ABSTRACT

Title of dissertation: APPLICATIONS OF SUPERSPACE TECHNIQUES TO EFFECTIVE ACTIONS, COMPLEX GEOMETRY, AND T DUALITY IN STRING THEORY

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Doctor of Philosophy, 2007

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We describe the use of superspace techniques to discuss some of the issues in string theory. First we use superspace techniques to derive the effective action for the 10D $\mathcal{N} = 1$ Heterotic string perturbatively to first order in the parameter α'. Next we demonstrate how to use the superspace description of the supersymmetric gauge multiplet for chiral superfield in 2d $\mathcal{N} = (2, 2)$ to discuss T duality for sigma models that realizes a particular case of generalized Kähler geometry. We find that the salient features of T duality are captured but at the cost of introducing unwanted fields in dual sigma model. Fortunately the extra fields decouple from the relevant fields under consideration. This leads us to introduce a new supersymmetric gauge multiplet that will eliminate the need to introduce extra fields in the dual sigma model.

APPLICATIONS OF SUPERSPACE TECHNIQUES TO EFFECTIVE ACTIONS COMPLEX GEOMETRY, AND T DUALITY IN STRING THEORY

by

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Dissertation submitted to the Faculty of the Graduate School of the
University of Maryland, College Park in partial fulfillment
of the requirements for the degree of
Doctor of Philosophy
2007

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Dedication

This work is dedicated to those that have been instrumental in my life and career, to my family and friends, mentors, and Jesus Christ.

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Chapter 1

Introduction

The use of symmetry is ubiquitous in the study of physics. Understanding the symmetry of a system eases one's ability to describe the system and capture the physics. Simple examples include using translation symmetry to choose a convenient origin for a coordinate system and rotational symmetry to choose a convenient orientation for a coordinate system. In terms of capturing the physics we have learned to relate conserved quantities with continuous symmetries. Continuous time translation symmetry corresponds to conservation of energy. Continuous spatial translation symmetry gives conservation of momentum. Continuous rotation symmetry corresponds to conservation of angular momentum. These symmetries are used to make solving certain problems as easy as possible. Imagine solving for the orbits of planets or the wave function of the hydrogen atom using Cartesian coordinates! Sometimes, we impose a symmetry just to make finding solutions to a problem easier. Consider imposing spherical symmetry to find the Schwarzschild solution in general relativity. Symmetries also show up in a more abstract way. We have learned to relate the rotation symmetry of the unit circle to the existence and conservation of electric charge for the electromagnetic interaction. A similar, though not as intuitive description can be given for the weak, strong, and gravitational interactions. Formally symmetries are naturally described in terms of the mathematical topic called group theory. Just as using symmetries gives one better control over describing the physics of a system, understanding the natural description of the symmetry itself can be of great value in using the symmetry. The study of group theory is very helpful for particle physicists to understand the relationships between masses of particles, write actions that realize a symmetry, and via the gauging prescription construct the interactions that correspond to the known interactions. Group theory not only helps us to describe symmetries, it also helps us characterize how symmetries are broken and the effects of symmetry breaking on physics. Our current understanding of the origin of mass requires an understanding of how the symmetry corresponding to the weak interaction is broken. The description of this process relies on a group theoretic framework. A similar description of symmetry breaking explains the existence of Nambu-Goldstone particles that arise from symmetry breaking. The breaking of a symmetry by the quantization procedure, called an anomaly, is responsible for our understanding of the decay $\pi^0 \rightarrow 2\gamma$. In string theory, group theory is useful in understanding the spectrum of the string and even the symmetries that can be used to define consistent string theories , for example the heterotic $SO(32)$ and $\mathrm{E}_8 \otimes \mathrm{E}_8$ string. There are countless other examples that can be cited, but the point is hopefully made. In this work will consider a particular symmetry, supersymmetry, that has various applications in physics. The natural mathematical framework for describing supersymmetry is called superspace. Superspace is the natural mathematical framework in which the group theory description of supersymmetry is realized. We will demonstrate the utility of using this framework in two calculations in the context of string theory.

Supersymmetry is a symmetry that relates bosonic and fermionic degrees of freedom. Intuitively one can think of supersymmetry as imposing the condition that for every bosonic degree of freedom, there is a corresponding fermionic degree of freedom. The more supersymmetry one has, the more relations one has between the bosonic and fermionic degrees of freedom. On a more formal level, supersymmetry is an extension of the Poincare group that includes fermionic generators Q whose anti commutators close on translation generators.

$$\{Q, \bar{Q}\} = P, \tag{1.0.1}$$

where $\{A, B\} = AB + BA$ is the anti commutator, not the Poisson bracket. There are many reasons to consider supersymmetric theories. A direct reason is that supersymmetry is realized in the tri-critical Ising model in condensed matter physics. While supersymmetry hasn't been observed in nature in the context of particle physics, supersymmetric extensions of the standard model of particle physics are leading candidates to explain the hierarchy problem in the standard model. Supersymmetry provides a symmetry to protect the Higgs mass from being renormalized up to the Plank scale. From another perspective, supersymmetry gives more theoretical control over certain calculations making them much easier. This allows theorists to obtain exact results in supersymmetric theories that are prohibitively hard to obtain in non supersymmetric theories. Supersymmetry can lead to non-remormalization theorems in perturbative field theory. Another example of this is the computation of an exact fermion condensate that leads to chiral symmetry breaking. The first example of this calculation was done in the context of supersymmetric Yang-Mills theories when the exact gaugino condensate was calculated in [1]. Solitons in supersymmetric Yang-Mills theories have helped develop our understanding of how confinement might occur in an analytic setting. It is a general feature of supersymmetric field theories that the high energy behavior of the theories is better than the corresponding non supersymmetric field theory. Supersymmetry has also provided the first example of a finite quantum field theory, 4D $\mathcal{N} = 4$ super Yang-Mills theory. It is a general feature that when the superspace description is known, it greatly helps in the analysis of these theories.

Superspace can be understood as a space where the usual coordinate directions have been extended to include Grassmann valued coordinates i.e.

$$Z^m(x^m) \to Z^M = (x^m, \theta^\mu), \tag{1.0.2}$$

where $m = 1 \cdots d$ and θ^μ transforms as a spinor under $SO(1, d-1)$ and $\{\theta^\mu, \theta^\nu\} = 0$. The concept of a field on a manifold is extended to superspace in terms of superfields. Superfields are functions on superspace i.e. $F(Z^M)$ that are analytic in the variables θ^μ. The analyticity property allows us to consider Taylor expanding a superfield in the Grassmann variables and the nilpotency of the Grassmann variables implies that the series expansion will terminate at some finite order. For example, if $\mu = 1, 2$ then

$$F(Z^M) = f(x^m) + \theta^\mu \psi_\mu(x^m) + \theta^1\theta^2 g(x^m) \quad . \tag{1.0.3}$$

The x^m dependent fields at each order of the expansion are the component fields of the superfields. A given superfield contains both bosonic and fermionic component fields. This simple example already shows an important feature in supersymmetry, the equality of bosonic and fermionic degree's of freedom i.e. (f, g) are the bosons and (ψ_μ) are the fermions. Just as the action of the Poincare group can realized on fields by differential and matrix operators on flat space realizing the Poincare algebra, supersymmetry can be realized on superfields by differential and matrix operators in superspace realizing the supersymmetry algebra. One can think of a superfield as a vector in the space of fields with the Grassmann variables (and non vanishing products of the Grassmann variables) as basis vectors. The action of supersymmetry is to perform a rotation of the vector mixing the bosonic and fermionic components just as the rotation of the position vector in $\mathbb{R}^2$ mixes the x and y components. Because the components of a superfield rotate amongst themselves under supersymmetry, a superfield generically contains all of the fields necessary to have a complete representation of a supermultiplet[1]. When a supersymmetric system is expressed in superspace, it often simplifies the notation and allows the supersymmetry to be manifest. The cancellations of divergences in Feynman graphs between bosons and fermions due to supersymmetry is handled automatically when working in superspace. Non renormalization theorems are much easier to prove in superspace and the most elegant proof [2] comes from using very simple concepts required by superspace. As we will see in a later section, superspace encodes the relationship between scalar field theories with four supercharges and complex geometry in a very

[1]The representation is often reducible but that doesn't impact the spirit of the comments made based on this property. We are not arguing however, that the problem of finding complete irreducible representations is in general a simple exercise. Finding superspace formulations of irreducible multiplets with more than four supercharges is an open challenge.

beautiful and elegant way. The Green Schwarz formulation of the superstring can be understood as describing the embedding of a string into superspace. This list of examples is by no means exhaustive in terms of the applications of superspace. It is only meant to indicate that there are many places where superspace can be used to ease the study supersymmetry. In this work we will use superspace to derive results relevant to string theory in the area of effective actions and generalized Kähler geometry. Since any decent introduction to superspace is an entire volume of work by itself, we won't attempt to provide one here. Instead we refer the reader to books written on the topic [3–5]. A good review is given in [6].

Outline of the dissertation

In chapter 2, we will use a superspace approach to derive the lowest order string corrections to the 10D $\mathcal{N} = 1$ supergravity low energy effective action for the heterotic string for both the gauge 2-form and gauge 6-form formulations. We will describe how the input from string theory is used and discover an interesting geometric property for one of the tensors in the solution. The content in chapter 2 is published [7]. We present the calculations that demonstrate the desired results and refer the reader to [7] for references to the historical development of the approach.

In chapter 3, we will review the relevant features of generalized Kähler geometry and how they are connected to two dimensional $\mathcal{N} = (2, 2)$ supersymmetric non linear sigma models. We also review how $(2, 2)$ superspace captures the conditions on the sigma model target space derived from non-linearly realizing an extra $(1, 1)$ supersymmetry in $(1, 1)$ superspace.

In chapter 4, we perform the reduction of the $(2, 2)$ non linear sigma model with semi chiral superfields from $(2,2)$ superspace to $(1,1)$ superspace with the covariant derivative algebra for the chiral vector multiplet and identify the moment map and 1-form u associated to the isometry of the target space.

In chapter 5, we explore the relationship between the sigma model derivation of the moment map and a definition of the moment map given in terms of generalized Kähler geometry.

In chapter 6, we give a formulation of T duality for sigma models with semi chiral superfields using the chiral vector multiplet. We work out a simple example to see the characteristic $R \to \frac{1}{R}$ property of T duality on circle where R is the radius of the circle. The content in chapers 4, 5, and 6 was published in [8].

In chapter 7, we describe a previously unknown $\mathcal{N} = (2,2)$ supersymmetric vector multiplet, the semi chiral vector multiplet. We argue that this multiplet is the proper multiplet to use in formulating T duality for sigma models with semi chiral superfields.

In chapter 8, we give our conclusions. This is followed by appendices that give the conventions used and that describe details that were omitted in the main body of the disseration.

Chapter 2

Effective Action for the Heterotic String

2.1 String Theory Effective Actions

The effective action for superstring theory provides a useful approach for connecting string theory, which lives in 1+9 dimensions, to our 1+3 dimensional experience. It is important to understand as much about the effective action as possible. The world sheet theory for the supersting gives the spectrum of massless fields that go into the effective action. The spectrum is that of 10D $\mathcal{N}=1$ supergravity plus super Yang-Mills for type I or heterotic strings and 10D $\mathcal{N}=2$A and $\mathcal{N}=2$B for the type IIA and type IIB strings respectively. At the lowest order in the perturbation parameter α' (the inverse string tension), we know the supergravity plus super Yang-Mills actions corresponding to each superstring theory. We expect superstring theory to introduce extra terms to the known actions for the corresponding multiplets to encode stringy effects. To determine the effective action at higher orders in α', four methods are used. The most fundamental approach is to compute string scattering amplitudes for the appropriate supergravity or super Yang-Mills excitations and reconstruct the action that produces the amplitudes. This has the advantage that stringy effects are automatically incorporated. This approach however is difficult to perform in practice because one can only consider the bosonic fluctuations so that the information about the fermions is absent and thus supersymmetry is unclear. Another approach is to calculate loop corrections in the non linear sigma model and require the β functions to vanish. This approach also lacks the ability to capture information about the fermions. The last two methods use supersymmetric field theory as the starting point and use some input from the scattering amplitude method to incorporate stringy effects. The extra input normally comes in the form of a higher order α' correction to the action. This has the advantage that the fermion information is obtained along with the information about the bosons and supersymmetry remains clear. The two methods are the Noether procedure and superspace methods. The Noether procedure works directly with the action by proposing new α' dependent terms to the supersymmetry transformations and action in order to obtain the terms necessary to regain supersymmetry after including the scattering amplitude input. The superspace method involves embedding the scattering amplitude input into the supergeometry and using superspace to derive the complete effective action. The superspace approach has the advantage that its consistency conditions give a systematic way to obtain the supersymmetric completion to the scattering amplitude input. In this work we will describe the derivation of the supersymmetric effective action at first order in α' for the heterotic string. The embedding of the scattering amplitude input was worked out in [9]. For ease of presentation we will only give the bosonic terms of the effective action, however the procedure we describe will clearly demonstrate that the fermion information is under control.

The scattering amplitude input that we will use comes from the Green Schwarz anomaly cancelation mechanism [10]. It is know that this mechanism requires the inclusion of the local Lorentz Chern Simons form with the exterior derivative of the gauge two form as

$$H = dB + \alpha' \Omega^{LL} \tag{2.1.1}$$

where Ω^{LL} is the local Lorentz Chern Simons form

$$\Omega^{LL} = Tr(\omega \wedge R + \frac{1}{3}\omega \wedge \omega \wedge \omega). \tag{2.1.2}$$

The Chern Simons form is written in terms of ω, the local Lorentz connection one form and R, the Ricci two form. In [9], the local Lorentz Chern Simons form was included in the 10D $\mathcal{N} = 1$ supergeometry. In this work we describe the derivation of the effective action based on that stringy input into the supergeometry and demonstrate that what we find is consistent with what is known for the effective action of the heterotic string to first order in α'.

The perturbative expansion is organized terms of the superspace torsions and curvatures which are obtained from the graded commutator of the supergravity covariant derivatives. The supergravity gauge covariant derivatives are

$$\nabla_A = E_A{}^M D_M - \frac{1}{2}\omega_A{}^{\underline{b}\,\underline{c}} M_{\underline{b}\,\underline{c}} \tag{2.1.3}$$

where $E_A{}^M$ is the super frame field,D_M are the flat superspace covariant derivatives, $\omega_A{}^{bc}$ is the spin connection, and M_{bc} is the abstract generator of local Lorentz transformations. The commutator of supergravity covariant derviatives gives the super-torsions and super-curvatures.

$$[\nabla_A, \nabla_B\} = T_{AB}{}^C \nabla_C - \frac{1}{2} R_{AB}{}^{\underline{c}\,\underline{d}} M_{\underline{c}\,\underline{d}} \tag{2.1.4}$$

The expansion in α' coming from string theory corresponds to an expansion of the super-torsions and super-curvatures of the form

$$T_{AB}{}^{C} = \sum_{n} (\alpha')^{n} T^{(n)C}_{AB}$$

$$R_{AB}{}^{\underline{c}\underline{d}} = \sum_{n} (\alpha')^{n} R^{(n)}_{AB}{}^{\underline{c}\underline{d}} \qquad (2.1.5)$$

Solving the supergravity Bianchi identities[1] for the super-torsions and super-curvatures subject to the appropriate constraints gives a consistent description of the supergravity multiplet at least in terms of the physical multiplet spectrum. If the super Bianchi identities are solved without directly imposing dynamical equations, then the algebra is said to be off shell and the full super multiplet content is captured. If dynamical equations are imposed, then the multiplet is said to be on shell. On shell multiplets correspond to multiplets whose auxiliary fields, (non dynamical fields that are required for the supersymmetry variations of the fields in the multiplet to close without the need of dynamical equations), have been replaced by some combination of dynamical fields. In some cases we can use the dynamical equations to derive an action that is consistent with the supersymmetry described by the supergravity covariant derivatives. It is a general feature of supersymmetric theories that dynamic equations imposed by closure of a supersymmetry algebra acting on fields can be consistent with equations of motion derivable from an action. Since the description of the supergravity theory in [9] is on shell, dynamical equations are implied by the super Bianchi identities. We use these dynamical equations to derive the action

[1] Just as in G.R. the supergravity Bianchi identities are given by the superspace Jacobi identity for the supergravity covariant derivatives.

associated to the supersymmetric multiplet. It is this action that is the $\mathcal{O}(\alpha')$ string corrected effective action.

2.2 A Review of First-Order Corrected 10D, $\mathcal{N}=1$ Superspace Supergravity Geometry

Let us begin by reviewing the results in [9]. There a solution was given to the 10D, $\mathcal{N}=1$ supergravity plus super Yang-Mills Bianchi identities that is correct to first order in the perturbative parameters β' and[2] γ'. Here we recall the Yang-Mills truncated version (eliminating the Yang-Mills fields or equivalently putting $\beta'=0$) of this solution and we complete the results by the presentation of the bosonic equations of motion. The solution depends on two assumptions, both valid at first order in γ'. The first assumption is a constraint on the 0 dimensional torsion component $T_{\delta\gamma}{}^{\underline{a}}$ and the second is a choice for the usual 1 dimensional auxiliary field denoted by $A_{\underline{a}\,\underline{b}\,\underline{c}}$. The choice for the auxiliary field will contain the input from string theory. After a short digression we will discuss these two inputs and their consequences in order.

Before considering any non trivial constraints on torsion components it is always worthwhile to study purely conventional constraints, which do reduce the number of independent torsion and curvature components, but do not have any consequence on the dynamics. Using standard methods (see the work in [11]) one

[2]The parameter γ' is proportional to α' with the proportionality coefficient to be determined.

can show that the following set of constraints is purely conventional:

$$
\begin{aligned}
& i\,(\sigma_{\underline{a}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{b}} \;=\; 16\,\delta_{\underline{a}}{}^{\underline{b}} \quad , \quad i\,(\sigma_{\underline{c}})^{\alpha\beta}\,T_{\alpha\,\underline{b}}{}^{\underline{c}} \;=\; 0 \quad , \\
& i\,(\sigma_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{e}} \;=\; 0 \quad , \quad T_{\alpha\,[\underline{d}\,\underline{e}]} \;=\; 0 \qquad , \\
& T_{\underline{d}\,\underline{e}\,\underline{b}} \;=\; \frac{1}{8}\,(\sigma_{\underline{d}\,\underline{e}})_{\alpha}{}^{\beta}\,T_{\beta\,\underline{b}}{}^{\alpha} \;+\; i\,\frac{1}{16}\,(\sigma_{\underline{b}})^{\alpha\,\beta}\,\mathcal{R}_{\alpha\,\beta\,\underline{d}\,\underline{e}} \quad .
\end{aligned}
\tag{2.2.1}
$$

The role of each of these respective constraints is easy to understand. The first equation removes $\mathrm{E}_{\underline{a}}{}^{\underline{m}}$ as an independent variable. The second equation removes $\mathrm{E}_{\underline{a}}{}^{\mu}$ as an independent variable. The third constraint is a coset conventional constraint that removes part of $\mathrm{E}_{\alpha}{}^{\mu}$ as an independent variable. The fourth constraint removes $\omega_{\alpha\,\underline{b}\,\underline{c}}$ as an independent variable and the final constraint removes $\omega_{\underline{a}\,\underline{b}\,\underline{c}}$ as an independent variable. It is a simple matter to show that the torsion and curvature super tensors in [9], satisfy these conditions. Since these are purely conventional constraints, they may be imposed to all orders in the string slope-parameter expansion.

The first assumption is recalled by noticing that the first and third equations of (2.2.1) imply that the most general structure of the zero dimensional torsion is

$$
T_{\delta\gamma}{}^{\underline{a}} \;=\; i\,(\sigma^{\underline{a}})_{\delta\gamma} \;+\; i\,\frac{1}{5!}\,(\sigma^{[5]})_{\delta\gamma}\,X_{[5]}{}^{\underline{a}} \quad , \tag{2.2.2}
$$

with $X_{[5]}{}^{\underline{a}}$ in the appropriate 1050 dimensional irrep of $SO(1,9)$. However, in [9] the following zero dimensional torsion constraint was used

$$
T_{\delta\gamma}{}^{\underline{a}} \;=\; i\,(\sigma^{\underline{a}})_{\delta\gamma} + \mathcal{O}(\,(\gamma')^2\,) \quad . \tag{2.2.3}
$$

With regards to potential extensions of this work, we note that Nilsson advocated in [12] that the assumption $X_{[5]}{}^{\underline{a}} = 0$ is incompatible with the inclusion of higher than second order curvature terms in the effective action. This implies that the vanishing of this 0 dimensional superfield can be valid only at first order in γ' – which is consistent with the limit in which [9] was written. Note the possibility that $X_{[5]}{}^{\underline{a}}$ can contribute at higher order in γ',

$$X_{[5]}{}^{\underline{a}} \;=\; \mathcal{O}(\,(\gamma')^2\,) \quad . \tag{2.2.4}$$

Now let us begin to write the first order in γ' solution of the Bianchi identities for the torsion subject to the conventional constraints (2.2.1) and assumption (2.2.3):

$$T_{\alpha\beta}{}^{\gamma} \;=\; -\Big[\;\delta_{(\alpha}^{\gamma}\delta_{\beta)}^{\delta} \;+\; (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\;\Big]\,\chi_{\delta} \quad , \tag{2.2.5}$$

$$T_{\alpha\underline{b}}{}^{\gamma} \;=\; \frac{1}{48}(\sigma_{\underline{b}}\,\sigma^{[3]})_{\alpha}{}^{\gamma}A_{[3]} \quad , \tag{2.2.6}$$

$$T_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; -2\,L_{\underline{a}\,\underline{b}\,\underline{c}} \quad , \tag{2.2.7}$$

$$\mathcal{R}_{\alpha\,\beta\;\underline{a}\,\underline{b}} \;=\; i\,2(\sigma^{\underline{c}})_{\alpha\beta}(\;L_{\underline{a}\,\underline{b}\,\underline{c}} \;-\; \frac{1}{8}A_{\underline{a}\,\underline{b}\,\underline{c}}\;) \;-\; i\,\frac{1}{24}(\sigma_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,\underline{e}})_{\alpha\beta}A^{\underline{c}\,\underline{d}\,\underline{e}} \quad , \tag{2.2.8}$$

$$\nabla_{\alpha}\chi_{\beta} \;=\; -i\,(\sigma^{\underline{a}})_{\alpha\beta}\nabla_{\underline{a}}\Phi \;+\; i\,\frac{1}{48}(\sigma^{[3]})_{\alpha\beta}\left(\,4L_{[3]} \;+\; A_{[3]} \;-\; i\frac{1}{2}(\chi\sigma_{[3]}\chi)\right) \tag{2,2.9}$$

$$\mathcal{R}_{\alpha\underline{c}\,\underline{a}\,\underline{b}} \;=\; i\,(\sigma_{[\underline{a}|})_{\alpha\beta}T_{\underline{c}|\,\underline{b}]}{}^{\beta} + i\gamma'\,(\sigma_{[\underline{c}|})_{\alpha\beta}\mathcal{R}^{\underline{k}\,\underline{l}}{}_{|\underline{a}\,\underline{b}]}T_{\underline{k}\,\underline{l}}{}^{\beta} \quad , \tag{2.2.10}$$

with Φ a scalar superfield (the dilaton) transforming into χ_{α} (the dilatino) under supersymmetry,

$$\chi_{\alpha} \;=\; -2\,\nabla_{\alpha}\Phi \quad , \tag{2.2.11}$$

and $A_{\underline{a}\,\underline{b}\,\underline{c}}$ an auxiliary superfield.

The string theory input can be included in two different but equivalent ways. The first approach is to include the superspace local Lorentz Chern Simons form directly in the constraints for the super Bianchi identities for the super gauge 2-form. This is the approach taken in [9]. This results in the auxiliary field $A_{\underline{a}\,\underline{b}\,\underline{c}}$

being put on shell and taking a specific form that is related to open-string/closed-string duality. This conjectured property of the low energy effective action of the superstring was made even before this property had a name based on an interesting observation. Bergshoeff and Rakowski [13] noted that in 6D simple superspace the quantities

$$T^{\underline{c}\,\underline{d}\,\gamma} \quad , \quad \mathcal{R}_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \tag{2.2.12}$$

share many common properties with the fields of a vector multiplet

$$\lambda^{\gamma\,\hat{I}} \quad , \quad F_{\underline{a}\,\underline{b}}{}^{\hat{I}} \tag{2.2.13}$$

and thus asserted that large numbers of higher derivative supergravity terms may be treated as if one were coupling a vector multiplet to the supergavity multiplet. A similar relationship exists for the same quantites in 10 dimensions.

We can instead start by using this relationship between the supergravity multiplet and super Yang-Mills multiplet to make a choice for the form of the auxiliary field. We would then find that the local lorentz Chern Simons form is included with the exterior derivative of the gauge 2-form. The advantage of this approach is that it doesn't require one to start the analysis by choosing to have a 2-form gauge field in the spectrum instead of considering the dual theory which has a gauge 6-form. In the presentation that follows we will use the open-string/closed string duality starting point i.e. we make the following choice for the auxiliary field

$$A_{\underline{a}\,\underline{b}\,\underline{c}} \doteq -i\,\gamma'\,(T_{\underline{k}\,\underline{l}}\,\sigma_{\underline{a}\,\underline{b}\,\underline{c}}\,T^{\underline{k}\,\underline{l}}) \quad , \tag{2.2.14}$$

With this choice the theory is put completely on shell. This means that all torsion and curvature components, as well as the spinorial derivatives of all objects in the geometry can be expressed as a function of the dilaton Φ, the dilatino χ_α, the gravitino Weyl tensor sitting in its field strength $T_{\underline{a}\,\underline{b}}{}^\gamma$, the Weyl tensor sitting in the curvature $\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}$ together with the supercovariant object $L_{\underline{a}\,\underline{b}\,\underline{c}}$ appearing in the spacetime torsion.

The object $L_{\underline{a}\,\underline{b}\,\underline{c}}$ was introduced[3] for the ten dimensional theory [9] in order to permit the simple passage between the 2-form and 6-form formulation of the 10D, $\mathcal{N} =$ 1 supergravity theory. It is not an independent variable but its explicit form as a function of the component fields is determined only by specifying which of the two (2-form vs. 6-form) gauge fields is in the supergravity multiplet. This will be discussed in subsequent sections.

In particular, $L_{\underline{a}\,\underline{b}\,\underline{c}}$ must satisfy the following conditions

$$\begin{aligned}
\nabla_\alpha L_{\underline{a}\,\underline{b}\,\underline{c}} &= i\,\frac{1}{4}(\sigma_{[\underline{a}})_{\alpha\beta}\,(\;T_{\underline{b}\,\underline{c}]}{}^\beta \;-\; \gamma' \mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{b}\,\underline{c}]}\,T_{\underline{k}\underline{l}}{}^\beta\;) \quad , \qquad (2.2.15)\\
\nabla_\alpha T_{\underline{a}\,\underline{b}}{}^\beta &= \frac{1}{4}(\sigma^{\underline{c}\,\underline{d}})_\alpha{}^\beta\,\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}} \;-\; T_{\underline{a}\,\underline{b}}{}^\gamma\,T_{\gamma\alpha}{}^\beta \\
&\quad + \frac{1}{48}\left[\;2L_{\underline{a}\,\underline{b}\,\underline{c}}(\sigma^{\underline{c}}\sigma^{[3]})_\alpha{}^\beta \;-\; (\sigma_{[\underline{a}|}\sigma^{[3]})_\alpha{}^\beta\nabla_{|\underline{b}]}\;\right]\;A_{[3]} \quad , \qquad (2.2.16)
\end{aligned}$$

in order for the Bianchi identities on the superspace torsions and curvatures to be satisfied. These same Bianchi identities require

$$\begin{aligned}
\nabla_{\underline{a}}\chi_\beta &= -i\,\frac{1}{2}(\sigma^{\underline{b}})_{\alpha\beta}(T_{\underline{a}\,\underline{b}}{}^\alpha \;-\; 2\gamma'\mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{a}\underline{b}}\,T_{\underline{k}\,\underline{l}}{}^\alpha) \quad , \qquad (2.2.17)\\
(\sigma^{\underline{a}\,\underline{b}})_\beta{}^\alpha T_{\underline{a}\,\underline{b}}{}^\beta &= -i\,8\,(\sigma^{\underline{a}})^{\alpha\beta}\chi_\beta\nabla_{\underline{a}}\Phi \;-\; i\frac{1}{24}(\sigma^{[3]})^{\alpha\beta}\chi_\beta\,(\;16L_{[3]} \;+\; A_{[3]}\;) \\
&\quad + 3\gamma'(\sigma^{\underline{a}\,\underline{b}})_\beta{}^\alpha\mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{a}\,\underline{b}}\,T_{\underline{k}\,\underline{l}}{}^\beta \quad . \qquad (2.2.18)
\end{aligned}$$

[3]The first appearance of the L-type variable in the physics literature occurred in the work of [16]. It was introduced to permit a unified superspace description of theories related one to another by Poincaré duality.

