

## ABSTRACT

Title of Document: IMPLEMENTING EIGHTH GRADE  
MATHEMATICS PROBLEMS IN SIX  
COUNTRIES: A SECONDARY ANALYSIS  
OF THE TIMSS 1999 VIDEO DATA

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This study examined transcripts of videotaped lessons from the U.S. and five high performing countries participating in the Third International Mathematics and Science Study (TIMSS) 1999 Video Survey to investigate how eighth-grade teachers implemented mathematics problems. A coding system was developed to describe how teachers maintained or altered the potential of problems to “make connections” as they led public discussions of these problems. An analysis of the transcripts of 82 problem implementations found that when teachers or students made connections during problem discussions they most frequently did so by addressing mathematical justification, examining concepts more deeply than simply recalling or applying them, and connecting representations. Teachers most frequently led such discussions by drawing conceptual connections, taking over challenging aspects of the problems, and stepping students through arguments. Teachers much less frequently developed generalizations, compared solution methods, built on student ideas, provided scaffolding, or pressed students for

justification. When connections were lost, teachers most often took over challenging aspect of the problems or shifted the focus to procedures, answers, or superficial or vague treatment of concepts. Regardless of whether or not connections were made, in about half of all implementations, teachers did most of the mathematical work, in about 8% of implementations students did it, and in the remainder, the work was shared more or less equally. This study suggests that teachers in high performing countries often make connections using approaches American mathematics educators associate with traditional teaching. Teachers in other countries may not share the assumption held by some American educators that teacher-centered instruction is ineffective for improving students' conceptual understanding and abilities in problem solving and mathematical reasoning.

IMPLEMENTING EIGHTH GRADE MATHEMATICS PROBLEMS IN SIX  
COUNTRIES: A SECONDARY ANALYSIS OF THE TIMSS 1999 VIDEO DATA

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## Chapter 1: Introduction

Mathematics teaching in the U.S. has long been criticized for placing an emphasis on recall of isolated facts and procedural skills rather than conceptual understanding, mathematical reasoning, and problem solving (Brownell, 1935/1970; Hoetker & Ahlbrand, 1969; Schmidt et al., 1996). Evidence suggests that mathematics teaching in some other countries may more successfully develop students' abilities to think mathematically and solve challenging problems through the way teachers use mathematics problems in class (Hiebert et al., 2003). The purpose of this study was to examine more closely how teachers in the U.S. and five countries with high scores on the mathematics achievement test of the Trends in International Mathematics and Science Study (TIMSS) implement mathematical problems in ways that may promote or inhibit higher-order thinking.

### Rationale

The current wave of reform in mathematics education was at least in part provoked by the publication of *A Nation At Risk* by the National Commission on Excellence in Education (1983) and by the low achievement of American students compared to those in other nations in the Second International Mathematics Study (SIMS; McKnight et al., 1987). The SIMS researchers placed the blame on an “underachieving curriculum” (p. 85) characterized by less sophisticated content and more superficial coverage than that found in other countries. Subsequent calls for reform in the U.S. advocated for changed emphases in curriculum, teaching practices, and assessment so that they focused on “seeking solutions, not just memorizing procedures; exploring

patterns, not just memorizing formulas; formulating conjectures, not just doing exercises” (National Research Council [NRC], 1989, p. 84). The National Council of Teachers of Mathematics (NCTM) recommended that teachers assign their students tasks that “stimulate students to make connections and develop a coherent framework for mathematical ideas; call for problem formulation, problem solving, and mathematical reasoning; [and] promote communication about mathematics” (NCTM, 1991, p. 25).

However, in spite of two decades of reform efforts since *A Nation at Risk*, recent international comparisons have continued to find disappointing levels of mathematics achievement among U.S. students (Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education, 1998a, 1998b). Of particular concern is evidence that, compared with their peers in other countries, fewer American students achieve proficiency with problems requiring them to integrate representations, construct arguments, select strategies in unfamiliar situations, formulate generalizations, model complex phenomena, and demonstrate advanced thinking (Lemke et al., 2004). Reformers argue that these are the kinds of thinking that will be needed by students as they enter a workforce in which occupations increasingly require analytical thinking and problem solving, as they compete internationally with those educated in systems with higher expectations, and as they participate as citizens in a society progressively more influenced by technology, statistics, and quantitative argumentation (Mathematical Sciences Education Board, 1990; National Council of Teachers of Mathematics, 2000).

Researchers trace the lack of improvement in U.S. student achievement to the fact that reform efforts seem to have had little widespread effect on classroom practice (Schmidt et al., 1996; Stigler & Hiebert, 1999). International studies continue to

characterize American curricula and teaching practices as unfocused and superficial, emphasizing recall of facts and mechanical use of memorized algorithms disconnected from meaning (Schmidt et al., 1996; Schmidt et al., 1997; Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education, 1998a). For example, while Japanese and German teaching approaches have been described as “structured problem solving” and “developing advanced procedures” respectively, American teaching has been characterized as “learning terms and practicing procedures” (Stigler & Hiebert, 1999).

Thus, the NCTM (2000) continues to call for changes in curricula and teaching practices in order to emphasize problem solving, mathematical reasoning, communication, connections among ideas, and representations of concepts. In its vision for school mathematics, it focuses on the nature of tasks that teachers assign to their students, imagining that instead of practicing routine procedures demonstrated by their teachers,

[s]tudents confidently engage in complex mathematical tasks chosen carefully by teachers. They draw on knowledge from a wide variety of mathematical topics, sometimes approaching the same problem from different mathematical perspectives or representing the mathematics in different ways until they find methods that enable them to make progress. Teachers help students make, refine, and explore conjectures on the basis of evidence and use a variety of reasoning and proof techniques to confirm or disprove those conjectures. Students are flexible and resourceful problem solvers. (p. 3)

The problems or exercises that teachers have students work on in class can emphasize either recall of facts and use of procedures they have been told how to perform, or conceptual understanding and mathematical reasoning. Evidence suggests that implementing the latter kind of problem can lead to significantly increased levels of achievement in the areas desired by reformers (Ben-Chaim et al., 1998; Boaler, 2004; Kilpatrick, Swafford, & Findell, 2001; Newmann, Marks, & Gamoran, 1995; Stein & Lane, 1996; Talbert & McLaughlin, 2002).

Paradoxically, studies suggest that American curricula already include, and teachers assign, these kinds of problems at rates not dissimilar from those in other countries (Hiebert et al., 2003). Yet the achievement of U.S. students seems not to reflect this fact. Hiebert et al. (2003) suggest that the resolution of this apparent paradox can be found in how teachers and students actually work through these problems. According to the TIMSS 1999 Video Survey, in American eighth-grade classrooms, problems that appear to ask students to engage in special forms of mathematical reasoning—those called “making connections” problems—are almost always implemented in ways that require only the statement of answers, recall of facts, or use of routine skills (Hiebert et al., 2003). Although this also occurs in other countries, it does not occur at rates nearly as high as in the U.S.; at least some proportion of making connections problems are implemented in ways that preserve their original nature. In fact, in some countries, according to the TIMSS Video Survey, some exercises initially requiring only recall or the use of previously given procedures—that is, “non-making connections” problems—are implemented in ways that “make connections.”

However, little has been written about what this looks like in actual classrooms. What are typical teacher behaviors that occur in the U.S. and other countries that do or do not change the nature of problems in terms of their potential to engage students in more sophisticated thinking? Hiebert et al. (2003) did not systematically examine this question, and others who have done so have limited their work to American teachers undergoing intensive professional development programs specifically targeted at raising the cognitive level of tasks they give their students (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). An examination of teaching in a wider sample of classrooms in the U.S. and abroad could suggest how teachers may commonly implement problems in ways that do not make full use of their power to develop student thinking. In addition, international studies of teaching can reveal alternative practices and assumptions about how to promote conceptual understanding and mathematical reasoning that may not otherwise occur to American teachers, researchers, and reformers (Hiebert et al., 2003).

One reason in particular suggests that such an international perspective on teaching may be especially useful to American mathematics educators. Current reform efforts call for quite sweeping changes in teachers' practice along with corresponding shifts in their understanding of mathematics as a field, how students learn it, and how it should be taught. Research on professional development has documented how difficult it is to achieve such changes (Spillane, 2004; Thompson & Zeuli, 1999). Teachers tend to re-interpret reform recommendations to fit their pre-existing understandings, then believe they are implementing reform ideas when in reality they have only changed surface features of their teaching (Hiebert et al., 2003; Spillane, 2004; Stigler & Hiebert, 1999).

On the other hand, most teaching in high performing countries does not conform to American reform recommendations (Hiebert et al., 2003; TIMSS Video Mathematics Research Group, 2003). This fact may challenge the assumption that reform practices are the only way to promote conceptual understanding and mathematical reasoning. While there may be contextual and cultural differences that underlie achievement differences or that could render instructional practices imported from high performing countries ineffective in American classrooms, there is no evidence to suggest that American educators could not learn from the practices found in other countries. By providing data regarding such practices, the study of classrooms in other countries may supplement the research on teaching in the U.S. Such information could be very valuable to researchers, policymakers, and professional development providers in their efforts to improve American mathematics teaching.

### Definitions of Terms

To make it easier to describe the purposes, methods, and findings of this study, I now define the following terms which will be used throughout this paper:

*A problem* is any question a teacher assigned to students that necessitated the use of a mathematical operation and required “some degree of thought by an eighth-grade student” (Jacobs et al., p. 91), and was therefore identified as a problem by the TIMSS Video Study researchers. This contrasts with the more restrictive use of the term in much of the mathematics education literature, where it refers only to a question for which students have not been specifically told how to find an answer. In this study, then, a problem refers to any exercise or question, regardless of whether students can rely on a previously known procedure to find an answer.



A *making connections problem* is a problem whose statement during class seemed to ask students to “engage in special forms of mathematical reasoning such as conjecturing, generalizing, and verifying” (Jacobs et al., 2003, p. 122) and was therefore classified as “making connections” by the TIMSS Video Study researchers.

A *non-making connections problem* is a problem whose statement was not classified as “making connections;” that is, the TIMSS researchers interpreted it as asking students only to use “routine algorithms such as calculations, symbol manipulation, and practicing of formulae” or “recall information regarding a mathematical definition, formula, or property” (Jacobs et al., 2003, pp. 121-122).

*Problem implementation* is defined as the public, whole-class discussion of a problem. It includes all teacher and student talk, as well as any actions taken by either.

A *making connections implementation*, or an *implementation that makes connections*, is an implementation that was classified by either the TIMSS Video Study researchers or myself as “making connections” because it was judged to “include mathematically rich discussions” which for example may have “included describing connections between multiple representations, making and justifying generalizations, comparing the mathematics of different solution methods, and considering why a particular process was mathematically appropriate” (Jacobs et al., 2003, p. 124). Who the coder is will be apparent from the context.

Similarly, a *non-making connections implementation*, or an *implementation that does not make connections*, is an implementation that was not classified as “making connections.” In general, that means it was judged to involve only the use of a routine algorithm, the recall and statement of concepts, or the statement of the answer.

The combination of problem statement and implementation classifications yields an *implementation trajectory*. Only problems in three of the four possible trajectories were examined in this study: (a) *maintaining connections*: both problem statement and problem implementation were coded as making connections, (b) *losing connections*: the problem statement was coded as making connections but the implementation was not, and (c) *gaining connections*: the problem statement was not coded as making connections, but the problem implementation was.

In describing teacher behaviors, I found it helpful to use Good and Brophy's (1987) categories of product and process questions. *Product questions* are those that seek to elicit a single correct answer that can be expressed in a single word or short phrase. They usually begin with "what," "where," or "how much." I further defined them in terms of the kind of thinking they require of students by specifying that they can be answered by memory, observation, or performing a procedure or step as instructed by the teacher.

*Process questions* are those that seek to elicit an explanation which requires students to integrate information or show knowledge of their interrelationships. They usually begin with "why" or "how."

#### Statement of the Problem

American reformers seem to assume that improving students' conceptual understanding of mathematics and their abilities to engage in reasoning and problem solving requires drastic changes in teachers' knowledge, beliefs, and practice—changes that are difficult to achieve on a widespread basis. However, teaching in other countries suggests the possibility that such changes may not be necessary to achieve significant

improvement in student learning. The TIMSS 1999 Video Survey implies that while teaching in the participating countries does not meet many of the standards of reform teaching, it does involve the implementation of problems in ways that may help students develop the abilities about which mathematics educators are concerned (Hiebert et al., 2003).

Although the video study gives us some idea of the frequency with which typical eighth grade teachers in the U.S. and five high performing countries make use of making connections problems, and the frequency with which the teachers do or do not implement them in ways that make connections, it does not provide a sense of what teachers actually do that may influence whether these connections are made. In addition, examining this in a systematic way is made difficult by the lack of a coding system specifically developed for this purpose. Currently available classroom observation instruments assess the extent to which the observed lessons exhibit elements of instruction recommended by American reformers, rather than on actions teachers take to implement problems in their classes—actions that may not conform to American reform recommendations. Therefore, these instruments may not capture the ways that teachers in other countries promote conceptual understanding and mathematical reasoning through the implementation of problems.

### Research Questions

The purpose of this study, then, is to address the following research questions with regard to eighth grade mathematics teaching in the U.S. and the five other countries participating in the TIMSS 1999 Video Study:

1. What teacher behaviors are associated with making connections implementations?

2. What similarities and differences are there between these behaviors when teachers maintain connections versus when they add connections to non-making connections problems?
3. What teacher behaviors are associated with losing connections?
4. When implementing making connections problems, what similarities and differences are there between behaviors when teachers maintain connections versus when they lose connections?

### Overview of Research Design

This study consisted of a re-analysis of data from the TIMSS 1999 Video Study, which was conducted by LessonLab, Inc. under contract to the U.S. Department of Education (Hiebert et al., 2003). That study involved the videotaping of 638 eighth-grade mathematics lessons from the United States and six countries with high scores on the TIMSS mathematics achievement test: Australia, the Czech Republic, Hong Kong, Japan, the Netherlands, and Switzerland. The lessons were randomly selected to be representative of teaching in those countries. My study used problems randomly selected from among all but four of the 54 videotaped lessons from Japan, and a randomly selected sub-sample of 20 lessons from each of the remaining countries except for Switzerland, whose transcripts had not been translated to English.

In the original TIMSS Video Study, a “problem implementation team” analyzed the mathematics problems that were publicly completed during the videotaped lessons from all countries except Switzerland (Jacobs et al., 2003). Working from the videos and translated transcripts, members of this team coded each problem at two stages—first, according to how it was stated, and second, according to how it was implemented. At the

problem statement stage, a problem was classified as either “using procedures,” “stating concepts,” or “making connections,” depending on which it seemed to ask students to do. At the implementation stage, it was classified into one of the same categories, or as “giving answers only,” depending on how it was publicly completed by the students and/or teacher.

The TIMSS Video Study did not describe the behavior of teachers as they implemented these problems in the various ways. Prior research in the U.S. has identified teacher behaviors, classroom norms, and task characteristics associated with implementing tasks at “high cognitive level” versus implementing tasks at “low cognitive level” (Henningsen, 2000; Henningsen & Stein, 1997). Teacher behaviors associated with high cognitive level implementations included scaffolding, pressing students for justification and explanation, and modeling high level performance. Behaviors associated with low cognitive level implementations included using inappropriate tasks, specifying procedures, and shifting the focus away from meaning and to accuracy of answers. In a similar manner, my study sought to identify such behaviors, as well as any others that may not have been noted in prior American research.

However, the instrument used in the above-mentioned research and others that were available for observing classroom practice were inappropriate for the purposes of my study for two reasons. First, some of the factors they assessed were not observable in the TIMSS video data, such as the extent to which tasks aligned with students’ prior knowledge and interests (Henningsen & Stein, 1997). This was especially true since only transcripts translated to English were analyzed; due to legal restrictions, the original TIMSS videos were unavailable. Second, the instruments were tailored to assess the

presence of approaches advocated by American proponents of reform, and, in the case of Henningsen and Stein (1997), elicited by the professional development program related to the study. For example, many instruments assessed the extent to which teachers pressed students for justification and had them construct mathematical arguments, but not the extent to which teachers provided mathematical arguments or justifications, a much more common occurrence in the TIMSS videos, and one which TIMSS researchers interpreted as making connections implementations. Thus, such instruments might not have fully captured the ways that teachers in the TIMSS videos implemented problems.

Therefore, an important part of my study was to develop a coding system for characterizing the nature of teacher behavior as problems were implemented. I used an iterative method to develop and refine this coding system. Each of three iterations involved examination of a successively larger sample of transcript segments constituting the implementation of problems from the video sample, with the goal of identifying important teacher behaviors that seemed to influence whether connections were made. Identification of behaviors to be coded was informed by relevant literature and observation instruments. During each iteration, I tested the previously developed set of codes on additional problem implementations from the video sample, and added, refined, or deleted codes as necessary. In addition, during each iteration, I selected transcripts for double coding to check for reliability, and worked with the other coder to resolve discrepancies and refine code definitions.

Once I developed the set of codes and used them to characterize problem implementations in the corpus of lessons, I tabulated frequency counts for these codes in each of the three implementation trajectories relevant to this study: maintaining, losing,

and gaining connections. To address research question 1, I examined the sums of the frequency counts for maintaining connections and gaining connections implementations. To address question 2, I compared frequency counts for those two trajectories. To address research question 3, I examined frequency counts for those implementations that lost connections. To address question 4, I compared frequency counts for maintaining connections and losing connections implementations. In addition, I selected vignettes to illustrate each of the codes I had developed, and to show more concretely the ways in which teachers implemented problems in ways that did or did not make connections.

### Significance

At present, we have data to suggest that while U.S. teachers give their students problems that have the potential to promote mathematical thinking and problem solving, they rarely implement these problems in ways that do so (Hiebert et al., 2003). Based on one research program (Henningsen & Stein, 1997), we have some idea of what teachers may do that may inhibit the potential of tasks to develop student reasoning ability and conceptual understanding. However, the teaching analyzed by those researchers cannot be said to be typical of that found in U.S. schools, since it was conducted in the context of intensive reform-oriented professional development. My study was based on lessons sampled to be more typical of teaching found in the U.S. and other countries. Therefore, it suggests how generalizable Henningsen and Stein's (1997) findings may be beyond professional development classrooms, and outside the U.S.

We also have data suggesting that teachers in high performing countries more often make connections as they implement problems with their students, but prior research has not shown how they do this. Again, their methods may or may not be similar

to those found in the limited research on American teachers when they implement tasks at a high cognitive level after undergoing professional development specifically designed to help them do so (Henningsen & Stein, 1997). This study attempted to fill in this gap. By providing an international perspective, it may suggest approaches not currently advocated by U.S. reformers. It may also suggest that educators in other countries may not adhere to American assumptions about teaching practices; for example, the assumption that teacher-centered direct instruction is primarily only effective at transmitting basic facts and skills, while student-centered reform approaches are required to effectively develop students' higher-order thinking abilities (e.g., see Stein, Grover, & Henningsen, 1996, p. 462).

This study may also suggest important lines of research. Any alternative practices or assumptions it identifies may or may not be transportable to American education. Therefore, further research might be needed to determine whether changes in practice suggested by this study might be effective at developing American students' conceptual understanding and mathematical reasoning abilities. If so, further research could also examine the factors influencing the realization of such improvements and whether professional development could help teachers carry them out. This research would need to examine what other contextual features (e.g., students, schools, community, or district) and teacher characteristics (e.g., knowledge, experience, beliefs, or attitudes) might also contribute to or hinder such changes. Professional developers, teacher educators, and curriculum developers would find the results of such research helpful as they work to support teachers in developing their students' ability to engage in higher order thinking.



## Limitations

This study was limited by the data gathered and definitions used in the TIMSS 1999 Video Study. Videos and transcripts cannot capture the individual thinking of students. In addition, the implementation of problems was defined by both TIMSS and this study as the way in which problems were publicly discussed by the teacher or students, rather than according to how students engaged with them on their own. My study did not examine student thinking as evidenced by their written work or other products, or by private or small group dialog they may have had with the teacher or other students. In fact, there were no data on student learning associated with any particular classrooms. Thus, I cannot make definitive conclusions about whether any particular teaching practices were effective at promoting student engagement in higher order thinking or increasing student learning. However, by examining the knowledge that teachers made public, this study can describe the kind of information to which all students were exposed, as well as some of the practices used by typical teachers in high performing countries to make that information public.

Also, my study relied to a great extent on the coding by the TIMSS problem implementation team. I only examined problems that were stated and/or implemented as making connections according to the TIMSS problem implementation team; any other problems whose statements or implementations might have been coded as “making connections” or some other category of “higher order thinking” by myself or others was excluded from this study. Therefore, I can only make conclusions about problem statements and implementations identified by TIMSS researchers as involving the kinds of mathematical processes referred to in the TIMSS definition of “making connections,”

which excludes some practices advocated by reformers and which could have important implications for student learning (e.g., inventing algorithms and abstracting). In addition, although the process of analyzing problem implementations had the effect of reclassifying some of them according to whether or not they made connections, recoding initial problem statements was beyond the scope of this study. Therefore, answers to research questions 2 - 4, which depend on the coding of initial problem statements, reflect coding decisions made by the TIMSS problem implementation team.

Factors that influence how teachers implement tasks also were not within the scope of this study. These include teacher characteristics such as knowledge, beliefs, attitudes, education, experiences with learning and teaching, or engagement with mathematics, as well as departmental and school culture, and district, state, and national-level contextual factors. Similarly, factors that influence the ways in which students engage in tasks—such as students’ backgrounds and attitudes, and the social make-up and culture of the class, school, and community—were not considered. Furthermore, only one lesson was videotaped for each teacher, so this study did not examine what classroom norms were developed or established, how they were established, or how prior knowledge was developed among students—all factors that influence what the teacher can do and how it may impact students’ thinking. Thus, this study cannot make causal claims suggesting that if teachers tried to engage in certain behaviors, then students would engage in more sophisticated reasoning and gain deeper conceptual understanding. It may only point out the kinds of teacher behaviors that might warrant further research in terms of their potential to provide opportunities for students to engage in such thinking.

Although teachers and problems were selected randomly from each country and within problem statement and implementation types, the sample size was rather small. Code development was performed by people steeped in American culture, so this study may not have captured all of the important ways, or even the most important ways, that teachers implemented problems. In addition, student and teacher behavior may have been affected by the presence of videotaping. Although Hiebert et al. (2003) found that students' reactions made it clear when class was conducted in a manner than was out of the ordinary, the possibility that subtle changes occurred cannot be ruled out.

For these reasons, while the approaches to implementing problems described in this study can probably be seen as rather typical in the participating countries, frequency counts of behaviors can only suggest in a rough way how common they are. They cannot provide accurate estimates of how frequently such behaviors occur in classrooms beyond the sample, nor can they indicate the full range of teaching that exists in the countries participating in the TIMSS Video Study. The small sample does not allow us to determine that certain patterns of teaching are typical in certain countries or to compare teaching approaches in different countries. Because the study focused on eighth-grade mathematics teaching, it does not tell us how teachers implement tasks at the elementary or high school level.

Despite these limitations, this study does show some of the behaviors exhibited by typical eighth-grade teachers in five high performing countries as they lead discussions of mathematics problems in ways that “make connections,” suggesting approaches that may merit further examination by American mathematics educators. It also shows behaviors exhibited by typical eighth-grade teachers in the U.S. and five other countries as they lead

discussions that fail to make these connections, providing mathematics educators with information about some common ways that teachers may overlook the potential for problems to promote higher order thinking in the classroom.

## Chapter 2: Review of Literature

American reform recommendations have been shaped by theories about what is often called “learning with understanding” as well as by current thinking about the nature of mathematics as a field. However, these two areas of thought do not imply that current recommendations constitute the only route to improvement. There is evidence that these ideas can inform other “non-reform” teaching practices, including those seen in other countries. We cannot rule out the possibility that such practices could result in improvements that may be less difficult to put into place than reform recommendations, although it is not settled how effective they would be.

Therefore, this chapter begins by reviewing some of the literature on learning with understanding and on the nature of mathematics. Then it moves on to describing both reform and non-reform approaches as different ways of putting these ideas into practice, and reviews the evidence that they show promise for improving student learning. Descriptions of both kinds of instruction suggest potentially important teacher behaviors to look for and therefore inform the development of the coding system used in this study. Along the way, the review also describes the difficulties of implementing reform, which provide part of the rationale for this study.

A common characteristic of both reform and non-reform teaching approaches discussed in this chapter is a focus on the kinds of problems teachers give their students and how teachers implement them in class. The academic task literature has much to say about this, and lies behind both the rationale and the design of this study in its examination of the ways that teachers implement mathematics problems. Therefore, the last part of this chapter reviews the academic task literature.

## Learning with Understanding

Reform recommendations are often based on current understandings of how students learn “with understanding”—perspectives that also contribute to knowledge about the nature of tasks teachers assign to students, how teachers implement them, and the kind of learning they promote (NCTM, 1991, 2000). Thus, these perspectives provide the rationale for the classification of problem statements and implementations according to whether they ask students to use procedures, state concepts, or make connections.

Much of the literature on “learning with understanding” starts from a distinction between procedural and conceptual knowledge; that is, “knowing how” and “knowing what” (Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986). Conceptual knowledge is conceived of as knowledge that is rich in relationships. It can be thought of as a connected web of individual facts and propositions which are given meaning by their connections to other facts. Procedural knowledge, on the other hand, consists of rules, algorithms, and methods that are triggered by some recognizable input, and that produce some kind of result after a predetermined sequence of actions.

Both kinds of knowledge are theorized as residing in long-term memory, a seemingly limitless repository of permanent knowledge and skills. Recalling information means moving it from long-term memory to working memory, where most cognitive operations—thinking—take place. Working memory also receives information from the senses. Learning occurs when information from both sources is combined, or when knowledge from long-term memory is reflected upon, and new knowledge representations are created or old ones are altered in working memory, then placed back into long-term memory for permanent storage. Learning, then, occurs not by direct

placement of information into long term memory, but by cognitive processes which combine and operate on new information and pre-existing information (Hiebert & Lefevre, 1986; Schoenfeld, 1992; Silver, 1987).

Both facts and procedures can be learned either with or without understanding. Learning without understanding involves acquiring new information with few connections to other information other than to the context in which it is learned. In fact, the new facts or procedures may be tied strongly only to the surface features of the context. Lacking connections to other knowledge, isolated procedures and facts have little meaning and are fragile and difficult to access, they tend to deteriorate quickly, and students cannot apply them to situations different from the original context (Carpenter & Lehrer, 1999; Hiebert & Lefevre, 1986).

