x_2)\Big) \\ &= 2\left(\frac{1}{\pi + T} + \frac{1}{T} - \frac{1}{\pi - T}\right) \\ &= 2\left(\frac{T(\pi - T) + (\pi^2 - T^2) - T(\pi + T)}{T(\pi^2 - T^2)}\right) \end{aligned}$$

$$= 2\left(\frac{\pi T - T^2 + \pi^2 - T^2 - T\pi - T^2}{T(\pi^2 - T^2)}\right)$$
$$= 2\left(\frac{\pi^2 - 3T^2}{T(\pi^2 - T^2)}\right).$$

At this point,

$$D_{\tilde{w}_1}(s) = \begin{cases} 0, \;\; s > 1 & (7) \\ \frac{4}{3\pi}, \;\; s = 1 & (8) \\ \frac{2(\pi^2 - 3T^2)}{T(\pi^2 - T^2)}, \;\; 0 < s < 1 & (9) \end{cases}$$

The value for (7) is due to the fact that $|\sin z| \leq 1$. Now we will approximate (9) for s close to 0. Note that as $s \to 1$, then $s^{1/a} \to 1$ and $T = \arcsin s^{1/a} \to \frac{\pi}{2}$. Let $q(T) = \frac{2(\pi^2 - 3T^2)}{T(\pi^2 - T^2)}$. Hence,

$$q(T) \to \frac{2(\pi^2 - 3(\frac{\pi}{2})^2)}{\frac{\pi}{2}(\pi^2 - (\frac{\pi}{2})^2)} = \frac{2(\frac{1}{4})\pi^2}{\frac{\pi}{2}(\frac{3}{4})\pi^2} = \frac{4}{3\pi}.$$

As $s \to 0^+$, then $s^{1/a} \to 0^+$, $T \to 0$, $q(T) \sim \frac{2}{T} \to +\infty$. For $t \sim 0$, then by Taylor series we have for $f(t) = \arcsin t$, $f'(t) = \frac{1}{\sqrt{1-t^2}}$ and $f'(0) = 1$. The Taylor expansion is

$$f(t) = f(0) + f'(0)t + f''(0)\frac{t^2}{2!} + \dots \; = t + \dots,$$

and hence $\arcsin t \sim t$. Then for $0 < s < \delta$, then $T = \arcsin s^{1/a} \approx s^{1/a}$. Now the expression for $q(T)$ can be simplified to

$$q(T) = \frac{2}{T}\left(\frac{\pi^2 - 3T^2}{\pi^2 - T^2}\right) \leq \frac{2}{T}$$

because $\left(\frac{\pi^2 - 3T^2}{\pi^2 - T^2}\right) \leq 1$. Hence, $T \geq s^{1/a}$ which implies that $q(T) \leq \frac{2}{s^{1/a}}$ for $0 < s < 1$. Since $D_{\tilde{w}_1}(s)$, then $q(T)$ is decreasing.

We can now rewrite $D_{\tilde{w}_1}(s)$ as $D_{\tilde{w}_1}(s) \leq Q_1(s)$ where

$$Q_1(s) = \begin{cases} 0, \;\; s > 1 & (10) \\ \frac{4}{3\pi}, \;\; s = 1 & (11) \\ \frac{2}{s^{1/a}}, \;\; 0 < s < 1 & (12) \end{cases}$$

Now we can write the nonincreasing rearrangements $\tilde{w}_1^*(t)$ and $Q_1^*(t)$ as

$$\tilde{w}_1^*(t) = \inf\{s | D_{\tilde{w}_1}(s) \leq t\} \text{ and } Q_1^*(t) = \inf\{s | Q_1(s) \leq t\}$$

and by (6) we have that $\tilde{w}_1^*(t) \leq Q_1^*(t)$ where

$$Q_1^*(t) = \begin{cases} 1, & 0 \leq t \leq 2 \\ (\frac{2}{t})^a, & 2 < t < \infty. \end{cases}$$

Now consider

$$(\int_0^{1/s} \tilde{w}_1^*(t)\, dt)^{1/2} (\int_0^s (1/w_1)^*(t)\, dt)^{1/2},$$

where $0 < s$. The supremum of this product must be bounded in order to have $(\tilde{w}_1, w_1) \in F_{2,2}^*$ and moreover that $\tilde{w}_1$ satisfies inequality (1). To demonstrate that this is indeed true, we only need to show that

$$(\int_0^{1/s} \tilde{w}_1^*(t)\, dt)^{1/2} (\int_0^s (1/w_1)^*(t)\, dt)^{1/2}$$

is bounded. Let

$$\int_0^{1/s} \tilde{w}_1^*(t) = C_1 \text{ and } \int_0^s \left(\frac{1}{w_1}\right)^*(t) = D_1.$$