The results given above are sufficient to derive the equations of motion for the spinors, already presented in [9], and we will now use them in order to derive the bosonic equations of motion. A detailed presentation of using superspace techniques for deriving equations of motion can be found in [14, 15]

In order to find the equation of motion of the scalar let us begin with the relation (2.2.17) multiplied by a sigma matrix $(\sigma^{a})^{\gamma\alpha}$ and differentiate it with ∇_{β},

$$\begin{aligned}(\sigma^{\underline{a}})^{\gamma\alpha}\nabla_{\underline{a}}\left(\nabla_{\beta}\chi_{\alpha}\right) = \ & i\,\frac{1}{2}(\sigma^{\underline{a}\,\underline{b}})_{\alpha}{}^{\gamma}\nabla_{\beta}\left(T_{\underline{a}\,\underline{b}}{}^{\alpha} \ - \ 2\gamma'\mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{a}\,\underline{b}}T_{kl}{}^{\alpha}\right) \\ & + \ (\sigma^{\underline{a}})^{\gamma\alpha}\left[\nabla_{\underline{a}},\nabla_{\beta}\right]\chi_{\alpha} \quad .\end{aligned} \tag{2.2.19}$$

Notice that the LHS contains the spacetime derivatives of both $(\sigma^{\underline{b}})_{\beta\alpha}\nabla_{\underline{b}}\Phi$ and $(\sigma^{[3]})_{\beta\alpha}L_{[3]}$, while the RHS can be computed using at most three-half dimensional results recalled above. Therefore, one obtains the equation of motion of the scalar from (2.2.19) by taking the trace $\delta_{\gamma}{}^{\beta}$

$$16\nabla^{\underline{a}}\nabla_{\underline{a}}\Phi \ = \ 4\mathcal{R} \ - \ 8\gamma'\mathcal{R}^{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}}\mathcal{R}_{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}} \ + \ \text{fermions} \quad . \tag{2.2.20}$$

Moreover, the same relation (2.2.19), if multiplied by $(\sigma_{\underline{e}\,\underline{f}})_{\gamma}{}^{\beta}$, yields

$$\nabla^{\underline{a}}L_{\underline{a}\,\underline{e}\,\underline{f}} \ = \ -4L_{\underline{a}\,\underline{e}\,\underline{f}}\nabla^{\underline{a}}\Phi \ + \ \text{fermions} \quad . \tag{2.2.21}$$

The remaining independent part of (2.2.19) can be projected out if one multiplies it by $(\sigma_{efgh})_{\gamma}{}^{\beta}$. The obtained relation together with the Bianchi identity for the torsion with only vectorial indices gives

$$\nabla_{[\underline{e}}L_{\underline{f}\,\underline{g}\,\underline{h}]} \ = \ -3L_{[\underline{e}\,\underline{f}}{}^{\underline{a}}L_{\underline{g}\,\underline{h}]\,\underline{a}} \ - \ \frac{3}{2}\gamma'\mathcal{R}_{\underline{k}\,\underline{l}\,[\underline{e}\,\underline{f}}\mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{g}\,\underline{h}]} \ + \ \text{fermions} \quad . \tag{2.2.22}$$

Notice that (2.2.21) and (2.2.22) suggest that the object $L_{\underline{abc}}$ might be *either* related to the field strengths of a two-form *or* dual field strength of a six-form depending on which of these two equations is interpreted as the Bianchi identity and which is as the equation of motion.

Assuming that (2.2.21) gives the equation of motion for a two-form gauge field, then (2.2.22) must correspond to its Bianchi identity. Searching for a closed three-form in the geometry, in which the field strengths of this two-form can be identified, one might want to use the identity satisfied by a Lorentz Chern-Simons three-form Q[4]

$$\nabla_{[\underline{e}} Q_{\underline{f}\,\underline{g}\,\underline{h}]} \; - \; \frac{3}{2} T_{[\underline{e}\,\underline{f}}{}^{\underline{a}} Q_{\underline{g}\,\underline{h}]\,\underline{a}} \; = \; - \; \frac{3}{2} \mathcal{R}_{[\underline{e}\,\underline{f}|\underline{k}\,\underline{l}} \mathcal{R}_{|\underline{g}\,\underline{h}]}{}^{\underline{k}\,\underline{l}} + \text{fermions} \quad . \qquad (2.2.23)$$

in order to "absorb" the curvature squared term in the RHS of (2.2.22) . However this doesn't quite work. It is interesting to consider how this fails.

Observe that the structure of the equations (2.2.22) and (2.2.23) is almost the same, with the only difference that in the RHS of (2.2.21) the role of the "group" indices and "form" indices of the curvature are exchanged with respect to one another. Since the curvature is defined by a connection with torsion, it is not symmetric with respect to the exchange of its pairs of indices. Therefore, $(L - \gamma' Q)_{\underline{abc}}$ cannot be equal exactly to the vectorial component of a closed three-form, but their difference is an object which serves as a link between the two curvature squared expressions we have in (2.2.22) and (2.2.23). This object (called "$Y_{\underline{abc}}$" in the next chapter) does exist as was first demonstrated in [9]. After it has been properly identified, we

[4]The second two indices on the Riemann curvature tensor may be thought of as the Lie algebraic "group" indices for SO(1,9).

can use $Y_{\underline{abc}}$ to show

$$\nabla_{[\underline{e}}(L - \gamma' Q - \gamma' Y)_{\underline{f}\,\underline{g}\,\underline{h}]} - \frac{3}{2} T_{[\underline{e}\,\underline{f}}{}^{\underline{a}}(L - \gamma' Q - \gamma' Y)_{\underline{g}\,\underline{h}]\,\underline{a}} = \text{fermions} \quad (2.2.24)$$

at first order in γ'. This is the relation, which shows that (at least modulo fermionic contributions) $(L - \gamma' Q - \gamma' Y)_{\underline{abc}}$ can be identified as the vectorial component of a closed three-form.

Conversely assuming that (2.2.22) gives the equation of motion for a two-form gauge field, then (2.2.21) must correspond to its Bianchi identity in the dual theory. This theory is slightly easier to construct because although it contains the first order superstring corrections, it does not require a dual Chern-Simons term for its consistency.

Finally, the Ricci tensor and the scalar curvature can be derived from (2.2.18) using the dimension three-half results

$$\frac{1}{2}\mathcal{R}^{(\underline{d}\,\underline{c})} = 2\nabla^{(\underline{d}}\nabla^{\underline{c})}\Phi + 2\,\gamma'\,\mathcal{R}^{\underline{k}\,\underline{l}\,\underline{d}\,\underline{b}}\,\mathcal{R}_{\underline{k}\,\underline{l}}{}^{\underline{c}}{}_{\underline{b}} + \text{fermions} \quad , \quad (2.2.25)$$

$$\mathcal{R} = -16\nabla^{\underline{a}}\Phi\nabla_{\underline{a}}\Phi + \frac{2}{3}L^{\underline{a}\,\underline{b}\,\underline{c}}L_{\underline{a}\,\underline{b}\,\underline{c}} + 3\gamma'\,\mathcal{R}^{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}}\,\mathcal{R}_{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}} + \text{fermions} \quad (2.2.26)$$

Throughout our discussion up to this point, we were working directly with the superfields of 10D, $\mathcal{N} = 1$ superspace supergravity. So all equations were superspace equations. For the rest of this discussion, we will set *all* fermions to zero. We will utilize the same symbols to denote the various quantities however. We use the following notation for the purely bosonic equations found from the superspace

Bianchi identities,

$$\hat{\mathcal{E}}_{\Phi} \doteq 4\nabla^{\underline{a}}\nabla_{\underline{a}}\Phi - \mathcal{R} + 2\gamma' \mathcal{R}^{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}}\mathcal{R}_{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}} \quad , \tag{2.2.27}$$

$$\hat{\mathcal{E}}_{B_{\underline{e}\underline{f}}} \doteq \nabla_{\underline{a}}\left(e^{4\Phi}L^{\underline{a}\,\underline{e}\,\underline{f}}\right) \quad , \tag{2.2.28}$$

$$\hat{\mathcal{E}}_{\widetilde{B}^{\underline{e}\,\underline{f}\,\underline{g}\,\underline{h}}} \doteq \nabla_{[\underline{e}}(L - \gamma'Q - \gamma'Y)_{\underline{f}\,\underline{g}\,\underline{h}]} - \frac{3}{2}T_{[\underline{e}\,\underline{f}}{}^{\alpha}(L - \gamma'Q - \gamma'Y)_{\underline{g}\,\underline{h}]\alpha} \tag{2.2.29}$$

$$\hat{\mathcal{E}}_{\eta_{\underline{d}\underline{c}}} \doteq \frac{1}{2}\mathcal{R}^{(\underline{d}\,\underline{c})} - 2\nabla^{(\underline{d}}\nabla^{\underline{c})}\Phi - 2\gamma'\mathcal{R}^{\underline{k}\,\underline{l}\,\underline{d}\underline{b}}\mathcal{R}_{\underline{k}\,\underline{l}}{}^{\underline{c}}{}_{\underline{b}} \quad , \tag{2.2.30}$$

$$\hat{\mathcal{E}}_{\eta} \doteq \mathcal{R} + 16\nabla^{\underline{a}}\Phi\nabla_{\underline{a}}\Phi - \frac{2}{3}L^{\underline{a}\,\underline{b}\,\underline{c}}L_{\underline{a}\,\underline{b}\,\underline{c}} - 3\gamma'\mathcal{R}^{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}}\mathcal{R}_{\underline{k}\,\underline{l}\,\underline{a}\,\underline{b}} \quad . \tag{2.2.31}$$

In order for the superspace Bianchi identities to be satisfied all of the $\hat{\mathcal{E}}$-quantities are required to vanish. The question we shall address in this work is, "Does there exist a component level action whose variations lead to equations of motion that are compatible with (2.2.27) - (2.2.31)?" This same action must also contain a field such that either (2.2.28) or (2.2.29) can be interpreted as a Bianchi identity.

2.3 Bosonic Terms of a Component Action for Two-form Formulation

The non-vanishing components of the modified 3-form field strength to this order can be written as (below we have used a slightly different set of conventions from [9] as discussed in an appendix)

$$H_{\alpha\beta\,\underline{c}} = i\frac{1}{2}(\sigma_{\underline{c}})_{\alpha\beta} + i\,4\,\gamma'(\sigma_{\underline{a}})_{\alpha\beta}\,G^{\underline{a}\,\underline{e}\,\underline{f}}G_{\underline{c}\,\underline{e}\,\underline{f}} \quad , \tag{2.3.1}$$

$$H_{\alpha\underline{b}\,\underline{c}} = i\,2\,\gamma'\left[\,(\sigma_{[\underline{b}|})_{\alpha\beta}T_{\underline{e}\underline{f}}{}^{\beta} - 2(\sigma_{[\underline{e}]})_{\alpha\beta}T_{\underline{f}\,[\underline{b}|}{}^{\beta}\,\right]\,G_{\underline{c}]}{}^{\underline{e}\,\underline{f}} \quad , \tag{2.3.2}$$

$$H_{\underline{a}\,\underline{b}\,\underline{c}} = G_{\underline{a}\,\underline{b}\,\underline{c}} + \gamma'Q_{\underline{a}\,\underline{b}\,\underline{c}} \quad . \tag{2.3.3}$$

In the limit where $\gamma' = 0$ these equations correspond to the superspace geometry in a string-frame description of the pure supergravity theory. As was pointed out some time ago [17], the field *independence* of the leading term in the $G_{\alpha\beta\underline{c}}$ component of the 3-form field strength is indicative of this.

The quantity $L_{\underline{a}\,\underline{b}\,\underline{c}}$ in this formulation is defined by,

$$L_{\underline{a}\,\underline{b}\,\underline{c}} \;\doteq\; G_{\underline{a}\,\underline{b}\,\underline{c}} + \gamma' Q_{\underline{a}\,\underline{b}\,\underline{c}} + \gamma' \, Y_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \mathcal{O}((\gamma')^2) \quad , \tag{2.3.4}$$

where $G_{\underline{a}\,\underline{b}\,\underline{c}}$ is the supercovariantized field strength of a two-form, $Q_{\underline{a}\,\underline{b}\,\underline{c}}$ is the Lorentz Chern-Simons form and

$$Y_{\underline{a}\,\underline{b}\,\underline{c}} \;\doteq\; - \left(\mathcal{R}^{\underline{e}\,\underline{k}}{}_{[\underline{a}\,\underline{b}|} + \mathcal{R}_{[\underline{a}\,\underline{b}|}{}^{\underline{e}\,\underline{k}} + \frac{8}{3} G_{\underline{d}}{}^{\underline{e}}{}_{[\underline{a}|} G^{\underline{d}\,\underline{k}}{}_{|\underline{b}|} \right) G_{|\underline{c}]\,\underline{e}\,\underline{k}} \quad . \tag{2.3.5}$$

This quantity, (which to our knowledge first appeared in [9]) has a remarkable property. It is a straightforward calculation to show

$$\nabla_{[\underline{e}} Y_{\underline{f}\,\underline{g}\,\underline{h}]} \;-\; \frac{3}{2} T_{[\underline{e}\,\underline{f}}{}^{\underline{a}} Y_{\underline{g}\,\underline{h}]\,\underline{a}} \;=\; -\,\frac{3}{2} \left(\mathcal{R}_{\underline{k}\,\underline{l}[\underline{e}\,\underline{f}} \mathcal{R}^{\underline{k}\,\underline{l}}{}_{\underline{g}\,\underline{h}]} \;-\; \mathcal{R}_{[\underline{e}\,\underline{f}|\underline{k}\,\underline{l}} \mathcal{R}_{|\underline{g}\,\underline{h}]}{}^{\underline{k}\,\underline{l}} \right) \;+\; \mathcal{O}(\gamma') \quad . \tag{2.3.6}$$

By keeping terms *only* up to first order in γ' we find that a Lagrangian density of the form

$$\mathcal{L} \;=\; \mathrm{e}^{-1} e^{4\Phi} \left[\mathcal{R}(\omega) \;+\; 16\, (\mathrm{e}^{\underline{a}}\Phi)\, (\mathrm{e}_{\underline{a}}\Phi) \;-\; \frac{1}{3} L^{\underline{a}\,\underline{b}\,\underline{c}} L_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \gamma' \mathrm{tr}(\mathcal{R}^{\underline{a}\,\underline{b}} \mathcal{R}_{\underline{a}\,\underline{b}}) \right] \quad , \tag{2.3.7}$$

where ω is the torsion-less spin connection, is compatible with the set of equations of motion (2.2.25), (2.2.26),(2.2.28), (2.2.29) and Bianchi identity (2.2.27), If we expand the penultimate term to first order in γ' we find

$$\begin{aligned} \mathcal{L} \;=\; \mathrm{e}^{-1} e^{4\Phi} \Big[\, & \mathcal{R}(\omega) \;+\; 16\, (\mathrm{e}^{\underline{a}}\Phi)\, (\mathrm{e}_{\underline{a}}\Phi) \;-\; \frac{1}{3} G^{\underline{a}\,\underline{b}\,\underline{c}} \left(G_{\underline{a}\,\underline{b}\,\underline{c}} + 2\gamma' \, Q_{\underline{a}\,\underline{b}\,\underline{c}} \right) \\ & -\; \frac{2}{3} \gamma' \, G^{\underline{a}\,\underline{b}\,\underline{c}} Y_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \gamma' \mathrm{tr}(\mathcal{R}^{\underline{a}\,\underline{b}} \mathcal{R}_{\underline{a}\,\underline{b}}) \, \Big] \quad . \end{aligned} \tag{2.3.8}$$

It is easily seen that the action to first order in γ' when written using the Y variable takes a simple and elegant form.

Variation of this Lagrangian with respect to the dilaton gives

$$\delta_\Phi \mathcal{L} \;\sim\; -4\mathrm{e}^{-1} e^{4\Phi} \, [\; \mathcal{E}_\eta \;+\; 2\mathcal{E}_\Phi \;] \; \delta\Phi \quad . \tag{2.3.9}$$

where $\mathcal{E}_\eta$ and $\mathcal{E}_\Phi$ are given by (2.2.31) and (2.2.27).

The variation with respect to the antisymmetric tensor at first seems very complicated due to the fact that its field strength appears in the Lorentz connection. However, one can write it simply as

$$\delta_B \mathcal{L} \;=\; \mathrm{e}^{-1} e^{4\Phi} \left(-\frac{2}{3} L^{\underline{a}\,\underline{b}\,\underline{c}} \, \delta L_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \gamma' \delta \mathrm{tr} \, (\mathcal{R}^{\underline{a}\,\underline{b}} \, \mathcal{R}_{\underline{a}\,\underline{b}}) \right) \quad . \tag{2.3.10}$$

Replacing now (2.3.4) into the first term, we obtain the form

$$\begin{aligned} \delta_B \mathcal{L} \;\sim\; & 2\mathcal{E}_{B_{\underline{a}\underline{b}}} \delta B_{\underline{a}\,\underline{b}} - \frac{2}{3} \mathrm{e}^{-1} \gamma' \, e^{4\Phi} L^{\underline{a}\,\underline{b}\,\underline{c}} \delta_L \, (Q + Y)_{\underline{a}\,\underline{b}\,\underline{c}} \\ & + \; \mathrm{e}^{-1} \, e^{4\Phi} \gamma' \delta_L \mathrm{tr} \; \big(\; \mathcal{R}^{\underline{a}\,\underline{b}} \, \mathcal{R}_{\underline{a}\,\underline{b}} \; \big) \quad . \end{aligned} \tag{2.3.11}$$

The last terms in fact form a combination of variations, (that will appear repeatedly), which can be expressed in terms of zero order equations of motion for arbitrary variations of the object $L_{\underline{a}\,\underline{b}\,\underline{c}}$. This is shown in appendix B, where this combination is denoted symbolically by $f(\mathcal{E})$. In terms of $f(\mathcal{E})$, the variation of the Lagrangian with respect to the antisymmetric tensor is

$$\delta_B \mathcal{L} \;\sim\; 2\mathcal{E}_{B_{\underline{a}\underline{b}}} \delta B_{\underline{a}\,\underline{b}} \;+\; \gamma' \, f(\mathcal{E}) \tag{2.3.12}$$

with

$$
\begin{aligned}
f(\mathcal{E}) \sim\ & 4\mathcal{E}_{B_{\underline{kl}}}\Phi_{\underline{k}}{}^{\underline{a}\,\underline{b}}\delta L_{\underline{l}\,\underline{a}\,\underline{b}} \\
& +\ 8\left[e^{4\Phi}\nabla^{\underline{a}}\left(e^{-4\Phi}\hat{\mathcal{E}}_{B_{\underline{bc}}}\right)\ +\ \left(e^{4\Phi}\hat{\mathcal{E}}_{\eta_{\underline{ak}}}\ -\ \hat{\mathcal{E}}_{B_{\underline{ak}}}\right)L_{\underline{k}}{}^{\underline{b}\,\underline{c}}\right]\delta L_{\underline{a}\,\underline{b}\,\underline{c}} \\
& -\ \frac{2}{3}\,\frac{1}{4!}\,e^{4\Phi}\,\mathcal{E}_{\widetilde{B}^{\underline{abcd}}}\delta_L\left(\mathcal{E}_{\widetilde{B}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}}\right)\ +\ \mathcal{O}(\gamma')\quad .
\end{aligned}
\tag{2.3.13}
$$

2.4 Bosonic Terms of a Component Action for Six-form Formulation

Retaining the same current $A_{\underline{abc}}$ specified by (2.2.10) we can introduce a seven-form N [18] satisfying an appropriate Bianchi identity. At the component level similar considerations have been carried out for the six-form formulation [19]. One of the remarkable things about this formulation is that in order to describe lowest order perturbative contributions to the effective does *not* require a Chern-Simons like modification to the seven-form field strength.

$$
N_{\alpha\beta[5]} \ = \ i\,\frac{1}{2}\,e^{4\Phi}\,(\sigma_{[5]})_{\alpha\beta} \quad , \tag{2.4.1}
$$

$$
N_{\alpha[6]} \ = \ -\frac{1}{4!}\epsilon_{[6][4]}\,e^{4\Phi}\,(\sigma^{[4]})_{\alpha}{}^{\beta}\chi_{\beta} \quad , \tag{2.4.2}
$$

$$
N_{[7]} \ = \ \frac{1}{3!}\,e^{4\Phi}\,\left(L^{[3]} - \frac{13i}{8}\chi\sigma^{[3]}\chi\right)\epsilon_{[3][7]} \quad . \tag{2.4.3}
$$

In particular, it is the equation (2.2.27) which insures that the purely vectorial component of the N Bianchi identity is satisfied. Equations (2.2.27), (2.2.28), (2.2.30) and (2.2.31) contain the bosonic equations of motion for the component fields of the dual theory. Notice that in this case (2.2.27) identifies $L_{\underline{abc}}$ as the following function of the component fields of the dual theory

$$
L_{\underline{a}\,\underline{b}\,\underline{c}} \ = \ -\frac{1}{7!}\,\epsilon_{\underline{a}\,\underline{b}\,\underline{c}[7]}\,e^{-4\Phi}\,N^{[7]} \quad . \tag{2.4.4}
$$

upon setting the fermions to zero. In the following we show that the Lagrangian density

$$\mathcal{L}_d \;=\; \mathrm{e}^{-1}\, e^{4\Phi} \Big[\; \mathcal{R}(\omega) \;+\; 16\,(\mathrm{e}^{\underline{a}}\Phi)\,(\mathrm{e}_{\underline{a}}\Phi) \;+\; \frac{1}{3}(\; L \;-\; \gamma'(Q+Y))^2_{\underline{a}\,\underline{b}\,\underline{c}} \\ +\; \gamma'\,\mathrm{tr}(\mathcal{R}^{\underline{a}\,\underline{b}}\mathcal{R}_{\underline{a}\,\underline{b}}) \;\Big] \quad , \tag{2.4.5}$$

is compatible with the set of equations of motion and Bianchi identity. Since our results are only valid to first order in γ' it follows that (2.4.5) should be more properly written as

$$\mathcal{L}_d \;=\; \mathrm{e}^{-1}\, e^{4\Phi} \Big[\; \mathcal{R}(\omega) \;+\; 16\,(\mathrm{e}^{\underline{a}}\Phi)\,(\mathrm{e}_{\underline{a}}\Phi) \;+\; \frac{1}{3}\, L^{\underline{a}\,\underline{b}\,\underline{c}}\, L_{\underline{a}\,\underline{b}\,\underline{c}} \\ -\; \frac{2}{3}\,\gamma'\, L^{\underline{a}\,\underline{b}\,\underline{c}}\, Q_{\underline{a}\,\underline{b}\,\underline{c}} \;-\; \frac{2}{3}\,\gamma'\, L^{\underline{a}\,\underline{b}\,\underline{c}}\, Y_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \gamma'\,\mathrm{tr}(\mathcal{R}^{\underline{a}\,\underline{b}}\mathcal{R}_{\underline{a}\,\underline{b}}) \;\Big] \quad , \tag{2.4.6}$$

and in this expression L is replaced by the expression in (2.4.4). When this is done two points are made obvious. Firstly, this action is not in the string-frame formulation. This follows in particular since the object $L_{\underline{abc}}$ depends on the dilaton through (2.4.4). From the superspace point of view this was already obvious due to the field dependence exhibited by (2.4.1). A string-frame formulation of the dual theory does exist after additional field redefinitions are applied to (2.4.5) and (2.4.6).

Secondarily, the Chern-Simons term does not actually appear in this action. One can perform an integration-by-part on the first term on the second line of (2.4.6) and this leads to a term

$$L^{\underline{a}\,\underline{b}\,\underline{c}}\, Q_{\underline{a}\,\underline{b}\,\underline{c}} \;\propto\; \epsilon^{\underline{a}_1 \cdots \underline{a}_6\, \underline{b}_1\, \underline{b}_2\, \underline{c}_1\, \underline{c}_2}\; M_{\underline{a}_1 \cdots \underline{a}_6}\, \mathrm{tr}(\mathcal{R}_{\underline{b}_1\,\underline{b}_2}\mathcal{R}_{\underline{c}_1\,\underline{c}_2}) \quad , \tag{2.4.7}$$

which can be seen to be precisely the term required by the dual Green-Schwarz mechanism for anomaly cancellation first given in [18]. Notice the change of sign of the L-squared term in (2.4.5) and (2.4.6) compared to (2.3.7) and (2.3.8). This is the usual sign-flip seen between theories connected by Poincaré duality.

Indeed, now even the variation with respect to the dilaton becomes complicated since $L_{\underline{abc}}$ appears in the connection. However, just marking the variation and using $\delta L = -4L\delta\Phi$ only in the most obvious terms, one ends up again with the combination of variations $f(\hat{\mathcal{E}})$ with the terms for the equation for the dilaton in the theory with two-form (2.3.13),

$$\delta_\Phi \mathcal{L}_d \sim -4\, e^{-1}\, e^{4\Phi} \Big[\, \hat{\mathcal{E}}_\eta \,+\, 2\hat{\mathcal{E}}_\Phi \,\Big]\, \delta\Phi \,+\, \gamma' f(\hat{\mathcal{E}}) \quad . \tag{2.4.8}$$

The variation with respect to the six-form M is computed in the same manner. The combination $f(\hat{\mathcal{E}})$ surprisingly appears again and one simply obtains,

$$\delta_M \mathcal{L}_d \sim -\frac{2}{3}\frac{1}{4!6!}\epsilon^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}\,[6]}\hat{\mathcal{E}}_{\widetilde{B}_{\underline{abcd}}}\delta M_{[6]} + \gamma' f(\mathcal{E}) \quad . \tag{2.4.9}$$

So the final conclusion is that in the dual theory, the component action in (2.4.6) is compatible with the equations of motion derived from superspace for the dual theory.

2.5 Comparison with a Component Level Investigation

Next, we study the relationship of the Lagrangian (2.3.8) with the component Lagrangian in [20]. A quick look to the component Lagrangian in [20] convinces us that using just rescalings of the fields it can be written in the form

$$\begin{aligned}\hat{\mathcal{L}} \,=\, \mathrm{e}^{-1}\, e^{4\Phi} \Big[\, & \mathcal{R}(\omega) \,+\, 16\, (\mathrm{e}^{\underline{a}}\Phi)\, (\mathrm{e}_{\underline{a}}\Phi) \,-\, \frac{1}{3} G^{\underline{a}\,\underline{b}\,\underline{c}}\, (\, G_{\underline{a}\,\underline{b}\,\underline{c}} \,+\, 2\gamma'\, Q_{\underline{a},\underline{b}\,\underline{c}}\,) \\ & +\, \gamma' \mathrm{tr}(\hat{\mathcal{R}}^{\underline{a}\,\underline{b}}\, \hat{\mathcal{R}}_{\underline{a}\,\underline{b}})\,\Big] \quad ,\end{aligned} \tag{2.5.1}$$

where hatted objects are defined using a Lorentz connection $\hat{\Omega}$, which may differ from ours by its torsion. In order to compare this to our Lagrangian (2.3.8), let us write the difference as

$$\begin{aligned} \mathrm{e}^{-1}\, e^{-4\Phi}\Big(\, \mathcal{L} - \hat{\mathcal{L}}\Big) = \; & -\frac{2}{3}\gamma' G^{\underline{a}\,\underline{b}\,\underline{c}}\, \Big(Q - \hat{Q}\Big)_{\underline{a}\,\underline{b}\,\underline{c}} \; - \; \frac{2}{3}\gamma' G^{\underline{a}\,\underline{b}\,\underline{c}}\, Y_{\underline{a}\,\underline{b}\,\underline{c}} \\ & + \; \gamma' \,\mathrm{tr}\,\Big(\, \mathcal{R}_{\underline{a}\,\underline{b}}\, \mathcal{R}^{\underline{a}\,\underline{b}} \; - \; \hat{\mathcal{R}}_{\underline{a}\,\underline{b}}\, \hat{\mathcal{R}}^{\underline{a}\,\underline{b}} \,\Big) \quad . \end{aligned} \tag{2.5.2}$$

Observe that the difference is in fact a GY term. The question is whether this additional term can be removed by field redefinitions.