In contrast, learning with understanding is seen as making connections between the new facts or procedures and pre-existing conceptual knowledge. New facts become part of a more extended body of conceptual knowledge. New procedures gain strong connections with the conceptual knowledge on which they are based. Connections are paramount; the degree to which a person understands a fact, concept, or procedure is determined by the number and strength of connections it has with other facts, concepts, and procedures in the person's internal mental representation (Hiebert & Carpenter, 1992). Two kinds of connections are considered especially important for understanding: those that represent similarity and difference relations, and those that represent inclusion relations (e.g., the concepts of addition and subtraction are included within the part-whole concept). Inclusion relations are thought to be especially crucial in the development of structured, general, abstract knowledge.

This is not to say that students should never develop proficiency in using routine procedures, or that all learning tasks should involve the formation of new connections among concepts, procedures, and facts. The development of procedural proficiency is still important (Kilpatrick, Swafford, & Findell, 2001). However, the learning of procedures with understanding means that these procedures are closely connected to conceptual understanding. Instruction that focuses on procedures and facts without attention to conceptual connections and mathematical reasoning, which the TIMSS Video Study suggests is common in U.S. education, provides little opportunity for such learning.

Researchers cite several reasons for working to develop the kind of connected understanding described above (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Hiebert & Wearne, 2003; Hiebert et al., 1997; Kahan & Wyberg, 2003). First, it improves memory and at the same time reduces the amount of material that needs to be remembered. Second, it enhances the transfer of knowledge to new contexts. Third, it is generative and flexible; that is, it leads to more productive adaptation, invention, and learning of ideas and procedures in novel situations. Fourth, it improves the effectiveness of future learning by providing a more extensive mental network to which new concepts can be attached. Fifth, it leads to a more accurate and positive conception of mathematics. Finally, it is engaging and intellectually satisfying for students.

The implication is that desirable learning results from students actively constructing an interconnected network of knowledge through integrating previous knowledge with new information, rather than by memorizing facts and practicing procedures demonstrated by the teacher. This implication underlies researchers' and many professional leaders' recommendations that curricula, teachers, and assessments



emphasize connections and relationships among ideas. The NCTM's Connections Standard exhorts educators to “enable all students to recognize and use connections among mathematical ideas” and “understand how mathematical ideas interconnect and build on one another to produce a coherent whole” (NCTM, 2000, p. 64).

Mathematics educators often also assert that students construct these relationships or connections in their minds by struggling with novel situations. Hiebert et al. (1996, 1997) write that students should engage in tasks that are problematic for them—that is, that they see as interesting and having something they must figure out on their own. The NCTM's Problem Solving Standard defines problem solving as “engaging in a task for which the solution method is not known in advance” and asserts that “students should have frequent opportunities to formulate, grapple with, and solve complex problems that require a significant amount of effort and should then be encouraged to reflect on their thinking” (NCTM, 2000, p. 52). Thus, reformers advocate a shift in emphasis from tasks that require memory of facts and use of routine procedures, to those that require students to solve novel problems, justify solution methods and assertions, construct arguments, and make connections among ideas.

This literature motivates the distinctions among the three categories of problem statements and implementations used in the TIMSS Video Study and also suggests potentially important teacher behaviors. Problems that ask students to use procedures or state concepts do not necessitate (although for some students may still involve) the formation or use of the kind of conceptual understanding described above, as making connections problems are thought to do. The literature also suggests teacher behaviors that may promote the kind of understanding described above, such as drawing attention to

conceptual connections, using and connecting multiple representations, and developing, comparing, and justifying solution methods.

### The Nature of Mathematics

Both reform recommendations and the definition of making connections developed by TIMSS and elaborated in this study include references to habits and dispositions that characterize the work of mathematicians, such as making generalizations and formulating mathematical arguments. They reflect the view that mathematics learning means taking on the kinds of collaborative discourse, thinking, activities, and perspectives that mathematicians do (Schoenfeld, 1992). According to this view, students learn mathematics when they come to see the world as the mathematician does, and when they “do mathematics” as the mathematician does.

This raises the question of what it means to “do mathematics,” a question on which mathematicians and philosophers of mathematics have written extensively. According to Devlin (1994), mathematics is the science of abstract patterns (e.g., of number, shape, or motion) found in nature, in man-made creations, or in the human mind. Doing mathematics means discovering and investigating patterns, making initial simplifications, identifying and analyzing key concepts, using abstract notation, axiomatizing, increasing the level of abstraction, formulating and proving theorems, uncovering connections with other parts of mathematics, and generalizing theory.

Similarly, Steen (1990) writes, “Seeing and revealing hidden patterns are what mathematicians do best” (p. 1). He describes mathematical thinking as investigating, visualizing, classifying, inferring (from both axioms and data), finding connections, developing and using algorithms, and applying the tools of mathematics to other fields.

Cuoco, Goldenberg, and Mark (1996) list 28 mathematical “habits of mind,” including such behaviors as experimenting, finding hidden patterns, describing, inventing, visualizing, conjecturing, guessing, generalizing, deducing, using functions, and using multiple points of view.

Formulating arguments, or proofs, to establish truth and understanding is commonly seen as of central importance to doing mathematics. Philosophers Davis and Hersh (1981) write:

Proof, in its best instances, increases understanding by revealing the heart of the matter. Proof suggests new mathematics. The novice who studies proofs gets closer to the creation of new mathematics. Proof is mathematical power, the electric voltage of the subject which vitalizes the static assertions of the theorems. (p. 151)

They write that the ingredients of proof are abstraction, formalization, axiomatization, and deduction, but that “proof is subject to a constant process of criticism and revalidation” (p. 151). Mathematics is typically presented as a systematic list of definitions and axioms followed by theorems proved through the application of logic. However, the process of actually deciding on definitions and axioms, discovering theorems to prove, and finding proofs for them is quite messy. According to Hanna (1983), the creation of mathematics depends to a significant extent on creativity and intuition as forces which guide the generation of concepts, conjectures, and proofs. Lakatos (1976) describes what he calls the “quasi-empirical” nature of mathematics which involves a process of give and take whereby definitions, theorems, and proofs are

proposed, then successively refined by the consideration of counterexamples in the context of social interaction.

Thus, reformers and researchers promote the use of curricula and pedagogy that give students the opportunity to interact with each other while “doing mathematics” (Cuoco, Goldenberg, & Mark, 1996; Hiebert et al., 1996; Hiebert et al., 1997; NCTM, 2000; NRC, 1989 & 1990; Schoenfeld, 1992). Students should “make and investigate mathematical conjectures; [and] develop and evaluate mathematical arguments and proofs” (NCTM, 2000, p. 56). Furthermore, they should “communicate their mathematical thinking coherently and clearly to peers, teachers, and others; [and] analyze and evaluate the mathematical thinking and strategies of others” (NCTM, 2000, p. 60). Rather than the teacher (or textbook) being the authority who tells students what is correct or not, the class should arrive at agreement through negotiation and judgment of ideas against shared understandings of what constitutes mathematical validity. This implies that classroom tasks should engage students in exercising mathematical habits of mind and creating mathematics through interaction with others, rather than in practicing procedures given to them by the teacher.

Like the literature on learning theory, this literature lies behind the inclusion of mathematical activities such as looking for patterns, making conjectures, generalizing, and justifying assertions in the definition of making connections. It also implies that the code development process used in this study should be sensitive to teacher behaviors that involve or contribute to such activities.

## Reform Teaching

Theories regarding learning with understanding and prevalent thinking on the nature of mathematics have contributed to the current reform vision for teaching. This section reviews the literature on reform teaching, including what it is, evidence of its effectiveness, classroom observation instruments that assess the extent to which it has been implemented, and difficulties that arise when attempting to implement it.

### *Characterizing Reform Teaching*

Teaching models advocated by several researchers and reformers have been referred to as “teaching for understanding,” “teaching through problem solving,” or “problem-based teaching.” These instructional approaches attempt to improve student performance on assessments of problem solving and reasoning and to address the difficulty they have connecting skills with concepts—a difficulty which manifests itself in procedures that are flawed or easily forgotten, and that students cannot adapt to slightly different situations. They attempt to help students form well-connected mental networks of facts, concepts, and procedures by centering instruction around problems that are truly problematic—that is, problems for which students have not previously been taught solution procedures—but which they can approach by using knowledge and tools (perhaps in new ways) that they have at their disposal (Carpenter & Lehrer, 1999).

The problems are structured so that they lead students through new and significant mathematical territory, illustrate new ideas, and cause students to develop new connections among these new ideas and their pre-existing knowledge. Activities require students to engage in reflection on and communication of their ideas, while the teacher provides information only when needed (Carpenter & Lehrer, 1999; Hiebert & Carpenter,

1992; Hiebert et al., 1996; Hiebert et al., 1997; Kahan & Wyberg, 2003; Romberg & Kaput, 1999). Students learn the mathematics by working through genuine problems that require them, rather than the teacher, to do the mathematical work. Students frequently discuss alternative strategies in order to make connections with other methods and concepts. They compare methods, explain why particular methods work and others do not, and justify their choice of approaches. Errors are conceived of as sites for learning rather than as events to avoid (Carpenter & Lehrer, 1999; Hiebert & Wearne, 2003; Hiebert et al., 1997). Thus, problem solving becomes the heart of the curriculum, not simply part of it. Rather than learning *about* problem solving, students learn *through* problem solving (Stein, Boaler, & Silver, 2003).

Mathematics is seen as a mode of inquiry and a language for understanding patterns. According to this view, knowing mathematics means being able to investigate, understand, and express patterns and relationships among patterns, through the use of mathematical methods of inquiry: examining examples, abstracting common features, making conjectures, constructing logical arguments, making generalizations, devising solutions (Devlin, 1994; MSEB, 1990; NCTM, 2000). In this way, students engage in purposeful activities similar to those of actual creators or users of mathematics (Cuoco et al., 1996; Schoenfeld, 1992).

This vision of teaching contrasts with much American teaching which is based on the assumption that knowing mathematics means being able to select the correct procedures and use them accurately to calculate the correct answers to certain types of exercises (MSEB, 1990; Schoenfeld, 1992). Such teaching consists of telling and showing students—through clear exposition and worked out examples—how to perform

these techniques, and having students practice them on extensive sets of stereotypical exercises. When problem solving is addressed, it is assumed to be an advanced skill that cannot be taught until students master simpler pre-requisite skills (Carpenter & Lehrer, 1999). The literature in this area points to evidence contradicting this assumption. It also argues that traditional teaching seems to be especially ineffective with students from minority groups, who will continue to make up more and more of the American population (MSEB, 1990; Talbert & McLaughlin, 2002).

Researchers describe several characteristics problems must possess in order to be effective as the centerpieces of instruction (Carpenter & Lehrer, 1999; Hiebert & Wearne, 2003; Hiebert et al., 1997; Kahan & Wyberg, 2003; Marcus & Fey, 2003; Romberg & Kaput, 1999). First, they must be genuinely problematic to students—not amenable to solution by previously taught procedures—and the problematic aspect must be the mathematics. Second, they must be engaging to students. They must have something perplexing about them, or involve something that students want to make sense of or figure out. Third, they must be accessible to students. Although students should not have been told how to solve them, the problems must be just within their reach. They must connect to students' prior knowledge (formal or informal), so that students have some way to approach them. Since the problems may be challenging, instruction should provide adequate scaffolding and hints so students don't flounder, without removing the problematic aspect of the problem, short-circuiting students' opportunities to think through the ideas (Hiebert et al., 1997; Hiebert & Wearne, 2003). Fourth, the problems must engage students in thinking about important mathematical ideas, so that students will be left with a "residue" of important concepts. That residue can be insights into

structure or relationships, techniques for solving certain kinds of problems, or general approaches for adapting or inventing procedures. The problems should involve significant mathematical processes, such as model building, invention, inquiry, justification, and/or abstracting essential features.

### *Effects on Learning*

Preliminary research has found that instruction centered around problem solving can lead to higher achievement on conceptual understanding and more positive attitudes about mathematics, with little or no loss (even some gains) on procedural skills (Ben-Chaim et al., 1998; Carpenter, Ansell, & Levi, 2001; Cobb, Wood, & Yackel, 1993; Hiebert & Wearne, 1993; Kilpatrick, Swafford, & Findell, 2001; Newmann, Marks, & Gamoran, 1995). When elementary school students are allowed to devise and refine their own algorithms for solving problems involving multi-digit computation, their methods are tightly connected to conceptual understanding and they can more easily extend them to a larger variety of problems than can students taught with low-level tasks (Kilpatrick, Swafford, & Findell, 2001). For example, Hiebert and Wearne (1993) observed six second-grade classrooms during 12 weeks of instruction on place value and multi-digit arithmetic. In two of the classrooms, students engaged in tasks that required them to develop and explain their own approaches and construct relationships between place value and computation methods. Compared with students in the other four classes, which emphasized practice of routine procedures explained by the teacher, these students showed higher gains in performance, especially on tasks assessing place value concepts and ability to solve story problems.



Hiebert and Wearne (1993) suggested that both the kinds of tasks given to students and the nature of classroom discourse influenced learning by affecting the kinds of cognitive processes in which students engaged. (However, the use of specially employed teachers for these two classes meant a Hawthorne effect could have occurred.) Similarly, Cobb, Wood, and Yackel (1993) studied a second grade classroom where students constructed their own understandings of arithmetic by developing and verbalizing their own solutions and resolving conflicting points of view, without the teacher expecting any particular pre-determined solution methods. These students outperformed (by one standard deviation) traditionally taught students with respect to conceptual understanding, and did as well in computation.

Research conducted on Cognitively Guided Instruction (CGI) also supports the idea that work on novel tasks contributes to increased understanding among elementary students (Carpenter, Ansell, & Levi, 2001). CGI centers instruction on tasks that students are not told how to approach, but which encourage them to construct relationships, extend their knowledge, reflect on their experiences, and articulate what they know. A case study of two first-grade CGI classes found that most students became adept at calculations not usually expected until second or third grade, and that these skills were grounded in an understanding of base-ten concepts and operations. A three-year study of 14 first- through third-grade teachers learning to implement CGI also found consistent increases in conceptual understanding and problem solving performance among students as compared with corresponding students before CGI was implemented (Fennema et al., 1996). For students who remained in CGI classrooms for more than one year, such gains seemed to be cumulative. They also seemed to be tied to teachers' increased use of CGI

tasks and approaches. However, it must be kept in mind that other aspects of instruction, besides the tasks themselves, differed from typical instruction—most notably learning goals that were individually tailored to students.

At the middle school level, Ben-Chaim et al. (1998) found that a curriculum that encouraged seventh graders to construct their own conceptual and procedural knowledge of proportionality through the solving of novel, contextualized problems performed significantly better on an assessment of proportional thinking than students in traditional instruction. Mack (1990) studied eighth graders who originally had difficulty using symbolic algorithms to solve de-contextualized fraction problems. They were able to solve real-world versions of these problems using their own informal but conceptually-based methods. Through instruction they were eventually able to connect their conceptual understanding to symbolic representations, but the previously taught isolated knowledge interfered with this process.

In a large study involving teaching at several grade levels, Newmann, Marks, and Gamoran (1995) studied 23 schools (8 elementary, 7 middle, and 8 high schools) in the process of restructuring to examine links between what they called “authentic pedagogy” in mathematics and social studies, and student achievement on complex tasks. They defined authentic pedagogy as instruction that used tasks emphasizing higher-order thinking (organizing and synthesizing information, generalizing, explaining, hypothesizing, generating new meanings), alternative solutions, central ideas, and exploration of connections and relationships. They found that such instruction was associated with higher achievement on complex tasks across all grade levels, for both boys and girls, and for both white and African American students. However, their

measures of achievement were based on teacher-developed assessments, which therefore varied across classes, so that the performance of students whose teachers used less authentic pedagogy was limited by the assessments used. Other researchers have also found that formerly low-achieving students from minority and low socioeconomic groups develop significantly increased levels of ability in mathematical reasoning and problem solving when teachers implement tasks that require them to engage in such sophisticated mathematical thinking (Boaler, 2004; Talbert & McLaughlin, 2002).

### *Assessing Implementation*

Several research projects have developed classroom observation instruments intended to assess the extent to which mathematics teaching conforms to the teaching approach described above. These instruments ask observers to look for teacher practices thought to promote or hinder higher-order thinking—behaviors that the coding system developed in this study needed to attend to.

In a study of the relationship between reform teaching and mathematics achievement among middle-grade students, Milloy (2006) used an instrument that assessed the extent to which teachers engaged in teaching behaviors such as soliciting student solution ideas, making connections to other disciplines and the real world, providing a variety of representations, and having students critically assess procedures, challenge ideas constructively, and reflect on their learning. The instrument also examined whether teachers engaged students in mathematical processes such as testing hypotheses, making predictions, abstracting (symbolizing and building theory), generating conjectures, developing alternative solutions, and interpreting evidence.

Similarly, the Looking Inside the Classroom study of K-12 mathematics teaching across the United States used an instrument that examined the degree to which teachers connected material to students' experience and knowledge, promoted sense-making, used higher-order questioning, solicited student-generated ideas, promoted interaction among students, had students conjecture, investigate, prove, justify, abstract, constructively challenge ideas, and make connections to other areas of mathematics, other fields, or the real world (Horizon Research, 2000).

While the above instruments asked observers to rate the extensiveness of particular lesson characteristics at the end of the lesson, based on their general impressions, another approach has been to ask the observer to code the actions of the teacher repeatedly during the lesson, at regular times intervals. In its study of how teachers help struggling fourth and fifth graders succeed in mathematics, the High Quality Teaching Project (n.d.) used a computerized instrument which prompted the observer every three minutes to record whether the teacher was engaging in certain behaviors, such as having students reflect on their learning, soliciting alternative solution methods, elaborating on student responses, attending to student ideas, posing higher-order (or routine) questions, elaborating on a previous higher-order (or routine) question, having students engage in self-assessment, evaluating a student answer, providing an extrinsic reward, redirecting the conversation, modeling a procedure, defining a term, posting a key idea, and lecturing on content.

The Oregon Mathematics Leadership Institute's five-year study of K-12 mathematics teaching in ten Oregon school districts used an instrument that focused on student discourse rather than teacher behavior (Weaver, Dick, & Rigelman, 2005). It

required observers to code each “incident” of student mathematical discourse in real-time. Only student talk was coded; teacher talk was not examined. Each incident was classified according to whether it was a short answer, the statement of an assertion, an explanation of how to perform a task, a question to clarify understanding, a challenge to the validity of an idea or procedure, the description of a relationship or connection to prior knowledge, a prediction or conjecture, a justification of an idea or procedure, or a generalization.

Stein, Grover, and Henningsen (1996; see also Henningsen & Stein, 1997) developed an instrument specifically for identifying factors that previous literature had found may be correlated with the maintenance or lowering of cognitive level. Some of these were teacher actions that directly caused students to engage with tasks in certain ways, while others were task characteristics or student characteristics or behaviors. Teacher actions included the use of scaffolding vs. routinizing problematic aspects of the task; placing emphasis on meaning and understanding vs. accuracy, speed, or form of answer; providing an appropriate amount of time for students to work on the task; pressing students for justification and meaning; modeling higher-order thinking; encouraging students to engage in meta-cognition (self-monitoring, self-questioning); and implementing an appropriate accountability system.

Henningsen (2000) expanded this instrument in her study of the implementation of tasks that took more than one day. She added the following to the list of teacher behaviors examined: making references to previous knowledge, clarifying the task, talking about social and sociomathematical norms (e.g., stating that learning requires time and persistence, indicating that students’ ideas are valued, discussing criteria for

mathematical solutions and justifications), establishing accountability for focusing on and articulating significant ideas and relationships, having students compare alternative solutions, asking them to draw connections, having students engage in argumentation or validation, recording and using student ideas, and encouraging students to reflect on their thinking.

### *Difficulties of Implementation*

Studies using such instruments, as well as other surveys of teaching, consistently document the difficulty of implementing reform teaching on a widespread basis, or even among teachers involved in a professional development program intended to promote such reform (Stein, Grover, & Henningsen, 1996; Stigler & Hiebert, 1999; Weiss et al., 2003). Calls for a style of teaching that replaces teacher telling and student practice with group work and whole class discourse during which students invent solutions, justify assertions, discover patterns, make conjectures, and construct arguments have had little widespread effect on classroom practice (Mid-Atlantic Eisenhower Consortium for Mathematics and Science Education, 1998a; Schmidt et al., 1996; Schmidt et al., 1997; Stigler & Hiebert, 1999). Several obstacles to implementing reforms have been identified, including inherent difficulties of the new style of teaching, teachers' knowledge and beliefs, the cultural nature of teaching, teachers' sense of efficacy, and a large number of contextual factors.

Putnam and Reineke (1993) identify a fundamental difficulty that makes reform teaching decidedly more challenging than traditional teaching: the tension between centering instruction on students' thinking and ensuring that they come to accept conventional understandings regarding content they are expected to cover. In order to

achieve the latter goal, teachers sometimes press for convergence and offer their own explanations, rather than eliciting students' constructions and inventions.

But before such tensions can even occur, reform teaching requires quite sweeping changes in teachers' understanding of mathematics as a field, how students learn it, and how it should be taught. Several researchers have documented how teachers' current knowledge and beliefs about teaching, learning, and the nature of mathematics have hindered their ability to have students invent and compare multiple solutions, make connections among concepts, and construct mathematical arguments (Borko et al., 1992; Cooney, 1985; Putnam et al., 1992; Silver et al., 2005). Research on professional development has documented how difficult it is to achieve the changes in teachers' knowledge and beliefs necessary for teachers to implement such approaches (Spillane, 2004; Thompson & Zeuli, 1999). As already indicated, teachers tend to believe they are implementing innovative ideas when in reality they have only changed surface features of their teaching (Hiebert et al., 2003; Stigler & Hiebert, 1999).

Hiebert and Stigler (2000) argued that this difficulty is due to the fact that teaching is a cultural activity and therefore involves deeply ingrained practices based on implicit assumptions. Thus, due to the cultural assumption that good teaching consists of a set of particular techniques, American teachers may interpret recommendations for improvement as limited to practices such as cooperative group work, use of manipulatives, and use of real-life applications. Thus, lengthy and comprehensive professional development is essential for reforms to take root (Spillane & Thompson, 1997).

It may not only be teachers' perceptions about teaching that must be changed, but also their perceptions of themselves. Many teachers' sense of efficacy rests on their own perceived ability to explain material in ways they believe are effective and thorough. Reform teaching threatens that source of their sense of efficacy, and requires teachers to find a new one (Smith, 1996).

Not only must teachers' knowledge, beliefs, and perceptions change in order for reforms to be implemented, but so must contextual factors that impact teachers' practice. Such significant changes in teaching approaches require the alignment of all aspects of the educational system—policy documents, curriculum frameworks, instructional materials, and assessments (Cohen & Hill, 2001; Garet et al., 2001; O'Day and Smith, 1993). In addition, this alignment must encompass all students, including minority and low socioeconomic status students (O'Day and Smith, 1993). To achieve such alignment, professional development must involve not only teachers, but all those involved in carrying out policies, including administrators and supervisors, in order to insure their understanding of, and commitment to, recommended changes in practice (Knapp, 1997; Spillane & Thompson, 1997).

Such understanding and support require an extraordinary amount of learning by district administrators and teachers, but without them reform attempts are unlikely to succeed (Burch & Spillane, 2003; Spillane & Thompson, 1997). Spillane (2004) demonstrated how district policymakers and teachers can misinterpret reform policies when they are not given the time and opportunity to construct deeper understandings of the intended reforms. Lasting and widespread change also requires public support and the institutionalization of reforms. In summary, successful implementation of reform requires



substantial investment in human, social, and financial capital (Garet et al., 2001; Knapp, 1997; O'Day and Smith, 1993; Putnam & Borko, 2000; Spillane & Thompson, 1997).

### Conceptual Non-Reform Teaching

In light of the fact that changing teachers' practice so dramatically and on a large scale is exceedingly difficult, it is important to ask whether the kind of learning desired by reformers necessarily requires such change. In the high performing countries that participated in the TIMSS Video Study (except for Japan), teaching did not seem to reflect the characteristics of "reform" teaching: most of class time was spent with students listening to the teacher's explanations, listening and responding to the teacher's questions, or working practice exercises (Hiebert et al., 2003). However, this kind of teaching did not necessarily conform to the presumed model of traditional teaching in which teachers' explanations and student practice focused only on recalling facts and using procedures correctly (e.g., MSEB, 1990; Schoenfeld, 1992). In the high performing TIMSS countries, teachers implemented 37 to 52% of making connections problems in ways that actually made connections, while in the lower performing Australia and U.S., the corresponding figures were only 8 and 0%, respectively (TIMSS Video Mathematics Research Group, 2003).

This suggests that it may be possible for teachers to discuss problems in ways that promote deeper conceptual understanding, problem solving abilities, and mathematical reasoning without as drastic a change in teachers' practice as reformers recommend. In fact, several studies of teaching both abroad and in the U.S. have documented teaching practices that appear rather traditional in form, but whose content seem to involve ideas

similar to those suggested by theories of learning with understanding and the literature on the nature of mathematics.

### *Characterizing Conceptual Non-Reform Teaching*

For example, Hiebert and Handa (2004) took a closer look at one Hong Kong lesson that was a part of the TIMSS Video Study, arguing that teaching that appears traditional to American researchers may actually promote conceptual understanding through the use of carefully selected and sequenced problems that systematically explore different areas of a mathematical terrain, making connections among them, and linking the ideas that arise back to basic definitions presented early in the lesson. In fact, the primary problems in this particular lesson were coded by TIMSS researchers as “using procedures” when stated, but as “making connections” when implemented.

Some researchers analyzing Chinese teaching have made similar points. Huang and Leung (2004) tried to resolve what they called the “paradox of Chinese learners”—the high performance of Chinese students in spite of teaching that appears to use traditional methods judged ineffective by American researchers. They argued that the perception that students are passive during instruction is inaccurate and results from the limitations of Western theoretical perspectives. They described how 19 eighth-grade teachers in Hong Kong and Shanghai used questioning to lead students through proofs of the Pythagorean Theorem, noting that four of the teachers in Shanghai used open-ended questioning to promote student-teacher dialog and student reflection as they constructed rather abstract symbolic proofs. Huang and Leung admitted that lessons were teacher-dominated, but claimed that students were actually engaged in sophisticated thinking and problem solving, with the teacher strategically choosing to solicit student ideas at certain

times and choosing to assign more routine practice exercises at other times. Lopez-Real et al. (2004) made a similar argument in their case study of a Shanghai teacher who taught a coherent sequence of lessons on coordinate graphing and equations in two variables, varying his approach from exploratory to directive as necessary, and using both open-ended tasks and guided practice strategically.

Similarly, Wang and Paine (2003) described a beginning sixth-grade teacher in China whose lesson looked much like direct instruction, with the teacher leading the whole class through a development of fraction multiplication, followed by a set of practice exercises. However, she developed the concept by leading the class through a problem which required students to engage in the kinds of processes recommended by American reformers: combining ideas they had previously learned about area and multiplication to discover the theorem for fraction multiplication, developing an alternative proof for the theorem, justifying each step in the proof, judging persuasiveness of an argument, developing alternative methods for solving problems, and comparing methods according to efficiency. In addition, one of the tasks students completed during practice exercises could not be done by simply following the procedure students learned, but required them to extend the procedure to a new situation.