Now compute $C_1 D_1$ for $0 < s$ and determine if $C_1 D_1$ is bounded. Since $\tilde{w}_1^*(t) \leq Q_1^*(t)$ and $(\frac{1}{w_1})^*(t) \leq P_1^*(t)$, then

$$C_1 D_1 \leq \left(\int_0^{1/s} Q_1^*(t)\, dt \right) \left(\int_0^s P_1^*(t)\, dt \right).$$

Because of the definition of $(\tilde{w}_1, w_1)$ belonging to $F_{2,2}^*$, it is enough to show that for any $s > 0$,

$$\left(\int_0^{1/s} Q_1^*(t)\, dt \right) \left(\int_0^s P_1^*(t)\, dt \right)$$

is bounded for all fixed a where $0 < a < 1$.

Based on Q_1^* and P_1^*, there are four cases of s to consider. Further work will illustrate that these four cases can be combined into three cases.

Case 1 : $\frac{1}{s} \leq 2$ or $s \geq \frac{1}{2}$. Then

$$\int_0^{1/s} Q_1^*(t)\, dt = \int_0^{1/s} 1\, dt = \frac{1}{s}.$$

Case $2 : \frac{1}{s} > 2$ or $s < \frac{1}{2}$. Then

$$\begin{aligned}
\int_0^{1/s} Q_1^*(t)\,dt &= \textstyle\int_0^2 Q_1^*(t)\,dt + \int_2^{1/s} Q_1^*(t)\,dt \\
&= \textstyle\int_0^2 1\,dt + \int_2^{1/s} (\frac{2}{s})^a\,dt \\
&= 2 + 2^a (\textstyle\int_2^{1/s} t^{-a}\,dt) \\
&= 2 + 2^a \left[\tfrac{1}{-a+1} t^{-a+1}\right]_2^{\frac{1}{s}} \\
&= 2 + 2^a \left(\tfrac{t^{1-a}}{1-a}\right)\Big|_2^{\frac{1}{s}} \\
&= 2 + 2^a \left(\tfrac{(\frac{1}{s})^{1-a} - 2^{1-a}}{1-a}\right) \\
&= \tfrac{2(1-a)}{1-a} + \tfrac{2^a s^{a-1} - 2}{1-a} \\
&= \tfrac{2^a s^{a-1} - 2a}{1-a}
\end{aligned}$$

Case $3 : s < 3\pi$. Then

$$\int_0^s P_1^*(t)\,dt = \int_0^s \left(\csc\frac{t}{6}\right)^a dt.$$

In general, since one cannot exactly evaluate $\int(\csc x)^a\,dx$ where $0 < a < 1$ in closed form, then we must perform the following calculations to obtain an estimate for $\int(\csc x)^a\,dx$.

Consider the function $\phi(x) = x - \tan x$, where $\phi(0) = 0$. Then

$$\phi^{'}(x) = 1 - \sec^2 x < 0 \text{ for } x > 0.$$

Consider another function $\psi(x) = \tan x - x$, where $\psi(0) = 0$. Then

$$\psi^{'}(x) = \sec^2 x - 1 > 0.$$

Note also that $\psi(x) \geq 0$ for all $x > 0$.

If $0 \leq x \leq \frac{\pi}{2}$, then $x \leq \tan x$. Now consider the function $F(x) = \frac{\sin x}{x}$. Then

$$F^{'}(x) = \frac{\cos x[x - \tan x]}{x^2}.$$

On the interval $\left(0, \frac{\pi}{2}\right)$, $\mathrm{F}'(x) < 0$. Hence, for $0 < x < \frac{\pi}{2}$, then

$$\frac{2}{\pi} \leq \frac{\sin x}{x} \leq 1,$$

but letting $x \longrightarrow \frac{\pi}{2}$ or $x \longrightarrow 0$, then the ineqality is true for $0 \leq x \leq \frac{\pi}{2}$.