First of all, notice, that only redefinitions at zero order of the Lorentz connection can affect this difference at first order. For example, let us redistribute the torsion in the connection using a real parameter k in the simplest way,

$$\Omega_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; \omega_{\underline{a}\,\underline{b}\,\underline{c}} \;-\; L_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; \hat{\Omega}_{\underline{a}\,\underline{b}\,\underline{c}} \;+\; \chi_{\underline{a}\,\underline{b}\,\underline{c}} \quad , \tag{2.5.3}$$

$$\hat{\Omega}_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; \omega_{\underline{a}\,\underline{b}\,\underline{c}} \;-\; (1-k) L_{\underline{a}\,\underline{b}\,\underline{c}} \quad , \tag{2.5.4}$$

$$\chi_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; -k L_{\underline{a}\,\underline{b}\,\underline{c}} \quad . \tag{2.5.5}$$

This can be seen as a shift in the connection of type (A.0.13), which we use to find conventional constraints in supergravity. For $k = 0$ in fact there is “no redefinition”, for $k = 1$ the new connection $\hat{\Omega} = \omega$ is torsionfree, while for $k = 2$ the sign of the torsion flips.

How does this shift in the connection affect the form of the Lagrangian? One computes the changes in the Chern-Simons term and the curvature squared term using (A.0.16) and respectively (A.0.15)

$$\frac{2}{3}\,G^{\underline{a}\,\underline{b}\,\underline{c}}\Big(Q\,-\,\hat{Q}\Big)_{\underline{a}\,\underline{b}\,\underline{c}} \;\sim\; -4k\left[\,\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,+\,2k\left(1-\frac{k}{3}\right)\,G_{\underline{a}\,\underline{c}}{}^{\underline{k}}\,G_{\underline{b}\,\underline{d}\,\underline{k}}\right]\,G^{\underline{a}\,\underline{b}\,\underline{l}}G^{\underline{c}\,\underline{d}}{}_{\underline{l}}$$
$$-4k\,e^{-4\Phi}\hat{\mathcal{E}}_{B_{\underline{a}\underline{b}}}\,\hat{\Omega}_{\underline{b}\,\underline{e}\,\underline{k}}\,G_{\underline{a}}{}^{\underline{e}\,\underline{k}}\;+\;\mathcal{O}(\gamma')\quad, \tag{2.5.6}$$

$$-\frac{2}{3}G^{\underline{a}\,\underline{b}\,\underline{c}}\,Y_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; 8\,\left[\,\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,+\,\frac{4}{3}\,G_{\underline{a}\,\underline{c}}{}^{\underline{k}}\,G_{\underline{b}\,\underline{d}\,\underline{k}}\right]\,G^{\underline{a}\,\underline{b}\,\underline{l}}\,G^{\underline{c}\,\underline{d}}{}_{\underline{l}}\quad, \tag{2.5.7}$$

$$\mathrm{tr}\Big(\,\mathcal{R}_{\underline{a}\,\underline{b}}\,\mathcal{R}^{\underline{a}\,\underline{b}}\,-\,\hat{\mathcal{R}}_{\underline{a}\,\underline{b}}\,\hat{\mathcal{R}}^{\underline{a}\,\underline{b}}\Big) \;\sim\; 2k^2\,(k-2)^2\,\Big[\,\Big(G_{\underline{a}\,\underline{b}}{}^{\underline{k}}\,G_{\underline{c}\,\underline{d}\,\underline{k}}\,-\,G_{\underline{a}\,\underline{c}}{}^{\underline{k}}\,G_{\underline{b}\,\underline{d}\,\underline{k}}\Big)\,\Big]\,G^{\underline{a}\,\underline{b}\,\underline{l}}\,G^{\underline{c}\,\underline{d}}{}_{\underline{l}}$$
$$-4k(k-2)\left[\hat{\mathcal{E}}_{\eta_{\underline{k}\underline{l}}}G_{\underline{l}}{}^{\underline{c}\,\underline{d}}\,+\nabla^{\underline{k}}(\,e^{-4\Phi}\hat{\mathcal{E}}_{B_{\underline{c}\,\underline{d}}})\right]\,G_{\underline{k}\,\underline{c}\,\underline{d}}$$
$$+\,2k(k-2)\,\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,G^{\underline{a}\,\underline{b}\,\underline{k}}\,G^{\underline{c}\,\underline{d}}{}_{\underline{k}}\;+\;\mathcal{O}(\gamma')\quad, \tag{2.5.8}$$

and finally we find

$$e^{-4\Phi}\Big(\,\mathcal{L}\,-\,\hat{\mathcal{L}}\,\Big) \;\sim\; 2(k-2)^2\gamma'\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,G^{\underline{a}\,\underline{b}\,\underline{k}}\,G^{\underline{c}\,\underline{d}}{}_{\underline{k}}$$
$$+(k-2)^2\gamma'\left[2k^2G_{\underline{a}\,\underline{b}}{}^{\underline{k}}\,G_{\underline{c}\,\underline{d}\,\underline{k}}\,+\,\frac{2}{3}(k+4)G_{\underline{a}\,\underline{c}}{}^{\underline{k}}\,G_{\underline{b}\,\underline{d}\,\underline{k}}\right]G^{\underline{a}\,\underline{b}\,\underline{l}}\,G^{\underline{c}\,\underline{d}}{}_{\underline{l}}$$
$$-4k(k-2)\gamma'\left[\,\hat{\mathcal{E}}_{\eta_{\underline{c}\underline{l}}}G_{\underline{l}}{}^{\underline{k}\,\underline{d}}\,+\,\nabla^{\underline{k}}(\,e^{-4\Phi}\hat{\mathcal{E}}_{B_{\underline{c}\,\underline{d}}})\,\right]\,G_{\underline{k}\,\underline{c}\,\underline{d}}$$
$$-4k\gamma'\,e^{-4\Phi}\,\hat{\mathcal{E}}_{B_{\underline{c}\,\underline{d}}}\,\hat{\Omega}_{\underline{c}\,\underline{e}\,\underline{k}}\,G_{\underline{d}}{}^{\underline{e}\,\underline{k}}\quad. \tag{2.5.9}$$

Observe, that for $k=0$, indeed, the difference is equal to the GY term, while for $k=2$, the difference is a term proportional to the equation of motion for the antisymmetric tensor at zero order:

$$\mathcal{L}-\hat{\mathcal{L}}\sim -8\gamma'\hat{\mathcal{E}}_{B_{\underline{c}\underline{d}}}\hat{\Omega}_{\underline{c}\underline{e}\underline{f}}G_{\underline{d}}{}^{\underline{e}\underline{f}}\quad. \tag{2.5.10}$$

At first sight it seems that the change of sign of the torsion in the Lorentz connection just exchanges the GY term to another "unwanted" one. However, correction terms which are propotional to equations of motion can be absorbed by field redefinitions involving the perturbation parameter and therefore $\mathcal{L}$ and $\hat{\mathcal{L}}$ are equivalent.

Indeed, let us consider the expression

$$S[\phi] + \gamma' \int dx^n \frac{\delta S}{\delta \phi} \mathcal{F}(\phi), \tag{2.5.11}$$

with $S[\phi]$ an action for the fields ϕ, $\frac{\delta S}{\delta \phi} = 0$ the equations of motion for the fields ϕ, $\mathcal{F}(\phi)$ an arbitrary function of the fields ϕ and γ' an infinitesimal parameter. Now consider the field redefinitions

$$\phi' = \phi + \gamma' \mathcal{F}(\phi), \tag{2.5.12}$$

and expand $S[\phi']$ around ϕ using that γ' is infinitesimal. Then one obtains

$$S[\phi'] = S[\phi] + \gamma' \int dx^n \frac{\delta S}{\delta \phi} \mathcal{F}(\phi) + \mathcal{O}(\, \gamma'^2 \,). \tag{2.5.13}$$

We have demonstrated here that the bosonic Lagragian (2.3.8), based on the superspace geometry proposed in [9] is equivalent to the component-level first-order corrected supergravity Lagrangian of [20]. While it is a matter of taste to say that the superspace approach is preferable to the Noether approach for constructing effective actions, we have demonstrated that the way superspace keeps supersymmetry manifest at each step in the calculation makes the computation of the effective action very straighforward.

Chapter 3

Generalized Kähler Geometry and $\mathcal{N} = (2, 2)$ Supersymmetric Sigma Models

Two dimensional supersymmetric sigma models play an important role in superstring theory. They are used to provide descriptions of the types of spaces on which superstrings can propagate. They are also useful for providing descriptions of constructs used in string theory like T duality and quotients. Interest in $(2, 2)$ sigma models[1] has risen recenly due to the work of Hitchin [21] and Gaultieri [22], which has established a connection between $(2, 2)$ sigma models and generalized Kähler geometry which contains generalized CY manifolds that appear in string theory compactifications with H flux. The purpose of this section is to give a description of the connection between generalized Kähler geometry and $\mathcal{N} = (2, 2)$ supersymmetric non linear sigma models. It would be of value to understand properties of such backgrounds in relation to string theory such as what is the effect of T duality on the background. The construction of quotients, which has a natural sigma model description, is a useful tool for describing string backgrounds. A complete description of T duality and the construction of quotients is understood for some cases of generalized Kähler geometry, the cases of bi-hermitian geometries with commuting complex structures. It is not known how to deal with the more general case i.e. when

[1]See Appendix C for definition of (p, q) supersymmetry.

the complex structures do not commute. We would like to be able to understand whether the T dual or quotient of a generalized Kähler geometry is still generalized Kähler. In this dissertation we will describe an approach to answer that question. However the approach we will take will not be in the regular language used to discuss generalized Kähler geometry. We will discuss the approach in the language of $\mathcal{N} = (2,2)$ supersymmetric non linear sigma models. In this language the questions can be rephrased in terms of more tractable questions about preserving $(2,2)$ supersymmetry. These questions are easiest to answer when working with the full power of $(2,2)$ superspace. We will therefore give a review of the relevant features of generalized Kähler geometry and how they relate to $(2,2)$ supersymmetric non linear sigma models in $(2,2)$ superspace.

3.1 Complex Geometry

It is easiest to review generalized complex geometry which contains generalized Kähler geometry by reminding ourselves about complex geometry. An real even dimensional manifold M of dimension d is said to have an almost complex structure if it possess a rank $(1,1)$ tensor, $J^a{}_b$, such that

$$J^a{}_b J^b{}_c = -\delta_c{}^a \tag{3.1.1}$$

From this relation we can see that J has eigenvalues $\pm i$. With such a tensor one can construct projectors $\pi_\pm = \frac{1}{2}(1 \pm iJ)$ that can be used to define the notion of holomorphic and anti holomorphic. Locally one can always define such a tensor. If one can find such a tensor that satisfies the property (3.1.1) globally then the manifold is said to possess a complex structure or is called complex. The condition for the almost complex structure to be a complex structure i.e. to be integrable, is the vanishing of the Nijenhuis tensor

$$N_{ij}(J)_{ab}{}^{c} = J^{d}{}_{a}\partial_{[d}J^{c}{}_{b]} - (a \leftrightarrow b) = 0. \tag{3.1.2}$$

This is equivalent to the condition that

$$\pi_{\mp}[\pi_{\pm}X, \pi_{\pm}Y] = 0 \;\; \forall \;\; X, Y \in TM \tag{3.1.3}$$

where [,] is the Lie bracket. There is additional structure of interest in discussing complex manifolds. A manifold is said to be hermitian if its metric satisfies the property

$$J_{a}{}^{c} g_{cd} J^{d}{}_{b} = g_{ab} \tag{3.1.4}$$

This implies that in the coordinate basis that diagonalizes the complex structure the only non vanishing components of metric are the mixed holomorphic and anti holomorphic components.

$$g = \begin{pmatrix} 0 & g_{\imath\bar{\jmath}} \\ g_{\bar{\imath}j} & 0 \end{pmatrix} \tag{3.1.5}$$

Since the hermiticity conditon implies that

$$g_{ac}J^{c}{}_{b} = -g_{bc}J^{c}{}_{a} \tag{3.1.6}$$

the manifold possess a two form $\omega_{ab} = g_{ac}J^{c}{}_{b}$ i.e. $\omega \in \Lambda^{2}T^{*}M$. If the two form is closed i.e. $d\omega = 0$, where here d is the exterior derivative, then the manifold is said to be Kähler and ω is called the Kähler form. A Kähler manifold has the interesting property that all of the geometry is locally determined by a single real function,

$K(z,\bar{z})$, called the Kähler potential. Let's see how this works. In the coordinate basis that diagonalizes the complex structure the two form ω takes the form

$$\omega = 2ig_{i\bar{j}}dz^i \wedge d\bar{z}^{\bar{j}} \tag{3.1.7}$$

The condition

$$d\omega = (\partial + \bar{\partial})\omega = 2i\partial_{[l}g_{i]\bar{j}}dz^l \wedge dz^i \wedge d\bar{z}^{\bar{j}} + 2i\bar{\partial}_{[\bar{l}|}g_{i|\bar{j}]}d\bar{z}^{\bar{l}} \wedge dz^i \wedge d\bar{z}^{\bar{j}} = 0 \tag{3.1.8}$$

implies that

$$\begin{aligned} g_{i\bar{j}} &= \partial_i g_{\bar{j}} \\ g_{i\bar{j}} &= \bar{\partial}_{\bar{j}} g_i \end{aligned} \tag{3.1.9}$$

These two conditions together imply that

$$g_{i\bar{j}} = K_{i\bar{j}} = \partial_i \bar{\partial}_{\bar{j}} K \tag{3.1.10}$$

for some function K. This Kähler potential will provide a beautiful geometric interpretation for supersymmetric non linear sigma models.

3.2 Generalized Complex Geometry

The discussion of complex geometry above places emphasis on the role of the tangent bundle TM to the manifold. The complex structure is a tangent bundle endomorphism $J\colon TM \to TM$, whose projectors,$\pi_\pm = \frac{1}{2}(1 \pm iJ)$, define integrable distributions when (3.1.3) is satisfied. Generalized complex geometry was first proposed by Hitchin [21] and later formalized by Gualtieri [22]. We refer the reader to [22] for a complete discussion of generalized complex geometry. This review follows the presentation given in [23]. Generalized complex geometry extends the idea of complex geometry to the direct sum of the tanget bundle and cotangent bundle $TM \oplus T^*M$.

A generalized vector in $TM \oplus T^*M$ is represented by

$$V^I = \begin{pmatrix} X^i \\ \eta_j \end{pmatrix} \tag{3.2.1}$$

or as a formal sum $V = X + \eta$ with $X \in TM$ and $\eta \in T^*M$. There is a natural inner product on the vector space given by $< (X+\eta)|(Y+\rho) > \;=\; \frac{1}{2}(\rho(X) + \eta(Y))$. The isometry group preserving the inner product and its canonical orientation is $SO(d,d)$ where d is the real dimension of M. In a coordinate basis, (∂_i, dx^j) the inner product has the representation

$$\mathcal{I} = \begin{pmatrix} 0 & 1_d \\ 1_d & 0 \end{pmatrix} \tag{3.2.2}$$

One of the $SO(d,d)$ transformations that we will use is the B transform which acts on a generalized vector as

$$e^b(X+\eta) = X + \eta + \mathrm{i}_X b \tag{3.2.3}$$

where $b\colon TM \to T^*M$ is a closed 2-form, i.e. $b = b_{ij} dx^i \wedge dx^j$ with $db = 0$. This is also represented as

$$\begin{pmatrix} 1_d & 0 \\ b & 1_d \end{pmatrix} \begin{pmatrix} X \\ \eta \end{pmatrix} = \begin{pmatrix} X \\ \eta + \mathrm{i}_X b \end{pmatrix} \tag{3.2.4}$$

A generalized almost complex structure is a $TM \oplus T^*M$ endomorphism, $\mathcal{J}: TM \oplus T^*M \rightarrow TM \oplus T^*M$, that preserves the natural inner product

$$\mathcal{J}^t \mathcal{I} \mathcal{J} = \mathcal{I} \tag{3.2.5}$$

and squares to minus the identity

$$\mathcal{J}^2 = -1_{2d}. \tag{3.2.6}$$

We can represent $\mathcal{J}$ in terms of components as

$$\mathcal{J} = \begin{pmatrix} J & P \\ L & K \end{pmatrix} \tag{3.2.7}$$

where the components should be thought of as maps in the following sense.

$$J: TM \rightarrow TM, \quad P: T^*M \rightarrow TM, \quad L: TM \rightarrow T^*M, \quad K: T^*M \rightarrow T^*M \tag{3.2.8}$$

In a coordinate basis the components have index structure

$$J^a{}_b, \quad P^{ab}, \quad L_{ab}, \quad K_b{}^a \tag{3.2.9}$$

One can also think of B transformations of the generalized almost complex structure. In that case one would get

$$\mathcal{J}_b = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \mathcal{J} \begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix} \tag{3.2.10}$$

As before one can use the generalized almost complex structures to define projectors $\Pi_{\pm} = \frac{1}{2}(1_{2d} \pm i\mathcal{J})$. Integrability of the generalized almost complex structures is defined with respect to the Courant bracket [24]. The Courant bracket of two generalized vectors is

$$[X + \eta, Y + \rho]_c = [X, Y] + \mathcal{L}_X \rho - \mathcal{L}_Y \eta - \frac{1}{2} d(\mathrm{i}_X \rho - \mathrm{i}_Y \eta) \tag{3.2.11}$$

The Courant bracket has the property that it reduces the the Lie bracket on the tangent bundle and vanishes on the cotangent bundle. Under b transformations of the Courant bracket we get

$$[e^b(X+\eta), e^b(Y+\rho)]_c = e^b[X+\eta, Y+\rho]_c \tag{3.2.12}$$

From this we see that the automorphism group of the Courant bracket is the diffeomorphism group times the b transformations. If the generalized almost complex structure satisfies the integrability condition

$$\Pi_{\mp}[\Pi_{\pm}(X+\eta), \Pi_{\pm}(Y+\rho)]_c = 0, \quad \forall\ (X+\eta), (Y+\rho) \in TM \oplus T^*M \tag{3.2.13}$$

then $\mathcal{J}$ is called a generalized complex structure. Lets look at some examples. A complex structure also defines a generalized complex structure.

$$\mathcal{J}_J = \begin{pmatrix} J & 0 \\ 0 & -J^t \end{pmatrix} \tag{3.2.14}$$

A symplectic structure also defines a generalized complex structure.

$$\mathcal{J}_\omega = \begin{pmatrix} 0 & -\omega^{-1} \\ \omega & 0 \end{pmatrix} \tag{3.2.15}$$

A generalized Kähler geometry is a generalized complex geometry that possesses two commuting generalized complex structures $\mathcal{J}_1$ and $\mathcal{J}_2$ such that $G = -\mathcal{J}_1\mathcal{J}_2$ is a positive definite metric on $TM \oplus T^*M$. A generalized Kähler geometry can be specified by the geometric data (g, B, J_+, J_-) i.e. a metric, 2-form, and two hermitian almost complex structures. There are, as in the discussion on complex geometry, two 2-forms $\omega_\pm = gJ_\pm$. The generalized almost complex structures are given by

$$\mathcal{J}_{1/2} = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \tag{3.2.16}$$

The generalized almost complex structures are integrable if

$$N_{ij}(J_\pm) = 0$$

$$H = dB(X,Y,Z) = d\omega_+(J_+X, J_+Y, J_+Z) = -d\omega_-(J_-X, J_-Y, J_-Z) \tag{3.2.17}$$

These conditions are equivalent to the statements that H is a type $(2,1)+(1,2)$ form with respect to both J_+ and J_- and that

$$\nabla^\pm J_\pm = 0 \tag{3.2.18}$$

where $\nabla^\pm = \nabla^0 \pm g^{-1}H$. Here ∇^0 is the covariant derivative on tensors with the metric compatible connection. The H flux acts as torsion for the covariant derivatives. As an example we can consider Kähler geometry which has $J_+ = J_-$ and $B = 0$. The two generalized complex structures reduce the generalized complex structures given above for a regular complex structure and symplectic form. We then see that

$$G = -\mathcal{J}_J \mathcal{J}_\omega = \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \tag{3.2.19}$$

3.3 Generalized Kähler Geometry and $\mathcal{N} = (2,2)$ Non Linear Sigma Models

As before it is easiest to start with the connection between $\mathcal{N} = (2,2)$ Non Linear Sigma Models (NLSM) and Kähler geometry. This connection was first made in the seminal work of Zumino [25]. Important geometric structures that we will consider were understood in this context including the moment map as well as symplectic and Kähler quotients. Many of these structures were devoloped independently in the mathematical and physics literature. The use of a Legendre transform and a symplectic quotient in the study of hyperkähler geometry arose from their use in supersymmetric sigma models [26–28]. In the context of hyperkähler geometry a comprehensive review was presented in [29]. The connection to complex geometry was then furthered by discovery of new sigma models in the works [30,31] that include more general backgrounds than those in [25]. The goal of this discussion is to develop an understanding of the sigma models connection to these backgrounds in $\mathcal{N} = (2,2)$ superspace so that we can use the power of $(2,2)$ superspace to obtain our results. This is helpful because $(2,2)$ superspace hides the background data so that it isn't obviously present[2]. We will start our discussion of $(2,2)$ NLSMs in $(1,1)$ superspace with a non linearly realized extra $(1,1)$ supersymmetry where the geometric aspects of the theory are easy to see and the connection to generalized Kähler geometry is made. The $(1,1)$ susy algebra looks like

$$D_+^2 = i\partial_{\ddagger}, \quad D_-^2 = i\partial_{=}, \quad [D_+, D_-\} = 0 \tag{3.3.1}$$

[2]However it does so in a very beautiful way.

The action for a $(1,1)$ sigma model is

$$S = \int d^2x d^2\theta D_+ \phi^a E_{ab} D_- \phi^b \tag{3.3.2}$$

Here $E_{ab} = g_{ab} + B_{ab}$. We think of g_{ab} as the metric for the target space and B_{ab} is a gauge 2-form on the target space. There are no conditions on the metric or 2-form required by $(1,1)$ supersymmetry. The connection to complex geometry comes from imposing the conditions required to make the theory possess a hidden extra $(1,1)$ supersymmetry. This is done by proposing a second $(1,1)$ supersymmetry transformation for the field ϕ^a and requiring that the action remain invariant under the proposed transformation and that the extra transformation close on the usual $(1,1)$ algebra. The proposed second transformation takes the form

$$\delta\phi^a = \epsilon^+ J^a_{(+)b} D_+ \phi^b + \epsilon^- J^a_{(-)b} D_- \phi^b \tag{3.3.3}$$

The $(\pm)$ on the tensors $J_\pm$ are not indices but just labels to match the tensor to the supersymmetry transformation. Requiring that the second supersymmetry transformation close on the usual supersymmetry algebra requires that

$$\begin{aligned} J_\pm^2 &= -1 \\ N_{ij}(J_\pm) &= 0 \end{aligned} \tag{3.3.4}$$

These are the conditions for $J_\pm$ to be separately integrable complex structures. Requiring that the action is invariant under the second $(1,1)$ transformation implies that

$$g_{ac} J^c_{\pm b} = -g_{bc} J^c_{\pm a}$$

$$\nabla^{\pm} J_{\pm} = 0 \tag{3.3.5}$$

where once again $\nabla^{\pm} = \nabla^0 \pm g^{-1}H$ and $H = dB$. At this point we see the requirements for the extra supersymmetry transformations are equivalent to the integrability conditions for the data necessary to specifiy a generalized Kähler geometry. The last thing to check is that extra $(1,0)$ and $(0,1)$ transformations commute with each other. The commutator of the supersymmetry transformations is proportional to the commutator of the complex structures times the e.o.m for the field ϕ^a. This leave us with two choices. We either require that the complex structures commute or we consider the theory on shell. If we impose that the complex structures commute, then it can be shown that the both complex structures are simultaneously integrable. That also means that the algebra closes off shell and there is a manifestly $(2,2)$ action for the model. That model will be terms of what are called chiral and twisted chiral superfields.

If we don't impose the condition that commutator of the complex structures vanish, the algebra only closes on shell and there will be no manifest $(2,2)$ action. However, one can add auxiliary fields that allow one to account for the non commutativity of the complex structures and still have the algebra close off shell. The model will then have a manifest $(2,2)$ description in terms of what are called semi chiral superfields. The $(1,1)$ action for this model is

$$S = \int d^2x d^2\theta (S_{a+} E^{ab} S_{b-} - S_{a(-}D_{+)}\phi^a) \tag{3.3.6}$$

Where E^{ab} is the inverse of E_{ab}. Imposing the $S_{a\pm}$ equations of motion gives back the action (4.0.7). Once again one proposes extra supersymmetry transformations of the fields ϕ^a and $S_{a\pm}$ and consider the requirements for the extra transformations to close on the regular $(1,1)$ supersymmetry algebra. We think of $D_\pm\phi^a$ combining with $S_{a\pm}$ to form a generalized vector in $TM\oplus T^*M$ i.e. $L^A_\pm = (D_\pm\phi^a, S_{a\pm}) \in TM\oplus T^*M$. In terms of $L^A_\pm$ the general transformations,$\delta = \delta^{(+)} + \delta^{(-)}$, take the form

$$
\begin{aligned}
\delta^{(\pm)}\phi^a &= \epsilon^\pm L^B_\pm A^{(\pm)a}_B \\
\delta^{(\pm)}S_{a\pm} &= \epsilon^\pm(D_\pm L^B_\pm B^{(\pm)}_{aB} + L^B_\pm L^C_\pm C^{(\pm)}_{aBC}) \\
\delta^{(\pm)}S_{a\mp} &= \epsilon^\pm(D_\pm L^B_\mp M^{(\pm)}_{aB} + D_\mp L^B_\pm N^{(\pm)}_{aB} + L^B_\pm L^C_\pm X^{(\pm)}_{aBC})
\end{aligned}
\tag{3.3.7}
$$

Typically the three index tensors,$C^{(\pm)}_{aBC}$ and $X^{(\pm)}_{aBC}$ are solved for in terms of derivatives of the two index tensors $A^{(\pm)a}_B$, $B^{(\pm)}_{aB}$, $M^{(\pm)}_{aB}$ and $N^{(\pm)}_{aB}$. In [32] it was shown, for the case of $(2,0)$ supersymmetry, that the integrability conditions imposed by supersymmetry contained solutions that correspond to courant integrability of a generalized complex structure with the two index coefficient terms in the supersymmetry transformations interpreted as submatrices of the generalized complex structure.