### *Effects on Learning*

Some of the older literature on effective direct instruction (i.e., instruction dominated by teacher explanations and student practice) in the U.S., although sparse, also suggests that non-reform methods of instruction can make use of some of the ideas of learning with understanding and improve student learning. Unfortunately, however, the

measures of student learning have often been standardized tests which do not necessarily assess mathematical reasoning and problem solving abilities.

Anderson (1989) wrote that direct instruction can be effective at developing conceptual understanding when it involves well-organized lessons that make clear to students the links among main ideas. Based on a six-year study of seven elementary school teachers identified by their students' high growth scores on standardized tests, Leinhardt (1986) found that successful teachers emphasized building multiple representations for concepts or procedures, justifying procedures, and proving the legitimacy or consistency of concepts with those the students had already learned.

Good, Grouws, and Ebmeier (1983) conducted a series of studies examining what they called "active mathematics teaching." They began with a naturalistic study in which they compared the teaching of 41 effective and ineffective third- and fourth-grade teachers as defined by their students' scores on standardized achievement tests. They found that effective teachers, among other things, spent more time on concept development through clear, whole-class explanations, and used more product (short answer) questions and fewer process (how and why) questions.

Using a rather traditional model of instruction consisting of review, concept development, and practice, the researchers defined concept development to include teacher explanations and demonstrations that modeled procedures, described concepts, abstracted common features from concrete examples, made comparisons, helped students see patterns, pointed out relationships among concepts, attended to representations, and called attention to relevant attributes of objects and situations. Good et al. emphasized the importance of spending sufficient time on the development portion of the lesson to

promote conceptual understanding rather than memorization of isolated information. Again paradoxically, they cautioned teachers to use process questions sparingly, suggesting instead that teachers more frequently provide students with “process explanations” that describe procedures, integrate facts, and show relationships. Their rationale was that process questions are often ambiguous to students and can waste time when students have difficulty answering them.

Good et al. devised a very short training program to help teachers implement their model, which they tested in 40 fourth-grade classrooms in 27 schools. Students in the treatment classes showed significantly higher gains on a standardized test than students in control classes, but there was no significant difference on a test of problem solving. The researchers then revised the program to include attention to problem solving as part of concept development, and conducted a similar experiment in 36 sixth-grade classrooms. They found higher gains among students in treatment classes than those in control classes on both a standardized test and a test of problem solving, but differences were not statistically significant. They attributed this to a failure to insure fidelity of implementation, and to contamination of the control group (school administrators began promoting active teaching among all teachers in the participating schools). A third experiment conducted with 19 eighth- and ninth-grade classrooms in 12 schools, with modification of the intervention for older students, found higher gains among students in the experimental group on both a test of problem solving and a standardized test, although while the former were statistically significant, the latter were not.

Few studies have compared the effects of conceptually-oriented direct instruction with reform teaching. One study did so at the undergraduate level, in the context of a

mathematics course for pre-service elementary teachers (McLaren, 2005). It examined the effects of direct instruction where the teacher provided students with intuitive conceptual explanations, as compared with the effects of problem-based instruction where students worked on tasks in groups to develop their own solution methods and then explained them to the class. The study found no differences in the procedural or conceptual understanding of students at the end of the course.

Thus, in terms of promoting higher-order thinking among students, while there is some research on the effectiveness of reform teaching, there is less U.S. research on the effectiveness of direct instruction that emphasizes conceptual understanding, or how its affects on learning compare with those of reform teaching. There is some evidence from research involving instruction in other countries that non-reform teaching can emphasize conceptual understanding in ways not ordinarily considered by American educators, and international studies of achievement raise the possibility that these approaches may contribute to high achievement on both routine skills and higher order thinking. However, a direct link between this kind of teaching and student achievement has not been established. It is also not known whether there are other instructional features or contextual or cultural factors that contribute to the higher achievement in other countries, or to the effectiveness of direct instruction in other countries that may not apply in the U.S.

### Mathematics Tasks

Important characteristics of both reform and conceptually-oriented non-reform practices seem to include the kinds of tasks teachers give their students and how teachers use those tasks. This section will review the literature on mathematical tasks, examining

their role in student learning, how they can be characterized according to the type of thinking they engender, and how teachers implement tasks in ways that affect student learning. The characterization of tasks is relevant to the way in which the TIMSS Video Study and my study classified problems, and findings related to how teachers implement tasks form the basis for some of the coding categories I used.

### *The Role of Tasks*

The tasks teachers assign to students are where curricula, pedagogy, and assessment intersect students' experience. Tasks consist of products students are expected to produce (e.g., answers to questions, solutions to problems), operations they must perform to obtain those products (e.g., recalling information, applying a rule, inventing an algorithm), and resources they have at their disposal to perform those operations (e.g., notes, textbooks, sample solutions, peers; Doyle, 1988). Tasks can emphasize recall of memorized facts, use of previously provided procedures to obtain answers to exercises, guided exploration of new content, invention of solution strategies, or engagement in mathematical processes such as generalization and argumentation.

Tasks influence what information students attend to and how they process it. Research in cognitive psychology suggests that the nature of tasks affects the cognitive processes in which students engage as they recognize how new information fits into or contradicts their existing schema, as they encode and store new information, as they reorganize pre-existing information, and as they rehearse information or procedures to make recall automatic (Anderson, 1989). Working through tasks not only teaches students specific mathematical concepts and techniques, but also strongly influences the ways in which students connect their learning to pre-existing knowledge, and how they

come to view the discipline (Bennett & Desforges, 1988; Doyle, 1988; Schoenfeld, 1992).

### *Classifying Tasks*

Researchers, including those conducting the TIMSS Video Study, have developed various ways of characterizing tasks or problems according to the cognitive processes they involve or would be expected to involve. In his work examining classroom practice in a variety of subject areas, Doyle (1988) distinguished between two kinds of tasks: familiar and novel. The former are those that students carry out routinely by using methods specified by the teacher. They involve recall and rehearsal of previously acquired information and procedures. By contrast, novel tasks require students to make decisions regarding the methods and information they should use. Thus, they involve uncertainty, unpredictability, flexible application of knowledge from different sources, and nuanced judgment, making them much more cognitively and emotionally demanding than familiar routine tasks (Doyle, 1988; Resnick, 1987).

In other work, Doyle (1983) has elaborated this distinction to obtain four classifications of tasks:

- *memory tasks*, in which students are expected to recognize or reproduce previously encountered information;
- *procedural or routine tasks*, in which students are expected to apply previously taught methods to generate answers;
- *comprehension or understanding tasks*, in which students are expected to recognize transformed versions of previously encountered information, apply procedures to new



- problems, decide from among several procedures those applicable in a particular situation, or draw inferences from previously encountered information; and
- *opinion tasks*, in which students are expected to state preferences.

He adds that memory tasks generally require students to attend to surface features of information, while comprehension tasks call attention to conceptual structure.

Researchers and assessment designers have developed several other frameworks to characterize tasks according to the mathematical processes or cognitive levels they involve (College Board, 2002; Henningsen & Stein, 1997; Mullis et al., 2003; Organisation for Economic Co-operation and Development, 2003). These frameworks have generally been developed by expert consensus and tested for inter-rater reliability. They have been used to varying degrees to examine the cognitive level of tasks as intended by curriculum developers and implemented by teachers, to improve instruction through professional development, and to design assessments and report their results.

The QUASAR project (Quantitative Understanding: Amplifying Student Achievement and Reasoning), based at the University of Pittsburgh, used a modified version of Doyle's categories in its professional development and research program in middle school mathematics teaching (Stein & Lane, 1996). Researchers tailored Doyle's categories to mathematics, omitted his opinion category, and divided his comprehension category into two types, resulting in four of what they called "cognitive levels":

- *memorization* – reproducing previously learned facts, rules, formulas, or definitions; or committing them to memory; with no connection to underlying concepts or meaning

- *procedures without connections* – use of algorithmic procedures, either specified by the task or evident based on their placement in the curriculum or prior instruction; with no connection to underlying concepts or meaning; focused on producing correct answers rather than developing understanding; explanations, if required, involve only describing the procedures used
- *procedures with connections* – use of suggested, broadly applicable procedures requiring attention to underlying concepts and meaning, for the purpose of developing deeper understanding; usually with multiple representations
- *doing mathematics* – exploring situations, concepts, or procedures, without predictable or suggested approaches; requiring the analysis of tasks and constraints, selection of methods, understanding of relevant concepts and relationships, and self-monitoring

Although the second and third levels both involve procedures, from a cognitive perspective, levels 1 and 2 are more similar than are levels 2 and 3. They are considered to be low cognitive levels because they involve only remembering and repeating information with the goal of improving efficiency of recall, and are thought to promote only the acquisition of isolated facts and procedural skills, respectively. Although the third level involves procedures to the extent that students are told what to do and may perform previously learned procedures, the intent of such tasks is to help them construct new, meaningful understandings of the material—that is, well-connected conceptual knowledge. Level 4 is similar, except that the student is not told explicitly what to do. Tasks in both levels 3 and 4 have the potential to lead to procedural knowledge that is

well-connected to conceptual knowledge. Thus, they are considered to be high cognitive levels.

The QUASAR framework has been integrated into the Instructional Quality Assessment (IQA) developed by the National Center for Research on Evaluation, Standards, and Student Testing (Junker et al., 2006), an instrument intended for rating instructional quality in elementary reading and mathematics based on classroom observation and examination of student work. The academic rigor portion used to rate mathematics instruction uses the QUASAR framework to rate tasks four times: (a) according to their potential to engage students in high level thinking, (b) according to how they are implemented (based on how students engage with them), (c) according to the expectations the teacher expresses for how students are to engage with the task, and (d) according to the expectations expressed by the students. The framework has also been used in teacher education and professional development programs for high school teachers to develop their abilities to choose and create worthwhile mathematical tasks (Arbaugh & Brown, 2004).

Until recently, the National Assessment of Educational Progress (NAEP) classified items on its written test into one of three categories of “mathematical abilities”: procedural knowledge, conceptual understanding, and problem solving (College Board, 2002). For the 2005 NAEP, these three categories were replaced by three levels of “mathematical complexity” (low, medium, and high) because “the dimension of mathematical abilities proved somewhat difficult for experts to agree upon, relying as it does on inferences about students’ approaches to each particular item” (Wilson, 2001, p. 10).

Prior to 2003, TIMSS curriculum studies characterized cognitive complexity of tasks by classifying them into five categories of performance expectations: knowing, using routine procedures, investigating and problem solving, mathematical reasoning, and communicating (Schmidt et al., 1996). The coding system allowed for multiple classifications for each item as necessary to capture its nature. Researchers used this framework to classify tasks found in textbooks and curriculum guides in a comparison of expectations in different countries (the intended curriculum), and to relate these findings to differences in achievement (the attained curriculum) (Schmidt et al., 1996; Schmidt et al., 1997). In 2003, TIMSS classified the items on its mathematics achievement tests into four “cognitive domains”: knowing facts and procedures, using concepts, solving routine problems, and reasoning (Mullis et al., 2003).

For varying reasons, none of the frameworks described above were appropriate for the examination of classroom implementation of problems seen in the TIMSS 1999 Video Study (M. Smith, personal communication, October 9, 2006). Frameworks used for NAEP and TIMSS achievement tests and curriculum studies attempted to characterize the ways students were expected to solve the problems in writing, while the main objective of the TIMSS Video Study and my study was to describe the discussion of problems in class. Although the QUASAR levels of cognitive demand were tailored to classroom implementation, the definition of implementation there was different, and their definition of cognitive level required knowledge of students and context (e.g., what had occurred in class during prior sessions) and thus was inappropriate for one-shot observations of a large number of classrooms.

Instead, through studying the problems as stated and publicly discussed in class, the TIMSS problem implementation team developed the three categories shown in Table 1. The TIMSS researchers did not attempt to indicate “levels of cognitive demand,” but rather attempted to describe first the kind of mathematical behaviors implied by the problem statement, and second the kind of mathematical behaviors that actually transpired during problem implementation (Jacobs et al., 2003).

Table 1: Categories Used by the TIMSS Problem Implementation Team

| <b>Category</b>    | <b>Problem Statement</b>   | <b>Problem Implementation</b>  |
|--------------------|--|--|
| Using procedures   | Use a routine algorithm, process, or set of steps  | Use a routine algorithm; talk only about how to progress to find an answer   |
| Stating concepts   | Recall information regarding a mathematical definition, formula, or property   | Allude to a mathematical concept without describing mathematical relationships or noting why the concept is appropriate  |
| Making connections | Engage in special forms of mathematical reasoning such as conjecturing, generalizing, and verifying; or think about, develop, or extend a mathematical concept | Include mathematically rich discussions involving mathematical relationships, properties, concepts, or mathematical justification stated as logically necessary consequences |

The TIMSS definition of “making connections” for problem implementation was used in this study, although it was more precisely specified as needed to resolve coding issues, as will be explained in Chapter 3.

### *Implementing Tasks*

The coding system developed by the TIMSS Video Study reflects the finding from the academic task literature that tasks are often not implemented in ways that reflect

how they are stated. As a result, simply classifying the problems teachers give students does not capture what actually happens in the classroom. For example, Doyle's (1988) study of an average-ability eighth-grade class found that the teacher presented students with a variety of novel word problems involving fractions, decimals, and percents. However, she then proceeded to routinize their solutions by giving students several computational algorithms and telling them how to match the problems with the correct algorithm. In effect, she changed "comprehension or understanding" tasks into "procedural or routine" tasks.

In a larger study, the QUASAR researchers observed the teaching of 12 urban middle school teachers who had participated in their professional development program focusing on the use of high-level tasks (Henningsen & Stein, 1997; Stein et al., 1996). The researchers examined only one task from each of 144 lessons—the task that took up the largest amount of time. They classified each task according to their categories of cognitive level twice: first according to how the teacher set it up, and second according to how students worked on it.

They found that about 22% of all tasks were set up at low cognitive levels (most were "procedures without connections"), and that, not surprisingly, virtually all of those tasks remained at low levels when implemented by students. Probably due to the professional development the teachers had received, 74% of tasks were set up at high levels (slightly more "doing mathematics" than "procedures with connections"). However, almost 60% of these tasks declined to low levels when carried out by students. Not all of them could be classified as "memorization" or "procedures without connections"; researchers found it necessary to add two more low-level implementation

categories: “unsystematic exploration” (exploring around the edges of significant ideas but failing to make progress in developing understanding) and “no mathematical activity” (off-task behavior).

The TIMSS Video Study examined more representative samples of eighth-grade mathematics teaching in the U.S. and other countries (Hiebert et al., 2003). Like the QUASAR researchers, the TIMSS researchers classified problems both as originally stated and as implemented. However, their approach differed from the QUASAR approach in two ways. First, they included in their analysis all problems assigned by the teacher whose solutions or answers were publicly discussed (Jacobs et al., 2003). Second, problems at the implementation stage were classified according to how the solutions or answers were publicly discussed by teacher or students, rather than according to how students appeared to work on the problems. An analysis of the 1995 TIMSS Video Study of eighth-grade mathematics lessons from the U.S., Japan, and Germany found that 40% of all problems in the U.S. were stated as “making connections”—more than in Germany (31%; Smith, 2000). However, only 5% of these problems were actually implemented as “making connections,” in contrast to Japan and Germany where the figures were 34% and 15% respectively.

The 1999 TIMSS Video Study of lessons in seven countries found even fewer “making connections” implementations in the United States. There, 17% of all problems were initially stated as “making connections,” which was in line with the figure for the high performing countries where the figure ranged from 13% to 24% (except in Japan, where it was 54%; Hiebert et al., 2003). Almost none (less than 0.5%) of those problems were implemented as making connections in the U.S., while in the high performing

countries the figure was between 37% and 52%. Fifty-nine percent of these problems in the U.S. were implemented as using procedures, while in the other countries the figure was between 16% and 20% (except Australia where it was 31%).

### *Task Implementation and Learning*

Of course, concerns over the implementation of tasks or problems would be unwarranted if there were no relationship to student learning. Both theory and empirical research, however, suggests the existence of such a relationship. Researchers claim that recall and procedural tasks do not provide meaning for students, nor give them the cognitive structures to which they can attach new learning (Koehler & Grouws, 1992). Reliance on such tasks prevents students from developing an adequately connected network of concepts and methods based on them.

That is, different types of problems and tasks cause students to engage in different cognitive processes, which in turn result in differences in learning. Anderson's (1989) review of the literature from cognitive psychology supports the first part of this assertion. The second part is supported by research suggesting that students' self-reports of cognitive processes (such as trying to understand the task and linking information to prior knowledge) is related more to achievement than is listening or time on task (Marx & Walsh, 1988; Peterson et al., 1984). Thus, the products that students produce while working on tasks would seem to be less important than the cognitive processes they engage in.

The QUASAR research itself points to a strong relationship between cognitive level of tasks and student learning. In their study of twelve classrooms in four urban middle schools, the QUASAR project administered an assessment of reasoning and



problem solving based on NCTM reform recommendations and findings from cognitive psychology (Stein & Lane, 1996). They examined average gain scores on this assessment over the three-year period that a cohort of students went through middle school. They found that larger gains in student achievement were associated with larger percentages of tasks set up at a high cognitive level, and that gains tended to be even larger when more tasks were implemented at a high level.

### *Changing the Nature of Tasks*

If the implementation of problems affects learning, then it is important to determine how and why teachers in the U.S. implement them in the way they do. How teachers implement tasks is a central focus of my study, so that findings from this literature shaped my code development process. Especially important were teacher behaviors identified by the QUASAR research (Stein, Grover, & Henningsen, 1996) involving middle school mathematics teachers undergoing professional development. In that study, teachers lowered the cognitive level of tasks by telling students what procedures to perform, performing them for students, or shifting the focus away from meaning to correct answers.

Several researchers have pointed to various reasons that teachers may do this, including their beliefs and knowledge, time constraints, mismatches between tasks and students' background, inappropriate assessment, the inherent difficulty students face in tackling high-level tasks, and general student resistance to engaging in such tasks. Teachers' beliefs and knowledge about teaching and learning, mathematics content, and the nature of mathematics as a field can contribute to their tendency to change the nature of tasks (Putnam et al., 1992). Teachers may remove the task features that promote

higher-order thinking because they do not see their value for learning, because they believe that students are incapable of dealing with them, because they are unfamiliar with the mathematics content they explore, or because their conceptions of mathematics prevent them from seeing the significance of these features.

The QUASAR research found a simpler reason was at work at least some of the time: declining cognitive levels were associated with the amount of time provided to complete tasks (Henningsen & Stein, 1997). When there was insufficient time, teachers tried to help students finish tasks by telling them what procedures to perform, solving the problems for the students, or simply focusing on answers to the exclusion of justifications and conceptual connections. Although this explanation may be a simple one, it can be difficult to rectify, for example, if it results from external pressure to cover curriculum in a short amount of time.

Another culprit identified by the QUASAR Project was the inappropriateness of tasks in terms of students' interests or prior knowledge. Inappropriate tasks caused students to fail to engage with them, a difficulty documented in other literature. In two studies involving a total of 57 elementary school teachers in Britain, Bennett and Desforges (1988) found that for high achievers, 41% of the tasks given to them underestimated their attainments, and for low achievers, 44 to 50% of the tasks overestimated their attainment. Both situations led to wasted time and low engagement with tasks. The researchers related this to teacher knowledge about teaching and learning, blaming the mismatch on teachers' unawareness of student thinking due to teaching, assessment, and classroom management approaches that focused exclusively on the

products students produced rather than on their thinking, and that were oriented toward rapid responses by teachers to students' answers.

Bennett and Desforges (1988) also found that mismatches between task specifications and assessment methods contributed to the lowering of cognitive level. Although tasks might have asked for higher-order thinking, formal and informal (e.g., teacher praise) assessment often emphasized following procedures and using correct formats.

While such mismatches could be due to teachers' knowledge and beliefs, Doyle and Carter (1984) proposed another explanation (see also Doyle, 1988). They described a process whereby teachers negotiate with students to change the nature of tasks in several ways, including by changing the standards of assessment. This process results from the fact that novel tasks present considerable challenge not only for students, but for the teacher in terms of classroom management. Such tasks involve uncertainty, unpredictability, flexible application of knowledge from different sources, and nuanced judgment, making them much more cognitively and emotionally demanding for students than familiar routine tasks (Doyle, 1988; Resnick, 1987). This kind of thinking is non-algorithmic, complex, and effortful; it involves self-regulation, the application of multiple criteria, and imposing meaning or order; and it often yields multiple solutions.

These characteristics cause considerable difficulty for students, so they ask numerous questions to reduce ambiguity, unfamiliarity, and risk. Maintaining the flow of classroom activity becomes very difficult as student error rates are high and productivity is low. Students engage in off-task behavior, or they press the teacher to remove problematic aspects of the task and/or reduce grading standards. To restore order and

pace, the teacher reduces ambiguity by simplifying tasks, providing procedures connected to surface features of the task, and reducing accountability, for example, by easing grading standards or accepting answers without justification. This reduces risk for the students and encourages them to work, but it also reduces opportunities for students to engage in more sophisticated reasoning. From this perspective, this renegotiation of expectations is not due to teacher's deficiencies in knowledge or classroom management, but rather to the inherent tension between teachers' desire to engage students in higher-order thinking and students' desire for guidance and predictability.

Powell, Farrar, and Cohen (1985) and Sedlak et al. (1986) provided lucid illustrations of this negotiation process based on extensive studies of American high schools and reviews of the literature on student engagement. They described the resulting “treaties” or “bargains”—some tacit, and some explicit—in which teachers agreed to limit their expectations of students to completing routine and undemanding tasks emphasizing recall of facts, in return for cooperative behavior from students:

[S]tudents marshaled unimaginable resources to challenge incessantly, and often spuriously, their teachers' authority to impose academic standards. Teachers accepted unacceptable work, forgave confusion, and struggled constantly with students determined to impose their own definitions of knowledge on the class or at least to demoralize teachers who sought to preserve the integrity of requirements and expectations... teachers often cope by making the acquisition of knowledge “easier,” less painful, and therefore less threatening, through unchallenging instructional methods: lecturing, assigning more seatwork, reducing complex conceptual

problems to factual lists, diluting or omitting essential content knowledge, refusing to challenge students seriously, requiring little reading, minimizing writing assignments, changing instructional and classroom goals on the spot by attending to personal matters, or conversing with students. (Sedlak et al., 1986, pp. 102-103)

Powell et al. (1985) provided two motivations for teachers to enter into such treaties: a need to avoid conflict with disengaged, resistant, or defiant students, and a belief that demanding more from students would interfere with teachers' therapeutic goals of helping students feel good about themselves and about school.

Along with Sedlak et al. (1986), they painted a pessimistic picture, arguing that, at least at the high school level, this state of affairs was one aspect of the larger phenomenon of low academic expectations—a phenomenon that has resulted from fundamental American educational values. They traced students' (and in some cases, teachers') lack of interest in academics back to long-standing and widespread American attitudes that have placed a high value on social and "life adjustment" skills while devaluing academic work. Powell et al. (1985) cited literature from throughout the twentieth century showing that these attitudes have been common in the U.S. since the advent of mass schooling and have involved the belief that most students neither need nor are capable of serious academic work: "American educators quickly built a system around the assumption that most students didn't have what it took to be serious about the great issues of human life, and that even if they had the wit, they had neither the will nor the futures that would support heavy-duty study (p. 245)."

### *Preserving the Nature of Tasks*

Yet the literature documents cases where teachers are able to strike bargains with students that include engagement with tasks in ways that maintain a high level of challenge or emphasis on conceptual understanding (Stein et al., 1996). How are they able to do this? Again, answers to this question in the literature shaped the code development process in my study.

Based on a reading of the literature on cognitive psychology and instructional research, Anderson (1989) suggested ways that teachers can reduce ambiguity and risk for students without lowering the cognitive level of tasks. One way is through scaffolding; that is, providing information that serves as a resource for students to use as they approach tasks, without diminishing their problematic nature, so that the students will still perform cognitive operations that create conceptual connections and make decisions regarding selection of information and methods. Repeated use of scaffolding is thought to help students develop powers of metacognition as they internalize the scaffolding over time (Holton & Clarke, 2006).

In addition, three teacher behaviors appear to help students handle challenging tasks and develop their problem solving abilities: (a) modeling, in which the teacher shows students how to think about problems by thinking out loud, pointing out crucial information, drawing conceptual links to clarify the relevant dimensions of the task, and calling attention to alternatives, (b) coaching, in which the teacher provides hints and cues, and (c) fading, in which the teacher provides less and less support over time. Confirming such claims, the QUASAR research found that factors associated with maintaining high cognitive level were high-level performance modeled by the teacher or

students, sustained press by the teacher for student explanations, and scaffolding by the teacher, as well as the use of tasks that built on students' prior knowledge and an appropriate amount of time provided for students to complete the task (Henningsen & Stein, 1997).

Anderson (1989) acknowledged the dilemma teachers face regarding accountability when implementing challenging or novel tasks. The literature on classroom management stresses the importance of holding students accountable for their performance on tasks in order to prevent disengagement. On the other hand, the literature on academic tasks shows that accountability can cause students to focus on extrinsic rewards and concentrate their energy on reducing the ambiguity and risk of tasks by pressing the teacher to lower expectations. Thus, based on the literature on motivation, she suggested ways teachers can refocus students' attention on intrinsic rather than extrinsic sources of motivation in order to reduce their tendency to pressure the teacher to reduce ambiguity: communicating the assumption that students are eager learners, inducing curiosity or dissonance, and making content more personally meaningful or concrete. Teachers can also establish classroom environments that minimize performance anxiety and promote positive views by students of themselves in relation to the task by minimizing competition and by utilizing tasks that make use of a variety of student abilities and that are novel or challenging yet do not present too great a risk of failure.

The role of teachers' knowledge and beliefs in inhibiting their ability to maintain the level of challenge or conceptual emphasis of tasks has already been mentioned; the converse would suggest that an understanding of how such tasks promote learning, a belief that students can handle such tasks, and a sufficiently deep understanding of

mathematics may help teachers implement these tasks in ways that preserve their nature. Warfield (2001) presented the case study of a fifth-grade CGI teacher whose knowledge of research-based information on children's thinking, along with deep understanding of the mathematics she taught, supported her in using tasks in ways that helped students make connections between solution strategies and their mathematical bases. Warfield argued that such knowledge helps teachers create tasks that enable students to extend their thinking to novel situations.