Now we have all the information needed to determine a lower and an upper bound for $\int_0^s (\csc(\frac{t}{6}))^a \, dt$. Recall that for $0 < x \leq \frac{\pi}{2}$, we have

$$\frac{2}{\pi} < \frac{\sin x}{x} \leq 1,$$

$$\frac{1}{x} < \frac{1}{\sin x} \leq \frac{\pi}{2x},$$

$$\frac{1}{x} < \csc x \leq \frac{\pi}{2x}.$$

For $0 < t \leq 3\pi$ or $0 \leq \frac{t}{6} \leq \frac{\pi}{2}$,

$$\frac{6}{t} \leq \csc \frac{t}{6} \leq \frac{6\pi}{2t},$$

$$\left(\frac{6}{t}\right)^a \leq \left(\csc \frac{t}{6}\right)^a \leq \left(\frac{6\pi}{2t}\right)^a.$$

Now

$$\int_0^s \left(\frac{6}{t}\right)^a dt \leq \int_0^s \left(\csc \frac{t}{6}\right)^a dt \leq \int_0^s \left(\frac{3\pi}{t}\right)^a dt,$$

$$\frac{6^a s^{1-a}}{1-a} \leq \int_0^s \left(\csc \frac{t}{6}\right)^a dt \leq \frac{(3\pi)^a s^{1-a}}{1-a},$$

that is for $s \leq 3\pi$,

$$\frac{6^a s^{1-a}}{1-a} \leq \int_0^s P_1^*(t) \, dt \leq \frac{(3\pi)^a s^{1-a}}{1-a}.$$

Case 4 : $s \geq 3\pi$. Then

$$\begin{aligned}\int_0^s P_1^*(t)\,dt = & \int_0^{3\pi} P_1^*(t)\,dt + \int_{3\pi}^s P_1^*(t)\,dt \\ = & \int_0^{3\pi} \left(\csc \tfrac{t}{6}\right)^a dt + \int_{3\pi}^s 1\,dt\end{aligned}$$

From the previous four cases, one can conclude that there are actually three cases (I, II and III) of s to consider. In each of these cases, let

$$\left(\int_0^{1/s} Q_1^*(t)\,dt\right)\left(\int_0^s P_1^*(t)\,dt\right) = H.$$

We will disregard the lower bounds for Cases I, II and III because the integrals in each of these cases are positive.

Case I : $0 < s < \frac{1}{2}$. Then

$$\begin{aligned}H \leq & \left(\tfrac{2^a s^{a-1} - 2^a}{1-a}\right)\left(\tfrac{(3\pi)^a s^{1-a}}{1-a}\right) \\ = & \tfrac{2^a(3\pi)^a - 2^a(3\pi)^a s^{1-a}}{(1-a)^2} \\ \leq & \tfrac{2^a(3\pi)^a}{(1-a)^2} \\ = & \tfrac{(6\pi)^a}{(1-a)^2} \\ = & Z_1\end{aligned}$$

Recall that $C_1 D_1 \leq H$. Since $0 < a < 1$, then $H \leq Z_1$ where $Z_1 \in R^+$ and hence H is bounded above.

Case II : $\frac{1}{2} \leq s < 3\pi$. Here,

$$\begin{aligned}H \leq & \tfrac{1}{s}\left(\tfrac{(3\pi)^a s^{1-a}}{1-a}\right) \\ = & \tfrac{(3\pi)^a}{(1-a)s^a} \\ \leq & \tfrac{(3\pi)^a}{(1-a)(\frac{1}{2})^a} \\ = & \tfrac{(6\pi)^a}{(1-a)} \\ = & Z_2\end{aligned}$$

Recall that $C_1D_1 \leq H$. Since $0 < a < 1$, then $H \leq Z_2$ where $Z_2 \in R^+$ and hence H is bounded above.

<u>Case III : $s \geq 3\pi$</u>.

$$\begin{aligned}
H \leq & \quad \frac{1}{s}\left(\frac{(3\pi)^a(3\pi)^{1-a}}{1-a} + s - 3\pi\right) \\
= & \quad \frac{1}{s}\left(\frac{(3\pi)^a(3\pi)(3\pi)^{-a}}{1-a} + s - 3\pi\right) \\
= & \quad \frac{1}{s}\left(\frac{3\pi}{1-a} + s - 3\pi\right) \\
= & \quad \frac{3\pi}{(1-a)s} + 1 - \frac{3\pi}{s} \\
\leq & \quad \frac{3\pi}{(1-a)s} + 1 \\
\leq & \quad \frac{3\pi}{(1-a)(3\pi)} + 1 \\
= & \quad \frac{1}{1-a} + 1 \\
= & \quad Z_3
\end{aligned}$$

Recall that $C_1D_1 \leq H$. Since $0 < a < 1$, then $H \leq Z_3$ where $Z_3 \in R^+$ and hence H is bounded above. Now, $C_1D_1 \leq \max(Z_1, Z_2, Z_3)$. This means C_1D_1 is bounded above which implies that $(\tilde{w}_1, w_1) \in F^*_{2,2}$. Thus, $\tilde{w}_1$ satisfies inequality (1).