3.4 $\mathcal{N} = (2,2)$ Non Linear Sigma Models In $\mathcal{N} = (2,2)$ Superspace

In this section we will review how the content of the previous section is captured by $(2,2)$ superspace. The review will follow the content and presentation of [23, 30, 33]. We'll start our consideration of the $(2,2)$ formulation of the sigma models by recalling a property of the fermionic measure, $d^4\theta$. On dimensional grounds,$[d^4\theta] = 2$, the only contribution we can consider to the action is a real scalar potential function of dimensionless superfields[3]. We will consider scalar superfields for the action.

$$S = \int d^2x d^4\theta K(\{\Phi, \bar{\Phi}\}) \tag{3.4.1}$$

where $\{\Phi\}$ is the set of superfields considered for the action. We interpret the lowest component of the superfield to be the coordinates on the manifold. If the superfields are unconstrained, then the action will contain no dynamics. Therefore we need to understand what constraints we can place on scalar superfields. The constraints are made using the supercovariant derivatives $(D_\alpha, \bar{D}_\alpha)$. The algebra of supercovariant derivatives is

$$\begin{aligned} [D_\alpha, D_\beta\} &= 0 \\ [D_\alpha, \bar{D}_\beta\} &= 2i(\gamma^a)_{\alpha\beta}\partial_a \end{aligned} \tag{3.4.2}$$

The constraints that can placed consistently are

Chiral	$\bar{D}_\alpha \phi = 0$
anti Chiral	$D_\alpha \Theta = 0$
Twisted Chiral	$\bar{D}_- \chi = D_+ \chi$
Twisted anti Chiral	$D_- \psi = \bar{D}_+ \psi$
Left Semi Chiral	$\bar{D}_+ X = 0$
Left Semi anti Chiral	$D_+ P = 0$
Right Semi Chiral	$\bar{D}_- W = 0$
Right Semi anti Chiral	$D_- Y = 0$

[3]By [ ] here we mean the mass dimension. We are also ignoring superpotential terms for the moment.

$$(3.4.3)$$

It is the constraints on the superfields that will determine the type of geometry that we have. They will determine the complex structures and, along with the potential function, the metric and B field. This is done by reducing the action from $(2,2)$ superspace to $(1,1)$ superspace. The details of this reduction has been worked out for all combinations of fields in [23, 30, 33]. We will describe the simplest case here to give a flavor for how things are done and then we will give the main results of the other cases before preceding to a discussion of gauging the sigma models which is the focus our research. The simplest choice to consider is the case were only chiral superfields are used. This was the model considered by Zumino [25]. In that work, Zumino showed that the potential function of superfields corresponds to the Kähler potential from Kähler geometry discussed above[4]. The metric for the sigma model is determined in terms of second derivatives of the Kähler potential just as in Kähler geometry. Lets see how we can determine the properties of geometry starting from $(2,2)$ superspace. This is done by reducing the $(2,2)$ description to a $(1,1)$ description which is done in two part. The first is to reorganize the supercovariant derivatives into two copies of $(1,1)$. The appropriate combinations are

$$\begin{aligned}\hat{D}_\alpha &= \frac{1}{\sqrt{2}}(D_\alpha + \bar{D}_\alpha) \\ \tilde{D}_\alpha &= \frac{i}{\sqrt{2}}(D_\alpha - \bar{D}_\alpha)\end{aligned} \tag{3.4.4}$$

[4] Álvarez-Gaumé and Freedman extended Zumino's work by showing that further extensions of supersymmetry to $\mathcal{N} = (4,4)$ required the sigma model metric to be hyperkähler [34].

The constaints on the chiral field expressed in terms of these derivatives are

$$\begin{aligned} \tilde{D}_\alpha \phi^u &= i\hat{D}_\alpha \phi^u \\ \tilde{D}_\alpha \bar{\phi}^{\bar{u}} &= -i\hat{D}_\alpha \bar{\phi}^{\bar{u}} \end{aligned} \tag{3.4.5}$$

or defining

$$Z^A = \begin{pmatrix} \phi^u \\ \\ \bar{\phi}^{\bar{u}} \end{pmatrix} \tag{3.4.6}$$

the constraints are

$$\tilde{D}_\alpha Z^A = J^A{}_B \hat{D}_\alpha Z^B \tag{3.4.7}$$

where

$$J^A{}_B = \begin{pmatrix} i\delta^u_v & 0 \\ \\ 0 & -i\delta^{\bar{u}}_{\bar{v}} \end{pmatrix} \tag{3.4.8}$$

At this point we have made contact with one part of the $(1,1)$ approach. For the reduction we will "hide" one of the $(1,1)$ supersymmetries, the one described by $\tilde{D}_\alpha$. From this perspective the constraints described in (3.4.7) become the second $(1,1)$ supersymmetry transformation as in (3.3.3). The tensor $J^A{}_B$ satisfies $J^A{}_B J^B{}_C = -\delta_C{}^A$. This is how the complex structure arises starting from the $(2,2)$. It should be noted that when comparing(3.4.7) with (3.3.3) the two a priori independent almost complex structures in (3.3.3) are identified in (3.4.7). The next point is to see how the metric arises. For that we consider the evaluation of the fermionic measure. The fermionic measure is normally evaluated as

$$\int d^4\theta = \bar{D}^2 D^2 \tag{3.4.9}$$

Using the supercovariant derivative algebra, we can show that up to total derivatives

$$\bar{D}^2 D^2 = \hat{D}^2 \tilde{D}^2 \tag{3.4.10}$$

Using this we can write

$$\begin{aligned} \int d^4\theta K(\phi, \bar{\phi}) &= \bar{D}^2 D^2 K = \hat{D}^2 \tilde{D}^2 K = \hat{D}^2 (K_{u\bar{v}} D^\alpha \phi^u D_\alpha \bar{\phi}^{\bar{v}}) \\ &= \int d^2\hat{\theta} K_{u\bar{v}} D^\alpha \phi^u D_\alpha \bar{\phi}^{\bar{v}} \end{aligned} \tag{3.4.11}$$

where we have used the form of the $(1,1)$ action that is more compact for working with complex fields. Comparison with (4.0.7) allows us to identify the target space metric.

$$g_{AB} = \begin{pmatrix} 0 & K_{u\bar{v}} \\ K_{\bar{u}v} & 0 \end{pmatrix} \tag{3.4.12}$$

which is the same as what one gets for Kähler geometry. We can verify that the metric is hermitian with respect to the complex structure.

$$\omega_{AB} = g_{AC} J^C{}_B = \begin{pmatrix} 0 & -iK_{u\bar{v}} \\ iK_{\bar{u}v} & 0 \end{pmatrix} = -g_{BC} J^C{}_A = -\omega_{BA} \tag{3.4.13}$$

If we considered the theory with only twisted chiral superfields, we would find the same result as that for pure chiral superfields. The target space geometry would be Kähler. A classification of backgrounds consistent with extended supersymmetry[5] was given in [35] based on using only chiral superfield representations.

The next case we will consider is what happens when we include both chiral and twisted chiral superfield representations. We will start as before by considering the constraints on the superfields from the point of view of the two $(1,1)$ derivatives.

[5] By extended we mean greater than $(1,1)$ supersymmetry.

In this way we will see the complex structures arise again. The $(1,1)$ derivatives are the same as in (3.4.4). The constraints on the chiral superfields are the same as in (3.4.5). The constraints for the twisted chiral superfieds are

$$\begin{aligned}
\tilde{D}_+\chi^p &= -i\hat{D}_+\chi^p \\
\tilde{D}_-\chi^p &= i\hat{D}_-\chi^p \\
\tilde{D}_+\bar{\chi}^{\bar{p}} &= i\hat{D}_+\bar{\chi}^{\bar{p}} \\
\tilde{D}_-\bar{\chi}^{\bar{p}} &= -i\hat{D}_-\bar{\chi}^{\bar{p}}
\end{aligned} \tag{3.4.14}$$

We define

$$Z^A = \begin{pmatrix} \phi^u \\ \bar{\phi}^{\bar{u}} \\ \chi^p \\ \bar{\chi}^{\bar{p}} \end{pmatrix} \tag{3.4.15}$$

The combined (3.4.5) and (3.4.14) can be expressed as

$$\begin{aligned}
\tilde{D}_+Z^A &= J^A_{(+)B}\hat{D}_+Z^B \\
\tilde{D}_-Z^A &= J^A_{(-)B}\hat{D}_-Z^B
\end{aligned} \tag{3.4.16}$$

where

$$J^A_{(+)B} = \begin{pmatrix} i\delta^u_v & 0 & 0 & 0 \\ 0 & -i\delta^{\bar{u}}_{\bar{v}} & 0 & 0 \\ 0 & 0 & -i\delta^p_q & 0 \\ 0 & 0 & 0 & i\delta^{\bar{p}}_{\bar{q}} \end{pmatrix}$$

$$J^A_{(-)B} = \begin{pmatrix} i\delta^u_v & 0 & 0 & 0 \\ 0 & -i\delta^{\bar{u}}_{\bar{v}} & 0 & 0 \\ 0 & 0 & i\delta^p_q & 0 \\ 0 & 0 & 0 & -i\delta^{\bar{p}}_{\bar{q}} \end{pmatrix} \tag{3.4.17}$$

Using the same evaluation of the fermionic measure as before we find a metric and a b field.

$$g_{AB} = \begin{pmatrix} 0 & K_{u\bar{v}} & 0 & 0 \\ K_{\bar{u}v} & 0 & 0 & 0 \\ 0 & 0 & 0 & -K_{p\bar{q}} \\ 0 & 0 & -K_{\bar{p}q} & 0 \end{pmatrix}$$

$$B_{AB} = \begin{pmatrix} 0 & 0 & 0 & K_{u\bar{p}} \\ 0 & 0 & K_{\bar{u}p} & 0 \\ 0 & -K_{p\bar{u}} & 0 & 0 \\ -K_{\bar{p}u} & 0 & 0 & 0 \end{pmatrix} \tag{3.4.18}$$

One can check that the metric is hermitian with respect to both complex structures. The geometry is called bi-hermitian with almost product structure. The complex structures commute and the metric almost factorizes into product form. What we have described so far are the cases described in the previous section that could satisfy the off shell closure of the algebra by requiring that the complex structures commute. We can us the data to construct generalized complex structures via the perscription given in [22]. To describe the more general case, we need to consider the $(2,2)$ superfield representations that incorporate the extra auxiliary fields necessary for the $(1,1)$ action to possess extended supersymmetry without needing the complex structures to commute. Those representations are the left and right semi chiral superfields.

This portion of the review follows directly from [23]. The discussion will go the same as before. The extra auxiliary fields will show up when we consider the constraints on the semi chiral superfields. For technical reasons, we need to include both left and right semi chiral superfields in order to obtain a sigma model we will consider them both. In terms of the $(1,1)$ derivatives, the constraints on the semi chiral superfields are expressed as

$$\begin{aligned}
\tilde{D}_- X^a &= i\hat{D}_- X^a \\
\tilde{D}_- \bar{X}^{\bar{a}} &= -i\hat{D}_- \bar{X}^{\bar{a}} \\
\tilde{D}_+ Y^{a'} &= -i\hat{D}_+ Y^{a'} \\
\tilde{D}_+ \bar{Y}^{\bar{a}'} &= i\hat{D}_+ \bar{Y}^{\bar{a}'}
\end{aligned} \tag{3.4.19}$$

We can express these in the compact form by defining, as before

$$\begin{aligned}
Z^A &= \begin{pmatrix} X^a \\ \bar{X}^{\bar{a}} \end{pmatrix} \\
Z^{A'} &= \begin{pmatrix} Y^{a'} \\ \bar{Y}^{\bar{a}'} \end{pmatrix}
\end{aligned} \tag{3.4.20}$$

Then we have

$$\begin{aligned}
\tilde{D}_- Z^A &= J^A{}_B \hat{D}_- Z^B \\
\tilde{D}_+ Z^{A'} &= -J^{A'}{}_{B'} \hat{D}_- Z^{B'}
\end{aligned} \tag{3.4.21}$$

Where

$$J^{A}{}_{B} = \begin{pmatrix} i\delta^{a}_{b} & 0 \\ 0 & -i\delta^{\bar{a}}_{\bar{b}} \end{pmatrix}$$

$$J^{A'}{}_{B'} = \begin{pmatrix} i\delta^{a'}_{b'} & 0 \\ 0 & -i\delta^{\bar{a}'}_{\bar{b}'} \end{pmatrix} \tag{3.4.22}$$

The constraints don't tell us how to relate the action of all of the $(1,1)$ derivatives on the semi chiral superfields. When we reduce to $(1,1)$ superspace we need to hide all reference to the extra supersymmetry generated by $\tilde{D}_\alpha$. So the actions not specified by constraints must define extra $(1,1)$ superfields. We denote them as.

$$\begin{aligned} \tilde{D}_+ X^a &= \Psi^a_+ \\ \tilde{D}_+ \bar{X}^{\bar{a}} &= \bar{\Psi}^{\bar{a}}_+ \\ \tilde{D}_- Y^{a'} &= \Upsilon^{a'}_- \\ \tilde{D}_- \bar{Y}^{\bar{a}'} &= \bar{\Upsilon}^{\bar{a}'}_- \end{aligned} \tag{3.4.23}$$

These are related to the auxiliary fields added to the $(1,1)$ action in (3.3.6). We perform the reduction as before evaluating the fermionic measure using (3.4.10), pushing the $\tilde{D}_\alpha$ derivatives onto the potential. This time considering the supersymmetry transformations of the $(1,1)$ superfields, we'll be able to see the full generalized complex structures. It is easiest to describe the resulting action and generalized complex structures in terms of the following matrices

$$\begin{aligned} m_{AA'} &= J^{B}{}_{A} K_{B'B} J^{B'}{}_{A'} \\ n_{A'A} &= K_{A'A} \\ \omega_{AB} &= \frac{1}{2} J^{C}{}_{[A} K_{B]C} \end{aligned}$$

$$\begin{aligned} \omega_{A'B'} &= \frac{1}{2} J^{C'}{}_{[A'} K_{B']C'} \\ p_{AA'} &= -iJ^{C}{}_{A} K_{CA'} \\ q_{A'A} &= iJ^{C'}{}_{A'} K_{C'A} \end{aligned} \tag{3.4.24}$$

The reduced action to $(1,1)$ superspace is

$$S = -\frac{i}{4} \int d^2x d^2\hat{\theta} (D_+ Z^t E D_- Z + S_{A+} u^{AA'} S_{A'-}) \tag{3.4.25}$$

Where $u^{AA'}$ is the inverse of $n_{A'A}$,

$$E = g + B = \begin{pmatrix} 2i\omega uq & m - 4\omega u\omega' \\ p^t uq & 2ip^t u\omega' \end{pmatrix} \tag{3.4.26}$$

and

$$\begin{aligned} u^{AA'} S_{A'-} &= \Psi^A_- - iu^{AA'} (q_{A'B} D_- Z^B + 2i\omega_{A'B'} D_- Z^{B'}) \\ u^{AA'} S_{A+} &= \Upsilon^A_+ - iu^{AA'} (-2i\omega_{AB} D_+ Z^B + p_{AB'} D_+ Z^{B'}) \end{aligned} \tag{3.4.27}$$

The generalized complex structures are read off of the supersymmetry variations of Z, S_{A+}, and $S_{A'-}$. The generalized complex structures are

$$\mathcal{J}_+ = \begin{pmatrix} J & 0 & 0 & 0 \\ 2u^t\omega & iu^t p & u^t & 0 \\ -2(\omega J + ipu^t\omega) & -(n - pu^t p) & -ipu^t & 0 \\ i(-nJu^t q + qJ + 4i\omega' u^t \omega) & 2(nJu\omega' - i\omega' u^t p) & -2\omega' u^t & nJu \end{pmatrix}$$

$$\mathcal{J}_- = \begin{pmatrix} iuq & -2u\omega' & 0 & -u \\ 0 & -J' & 0 & 0 \\ -2(n^t J' u^t \omega + i\omega uq) & -i(n^t J' u^t p + 4i\omega u\omega' - pJ') & -n^t J' u^t & 2\omega u \\ n - quq & -2(iqu\omega' - \omega' J') & 0 & -iqu \end{pmatrix} \tag{3.4.28}$$

The upper left blocks satisfy the conditions for being almost complex structures independently of the form of the submatices (3.4.24). One can verify that

$$
\begin{aligned}
(J_+)^2 &= \begin{pmatrix} J & 0 \\ 2u^t\omega & iu^t p \end{pmatrix}^2 = -\begin{pmatrix} \delta^a_b & 0 & 0 & 0 \\ 0 & \delta^{\bar a}_{\bar b} & 0 & 0 \\ 0 & 0 & \delta^{a'}_{b'} & 0 \\ 0 & 0 & 0 & \delta^{\bar a'}_{\bar b'} \end{pmatrix} \\
(J_-)^2 &= \begin{pmatrix} iuq & -2u\omega' \\ 0 & -J' \end{pmatrix}^2 = -\begin{pmatrix} \delta^a_b & 0 & 0 & 0 \\ 0 & \delta^{\bar a}_{\bar b} & 0 & 0 \\ 0 & 0 & \delta^{a'}_{b'} & 0 \\ 0 & 0 & 0 & \delta^{\bar a'}_{\bar b'} \end{pmatrix}
\end{aligned}
\tag{3.4.29}
$$

This is to be expected. When S_{A+} and $S_{A'-}$ are put on shell then we must recover the regular $(1,1)$ sigma model with second supersymmetry transformation (3.3.3). It was shown in [33] that all generalized Kähler geometries are locally describable in terms of $\mathcal{N} = (2,2)$ with chiral, twisted chiral and semi chiral superfields.

Chapter 4

Generalized Kähler Geometries With Isometries And Gauged Sigma Models

The focus of research in this dissertation will be on target spaces corresponding to $\mathcal{N} = (2,2)$ supersymmetric sigma models with $U(1)$ isometries. When such target spaces have isometries, there is extra geometric data that characterizes the manifold. This data includes the Killing vector, the moment map [36–38], and a one form [39] if the background has non trivial three form flux. For supersymmetric sigma models corresponding to bi-hermitian geometries with commuting complex structures, the descriptions of the extra data is clear at both the $(1,1)$ level and manifest $(2,2)$ level. However, for the case with non commuting complex structures, only the $(1,1)$ level description is known. Since the entire background is determined by the potential function K, the generalized Kähler potential, the extra geometric data will be determined using the generalized Kähler potential. We will also investigate the role that this extra geometric data in generalized Kähler geometry. There are other issues concerning such target spaces with isometries. Specifically target space duality, or T duality, and quotient constructions. Since quotients are a special case of T duality, we will focus on developing a manifestly $(2,2)$ description of T duality with the question in mind "Is the T dual background to a generalized Kähler geometry still generalized Kähler?". This amounts to showing showing that the procedure for constructing the T dual geometry preserves $(2,2)$ supersymmetry. The main tool we will use in the investigation of target spaces with isometries are the known $(2,2)$ gauge multiplets.

The presentation will go as follows. We will begin with a description of how the geometric data arises in the sigma model. Then we will identify the same data in terms of the generalized Kähler potential and investigate to role of the moment map and killing vector in generalized Kähler geometry. We will find some interesting structures that hint at something new in $(2,2)$ gauge multiplets. Then we will review T duality in the known cases and describe a proposed description of T duality for sigma models parameterized by semi chiral superfields. The prescription will have some very undesirable features that we can trace to a root cause, an insufficiency of the known $(2,2)$ gauge mutiplets. This insufficiency of the known $(2,2)$ gauge multiplets along with the hints obtained from studying the geometric data associated to the isometry will lead us to a new $(2,2)$ gauge multiplet

If the sigma model target space has an isometry group, then a generic Killing vector can be decomposed in a basis of the Killing vectors k_A which generate the Lie algebra of the isometry group

$$\xi = \xi^A k_A = \xi^A k_A^i \partial_i, \qquad [k_A, k_B] = f_{AB}{}^C k_C, \qquad \mathcal{L}_\xi g = 0. \tag{4.0.1}$$

The infinitesimal transformation of the sigma-model fields is given by

$$\delta\phi^i = \epsilon^A k_A^i, \tag{4.0.2}$$

where ϵ^A are rigid infinitesimal parameters. For a sigma model with isometries, there is additional geometric data. These follow from the integrability conditions associated with the additional requirements that the action of the Killing vector leave invariant not just the metric, but the field strength of the B-field H, and the 2-forms $\omega_\pm = gJ_\pm$:

$$\mathcal{L}_\xi H = 0, \qquad \mathcal{L}_\xi \omega_\pm = 0. \tag{4.0.3}$$

From the condition that H is invariant, it follows that

$$\mathcal{L}_\xi H = d\mathrm{i}_\xi H + \mathrm{i}_\xi dH = d\mathrm{i}_\xi H = 0. \tag{4.0.4}$$

Since the two-form $i_\xi H$ is closed, locally it can be written as

$$\mathrm{i}_{\xi_A} H = du_A, \tag{4.0.5}$$

where the one-form u is determined up to an exact, Lie-algebra valued one-form. The ambiguity in u can be fixed, up to $U(1)$ factors in the Lie algebra, by requiring that it is equivariant $\mathcal{L}_A u_B = f_{AB}{}^C u_C$.

Besides this one form u, the other geometric data associated with the existence of an isometry group is the moment map (also known as the Killing potential). From the condition that the symplectic form is invariant under ξ, and from $d\omega_\pm(J_\pm X, J_\pm Y, J_\pm Z) = \pm H(X,Y,Z)$, it follows that $\omega_\pm \xi \mp J^T_\pm u$ is closed. Therefore, locally one finds

$$d\mu_\pm = \omega_\pm \xi \mp J^t_\pm u, \tag{4.0.6}$$

where $\mu_\pm$ are the moment maps. This expression is the generalization for a manifold with torsion of the integrability condition for the vector field ξ which satisfies, in additon to (4.0.3),: $\mathcal{L}_\xi J_\pm = 0$.

The relevance of these two quantities, the one-form u and the moment map μ, becomes clear when constructing the gauged sigma model, by promoting the rigid (global) isometries 4.0.2 to local ones. This is accompanied, in the usual manner, by introducing a compensating connection (gauge potential) $\partial_\mu \phi^i \longrightarrow \nabla_\mu \phi = \partial_\mu \phi^i + A_\mu^A k_A^i$, which transforms as $\delta A_\mu^A = \partial_\mu \epsilon^A + f_{AB}{}^C A_\mu^B \epsilon^C$. For $(1,0)$ or $(1,1)$ supersymmetric sigma models, the bosonic gauge connection becomes part of a corresponding $(1,0)$ or $(1,1)$ vector multiplet. Promoting the partial derivatives to gauge covariant derivatives is not enough in the presence of a B-field [38–40]. New terms, which depend on the one-form u and the moment map, must be added to the sigma model action. For a bosonic, $(1,0)$ or $(1,1)$ supersymmetric sigma-model, adding only u-dependent terms is sufficient.

$$\mathcal{S} = \int d^2x d^2\theta \Big(g_{ij} \nabla_+ \phi^i \nabla_- \phi^j + B_{ij} D_+ \phi^j D_- \phi^j - 2u_{iA} A^A_{(+} D_{-)} \phi^i + A_+^A A_-^B c_{[AB]} \Big), \tag{4.0.7}$$

where $D_\pm$ are flat superspace covariant derivatives and $\nabla_\pm$ are superspace gauge covariant derivatives, and $c_{[AB]} = k^i_{[A} u_{iB]}$.

When the sigma model has additional supersymmetries, then the gauged sigma model action acquires new terms, which are moment map dependent. The gauged $(2,2)$ sigma model action typically contains a term

$$\delta\mathcal{S} = \int d^2x d^2\theta S \mu, \tag{4.0.8}$$

where μ is the moment map, and S is a super-curvature that appears in the $(2,2)$ gauged superalgebra (more precisely in the super-commutator $\{\nabla_+, \bar{\nabla}_-\}$).

Alternatively, one could chose to perform the gauging directly in $(2,2)$ superspace. That is the approach we will use in this work. We shall be interested in gauging $(2,2)$ sigma models whose target space has a bihermitian structure, with non-commuting almost complex structures. The natural starting point for us then is the $\mathcal{N} = (2,2)$ superspace formulation of a sigma-model written in terms of $(2,2)$ semi-chiral superfields,

$$\begin{aligned} &\text{left chiral:} \qquad \bar{D}_+ X = 0, \\ &\text{right antichiral:} \qquad D_- Y = 0. \end{aligned} \tag{4.0.9}$$

We begin by making the observation that the following transformations are consistent with the contraints[1] on X and Y.

$$\begin{aligned} X &\to (A+B)X + C + D, \\ Y &\to (F+G)Y + W + Z, \end{aligned} \tag{4.0.10}$$

where A, C are chiral superfields, B, D are twisted anti chiral superfields, F, W are anti chiral, and G, Z are twisted chiral. When these transformations correspond to gauge transformations they can be properly accounted for using both the chiral and twisted chiral vector multiplets.

For simplicity we will consider only the gauge transformations where the semi-chiral superfields are multiplied and shifted by chiral and anti-chiral superfields. In this work we shall follow two complementary approaches to constructing the gauged action in $(2,2)$ superspace. The first method involves descending to the level of $(1,1)$ superspace by following the usual route of substituting the Grassmann integration by differentiation $\int d\theta d\bar{\theta} \to D\bar{D}$, and by the subsequent replacement of the ordinary superspace covariant derivatives by gauge covariant derivatives $D\bar{D} \to \nabla\bar{\nabla}$. This

[1] This is not the most general set of transformations consistent with the constraints on X and Y. However, these are the only transformations relevant to our considerations.

is equivalent to gauging by minimal coupling, if the generalized Kähler potential is invariant under the action of the isometry generators. The second method [37] uses the prepotential of the gauge multiplet V explicitly in the generalized Kähler potential to restore the invariance of the action under local transformations.

For simplicity we restrict ourselves to $U(1)$ isometries. As such we can go to a coordinate system in which the isometry is realized by a shift of some coordinate. This implies that the generalized Kähler potential $K(X, \bar{X}, Y, \bar{Y})$ will be independent of a certain linear combination of the left and right semi-chiral superfields. For example, for

$$K = K(X + \bar{X}, Y + \bar{Y}, X + Y). \tag{4.0.11}$$

we can immediately read off the Killing vector associated with the isometry. In this case it takes the form

$$\xi = i\frac{\partial}{\partial X} - i\frac{\partial}{\partial \bar{X}} - i\frac{\partial}{\partial Y} + i\frac{\partial}{\partial \bar{Y}}. \tag{4.0.12}$$

From (4.0.10) we see that this is an example of a generalized Kähler potential, with a U(1) isometry which can be gauged using the chiral $(2,2)$ vector multiplet.

4.1 Gauging and the reduction to (1,1) superspace

Let us consider the first of the two approaches to gauging which we have outlined before. Since we are interested in extracting the geometric data (including those associated with isometries) from the sigma model, and these are most easily seen in the language of $(1,1)$ superspace, here we describe the bridge from $(2,2)$ to $(1,1)$ superspace, following [30] closely.

We begin by giving the $(2,2)$ gauge covariant supersymmetry algebra for the chiral vector multiplet.

$$\begin{aligned}
[\nabla_\alpha, \nabla_\beta\} &= 0, \\
[\nabla_\alpha, \bar{\nabla}_\beta\} &= 2i(\gamma^c)_{\alpha\beta}\nabla_c + 2g[C_{\alpha\beta}S - i(\gamma^3)_{\alpha\beta}P]t, \\
[\nabla_\alpha, \nabla_b\} &= g(\gamma_b)_\alpha{}^\beta \bar{W}_\beta t, \\
[\nabla_a, \nabla_b\} &= -ig\epsilon_{ab}\mathcal{W}t,
\end{aligned} \tag{4.1.1}$$

With the Bianchi identites

$$\begin{aligned}
\nabla_\alpha S &= -i\bar{W}_\alpha, \qquad \nabla_\alpha P = -(\gamma^3)_\alpha{}^\beta \bar{W}_\beta, \\
\nabla_\alpha \bar{W}_\beta &= 0, \qquad \nabla_\alpha d = (\gamma^c)_\alpha^\beta \nabla_c \bar{W}_\beta, \\
\nabla_\alpha W_\beta &= iC_{\alpha\beta}d - (\gamma^3)_{\alpha\beta}\mathcal{W} + (\gamma^a)_{\alpha\beta}\nabla_a S - i(\gamma^3\gamma^a)_{\alpha\beta}\nabla_a P.
\end{aligned} \tag{4.1.2}$$

Having in mind the gauging of a certain isometry of a $(2,2)$ sigma model, the abstract $U(1)$ generator t will be related to the Killing vector for the isometry. The relation takes the form $t = -i\mathcal{L}_\xi$. According to our previous discussion on gauging methods, we begin constructing the gauged $(2,2)$ sigma model by evaluating the fermionic measure using the gauged supercovariant derivatives,

$$\int d^2\bar{\theta}d^2\theta = \frac{1}{8}[\nabla^\alpha\nabla_\alpha\bar{\nabla}^\beta\bar{\nabla}_\beta + \bar{\nabla}^\beta\bar{\nabla}_\beta\nabla^\alpha\nabla_\alpha], \tag{4.1.3}$$

where we have used the conventions of [41].