Fennema et al. (1996) also identified several types of knowledge and beliefs that assisted CGI teachers as they implemented tasks: knowledge of problem classifications, the belief that students were capable of solving problems with their own strategies, an understanding of student thinking, and knowledge about how to build on that thinking. Carpenter, Ansell, and Levi (2001) also found that CGI teachers' knowledge of learning trajectories for the content students were learning, and their ability to provide scaffolding, were essential.

Some of the literature has focused on sociomathematical norms that support teachers in their efforts to help students develop conceptual understanding, construct mathematical arguments, and solve non-routine problems—again, the kinds of thinking involved when engaging in challenging or novel tasks. Based on their study of a second-grade classroom using an inquiry approach to mathematics instruction, Yackel and Cobb (1996) noted that examining generic social norms (such as whether or not students challenged each other's thinking, developed explanations, and justified their own thinking) was insufficient for understanding how effectively students and teachers dealt with tasks that promoted mathematical reasoning and conceptual understanding. Those



norms that are specific to mathematics—sociomathematical norms—influenced whether such tasks were implemented as intended. These included agreement as to what counts as a convincing justification, a complete explanation, a mathematically sophisticated solution, or an efficient or elegant solution. These norms determined learning opportunities, for example, by influencing the extent to which statements were justified mathematically versus accepted due to the social status of the speaker, or whether explanations were accepted because they were connected to actions on mathematical objects that were meaningful to students or because they simply relied on procedural instructions. Yackel and Cobb (1996) argued that these norms are continually constructed and modified through the interaction of the teacher and students, and they act to support or hinder problem solving and mathematical reasoning.

In their study of four fourth and fifth-grade classrooms in three schools, Kazemi and Stipek (2001) identified four sociomathematical norms that contributed to a “press for conceptual learning.” While all four classrooms were characterized by the social norms of explaining strategies, finding multiple solutions, accepting errors as a part of learning, and working collaboratively, they did not have equal amounts of press for conceptual learning. Those classrooms that exhibited the most were characterized by the following sociomathematical norms: agreement that explanations go beyond procedural descriptions to include mathematical arguments, that mathematical thinking includes understanding connections among multiple strategies, that errors provide opportunities for developing deeper understanding and even reconceptualization of content, and that collaborative work involves individual accountability and mathematical justification.

## Summary

The literature described in this chapter shaped both the rationale for and method used in this study. Current thinking on how students “learn with understanding” and on the nature of mathematics provide perspectives on the kind of learning that mathematics educators wish to promote. The academic task literature shows that focusing on how teachers implement mathematics problems may yield understandings of how such learning can be supported. The literature on reform teaching provides a description of one way such understandings may be put into practice, but it also shows how difficult this approach is to implement. The literature on conceptually-oriented non-reform teaching suggests a possible alternative that seems to be commonly used in other countries, and which may be worthy of study. The literature on both reform teaching (including classroom observation instruments) and conceptually-oriented non-reform teaching describes problem characteristics and teacher actions thought to develop students’ problem-solving and mathematical reasoning abilities—factors that, along with behaviors identified in the academic task literature—influenced the development of coding categories as I carried out this study. Table 2 summarizes this literature.

Table 2: Summary of Literature on Reform and Conceptual Non-reform Teaching

| <b>Authors</b>  | <b>Type &amp; scope</b>   | <b>Claims/findings</b>  |
|---|---|---|
| Schoenfeld, 1992; Cuoco et al., 1996  | Philosophical;<br>Reform  | “Doing math” as mathematicians do (exploring, conjecturing, proving) helps students come to “know math.”  |
| Ben-Chaim et al., 1998; Carpenter & Lehrer, 1999; Carpenter et al., 2001; Cobb et al., 1993; Fennema et al., 1996; Hiebert et al., 1997; Hiebert & Wearne, 1993; Kahan & Wyberg, 2003; Kilpatrick et al., 2001; Romberg & Kaput, 1999; Stein et al., 2003 | Theoretical & empirical;<br>Reform;<br>Primary grades & 7 <sup>th</sup> grade | Problem-based teaching (students invent, justify, and compare multiple solution methods) and scaffolding lead to improved problem-solving and reasoning abilities, conceptual understanding, and more robust and flexible procedural knowledge. |
| Boaler, 2004; Talbert & McLaughlin, 2002  | Empirical;<br>Reform;<br>Minority HS students                                 | Traditional methods (teacher tells procedures, students practice) are ineffective; higher order tasks lead to increased reasoning and problem solving abilities.  |
| Mack, 1990  | Empirical;<br>Reform;<br>8 <sup>th</sup> grade                                | Building on students’ informal methods leads to connected conceptual and procedural knowledge.  |
| Newmann et al., 1995  | Empirical;<br>Reform;<br>elementary, middle, & high school                    | Tasks emphasizing generalizing, explaining, hypothesizing, generating new meanings, alternative solutions, and exploration of connections and relationships leads to higher achievement on complex tasks.                                       |
| Stein & Lane, 1996  | Empirical;<br>Reform;<br>8 <sup>th</sup> grade                                | Higher gains in problem solving and reasoning associated with use of high-level tasks, especially when implemented at high level  |
| Anderson, 1989; Henningsen & Stein, 1997  | Empirical;<br>Reform;<br>8 <sup>th</sup> grade                                | Scaffolding, modeling, and press for explanation are associated with high-level implementation of tasks   |

| <b>Authors</b>   | <b>Type &amp; scope</b>   | <b>Claims/findings</b>   |
|--|---|--|
| Hiebert et al., 2003   | Empirical;<br>Non-reform;<br>8 <sup>th</sup> grade                    | TIMSS Video Study countries with higher achievement than the U.S. implement making connections problems as making connections more often, but most involve listening to teachers' explanations and practicing exercises  |
| Hiebert and Handa, 2004;<br>Huang and Leung, 2004;<br>Lopez-Real et al., 2004;<br>Wang & Paine, 2003 | Empirical;<br>Non-reform;<br>6 <sup>th</sup> & 8 <sup>th</sup> grades | Chinese and Hong Kong lessons are teacher-dominated but lead students through carefully selected problems to systematically explore and connect ideas, construct arguments, develop concepts, develop and compare methods, and provide practice. Students are actively involved. |
| Leinhardt, 1986; Good et al., 1983   | Empirical;<br>Non-reform;<br>Elementary through 9 <sup>th</sup> grade | Direct instruction with product questions emphasizing multiple representations, patterns, justifying procedures, concept development, clear explanations, and connecting concepts is associated with higher scores on standardized tests and sometimes tests of problem solving. |
| McLaren, 2005  | Empirical,<br>Non-reform &<br>reform;<br>Undergraduates               | When comparing problem-based instruction with direct instruction using conceptual explanations, there were no differences in procedural or conceptual understanding.   |

## Chapter 3: Method

I describe the method for my study in three sections. First, I describe the pool of problems from which I drew successively larger samples. Then I explain the procedure I used for developing the coding system and coding the data. Lastly, I present the method I used for analyzing the data.

### The Problems

For the 1999 Video Study, TIMSS researchers selected a national random sample of 90 to 140 schools containing eighth-grade in each of Australia, Hong Kong, the Czech Republic, the Netherlands, Switzerland, and the United States (Jacobs et al., 2003). They randomly selected one eighth-grade mathematics teacher from each school, and one lesson was videotaped for each teacher. In addition, the 54 Japanese lessons videotaped for the 1995 study (also randomly selected from across the country) were added to this sample for the 1999 study. This yielded a sample of 638 lessons intended to be representative of lessons in the participating countries.

Because I could not obtain actual videos for this study, my analysis was based on lesson transcripts which had been translated to English. In the case of Japan, I obtained transcripts of 50 of the videotaped lessons. For each of Australia, the Czech Republic, the Netherlands, and the United States, I obtained transcripts of the 20 lessons that had been randomly selected for analysis by the TIMSS mathematics quality analysis group. In the case of Hong Kong, I obtained 19 of the 20 lessons used by that group, but two were missing lesson tables (to be described shortly), so I used 17. Transcripts from Switzerland

had not been translated to English and so could not be analyzed. This resulted in a total of 147 lesson transcripts for this study.

A TIMSS problem implementation team had examined all mathematics problems that were publicly discussed during the lessons. Each problem was coded at two stages—first, according to how it was initially stated, and second, according to how it was implemented; that is, publicly discussed (Jacobs et al., 2003). At the initial problem statement stage, it was classified as using procedures, stating concepts, or making connections (as defined in Chapter 2), depending on what it seemed to ask students to do. In particular, making connections problem statements were defined as follows:

Problem statements coded as making connections were those that asked students to engage in special forms of mathematical reasoning such as conjecturing, generalizing, and verifying. They were situations that asked students to think about mathematical concepts, develop mathematical ideas, or extend concepts and ideas...Some other examples of making connections problem statements included those that asked students to find a pattern, describe a relationships, generalize, compare results and methods, find examples of a mathematical principle, or write a problem with given conditions. (p. 122)

At the implementation stage, each problem was classified into one of the same three categories, or alternatively as giving results only, depending on how it was publicly discussed by the students and teacher. The definition for a making connections implementation was slightly different from that for a making connections statement:

Problem implementations were coded as making connections when the completion of such problems included mathematically rich discussions. Such discussions might focus on mathematical relationships, and include descriptions of properties and concepts containing mathematical justifications that were not stated as rules but as logically necessary consequences. If applicable, relationships between examples and principles might be demonstrated. Moreover, these mathematical ideas and relationships needed to be made explicit for all members of the class to see and think about the connections.

Some examples of making connections problem implementations included: describing connections between multiple representations (i.e., pictorial and numeric), making and justifying generalizations, comparing the mathematics of different solution methods, and considering why a particular process was mathematically appropriate. (p. 124)

Along with transcripts, I obtained lesson tables for all countries except Japan. These tables indicated, for each problem assigned to students, the initial statement of the problem, the answer accepted or presented by the teacher (if available), the problem's starting time (when it was first assigned), its ending time (when its public discussion ended), and if publicly discussed, its statement and implementation codes (answers only, using procedures, stating concepts, or making connections). Each lesson table also included an outline of the lesson listing major actions by the teacher and ideas discussed. For Japan, I obtained a spreadsheet that listed start and end times for all problems

assigned in the lessons and statement and implementation codes for all publicly discussed problems.

Altogether in the 147 lessons, there were 298 publicly discussed problems which were either stated or implemented as making connections. Based on the statement and implementation codes determined by the TIMSS researchers, they fell into the following three implementation trajectories: (a) in 69 problems connections were maintained, (b) in 179 problems connections were lost, and (c) in 50 problems connections were gained. The unit of analysis for this study was the problem discussion segment of dialog; that is, the portion of transcript between problem start time and problem end time that constituted public discussion of the problem.

### Development of the Coding System

My goal was to create a coding system that characterized important teacher behaviors during public discussion of a problem that influenced whether or not that discussion occurred in a way that made connections. This was to be done through a process that examined transcripts in light of the literature and instruments that examine teacher behaviors believed to either facilitate or inhibit higher-level thinking. I selected a subsample of problem discussions to develop a preliminary set of codes, which would then be used to code successively larger samples. During each stage, I calculated inter-rater agreement as a measure of reliability.

#### *Initial Code Development*

For initial code development, I randomly selected 37 of the problems (12%) stratified by the three implementation trajectories and by country. In nine of the



implementations, connections were maintained, in 22 connections were lost, and in six connections were gained, according to the coding performed by the TIMSS problem implementation team.

For each selected problem, I read the lesson table for the lesson in which it occurred to understand the context of the problem. In the case of Japanese lessons, since lesson tables were not available, I read the entire lesson transcripts. Then I read the portion of the transcript between the problem's starting and ending times to gain familiarity with the problem and its implementation. I read it a second time, more carefully, to try to identify the important actions of the teacher that influenced whether or not the implementation made connections.

I hypothesized that some of the behaviors previously seen to affect cognitive level during task implementation (Henningsen & Stein, 1997) might be observable in the transcripts and seen to significantly affect implementation. In addition, I thought that behaviors listed in classroom observation instruments might be similarly significant (e.g., Horizon Research, 2000; Milloy, 2006; Weaver, Dick, & Rigelman, 2005). However, I also thought it important to be alert to other practices that may not have been noted in prior American research. In particular, I hypothesized that whereas many U.S. reformers recommend that the teacher have students engage in particular mathematical behaviors (e.g., formulating arguments or reflecting on their thinking), the teachers in this sample may instead have performed these behaviors themselves as students watched and listened.

With these considerations in mind, I read transcript segments repeatedly with an eye toward identifying such key teacher behaviors and developing definitions for them.

The development of coding categories and definitions occurred simultaneously with the coding of the 37 transcript segments.

Portions of some transcript segments between problem start and end times did not represent public discussion of the problems. Instead, teachers were sometimes circulating about the room, speaking with individual students as they worked. Much of this dialog was not understandable, as it was either inaudible or made reference to things students had written on their papers but could not be determined from the transcripts. In addition, this dialog was not part of the public discussion of the problems. Therefore, I excluded these portions from the coding.

It became apparent that to identify the relevant teacher behaviors, it was necessary to understand more clearly what characteristics of a discussion qualified it as making connections. Through the reading of transcript segments and examination of the TIMSS definition of making connections implementations, I identified eight “making connections features”:

- comparing solution methods
- connecting representations
- developing, extending, or thinking about a concept
- describing a relationship between an example and a principle
- describing a mathematical relationship or pattern
- making a generalization
- justifying an assertion or solution method
- problem solving

Although the last feature (“problem solving”) was not found in the TIMSS definition, it seemed to characterize some making connections implementations that contained none of the other features. (Final definitions for each of these features will be given at the end of this section.)

If any one of the above features was present in an implementation, I considered it to qualify as making connections, regardless of whether it was present in teacher or student talk. I considered the absence of all of them to mean that the implementation did not make connections. Thus, these feature codes essentially “unpacked” the TIMSS definition of making connections, and described features of a problem that teachers brought out in a making connections discussion. They were also frequently mentioned in reform documents and observation protocols (Horizon Research, 2000; Milloy, 2006; MSEB, 1990; NCTM, 2000; Weaver, Dick, & Rigelman, 2005).

While these features gave a partial description of what teachers were doing to make connections (e.g., emphasizing justification), they did not present the whole picture. For example, how did teachers emphasize justification? Were they simply explaining a mathematical argument, or were they pressing students to formulate it? Therefore, I added a code to indicate who was doing most of the mathematical work during the discussion: teacher or students. However, in some implementations, the work seemed to be shared by both, and in others (some of the non-making connections implementations) there was no mathematical work being done; that is, only answers were given and/or non-mathematical topics were discussed. This resulted in four possibilities for who did most of the mathematical work: teacher, students, both, or no mathematical work.

I still needed to describe what the teacher did to influence who was doing the mathematical work, as well as what the teacher did to bring out or inhibit the making connections features. Again, I read transcripts to identify such behaviors. At this point, some of the QUASAR Project's "classroom based factors" that influenced cognitive level of implementation seemed to provide explanatory power. I used those factors that were teacher behaviors and that I expected to be observable in transcripts (e.g., that did not require knowledge of unavailable contextual information) to create a list of teacher behavior codes. Some of these behaviors occurred with varying frequencies in the transcripts, and seemed to be key behaviors that affected the type of discussion that occurred. For two reasons, I also included potentially relevant behaviors that I did not see in this initial sample: I anticipated that I might see such behaviors in the larger sample, and I wanted to be able to compare some of my findings to those of Henningsen and Stein (1997). The resulting list of teacher behaviors included the following:

- Lack of accountability for high level products or processes
- Shifting the focus to the correct answer
- Routinizing
- Modeling high level performance
- Pressing for justification and explanation
- Drawing conceptual connections
- Scaffolding

Henningsen and Stein (1997) had found that the first three of these were associated with implementations of low cognitive level, while the remaining four were associated with implementations of high cognitive level.

It should be noted that my coding procedure differed from that of Henningsen and Stein (1997) in an important way. Those researchers used one set of classroom based factors to account for implementations classified as high cognitive level, and another for those classified as low cognitive level. After coding each implementation as either high or low cognitive level, they limited their coding to only those found in the corresponding list. However, in my study, behaviors of both kinds seemed to coexist in the same transcript. I found instances, for example, where teachers set the direction of the discussion by consistently routinizing the problem, but also drew important conceptual connections. This coexistence of such presumed contradictory behaviors was at least in part due to differing definitions of implementation; in Henningsen and Stein (1997), implementation referred to how students dealt with tasks as they worked on them, but in my study (constrained by the TIMSS Video Study) it referred to how the teacher and students publicly discussed problems. Therefore, in order to provide an accurately nuanced description of what was happening in the classroom, I considered all of the codes to be eligible for use whether or not an implementation was judged as making connections.

In addition to the teacher behaviors taken from the QUASAR literature, I saw others in the transcripts that seemed to set or change the direction of implementation:

- Failing to build on student contributions (both ideas and questions)
- Skimming the mathematical surface
- Shifting the focus to a procedure
- Building on student contributions

This yielded a coding system consisting of 8 making connections features, 11 teacher behaviors, and an indication of who did most of the mathematical work, which was to be applied to each problem implementation. As coding proceeded, I wrote definitions for both features and behaviors, and developed threshold criteria for both to determine when they were significant enough to code. I refined these definitions and criteria as new issues arose during the coding process, and their final versions will be presented at the end of this section.

Because my goal was different from that of the original TIMSS Video Study, and because my coding relied only on transcripts and involved the interpretation of the TIMSS definition of making connections, the above process narrowed the sample to 32 problem implementations. First, in the case of implementations where connections were maintained or gained, the sample included only those implementations in which I interpreted the transcripts to show evidence of the making connections features I had identified.

Second, in the case of implementations that lost connections, the sample included only those implementations where I was able to determine the potential connections implied by the problem statement that were not followed up by the teacher, so that I could code teacher behaviors that seemed to be associated with the loss of connections. For example, in a lesson on area, after the teacher had students find areas of rectangles by multiplying length and width, and areas of irregular figures by superimposing them on grids, she drew an L-shaped region on the chalkboard and asked students, “Can anyone suggest how you might do something like that?” My assumption in this instance was that the problem statement implied that students would think about the concept of area to

develop a way of solving this new kind of problem, and my task was to determine what the teacher did to contribute to the lack of discussion that reflected that thinking.

Once the preliminary coding system was established, I recruited a post-doctoral fellow and a doctoral student to perform reliability checks. I conducted a two and a half-hour meeting with them in which we discussed the coding procedure and definitions, and I had each of them code a transcript segment individually. To help them understand the context of the problem implementation, I gave them a summary of what had transpired in the lesson prior to discussion of this particular problem. We then discussed the coding of this segment to resolve any disagreements.

Then I assigned each of the two coders five problem implementation transcript segments to code independently over the next week. Again, each segment included a description of context. Upon completion, inter-coder agreement ranged from 66 to 94%, averaging 79%. We resolved discrepancies during two meetings, one with each coder, that lasted three to four hours. These discussions led to refined code definitions and threshold criteria.

### *Coding of the Second Sample*

I randomly selected additional transcript segments in order to obtain a larger total sample containing ten problems from each country—five that were implemented as making connections, and five that were not. I had to make an exception for the U.S. problems since only one was implemented as making connections. This yielded a sample of 56 problems, and I proceeded to code the new problems according to the same procedure as before.

I selected five transcript segments (including two that were difficult to code) and had them additionally coded by the post-doctoral fellow. Agreement was 86% and discrepancies were resolved. This had the effect of further refining definitions and eliminating two codes: one making connections feature and one teacher behavior:

- describing a relationship between an example and a principle
- failing to build on student contributions

These codes were eliminated due to an inability to develop definitions that could be applied reliably. In addition, removal of the making connections feature above did not in itself change the classification of any of the implementations in the current sample.

I also added one code; a common way for teachers to enact the justification feature was to step students through a mathematical argument as opposed to pressing for justification—an approach noted in the literature on non-reform Chinese teaching (Huang & Leung, 2004; Wang & Paine, 2003).

### *Coding of the Final Sample*

I randomly selected additional problems in an attempt to obtain a sample of roughly 100 problems approximately evenly distributed among countries and implementation trajectories. In order to meet this condition, in four cases I included a pair of problems from the same teacher. I considered this as acceptable as long as they were not in the same implementation trajectory, which may have caused certain teachers to have more influence on the findings for a particular trajectory than others. After I had coded a total of 119 problem implementations, the narrowing process as described earlier resulted in a final sample of 82 problem implementations distributed as shown in Table 3.



Table 3: Distribution of Problems in Final Sample

| <b>Country</b> | <b>Maintaining Connections</b> | <b>Losing Connections</b> | <b>Gaining Connections</b> | <b>Total</b> |
|----------------|--------------------------------|---------------------------|----------------------------|--------------|
| Australia      | 2                              | 8                         | 1                          | 11           |
| Czech Republic | 5                              | 5                         | 3                          | 13           |
| Hong Kong      | 2                              | 5                         | 3                          | 10           |
| Japan          | 10                             | 7                         | 4                          | 21           |
| Netherlands    | 5                              | 7                         | 4                          | 16           |
| USA            | 0                              | 10                        | 1                          | 11           |
| Total          | 24                             | 42                        | 16                         | 82           |

Finally, I identified four more difficult-to-code transcripts and had them coded by the post-doctoral fellow, and two more meetings occurred to discuss discrepancies and refine definitions. At this point, one making connections feature was eliminated—discussion of mathematical relationships or patterns—because a workable definition of “mathematical relationship or pattern” could not be found, resulting in unreliable coding and disagreements which could not be resolved. This may have changed the classification with respect to making connections of only one problem implementation since all others that had been coded with this feature had also been coded with another making connections feature.

In addition, I eliminated one teacher behavior from the coding system: modeling high level performance. Henningsen and Stein (1997) defined it as engaging in a high level performance such as the presentation of a solution using multiple representations or solution methods, meaningful exploration, or appropriate justification. Although transcripts often appeared to show teachers performing at what could perhaps be called a high level since they were well versed with the material—developing a proof or solving a

problem using multiple representations—it was too difficult to determine when this constituted “modeling,” especially considering the lack of contextual information. In addition, I felt that simply saying that a teacher was performing at a high level did not give me much information about what the teacher was actually doing.

Once the coding system was finalized and applied to all transcript segments, I randomly selected five problem implementations for double coding in order to calculate inter-rater agreement. This resulted in a final inter-rater agreement of 88%.

### *Final Coding System*

The final version of the coding system consisted of the two groups of codes for making connections features and teacher behaviors, and a code specifying who did most of the mathematical work.

#### *Making Connections Features*

Each making connections feature was coded only if it involved substantive mathematics as judged by the coders. If more than one feature occurred in a particular implementation, they were all coded. Final definitions of the features were as follows:

- *Comparison of the mathematics of solution methods.* Either a relationship between solution methods is explained (e.g., why one solution method is more elegant than, or a general case of, another one), or a correspondence between steps or aspects of different solution methods is described (e.g., subtracting from both sides of an equation in the symbolic method corresponds to undoing the last addition step while working backwards in an informal method).

- *Connection between representations.* A representation is an algebraic symbol string, table, graph, diagram, or physical object(s) used to represent a problem situation, quantity, object, concept, or relationship among them. The representations must provide different perspectives of some common idea, situation, or object. A connection between representations means the way in which aspects of different representations correspond to each other. For example, “a negative linear coefficient in a linear equation corresponds to a downward slant in the graph.”
- *Examining a concept.* A concept or property is examined more deeply than simply recalling or applying it. This may involve describing some component, aspect, representation, or example of the concept, or some connection to another concept. It may involve extending a concept or developing a new concept.
- *Generalization.* There is explanation of a mathematical problem, assertion, solution method, concept, or argument that is more general than that previously stated or discussed; the object of the earlier discussion is a specific case of that of the later discussion.
- *Justification.* Mathematical knowledge is used to explain why a solution method, step, problem-specific claim, or general mathematical assertion (e.g., theorem) is or is not correct, valid, or appropriate. Justification does not include procedural explanations, strategic reasons for choosing a particular solution step or approach, or non-mathematical rationales.
- *Problem-solving.* There is explicit examination (not just carrying out) of an overall solution plan, not just pieces of a plan. This might include discussion of how one arrives at a solution plan, strategic justification of a plan (i.e., explanation of why the

overall plan is chosen), discussion of intermediate goals beyond those already stated in the problem, or monitoring progress toward meeting those goals.

### *Teacher Behaviors*

Each teacher behavior code was coded only if it was consistently enacted or enacted at key moments, so that it seemed to set or change the direction of problem implementation. Again, if more than one teacher behavior was observed, all were coded:

- *Lack of accountability for high level product or processes.* Student(s) contribute incorrect or insufficient (e.g., unclear or incomplete) answers, explanations, or ideas, but the teacher does not make a significant effort to probe them (i.e., ask for more detail or justification) or press for more adequate contributions. This definition was adapted from Henningsen and Stein (1997). It originally also included the teacher's lack of expectation that students justify their methods, but in my study this would have resulted in this code being applied to all losing connections implementations by definition. Also, Henningsen and Stein's (1997) original definition simply said that unclear or incorrect explanations were *accepted*; I interpreted the word "accepted" to mean that there was no press by the teacher for a more adequate contribution from the student. I did not consider the teacher simply correcting the student to be holding him or her accountable for high level processes.
- *Skims the mathematical surface.* The original problem statement implies, or initial discussion includes, a focus on concepts, meaning, or understanding, but the teacher fails to delve sufficiently into the mathematics of the problem, resulting in a discussion which refers to a concept or meaning but only at a superficial or vague level.

- *Shifts to focus on answer.* The original problem statement implies, or initial discussion includes, a focus on meaning, concepts, or understanding, but the teacher shifts the focus away from it and to the accuracy or completeness of the answer.
- *Shifts to focus on procedure.* The original problem statement implies, or initial discussion includes, a focus on meaning, concepts, or understanding, but the teacher shifts the focus away from it and to a procedure.
- *Routinizes problematic aspects.* The teacher reduces ambiguity or complexity by specifying explicit procedures or steps to perform, or takes over challenging aspects by telling students how to perform them or performing them for students. The teacher takes away opportunities for students to discover and make progress on their own. This may occur from the beginning of problem implementation, or later in the discussion but soon enough to affect the direction of the discussion. This definition came from Henningsen and Stein (1997), and is somewhat different from the dictionary definition, which may align more with “shifts to focus on procedure.” The difference between these two codes will be elaborated in Chapter 4, where specific examples are given.
- *Steps through argument.* The teacher steps students through an argument by telling or using product questions. An argument is a sequence of justified assertions leading to a mathematical claim.
- *Presses for justification.* The teacher repeatedly asks students for justification, meaning, or explanation (beyond recounting a procedure) through questioning, comments, or feedback. Clear and consistent messages are sent to students that

explanations and justifications are as much a part of classroom mathematical activity as are correct answers.

