ABSTRACT

Title of dissertation: TWO EQUIVALENCE RELATIONS
IN SYMBOLIC DYNAMICS

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A G-shift of finite type (G-SFT) is a shift of finite type which commutes with
the continuous action of a finite group G. We classify irreducible G-SFTs up to
right closing almost conjugacy, answering a question of Bill Parry.

Then, we derive a computable set of necessary and sufficient conditions for the
existence of a homomorphism from one shift of finite type to another. We consider
an equivalence relation on subshifts, called weak equivalence, which was introduced
and studied by Beal and Perrin. We classify arbitrary shifts of finite type up to
weak equivalence.

TWO EQUIVALENCE RELATIONS IN SYMBOLIC DYNAMICS

by

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Dedication

To my parents Tim and Nancy, and to my partner John.

Acknowledgements

I wish to thank a number of people whose support was essential to the writing of this dissertation.

It was my great fortune to have Dr. Mike Boyle as my thesis advisor. Mike always had good suggestions for problems to work on, as well as possible approaches to solving those problems. Often I would come to him feeling discouraged, having made only feeble progress on a particular problem; he would instantly suggest a way to move forward with my ideas, or perhaps a new train of thought altogether. For a couple of years we worked on some truly challenging problems, but were unable to solve any of them completely. In this time, Mike taught me never to lose heart, and to learn from my failures—failure is part of being a mathematician. Leading by example, he taught me that success comes with optimism and perseverance. Thank you, Mike, for believing in me more than I did.

My work at Maryland was also shaped by my colleagues in the Research Interaction Team (RIT) in symbolic dynamics. Led by Mike Boyle, the RIT consisted of a small group of students and postdocs. Each week, we would meet to discuss our research and significant papers from our field. I would especially like to thank RIT participants Angela Desai, Todd Fisher, and Nick Long, who provided a sounding board for my ideas and taught me about the problems they were working on. Angela was especially helpful when I first joined the RIT. She had already mastered the fundamentals of symbolic dynamics and was willing to bring me up to speed. She initiated regular meetings with me outside of the RIT, in which we read and discussed the book, *An Introduction to Symbolic Dynamics and Coding*, by Lind and

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Chapter 1

Introduction

We work in symbolic dynamics, focusing on the classification theory of one-dimensional shifts of finite type. This classification theory is built on invariants, which enable us to describe the similarities and differences between systems.

We begin with a discussion of Bernoulli shifts. For a natural number n, assign to each element i of the set $\{0, \ldots, n-1\}$ a number p_i, such that $p_0 + \cdots + p_{n-1} = 1$. Let X be the set of sequences $x = (x_i)_{i \in \mathbb{Z}}$ where each $x_i \in \{0, \ldots, n-1\}$. Give to X the sigma algebra $\mathcal{A}$ which is the product of the algebra of subsets of $\{0, \ldots, n-1\}$. Define a probability measure μ on X, given by $\mu = \{p_0, \ldots, p_{n-1}\}^{\mathbb{Z}}$. The triple $(X, \mathcal{A}, \mu)$ is a probability space.

Define the *shift map* $\sigma : X \to X$ by $\sigma(x)_i = x_{i+1}$ for each $i \in \mathbb{Z}$. Then σ is a measure-preserving transformation of $(X, \mathcal{A}, \mu)$, and the dynamical system $(X, \mathcal{A}, \mu, \sigma)$ is a *Bernoulli shift*. As an example, the Bernoulli shift with $n = 2$ and $p_0 = p_1 = \frac{1}{2}$ corresponds to tossing a fair coin infinitely often (indexed by $\mathbb{Z}$).

In the 1950's, Kolmogorov and Sinai introduced measure-theoretic entropy as an isomorphism invariant for measure-preserving transformations on a probability space. They showed that the measure-theoretic entropy of a Bernoulli shift (as above) is $-\sum_{i=0}^{n-1} p_i \log p_i$. It follows that not all Bernoulli shifts are measurably isomorphic, as Bernoulli shifts defined by different probabilities p_i typically do not

have the same measure-theoretic entropy. Then in 1970, Ornstein [36] proved that Bernoulli shifts of equal measure-theoretic entropy are measurably isomorphic. In other words, not only is measure-theoretic entropy an isomorphism invariant for Bernoulli shifts—it is a complete isomorphism invariant.

Often we think of the elements of $\{0, \ldots, n-1\}$ as states, and each coordinate x_k of a point $x \in X$ represents one unit of time spent at state x_k. For $k > 0$, we think of the state x_k as occurring in the future, x_{-k} as having occurred in the past, and x_0 as the present state of x. A Bernoulli shift X has the property that, given a state $i \in \{0, \ldots, n-1\}$, the probability that a point $x \in X$ has i as its present state is independent of the past or future states of x (the probability that $x_0 = i$ is simply p_i).