In order to reduce the action written in $(2,2)$ superspace to $(1,1)$ superspace we need to express the $(2,2)$ gauge covariant derivatives in terms of two copies of the $(1,1)$ derivatives:

$$\hat{\nabla}_\alpha = \frac{1}{\sqrt{2}}(\nabla_\alpha + \bar{\nabla}_\alpha), \qquad \tilde{\nabla}_\alpha = \frac{i}{\sqrt{2}}(\nabla_\alpha - \bar{\nabla}_\alpha). \tag{4.1.4}$$

The $(1,1)$ derivatives satisfy the following algebra:

$$\begin{aligned}
[\hat{\nabla}_\alpha, \hat{\nabla}_\beta\} &= 2i(\gamma^c)_{\alpha\beta}\nabla_c - 2i\lambda(\gamma^3)_{\alpha\beta}Pt, \\
[\hat{\nabla}_\alpha, \nabla_b\} &= \frac{\lambda}{\sqrt{2}}(\gamma_b)_\alpha{}^\beta \hat{W}_\beta t, \\
[\tilde{\nabla}_\alpha, \tilde{\nabla}_\beta\} &= 2i(\gamma^c)_{\alpha\beta}\nabla_c - 2i\lambda(\gamma^3)_{\alpha\beta}Pt, \\
[\tilde{\nabla}_\alpha, \nabla_b\} &= \frac{\lambda}{\sqrt{2}}(\gamma_b)_\alpha{}^\beta \tilde{W}_\beta t, \\
[\hat{\nabla}_\alpha, \tilde{\nabla}_\beta\} &= -2i\lambda C_{\alpha\beta}St,
\end{aligned} \tag{4.1.5}$$

where $\tilde{W}_\beta = \frac{i}{\sqrt{2}}(\bar{W}_\beta - W_\beta)$ and $\hat{W}_\beta = \frac{1}{\sqrt{2}}(\bar{W}_\beta + W_\beta)$.

Next we consider the measure of the $(2,2)$ action (4.1.3) and we rewrite it in terms of the $(1,1)$ derivatives

$$\hat{\nabla}^\alpha\hat{\nabla}_\alpha\tilde{\nabla}^\beta\tilde{\nabla}_\beta = 2\nabla^\alpha\nabla_\alpha\bar{\nabla}^\beta\bar{\nabla}_\beta + 2\bar{\nabla}^\beta\bar{\nabla}_\beta\nabla^\alpha\nabla_\alpha + (...)t + \text{total derivative} \tag{4.1.6}$$

Therefore $\hat{\nabla}^\alpha\hat{\nabla}_\alpha\tilde{\nabla}^\beta\tilde{\nabla}_\beta$ and $2\nabla^\alpha\nabla_\alpha\bar{\nabla}^\beta\bar{\nabla}_\beta+2\bar{\nabla}^\beta\bar{\nabla}_\beta\nabla^\alpha\nabla_\alpha$ are equivalent when acting on a potential which is invariant under the isometry, that is, satisfies $tK = 0$.

Reducing the $(2,2)$ Lagrangian amounts to evaluating

$$L = \int d^2\bar{\theta}d^2\theta K = \frac{1}{4}\hat{\nabla}^2\tilde{\nabla}^2 K(X,\bar{X},Y,\bar{Y}). \tag{4.1.7}$$

Then, using the relation

$$\tilde{\nabla}^\alpha\tilde{\nabla}_\alpha = -2i\tilde{\nabla}_+\tilde{\nabla}_- - 2i\lambda Pt, \tag{4.1.8}$$

we only need to evaluate $\tilde{\nabla}_+\tilde{\nabla}_- K$. Additionally, we must decompose the $(2,2)$ left and right semi-chiral superfields into $(1,1)$ superfields

$$\varphi = X|, \qquad \Psi = \tilde{\nabla}_- X|, \qquad \chi = Y|, \qquad \Upsilon = \tilde{\nabla}_+ Y|. \tag{4.1.9}$$

We end up with:

$$
\begin{aligned}
\tilde{\nabla}_+\tilde{\nabla}_- K &= \hat{\nabla}_+\varphi^I m_{II'}\hat{\nabla}_-\chi^{I'} + \Upsilon_+^{I'} n_{I'I}\Psi_-^I + \Psi_-^I(2\omega_{IJ}\hat{\nabla}_+\varphi^J + ip_{II'}\hat{\nabla}_+\chi^{I'}) \\
&\quad +\Upsilon_+^{I'}(2\omega_{I'J'}\hat{\nabla}_-\chi^{J'} - iq_{I'I}\hat{\nabla}_-\varphi^I) - 2i\lambda S K_{i'}(tY^{i'}) + 2igS K_{\bar{i}'}(t\bar{Y}^{\bar{i}'}) \\
&\quad +2i\lambda(S+iP)K_i(tX^i) - 2i\lambda(S-iP)K_{\bar{i}}(t\bar{X}^{\bar{i}}) \\
&= \hat{\nabla}_+ Z^t\cdot E\cdot\hat{\nabla}_- Z + S_{+I}u^{II'}S_{-I'} - 2i\lambda S K_{i'}(tY^{i'}) + 2i\lambda S K_{\bar{i}'}(t\bar{Y})^{\bar{i}'} \\
&\quad +2i\lambda(S+iP)K_i(tX^i) - 2i\lambda(S-iP)K_{\bar{i}}(t\bar{X}^{\bar{i}}), \qquad (4.1.10)
\end{aligned}
$$

where we have used the notation $K_i = \partial_{\varphi^i}K, K_{i'} = \partial_{\chi^{i'}}K$. The index I is a collective index: $I = \{i, \bar{i}\}$, and $\Phi = \{\phi, \bar{\phi}, \chi, \bar{\chi}\}$. The matrices m, n, ω, p, q, expressed in terms of the second order derivatives of the generalized Kähler potential, are the same as in (3.4.24) [23]. Also, analogous to (3.4.24) [23]

$$
\begin{aligned}
S_{+I}u^{II'} &= \Upsilon_+^{I'} - 2u^{II'}\omega_{IJ}\hat{\nabla}_+\varphi^J - iu^{II'}P_{IJ'}\hat{\nabla}_+\chi^{J'} \\
u^{II'}S_{-I'} &= \Psi_-^A + 2u^{II'}\omega_{I'J'}\hat{\nabla}_-\chi^{J'} - iu^{II'}q_{I'J}\hat{\nabla}_-\varphi^B \\
E &= g + B = \begin{pmatrix} 2i\omega u q & m - 4\omega u\omega' \\ p^t u q & 2ip^t u\omega' \end{pmatrix}. \qquad (4.1.11)
\end{aligned}
$$

At a first glance it appears that we have an asymmetric coupling of the field strength P between the fields φ^I and $\chi^{I'}$. However, this is just an artifact of our choice in evaluating the covariant derivatives. Note that

$$
tK(\varphi,\chi) = 0 \to K_i(tX^i) + K_{\bar{i}}(t\bar{X}^{\bar{i}}) + K_{i'}(tY^{i'}) + K_{\bar{i}'}(t\bar{Y}^{\bar{i}'}) = 0. \qquad (4.1.12)
$$

This means that the reduced Lagrangian is given by

$$
\begin{aligned}
L &= -2i\hat{\nabla}^\alpha\hat{\nabla}_\alpha\Big(\hat{\nabla}_+ Z^t\cdot E\cdot\hat{\nabla}_- Z + S_{+I}u^{II'}S_{-I'} \\
&\quad +2i\lambda(S+\frac{i}{2}P)K_i(tX^i) - 2i\lambda(S-\frac{i}{2}P)K_{\bar{i}}(t\bar{X}^{\bar{i}})
\end{aligned}
$$

$$
\begin{aligned}
& -2ig(S+\frac{i}{2}P)K_{i'}(tY^{i'})+2ig(S-\frac{i}{2}P)K_{\bar{i}'}(t\bar{Y}^{\bar{i}'})\Big) \\
= \quad & -2i\hat{\nabla}^\alpha\hat{\nabla}_\alpha\Big(\hat{\nabla}_+Z^t\cdot(g+B)\cdot\hat{\nabla}_-Z+S_{+I}u^{II'}S_{-I'} \\
& +2i\lambda S(K_i(tX^i)-K_{\bar{i}}(t\bar{X}^{\bar{i}})-K_{i'}(tY^{i'})+K_{\bar{a}'}(t\bar{Y}^{\bar{i}'})) \\
& -\lambda P(K_i(tX^i)+K_{\bar{i}}(t\bar{X}^{\bar{i}})-K_{i'}(tY^{i'})-K_{\bar{i}'}(t\bar{Y}^{\bar{i}'}))\Big). \qquad (4.1.13)
\end{aligned}
$$

This is the gauged sigma model we were after, and such it is one of our main results.

To understand the various terms that appear in (4.1.13), it is useful to compare this action with (4.0.7), given that both actions represent gauged sigma models with manifest $(1,1)$ supersymmetry. This explains the obvious common elements $\hat{\nabla}^\alpha Z^t\cdot g\cdot\hat{\nabla}_\alpha Z+\hat{D}^\alpha Z^t\cdot B\cdot\hat{D}_\alpha Z$. The gauging of the B-field terms is done in (4.0.7) by including the u-dependent terms. To see how this is reflected in (4.1.13) requires some extra consideration. The extra terms required for the gauging of the B-field terms can be combined into $\mathrm{i}_\xi B\cdot\hat{D}_{(-}\Phi A_{+)}$. As a consequence of the condition $tK=0\to\xi K=0$, we find that $\mathcal{L}_\xi B=0$. This is a stronger condition than $\mathcal{L}_\xi H=0$, and it implies the latter. Since $\mathcal{L}_\xi B=0$, we find

$$
u=-\mathrm{i}_\xi B+d\sigma, \qquad (4.1.14)
$$

where $d\sigma$ is an exact one-form invariant under the action of the isometry group. This is exactly what is required to match the minimal coupling of the B-field terms against the u-terms in (4.0.7). The c_{AB} terms in (4.0.7) vanish in the case of a $U(1)$ gauging. Otherwise, they, too, could be recognized in the minimal coupling gauging of (4.0.7).

We shall see that the ambiguity in defining u, namely the exact one-form $d\sigma$, is reflected in (4.1.13) in the term which multiplies the field strength P. The expression $-gd(K_I\xi^I - K_{I'}\xi^{I'})$ is $d(\sigma)$. We verify that it is invariant under the $U(1)$ action:

$$\begin{aligned}\mathcal{L}_\xi d\sigma &= d(i_\xi d\sigma) \sim d\Big((\xi^I\partial_I + \xi^{I'}\partial_{I'})(\xi^J\partial_J - \xi^{J'}\partial_{J'})K\Big) = \\ &= 2d\Big((\xi^I\partial_I + \xi^{I'}\partial_{I'})\xi^I\partial_J\ K\Big) = 2d\Big((-\xi^I\partial_I\xi^{I'}\partial_{I'} + \xi^{I'}\partial_{I'}\xi^I\partial_I)K\Big) = 0,\end{aligned} \tag{4.1.15}$$

where in the last step we used that we can go to a coordinate system where the U(1) action is realized by a shift of some coordinate, which implies $[\xi^I\partial_I, \xi^{I'}\partial_{I'}] = 0$.

The remaining terms in (4.1.13), such as those dependent on the auxiliary superfields $S_\pm$ and which have no counterpart in (4.0.7), are present because our starting point was a $(2,2)$ supersymmetric action with off-shell (2,2) superfields. Lastly, we recognize in the terms proportional to the superfield strength S, a linear combination of the moment maps. Their presence is required to insure the invariance of the gauged sigma model action. While the expression proportional to S in (4.1.13) is not immediately relatable to the moment map given in (4.0.6),it does have a form similar to that given in [38, 42, 43] for the moment map. There the moment map is identified as the imaginary part of the holomorphic transformation of the generalized Kähler potential under the action of the Killing vector.

Thus we conclude with the identifications:

$$\text{Moment map} \sim\ i(K_i\xi^i - K_{\bar{i}}\xi^{\bar{i}} - K_{i'}\xi^{i'} + K_{\bar{i}'}\xi^{\bar{i}'}) \tag{4.1.16}$$

$$\sigma \sim\ K_i\xi^i + K_{\bar{i}}\xi^{\bar{i}} - K_{i'}\xi^{i'} - K_{\bar{i}'}\xi^{\bar{i}'}. \tag{4.1.17}$$

These identifications, and especially the rapport between (4.1.16) and (4.0.6), will be verified in the next section. At this point we note a peculiarity. The two function we have identified are derivable from the condition $\xi K = 0$. A third function,

$$i(K_i\xi^i - K_{\bar{i}}\xi^{\bar{i}} + K_{i'}\xi^{i'} - K_{\bar{i}'}\xi^{\bar{i}'}) \tag{4.1.18}$$

is also derivable from this condition yet doesn't seem to play a role in the gauged sigma model. We speculate that this function does play a role that will be hinted at in the last chapter.

4.2 An example: the $SU(2) \times U(1)$ WZNW model

In this section we apply our previous construction of a $(2,2)$ gauged sigma model to a concrete example: the $SU(2) \times U(1)$ WZNW model. The $(2,2)$ supersymmetric $SU(2) \times U(1)$ WZNW sigma model was first formulated in terms of semi-chiral superfields in [44]. The authors discovered non-commuting complex structures on $SU(2)\times U(1)$ and constructed a duality functional that maps between the known description in terms of chiral and twisted chiral superfields and a description in terms of semi-chiral superfields. The explicit form of the generalized Kähler potential was given in [45,46]. A discussion on the various dual descriptions which can be obtained by means of a Legendre transform can be found in [46]. The $SU(2) \times U(1)$ generalized Kähler potential is

$$K = -(\bar{\phi} + \eta)(\phi + \bar{\eta}) + \frac{1}{2}(\bar{\eta} + \eta)^2 - 2\int^{\bar{\eta}+\eta} dx \ln(1 + \exp(x/2)), \tag{4.2.1}$$

where $\bar{D}_+\phi = D_-\eta = 0$. Because $K = K(\bar{\phi}+\eta, \phi+\bar{\eta}, \eta+\bar{\eta})$ we cannot directly gauge the theory, using only the coupling with the chiral $(2,2)$ vector multiplet. However, there is an easy remedy to this problem, namely we shall use a dual description, found via a Legendre transform [46]:

$$K(r,\bar{r},\eta,\bar{\eta}) = K(\phi,\bar{\phi},\eta,\bar{\eta}) - r\phi - \bar{r}\phi, \tag{4.2.2}$$

where r is semi-chiral, $\bar{D}_+ r = 0$, and ϕ is unconstrained. By integrating over r, we recover the previous generalized Kähler potential. On the other hand, by integrating over ϕ, that is eliminating it from its equation of motion, we find a generalized Kähler potential $K = K(r+\eta, \bar{r}+\bar{\eta}, \eta+\bar{\eta})$ (up to terms that represent a generalized Kähler transform $\frac{1}{2}\eta^2 + \frac{1}{2}\bar{\eta}^2$). This is an example of a "duality without isometry" [46], where the generalized Kähler potential of a semi-chiral superfield sigma model can be mapped via Legendre transforms into four different, but equivalent expressions, all involving only semi-chiral superfields.

The new form taken by the $SU(2) \times U(1)$ generalized Kähler potential

$$\tilde{K} = (\bar{r}+\bar{\eta})(r+\eta) - 2\int^{\bar{\eta}+\eta} dx \ln(1+\exp(x/2)) \tag{4.2.3}$$

indicates that the $U(1)$ isometry is realized by the transformations

$$r \to r + i\epsilon, \quad \eta \to \eta + \overline{(i\epsilon)}, \tag{4.2.4}$$

where ϵ is a constant real parameter. However, when promoting this symmetry to a local one, according to our previous discussion, ϵ is to be interpreted as a chiral superfield, and $\bar{\epsilon}$ as an anti-chiral superfield.

The generalized Kähler potential is left invariant under the action of the $(2,2)$ Killing vector

$$\xi = i\frac{\partial}{\partial r} - i\frac{\partial}{\partial \bar{r}} - i\frac{\partial}{\partial \eta} + i\frac{\partial}{\partial \bar{\eta}}. \tag{4.2.5}$$

From (4.1.11) we can now calculate the B field, its field strength and their contractions with the Killing vector:

$$B = (1 - 2f)(dr \wedge d\bar{\eta} + d\bar{r} \wedge d\eta)$$

$$\mathrm{i}_\xi B = i(1-2f)d\bar{\eta} - i(1-2f)d\eta + i(1-2f)d\bar{r} - i(1-2f)dr$$

$$H = dB = 2(\frac{\partial f}{\partial \eta}dr - \frac{\partial f}{\partial \bar{\eta}}d\bar{r}) \wedge d\eta \wedge d\bar{\eta}$$

$$\mathrm{i}_\xi H = d(2if[-dr + d\bar{r} - d\eta + d\bar{\eta}]) = du, \tag{4.2.6}$$

where

$$f = f(\bar{\eta} + \eta) = \frac{\exp[\frac{1}{2}(\bar{\eta}+\eta)]}{1+\exp[\frac{1}{2}(\bar{\eta}+\eta)]}. \tag{4.2.7}$$

We also find that $\mathcal{L}_\xi B = 0$, in accord to the expectation that the gauging is done via minimal coupling [39, 40]. As discussed before, it implies that $u = -\mathrm{i}_\xi B + d\sigma$ where $d\sigma$ is an exact one-form, invariant under the action of the Killing vector. As to the term proportional to P in (4.1.13) we find that is equal to $2i\lambda\sigma$, where $d\sigma = d(\bar{r} - r + \bar{\eta} - \eta)$. Indeed, this one-form satisfies the condition $\mathrm{i}_\xi d\sigma = 0$.

Next, we show how the term proportional to S corresponds to the moment map.

4.2.0.1 The Moment Map

Here we verify that the term proportional to the super-curvature S in (4.1.13)

$$i(K_r - K_{\bar{r}} - K_\eta + K_{\bar{\eta}}) = 2i\left[r + \bar{r} + \eta + \bar{\eta} - 2\ln(1 + \exp(\frac{\eta + \bar{\eta}}{2}))\right] \equiv M \quad (4.2.8)$$

is a certain linear combination of the two moment maps of the bihermitian geometry. We recall their definition

$$g_{ij}\xi^j \pm u_i = I^j_{\pm i}\partial_j\mu_\pm. \quad (4.2.9)$$

Before we consider (4.2.9) we must first address the ambiguity in the expression for the one form u. The one-form u is defined only up to an exact one form that satisfies $\mathcal{L}_\xi d\sigma = 0$: $u = 2if[-dr + d\bar{r} - d\eta + d\bar{\eta}] + di(C_r r + C_{\bar{r}}\bar{r} + C_\eta\eta + C_{\bar{\eta}}\bar{\eta})$ with $C_{r,\bar{r},\eta,\bar{\eta}}$ constants, constrained only by $C_r - C_{\bar{r}} - C_\eta + C_{\bar{\eta}} = 0$. However, our previous considerations have eliminated most of the freedom in $d\sigma$, given that, from the gauged action we have identified $C_r = -1, C_{\bar{r}} = 1, C_\eta = -1, C_{\bar{\eta}} = 1$. Armed with the concrete expressions of the moment maps we find the following relationship with M:

$$M = -(\mu_+ + \mu_-). \quad (4.2.10)$$

We speculate that had one chosen to gauge the $U(1)$ isometry using instead the twisted chiral vector multiplet, the term proportional to the super-curvature S in (4.1.13) would have involved other linear combination of the two moment maps $\mu_+ - \mu_-$.

4.3 Alternative gauging procedure: the prepotential

In section 4.1 we gauged the sigma model by replacing the Grassmann integration measure with gauge supercovariant derivatives and thus reducing the $(2,2)$ action to a gauged action with $(1,1)$ manifest supersymmetry. Here we take the alternative approach of using the gauge prepotential superfield V to arrive at a gauge-invariant generalized Kähler potential. This procedure is done in $(2,2)$ superspace, and all supersymmetries remain manifest. Therefore this gauging method has the advantage of facilitating the discussion of duality functionals, which we will address in the next section.

In simple cases, the gauging is done by adding the prepotential V to the appropriate combination of superfields in the generalized Kähler potential. For the example $K = K(X+\bar{X}, Y+\bar{Y}, X+Y)$, the global symmetry is promoted to a local one by

$$K(X+\bar{X}, Y+\bar{Y}, X+Y) \to K(X+\bar{X}+V, Y+\bar{Y}+V, X+Y+V), \quad (4.3.1)$$

if the gauging is done using the prepotential for the chiral vector multiplet, i.e. if the gauge parameter is a chiral superfield. On the other hand, if the gauge parameter is a twisted chiral superfield, then we must use the gauge prepotential associated with the twisted chiral vector multiplet V_t. For example, we could gauge $K = K(X+\bar{X}, Y+\bar{Y}, X+\bar{Y})$ by

$$K(X+\bar{X}, Y+\bar{Y}, X+\bar{Y}) \to K(X+\bar{X}+V_t, Y+\bar{Y}+V_t, X+\bar{Y}+V_t). \quad (4.3.2)$$

For concreteness we continue to address only the gauging done using the coupling to the chiral vector multiplet. In general, the isometry transformations of a given superfield are given by:

$$X \to e^{i\epsilon t} X \Rightarrow \bar{X} \to e^{i\bar{\epsilon} t} \bar{X}, \tag{4.3.3}$$

where t denotes the isometry generator and ϵ is a real valued constant parameter. For our purposes, it is better to think of the gauging in terms of the Killing vector that generates the isometry. That means we make the following replacement in the above expression for the global transformations of the fields

$$X \to e^{\epsilon \xi} X \Rightarrow \bar{X} \to e^{\bar{\epsilon} \xi} \bar{X}, \tag{4.3.4}$$

When promoting this global symmetry to a local one, the gauge parameter ϵ becomes a chiral superfield, and $\bar{\epsilon}$ an anti-chiral superfield. The invariance of the potential is lost because the fields no longer transform with the same parameter. The invariance is restored by introducing the gauge prepotential superfield V, transforming as

$$V \to V + i(\bar{\epsilon} - \epsilon). \tag{4.3.5}$$

We include V through the replacement:

$$\bar{X} \to e^{iV\xi} \bar{X}. \tag{4.3.6}$$

Now $\bar{X}$ transforms in the same way as in the global case and thus the invariance has been restored.

Although we have used the whole Killing vector ξ in constructing the field that transforms properly (4.3.6), to be more specific, it is only the part of the Killing vector that induces a transformation with the anti-chiral gauge parameter which contributes to this definition. In the example that we gave, $K = K(X + \bar{X}, Y +$

$\bar{Y}, X+Y)$, $X, \bar{Y}$ transform with a chiral gauge parameter, and $\bar{X}, Y$, with an anti-chiral parameter. The Killing vector will generally factorize $\xi = \xi_c + \xi_{\bar{c}}$ such that ξ_c and $\bar{\xi}_c$ induce a chiral parameter, respectively an anti-chiral parameter gauge transformation. In the $SU(2) \otimes U(1)$ example we have $\xi_{\bar{c}} = -i\frac{\partial}{\partial \bar{r}} - i\frac{\partial}{\partial \eta}$.

Therefore we define

$$\tilde{X} = e^L \bar{X}, \quad L = iV\xi_{\bar{c}}. \tag{4.3.7}$$

The new field, $\tilde{X}$, transforms under the gauge transformation in the exact same way as $\bar{X}$ did under the global isometry. Therefore by replacing $\bar{X}$ in the generalized Kähler potential by $\tilde{X}$ we insure that the transformation of the generalized Kähler potential under the local transformation is the same as for the global isometry, namely it is a generalized Kähler transformation. Of course, the other semi-chiral superfield Y undergoes a similar treatment:

$$\tilde{Y} = e^L Y. \tag{4.3.8}$$

If the generalized Kähler potential remains invariant under the action of the Killing vector i.e. $\xi K(X, \bar{X}, Y, \bar{Y}) = 0$, the minimal coupling perscription is given by replacing $\bar{X}$ with $\tilde{X}$ and Y with $\tilde{Y}$. Specifically, the gauged (2,2) Lagrangian is given by the replacement

$$K(X, \bar{X}, Y, \bar{Y}) \to K(X, \tilde{X}, \tilde{Y}, \bar{Y}). \tag{4.3.9}$$

At this point we can use the relation $K(X, \tilde{X}, \tilde{Y}, \bar{Y}) = e^L K(X, \bar{X}, Y, \bar{Y})$ to rewrite the Lagrangian as

$$K(X, \tilde{X}, \tilde{Y}, \bar{Y}) \;\; = \;\; e^L K(X, \bar{X}, Y, \bar{Y}) = K(X, \bar{X}, Y, \bar{Y}) + \frac{e^L - 1}{L} LK$$

$$= \quad K(X,\bar{X},Y,\bar{Y}) + \frac{e^L - 1}{L} VM, \qquad (4.3.10)$$

where in $M = i\xi_{\bar{c}}K$ we recognized the same object which we have identified from the gauged (1,1) action as the moment map (5.2.4).

Next, we address the case of a generalized Kähler potential which under the action of the isomtery generator transforms with terms that take the form of generalized Kähler transformations

$$\xi K = f(X) + \bar{f}(\bar{X}) + g(Y) + \bar{g}(\bar{Y}). \qquad (4.3.11)$$

The trick is to introduce new coordinates and add them to the generalized Kähler potential in such a way that the new generalized Kähler potential is invariant under the transformation generated by the new Killing vector. Specifically we introduce α, β with $\bar{D}_+\alpha = D_-\beta = 0$. We construct the new generalized Kähler potential and Killing vector

$$K'(X,\bar{X},Y,\bar{Y},\alpha,\bar{\alpha},\beta,\bar{\beta}) = K(X,\bar{X},Y,\bar{Y}) - \alpha - \bar{\alpha} - \beta - \bar{\beta}$$

$$\xi' = \xi + f(X)\frac{\partial}{\partial\alpha} + \bar{f}(\bar{X})\frac{\partial}{\partial\bar{\alpha}} + g(Y)\frac{\partial}{\partial\beta} + \bar{g}(\bar{Y})\frac{\partial}{\partial\bar{\beta}}. \qquad (4.3.12)$$

Now the new generalized Kähler potential K' is invariant under the new Killing vector $\mathcal{L}_{\xi'}K' = 0$ and we can proceed as before. We replace all fields which transform with the parameter $\bar{\epsilon}$ with the combination which transforms with the field ϵ by using $e^{L'}$ where $L' = iV\xi'_{\bar{c}}$. Next we define the tilde versions of $\bar{X}, Y, \bar{\alpha}, \beta$ as follows

$$\tilde{X} = e^{L'}\bar{X}, \qquad \tilde{Y} = e^{L'}Y, \qquad \tilde{\alpha} = e^{L'}\bar{\alpha} \qquad \tilde{\beta} = e^{L'}\beta. \qquad (4.3.13)$$

The gauged Lagrangian is obtained by the same substitution as before. Finally we get

$$\begin{aligned}
K'(X,\tilde{X},\tilde{Y},\bar{Y},\alpha,\tilde{\alpha},\tilde{\beta},\bar{\beta}) &= K(X,\tilde{X},\tilde{Y},\bar{Y})-\alpha-\tilde{\alpha}-\tilde{\beta}-\bar{\beta} \\
&= e^{L}K(X,\bar{X},Y,\bar{Y})-i\frac{e^{L}-1}{L}V(\bar{f}(\bar{X})+g(Y)) \\
&= K(X,\bar{X},Y,\bar{Y})+\frac{e^{L}-1}{L}(LK-iV\bar{f}(\bar{X})-iVg(Y)) \\
&= K(X,\bar{X},Y,\bar{Y})+\frac{e^{L}-1}{L}VM. \qquad (4.3.14)
\end{aligned}$$

Chapter 5

Eigenspaces of Generalized Complex Structures

5.1 Hamiltonian action and moment map in the mathematical literature

In the context of generalized complex geometry, the origin of subsequent definitions of the Hamiltonian action can be found in Gualtieri's thesis [22] where it was shown that certain infinitesimal symmetries preserving the generalized complex structure $\mathcal{J}$ can be extended to second order.