- *Draws conceptual connections.* The teacher draws attention to connections between a concept and a representation, procedure, or other concept. This does not include justification.
- *Scaffolds.* The teacher provides assistance by providing information or asking a series of questions other than product questions that assists student(s) in answering a question or solving the problem without reducing complexity or challenge. Assistance is just enough to allow students to make progress.
- *Builds on student idea.* The teacher builds on student contribution(s) (perhaps erroneous) by having the student explain more, asking student(s) questions about it, discussing it, relating it to other ideas, or otherwise using it in his or her teaching. This brings some new mathematics or higher level of understanding to the discussion that was present in neither the prior discussion nor the student's original contribution.

#### *Who Does the Mathematical Work*

In addition, each transcript segment was coded according to who did most of the mathematical work. The coder selected one of four choices:

- The teacher did most of the mathematical work.
- Students did most of the mathematical work.
- The mathematical work was shared by both.
- No mathematical work occurred (e.g., only the problem or answer were given and any other dialog was non-mathematical).

## Method of Analysis

My first research question asked what teacher behaviors were associated with implementations that made connections. I addressed this question by generating frequency counts for the three types of codes (making connections features, teacher behaviors, and who did the mathematical work) for all making connections implementations, regardless of whether or not the problems were originally stated as making connections. In addition, I chose example transcript excerpts to illustrate each feature and behavior to give the reader a clearer picture of what the codes meant teachers were doing.

The second question asked what similarities and differences in behaviors existed between implementations that maintained connections and those that gained connections. To address this question, I tabulated and compared frequencies of the codes for those two implementation trajectories.

The third question asked what teacher behaviors were associated with losing connections. I addressed this question by generating and reporting frequency counts for teacher behaviors and who did the mathematical work for such implementations. Again, I chose examples from the transcripts to illustrate each behavior.

Finally, the last question asked what similarities and differences existed in teacher behaviors that occurred while maintaining versus losing connections. I addressed this by tabulating teacher behavior and mathematical work codes for the two implementation trajectories and comparing them.

## Chapter 4: Results

The purpose of this study was to describe what teachers in a small sample of eighth-grade classrooms from six countries did to lead the discussion of mathematics problems in ways that did or did not make connections. Results will be presented by research question. First, I will describe what teachers did during implementations that made connections: the features of the problems that teachers focused on, the behaviors they exhibited, and whether the teacher or students did most of the mathematical work. I will also provide excerpts from transcripts as examples. Then I will compare these results by problem statement type to determine whether there were any differences between maintaining and gaining connections.

Then I will present results for implementations that lost connections—the behaviors teachers exhibited and who did the mathematical work—again providing examples from the transcripts. Finally, I will compare these results to the corresponding results for implementations that maintained connections to determine what differences in teacher behaviors may have contributed to whether connections were maintained or lost.

It is important to remember that the sample of 82 problem implementations examined here is not representative of those that occur throughout the various countries, so generalizations can not be made beyond this sample. This study is meant to be exploratory and to suggest common ways teachers may implement problems.



## Making Connections Implementations

### *Features Exhibited*

Figure 1 shows the percent of making connections implementations that contained each feature identified through the code development process. Percents add up to more than 100 because some problem implementations exhibited more than one feature.

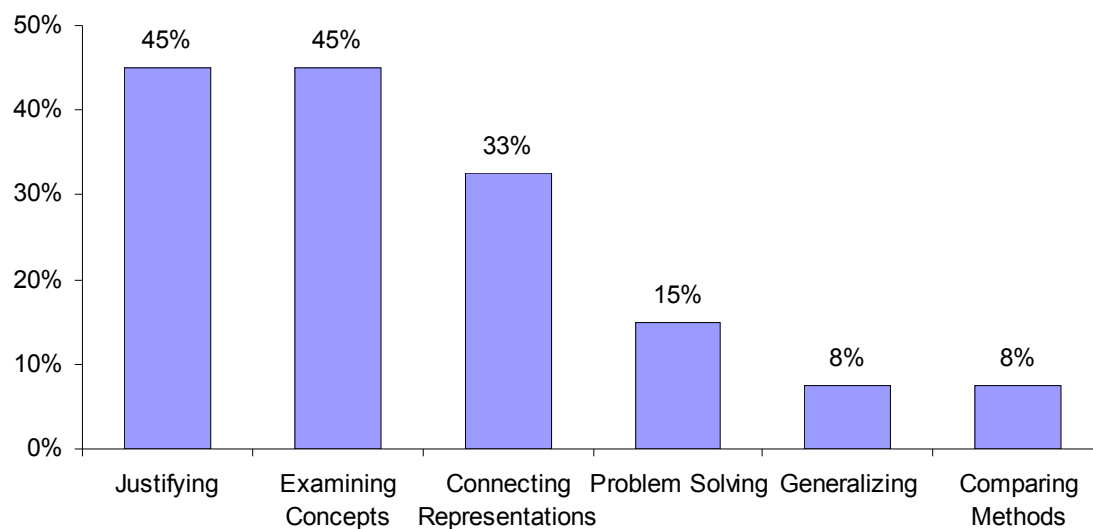


Figure 1: Features Observed in Making Connections Implementations (n = 40)

The most common features occurring in this sample were justifying and examining concepts, each of which occurred in almost half of the implementations. Connecting representations occurred in about one third of the implementations. Problem solving, generalizing, and comparing solution methods were not very common. Each of these features will now be described in more detail and illustrated by excerpts from transcripts of implementations.

### *Justification*

Justification was defined as the use of mathematical knowledge to explain why a solution method, procedure, result specific to the problem, or general mathematical assertion was or was not correct, valid, or appropriate. Justification did not include descriptions of procedures, strategic reasons for choosing a particular solution step or approach, or non-mathematical rationales. Justification could occur at a particular point in the discussion, or throughout a discussion. It was coded as long as, and only if, it was judged as mathematically substantive.

The excerpt below, taken from a Hong Kong lesson, illustrates how one teacher used justification to add connections to a problem whose statement had not been originally coded as making connections. The teacher had just reviewed squaring integers and had worked through problems asking students to find positive and negative integers given their squares. She explained the use of the radical sign and assigned several exercises where students had to find positive and negative square roots and solve equations similar to  $x^2 = 25$ , where the number after the equal sign was a perfect square. After going over the answers to these exercises, she asked students to determine whether the square root of -4 is equal to 2 or -2, or there is no solution.

25:02 T So, this uh square root of a- sorry, uh negative- sorry, square root of negative four. Is it equal to two, or negative two, or no solution?

25:13 Ss No solution, no solution.

25:16 T Okay. Who say that it is equal to two?

25:19 Ss Hahaha.

25:19 T Put up your hand.

25:24 T Why? Because take square root means- what is the number times itself equal to negative four? So two times two is four.

25:33 T So it- it does not equals to negative four. It is- it does not equal to negative four.

25:38 T How about negative two? So this answer is incorrect. How about this?

25:45 T Is it correct?

25:46 Ss Haha.

25:46 T Do you think this is correct? Please put up your hand.

25:49 T Why? Because negative two times negative two ... equals to ...

25:56 Ss Four.

25:56 T Four. So it does not equal to negative four.

26:00 T So, there is ... //no solution.

26:02 Ss //No solution.

26:04 T Why? Because uh, if you find that it is uh, just like that, A squared equal to negative four.

26:11 T So which number times itself is equal to a negative number?

26:15 Ss No.

26:15 T No. Because a number A, all number can be uh divides- or looked into three, uh, must be one of these. One is positive.

26:26 T One is ...

26:27 Ss Negative.

26:28 T Negative. Or ...

26:30 Ss No solution.

- 26:31 T No solution.
- 26:33 T A number must be one of- one of- one of these. Maybe it is a positive- maybe it is negative, or ...
- 26:42 Ss Zero.
- 26:43 T Zero, yes. Very good.
- 26:44 T So uh, if A squared- we see uh, by case- so if A is positive, then what is the value of A squared? Positive? Negative? Or zero?
- 26:58 Ss Positive.
- 26:59 T Positive. And then if it is negative, what is the result of the square of A?
- 27:05 Ss Positive.
- 27:05 T Positive. If it is zero, then what is the squ- //square of zero?
- 27:09 Ss //Zero.
- 27:10 T Zero. So is there any answer equal to negative?
- 27:13 Ss No.
- 27:14 T No. So, this answer- this neg- uh negative four, you cannot find the answers. Okay? Because, all the square ...
- 27:24 T All the square, you cannot find the negative result.

[HK-008, IP 20]

The teacher repeatedly used mathematical knowledge to explain why a claim or assertion was valid (25:24-25:33, 25:49-26:00, 26:15-27:24). The last portion of dialog (beginning at 26:15) is particularly interesting; it constitutes a more general proof by cases that no negative number has a square root.

Although justification ran through much of the discussion of the problem above, the following excerpt from a Czech lesson illustrates how it could be limited to one point in the dialog but still be sufficiently significant to be coded. In this case, a step in a geometric construction procedure was justified. The problem asked students to construct triangle ABC, given that  $AB = 7$  cm,  $BC = 4.5$  cm, and the height measured along a segment perpendicular to AB was 3.5 cm. The teacher had a student describe the procedure: first construct segment AB 7 cm long, then make an arc with center B and radius 4.5 cm. The student's next step was to draw a parallel line 3.5 from segment AB, and the teacher asked her to justify that step:

18:34 S Then we draw a line which is in the distance three and half centimeters from side AB and it is parallel to it.

18:53 T Why should it be a line that is parallel? Can you give a reason for it?

19:01 T We search a set of points, of all points, that meets the property that their distance from line segment AB is three and half centimeters. And we know that such a set of points is?

19:18 T Well?

19:19 S A triangle?

19:21 T ( ) you sketched- sketched a line//

19:23 S //A line.//

19:23 T //A parallel line.

[CZ-041 IP 31]

Although the teacher asked the student to justify the step in the procedure, the teacher actually did the justification (19:01, 19:23). In fact, in both of the examples given

here, justification was accomplished primarily through teacher talk rather than student discourse. This was typical of the enactment of making connections features in this sample, a point that will be addressed further.

### *Examining Concepts*

Most, if not all, mathematics problems involve one or more mathematical concepts in some way, whether these concepts are simply recalled and applied, or whether they exist in the background as the basis for a procedure that is to be executed. However, for this category to be coded, the discussion must have explicitly included the examination of a concept or property more deeply than simply recalling or applying it. The discussion had to have involved the examination of some component, aspect, or representation of a concept, the extension of the concept, or the development of a new concept.

The following excerpt from a lesson in the Czech Republic illustrates the precise examination of the intuitive concept of the exterior of a circle through the use of distance. The problem was to describe one of the relative positions of a circle and a line. The teacher asked a student go to the chalkboard to draw one possibility, and the student drew a line  $p$  that did not intersect the circle (with center  $S$ ). The teacher then proceeded as follows:

18:17 T And now then, I would like us to draw there the distance of the straight line  $p$   
-- I would mark it lower case  $p$  -- from the center  $S$  of the circle.

18:30 T How will you do it? What will you use?

18:32 Ss A perpendicular line.

- 18:34 T A perpendicular straight line, correct. So, then using the pivot of a right angle ruler.
- 18:59 T I will mark the distance  $h$ , lower case  $h$ .
- 19:05 T And now I will mark there for you the radius, I will mark it in green. You can also use a colored pencil. A radius  $r$ .
- 19:17 T So we can see instantly that...that the  $h$  is longer than  $r$ . But now we have to prove it. So I'll mark this point  $P$ .
- 19:34 T Do not forget to mark the foot of the perpendicular straight line, yes?
- 19:41 T And the the right angle.
- 19:51 T So, not to forget the right angle! So that it was sure that this is the shortest distance. So.
- 20:04 T So, the segment line  $SP$  is larger than the radius. Our segment line  $SP$ , as you see, is larger than the radius  $r$ . I can record that as...
- 20:20 T The distance  $h$  is larger than  $r$ . Now, we'll also on the straight line  $p$ , find an arbit... or choose an arbitrary point  $X$ . And add there the segment line  $SX$ .
- 20:45 T What have these three points created for us? Points  $P, X, S$  or  $S, P, X$ ? What have we got there?
- 20:53 Ss A right angled triangle.
- 20:53 T Len.
- 20:54 SN A right angled triangle.
- 20:55 T A right angled triangle. Ehm...The segment line  $SP$ , what function does it have in the right angled triangle?
- 21:03 SN Cathetus.

- 21:04 T SP is a cathetus. Excellent. And what about the segment line SX, Terri.
- 21:11 SN A hypotenuse.
- 21:12 T A hypotenuse, yes. And what do you know about the sides in the right angled triangle? What is the relationship between a cathetus and a hypotenuse? Which one is the longer? Iris.
- 21:25 SN A hypotenuse.
- 21:25 T A hypotenuse. A hypotenuse is the longest.
- 21:28 T So definitely the point X as it is further from the center S and the circle k than the point P, then definitely with the circle also, actually...is further from the center S than the radius. Is it so?
- 21:45 T So, even the point...because it is, actually, a hypotenuse in the right angled triangle.
- 21:53 T So we can state that the straight line p has no common point with the circle k . So zero common...
- 22:06 T ...points.
- 22:07 T And how do we call such a straight line? It does not intersect the circle. It has not have any common point with it.
- 22:20 T How do we call such a straight line? It is an external straight line of a circle. p is an external straight line- write all this down, yes-
- 22:39 B -of a circle k.
- 22:48 T So by this we have exhausted one, the first possibility of the relative position of a straight line and a circle.

[CZ-011, IP 3]



Although the teacher did not prove what she said she was going to prove since it was already given (namely, that  $h > r$ ), she did work through a rather precise proof that in this case the line did not intersect the circle. It was her translation of the informal concept of “exterior” into a precise distance formulation that constituted examining a concept.

The excerpt from the Hong Kong lesson given earlier also provides an example of examining a concept because the discussion involved development of an aspect of squares and square roots that had previously not been addressed; namely the fact that square roots of negative numbers do not exist (in the set of real numbers).

### *Connecting Representations*

A representation was defined as an algebraic symbol string, table, graph, diagram, or physical object(s) used to represent a problem situation, quantity, object, concept, or relationship among them. To be considered “different” representations, they had to provide different perspectives of some common idea, situation, or object. Many problem discussions involve multiple representations, but this code required that a connection between those representations be explicitly discussed. A connection was defined as a description of the way in which aspects of the representations correspond to each other.

A portion of dialog from a Dutch lesson illustrates this feature. The problem was to find an equation for  $y$  in terms of  $x$ , given the table of values shown in Figure 2, where  $x$  represented the number of months elapsed, and  $y$  represented the amount of money accumulated.

| $x$ | $y$ |
|-----|-----|
| 0   | 5   |
| 1   | 8   |
| 2   | 11  |
| 3   | 14  |

Figure 2: Finding an Equation for  $y$  in Terms of  $x$ .

The teacher began by telling students that for this kind of problem, there is always a certain number that is added at each step. He explained that he would graph the points before generating the formula. The following dialog began after he set up axes for the graph:

28:34 T And at the zero is an important point. Because that's five, it is five high.

28:40 T Well I'll make steps here too of one, two, three, four, five and at one it is eight, six, seven, eight.

28:51 T Here, this is the line. Are these two points enough to draw the rest of the line?

28:57 SN Yes.

28:57 SN Yes.

28:57 T They're all on top of each other anyway and is that? That's because these steps are the same every time. How much is added per step every time?

29:04 Ss Three.

29:04 SN Thr//ee.

29:05 SN //Three

29:04 T Three.

29:06 SN Oh, I thought that there were two.

29:08 T Well almost. Okay this line runs say continuously up like this. And always a straight line.

[Students work.]

- 29:56 T Boys, uh, a side issue, because we still have to look at the uh, formula a moment.
- 30:03 T And the book says, and that's very important.
- 30:06 T It says like what do we get per step, per one month, here, I'll just write this down for clarity, per one month, that is also per one added step.
- 30:20 SN Can I explain it?
- 30:20 T Yes, //definitely.
- 30:21 S //Well, in at - um - if you still, um, the zero the month let's say then you already have five - uh - five guilders.
- 30:30 S So - but also three guilders is added every time and so you have to do the number of months times three plus five.
- 30:37 T That's a perfect explanation. I try to tell it just as well. It - he is five already at zero (times/months).
- 30:46 T So you already have for example five thousand, Just saying something, in your bank account. After one month how many thousand is added?
- 30:54 SN Thr//ee.
- 30:54 T //Three thousand. So per month every time... three is added. And that's why you have to multiply that with each other.

[NL-002, IP 53]

Representations were connected at three places in this portion of dialog. At 28:34 the teacher connected the starting point in the table with the starting point (y-intercept) on the graph, at 28:57 he connected the pattern of increase seen in the table with the

alignment of the points on the graph, and in 30:06-30:54 he and a student connected this pattern with the coefficient in the equation they developed. In addition, because the concept of linearity was examined by studying its manifestation in three representations, examining concepts was coded for this dialog.

### *Problem-Solving*

Although this study followed the TIMSS convention of using the word “problem” to refer to any mathematical question for which students were expected to find an answer, for the purposes of this code, problem solving was not defined simply as applying a procedure or method to find an answer to a question. This code was reserved for the explicit development or examination of an overall solution approach to a problem beyond recall and application of the method. It could include discussion of how one arrived at a solution plan, strategic justification of a plan (i.e., explanation of why the overall plan is chosen), setting intermediate goals beyond those already stated in the problem, or monitoring progress toward meeting those goals. Only six implementations exhibited this feature.

The geometric construction problems found in some of the Czech lessons provide an interesting case in point. Three of the 13 Czech implementations in the sample involved construction of triangles or quadrilaterals given particular side lengths, angle measures, and/or altitudes. In these problems, teachers followed a standard four-part sequence: statement of the problem, analysis, construction, and conclusion. The analysis portion was presumably the point at which a solution method was to be developed. It consisted of drawing a rough sketch, developing a solution plan, and writing it down step-by-step, apparently according to a rather formal format.

In only one of the three cases did the analysis portion of the implementation meet the definition of problem solving used in this study, so that it constituted a case of gaining connections. The problem was to construct quadrilateral ABCD with  $a = 5.8$  cm,  $b = 3.4$  cm,  $c = 3.8$  cm, angle B = 75 degrees, and angle C = 115 degrees. (It was understood that  $a = AB$ ,  $b = BC$ , and  $c = CD$ .) The analysis began with the teacher sketching a figure:

15:59 T Let's not try to choose, perhaps, the shape of a rectangle or of a square or of a trapezoid because the dimensions are such that it probably won't be any of the shapes I named.

16:39 Ss ( )

16:41 T Well, exactly. It is probably a consequence of a triangle construction, right?  
So ... side A is the segment AB.

17:00 T Side B ... now is valid three point four tenths of a centimeter. Side C ... three point eight tenths of a centimeter. ...The angle beta at the vertex B ... //and angle

17:25 SN Seventy- alfa- I mean gamma.

17:28 T Gamma, or at the vertex C.

17:36 T The whole point is in this that we have to cleverly divide the whole quadrangle by a diagonal into two triangles. From the two triangles we'll be able to construct one and when we construct it we'll look for the last, the fourth vertex, right?

17:57 T When we look at the dimensions and the data ...

18:00 SN Then we can do the ABC - the triangle.

18:04 T Correct. So mark this way that it is possible to construct the triangle ABC because we know its one side, //second side and the angle enclosed by them.

18:15 SN //( )

18:18 T We wrote it in an abbreviation ... uh ... when two sides //and an angle were given

18:25 Ss //SUS

18:26 T SUS, right? So, the procedure. How will we proceed?

18:30 SN First we will draw the angle BC- D ( )

18:35 SN AB.

18:36 SN BCD- we have an angle and two sides.

18:39 T That's an idea! Do you hear what he's saying? He noticed that we know the sides B and C and the angle gamma. So he could easily construct in the first three steps the triangle BCD.

18:53 T And he would construct it according to the same method, right? Because he also has a side, also an angle and also a side. I would rather stay with the triangle ABC because it seems to me more convenient, no? So, first ...

19:12 SN AB.

19:13 T AB.

19:15 S //Five point eight.

19:16 T //That's the A. Five point eight tenths of a centimeter. See- help, help!

19:23 S The angle beta.

19:24 T The angle beta ... seventy five degrees. And right away we can put on the leg ...

19:31 S BX.

19:32 SN BC

19:34 T No?

19:35 SN Three point four.

19:37 T //Because

19:37 S //Three point four.

19:38 T Well, you see. That's the B, right? Three point four tenths of a centimeter.

19:43 S ( )

19:47 T Now, let's imagine what we have already, yes? Come, take a look. We have AB, angle beta and the vertex C already on the leg, right? So the triangle ABC is finished.

19:57 S Uh-huh.

19:58 T So, we need //only the vertex- D.

19:59 S? D.

20:00 S? ( ) make an angle one hundred fifty degrees.

20:03 T Yes, this way, right ? Perhaps ... and on its //leg we'll lay

20:09 Ss A circle

20:12 T Three point //eight tenths and it is ready.

20:13 S? // Eight ... three point

20:14 T Well you see how you said it- the angle ... gamma, ... one hundred fifteen degrees and right on the side of the angle I will measure CD- that's //the C.

20:29 SN //Three point eight.

20:30 T Three point eight tenths.

20:33 S ( )

20:35 T Yes, because by that we will get the vertex, right? So, ABCD ... it's ready. ...

So, we can perform the construction.

[CZ-061, IP 3]

This seemed to demonstrate the development of a solution method based on a general strategy given by the teacher at 17:36; namely partitioning the figure into two pieces, one of which could be constructed using a previously learned procedure. This strategy provided a rationale for the particular steps that would be developed, and was enough for a student to come up with a first step at 18:30. Although the teacher recognized the student's suggestion as valid, she chose instead to apply the strategy in a slightly different way. Constructing triangle ABC then became a subgoal, which the teacher noted was to be reached at the step described at 19:47.

By contrast, the analysis portion of another Czech implementation did not qualify as problem solving (although as previously described, it contained justifying so that it was still coded as making connections) because the procedure seemed to be only recalled, without any development or strategic rationale. This was the problem described earlier, where students were to construct a triangle ABC where  $AB = 7$  cm,  $BC = 4.5$  cm, and the height perpendicular to AB was 3.5 cm. The “analysis” portion of the discussion proceeded as follows:

17:42 S (We have) the side c ... the side a ... and the height  $v_c$ .

17:59 T Correct, a height is a perpendicular dropped from a vertex to the opposite side so the foot of the perpendicular is denoted there. Yes.



- 18:06 T What we know is denoted by the color. Excellent. We will return back to the white chalk and we will start with the strategy of the construction.
- 18:13 S I will (place the tip) on point B and I set the compass for four and half centimeters.
- 18:16 T First we draw- what line segment? Where do you have point B? First it has to (arise).//
- 18:24 S //( ) AB, seven centimeters.
- 18:26 T Yes.//
- 18:27 S //Then we place the compass tip on point B, we will transfer four and half centimeters and we will circumscribe an arc.
- 18:31 T Yes, the circumference K one will arise, correct.
- 18:34 S Then we draw a line which is in the distance three and half centimeters from side AB and it is parallel to it.
- 18:42 T Parallel, we will denote the parallelity.
- 18:50 T It is not to be seen well.
- 18:53 T Why should it be a line that is parallel? Can you give a reason for it?
- 19:01 T We search a set of points, of all points, that meets the property that their distance from line segment AB is three and half centimeters. And we know that such a set of points is?
- 19:18 T Well?
- 19:19 S A triangle?
- 19:21 T ( ) you sketched- sketched a line//
- 19:23 S //A line.//

19:23 T //A parallel line. Well. What next? How will point C arise? You see it there, already.

19:33 S By the intersection of the ( ) one and the (line), by ( ) and the line M.

19:37 T Excellent. Well, and how will we complete the triangle?

19:44 S We will connect A and C.

19:45 T Excellent. Yes. So this is the whole analysis.

### *Generalizing*

Many mathematics problems involve the statement or use of mathematical generalizations. For this feature to be coded, however, there needed to be the development or explanation of a mathematical problem, assertion, example, solution method, concept, or argument that was a more general version of one that had been previously stated or discussed. That is, the object of the earlier discussion had to be a specific case of that of the later discussion. Generalizing in this way was relatively rare, occurring in only three problem implementations, all of which gained connections.

In the following example, an Australian teacher implemented a non-making connections problem as making connections by having the class generalize examples students were giving to solve a problem, thus going beyond the original problem statement. The problem asked students to determine whether the following statement was sometimes, always, or never true: "The difference between two negative numbers is positive." The excerpt begins where a student provided both an example and a counterexample to the statement to support his answer of "sometimes":

32:36 S Negative five, take negative two and you get negative three or you could go negative five take negative six and you get positive one.

32:55 T Good, thank you, Norton. Now, can anybody take that a step further?

32:59 T Certainly Norton has shown us two separate cases where in the first instance the difference is negative and the second instance the difference is positive.

33:12 T Can anybody take that a little bit further and give us a description of why or when you're going to get a positive response and when you're going to get a negative response.

33:26 T Stan.

33:27 SN When the number's smaller like five into two, you got five that's larger than negative two, but, that, and that will turn out to be negative three, but for the positive the number's smaller than six-

33:41 SN So it would be a positive, it's still a negative, that's why it's positive.

33:45 T No.

33:45 Ss Ha ha ha.

33:51 T Did you understand that ( )?

33:53 SN No.

33:54 Ss Ha ha ha.

33:55 T I did, I did.

33:56 SN I mean like five is a negative but it's smaller than the six.

34:01 T Now I think one of the problems we're having is something that came up the other day. Which number is bigger, three or negative five?

34:15 T And I think if we answer that- that if we clearly state what we mean by bigger and smaller for positive and negative numbers it'll make Stan's answer a little clearer.

34:32 T Regina, you want to say something?

34:34 SN Um, yeah, if the number on the right is a negative, um, no, it's smaller than the number on the left then it's gonna be a negative first ( ).

34:44 T But what do we mean by smaller?

34:47 S Um, like um//smaller than the number on the left ( ).

34:50 T //Sh sh sh.

34:57 T Smaller, we usually talk about less than. When we talk about smaller we talk about less than. But in this case we need a - a wider understanding of what smaller means.

35:14 T Now somebody in here whispered something just now that I heard. Bud?

35:19 SN Closer to zero.

35:21 T Nice and loud.

35:22 S Closer to zero.

35:23 T So the - the idea of which number is closer to zero comes into it.

35:28 T Stan, if you were to explain your - give us your answer again and instead of using bigger and smaller you were to use the - the idea of closer to zero here -

35:41 T I think you might be a little bit, little bit clearer. You want to have another go?

35:46 SN All right. Okay, um, when it's negative, like you got negative three ( ) um negative two is closer to zero.

35:58 T Than.