This motivates us to consider objects, more general than Bernoulli shifts, for which we tolerate a certain amount of dependence on the past. Let A be an $n \times n$ stochastic matrix. Then A is the adjacency matrix for the directed labeled graph $\mathcal{G}$ which has n vertices $\{v_0, \ldots, v_{n-1}\}$ and, for each $A(i, j) \neq 0$, there is a directed edge in $\mathcal{G}$ beginning at v_i and ending at v_j, labeled $A(i, j)$. We let X be the set of sequences $x = (x_i)_{i \in \mathbb{Z}}$ such that each x_i is a vertex in $\mathcal{G}$, and each $A(x_i, x_{i+1}) \neq 0$. Then, with the appropriate sigma algebra $\mathcal{A}$ and measure μ, $(X, \mathcal{A}, \mu, \sigma)$ is a measure-preserving transformation called a *Markov chain*.

Example 1.0.1. Let
$$A = \begin{pmatrix} .3 & .7 \\ 1 & 0 \end{pmatrix}.$$
Then A is the adjacency matrix for the following directed graph.
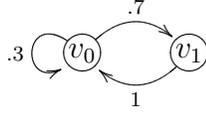
Notice here that the probability of a particular present state depends on the past. For example, if $x_{-1} = v_0$, then the probability of $x_0 = v_0$ is .3 whereas, if $x_{-1} = v_1$, then the probability of $x_0 = v_0$ is 1.

The basic building blocks of Markov chains are the *irreducible* Markov chains. Generalizing [36], Friedman and Ornstein [22] classify irreducible Markov chains up to measurable isomorphism in terms of measure-theoretic entropy and one additional invariant—*period*.

But not only are we interested in Markov chains as measurable objects. Also we may think of Markov chains topologically, as follows. For a Markov chain X and associated directed graph $\mathcal{G}$ having vertex set $\{v_0, \ldots, v_{n-1}\}$, give to $X \subset \{v_0, \ldots, v_{n-1}\}^{\mathbb{Z}}$ the topology $\mathcal{T}$ which is the relative of the product of the discrete topology on $\{v_0, \ldots, v_{n-1}\}$. Then $(X, \mathcal{T})$ is compact and metrizable, and the shift map $\sigma : (X, \mathcal{T}) \to (X, \mathcal{T})$ is a homeomorphism. $(X, \mathcal{T}, \sigma)$ is a topological dynamical system called a *shift of finite type* (SFT).

The topological analogue of measurable isomorphism is *topological conjugacy.* The problem of classifying SFTs up to topological conjugacy has turned out to be more difficult than the problem of classifying Markov chains up to measurable isomorphism. In fact, a complete set of invariants for topological conjugacy of irreducible SFTs has yet to be found. However there do exist a number of weaker equivalence relations up to which SFTs have been classified.

For an irreducible SFT X, there exists a Markov measure on X with respect

to which the measure-theoretic entropy of X is maximized. This maximal measure-theoretic entropy of X is also the *topological entropy* of X (topological entropy having been defined in 1965 by Adler, Konheim and McAndrew [3] as the topological analogue of measure-theoretic entropy). In light of Friedman and Ornstein's work [22], it is natural to ask what, if anything, can be said topologically about irreducible SFTs of equal period which are measurably isomorphic with respect to their maximal-entropy measures.

This question is answered in the seminal paper of Adler and Marcus [4]. Here they introduce an equivalence relation on SFTs, called *almost conjugacy* (AC), for which topological entropy and period are complete invariants. SFTs X and Y are *almost conjugate* if they admit a common SFT extension Z, where the maps $Z \to X$ and $Z \to Y$ giving the extension are surjective and one-to-one almost everywhere. Given irreducible SFTs X and Y of equal period, with measures of maximal entropy μ_X and μ_Y respectively, X and Y are almost conjugate if and only if there exists a measurable isomorphism $(X, \mu_X) \to (Y, \mu_Y)$.

To this point, all Markov chains we have considered have been two-sided, in the sense that they arise as $\mathbb{Z}$-products. But also it is natural to consider $\mathbb{N}$-products—the *one-sided* Markov chains. In contrast to the two-sided setting where the shift map is an isomorphism, the shift map on a one-sided Markov chain is merely an endomorphism (it is typically non-invertible). Surprisingly, whereas two-sided Bernoulli shifts are classified in terms of measure-theoretic entropy, it follows from [40], [42] and [43] that distinct one-sided Bernoulli shifts of equal measure-theoretic entropy are never measurably isomorphic. What invariants are necessary

and sufficient to classify one-sided Markov chains up to measurable isomorphism?