Intuitively, given a Hamiltonian action on a generalized complex manifold, the moment map is a quantity that is constant along the action of the group elements. More formal definitions of the moment map were given, for example, in [47–50]; in [49], Hu considered the Hamiltonian group globally. For concreteness here we will explore one of the definitions put forward by Lin and Tolman [47] in the simplest setting without H-twisting, namely, definition 3.4:

Let a compact Lie group G with Lie algebra g act on a manifold M, preserving a generalized complex structure $\mathcal{J}$. Let $L \subset T \oplus T^*$ denote the $\sqrt{-1}$-eigenbundle of $\mathcal{J}$. A generalized moment map is a smooth function $\mu : M \to g^*$ so that

(i) $\xi_M - \sqrt{-1}\, d\mu^\xi$ lies in L for all $\xi \in g$, where ξ_M denotes the induced vector

field on M.

(ii) μ is equivariant.

In subsequent works, the definition of Hamiltonian action was generalized to include the H-twisted case [48, 49]. In [51], the authors arrived at a definition of moment map in terms of the action of a Lie algebra on a Courant algebroid.

In what follows we will explore the particular definition cited above, and compare it with the expressions that we gave for the moment map in the previous sections. We leave for future work the issue of the equivalence of the various definitions given in the math literature, and their relationship with the physical point of view advocated in this paper, via the gauging of the $(2, 2)$ sigma model.

5.2 Generalized Kähler geometry and the eigenvalue problem

In a series of papers [23, 33] the authors established that chiral, twisted chiral, and semi-chiral superfields are the most generic off shell multiplets for D=2 $\mathcal{N} = (2, 2)$ supersymmetric non linear sigma models and that they give generalized Kähler geometries.

To practically use the above definition of moment map in the case of Kähler geometry we recall that according to Gualtieri (see Chapter 6 in [22]), the generalized complex structures of the generalized Kähler geometry take the following expressions:

$$\mathcal{J}_{1/2} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}\begin{pmatrix} J_+ \pm J_- & -(\omega_+^{-1} \mp \omega_-^{-1}) \\ \omega_+ \mp \omega_- & -(J_+^t \pm J_-^t) \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -B & 1 \end{pmatrix} \tag{5.2.1}$$

where g is a Kähler metric, which is bihermitian with respect to both almost complex structures $J_\pm$, while B is a 2-form field. We leave a discussion about its relationship with the B-field of the sigma model for section 3.4.

First, we shall derive the conditions for a generic element of $T \oplus T^*$ $(\xi, \pm id\mu)$ to be an eigenvector of the generalized complex structures. By identifying $\xi \in T$ with a Killing vector, we solve for the one form $d\mu \in T^*$. Next, after verifying that $d\mu$ is an exact one-form, we shall compare it with the the moment map and enquire whether these expressions are compatible. We discuss two concrete settings: the almost product structure spaces, with their commuting almost complex structures, and as an example of bihermitian geometry we turn to our favorite example, the $SU(2) \times U(1)$ sigma model.

We begin with some formal statements. The condition that an element of $T \oplus T^*$ lies in the eigenbundle of $\mathcal{J}_1$ is

$$\mathcal{J}_1 \begin{pmatrix} \xi \\ icd\mu \end{pmatrix} = ai \begin{pmatrix} \xi \\ icd\mu \end{pmatrix}, \tag{5.2.2}$$

where $c = \pm 1$, $a = \pm 1$. After a bit of massaging, we find that this eigenvalue problem is equivalent to the following linear homogeneous equation system[1]

$$\begin{aligned} (J_+ - ai)(\Gamma - \xi) &= 0 \\ (J_- - ai)(\Gamma + \xi) &= 0, \end{aligned} \tag{5.2.3}$$

[1] For the eigenvalue problem associated with the other generalized almost complex structure $\mathcal{J}_2$,

where

$$\Gamma = G^{-1}(B\xi - icd\mu) \tag{5.2.4}$$

Then, by solving (5.2.3) we find ξ and Γ. The number of independent solutions is equal to the number of zero eigenvalues of $J_\pm - ai$. However, after identifying ξ with a certain Killing vector, we generically find a corresponding Γ. This allows us to solve for μ

$$d\mu = ic(G\Gamma - B\xi). \tag{5.2.5}$$

To test the compatibility between this expression and the moment map (4.0.6) we explore in the next sections two concrete examples of bihermitian geometry.

5.3 Specialization to spaces with almost product structure

In the case of a space with almost product structure, which is realized by a (2,2) sigma model written in terms of chiral and twisted chiral superfields [30], we may choose to work in a coordinate system where the two commuting complex structures are diagonal:

$$J_+ = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix} \qquad J_- = \begin{pmatrix} J_1 & 0 \\ 0 & -J_2 \end{pmatrix}. \tag{5.3.1}$$

---

we find a similar linear homogeneous system:

$$(J_+ - ai)(\Gamma - \xi) = 0, \qquad (J_- + ai)(\Gamma + \xi) = 0.$$

In the same coordinate system, the metric and B-field are also block-diagonal:

$$g = \begin{pmatrix} g_1 & 0 \\ 0 & g_2 \end{pmatrix} \quad B = \begin{pmatrix} 0 & b \\ -b^t & 0 \end{pmatrix}. \tag{5.3.2}$$

The expressions taken by G, B, J_+, and J_- suggest that we should consider a similar decomposition for ξ,Γ and $d\mu$. Specifically,

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \quad \Gamma = \begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \quad d\mu = \begin{pmatrix} d\mu_1 \\ d\mu_2 \end{pmatrix}. \tag{5.3.3}$$

Under this decomposition $\Gamma_{1,2}, \xi_{1,2}$ are solutions to (5.2.3):

$$\begin{aligned} &(J_1 - ai)\Gamma_1 = (J_1 - ai)\xi_1 = 0 \\ &ai\Gamma_2 = -J_2\xi_2. \end{aligned} \tag{5.3.4}$$

and (5.2.5) becomes

$$\begin{aligned} d\mu_1 &= icg_1\Gamma_1 - icb\xi_2 \\ d\mu_2 &= icg_2\Gamma_2 + icb^t\xi_1, \end{aligned} \tag{5.3.5}$$

How does this compare with the moment maps which are given by $d\mu_\pm = \omega_\pm\xi \mp J_\pm^T u$? When we specialize to the case where the Lie derivative of B with respect to ξ vanishes, $\mathcal{L}_\xi B = 0$, we can use that $u = -B\xi + d\sigma$. Taking the appropriate linear combinations that match up most closely with the generalized complex structures we get the following for the components of the moment map

$$d\tilde{M} = \frac{1}{2}(dM_+ + dM_-), \qquad d\hat{M} = \frac{1}{2}(dM_+ - dM_-) \tag{5.3.6}$$

where

$$\begin{pmatrix} d\tilde{M}_1 \\ d\tilde{M}_2 \end{pmatrix} = \begin{pmatrix} \omega_1\xi_1 \\ -J_2^t b^t \xi_1 \end{pmatrix}, \qquad \begin{pmatrix} d\hat{M}_1 \\ d\hat{M}_2 \end{pmatrix} = \begin{pmatrix} -J_1^t b\xi_2 \\ \omega_2\xi_2 \end{pmatrix}. \tag{5.3.7}$$

First we notice that the appropriate expression to match with (5.3.7) is $d\hat{M}$. Second, in order for (5.3.7) and (5.3.5) to match we need $\Gamma_1 = \xi_1 = 0$. The condition $\xi_1 = 0$ is automatically satisfied for almost product structure geometries, where $J_{1,2}$ are both diagonal. Then the requirement that ξ is holomorphic (i.e. it leaves invariant the complex structures) implies that a Killing vector is such that either ξ_1 or ξ_2 vanish [38]. Next to complete the matching of (5.3.7) and (5.3.5) we need $\Gamma_2 = \pm iJ_2\xi_2$, but is exactly the expression of Γ_2 which we get from (5.3.4).

Now that we have verified the compatibility of two moment map definitions, (5.2.5) and (4.0.6), for the almost product structure geometry, we want to investigate their compatibility in a more generic case of bihermitian structure. Since the complex structures do not commute in this case, it is difficult to analyze what happens in general. However we can consider the concrete $SU(2) \otimes U(1)$ example and see how things work out there.

5.4 The $SU(2) \otimes U(1)$ example

In this case the non-commuting complex structures, read off from the supersymmetry transformations of the non-linear sigma model [23, 44], are:

$$J_+ = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ -2i & 0 & -i & 0 \\ 0 & 2i & 0 & i \end{pmatrix} \quad J_- = \begin{pmatrix} i & 0 & 2i(1-f) & 0 \\ 0 & -i & 0 & -2i(1-f) \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \tag{5.4.1}$$

where $f = f(\eta + \bar{\eta})$. The U(1) Killing vector is $\xi = (i, -i, -i, i)$. The B-field was given in (4.2.6), and the metric takes the form

$$g = \begin{pmatrix} 0 & 2 & 0 & 2(1-f) \\ 2 & 0 & 2(1-f) & 0 \\ 0 & 2(1-f) & 0 & 2(1-f) \\ 2(1-f) & 0 & 2(1-f) & 0 \end{pmatrix}. \tag{5.4.2}$$

The moment map $d\mu_+ = \omega_+\xi - J_+^T u$ reads

$$\begin{aligned} d\mu_+ &= (-2f, -2f, 0, 0) - (-2f, -2f, -2f, -2f) - (iC_r - 2iC_\eta, -iC_{\bar{r}} + 2iC_{\bar{\eta}}, -iC_\eta, iC_{\bar{\eta}}) \\ &= (iC_r - 2iC_\eta, -iC_{\bar{r}} + 2iC_{\bar{\eta}}, 2f - iC_\eta, 2f + iC_{\bar{\eta}}). \end{aligned} \tag{5.4.3}$$

wheare the last term on the first line represents the ambiguity in u, $J_+^T d\sigma$. The constants $C_{r,\bar{r},\eta,\bar{\eta}}$ satisfy the constraint $C_r - C_{\bar{r}} - C_\eta + C_{\bar{\eta}} = 0$.

We find that the solution to (5.2.3), corresponding to a $+i$ eigenvector, $(a = 1)$, is given by $(\xi, \Gamma_{1,\pm})$, where $\xi = (i, -i, -i, i)$ and

$$\Gamma_{1,+} = \left(-i, -i, i, i\frac{1+f}{1-f}\right). \tag{5.4.4}$$

For a $-i$ eigenvector $(a = 1)$, we find

$$\Gamma_{1,-} = \left(i, i, -i\frac{1+f}{1-f}, -i\right), \tag{5.4.5}$$

for the same Killing vector ξ. For completeness we record the eigenvectors $(\xi, \Gamma_{2,\pm})$ of the second generalized almost complex structure $\mathcal{J}_2$: $\Gamma_{2,+} = (-i, -i, i, -i)$ corresponds to the $+i$ eigenvalue and $\Gamma_{2,-} = (i, i, i, -i)$ to the $-i$ eigenvalue.

From (5.3.7), substituting $\Gamma_{1,\pm}$ as well as the the metric, B-field, and Killing vector we get

$$icG\Gamma_{1,+} = c(-2f, 2f, -4f, 0), \qquad icB\xi = c(-1+2f, 1-2f, -1+2f, 1-2f), \quad (5.4.6)$$

where we recall that $c = \pm 1$. We have also identified the 2-form B in the generalized almost complex structure with the B-field. Notice that in order to be able to recover an expression compatible with (5.4.3), we must take the *sum* $ic(G\Gamma + B\xi)$, and not the difference of the two terms in (5.4.6)! The reason for an apparent discrepancy between the two expressions that we have for the moment map, (4.0.6) and (5.2.5) lies in the identification of the sigma model B field and the 2-form B that appears in the generalized almost complex structure (5.2.1). The agreement is restored upon *making the identification between minus the sigma model B-field and the object by the same name present in (5.2.1).* It is essential that in replacing $B \to -B$ in (5.2.1), with B the sigma model B-field, we haven't spoiled any of the properties of the generalized Kähler geometry objects.

To complete our argument, we have to make the following assignments for the constants which enter in the one-form $d\sigma$: $C_r = C_{\bar{r}} = C_\eta = C_{\bar{\eta}} = i$.

We still find it possible to obtain the moment map from the condition that together with the Killing vector it forms a pair $(\xi, icd\mu)$ which lies in the eigenbundle of the generalized almost complex structure. However, we must exercise caution and interpret the 2-form B in (5.2.1) as *minus* the sigma-model B-field. We have also seen that the matching between (5.2.5) and (4.0.6) requires making use of the ambiguity in defining the one-form u. The exact, U(1) invariant one-form $d\sigma$ required by the matching between the two moment map definitions led us to a different one-form $d\sigma$ than the one we identified in Section 2.2 by matching u with the gauged sigma model action. The understanding based on this result isn't clear because the choice of $d\sigma$ only satisfies the condition for one of the expressions in (4.0.6) to be a moment map.

Chapter 6

T Duality For Semi Chiral Superfields With The Chiral Vector Multiplet

Target space duality or T duality is a symmetry of string theory that relates the geometry and topology of different string backgrounds. This symmetry was first discovered by considering toroidal compactifications. It is easiest to see when considering strings in flat space with one direction compactified on a circle of radius R. The spectrum of the string is invariant under the change $R \rightarrow \frac{\alpha'}{R}$ along with the exchange of the momentum and winding modes of the string. At the sigma model level, this is best understood in terms of gauging the isometry transformations of the background in the sigma model without including gauge kinetic terms and constructing a duality functional that reduces to either sigma model by integrating out fields in a specific order. This understanding was first introduced in [52]. Consider a sigma model with a metric, b field and dilaton. The duality functional is constructed by adding a lagrange multiplier times the gauge field strength to the gauged action. Integrating out the lagrange multiplier forces the gauge field to be pure gauge and the sigma model reduces to the original theory. If instead one itegrates out the gauge field first, one obtains a new sigma model with new metric, b field and dilation given in terms of the original metric, b field and dilaton according to the Buscher rules [53]. An excellent review of the topic is given in [54]. The description of how this procedure works for $\mathcal{N} = (2,2)$ sigma models in $(2,2)$ superspace was first given in [52]. There they worked with chiral and twisted chiral superfields only and showed that T duality amounts to a Legendre transformation of the Kähler potential. An interesting point is that the duality exchanges superfield representations i.e. it exchanges chiral superfields for twisted chiral superfields. In the following we will present a $(2,2)$ superspace description of T duality when the sigma model includes semi chiral superfields. We will see that the description has some very serious drawbacks related to an insufficiency of the chiral vector multiplet. We will then present a new $(2,2)$ vector multiplet that should resolve the problems and give a satisfactory description of T duality for sigma models with semi chiral superfields in $(2,2)$ superspace. The discussion here will be entirely classical and we will leave the quantum description of T duality to future work. Since the transformation of the dilaton is a purely quantum mechanical effect, we will ignore it for now and consider only the metric and b field.

T-duality can be implemented for chiral and twisted chiral superfields while preserving the manifest $(2,2)$ supersymmetries of the sigma model, by performing a Legendre transformation of the Kähler potential. This procedure amounts to starting from the gauged sigma model, introducing a Lagrange multiplier that enforces the condition that the gauge field is pure gauge, and eliminating the gauge field from its equation of motion. In terms of the geometric data, it was shown in [55] by descending to the level of $(1,1)$ superspace, that under T-duality, the metric and b field transform according to the Buscher rules. Let us begin with some review material detailing the execution of T-duality in $(2,2)$ superspace. The simplest example of T-duality involves a non-linear sigma model written in terms of either chiral or twisted chiral superfields with an $U(1)$ isometry. Under T-duality the chiral multiplets are mapped into twisted anti-chiral and vice-versa. Specifically, we choose a coordinate system such that the isometry is realized by a shift in a particular coordinate. Then the Kähler potential has the form

$$K = K(\bar{\Phi} + \Phi, Z^a), \tag{6.0.1}$$

where Z^a are spectator fields that can be either chiral or twisted chiral. According to the discusion in Section 2.3, the gauged action is obtained by replacing $\bar{\Phi} + \Phi$ with $\bar{\Phi} + \Phi + V$ where V is the usual superfield prepotential for the gauge multiplet. The gauged Kähler potential is

$$K_g = K(\bar{\Phi} + \Phi + V, Z^a). \tag{6.0.2}$$

To construct the duality functional we introduce a Lagrange multiplier that forces the gauge multiplet field strength to vanish:

$$K_D = K_g + U(S + iP) + \bar{U}(S - iP). \tag{6.0.3}$$

Since $(S + iP) = \frac{i}{2}\bar{D}_+ D_- V$ we see that the U and $\bar{U}$ equations of motion force V to be pure gauge, i.e., $V = \Lambda + \bar{\Lambda}$, with Λ a chiral superfield. For the next step, by choosing a gauge such that $\Phi + \bar{\Phi}$ have been completely gauged away

$$K_g = K(V, Z^a) \tag{6.0.4}$$

we arrive at the duality functional

$$K_D = K(V, Z^a) - U(S + iP) - \bar{U}(S - iP). \tag{6.0.5}$$

The original Kähler potential is recovered by integrating out U and $\bar{U}$. The T-dual theory is obtained by integrating out the gauge field. Its equation of motion is

$$\frac{\partial K}{\partial V} - (\Psi + \bar{\Psi}) = 0, \tag{6.0.6}$$

where $\Psi = \frac{i}{2}\bar{D}_+ D_- U$ is a twisted anti-chiral superfield. This defines $V = V(\Psi + \bar{\Psi}, Z^a)$. The dual potential

$$\tilde{K} = K(V, Z^a) - (\Psi + \bar{\Psi})V \tag{6.0.7}$$

is the Legendre transform of the original potential (6.0.1).

When one introduces semi-chiral superfields the story becomes somewhat more complicated. In [46], Grisaru et al. gave a detailed discussion of the various descriptions of a $(2,2)$ sigma model, which can be obtained by means of a Legendre transform. Starting with a $(2,2)$ Kähler potential written in terms of semi-chiral superfields $K(X, \bar{X}, Y, \bar{Y})$, one constructs the duality functional

$$K(r, \bar{r}, s, \bar{s}) - Xr - \bar{X}\bar{r} - sY - \bar{s}\bar{Y} \tag{6.0.8}$$

where $r, \bar{r}, s, \bar{s}$ are unconstrained superfields. Depending which fields are integrated out $(X, Y), (r, s), (r, Y), (s, X)$ one finds four equivalent formulations. In the absence of isometries, this amounts to performing a sigma-model coordinate transformation. The authors of [46] investigated the consequences that the existence of an isometry have on the duality functional. For instance if the Kähler potential has a U(1) isometry $K = K(X + \bar{X}, X + \bar{Y}, \bar{X} + Y)$, the duality functional reads $K(r + \bar{r}, \bar{r} + s, r + \bar{s}) - (X + \bar{X} - Y - \bar{Y})(r + \bar{r})/2 + (X - \bar{X} + Y - \bar{Y})(r - \bar{r})/2 - (r + \bar{s})Y - (\bar{r} + s)\bar{Y}$. By integrating over $r - \bar{r}$, ultimately leads to expressing X and Y as the sum and difference of a chiral and twisted chiral superfield. In this case, the dual description of the sigma model involves chiral and twisted chiral superfields. The $SU(2) \times U(1)$ WZNW model has two such dual descriptions [44]. The geometry does not change as we pass from one description to the other, but the pair of complex structures does change, from non-commuting complex structures, to commuting ones.

On the other hand, not all the dualities following from (6.0.8) can be derived from gauging an isometry. The reason is that Lagrange multipliers in (6.0.8) are semi-chiral superfields. Following the discussion given at the beginning of this section, one would need a gauge multiplet with a semi-chiral field strengths, in order to cast the gauged action duality functional (6.0.5) into (6.0.8). However, no known $(2,2)$ gauge multiplet contains such field strengths.

Therefore we choose to pursue the construction of the T-dual action of a sigma model with semi-chiral multiplets following the steps which led to (6.0.5). We add Lagrange multiplier terms to the gauged action as described previously, and construct the duality functional as in [52]. However, a technical difficulty, related to gauge fixing, prevents a straightforward application of this procedure.

The U(1) invariant Kähler potential, which generically takes the form given in (4.0.11), can be gauged by adding the prepotential V to the appropriate field combinations. The gauged Kähler potential is $K_g = K(X + \bar{X} + V, Y + \bar{Y} + V, X + Y + V)$. Because the semi-chiral superfield is not generically reducible in terms of chiral and twisted chiral superfields[1] one cannot completely gauge away X or Y, as it was possible for the chiral and/or twisted chiral superfields. Trying to gauge

[1] We thank Martin Roček for explaining this point to us.

away X we could fix $X| = D_\alpha X| = D^2X| = 0$, where $|$ means evaluation with all the Grassmann variables set to zero. Since X has higher order components which are independent of the lower components we realized that we have not gauged away all the X components. The independent left over components form a $(1,1)$ Weyl spinor multiplet. We shall address the resolution to this question in the following section.

6.1 Dualizing With Chiral and Twisted Chiral Superfields.

For simplicity we will consider a Kähler potential, parameterized by chiral and twisted chiral superfields, which is strictly invariant under the isometry. The potential is given by (6.0.1). We begin in the slightly more general setting:

$$K_g = K(\bar{\Phi} + \Phi, Z^a) + \frac{e^L - 1}{L} VM. \tag{6.1.1}$$

The moment map, M, is given by $M = i\xi_{\bar{c}}K$, and in this case $\xi_{\bar{c}} = -i\frac{\partial}{\partial\bar{\Phi}}$. To construct the duality functional we add Lagrange multiplier terms that force the superfield strength to vanish. This gives the Lagrangian

$$K_D = K(\bar{\Phi} + \Phi, Z^a) + \frac{e^L - 1}{L} VM + (\bar{\Psi} + \Psi)V. \tag{6.1.2}$$

The final step is chosing a gauge. Instead of setting $\Phi + \bar{\Phi} = 0$, we choose the Wess-Zumino gauge for the prepotential V

$$V| = D_\alpha V| = D^2 V| = 0. \tag{6.1.3}$$

This gauge choice will allow a better comparison with the semi-chiral case. To see that we do get back the original Lagrangian, we integrate out Ψ and $\bar{\Psi}$. This implies that

$$V = \bar{\Lambda} + \Lambda, \tag{6.1.4}$$

where Λ is a chiral superfield. However, consistency with the gauge choice requires that $V = 0$ and this give us back the original Kähler potential. To find the dual potential we integrate out V. Since $(V)^3 = 0$ in the Wess-Zumino gauge, this allows us to solve for V explicity. We obtain

$$\begin{aligned} V &= i\frac{\bar{\Psi} + \Psi + M}{\xi_{\bar{c}} M} \\ \tilde{K} &= K(\bar{\Phi} + \Phi, Z^a) + \frac{i}{2}\frac{(\bar{\Psi} + \Psi + M)^2}{\xi_{\bar{c}} M}. \end{aligned} \tag{6.1.5}$$

The important thing to note here is that consistency of the solution for V with the gauge fixing conditions require that

$$V| = 0 = i\frac{\bar{\Psi} + \Psi + M}{\xi_{\bar{c}} M}| \Rightarrow (\bar{\Psi} + \Psi)| = -M| \tag{6.1.6}$$

It should be understood that this is a component equation, and not a superfield equation. With this in hand we can show the following;

$$\frac{\partial^2 \tilde{K}}{\partial\bar{\Phi}\partial\Phi}| = 0, \qquad \frac{\partial^2 \tilde{K}}{\partial Z^a \partial\Phi}| = 0, \qquad \frac{\partial^2 \tilde{K}}{\partial\bar{\Psi}\partial\Phi}| = -1. \tag{6.1.7}$$

The implication which follows from these equations is that the contribution of $\Phi|$ to the geometry has been replaced by $\Psi|$ up to a surface term that comes from the new b field. Let us demonstrate how this works with a simple example, specifically $R \to \frac{1}{R}$ for one of the cycles on T^2. The Kähler potential and moment map are:

$$K = \frac{R}{2}(\bar{\Phi} + \Phi)^2$$

$$M = R(\bar{\Phi} + \Phi). \tag{6.1.8}$$

The dual potential is

$$\tilde{K} = -\frac{1}{2R}(\bar{\Psi} + \Psi)^2 - (\bar{\Phi} + \Phi)(\bar{\Psi} + \Psi). \tag{6.1.9}$$

While this looks as though both directions of T^2 were dualized, one must remember that the real part of $\Psi|$ is proportional to R times the real part of $\Phi|$. Only the direction parameterized by the imaginary part of $\Phi|$ was dualized.

6.2 Dualizing with semi-chiral superfields

Now we can give a straightforward extension of the previous discussion to the case when we dualize an isometry of a sigma model parametrized by semi-chiral superfields. We start with equation (4.3.10), add the Lagrange multipliers enforcing that V is pure gauge, and choose the same gauge Wess-Zumino gauge as in the previous section. The dual Kähler potential is:

$$\tilde{K} = K(X, \bar{X}, Y, \bar{Y}, Z^a) + \frac{i}{2}\frac{(\bar{\Psi} + \Psi + M)^2}{\xi_{\bar{c}} M}. \tag{6.2.1}$$

The analogue of (6.1.7) reads:

$$(\xi_c)(\xi_{\bar{c}})\tilde{K}| = 0, \qquad \frac{\partial(i\xi_{\bar{c}}\tilde{K})}{\partial Z^a}| = 0 \qquad \frac{\partial(i\xi_{\bar{c}}\tilde{K})}{\partial \Psi}| = -1. \tag{6.2.2}$$

From (6.2.2) we see that the coordinates in the combination of semi-chiral superfields corresponding to ξ_c have been replaced by coordinates in a twisted chiral superfield in the dual geometry. This is analogous to what happened in the case of chiral and twisted chiral superfields. It was also expected from gauge fixing considerations, although it was not a propri clear exactly how it would happen. We now have an explicit description of the T dual of a theory with semi-chiral superfields at the manifest $(2,2)$ sigma model level.

6.3 An example: T-duality with semi-chiral superfields in flat space

In this section we try to develop some intuition about the dualization prescription described in the previous section. Given that we perform a duality transformation by gauging away *part* of a certain combination of semi-chiral superfields, and in doing so we trade it for a twisted chiral superfield, it is not a priori obvious that this is equivalent to the Buscher rules. In particular, we would like to check this in a simple example, namely flat space with a U(1) isometry.