- 35:59 S And negative five is farther away zero so that means it would be a negative three and the positive the- it's the other way around the five is closer to the zero and the six is further away from the zero.
- 36:15 T Okay, so when are we going to get a positive answer from a subtraction of two negatives?
- 36:19 T Come straight to the point, when are we going to get a positive answer when we're subtracting one negative number from another negative number?  
Regina.
- 36:30 SN When the number on the right is further away from zero.
- 36:33 T Good.
- 36:33 S And the number on the left ( ).
- 36:35 T Good good, nice and succinct. When the number that you're subtracting is further away from zero than the number you're subtracting from and don't get the giggles, Regina.

[AU-030, CP 13]

From the specific examples  $-5 - -2 = -3$  and  $-5 - -6 = 1$  the teacher led the class to make the generalization that, when subtracting negative numbers, if the number after the subtraction sign is closer to zero than the number before the subtraction sign, then the difference is negative, and if the number after the subtraction sign is farther from zero than the number before the subtraction sign, then the difference is positive. Note also that this dialog involved the development of the concept of absolute value (although it was not referred to by name), so that this excerpt also exemplified examining a concept.

Although in the example above, students seemed actively involved in the development of the generalization, a generalization could be developed primarily by the teacher. This can be seen in the discussion of the square root of negative numbers that occurred in the Hong Kong implementation given earlier (see p. 89). In that case, the teacher's proof in the second part of the dialog (beginning at 26:15) was coded as generalizing since it generalized the preceding argument (25:24-26:00).

### *Comparing Solution Methods*

As previously noted, like generalizing, comparing solution methods was not a common way of making connections, occurring in only three implementations. As specified in the TIMSS definition of making connections, this required not only the presentation of more than one solution method, but also the comparison of the mathematics in them. This could have been done by explaining a relationship between solution methods or a correspondence between steps or aspects of different solution methods.

One of the three implementations that included such a discussion was found in a Japanese lesson, where the implementation gained connections. The problem was to prove that three parallel lines divide two transversals proportionally (see Figure 3).

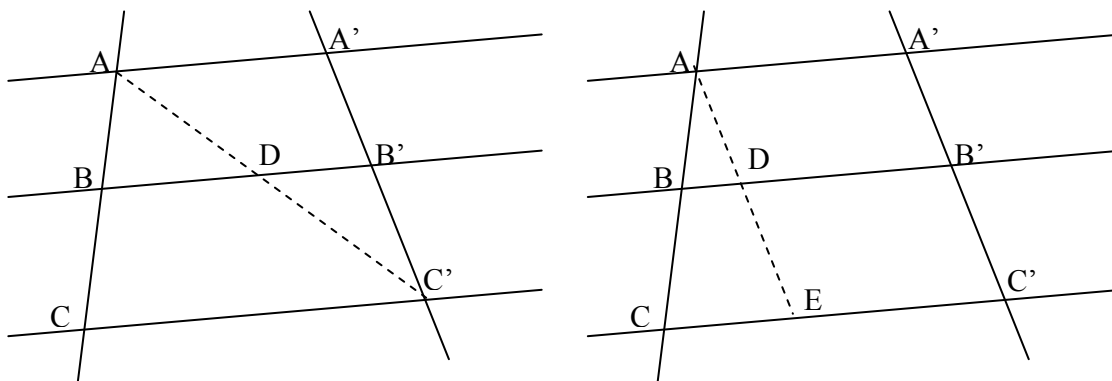


Figure 3: Two Ways of Proving  $AA' \parallel BB' \parallel CC'$  Implies  $AB/BC = A'B'/B'C'$

One solution method involved drawing an auxiliary line segment from A to C', while the other used an auxiliary segment drawn from A so that it was parallel to line A'C'. Both proofs relied on a theorem stating that a line segment drawn parallel to one side of a triangle (in this case segment BD) divides the other two sides of the triangle proportionally. In the excerpt below, the teacher summarized the first method (after noting that the ratio of AB to BC equaled the ratio of AD to DC'), and then summarized the second approach (“the case of N”) by comparing it to the first:

25:31 T Then next this triangle is inverted, but ... you just flip the other triangle, and if you were asked to find this over this it is equal to this over this. Next ... this over this what this means is that ...

25:52 T if you combine all three ... this over this is equal to this over this and this over this is equal to this over this ... therefore, this over this is equal to this over this ... and this expression is consistent.

26:08 T It still is true, right? So ... rather than write down this I want you to understand this with your eyes your sense ... this over this is equal to this over this ... this over this is equal to this over this ...

- 26:19 T therefore, please get a visual feel that this over this is equal to this over this.  
Thus, I won't be writing the reply all along here. Okay, for the case of N ...
- 26:29 T in the case of N we look at the triangle ACE just like before ... and in the  
same way we think of this over this. AB over BC.
- 26:42 T Just as before this time AB over BC is equal to AD over DE.
- 26:53 T There seemed to be a lot of people who understood this so I won't ask  
everyone, but at this point two sets of the opposite sides are parallel to each  
other ... so it is a parallelogram.
- 27:04 T This means the length of this segment and the length of this segment are  
equal, ... and the length of this segment and the length of this segment are  
equal,... this means ...
- 27:14 T this over this is equal to this over this ... and this length is equal to this length,  
... this length is equal to this length so ... this over this is equal to this over  
this and so ...
- 27:26 T this and this, and this and this are each equal ... and so this over this is finally  
equal to this over this. Right?

[JP-038 IP 2]

### *Teacher Behaviors*

Although the implementation features described so far indicate the content of problem discussions that qualified them as making connections implementations, and to some extent the kinds of mathematical thinking the teacher emphasized, they do not fully explain how teachers accomplished these emphases. The teacher behavior codes provide more information. Figure 4 shows the percent of these implementations in which each



teacher behavior captured by the coding system occurred either consistently or at a key moment, so that it was deemed to have set or changed the direction of problem implementation.

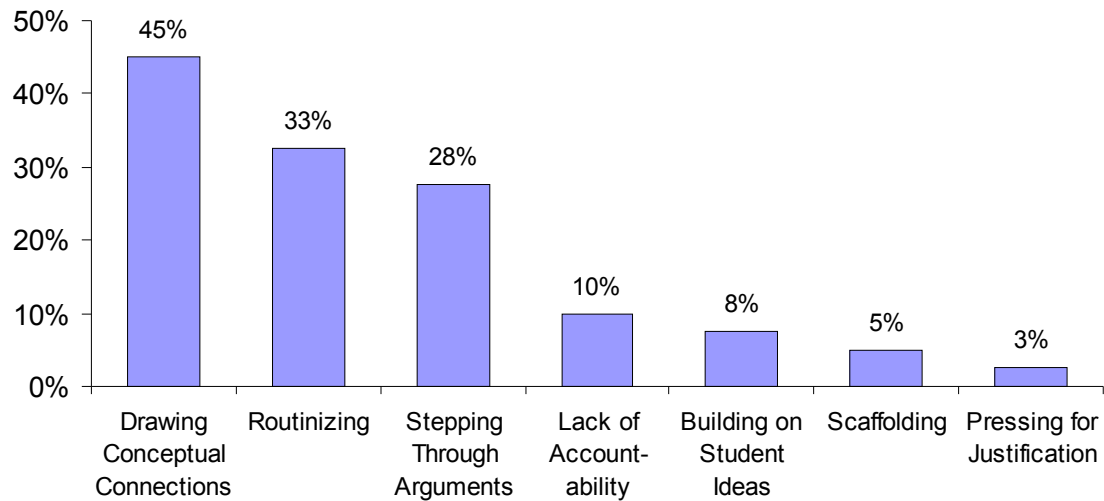


Figure 4: Teacher Behaviors in Making Connections Implementations (n = 40)

As can be seen from the figure, drawing conceptual connections was the most frequent teacher behavior during making connections implementations, followed by routinizing and stepping through arguments. Relatively rare were failing to hold students accountable, building on student ideas, scaffolding, and pressing students for justification.

#### *Drawing Conceptual Connections*

The most prevalent teacher behavior was drawing conceptual connections, taken from Henningsen and Stein (1997). It occurred in 18 out of the 40 making connections implementations. In these cases, teachers directed students' attention to a connection between a concept and a procedure, representation, or another concept. They did this by either explaining the connections or asking questions about them.

Most commonly, they made a connection between a concept and a representation, as in the Dutch lesson described earlier (NL-002, IP 53), where the teacher graphed data from a table where the y values increased by 3, described the resulting pattern in the graph, and developed the formula. In that discussion, the teacher drew connections between the concept of linearity and its manifestations in three representations: a constant increase in numeric values, a straight line graph, and a symbolic representation.

The connection between linearity and its tabular representation was examined even more closely in the following implementation from a Japanese lesson in which connections were gained. The problem was to graph  $y = 2x - 1$ . After graphing the y-intercept and the points (1, 1), (2, 3), (3, 5), and (4, 7), the teacher called attention to the role of the coefficient “2” in the equation:

04:37 T When X increases by one ... how about Y?

04:43 S Two.

04:45 T Increases by two. When the difference [between X values] is one the difference here [between Y values] is two.

04:54 T That's just because X is... multiplied by two the difference becomes doubled.

05:05 T As a matter of fact if we multiply two [to X] zero remains the same, but one becomes two. Two becomes four. At this point the difference becomes doubled.

05:16 T In lin- linear functions you multiply something and add something [to a function] but

05:20 S Uh huh.

05:24 T what we have to add here after that is ... we have to add minus one. So no matter what we add the difference does not change does it.

05:35 S No.

05:35 T Umm. The fact that the difference spreads here depends on the number multiplied to X. So this number and this number are the same.

05:46 S Uh huh.

05:46 T Um. So this number ... on linear functions the number multiplied to X agrees with the difference.

05:53 S Uh huh.

[JP-024, IP 1]

Here the teacher drew students' attention to the connection between the linear coefficient in the equation and the slope as seen in the difference between successive values in the y-coordinates. This was done by tracing the difference between x values as they are first multiplied by the coefficient (4:54-5:05, 5:35) then increased or decreased by the constant (5:24).

In some cases, a connection was drawn between concepts and solution methods, as shown in the following excerpt from another Japanese lesson. The teacher presented students with a drawing of three points in the plane, and asked them, "We would like to find one more point and draw a parallelogram. What kinds of methods are there to determine the fourth point?" After students had time to develop their solutions, the teacher had some of them present them on the board. Altogether seven students presented different solution methods, and the teacher connected five of them to conditions for a

parallelogram by asking other students to identify the conditions that validated each method. The following excerpt shows two of the solution methods being addressed:

30:24 T Umm then okay? [Please ] draw the second one- ... then, Itumo. Please introduce yours.

30:43 S Umm. First measure the length from here to here with the compass, and ... that. That is the length, and we put a mark here.

30:55 S Then we do over here in the same way, and we put mark here and then I connected them.

31:01 T Okay. Then the people who drew the quadrilateral ... in the same way.

31:09 T Okay. That's good. Then if we say this in words what kind of quadrilateral did she draw?

31:18 T Then, Okada Emi.

31:21 S The parallelogram's sides that face- face each other are equal.

31:26 T Oh. The sides that face each other are equal. Right?

31:38 T Umm. Now then ... next umm this. Okano.

32:09 S Well in the beginning, draw here a line like this and measure the angle here, and this. It's the same angle as here, and ... draw a line here and in the same way measure the angle over here,

32:25 S and put marks, and take the place where it intersects, and that's how I drew it.

32:32 T Okay. Then the people who drew it like this please raise your hands.

32:36 T One person two people three people fou-.

32:39 T Okay. Well there is only few huh? Then for this of what kind of conditioned quadrilateral did she draw?

32:46 T Somebody is mumbling it. Who is it? ( One more time ) please raise your hand with confidence. Who is it? Okay. Terashima.

32:54 S The alternate interior angles'

32:56 T The alternate interior angle is

32:57 S equal.

32:58 T That's right huh?

33:22 T She drew it with this idea right?

33:03 T Hmm. Um then let's go to the next one.

[JP-018 IP 1]

### *Routinizing*

Perhaps surprising is the presence of routinizing—a behavior associated with lowered cognitive level (Henningsen & Stein, 1997)—in roughly one out of every three implementations judged as making connections. Routinizing meant that teachers removed the challenge of the problem by giving students explicit procedures or steps to perform, by telling them how to perform them, or by actually performing them for students. For this to occur, of course, the original problem must have had ambiguity or challenge; for example, there must have been evidence that a solution procedure had not been previously given to students. The teacher must have taken away opportunities for students to make progress on their own, and this must have occurred soon enough in the discussion to affect its direction. The teacher did this by telling or using product questions.

There were two ways that this behavior could occur in an implementation that still made connections. First, a teacher could in one part of the dialog take over and tell

students what procedure to follow to solve a problem, but in another part of the dialog enact making connections features or behaviors, such as justifying or drawing conceptual connections. Second, both kinds of behavior could occur simultaneously. Teachers could make connections through telling or by asking product questions which implicitly made decisions for students about what procedure to follow. As a result, they enacted making connections features or behaviors while removing ambiguity and challenge from the problems.

This latter approach can be seen in a Japanese implementation of a problem involving a trapezoid with horizontal bases 12 and 18 and height 16 (see Figure 5). The students were asked to find the area of the portion above the segment connecting the midpoints  $M$  and  $N$  of the non-parallel sides.

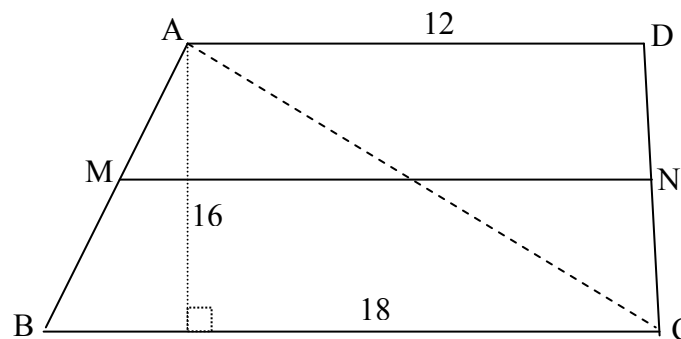


Figure 5: Finding the Area of AMND

12:13 T Uh, like the material we did before since it's one to one, one to one, the three lines are parallel.

12:20 T Therefore, both the top and the bottom are trapezoids.

12:22 T And, if it's like that in the top trapezoid (it's) the upper base plus the lower base; ... therefore, you want to know the length of MN.

12:32 T With that you use the Midpoint Connection Theorem.

- 12:35 T Okay to continue.  
[Students work]
- 14:04 T Okay, well were you able to do about half? Okay, stop.
- 14:09 T If we draw the supplementary line we can use the Midpoint Connection Theorem. There are various ways of drawing supplementary lines, for example, ... connect A and C.
- 14:21 T After doing [that] the whole figure is divided it into two triangles, triangle ABC and triangle ACD, uh again these are one to one; moreover, since they're parallel
- 14:32 T uh, these ones also are one to one ... and so this is the midpoint and this is also the midpoint.
- 14:38 T With that we can use the Midpoint Connection Theorem.
- 14:42 T Okay, after getting that ... about how much is this? Ninomiya.
- 14:50 S Nine centimeters.
- 14:52 T Right, half of eighteen is nine.
- 14:55 T About how much is here.
- [JP-045, CP 3]

The teacher removed problematic aspects of the problem by telling students how to solve it (12:22, 12:32, 14:09, 14:38) and leading them through the process (14:42-14:55), but he also justified the claims (12:13, 12:20, 14:21, 14:32) that made his procedure mathematically valid. Therefore, the teacher both routinized and justified, and did so almost simultaneously.

### *Stepping Through an Argument*

The third most frequently coded teacher behavior, stepping through an argument, occurred in a little over one-fourth of the making connections implementations. In these cases, the teacher used telling or a series of product questions to lead students through a sequence of mathematically justified assertions to make a conclusion. This was the primary way teachers enacted justification, occurring in 61% of implementations that involved that feature.

This approach can be seen in the Hong Kong excerpt given earlier, in which the teacher led students through an argument showing first that  $-4$  had no square root, and then that no negative number has a square root in the set of real numbers. The second, more general argument is repeated here:

26:33 T A number must be one of- one of- one of these. Maybe it is a positive- maybe it is negative, or ...

26:42 Ss Zero.

26:43 T Zero, yes. Very good.

26:44 T So uh, if A squared- we see uh, by case- so if A is positive, then what is the value of A squared? Positive? Negative? Or zero?

26:58 Ss Positive.

26:59 T Positive. And then if it is negative, what is the result of the square of A?

27:05 Ss Positive.

27:05 T Positive. If it is zero, then what is the squ- //square of zero?

27:09 Ss //Zero.

27:10 T Zero. So is there any answer equal to negative?



27:13 Ss No.

27:14 T No. So, this answer- this neg- uh negative four, you cannot find the answers.

Okay? Because, all the square ...

27:24 T All the square, you cannot find the negative result.

[HK-008, IP 20]

At 26:33, 26:44, 26:59, 27:05, and 27:10, the teacher provided steps of the argument but asked students to fill in pieces of information, then made the final conclusion at 27:14.

### *Lack of Accountability*

In this study, lack of accountability meant that the teacher failed to ask a student for more detail, justification, or a more adequate contribution when the student provided an incorrect or insufficient answer or idea. As with routinizing, this is a behavior that would seem to reduce chances that connections would be made, but coexisted with other behaviors that did make connections. This occurred in four implementations.

The following excerpt from an American lesson provides an example. The teacher had reviewed inequality symbols, talked about their use to describe real life situations, and had students translate English statements into simple algebraic inequalities (e.g.,  $p < 5$ ). Then she posed the problem: "Give me a number that would make this statement,  $y \geq -3$ , true." After soliciting several correct answers, all integers, and asking the class if they were correct, she proceeded as follows:

29:47 T How many numbers will make this a true statement?

29:58 T Gary, what do you think?

29:59 SN A lot.

29:59 T A lot, okay. Peter?

30:02 SN Six.

30:03 T Six, okay. Athena what are you thinking? You look like you disagree. All the numbers in the world. Okay, you're getting on the right track.

30:12 SN It's infinite.

30:13 T Infinite number, there is what we're looking for. Okay, any number- what's the smallest- how close can we get to this?

30:13 SN Zero.

30:20 T Okay.

30:21 SN Negative two.

30:22 Ss Negative three.

30:22 T Negative three because this may- it can be equal to a negative three, so negative three is greater than or equal to a negative three.

30:30 T So it can start at negative three, and everything that's greater than and then keep going on to infinity.

30:36 T So it starts at negative three and it keeps on going. If I wanted to show this answer on a piece of paper, I can't write numbers to infinity.

30:47 T So what would be a way that you can think of to show this answer? Can you think of one?

30:54 SN ( ) numbers and then put some dots.

30:56 T Okay, the numbers and put dots. Okay, what's another way Karl?

30:59 SN Um, draw a circle and equals negative three ( ).

31:04 T Okay, so you're using this term for infinity. Alright, those are all good ideas. Let's use a number line.

- 31:10 T And let me show you how to use it. If I have um, if I take the number line ...
- 31:19 T Okay. And we said that it could be equal to- if it- if it can be equal to a negative three, I'm going to put a circle here and I'm going to color it in.
- 31:31 T And when I color it in, that means that negative three is part of the answer. So it's negative three and everything to the right of it.
- 31:40 T And I'm going to put an arrow there to show that it keeps on going and doesn't stop. So the answer to this inequality is negative three and above.
- 31:50 T So put a dot on negative three, draw an arrow going in the um, greater than direction, and color it in.

[US-024 IP 11]

The teacher seemed to consistently accept insufficient or incorrect answers as correct, without probing or challenging them; this occurred at 29:59-30:03, 30:13, and 30:54-30:59. Although correct or complete answers were also given by other students or the teacher, the impression left was that all answers were correct. In spite of this, the discussion did involve the development of the concept of inequalities and the connection of symbolic and graphical representations, so it was coded as making connections.

### *Building on Student Ideas*

In only three of the making connections implementations did the teacher build on student ideas. This means that the teacher responded to a student's contribution, whether it was mathematically correct or not, in some way that involved an idea beyond the student's original contribution.

The following excerpt shows a rather simple way that a teacher in Hong Kong built on a student's thinking so that the problem gained connections: After having

students find the circumference of a circle given its diameter, the teacher asked them to find the circumference of another circle given that its radius was 33 cm. While going over this problem, the teacher used a student's solution to derive the formula  $C = 2\pi r$ :

12:45 T Now, this time we will try to- uh, we try to think about the special relationship between radius and diameter. Now, for example here, radius is thirty-three and Sandy tried to times two here.

12:59 T That means two radius- radius add another radius, but of course, uh, the same value- the same value. Radius add radius is just like two R. Radius add the same radius.

13:13 T Two R represent to the D or we can say that this formula can change to circumference is equal to R or two times R and also times the pi.

[HK-003 IP 2]

### *Scaffolding*

Scaffolding by teachers was rare in this sample, occurring in only two implementations. This behavior was defined as the teacher providing information or asking questions that assisted students in answering a question without reducing complexity or challenge. The assistance needed to be just enough to allow students to make progress. In general, this meant that teachers asked questions that directed students' attention to the issue at hand, or that suggested general heuristics, without telling them what to do. Thus, the use of product questions was excluded since they implicitly told students what steps to take.

The Australian discussion shown earlier (AU-030 CP 13) was one of the two implementations in which the teacher provided scaffolding. The teacher led the class in

generalizing about when the difference of two negative numbers is negative, and when it is positive. Only the portion of the discussion where the teacher scaffolded is given here.

33:12 T Can anybody take that a little bit further and give us a description of why or when you're going to get a positive response and when you're going to get a negative response.

33:26 T Stan.

33:27 SN When the number's smaller like five into two, you got five that's larger than negative two, but, that, and that will turn out to be negative three, but for the positive the number's smaller than six-

...

34:01 T Now I think one of the problems we're having is something that came up the other day. Which number is bigger, three or negative five?

34:15 T And I think if we answer that- that if we clearly state what we mean by bigger and smaller for positive and negative numbers it'll make Stan's answer a little clearer.

34:32 T Regina, you want to say something?

34:34 SN Um, yeah, if the number on the right is a negative, um, no, it's smaller than the number on the left then it's gonna be a negative first ( ).

34:44 T But what do we mean by smaller?

34:47 S Um, like um//smaller than the number on the left ( ).

34:57 T Smaller, we usually talk about less than. When we talk about smaller we talk about less than. But in this case we need a - a wider understanding of what smaller means.

- 35:14 T Now somebody in here whispered something just now that I heard. Bud?
- 35:19 SN Closer to zero.
- 35:23 T So the - the idea of which number is closer to zero comes into it.
- 35:28 T Stan, if you were to explain your - give us your answer again and instead of using bigger and smaller you were to use the - the idea of closer to zero here -
- 35:41 T I think you might be a little bit, little bit clearer. You want to have another go?
- 35:46 SN All right. Okay, um, when it's negative, like you got negative three ( ) um negative two is closer to zero.
- 35:58 T Than.
- 35:59 S And negative five is farther away zero so that means it would be a negative three and the positive the- it's the other way around the five is closer to the zero and the six is further away from the zero.
- 36:15 T Okay, so...when are we going to get a positive answer when we're subtracting one negative number from another negative number? Regina.
- 36:30 SN When the number on the right is further away from zero.
- 36:33 T Good.
- 36:33 S And the number on the left ( ).
- 36:35 T Good good, nice and succinct. When the number that you're subtracting is further away from zero than the number you're subtracting from.

The teacher's utterances were limited to managing the discourse (33:26, 34:32, 35:14), pointing to a barrier that needed to be overcome in order to make progress (34:01, 34:15, 34:44, 34:57), asking students to rephrase or clarify their ideas (35:28, 35:41,

35:58, 36:15), or emphasizing or repeating students' ideas (35:23, 36:33, 36:35). In this way, he helped students solve the problem (one that was more sophisticated than the one originally posed) without reducing the challenge of the task by doing any of the mathematical work for students.

### *Pressing for Justification*

Even rarer than scaffolding was pressing for justification, which occurred in only one implementation in the entire sample. Pressing for justification referred to the teacher repeatedly asking students to justify or explain their reasoning beyond description of a procedure. To receive this code, the teacher, through her questions, comments, and feedback had to consistently communicate to students that explanations and justifications were as much a part of classroom mathematical activity as were correct answers.

The single case of this behavior was a short discussion of a problem that occurred in a Dutch classroom. The problem statement asked students to determine, when rolling three dice, what outcome was just as likely as rolling a three. When a student replied, “eighteen,” the teacher began the following line of questioning:

15:41 T Eighteen, because?

15:43 SN ( ) three times six.

15:46 T Eighteen you can only throw by throwing three times six. Why is fifteen not correct?

15:53 SN Because you ( ).

15:54 T I can throw that by throwing three fives, can't I?

15:56 SN I can two fives and then six...

15:57 SN Six.

15:58 T I can also do that with two fives. Well, two fives is not a good example then, eh?

16:03 Ss Ha, ha, ha.

16:05 T Because then I have to throw another five to get fifteen.

16:08 SN Oh, yeah.

16:09 T But I also can?

16:10 Ss Five, four, six.

16:11 T Throw five four six. Or six five four. There are more possibilities. Yes? Well, that's exactly the point here. Just get the hang of what is equally difficult as those other situations.

16:28 T To throw three with three dice can be done in one way only. There is one other number you can throw in only one way.

The teacher began by asking the student to justify her answer (15:41), and when the response was rather short, the teacher rephrased it (15:46) and asked another question (15:54) to probe her understanding of the justification she just gave. When the student's explanation was inadequate, she pointed out that fact (15:58-16:05) and asked for a better explanation (16:09). Once it was obtained, the teacher elaborated on the explanation (16:11) and summarized the justification (16:28).

### *Who Did the Mathematical Work*

Figure 6 shows the percent of making connections implementations in which teachers, students, or both did most of the mathematical work. In half of these implementations, the teachers did most of the mathematical work during the discussions; that is, they made most of the decisions and did most of the talking that brought out the



making connections features. In only three of the implementations did students do most of the work. In the remaining 43% of implementations, the teacher and students shared in doing the bulk of the mathematical work.