3.2 The case for a weight function w_n with n zeros for $0 < x < \infty$

Consider the case for any natural number n where

$$w_n(x) = \begin{cases} |\sin x|^a & |x| \leq \frac{(2n+1)\pi}{2} \\ 1 & |x| > \frac{(2n+1)\pi}{2} \end{cases}$$

Then,

$$\tilde{w}_n(x) = w_n\left(\frac{1}{x}\right) = \begin{cases} |\sin \frac{1}{x}|^a & |x| \geq \frac{2}{(2n+1)\pi} \\ 1 & |x| < \frac{2}{(2n+1)\pi}, \end{cases}$$

and

$$\frac{1}{w_n}(x) = \begin{cases} \frac{1}{|\sin x|^a} & |x| \leq \frac{(2n+1)\pi}{2} \\ 1 & |x| > \frac{(2n+1)\pi}{2}. \end{cases}$$

As in the case for $n=1$, only consider the positive real axis for each function since they are all even. Recall also from this case that if $A(x) \le B(x)$, then the following is true:

$$D_A(s) \le D_B(s) \text{ and } A^*(t) \le B^*(t).$$

We will use similar reasoning to conclude that $(\tilde{w_n}, w_n)$ belongs to $F^*_{2,2}$ for every natural number n and hence $\tilde{w_n}$ satisfies inequality (1).

The distribution function of $\frac{1}{w_n}(x)$ is

$$D_{1/w_n}(s) = 2\Big[m\Big(0 \le x \le \frac{(2n+1)\pi}{2} : \frac{1}{|\sin x|^a} > s\Big) + m\Big(x > \frac{(2n+1)\pi}{2} : 1 > s\Big)\Big] \le p_n(s),$$

where

$$p_n(s) = \begin{cases} 2(2n+1)\arcsin\left(\frac{1}{s^{1/a}}\right), & s \ge 1 \\ \infty & 0 \le s < 1. \end{cases}$$

By definition of the nonincreasing rearrangement function,

$$\left(\frac{1}{w_n}\right)^*(t) = \inf\left\{s : D_{\frac{1}{w_n}}(s) \le t\right\}.$$

Using similar reasoning as in the case $n=2$,

$$\left(\frac{1}{w_n}\right)^*(t) \le \inf\left\{s : p_n(s) \le t\right\}.$$

Recall that if $s \longrightarrow 1$, $\frac{1}{s^{1/a}} \longrightarrow 1$, $\arcsin\left(\frac{1}{s^{1/a}}\right)$ approaches $\frac{\pi}{2}$ and

$$p_n(s) = 2(2n+1)\arcsin\left(\frac{1}{s^{1/a}}\right) \longrightarrow 2(2n+1)\Big(\frac{\pi}{2}\Big) = (2n+1)\pi.$$

Consider

$$\begin{aligned} 2(2n+1)\arcsin\left(\frac{1}{s^{1/a}}\right) &= t \\ \arcsin\left(\frac{1}{s^{1/a}}\right) &= \frac{t}{2(2n+1)} \\ \frac{1}{s^{1/a}} &= \sin\left(\frac{t}{2(2n+1)}\right) \end{aligned}$$

$$
\begin{aligned}
s^{1/a} &= \csc\left(\frac{t}{2(2n+1)}\right) \\
s &= \left(\csc\frac{t}{2(2n+1)}\right)^{a}.
\end{aligned}
$$

Hence,

$$\left(\frac{1}{w_n}\right)^{*}(t) \leq P_n^{*}(t)$$

where

$$P_n^{*}(t) = \begin{cases} \left(\csc\frac{t}{2(2n+1)}\right)^{a}, & 0 \leq t < (2n+1)\pi \\ 1, & t \geq (2n+1)\pi \end{cases}$$

As n increases, the domain of $w_n(\frac{1}{x})$ decreases. Hence, $w_n(\frac{1}{x}) \leq w_1(\frac{1}{x})$ for all natural numbers n. Moreover by (5),

$$\tilde{w}_n^{*}(t) \leq \tilde{w}_1^{*}(t).$$

Since $\tilde{w}_1^{*}(t) \leq Q_1^{*}(t)$, then $\tilde{w}_n^{*}(t) \leq Q_1^{*}(t)$. Let

$$\int_0^{1/s} \tilde{w}_n^{*}(t) = C_n \text{ and } \int_0^{s} \left(\frac{1}{w_n}\right)^{*}(t) = D_n.$$

Then we have

$$C_n D_n \leq \left(\int_0^{1/s} Q_1^{*}(t)\, dt\right)\left(\int_0^{s} P_n^{*}(t)\, dt\right).$$

Using analogous computations as those found in Section 3.1 show that $(\tilde{w}_n, w_n) \in F_{2,2}^{*}$ and thus, $\tilde{w}_n$ satisfies inequality (1).