We start with four-dimensional flat space as our simplest example because one needs both left and right pairs of chiral and anti-chiral superfields in order to be able to eliminate the auxiliary components of the semi-chiral superfields and obtain a sigma-model action. Therefore we begin with the following (2,2) Kähler potential

$$K = R(\bar{X} + \bar{Y})(X + Y) - \frac{R}{4}(Y + \bar{Y})^2 \tag{6.3.1}$$

where $\bar{D}_+ X = D_- Y = 0$. By descending to the level of (1,1) superspace using [33], we find the sigma model metric

$$G = \begin{pmatrix} 0 & 2R & R & 0 \\ 2R & 0 & 0 & R \\ R & 0 & 0 & R \\ 0 & R & R & 0 \end{pmatrix}, \tag{6.3.2}$$

where the rows and columns are labelled by $X|, \bar{X}|, \bar{Y}|, Y|$. This gives us the action for the bosonic components

$$S = \int d^2\sigma R(\partial^a X \partial_a \bar{X} + \partial^a (\bar{X} + \bar{Y})\partial_a (X + Y)), \tag{6.3.3}$$

where for simplicity we denoted by X the bosonic component of the (1,1) superfield $X|$. Denoting $Z = X + Y$ we notice that it is inert under the global shift symmetry. By performing a diffeomorphism transformation to $(X, \bar{X}, Z, \bar{Z})$, we obtain the metric in canonical form

$$G = \begin{pmatrix} 0 & R & 0 & 0 \\ R & 0 & 0 & 0 \\ 0 & 0 & 0 & R \\ 0 & 0 & R & 0 \end{pmatrix}. \tag{6.3.4}$$

The T-dual sigma model is obtained from the dual $(2,2)$ Kähler potential given in (6.2.1). In this particular case, (6.2.1) reads:

$$\tilde{K} = R(\bar{X}+\bar{Y})(X+Y) - \frac{R}{4}(Y+\bar{Y})^2 - \frac{1}{3R}\left(\psi+\bar{\psi}+R(\bar{X}+X+\frac{1}{2}(\bar{Y}+Y))\right)^2 \tag{6.3.5}$$

and the corresponding T-dual sigma-model metric is equal to:

$$\frac{9}{2}G = \begin{pmatrix} -4R & 5R & 4R & -5R & 5 & -4 \\ 5R & -4R & -5R & 4R & -4 & 5 \\ 4R & -5R & -4R & 5R & -5 & 4 \\ -5R & 4R & 5R & -4R & 4 & -5 \\ 5 & -4 & -5 & 4 & -\frac{4}{R} & \frac{14}{R} \\ -4 & 5 & 4 & -5 & \frac{14}{R} & -\frac{4}{R} \end{pmatrix}, \tag{6.3.6}$$

where the rows and columns are labelled by $X, \bar{X}, \bar{Y}, Y, \psi, \bar{\psi}$. At first sight this result is puzzling, because we claim that we found the T-dual of a sigma model whose target space is flat four-dimensional space. At the same time, the dual sigma-model involves six fields, and so, apparently the target space is six-dimensional. These two seemingly contradictory statements are reconciled when one takes a closer look at the T-dual metric, and finds that it actually describes a four dimensional subspace. This is obvious when expressing the previous T-dual metric in terms of the following coordinates: $(X, \bar{X}, W = Y - \bar{X}, \bar{W}, \psi, \bar{\psi})$

$$\frac{9}{2}G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4R & 5R & 4 & -5 \\ 0 & 0 & 5R & -4R & -5 & 4 \\ 0 & 0 & 4 & -5 & -\frac{4}{R} & \frac{14}{R} \\ 0 & 0 & -5 & 4 & \frac{14}{R} & -\frac{4}{R} \end{pmatrix}, \tag{6.3.7}$$

where we make the observation that $W = Y - \bar{X}$ is also inert under the global U(1) action. The final step in getting the metric in its canonical form is to make a coordinate transformation to $T = W - \frac{1}{R}\psi$:

$$\frac{9}{2}G = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4R & 5R & 0 & 0 \\ 0 & 0 & 5R & -4R & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{9}{R} \\ 0 & 0 & 0 & 0 & \frac{9}{R} & 0 \end{pmatrix}. \tag{6.3.8}$$

This form of the T-dual metric makes it clear that the T-dual geometry is four-dimensional and that the Buscher rules, which in this case amount to $R \to 1/R$, are obeyed.

Chapter 7

The Semi Chiral Vector Multiplet

The discussion of T duality given in the previous section has a very unpleasant feature, the need to introduce a higher dimensional auxiliary space in order to describe the T dual background. While the auxiliary space is definitely a generalized Kähler geometry, it is unclear if the effective geometry is generalized Kähler. These problems can be traced directly to the vector multiplet used to implement T duality. Using the chiral vector multiplet is inadequate for the purpose of describing T duality because one cannot entirely gauge away a semi chiral superfield whose gauge transformation is to shift by a chiral superfields. This leads us to ask the question "Is there a vector multiplet with prepotentials that shift by semi chiral superfields?". We begin our investigation by noticing that the known vector multiplets, the chiral and twisted chiral vector multiplets, have a direct relationship between the gauge transformation parameter representation and the constraints consistent with the gauged supercovariant derivative algebra. Specifically, the gauged supercovariant derivative algebra for the chiral vector multiplets is consistent with setting covariantly chiral constraints and has chiral gauge transformation parameters. The analogous statement is true for the twisted chiral vector multiplet. This suggests that we look for a gauged supercovariant derivative algebra that is consistent with setting covariantly semi chiral constraints. It should be noted that both the chiral and twisted chiral vector multiplets are already consistant with imposing covariantly semi chiral constraints. That is the covariant way of stating that one can couple semi chiral superfields to the chiral or twisted chiral vector multiplets. Since the semi chiral constraint is weaker than the chiral or twisted chiral constraint, we should look for an gauged supercovariant derivative algebra that isn't as constrained as the chiral or twisted chiral vector multiplet. Said another way, we want to look for a gauged supercovariant derivative algebra that is only compatible with imposing semi chiral constraints. In this section we will describe such a vector multiplet, the semi chiral vector multiplet. We will give the gauged supercovariant derivative algebra, the kinetic terms for the multiplet and describe how to couple the multiplet to semi chiral matter using the prepotentials. We will then give comments on describing T duality using the semi chiral vector multiplet.

7.1 The Algebra and B.I.'s

We start by introducing gauge supercovariant derivatives $\nabla_A = D_A - i\Gamma_A t$ where Γ_A is the supergauge field and t is the abstract generator of the $U(1)$ symmetry we wish to gauge. We then impose the following constraints on the gauge supercovariant derivative algebra. For conventional constraints we impose the condition

$$(\gamma_a)^{\alpha\beta}[\nabla_\alpha, \bar{\nabla}_\beta\} = -4i\nabla_a \tag{7.1.1}$$

The constraints that preserve semi chiral representations are

$$(\gamma_a)^{\alpha\beta}[\nabla_\alpha, \nabla_\beta\} = 0. \tag{7.1.2}$$

The algebra and bianchi identites for the above constraints are

$$
\begin{aligned}
&[\nabla_\alpha, \nabla_\beta\} = 4ig(\gamma^3)_{\alpha\beta}\bar{T}t \\
&[\nabla_\alpha, \bar{\nabla}_\beta\} = 2i(\gamma^c)_{\alpha\beta}\nabla_c + 2g[C_{\alpha\beta}S - i(\gamma^3)_{\alpha\beta}P]t \\
&[\nabla_\alpha, \nabla_b\} = g(\gamma_b)_\alpha{}^\beta \bar{W}_\beta t - g(\gamma^3\gamma_b)_\alpha{}^\beta \bar{\Omega}_\beta t \\
&[\nabla_a, \nabla_b\} = -ig\epsilon_{ab}\mathcal{W}t
\end{aligned}
\tag{7.1.3}
$$

and

$$
\begin{aligned}
\nabla_\alpha S &= -i\bar{W}_\alpha \\
\nabla_\alpha P &= -(\gamma^3)_\alpha{}^\beta \bar{W}_\beta \\
\bar{\nabla}_\alpha T &= 0 \\
\nabla_\alpha T &= \Omega_\alpha \\
\nabla_\alpha \Omega_\beta &= -C_{\alpha\beta}\sigma \\
\nabla_\alpha \bar{\Omega}_\beta &= 2i(\gamma^a)_{\alpha\beta}\nabla_a \bar{T} \\
\nabla_\alpha \bar{W}_\beta &= 0 \\
\nabla_\alpha W_\beta &= iC_{\alpha\beta}d - (\gamma^3)_{\alpha\beta}(\sigma_1 + \mathcal{W}) + (\gamma^a)_{\alpha\beta}\nabla_a S - i(\gamma^3\gamma^a)_{\alpha\beta}\nabla_a P \\
\nabla_\alpha d &= (\gamma^a)_\alpha{}^\beta \nabla_a \bar{W}_\beta \\
\nabla_\alpha \sigma &= 0 \\
\bar{\nabla}_\alpha \sigma &= 2i(\gamma^a)_\alpha{}^\beta \nabla_a \Omega_\beta
\end{aligned}
\tag{7.1.4}
$$

It is of interest to note that the B.I.s require that T is chiral and $\Pi = S - iP$ is twisted chiral. At first glance one might think that this algebra is direct sum of the algebra's for a chiral and twisted chiral vector multiplet. That this isn't the case can be seen in at least two different ways. The first is mixing of the auxiliarly field

σ in the B.I.'s, specifically in the $\nabla_\alpha W_\beta$ and $\nabla_\alpha \Omega_\beta$ terms. The discussion of the second argument is better suited to take place after the discussion of prepotentials.

7.2 Prepotentials

The description given above is an off shell description and thus the field strengths can be solved for in terms of unconstrained prepotentials. To find the prepotentials we consider the representation preserving constraint (7.1.2) and see what they imply for the potentials Γ_α. In terms of the super field strengths we have

$$\begin{aligned} F_{++} &= 2D_+\Gamma_+ = 0 \rightarrow \Gamma_+ = D_+\bar{V}_1 \\ F_{--} &= 2D_-\Gamma_- = 0 \rightarrow \Gamma_- = D_-\bar{V}_2 \end{aligned} \tag{7.2.1}$$

The prepotentials have two types of gauge transformation. Since the super field strengths are invariant under $\Gamma_A \rightarrow \Gamma_A + D_A L$ where L is an arbitrary real superfield, this implies that V_1 and V_2 share a common gauge transformation

$$V_1 \rightarrow V_1 + L, \qquad V_2 \rightarrow V_2 + L \tag{7.2.2}$$

V_1 and V_2 also have a priori independent gauge transformations. For $\bar{D}_+\Lambda = 0$ and $\bar{D}_- U = 0$ the super field strengths are invariant under the transformations

$$V_1 \rightarrow V_1 + \Lambda \tag{7.2.3}$$

and

$$V_2 \rightarrow V_2 + U \tag{7.2.4}$$

Here we see that we have found a vector multiplet with prepotentials that shift by semi chiral superfields under gauge transformations. At this point we can also give the second argument as to why the semi chiral vector multiplet can't be obtained as a direct sum of the chiral and twisted chiral vector multiplets. Recall for the chiral vector multiplet that after fixing the gauge symmetry parameterized by the analog of L gauge transformation, it has only one real prepotential. The same is true for the twisted chiral vector multiplet. One would expect that a direct sum of the chiral and twisted chiral vector multiplet would be described in terms of two real prepotentials. However, for the semi chiral vector multiplet given above, there are three real prepotentials after L gauge fixing.

The solutions for the field strengths in terms of the prepotentials are

$$\begin{aligned} T &= \frac{1}{4}\bar{D}^2(V_2 - V_1) \\ \bar{T} &= \frac{1}{4}D^2(\bar{V}_2 - \bar{V}_1) \\ \Pi &= S - iP = \frac{1}{2}D_+\bar{D}_-(V_2 - \bar{V}_1) \\ \bar{\Pi} &= S + iP = \frac{1}{2}D_-\bar{D}_+(\bar{V}_2 - V_1) \end{aligned} \tag{7.2.5}$$

7.3 Duality between Chiral and Twisted Chiral Vector multiplets

While the semi chiral vector multiplet isn't reducible in terms of a chiral and twisted chiral vector multiplet, it contains both the chiral and twisted chiral vector multiplet. This can be seen in the following way. Starting with equation (7.1.3) and setting the field strength $\bar{T} = 0$, one finds that the B.I.'s require that $\Omega_\alpha = \sigma = 0$. The resulting algebra and B.I.'s are identical to the those for the chiral vector multiplet [41]. Similarly if one sets $S = P = 0$ then the B.I.'s require that $W_\alpha = d = 0$ and $\sigma_1 = -\mathcal{W}$. This then gives the algebra and B.I.'s for the twisted chiral vector multiplet. In this way we can view the semi chiral vector multiplet as the parent multiplet that gives rise to the chiral and twisted chiral vector multiplet. This isn't very surprising in hind sight. The semi chiral constraint is weaker that the chiral or twisted chiral constraint. It is only the zero modes allowed for a massless representation that distinguishes a semi chiral superfield from the sum of a chiral and twisted chiral superfield. From this point of view, one could expect the semi chiral vector multiplet to incorporate both the chiral and twisted chiral vector multiplets in its structure. Then setting the field strengths to zero in the way described above is just how one enlarges the types of constraints that can be imposed on matter representations. The observed duality between the chiral and twisted chiral superfields can be seen as the origin of the mirror nature between chiral and twisted chiral vector multiplets described in [41].

7.4 The Gauge Field Action

At this point we have only established that the representation is irreducible. We need to find the action that governs the dynamics for the multiplet. We can guess the form of the action on dimensional grounds. Since $[d^4\theta] = 2$, then the action must be a function of dimensionless fields. Since the action must also be gauge invariant, this suggest that we can use the mass dimension zero field strengths from the algebra S, P, and T. What is particularly nice is that we will see the mechanism the theory uses to demonstrate that it is not a direct sum of the chiral and twisted chiral vector multiplets. Consider the following actions

$$S_1 = -\frac{1}{4}\int d^2x d^4\theta S^2 \tag{7.4.1}$$

and

$$S_2 = \frac{1}{2}\int d^2x d^4\theta \bar{T}T \tag{7.4.2}$$

Both actions are manifestly supersymmetric since they are written directly in superspace. However both terms are necessary in order to obtain the field strength squared term $\mathcal{W}^2$ in the action. Lets see how that works. Evaluating the Grassmann measure with

$$\int d^4\theta = \frac{1}{8}[\nabla^\alpha\nabla_\alpha\bar{\nabla}^\beta\bar{\nabla}_\beta + \bar{\nabla}^\beta\bar{\nabla}_\beta\nabla^\alpha\nabla_\alpha], \tag{7.4.3}$$

we get the component actions

$$S_1 = \frac{1}{2}\int d^2x[2i(\bar{\lambda}^\beta)(\gamma^a)_\beta{}^\alpha\nabla_a(\lambda_\alpha) + \nabla^a S\nabla_a S + \nabla^a P\nabla_a P + (\sigma_1 + \mathcal{W})^2 + d^2] \tag{7.4.4}$$

where $W_\alpha|=\lambda_\alpha$ and

$$S_2 = \frac{1}{2}\int d^2x[\bar{\sigma}\sigma + 2i\bar{\rho}^\beta(\gamma^a)_\beta{}^\alpha\nabla_a\rho_\alpha + 4\nabla^a\bar{T}\nabla_a T] \tag{7.4.5}$$

with $\Omega_\alpha| = \rho_\alpha$. If we just used S_1, we would see the e.o.m for σ_1 would eliminate the presence of $\mathcal{W}$ in the action and thus the gauge field wouldn't have kinetic terms. We would get the same result if we just used S_2 for the more simple reason that $\mathcal{W}$ doesn't appear in the action. It is only the sum of the two terms, $S_1 + c_0 S_2$, that will generate kinetic terms for the gauge field. Since each action is separately supersymmetric, the remaining issue to settle is what should the relative coefficient be. Looking at the kinetic terms for the scalars we see that c_0 must be positive with no extra restriction from requiring the appropriate sign for the gauge field kinetic terms in this case $+\mathcal{W}^2$. For simplicity we set $c_0 = 1$ and consider the action

$$
\begin{aligned}
S &= \int d^2x d^4\theta[-\frac{1}{4}S^2 + \frac{1}{2}\bar{T}T]. \\
&= \frac{1}{2}\int d^2x[2i(\bar{\lambda}^\beta)(\gamma^a)_\beta{}^\alpha\nabla_a(\lambda_\alpha) + \nabla^a S\nabla_a S + \nabla^a P\nabla_a P + (\sigma_1 + \mathcal{W})^2 + d^2 \\
&\qquad + \bar{\sigma}\sigma + 2i\bar{\rho}^\beta(\gamma^a)_\beta{}^\alpha\nabla_a\rho_\alpha + 4\nabla^a\bar{T}\nabla_a T]
\end{aligned}
\tag{7.4.6}
$$

7.5 Fayet-Iliopoulos terms

Since the semi chiral vector multiplet has three real prepotentials, it has room for three F.I. terms [56] in the action. They are given by

$$
\begin{aligned}
S_{FI} &= \int d^2x[aD^2T + \bar{a}\bar{D}^2\bar{T} + \bar{D}^\alpha D_\alpha(b\Pi + \bar{b}\bar{\Pi})] \\
&= 4\int d^2x[r_1(\sigma_1 + \frac{1}{2}\mathcal{W}) + r_2\sigma_2 + r_3 d].
\end{aligned}
\tag{7.5.1}
$$

The relations between the complex constants a, b and the real constants r_1, r_2, r_3 are

$$
\begin{aligned}
r_1 &= \frac{1}{4}(a + \bar{a}) + \frac{i}{2}(\bar{b} - b) \\
r_2 &= \frac{i}{4}(a - \bar{a}) \\
r_3 &= \frac{1}{2}(b + \bar{b})
\end{aligned}
\tag{7.5.2}
$$

7.6 Coupling to Matter

Coupling the new multiplet can be described in two ways. One is to evaluate the measure in terms of the covariant derivatives, push the derivatives onto the Kähler potential, and evaluate the fermionic derivatives acting on the matter superfields in terms of the covariantly defined components of the matter superfields. The other way is to use the prepotentials to adjust the local gauge transformations of the matter field to make the action invariant under the local transformations. Here we describe the second method because of its greater ability to describe the gauging of target space isometries in non linear sigma models. Lets recall the process for gauging chiral matter described in [37]. A chiral superfield transforms under the global transformation as

$$\Phi \to e^{i\epsilon t}\Phi, \quad \bar{\Phi} \to e^{i\epsilon t}\bar{\Phi} \tag{7.6.1}$$

where ϵ is the constant real transformation parameter. The kinetic terms for the chiral fields are given by the Kähler potential, $K = K(\bar{\Phi}, \Phi)$ which is invariant under the above transformations (and not to be confused with the gauge parameter discussed above). When the transformation is made local, the parameter ϵ is promoted to a chiral superfield and thus $\bar{\epsilon}$ is anti chiral. However, this means that $\bar{\Phi}$ no longer transforms with the same parameter as Φ and the invariance of the Kähler potential is lost. To restore the invariance we need to find a way to get $\bar{\Phi}$ and Φ to transform with the same transformation parameter. To do so we use the real prepotential[1] ,V, from the chiral vector multiplet which transforms as $\delta V = i(\bar{\epsilon} - \epsilon)$. We define a new field

$$\tilde{\bar{\Phi}} = e^{-Vt}\bar{\Phi}, \tag{7.6.2}$$

and replace $\bar{\Phi}$ in the Kähler potential with $\tilde{\bar{\Phi}}$, i.e. $K = K(\tilde{\bar{\Phi}}, \Phi)$ and we find that the potential is invariant under local transformations.

A similar procedure will be used to gauge the Kähler potential with left and

[1]This is actually the imaginary part of the complex prepotential for the chiral vector multiplet.

right semi chiral superfields, however we need to make a few adjustments. The Kähler potential is a function of the left and right semi chiral superfields, $K = K(\bar{X}, X, \bar{Y}, Y)$. It is invariant under the following transformations.

$$X \rightarrow e^{i\epsilon t}X, \quad \bar{X} \rightarrow e^{i\epsilon t}\bar{X}, \quad Y \rightarrow e^{i\epsilon t}Y, \quad \bar{Y} \rightarrow e^{i\epsilon t}\bar{Y}, \tag{7.6.3}$$

where once again ϵ is a constant real parameter. To make the transformation local we, as before, would look to promote ϵ a superfield. The issue is choosing the representation to use. The only consistent choice is to promote the parameter for each superfield to a parameter of the same representation. The transformations take the form

$$X \rightarrow e^{-i\Lambda t}X, \quad \bar{X} \rightarrow e^{-i\bar{\Lambda} t}\bar{X}, \quad Y \rightarrow e^{-iUt}Y, \quad \bar{Y} \rightarrow e^{-i\bar{U}t}\bar{Y}. \tag{7.6.4}$$

Once again the invariance of the Kähler potential is lost with the above local transformations. In order to restore the invariance we define new fields using the prepotentials, as before, that will transform properly to restore the invariance of the Kähler potential. This will happen in a way that looks different from chiral case.We recall that the prepotentials actually have two gauge transformations. We can use the left and right semi chiral transformation of the prepotentials to compensate for the local transformations and exchange them for L gauge transformations. We define new fields with the prepotentials transforming as in (7.2.3) and (7.2.4)

$$\begin{aligned}
\tilde{X} &= e^{iV_1 t}X \\
\tilde{\bar{X}} &= e^{i\bar{V}_1 t}\bar{X} \\
\tilde{Y} &= e^{iV_2 t}Y
\end{aligned}$$

$$\tilde{\bar{Y}} \;=\; e^{i\bar{V}_2 t}\bar{Y} \tag{7.6.5}$$

The new fields all transform with the same parameter and the invariance of the action is restored with the replacements

$$K(\bar{X}, X, \bar{Y}, Y) \to K(\tilde{\bar{X}}, \tilde{X}, \tilde{\bar{Y}}, \tilde{Y}). \tag{7.6.6}$$

The discussion of the gauged action via use of the prepotentials is completed by giving the gauge fixing conditions for the L gauge freedom and choosing the appropriate Wess Zumino gauge. To start we need to give the components for the left and right semi chiral transformation parameters.

$$\begin{aligned}
&\Lambda| = \lambda, && U| = u \\
&D_\alpha\Lambda| = \psi_\alpha, && D_\alpha U| = \chi_\alpha \\
&\bar{D}_-\Lambda| = \xi_-, && \bar{D}_- U| = 0 \\
&\bar{D}_+\Lambda| = 0, && \bar{D}_+ U| = \eta_+ \\
&D^2\Lambda| = F, && D^2 U| = G \\
&\bar{D}^2\Lambda = 0, && \bar{D}^2 U = 0 \\
&[D_+, \bar{D}_+]\Lambda = -i\partial_{\ddagger}\lambda, && [D_+, \bar{D}_+]U| = B_{\ddagger} \\
&[D_-, \bar{D}_-]\Lambda| = C_{=}, && [D_-, \bar{D}_-]U| = -i\partial_{=} u \\
&[D_-, \bar{D}_+]\Lambda| = 0, && [D_-, \bar{D}_+]U| = \theta' \\
&[D_+, \bar{D}_-]\Lambda| = \theta, && [D_+, \bar{D}_-]U| = 0 \\
&D^2\bar{D}_+\Lambda| = 0, && D^2\bar{D}_+ U| = \omega_+ \\
&D^2\bar{D}_-\Lambda| = \tau_-, && D^2\bar{D}_- U| = 0 \\
&\bar{D}^2 D_+\Lambda| = \partial_{\ddagger}\xi_-, && \bar{D}^2 D_+ U| = 0 \\
&\bar{D}^2 D_-\Lambda| = 0, && \bar{D}^2 D_- U| = \partial_{=}\eta_+
\end{aligned} \tag{7.6.7}$$

To perform the L gauge fixing we need to decompose the prepotentials into the linear combination of fields that transforms under the L gauge symmetry, and the orthogonal combinations that are inert under the L gauge symmetry. The combination that L gauge transforms is

$$\hat{V} = Re(V_1) + Re(V_2) \tag{7.6.8}$$

And the orthogonal combinations are

$$\begin{aligned} \tilde{V} &= Re(V_2) - Re(V_1) \\ \tilde{V}_1 &= Im(V_1) \\ \tilde{V}_2 &= Im(V_2) \end{aligned} \tag{7.6.9}$$

We use the L gauge to fix $\hat{V} = 0$. Then we consider the transformations of the remaining prepotentials components under the remaining gauge transformations to see which we can set to zero in the Wess Zumino gauge. We set to zero all of the fields that transform by a shift and here give the remaining components. The gauge field sits in $\tilde{V}_1$ and $\tilde{V}_2$ as

$$\begin{aligned} A_{\ddagger} &= -\frac{1}{4}(\gamma_{\ddagger})^{++}[D_+, \bar{D}_+]\tilde{V}_1| \\ A_{=} &= -\frac{1}{4}(\gamma_{=})^{--}[D_-, \bar{D}_-]\tilde{V}_2|. \end{aligned} \tag{7.6.10}$$

The remaining components that cannot be set to zero in the Wess Zumino gauge are related to the field strengths given in the algebra (7.1.3) and are given by

$$\begin{aligned} \frac{i}{4}\bar{D}^2(\tilde{V}_2 - \tilde{V}_1)| &= T| = T \\ \frac{i}{2}D_+\bar{D}_-(\tilde{V}_2 + \tilde{V}_1)| &= S - iP| = \pi \end{aligned}$$

$$
\begin{aligned}
\frac{i}{4}\bar{D}^2 D_\alpha(\tilde{V}_1+\tilde{V}_2)| &= -W_\alpha| = -\lambda_\alpha \\
\frac{i}{4}D_\alpha \bar{D}^2 \tilde{V}_2| &= \Omega_\alpha| = \rho_\alpha \\
\frac{1}{8}\{D^2,\bar{D}^2\}\tilde{V}| &= (\sigma_1 + \frac{1}{2}\mathcal{W})| \\
\frac{1}{8}\{D^2,\bar{D}^2\}(\tilde{V}_2-\tilde{V}_1)| &= \sigma_2| \\
\frac{1}{8}\{D^2,\bar{D}^2\}(\tilde{V}_2+\tilde{V}_1)| &= d| = d
\end{aligned} \tag{7.6.11}
$$

This completes the description of the Wess Zumino gauge.

7.7 Comments On T Duality With The Semi Chiral Vector Multiplet

The discussion of the coupling of the semi chiral vector multiplet above is the analog of the standard discussion of gauged sigma models for chiral and twisted chiral vector multiplets. In order to discuss T duality, we need to be able to gauge away the entire semi chiral superfield. The tranformations of the prepotentials, (7.2.3) and (7.2.4), allow the semi chiral vector multiplet to be gauged fixed in this way. This should allow for a formal discussion of T duality for semi chiral superfields analogous to the treatment given in [52]. We also have a description quotients since the construction of quotients is a special case of T duality. A formal development of this discussion is work in progress.

Chapter 8

Conclusion

In this work we have demonstrated the utility of using superspace techniques to study supersymmetric theories in two cases. In the first case we used superspace to derive the first order in α' corrections to the 10D $\mathcal{N} = 1$ supergravity low energy effective action for the heterotic string. We worked with both the gauge 2-form and gauge 6-form multiplets. This was accomplished by using the dynamical equations implied by the super Jacobi identities for the supergravity covariant derivatives to construct an action. We then demonstrated that this action is equivalent to what is obtained via the Noether procedure. Along the way, we saw an interesting tensor appear, the $Y_{\underline{a}\,\underline{b}\,\underline{c}}$ tensor, that played the role of exchanging the group and form indices on the wedge product of the Ricci 2-form in the exterior derivative of the Chern Simons form. Though we argued that at linear order in α' this new tensor doesn't affect the effective action, it remains to be seen if this stays true at second order. It would interesting to investigate the geometrical significance of the $Y_{\underline{a}\,\underline{b}\,\underline{c}}$ tensor on general grounds. One can potentially obtain other "R^2" terms by considering additional terms for the auxiliary field $A_{\underline{a}\,\underline{b}\,\underline{c}}$. This could generate the complete supersymmetric completion of the "R^2" terms. If one also turns on the Yang-Mills coupling, this procedure would generate the supersymmetric completion for this coupling as well.