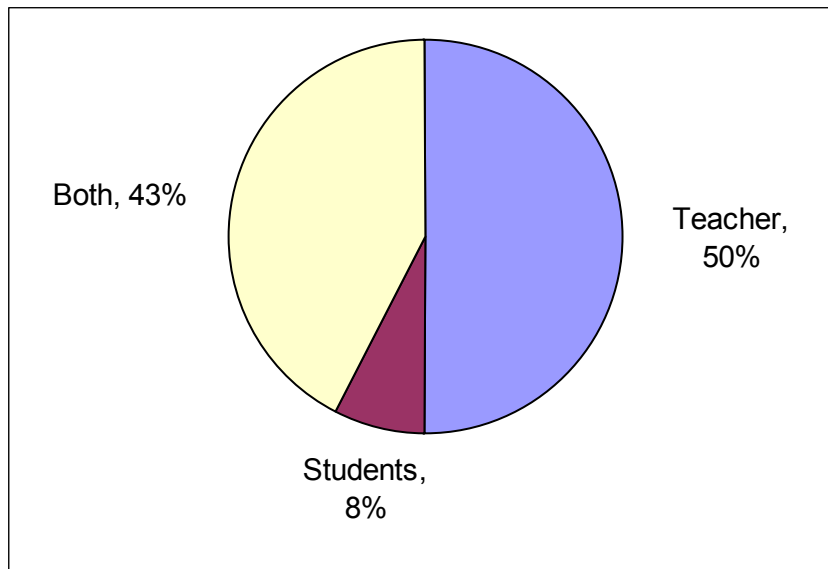


Figure 6: Who Did the Work in Making Connections Implementations (n = 40)

#### Maintaining vs. Gaining Connections

The second research question in this study asked what kinds of differences might be seen between problem implementations where connections were maintained and those where connections were gained. In other words, did teachers make connections differently depending on whether or not the problem was originally stated as making connections?

#### *Features Exhibited*

Figure 7 compares the making connections features addressed by teachers in the two implementation trajectories. It shows that for the most part, the features brought out during discussions of problems were similar regardless of trajectory. This suggests that

when teachers “added” to non-making connections problems in order to implement them as making connections, what they added was similar to what they addressed when problems were originally stated as making connections.

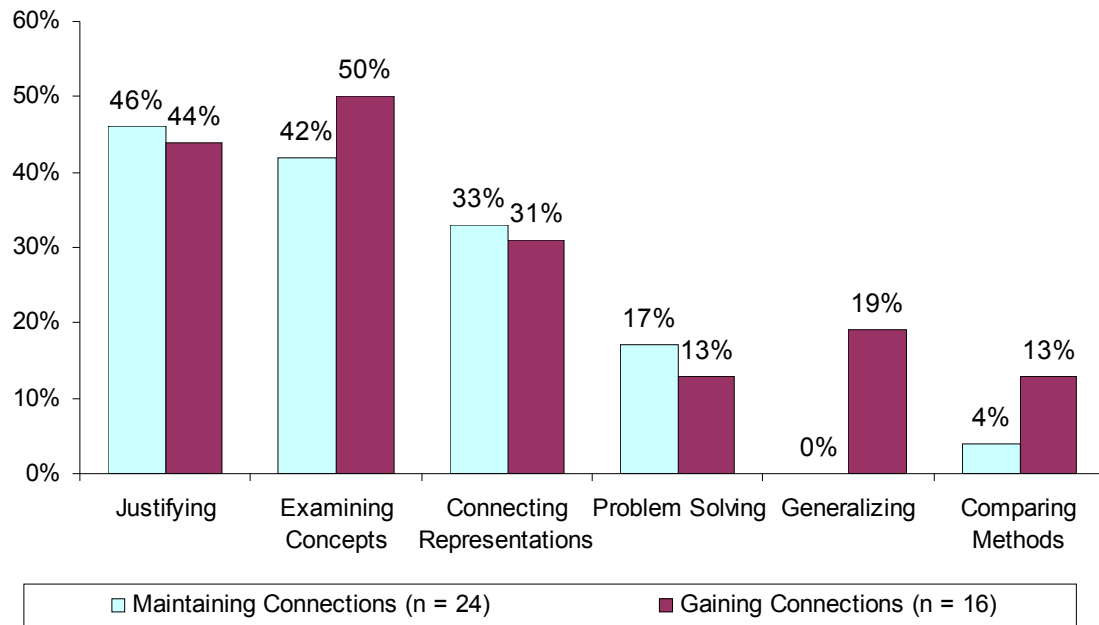


Figure 7: Features Observed when Maintaining vs. Gaining Connections

The only difference is that generalizing did not occur during discussion of any problems originally stated as making connections, but did occur during discussion of three of the problems where connections were gained. Therefore, in this sample, there were three instances in which teachers added to a problem by making (or having students make) generalizations beyond the original statement of the problem, but there were no instances in which teachers addressed generalization when problems were already stated as making connections.

### Teacher Behaviors

Figure 8 shows the corresponding results for teacher behaviors. Four of the behaviors seemed noticeably more common when connections were gained than when connections were maintained. Three of them are behaviors expected to contribute to making connections (drawing conceptual connections, stepping through arguments, and building on student ideas), while one would be expected to inhibit making connections, or at least contribute to the teacher doing most of the work (lack of accountability).

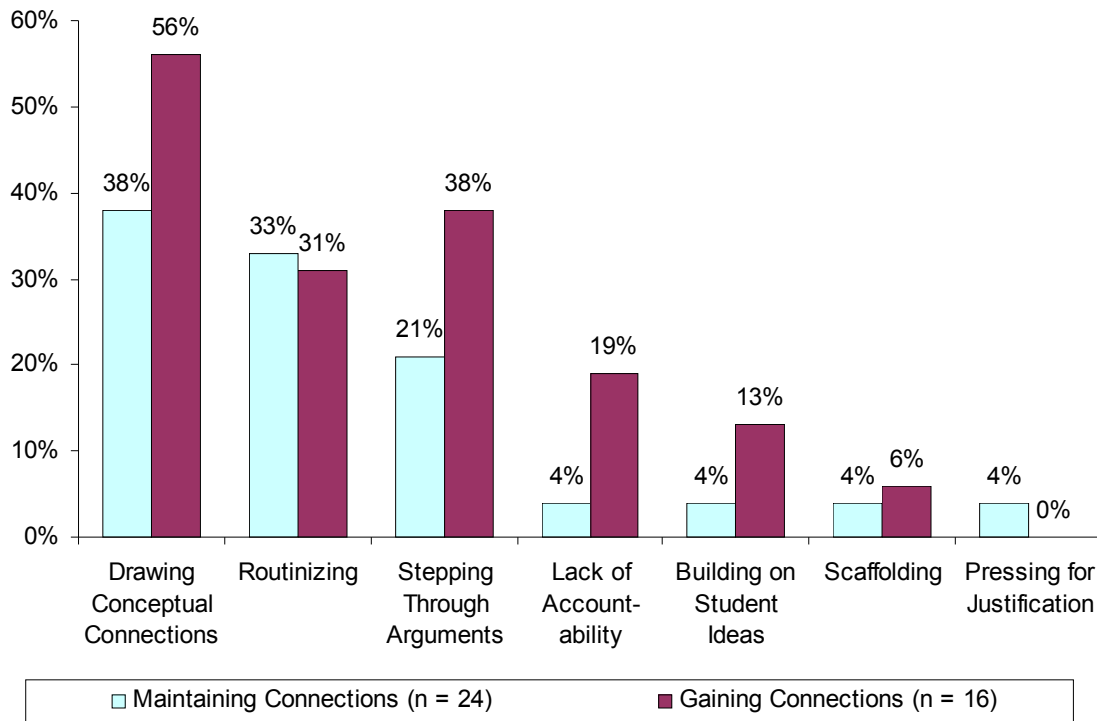


Figure 8: Teacher Behaviors when Maintaining vs. Gaining Connections

### Who Did the Mathematical Work

Table 4 shows who did most of the mathematical work when connections were maintained and when connections were gained. Again, there are no drastic differences; in general, the teacher did most of the work in half the instances, and most of the rest of the

time the work was shared by both. Students never did most of the mathematical work for problems originally stated as non-making connections, although this did occur occasionally when problems were originally stated as making connections.

Table 4: Who Did the Work when Maintaining vs. Gaining Connections

| <b>Who Did Most of the Mathematical Work</b> | <b>Maintaining Connections (n = 24)</b> | <b>Gaining Connections (n = 16)</b> |
|--|---|-------------------------------------|
| Teacher                                      | 50%                                     | 50%                                 |
| Students                                     | 13%                                     | 0%                                  |
| Both   | 38%                                     | 50%                                 |

#### Non-Making Connections Implementations

Turning now to the 42 problems that were judged as having been set up as making connections but not implemented as such, the goal was to identify what teacher behaviors may have contributed to the apparent loss of connections.

#### *Teacher Behaviors*

Figure 9 shows the percent of these implementations that contained each teacher behavior coded in this study. The majority (60%) of implementations were characterized by the teacher routinizing the problem, and almost half involved the teacher shifting the focus of the discussion to a procedure. Less frequent were the teacher skimming the mathematical surface, shifting the focus to the answer, or failing to hold students accountable for high level thinking. Perhaps surprising was the fact that teacher scaffolding occurred in one implementation.

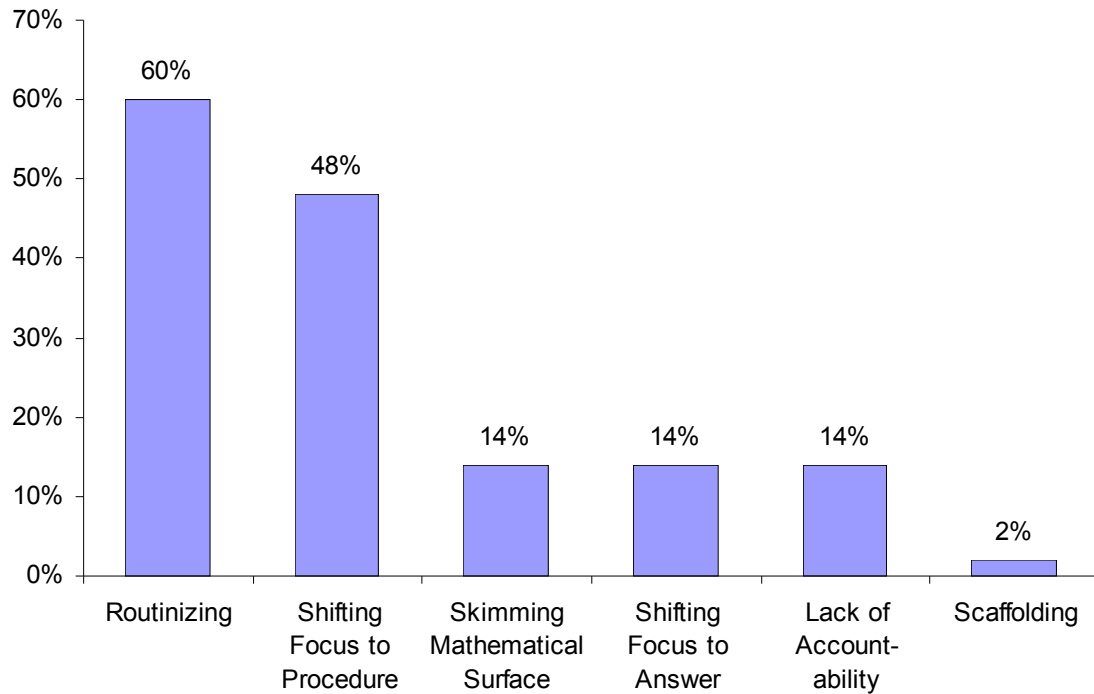


Figure 9: Teacher Behaviors when Connections Were Lost (n = 42)

### *Routinizing*

By far, the most common teacher behavior was routinizing, in which the teacher reduced ambiguity or complexity by specifying explicit procedures or steps to perform, or took over challenging aspects by telling students how to perform them or performing them for students. Earlier it was shown that routinizing could co-exist with other behaviors that made connections. In the cases here, none of these other behaviors occurred, resulting in implementations that did not make connections. This can be seen in the following Dutch implementation of a problem, considered to be making connections as stated, that asked students to find the measure of angle  $B_1$  given that angle  $C_1$  measures 20 degrees (see Figure 10).

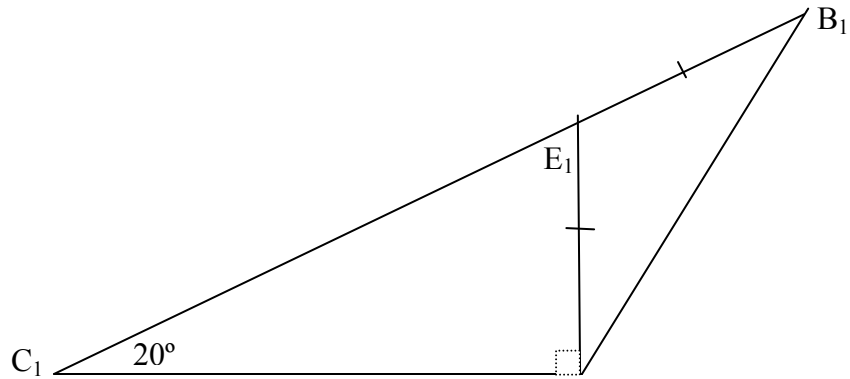


Figure 10: Finding the Measure of Angle B<sub>1</sub>

18:05 T If C one is twenty degrees, how much is angle E one then?

18:10 SN Seventy.

18:11 T Seventy, why?

18:12 SN ( )

18:15 T Okay, look here, that triangle...//.which is positioned like this.. If this one equals twenty and here is a right angle, then we must have here seventy degrees.

18:16 SN //Yes, okay.

18:16 SN Coughs.

18:22 T Okay? How much is its neighbor then?

18:26 S [Coughs.]

18:26 T This one.

18:27 SN One hundred eighty minus seventy.

18:28 T Hence, this one is?

18:29 SN One hundred and ten.

18:30 T Hundred and ten degrees. However, which angle was to be found?

18:32 S B.

18:32 T ( ) B one. Okay, if we make now a triangle here which is isosceles, right?

Then you have a hundred and eighty degrees minus that hundred and ten....

18:43 SN Divided by two ( )

18:44 T .. that's seventy.

18:45 T So how much is that..... ( ) angle B two, right? Hence, that is seventy divided by two and that is? Five and...?

18:50 Ss Thirty.

18:50 T Thirty-five.... Cecile, got it? ... Yes?

[NL-049, IP 6]

Rather than allowing students to make progress on their own, the teacher took over challenging aspects of the problem by using product questions to specify which angle measures to find in what order (18:05, 18:22, 18:30, 18:45). The teacher or students told how to find these angle measures, but the computations were not justified (18:15, 18:27, 18:32, 18:43). In addition, no strategic rationale was given for the steps in the solution procedure.

### *Shifting the Focus to a Procedure*

The second most common teacher behavior in implementations where connections were lost was a close relative of routinizing: shifting the focus away from concepts or meaning and onto a procedure, which occurred in almost half of the implementations. This can be seen in the following dialog from a lesson in the Netherlands. Earlier in the lesson, the teacher had students find volumes of beams (rectangular prisms) and cylinders, as well as solids formed by combining them. The

teacher had also had students find how the volume of a beam changed when one or two of its dimensions doubled. In this problem, students were to determine what happens to the volume when all three of its dimensions double.

17:32 T Well, what do you think will happen then?

17:34 SN But...but how is that possible - or can the height be done also?

17:36 T Yes, yes, so and the length, and the width and the height.

17:40 SN Yes, but it doesn't say (in the book).

17:41 T No, but we will just add those together. Because then we have all the possibilities together. Well, what happens then?

17:48 SN You get two times two times two.

17:49 T Yes, two times two times two. You have - this is not new to you, right?

17:52 SN That is twelve...

17:55 SN (...)

17:56 T This is for a beam. And this is actually also what they mean for assignment thirty-nine.

[NL-027, IP 6]

Although this problem could have led to a discussion of how doubling length, width, and height results in a prism made up of eight copies of the original, along with a diagram and a reference to the meaning of volume, the focus instead was on an arithmetic calculation. The key moment seemed to occur at 17:48, when the student responded to a rather open-ended question with a calculation. The teacher accepted this response, and as a result there was no explanation of why the twos should be multiplied together or why the product would imply any particular change in volume of the box. One could argue



that this may have been unnecessary if in fact the students were already familiar with the situation as the teacher suggested (17:49), but the fact remains that the resulting discussion focused on a calculational procedure without explicit reference to concepts or meaning. (There was however, an apparently incorrect conclusion drawn by a student at 17:52, whose reasonableness could have been addressed by a conceptual discussion.)

*Routinizing vs. Shifting the Focus to a Procedure*

As defined in this study, there is a subtle difference between routinizing and shifting the focus to procedure. The former means that the original problem involved some kind of challenge (complexity or ambiguity) that the teacher removed by specifying procedures or steps or by performing them for students, while the latter means that the original problem (and perhaps initial part of the discussion) implied a focus on meaning or conceptual understanding but the discussion shifted to an almost exclusive concentration on a procedure.

Although both often occurred together (such as in the Dutch discussion involving angle measures; NL-049, IP 6), one could occur without the other. Routinizing could occur without a shift in focus to a procedure if attention was paid to aspects of the problem other than a procedure. For example, in the illustration of routinizing given in the section on making connections implementations (JP-045, CP 3, where students were to find the area of a portion of a trapezoid), the teacher removed the problematic aspect of the question by telling students the steps they should perform to obtain the answer, but he provided justification so that the discussion did not focus on a procedure to the exclusion of conceptual meaning.

Conversely, a shift in focus to a procedure could occur without routinizing by the teacher. In the Dutch discussion of volume just described (NL-027, IP 6), the teacher allowed a student to shift attention to a procedure, but she did not remove the complexity of the problem by telling students what to do. In general, if such a shift in focus occurred, an implementation could not make connections, but if routinizing occurred without this shift, it was still possible for an implementation to make connections.

### *Skimming the Mathematical Surface*

A less common way for teachers to implement a problem so it did not make connections was by failing to delve sufficiently into the mathematics of the problem, resulting in a discussion of a concept which remained at a superficial or vague level. This occurred in six of the 42 cases. It can be seen in the following U.S. implementation of a problem that asked students to draw a mapping diagram and a graph for the relation  $\{(6,0), (6,-4), (4,-3), (5,-3)\}$ .

05:19 T Anybody have any questions on that one? Jeremy?

05:23 Sn On thirty-seven-

05:25 T Oh, thirty-seven was a mapping.

05:27 S Yeah.

05:28 T Yeah?

05:29 S (...) you didn't put the two sixes down or the two negative //threes.

05:34 T //You only have to put six one //time.

05:36 S //Okay.

05:36 T You only have to put negative three one time in the circle.

05:39 S Okay.

05:40 T But when I draw the lines, I have a six going to a zero and a six going to a negative four. So that means the six was used twice, right?

05:48 T So if I'm listing from a mapping, wouldn't I list both ordered pairs? Six zero and six negative four? Same thing with the other one.

[US-069 CP 11]

Here, the teacher's approach to the student's question was to state a (seemingly arbitrary) rule regarding notation—a rather superficial aspect of the problem—rather than the mathematical meaning of the situation. This meaning would have involved at least two ideas. First, the mapping diagram is a representation of two sets, and in a set of numbers, there is only one “6” and one “3”. Second, the mapping diagram shows the structure of the relation; that is, the way in which elements in the first set are linked (“mapped”) to elements in the second set. In this case, the 6 is mapped to both 0 and -4, while both 4 and 5 are mapped to the same number: -3. This can be seen in the very way two different arrows have the same head or tail. This structure would not be shown if the diagram were drawn by simply copying down each coordinate as many times as it appeared in the list of ordered pairs, and linking each separate pair with an arrow. By not addressing these ideas, the teacher did not draw students attention to the fact that the ordered pairs, the coordinate graph, and the mapping diagram all provided a different perspective on the same mathematical object—the whole purpose of examining multiple representations.

### *Shifting the Focus to the Answer*

Like shifting the focus to a procedure, this code meant that the original problem statement (and perhaps initial part of the discussion) focused on meaning or concepts, but

a shift in emphasis occurred. In this case, the result was that attention became focused on the accuracy or completeness of the answer, to the exclusion of conceptual meaning. This occurred in six of the implementations where connections were lost.

The following dialog from an Australian lesson provides an illustration. The teacher had students work in groups to assemble interlinking blocks into a solid figure and draw four views of the result: top, bottom, left, and right sides. She selected volunteers to put their sets of four drawings on the board, and asked each group to choose one set (not their own) and reconstruct the figure with the interlinking blocks. The following dialog began when the teacher had students show their reconstructions to the class:

38:45 T Who's made number one please? Can you hold- whose was number one?

38:53 Sn ( )

38:57 T Is that what it is?

38:59 Ss No.

39:00 Sn ( )

39:02 T But it's not the shape. Alright, anyone else make number one? Sam. Who made number two?

39:13 T Lee where is it? Whose is number two? Quiet. Whose is number two?  
That's yours.

39:23 Sn No it's ours.

39:23 T Whose? Yours. Number two, is that it?

39:28 Sn Yep.

39:30 Sn Mr. ( )

39:34 T So number two, here. So they've actually got it correct.

39:41 T What makes number two a bit easy? You want to tell me what makes-

39:45 Sn ( )

39:48 T Yes, that's right. Left- it's sort of only, because it's only one block thick, the left or right is actually the shape, so it makes it a bit easier like that. Alright, that's good.

39:57 T Now, wait a moment, who's made number three? Anyone done number three? You haven't. Hey. No one's made number three at the moment?

40:11 T What did I do with the duster?

40:14 T I'll rub two out. Four? No, who's made four? You've done four?

40:26 Sn Yeah.

40:27 T Hold it up.

40:28 Sn Wrong!

40:28 Sn Wrong!

40:29 Ss Why is everything wrong with, gosh I'm so.

40:33 T Five? //Don't-

40:35 Ss //( )

40:38 T Hold it up if you've made number five please. So we've got that right.

[AU-064, IP 4]

Although there was a brief discussion of what made one of the solids easy to determine (39:48), the overall focus was only on the answers, and there was no discussion of how the students arrived at them.

### *Lack of Accountability*

Also occurring in six of the implementations was the teacher's failure to respond to incorrect or insufficient student contributions by probing or pressing for more adequate responses. In the following example from Hong Kong, the teacher had first reviewed the right triangle definitions of cosine and had students use inverse cosine to find the measure of an angle in a right triangle. Then he presented the class with the following problem:

A hot-air balloon, at a height of 80 meters, is fixed to the ground by a rope AB 96 meters long. If the rope makes an angle along the vertical side, find theta. (Make sure your answer is correct to three significant figures.)

13:58 T How to find?

14:00 SN Adjacent side.

14:01 SN Adjacent side over.

14:03 T Uh, Elaine?

14:10 T You want to find the size of theta. Is this theta in a right-angled triangle?

14:19 Ss Yes.

14:20 T Yes. Okay. So that means maybe we can make use of cosine ratio, okay, to find the size of theta.

14:34 T Okay, C, B, A, theta. Eighty M [meter], ninety-six M [meter]. Okay, you want to use cosine ratio to find the size of theta. Then we must identify adjacent side and hypotenuse. Elaine, tell me, which side is adjacent side?

14:53 Ss AC.

14:55 T AC is adjacent side, very good. How about, uh, hypotenuse?

15:00 Ss AB.

15:02 T AB, very good. Okay, thank you, sit down.

15:06 T Okay, Jenny, Jenny. Okay, you tell me, how to make use of cosine ratio to find the size of theta?

15:15 SN Adjacent side over hypotenuse.

15:17 T So which side over which over?

15:19 Ss AC over AB.

15:20 T AC over?

15:22 SN //AB.

15:22 T //AB, very good. Okay, so cosine theta, what is the length of AC?

15:28 Ss Eighty.

15:29 T Eighty. Okay. The length of AB?

15:31 Ss //Ninety six.

15:32 T //Ninety six.

15:33 SN Isn't it thirty three?

15:34 T Okay. So theta, that is equal to the inverse of cosine, eighty over ninety six.  
Can you help me find the size of theta?

15:44 Ss Three ...

15:44 T //Correct to three significant figures.

15:44 Ss //Thirty three.

15:46 T Uh, Kelly?

15:49 SN Thirty three point six.

[HK-044 IP 5]

On two occasions, the teacher asked process questions (13:58 and 15:06); i.e., questions that sought an explanation requiring students to integrate information (Good & Brophy, 1987). Student responses, however, consisted of short definitions from memory. Rather than probing them or providing scaffolding to help them provide more coherent, complete responses, and thereby hold them accountable for higher level thinking, the teacher went on to use product questions to tell the class how to solve the problem.

Interestingly, in five out of the six cases of non-making connections implementations where teachers failed to hold students accountable after insufficient or erroneous contributions, these contributions were responses to teachers' open-ended or process questions. Thus, teachers asked questions that could have led to connections being made, but failed to follow up when student responses were insufficient. This also happened in three out of four cases of making connections implementations, but connections were made in other ways.

#### *Who Did the Mathematical Work*

As can be seen from Figure 11, in slightly more than one-half of the implementations, teachers did most of the mathematical work, and students did so in only three of the implementations. In over a quarter, teacher and students shared the work. In 10% (four) of the implementations, no mathematical work was done; all talk other than the problem statement and answer was non-mathematical in nature.



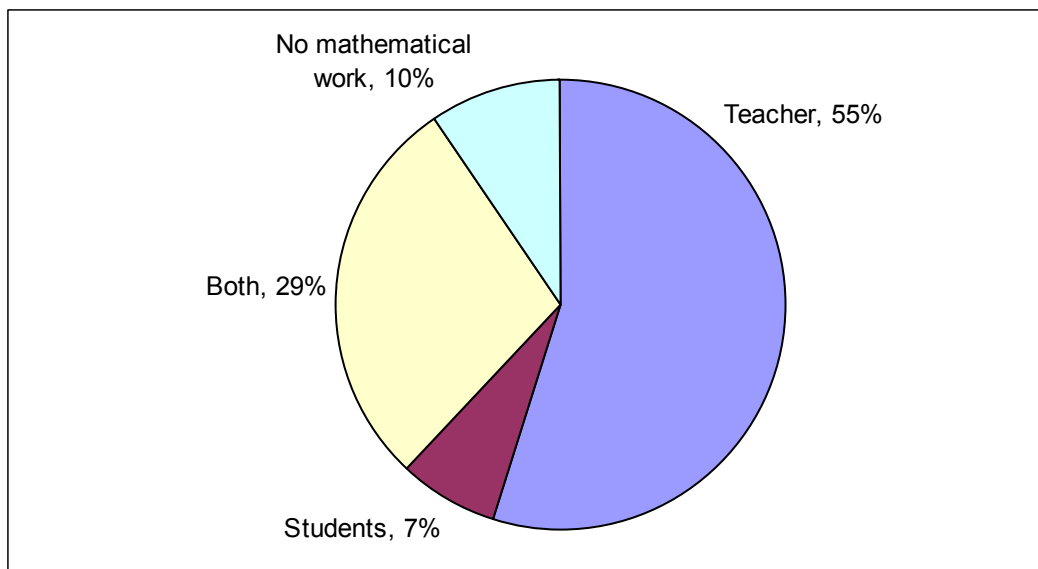


Figure 11: Who Did the Mathematical Work when Connections Were Lost

#### Maintaining vs. Losing Connections

When given a making connections problem, what did teachers do that seemed to make a difference between maintaining or losing those connections? Of course, focusing attention on making connections *features* is part of the answer; by definition, if a teacher did so in a mathematically substantive way, the implementation made connections. But what teacher *behaviors* seemed to be associated with focusing or not focusing on these features? Also, did it matter who did most of the mathematical work? For this section, only problems that were stated as making connections will be examined, and comparisons will be made between those implementations that made connections and those that did not.