4 Summary

Yayama's thesis used estimations of the distribution function and approximations for $\sin t$ and $\arcsin t$ to prove that a nonessentially increasing weight function $w(x)$ with n zeros can satisfy the inqequality (1). This paper provides more precise estimates of the distribution function and uses approximations as a last step.

List of Figures

Figure 1

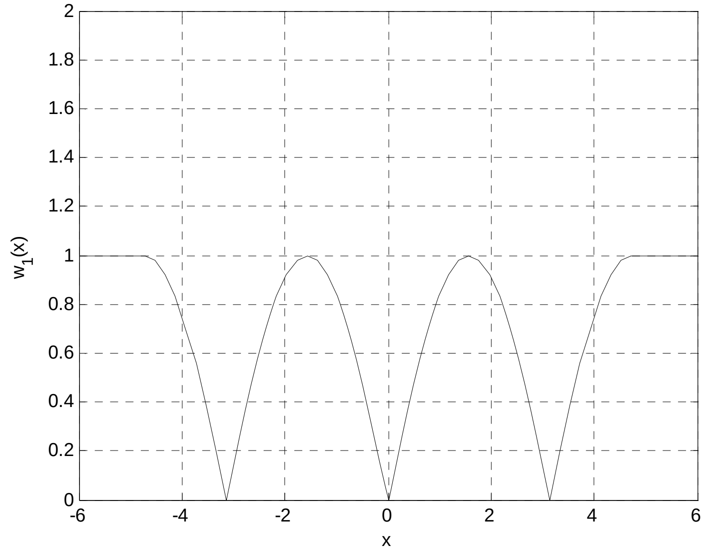


Figure 2

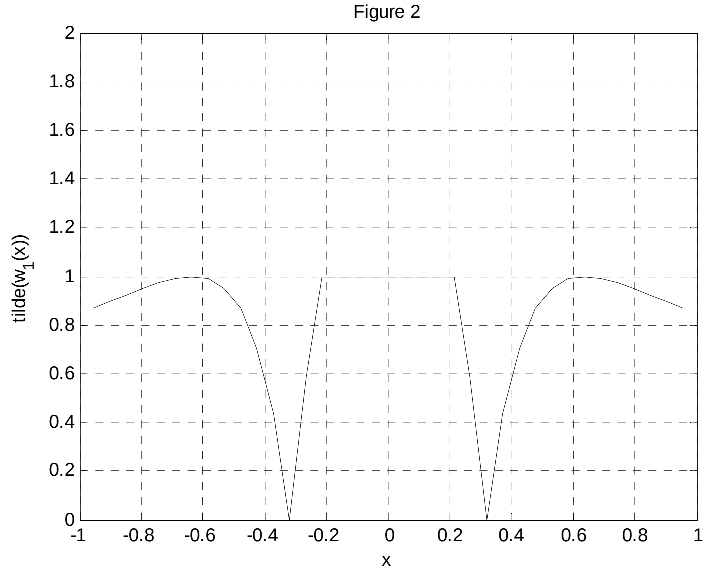
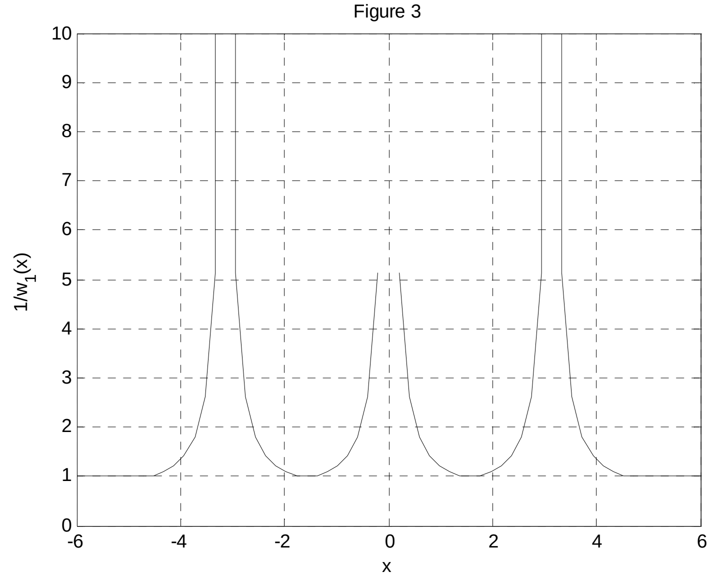


Figure 3



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