In the second case, we used superspace techniques to derive a formalism for discussing T duality in the context of generalized Kähler geometries via the use of $\mathcal{N} = (2,2)$ supersymmetric non linear sigma models with semi chiral superfields. There we saw the beautiful way that manifest $(2,2)$ superspace encoded all of the information concerning the background into a single potential function. This was done by starting in $(2,2)$ superspace and performing a reduction to $(1,1)$ superspace with a non linearly realized extra $(1,1)$ supersymmetry. All of the geometric information associated to generalized Kähler geometry with an isometry was captured very naturally by superspace without the need to give any input except for what scalar superfield representations we would use. The moment map and one form u associated to the presence of a B field were obtained. Working directly in $(2,2)$ superspace we used the chiral vector multiplet to give a formulation of T duality in terms of the sigma model. This formulation has the advantage of giving an explicit dual potential, but at the cost of introducing extra fields. These extra fields are a highly unwanted feature as they require us to think of the dual model as being embedded in degenerate auxiliary space. The problem of introducing extra fields was traced back to the fact that the chiral vector multiplet can not gauge away a semi chiral superfield. This lead to an investigation that resulted in a previously unknown $(2,2)$ vector multiplet, the semi chiral vector multiplet. This vector contains the degrees of freedom necessary to completely gauge away a semi chiral superfield. The formulation of T duality for sigma models with semi chiral superfields using the semi chiral vector multiplet is work in progress. After establishing a complete description of T duality using the semi chiral vector multiplet another important issue must be addressed. The discussion of T duality in this work has been entirely classical. However, for T duality to be a symmetry of string theory it must persist at the quantum level. Along the way, we also used the derivation of the moment map in the sigma model language to investigate a mathematical definition of the moment map in terms of eigenspaces of generalized complex structures. In the cases where the complex structures of the generalized Kähler geometries commute, we were able to show complete agreement with sigma model derivation of the moment map and the mathematical definiton. We were unable to obtain conclusive results in the case where the complex structures do not commute.

Chapter A

Appendix A: 10D Definitions & Conventions

The basic tool we use is ten dimensional chiral superspace with structure group $SO(1,9)$. Definitions and properties (such as multiplication table and Fierz identities) of ten dimensional chiral sigma matrices we adopted here can be found in [57]. Given the super frame $E^{\mathcal{A}} = (E^{\underline{a}},\ E^{\alpha})$, conventions for superforms and Leibniz rule for the exterior derivative are

$$\omega \;=\; \frac{1}{p!}E^{\mathcal{A}_1}...E^{\mathcal{A}_p}\,\omega_{\mathcal{A}_p...\mathcal{A}_1}\,, \tag{A.0.1}$$

$$\mathrm{d}(\omega_p\omega_q) \;=\; \omega_p(\mathrm{d}\omega_q) + (-)^q(\mathrm{d}\omega_p)\omega_q\,. \tag{A.0.2}$$

Representation matrices acting on the tangent space are blockdiagonal,

$$X \;=\; \begin{pmatrix} X_b{}^a & 0 \\ 0 & X_\beta{}^\alpha \end{pmatrix}, \tag{A.0.3}$$

and the vectorial and spinorial representations are related by the two-index sigma matrix,

$$X_\alpha{}^\beta \;=\; \frac{1}{4}(\sigma^{\underline{a}\,\underline{b}})_\alpha{}^\beta X_{\underline{a}\,\underline{b}}, \qquad X_{\underline{a}\,\underline{b}} \;=\; -\frac{1}{8}(\sigma_{\underline{a}\,\underline{b}})_\alpha{}^\beta X_\beta{}^\alpha. \tag{A.0.4}$$

As soon as the action of the structure group is fixed,

$$\delta E \;=\; \beta E X, \tag{A.0.5}$$

the covariant derivative

$$\nabla E \;=\; \mathrm{d}E \,+\, \alpha E \Omega \tag{A.0.6}$$

can be defined using the Lorentz connection Ω with transformation law

$$\delta\Omega \;=\; -\beta\left(\mathrm{d}X \,+\, \alpha X\cdot\Omega\right)\,. \tag{A.0.7}$$

The torsion T, the curvature $\mathcal{R}$ and field strengths F_p of an abelian $(p-1)$-form are defined by

$$\nabla E \;=\; \gamma T, \qquad \mathcal{R} \;=\; \mathrm{d}\Omega + \alpha\Omega\Omega, \qquad F_p \;=\; \mathrm{d}A_{p-1}, \tag{A.0.8}$$

and they satisfy the following Bianchi identities

$$\gamma\nabla T \;=\; \alpha E\mathcal{R}, \qquad \nabla\mathcal{R} \;=\; 0, \qquad \mathrm{d}F_p \;=\; 0. \tag{A.0.9}$$

The curvature in particular appears in the double covariant derivative of covariant vectors

$$\nabla\nabla u \;=\; \alpha u\mathcal{R}. \tag{A.0.10}$$

Dragon's theorem states that in supergravity the Bianchi identity for the torsion together with (A.0.10) implies that the Bianchi identity for the curvature is automatically satisfied.

The Chern-Simons form

$$Q \;=\; \mathrm{tr}\left(\Omega\mathcal{R} - \frac{\alpha}{3}\Omega\Omega\Omega\right) \tag{A.0.11}$$

satisfies

$$\mathrm{d}Q \;=\; \mathrm{tr}(\mathcal{R}\mathcal{R})\,. \tag{A.0.12}$$

Finally, let us consider a redefinition

$$\Omega \;=\; \hat{\Omega} + \chi \tag{A.0.13}$$

of the connection. This shift in the connection affects the torsion, the curvature and the Chern-Simons form in the following way:

$$\begin{aligned} \gamma(T-\hat{T}) &= \alpha E\chi \quad , && \text{(A.0.14)} \\ \mathcal{R}-\hat{\mathcal{R}} &= \nabla\chi-\alpha\chi\chi \quad , && \text{(A.0.15)} \\ Q-\hat{Q} &= \mathrm{tr}\left(2\mathcal{R}\chi-\chi\nabla\chi+\frac{2\alpha}{3}\chi\chi\chi+\mathrm{d}(\Omega\chi)\right) \quad . && \text{(A.0.16)} \end{aligned}$$

Let us display the above relations in terms of form-components. First of all, (A.0.10) gives the algebra of covariant derivatives acting on covariant vectors

$$(\nabla_P,\nabla_{\mathcal{B}})\,u^{\mathcal{A}} = -\gamma T_{P\mathcal{B}}{}^{\mathcal{F}}\nabla_{\mathcal{F}}u^{\mathcal{A}}+\alpha\mathcal{R}_{P\mathcal{B}\mathcal{F}}{}^{\mathcal{A}}u^{\mathcal{F}}\,. \tag{A.0.17}$$

The Bianchi identities become

$$\begin{aligned} \gamma\nabla_{(\nabla}T_{P\mathcal{B})}{}^{\mathcal{A}}+\gamma^2T_{(\nabla P|}{}^{\mathcal{F}}T_{\mathcal{F}|\mathcal{B})}{}^{\mathcal{A}}-\alpha\mathcal{R}_{(\nabla P\mathcal{B})}{}^{\mathcal{A}} &= 0 && \text{(A.0.18)} \\ \nabla_{(\mathcal{A}_1}F_{\mathcal{A}_2\ldots\mathcal{A}_{p+1})}+\gamma\frac{p}{2}T_{(\mathcal{A}_1\mathcal{A}_2|}{}^{\mathcal{F}}F_{\mathcal{F}|\mathcal{A}_3\ldots\mathcal{A}_{p+1})} &= 0\,. && \text{(A.0.19)} \end{aligned}$$

The components of the Chern-Simons form are

$$Q_{\mathcal{A}\mathcal{B}P} = \mathrm{tr}\left(\frac{1}{2}\Omega_{(\mathcal{A}}\mathcal{R}_{\mathcal{B}P)}+\frac{\alpha}{3}\Omega_{(\mathcal{A}}\Omega_{\mathcal{B}}\Omega_{P)}\right), \tag{A.0.20}$$

while the redefinitions take the form

$$\begin{aligned} \gamma(T-\hat{T})_{P\mathcal{B}}{}^{\mathcal{A}} &= \alpha\chi_{(P\mathcal{B})}{}^{\mathcal{A}} \quad , \\ (\mathcal{R}-\hat{\mathcal{R}})_{\mathcal{B}\mathcal{A}} &= \nabla_{(\mathcal{B}}\chi_{\mathcal{A})}+\gamma T_{\mathcal{B}\mathcal{A}}{}^{\mathcal{F}}\chi_{\mathcal{F}}+\alpha\chi_{(\mathcal{B}}\chi_{\mathcal{A})} \quad , \\ (Q-\hat{Q})_{P\mathcal{B}\mathcal{A}} &= \mathrm{tr}\Big[\,\mathcal{R}_{(P\mathcal{B}}\chi_{\mathcal{A})}-\chi_{(P}\left(\nabla_{\mathcal{B}}\chi_{\mathcal{A})}+\frac{\gamma}{2}T_{\mathcal{B}\mathcal{A})}{}^{\mathcal{F}}\chi_{\mathcal{F}}+\frac{2\alpha}{3}\chi_{\mathcal{B}}\chi_{\mathcal{A})}\right) \quad , \\ &\quad -\nabla_{(P}(\Omega_{\mathcal{B}}\chi_{\mathcal{A})}-\frac{\gamma}{2}\Omega_{(\mathcal{F}}\chi_{(\mathcal{A})}T_{P\mathcal{B})}{}^{\mathcal{F}}\Big] \quad . \end{aligned} \tag{A.0.21}$$

The conventions of Wess and Bagger correspond to the choice $\alpha = 1$, $\gamma = 1$, while the conventions in [9] correspond to $\alpha = -1$, $\gamma = -1$. Also, the Chern-Simons term denoted by X in [9] is $X = -Q$.

The graviton and gravitino is identified in the super frame $E^{\mathcal{A}} = (E^{\underline{a}},\, E^{\alpha})$,

$$E^{\underline{a}}| \;=\; \mathrm{d}x^{\underline{m}} \mathrm{e}_{\underline{m}}{}^{\underline{a}}, \qquad E^{\alpha}| \;=\; \frac{1}{2}\mathrm{d}x^{\underline{m}} \psi_{\underline{m}}{}^{\alpha}. \tag{A.0.22}$$

The torsion, $T \;=\; -\nabla E$, satisfies the Bianchi identity

$$\nabla T \;=\; E\mathcal{R}. \tag{A.0.23}$$

The two-form gauge potential of the pure 10 dimensional supergravity multiplet is identified in a two-form on the superspace

$$B| \;=\; \frac{1}{2}\mathrm{d}x^{\underline{m}}\mathrm{d}x^{\underline{n}} B_{\underline{n}\,\underline{m}}. \tag{A.0.24}$$

Its fieldstrengths $G = \mathrm{d}B$ satisfies the Bianchi identity

$$\mathrm{d}G \;=\; 0. \tag{A.0.25}$$

The Green-Schwarz mechanism teaches us that in order to deal with anomaly free supergravity the field strength of the antisymmetric tensor has to be accompanied by both the Yang-Mills and gravitational Chern-Simons terms. Here we consider only the gravitational part.

$$Q \;=\; \mathrm{tr}(\mathcal{R}\Omega + \frac{1}{3}\Omega\Omega\Omega), \qquad \mathrm{d}Q \;=\; \mathrm{tr}(\mathcal{R}\mathcal{R}). \tag{A.0.26}$$

Therefore, it is convenient in general to define a new object on superspace,

$$H \;\doteq\; G \;+\; \gamma' Q, \tag{A.0.27}$$

and consider the Bianchi identity satisfied by this three-form H,

$$\mathrm{d}H \;= \gamma' \mathrm{tr}(\mathcal{R}\mathcal{R}). \tag{A.0.28}$$

The six-form gauge potential of the dual pure 10 dimensional supergravity multiplet is identified in a six-form on the superspace

$$M_{||} \;=\; \frac{1}{6!}\mathrm{d}x^{\underline{m}_1}...\mathrm{d}x^{\underline{m}_6} M_{\underline{m}_6\cdots\underline{m}_1}. \tag{A.0.29}$$

Its fieldstrengths $N = \mathrm{d}B$ satisfies the Bianchi identity

$$\mathrm{d}N \;=\; 0. \tag{A.0.30}$$

Chapter B

Appendix B: 10D Variations

For arbitrary variation of the connection $\delta\Omega$ the curvature squared terms and the Chern-Simons form Q change according to

$$\delta\,\mathrm{tr}\left(\mathcal{R}^{\underline{ab}}\,\mathcal{R}_{\underline{ab}}\right) \;=\; -4\,\mathrm{tr}\left[(\nabla_{\underline{a}}\,\mathcal{R}^{\underline{ab}})\,\delta\Omega_{\underline{b}}\right] \;+\; 4\,\partial_{\underline{m}}\Big(\mathrm{e}_{\underline{a}}{}^{\underline{m}}\,\mathrm{tr}(\mathcal{R}^{\underline{ab}}\,\delta\Omega_{\underline{b}})\Big) \quad , \tag{B.0.1}$$

$$\delta Q \;=\; \mathrm{tr}\,[\,2\mathcal{R}\delta\Omega \,+\, \mathrm{d}(\Omega\delta\Omega)\,] \quad . \tag{B.0.2}$$

The scalar curvature transforms also:

$$\delta\mathcal{R} \;=\; \mathrm{e}_{\underline{a}}{}^{\underline{m}}\,\mathrm{e}_{\underline{b}}{}^{\underline{n}}\,\delta\mathcal{R}_{\underline{m}\,\underline{n}}{}^{\underline{a}\,\underline{b}} \tag{B.0.3}$$

$$\;=\; 2\mathrm{e}_{\underline{a}}{}^{\underline{m}}\,\partial_{\underline{m}}(\delta\Omega_{\underline{b}}{}^{\underline{a}\,\underline{b}}) \;-\; T_{\underline{a}\,\underline{b}}{}^{\underline{c}}\,\delta\Omega_{\underline{c}}{}^{\underline{a}\,\underline{b}} \quad . \tag{B.0.4}$$

In the case where $\delta\Omega_{\underline{a}\,\underline{b}\,\underline{c}} = \frac{1}{2}\delta T_{\underline{a}\,\underline{b}\,\underline{c}}$ with totally antisymmetric torsion this yields $\delta\mathcal{R} = -\delta(\frac{1}{4}T_{\underline{a}\,\underline{b}\,\underline{c}}T^{\underline{a}\,\underline{b}\,\underline{c}})$. In particular this implies also that the combination $\mathcal{R} + \frac{1}{4}T_{\underline{a}\,\underline{b}\,\underline{c}}T^{\underline{a}\,\underline{b}\,\underline{c}}$ is independent of a redefinition (A.0.13) provided that χ is totally antisymmetric.

Using the above formulae one may compute the following variations with respect to an object $L_{\underline{abc}}$ appearing in the Lorentz connection as

$$\Omega_{\underline{a}\,\underline{b}\,\underline{c}} \;=\; \omega_{\underline{a}\,\underline{b}\,\underline{c}} \;-\; L_{\underline{a}\,\underline{b}\,\underline{c}}, \tag{B.0.5}$$

with ω the torsion free spin connection:

$$
\begin{aligned}
\mathrm{e}^{-1}\, e^{4\Phi}\delta_L \mathrm{tr}\left(\mathcal{R}^{\underline{a}\,\underline{b}}\,\mathcal{R}_{\underline{a}\,\underline{b}}\right) \;&\sim\; -4\mathrm{e}^{-1}\nabla^{\underline{a}}(\, e^{4\Phi}\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}})\,\delta L^{\underline{b}\,\underline{c}\,\underline{d}} \\
&\quad +\; 4\mathrm{e}^{-1}\, e^{4\Phi}\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\, L^{\underline{a}\,\underline{b}}{}_{\underline{k}}\,\delta L^{\underline{k}\underline{c}\,\underline{d}} \\
&\quad +\; \mathcal{O}(\gamma') \quad , \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.0.6)} \\
-\frac{2}{3}\mathrm{e}^{-1}\, e^{4\Phi} L^{\underline{a}\,\underline{b}\,\underline{c}}\delta_L Q_{\underline{a}\,\underline{b}\,\underline{c}} \;&\sim\; 4\mathcal{E}_{B_{\underline{kl}}}\Omega_{\underline{k}}{}^{\underline{a}\,\underline{b}}\,\delta L_{\underline{l}\,\underline{a}\,\underline{b}} \;-\; 4\mathrm{e}^{-1}\, e^{4\Phi}\,\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\, L^{\underline{a}\,\underline{b}}{}_{\underline{k}}\delta L^{\underline{k}\,\underline{c}\,\underline{d}} \\
&\quad +\; \mathcal{O}(\gamma') \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.0.7)} \\
-\frac{2}{3}\mathrm{e}^{-1}\, e^{4\Phi} L^{\underline{a}\,\underline{b}\,\underline{c}}\delta_L Y_{\underline{a}\,\underline{b}\,\underline{c}} \;&=\; 4\, e^{4\Phi}(\mathcal{R}_{\underline{a}\,\underline{bc}\,\underline{d}} + \mathcal{R}_{\underline{c}\,\underline{da}\,\underline{b}})L_{\underline{k}}{}^{\underline{a}\,\underline{b}}\,\delta L^{\underline{k}\,\underline{c}\,\underline{d}} \\
&\quad -\frac{2}{3}\, e^{4\Phi} L_{\underline{a}\,\underline{b}}{}^{\underline{k}}\, L_{\underline{c}\,\underline{d}\,\underline{k}}\delta_L\left(\mathcal{E}_{\widetilde{B}_{\underline{abcd}}}\right) \;+\; \mathcal{O}(\gamma'). \qquad \text{(B.0.8)}
\end{aligned}
$$

However, the first term in the variation (B.0.6) may be recast in the form

$$
\begin{aligned}
-4\left[\nabla^{\underline{a}}(\, e^{4\Phi}\mathcal{R}_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}})\right]\delta L^{\underline{b}\,\underline{c}\,\underline{d}} \;&\sim\; -\,4\, e^{4\Phi}(\mathcal{R}_{\underline{a}\,\underline{bc}\,\underline{d}} \;+\; \mathcal{R}_{\underline{c}\,\underline{da}\,\underline{b}})L_{\underline{k}}{}^{\underline{a}\,\underline{b}}\,\delta L^{\underline{k}\,\underline{c}\,\underline{d}} \\
&\quad + 8\left[e^{4\Phi}\nabla^{\underline{a}}\left(e^{-4\Phi}\hat{\mathcal{E}}_{B_{\underline{b}\,\underline{c}}}\right)\right]\,\delta L_{\underline{ab}\,\underline{c}} \\
&\quad + 8\left[\left(e^{4\Phi}\hat{\mathcal{E}}_{\eta_{\underline{ak}}} \;-\; \hat{\mathcal{E}}_{B_{\underline{ak}}}\right)L_{\underline{k}}{}^{\underline{b}\,\underline{c}}\right]\delta L_{\underline{ab}\,\underline{c}} \\
&\quad + \frac{2}{3}\, e^{4\Phi}\left[\; L_{\underline{a}\,\underline{b}}{}^{\underline{k}}\, L_{\underline{c}\,\underline{d}\,\underline{k}} \;-\; \frac{1}{4!}\mathcal{E}_{\widetilde{B}^{\underline{abcd}}}\;\right]\,\delta_L\left(\mathcal{E}_{\widetilde{B}_{\underline{abcd}}}\right) \\
&\quad + \mathcal{O}(\gamma'). \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(B.0.9)}
\end{aligned}
$$

Now observe that the sum of the variations written above is expressed as a combination of the equations we derived from superspace geometry. We denote this combination of variations simbolically by $f(\mathcal{E})$:

$$f(\mathcal{E}) \doteq \mathrm{e}^{-1}\, e^{4\Phi} \delta_L \mathrm{tr}\left(\mathcal{R}^{\underline{a}\,\underline{b}} \mathcal{R}_{\underline{a}\,\underline{b}}\right) - \frac{2}{3} \mathrm{e}^{-1}\, e^{4\Phi}\, L^{\underline{a}\,\underline{b}\,\underline{c}}\, \delta_L\, (\, Q + Y\,)_{\underline{a}\,\underline{b}\,\underline{c}} \tag{B.0.10}$$

$$\begin{aligned} f(\mathcal{E}) \sim\;& 4\mathcal{E}_{B_{\underline{kl}}} \Omega_{\underline{k}}{}^{\underline{a}\,\underline{b}} \delta L_{\underline{l}\,\underline{a}\,\underline{b}} \\ &+ 8\left[e^{4\Phi} \nabla^{\underline{a}} \left(e^{-4\Phi} \hat{\mathcal{E}}_{B_{\underline{bc}}} \right) + \left(e^{4\Phi} \hat{\mathcal{E}}_{\eta_{\underline{ak}}} - \hat{\mathcal{E}}_{B_{\underline{ak}}} \right) L_{\underline{k}}{}^{\underline{b}\,\underline{c}} \right] \delta L_{\underline{a}\,\underline{b}\,\underline{c}} \\ &- \frac{2}{3}\, e^{4\Phi} \frac{1}{4!} \mathcal{E}_{\widetilde{B}^{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}} \delta_L \left(\mathcal{E}_{\widetilde{B}_{\underline{abcd}}} \right) \qquad \text{(B.0.11)} \\ &+ \mathcal{O}(\gamma'). \qquad \text{(B.0.12)} \end{aligned}$$

Therefore the superspace equations imply the vanishing of the above combination for an arbitrary variation of the object $L_{\underline{abc}}$. In particular, this is valid at zero order in γ' both for the anomaly free supergravity and for its dual.

Chapter C

Appendix C: 2D Spinor Conventions

$$\eta_{ab} = (1, -1) \quad , \quad \epsilon_{ab}\epsilon^{cd} = -\delta_{[a}{}^{c}\delta_{b]}{}^{d} \quad , \quad \epsilon^{01} = +1$$

$$(\gamma^a)_\alpha{}^\gamma(\gamma^b)_\gamma{}^\beta = \eta^{ab}\delta_\alpha{}^\beta - \epsilon^{ab}(\gamma^3)_\alpha{}^\beta \quad . \tag{C.0.1}$$

The last relation implies

$$\gamma^a\gamma_a = 2\mathbf{I} \quad , \quad \gamma^3\gamma^a = -\epsilon^{ab}\gamma_b \quad . \tag{C.0.2}$$

Some useful Fierz identites are

$$\begin{gathered} C_{\alpha\beta}C^{\gamma\delta} = \delta_{[\alpha}{}^{\gamma}\delta_{\beta]}{}^{\delta} \quad , \\ (\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} + (\gamma^3)_{\alpha\beta}(\gamma^3)^{\gamma\delta} = -\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \quad , \\ (\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} + (\gamma^3)_{(\alpha}{}^{\gamma}(\gamma^3)_{\beta)}{}^{\delta} = \delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \quad , \\ (\gamma^a)_{(\alpha}{}^{\gamma}(\gamma_a)_{\beta)}{}^{\delta} = -2(\gamma^3)_{\alpha\beta}(\gamma^3)^{\gamma\delta} \quad , \\ 2(\gamma^a)_{\alpha\beta}(\gamma_a)^{\gamma\delta} + (\gamma^3)_{(\alpha}{}^{\gamma}(\gamma^3)_{\beta)}{}^{\delta} = -\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \quad , \\ (\gamma_a)_\alpha{}^\delta\delta_\beta{}^\gamma + (\gamma^3\gamma_a)_\alpha{}^\gamma(\gamma^3)_\beta{}^\delta = (\gamma^3\gamma_a)_{\alpha\beta}(\gamma^3)^{\gamma\delta} \end{gathered} \tag{C.0.3}$$

For an explicit representation we chose to represent the γ-matrices in terms of the Pauli matrices as

$$(\gamma^0)_\alpha{}^\beta = (\sigma^2)_\alpha{}^\beta \quad , \quad (\gamma^1)_\alpha{}^\beta = -i(\sigma^1)_\alpha{}^\beta \quad , \quad (\gamma^3)_\alpha{}^\beta = (\sigma^3)_\alpha{}^\beta \tag{C.0.4}$$

The spinor metric $C_{\alpha\beta}$ and its inverse $C^{\alpha\beta}$ can be identified as

$$C_{\alpha\beta} \equiv (\sigma^2)_{\alpha\beta} \quad , \quad C^{\alpha\beta} \equiv -(\sigma^2)^{\alpha\beta} \tag{C.0.5}$$

The exlicit representation imply the following symmetry properties

$$\begin{aligned}
(\gamma^a)_{\alpha\beta} = (\gamma^a)_{\beta\alpha} \quad &, \quad (\gamma^3)_{\alpha\beta} = (\gamma^3)_{\beta\alpha} \quad , \quad C_{\alpha\beta} = -C_{\beta\alpha} \\
(\gamma^a)^{\alpha\beta} = (\gamma^a)^{\beta\alpha} \quad &, \quad (\gamma^3)^{\alpha\beta} = (\gamma^3)^{\beta\alpha} \quad , \quad C^{\alpha\beta} = -C^{\beta\alpha}
\end{aligned} \tag{C.0.6}$$

The complex conjugation rules follow from the explicit representation

$$[(\gamma^a)_\alpha{}^\beta]^* = -(\gamma^a)_\alpha{}^\beta \quad , \quad [(\gamma^3)_\alpha{}^\beta]^* = (\gamma^3)_\alpha{}^\beta \tag{C.0.7}$$

$$\begin{aligned}
[(\gamma^a)_{\alpha\beta}]^* = (\gamma^a)_{\alpha\beta} \quad &, \quad [(\gamma^3)_{\alpha\beta}]^* = -(\gamma^3)_{\alpha\beta} \quad , \quad [C_{\alpha\beta}]^* = -C_{\alpha\beta} \\
[(\gamma^a)^{\alpha\beta}]^* = (\gamma^a)^{\alpha\beta} \quad &, \quad [(\gamma^3)^{\alpha\beta}]^* = -(\gamma^3)^{\alpha\beta} \quad , \quad [C^{\alpha\beta}]^* = -C^{\alpha\beta}
\end{aligned} \tag{C.0.8}$$

The $\mathcal{N} = (2, 2)$ supercovariant derivative algebra in the complex basis is

$$\begin{aligned}
[D_\alpha, D_\beta\} &= 0 \\
[D_\alpha, \bar{D}_\beta\} &= 2i(\gamma^a)_{\alpha\beta}\partial_a
\end{aligned} \tag{C.0.9}$$

Meaning of (p, q) supersymmetry In two dimensions a Dirac spinor is a two component complex spinor. In two dimensions one can impose both the Weyl and Majoranna conditions on spinors. This means that the irreducible representations of spinors are one component real left or right handed spinors. Supersymmetric theories are labeled by the number of left handed and right handed supercharges they posses. This is usually denoted by saying the theory has (p, q) supersymmetry, where p is the number of left handed supercharges and q is the number of right handed supercharges.

Chapter D

Appendix D: Algebraic Aspects of Superspace

The supersymmetry algebra is the extension of the Lie algebra for the Poincare group to a $\mathbb{Z}_2$ graded Lie algebra. The grading is in terms of even (bosonic) elements and odd (fermionic) elements, where even(odd) refers to the elements commuting(anti-commuting) property. We use a collective index for graded tensors denoted by capital roman letter i.e. $A = (a, \alpha)$ where a is vector (even) index and α is a spinor (odd) index. The exchange of order for two elements is determined by the grading of the elements. The even elements are assigned weight 0 and the odd elements are assigned weight 1. Then the exchange of order of two elements is given by

$$\mathcal{O}_1\mathcal{O}_2 = (-)^{w(1)w(2)}\mathcal{O}_2\mathcal{O}_1 \tag{D.0.1}$$

Derivatives act on products via a graded product rule.

$$\nabla_A(\mathcal{O}_1\mathcal{O}_2) = (\nabla_A\mathcal{O}_1)\mathcal{O}_2 + (-)^{Aw(1)}\mathcal{O}_1(\nabla_A\mathcal{O}_2), \tag{D.0.2}$$

where A is also used to denote the weight of the tensor. The extension of the Lie bracket is just to chose the regular commutator if one of the elements under consideration is even and the anti-commutator if both elements under consideration is odd. This is easily denoted by defining the graded Lie bracket as

$$[\mathcal{O}_1, \mathcal{O}_2\} = \mathcal{O}_1\mathcal{O}_2 - (-)^{w(1)w(2)}\mathcal{O}_2\mathcal{O}_1 \tag{D.0.3}$$

The major tool used in studying both topics in this dissertation is the consistency of the gauge supercovariant derivative algebra, either for supergravity or super Yang-Mills. The consistency is determined requiring that the covariant derivative algebra satisfy the extension of the Jacobi identity for lie algebras, the super Jacobi identity

$$(-)^{AC}[[\nabla_A, \nabla_B\}, \nabla_C\} + (-)^{BA}[[\nabla_B, \nabla_C\}, \nabla_A\} + (-)^{CB}[[\nabla_C, \nabla_A\}, \nabla_B\} = 0 \tag{D.0.4}$$

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