#### *Teacher Behaviors*

Table 5 compares the relative frequencies of teacher behaviors for the two kinds of implementations. Even though some of these behaviors by definition led to features

being discussed (drawing conceptual connections, stepping through arguments, and pressing for justification) or not being discussed (shifting the focus to procedures or answers and skimming the mathematical surface), all are presented here to provide a complete picture of the behaviors that occurred.

Table 5: Teacher Behaviors when Implementing Making Connections Problems

| <b>Teacher behavior</b>        | <b>Maintaining connections (n = 24)</b> | <b>Losing Connections (n = 42)</b> |
|--------------------------------|---|------------------------------------|
| Drawing conceptual connections | 38%                                     |                                    |
| Stepping through arguments     | 21%                                     |                                    |
| Pressing for justification     | 4%                                      |                                    |
| Shifting focus to procedure    |   | 48%                                |
| Skimming mathematical surface  |   | 14%                                |
| Shifting focus to answer       |   | 14%                                |
| Routinizing                    | 33%                                     | 60%                                |
| Lack of accountability         | 4%                                      | 14%                                |
| Scaffolding                    | 4%                                      | 2%                                 |
| Building on student ideas      | 4%                                      | 0%                                 |

As seen earlier, the most frequent behaviors that maintained connections were drawing conceptual connections and stepping through arguments while the most frequent behavior that lost connections was shifting the focus to a procedure. There were four behaviors which theoretically could occur whether connections were maintained or lost, but only one occurred with significant frequency: routinizing, which occurred almost twice as often when implementations did not make connections as when they did. Similarly, although lack of accountability was not frequent, it occurred proportionally over three times as often when implementations did not make connections as when they did. Thus, it is important to note (again) that routinizing and lack of accountability did not necessarily prevent connections from being made. Conversely, scaffolding did not

necessarily lead to connections being maintained; it occurred in one case in which connections were lost.

*Who Did the Mathematical Work*

Table 6 shows that the person doing most of the mathematical work was not very different depending on whether or not the problem was implemented as making connections. In both cases, in at least half of the implementations, the teacher did most of the work. Problems that were implemented as making connections more frequently involved students doing most of the mathematical work, but in both cases the number of occurrences was small. Such implementations also slightly more frequently involved shared work by teacher and students. In four of the non-making connections implementations there was no mathematical work; of course this never occurred when connections were made.

Table 6: Who Did the Work when Implementing Making Connections Problems

| <b>Who did most of the mathematical work</b> | <b>Maintaining connections (n = 24)</b> | <b>Losing Connections (n = 42)</b> |
|--|---|------------------------------------|
| Teacher                                      | 50%                                     | 55%                                |
| Students                                     | 13%                                     | 7%                                 |
| Both   | 38%                                     | 29%                                |
| No mathematical work                         |   | 10%                                |

## Chapter 5: Discussion

The purpose of this study was to find out what teachers did to maintain, lose, or gain connections as they led the discussion of mathematics problems. When student learning is taken into account, this knowledge has implications for arguments about mathematics education reform in the U.S. and suggests directions for further research. Therefore, in this section I will discuss answers to the research questions suggested by this study, address their relationship to student learning, describe the implications of these findings for mathematics education reform, and describe avenues for further research.

### The Research Questions

#### *How did teachers make connections?*

The most common way that the class discussions seen here “made connections” (as defined by my interpretation of the TIMSS definition of “making connections”) was by including the justification of assertions and by developing or otherwise examining concepts more than simply recalling and applying them. Slightly less common was connecting representations, and relatively rare were focusing on problem solving, developing mathematical generalizations, and comparing solution methods. It may be that explicit attention to how solution methods are developed, developing generalizations, and comparing the mathematics of multiple methods are not part of the repertoire of many teachers in the countries in this study. It is perhaps noteworthy that while some of the making connections discussions were moderately long and in-depth, many were quite limited in duration and scope, especially by the standards of American reform recommendations (see for example the vignettes in NCTM, 1991).

Teachers most often led making connections discussions by drawing conceptual connections or stepping students through arguments, behaviors often accomplished by telling and using product questions. As previously mentioned, product questions are those that seek to elicit a single correct answer that can be expressed in a single word or short phrase (Good & Brophy, 1987) and can be answered by memory, observation, or performing a procedure or step as instructed by the teacher. These are probably the kinds of questions teachers feel the most comfortable using in their teaching. However, as teachers use them, they are implicitly making decisions about the solution path to be followed, rather than allowing students to do so. This was confirmed by the finding that it was rare for students to do most of the mathematical work and in half of the discussions, teachers did it. It also probably explains why routinizing and lack of accountability was found in 33% and 10% of making connections discussions, respectively.

These findings contrast with those found by Stein and colleagues, where teachers undergoing extensive professional development drew conceptual connections in less than 15% of “high cognitive level” implementations (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996). Instead, most high level implementations were associated with the teacher providing scaffolding and sustained press for justification and meaning, behaviors that rarely occurred in the sample studied here. This suggests that it may be quite difficult for teachers to learn to use these approaches without extensive professional development.

These approaches tend to be those identified with reform teaching, while those associated with making connections implementations in this study can be associated with more traditional teaching, confirming others' findings that teaching in countries with high

achievement in mathematics appears rather traditional (Huang & Leung, 2004; Lopez-Real et al., 2004; Wang & Paine, 2003). This should not be surprising; Stein and colleagues studied teachers undergoing professional development intended to help them implement reform ideas, while this study examined more typical teaching in the participating countries.

In addition, the definition of implementation here differed significantly from that used in the QUASAR Project. In the latter, it was defined as the way in which students worked on tasks, whereas in this study it was defined as whole class discussions led by the teacher. Therefore, while routinizing and lack of accountability may have prevented student thinking at what Henningsen and Stein (1997) would call high cognitive level, these behaviors could have been part of teachers' justifying and focusing on concepts. That is, while these behaviors lowered the cognitive level of implementation according to QUASAR's definition, they merely transferred the work of making connections from student to teacher, so that implementations were still making connections according to the TIMSS definition.

Thus, there are two dimensions at play here: the content of the implementation, defined by the making connections features, and the people who are enacting that content. One of the major differences between reform teaching and conceptually-oriented non-reform teaching may lie in who is enacting the making connections features. Some teacher behaviors as defined in this study—e.g., drawing conceptual connections—affect the content, while others—routinizing, lack of accountability, and scaffolding—affect who is enacting the content. Still others—stepping through arguments, pressing for

justification—affect both. Those teacher behaviors that affect who enacts the content seem to influence whether the teaching approach is considered reform or non-reform.

*Did teachers maintain and gain connections differently?*

With one exception, implementations gained connections by the teacher adding features similar to those they addressed when maintaining connections. The only exception was in the case of generalizing, which only occurred when connections were gained. It is not clear why this occurred; it may simply be an artifact of using a rather small sample of problem implementations.

Similarly, teachers seemed to engage in the same behaviors whether connections were maintained or gained, although they used four of the five most common ones (drawing conceptual connections, stepping through arguments, lack of accountability, and building on student ideas) more frequently when the problems were not originally stated as making connections. It makes intuitive sense that teachers would more frequently have to engage in particular behaviors to add connections than when maintaining connections; in the latter case teachers could just implement the problem as written. For example, a problem that asked students to justify assertions or connect multiple representations may not require the teacher to draw conceptual connections or step through an argument in order to bring out these features. This would be compatible with the fact that the rate of routinizing was the same in both cases, but it would not explain the higher frequency of lack of accountability, which would not seem to contribute to gaining connections any more than maintaining them. The differences, again, may be an artifact of the small sample size.

Another reason for not reading too much into these differences is the possibly artificial distinction between maintaining and gaining connections. Determining whether connections are maintained or gained depends on classifying a problem statement as making connections or not, which is an inexact process. It is based on assumptions about what the problem seems to imply that students do to solve it, and on assumptions about the curriculum and previous instruction. Few problem statements classified as making connections actually told students to connect representations or examine a concept. For example, one might assume that for the Dutch problem asking how the volume of a prism would change if all its dimensions doubled, students would have to think about the concept of volume in a deeper way than they had before. However, if the curriculum or teacher has previously presented an algorithm for solving this type of problem, it would effectively be a “using procedures” (i.e., non-making connections) problem when originally stated. This difficulty is exacerbated by the international scope of TIMSS; what is a challenging non-routine problem in one country may be a routine procedural problem in another.

Therefore, for the purposes of this study, it may not be helpful to assume that there is some inherent objective difference between those problems whose statements were classified as making connections and those whose statements were not. Apparent differences between teacher behaviors in maintaining and gaining connections may be spurious or unimportant. What is probably more important is the suggestion of these findings that regardless of how initial problem statements may be interpreted by coders, teachers may find it easier to make connections by connecting representations, drawing conceptual connections, and stepping students through arguments than by focusing on



problem solving, generalizing assertions or arguments, comparing solution methods, building on student ideas, scaffolding student thinking, and pressing students for justification.

*How did teachers fail to make connections?*

Turning now specifically to those problems that were not implemented as making connections, the single most common behavior was routinizing, although, as already mentioned, in this study routinizing did not necessarily lead to a lack of connections being made. The culprit seemed to be routinizing along with the lack of any of the other behaviors noted above, such as drawing conceptual connections or stepping students through arguments. In addition, a shift in focus to procedures or correct answers occurred in roughly 60% of the non-making connections implementations. This occurred when teachers primarily asked students to describe procedures or provide answers without justifications or connections to concepts. Skimming the mathematical surface also occurred as often as shifting the focus to the answer; this behavior can be seen as another kind of shift in focus, away from deeper conceptual meaning and to more superficial cues or vague statements. Altogether, then, these shifts in focus account for almost 80% of the non-making connections implementations.

These findings are similar to those of Henningsen and Stein (1997) who found that routinizing, shifting focus to answers, and lack of accountability occurred frequently in association with low cognitive level implementations. However, in their study, lack of accountability encompassed a larger range of actions by the teacher than could be observed here (e.g., implying that students' work on high-level tasks would not count). In

addition, in this study, lack of accountability, like routinizing, did not always lead to a non-making connections implementation.

*What made the difference between maintaining and losing connections?*

When teachers discussed problems originally stated as making connections, what actions did they take that influenced whether the connections afforded by the problems were maintained or lost? By definition, of course, it depended on whether they addressed the making connections features; in this sample, that meant justifying, examining concepts, connecting representations, and less frequently, problem solving. As far as teacher behaviors were concerned, also by definition, if teachers drew conceptual connections or formed mathematical arguments and stepped students through them, the implementations made connections. But if teachers shifted the focus to procedures, superficial cues, vague formulations, or answers, then of course they did not.

However, there were four additional behaviors coded in this study that did not by definition imply that connections were or were not made. Only two of them occurred with significant frequency: routinizing and failing to hold students accountable. Both of these behaviors seemed to increase the chance that connections would not be made, but did not guarantee it. They often shifted the mathematical work to the teacher, who made the connections rather than the students. Interestingly, in spite of the fact that these two behaviors were significantly more frequent when connections were lost than when they were maintained, the frequency with which the teacher did most of the work was only slightly higher. Perhaps when routinizing did not occur, the behaviors that made connections—stepping through procedures and drawing conceptual connections—shifted the work to the teacher to make up the difference. At any rate, having students do most of

the work, or sharing the work with the students, was only slightly associated with maintaining connections. In addition, scaffolding did not guarantee a making connections implementation; it occurred extremely infrequently whether connections were maintained or not, but it did occur in both cases.

### Student Learning

Ultimately our interest is in what bearing these teacher behaviors have on student learning. In my study, the link is not altogether clear. Although Stein and colleagues observed student engagement with tasks (Henningsen & Stein, 1997; Stein, Grover, & Henningsen, 1996), my study focused on whole class discussions through examining transcripts, so little information was available about student work and student thinking. Teachers in my study often made connections by doing most of the mathematical work through telling and the use of product questions. When teachers used product questions, they implicitly made decisions about the solution path to be followed, rather than allowing students to do so. During such discussions, it may not have been at all clear to students in which direction the sequence of facts and questions being presented was going, and students may not have been integrating them to construct coherent understandings of the concepts or arguments under discussion.

Furthermore, Stein and Lane (1996) had direct evidence that high-level task implementation was related to higher student achievement on an assessment of problem solving and reasoning. In the TIMSS study, there was no direct evidence that the achievement of students was higher in classrooms where making connections implementations were more frequent. In fact, we do not know whether teachers were consistent over the school year in the way they implemented problems. Furthermore, a

myriad of personal, contextual, and cultural factors could underly achievement differences.

For all of these reasons, the link between teacher behavior, student thinking during instruction, and student achievement is less clear in this study. The only evidence we have is that, to the extent that the TIMSS videos were representative of teaching in the participating countries and the coding of problem implementation is valid, teachers in the other five countries made connections in ways that were rare in the U.S., and the achievement of students in those countries was higher than in the U.S. We do not know to what extent the relationship between these two findings was causal.

However, the finding that U.S. teachers stand alone (as compared to those in the five other countries) in almost never implementing problems as making connections does suggest that the way that teachers discuss problems could be one factor that contributes to achievement differences. Implementations classified as making connections seem to be more mathematically substantive, and this additional substance seems to exist more frequently in classrooms in high performing countries. This possibility is supported by the TIMSS mathematics quality analysis group's study of 20 randomly selected lessons from each country,<sup>1</sup> which rated the U.S. lessons lower than the other countries' lessons on most aspects of mathematical quality, and concluded that on average, the U.S. lessons provided students with the least opportunity to construct important mathematical understandings (Hiebert et al., 2003).

Even instruction that is “traditional” in form but mathematically substantive may provide opportunities that would not otherwise be present for students to think mathematically. Both researchers in American and Chinese education have asserted that

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<sup>1</sup> Data from Switzerland was included in this analysis, but not from Japan.

students may actively construct conceptual understandings in classrooms where teaching occurs primarily through explanations and product questions, if that instruction emphasizes concept development, justification of procedures, links among ideas, comparison of methods, multiple representations, and mathematical proof (Anderson, 1989; Good, Grouws, & Ebmeier 1983; Huang & Leung, 2004; Leinhardt, 1986; Wang & Paine, 2003), some of which were seen in the problem implementations in this study. However, it is also possible that cultural and motivational factors, which vary across countries, could influence the effectiveness of such an approach, so that it might not be as effective in the U.S., particularly in light of the treaties that may operate between teachers and students in many American schools (Powell et al., 1985; Sedlak et al., 1986).

### Implications for Reform

When teachers in this international sample made connections, they did it by providing explanations and using product questions, suggesting that these are behaviors that teachers not only in the U.S., but also in several other countries, may feel comfortable with and find less difficult to enact than those advocated by U.S. reformers, such as scaffolding, pressing for justification, and building on students' ideas.

Traditional and reform teaching are sometimes described in terms of both particular pedagogical approaches and particular kinds of mathematics that are emphasized (e.g., Stein, Grover, & Henningsen, 1996, p. 462). Traditional teaching presumably consists of teacher explanations and demonstrations of procedures followed by student practice of those procedures, with an emphasis on basic facts and skills. Reform teaching is described as consisting of cooperative group work and student-formulated solutions and argumentation, with an emphasis on conceptual understanding

and more sophisticated mathematical reasoning. This study suggests that a more nuanced understanding may be more productive. Most of the making connections implementations seen in this study were centered around teacher explanations and demonstrations rather than group work or student explanations, yet they went beyond the statement of basic facts and execution of algorithmic procedures. Educators in other countries may not share the assumption that teacher-centered instruction is effective only for transmitting basic facts and skills and not for developing students' higher-order thinking abilities. The goal of emphasizing conceptual understanding and mathematical reasoning in itself may not necessarily imply the approaches currently advocated by the U.S. reform movement.

Conversely, the findings of this study suggest that scaffolding and any other reform techniques intended to help students do more of the mathematical work may not necessarily lead to making connections, at least as defined by the TIMSS problem implementation team. Thus, it is important for researchers to examine how teachers use these techniques in ways that do and do not make connections. It may be just as important, if not more important, to help teachers develop an orientation that prioritizes those features that characterize making connections (such as justification, conceptual connections, connections among representations, and the development of solution methods) as it is to concentrate on such pedagogical practices. Certainly it seems unhelpful to concentrate on reform practices without helping teachers notice and bring out the important mathematical features of problems.

#### Further Research

Results of this study suggest several lines of research. First, additional studies examining how teachers in other countries discuss mathematics problems might be in

order. This study was conducted from an American perspective, limited by lack of extensive knowledge of the teaching perspectives and approaches used in the countries being studied. As a result, it is quite possible that important behaviors were overlooked.

Regardless, research is still needed to determine the extent to which problem implementation (especially outside the context of intensive professional development) is related to student learning with respect to problem solving, conceptual understanding, and mathematical reasoning. If relationships are found, then additional research could examine factors that affect problem implementation, such as teacher characteristics and contextual factors. In addition, professional development interventions could be devised to help teachers improve the way they lead discussion of problems in class, and evaluation studies could be conducted to determine their effectiveness. It might also be helpful to study how teachers' use of reform techniques can still fail to address the features of problems that promote conceptual understanding, mathematical reasoning, and problem solving.

### Conclusion

This study confirms the literature's findings regarding ways middle school teachers often change the nature of problems and limit students' opportunities to engage in reasoning and problem solving. It shows that teachers in the U.S. and other countries may do this by shifting the focus of the discussion to procedures, answers, and superficial aspects of the problems. It also suggests that using reform pedagogy may not necessarily lead to addressing these opportunities if teachers do not consciously concentrate on important features of the problems.

This study also suggests that teachers in other countries may attempt to develop concepts and provide opportunities for students to engage in mathematical reasoning by using what appear to be rather direct teaching approaches—doing much of the mathematical work by explaining and using short-answer closed-ended questions—while still emphasizing some of the important features of mathematics problems; e.g., justification, examination of concepts, and connections between representations. It also suggests that some features are more commonly emphasized than others; for example, in this sample, generalizing and developing and comparing solution methods were less common.

Because this study was limited to examination of a small number of eighth grade classrooms, it cannot indicate the full range of teaching approaches that exists in any of the countries participating in the TIMSS Video Study. Neither can it describe certain ways of teaching that are typical in any particular country, or the ways in which teachers implement tasks at the elementary or high school level. Because it was conducted by an American, it may not have captured all of the important ways that teachers in other countries implement problems.

This study instead suggests that, if additional research can address some important issues, American teachers might profit from learning to bring out the potential of the problems and exercises they assign to emphasize conceptual connections, justification, and connections among representations.



## Appendix: Coding Instructions and Coding Form (Final Version)

### Part one: Making connections features

Read the transcript of public whole-class dialog, looking for any of the making connections features listed in the box below.

- Code only those that involve **substantive mathematics**.
- Indicate where in the dialog you find each feature that you code.
- If you feel like a feature is present, but you are somewhat unsure whether it involves substantive mathematics, code it with a question mark for later discussion.
- If you feel that something is missing that is necessary to convince you that the feature is present or involves substantive mathematics, then do not code it.

#### **Codes for making connections features**

**CMeth** – Comparison of the **mathematics** of solution methods. This includes either of the following:

- A relationship between solution methods is explained (e.g., why one solution method is more elegant than, or a general case of, another one), or
- A correspondence between steps or aspects of different solution methods is described (e.g., subtracting from both sides of an equation in the symbolic method corresponds to undoing the last addition step while working backwards in an informal method).

**CRep** – Connection between representations.

- A **representation** is an algebraic symbol string, table, graph, diagram, or physical object(s) used to represent a problem situation, quantity, object, concept, or relationship among them.
- The representations must provide different perspectives of some common idea, situation, or object. For example, exclude drawings of two triangles even if they are related in some way (e.g., congruent).
- A connection between representations means the way in which aspects of different representations correspond to each other. For example, “a negative linear coefficient in a linear equation corresponds to a downward slant in the graph.”

**Conc** – Examining a concept. A concept or property is examined **more deeply than simply recalling or applying it**. This may involve describing some component, aspect, representation, or example of the concept, or some connection to another concept. It may involve extending a concept or developing a new concept.

**Gen** – Generalization: a mathematical problem, assertion, solution method, concept, or argument that is **more general than that previously stated or discussed**; the latter is a specific case of the former.

**Jus** – Justification: use of **mathematical** knowledge to explain why a solution method, step, problem-specific claim, or general mathematical assertion (e.g., theorem) **is or is not correct, valid, or appropriate**. Justification **does not include**:

- procedural explanations,
- strategic reasons for choosing a particular solution step or approach, or
- non-mathematical rationales

**PS** - Problem-solving. Explicit examination (not just carrying out) of an overall solution plan, not just pieces of a plan. This includes **explicit** discussion or description of any of the following:

- How one arrives at a solution path
- Strategic justification of a plan (i.e., explanation of why the overall plan is chosen).
- Intermediate goals beyond those already stated in the problem and/or monitoring progress toward meeting those goals

**Oth** - Any other feature that suggests “making connections.” Specify what this behavior is.

### Part two: Who does the work

Indicate whether the mathematical work overall was done **mostly** by the teacher, the student(s), both, or there was no mathematical work done at all (e.g., during the entire dialog, only the problem and/or answer were given and any other dialog was non-mathematical).

### Part three: Teacher behaviors

Read the transcript of public whole-class dialog, looking for any of the teacher behaviors listed below, while referring to the definitions of question types below. Code only those behaviors that are either

- **consistently** enacted, or
- enacted at **key moment(s)**; i.e., such that they **change or set the direction** of the discussion. In this case, indicate where in the dialog this occurs.

#### **Question types**

**Product question:** a question that seeks to elicit a single correct answer that can be expressed in a single word or short phrase. Product questions usually begin with “what,” “where,” or “how much,” and can be answered **by memory, observation, or performing a procedure or step as instructed** by the teacher. They include questions such as, “What should you do next?” They also include statements that appear to be telling, but suggest that more information is to be supplied by the students, and to which students respond by doing so; for example, a teacher says, “A parallelogram has certain properties,” and a student responds, “Opposite sides are parallel.”

**Process question:** a question that seeks to elicit an explanation which requires students to **integrate** information or show knowledge of their **interrelationships**. Process questions are usually “why” or “how” questions, and include those that ask for explanations of multi-step procedures.

**Open-ended question:** a mathematical question that could have more than one valid answer, or is phrased in a way that suggests it has more than one valid answer, and for which students apparently have not previously been given the answer.

#### **Codes for teacher behaviors**

**NA** - No accountability on student for high-level product or processes: Students contribute **incorrect or insufficient** (e.g., unclear or incomplete) answers, explanations, or ideas, but the teacher **does not make a significant effort to probe** them (i.e., ask for more detail or justification) or press for more adequate contributions.

- Check additional blank if student contributions are responses to teacher's *open-ended* or *process* questions.

**SK** - Skim: The **original problem statement implies, or initial discussion includes, a focus on concepts**, meaning, or understanding, but the teacher fails to delve sufficiently into the mathematics of the problem, resulting in a discussion which refers to a concept or meaning but only at a **superficial or vague** level. Do not code this if there is no reference to any concept or meaning at all.

**SA** - Shift to focus on answer: The **original problem statement implies, or initial discussion includes, a focus on concepts**, meaning, or understanding, but the **teacher shifts the focus** away from it and to the accuracy or completeness of the answer.

**SP** - Shift to focus on procedure: The **original problem statement implies, or initial discussion includes, a focus on concepts**, meaning, or understanding, but the **teacher shifts the focus** away from it and to a procedure.

**RO** - Routinization: The teacher **routinizes problematic aspects** through

- reducing ambiguity or complexity by **specifying explicit procedures** or steps to perform, or
- taking over challenging aspects by **telling students how to perform** them, or
- taking over challenging aspects by **performing** them for students.

The original problem must have ambiguity or challenge; e.g., a solution procedure must not previously have been given to students (consider the context). The teacher takes away opportunities for students to discover and make progress on their own. This must occur soon enough in the discussion to affect the direction of the discussion. It is usually done by telling and/or using *product* questions.

- Check the additional blank if this is in response to student difficulty.

**ST** - Step through argument: The teacher steps students through an argument by telling and/or using *product* questions. An argument is a sequence of justified assertions leading to a mathematical claim.

**PJ** - Press for justification: The teacher **repeatedly asks** students for justification, meaning, or explanation beyond a procedure through questioning, comments, or feedback. Clear and consistent messages are sent to students that explanations and justifications are as much a part of classroom mathematical activity as are correct answers.

**CC** - Conceptual connections: The teacher draws attention to a connection between a concept and a representation, procedure, or other concept. This does not include justification.

**SC** - Scaffolding: The teacher **provides assistance** by providing information or asking a series of questions **other than *product* questions** that assists student(s) in answering a question or solving the problem **without reducing complexity or challenge**. Assistance is just enough to allow students to make progress.

- Check the additional blank if this is done in response to student difficulty.

**BU** - Builds: Teacher builds on student contribution(s) (perhaps erroneous) by having the student explain more, asking student(s) questions about it, discussing it, relating it to

other ideas, or otherwise using it in his or her teaching. This must bring some new mathematics or higher level of understanding to the discussion that wasn't there before the teacher did this.

**OT** – Other noteworthy teacher behavior that impacts the direction of the discussion.

### Coding Sheet

Lesson \_\_\_\_\_ Problem \_\_\_\_\_ Coder \_\_\_\_\_

#### Part one: Making connections features – **mathematically substantive**

\_\_\_ **CMeth** – Comparison of solution methods

\_\_\_ **CRep** – Connection between representations

\_\_\_ **Conc** – Examining a concept

\_\_\_ **Gen** - Generalization

\_\_\_ **Jus** – Justification

\_\_\_ **PS** - Problem solving

\_\_\_ **Oth** - Other. Specify: \_\_\_\_\_

#### Part two: Who does most of the mathematical work?

\_\_\_ T      \_\_\_ S      \_\_\_ both      \_\_\_ no mathematical work

#### Part three: Pedagogical behaviors – **consistent or at key moment**

\_\_\_ **NA** - No accountability    \_\_\_ S contribution is response to T's open/process question

\_\_\_ **SK** - Skims mathematical surface

\_\_\_ **SA** - Shift to focus on answer

\_\_\_ **SP** - Shift to focus on procedure

\_\_\_ **RO** – Routinization      \_\_\_ In response to S difficulty

\_\_\_ **ST** – Step through argument

\_\_\_ **PJ** - Press for justification

\_\_\_ **CC** - Drawing conceptual connections

\_\_\_ **SC** – Scaffolding \_\_\_ In response to S difficulty

\_\_\_ **BU** - Building on S contribution

\_\_\_ **OT** - Other. Specify: \_\_\_\_\_

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