The answer to this question involves a class of maps between SFTs called *closing* maps. The simplest examples of closing maps are *resolving* maps, which arise from certain natural matrix equations, and which play a key role in the construction of Adler and Marcus [4]. The (more general) closing maps are important to symbolic dynamics because of their connections to algebraic invariants, and also because all known general constructions of maps between SFTs involve closing maps. A map between SFTs is *right closing* if it is injective on each unstable set of the domain, and *left closing* if it is injective on each stable set of the domain. A map is *closing* if it is right or left closing. If X and Y are irreducible SFTs with the same topological entropy and period, then maps $\phi_X : Z \to X$ and $\phi_Y : Z \to Y$ giving an almost conjugacy between X and Y may be chosen to be closing maps. However, in the construction [4], it is not possible to construct ϕ_X and ϕ_Y that are either both right closing or both left closing. (If one of ϕ_X or ϕ_Y is right closing, then the other will be left closing.)

The importance of closing maps is emphasized by connections to *regular isomorphism*. Bill Parry introduces (and studies) regular ismomorphisms of two-sided Markov chains, first in [20] and also in [37]. Roughly speaking, a measurable isomorphism of two-sided Markov chains $\phi : (X, \mu) \to (Y, \nu)$ is *regular* if knowing the past and a bounded look into the future of a point $x \in X$ determines the present state of the point $\phi(x) \in Y$. Recall that almost conjugacies of SFTs X and Y correspond to measurable isomorphisms of Markov chains (X, μ_X) and (Y, μ_Y), where μ_X and μ_Y are the measures of maximal measure-theoretic entropy for X and Y respectively.

Perhaps there is some stronger form of almost conjugacy, corresponding to regular isomorphism?

Yes, there is, and it is called right closing almost conjugacy (RCAC): SFTs X and Y are *right closing almost conjugate* if they admit a common SFT extension Z, where the maps $Z \to X$ and $Z \to Y$ giving the extension are surjective, one-to-one almost everywhere, and right closing. In [15], irreducible SFTs are classified up to RCAC in terms of topological entropy, period, and an algebraic invariant called ideal class. Regular isomorphisms are exactly the measurable isomorphisms that right closing almost conjugacies induce.

In [8], Ashley, Marcus and Tuncel find computable invariants with which to classify one-sided Markov chains up to measurable isomorphism. As Boyle and Tuncel point out in [18], these include in particular all invariants of regular isomorphism of two-sided Markov chains.

In chapter 2, we investigate RCAC, but in a more general setting than was studied in the past. The objects we consider are called G-shifts of finite type (G-SFTs), and the maps between G-SFTs are called G-maps. For a finite group G, a G-SFT is an SFT equipped with a continuous and shift commuting G-action. A G-map $X \to Y$ between G-SFTs is a map of SFTs, equivariant with respect to the G-actions on both X and Y. So a G-SFT is in particular an SFT, and may be thought of as an SFT by simply ignoring the G-action. Similarly a G-map is in particular a map of SFTs. The equivalence relations AC and RCAC are defined in our new context by simply replacing 'SFT' with 'G-SFT' and 'map' with 'G-map' in the respective definitions.

Our initial motivation to classify RCAC for irreducible G-SFTs was to answer a question of Bill Parry [38]. Additionally, a classification of AC for irreducible G-SFTs had appeared in [1] (and also in [37]), so the problem of classifying RCAC for irreducible G-SFTs seemed natural. Also, there is a well-defined class of G-SFTs (the ones where the G-action is free) which arise as skew products. The study of skew products dates all the way back to von Neumann, and is of independent interest.

In section 2.3, we generalize a number of results from the SFT context to the G-SFT context, most noticeably with our $\mathbb{Z}_+G$ Masking Lemma and our $\mathbb{Z}G$ Replacement Theorem. Then in section 2.4 we show, somewhat surprisingly, that for mixing G-SFTs arising as skew products, there are no new invariants of RCAC that weren't already invariants in the SFT context. This is not the case in general, however, and in sections 2.5 and 2.6 we introduce some new invariants, and use them to classify arbitrary irreducible G-SFTs up to RCAC. We conclude chapter 2 with a discussion of regular isomorphism in the G-SFT context.

In chapter 3, my co-author Joseph Barth and I study another equivalence relation, called weak equivalence. As its name suggests, weak equivalence is a weak relation—significantly weaker than AC or RCAC. Subshifts S and T are *weak equivalent* if there exist full shifts Σ_n and Σ_m containing S and T, respectively, and maps $f : \Sigma_n \to \Sigma_m$ and $g : \Sigma_m \to \Sigma_n$ such that $f^{-1}(T) = S$ and $g^{-1}(S) = T$.

Weak equivalence was introduced and studied by Beal and Perrin in [9]. Their main result is a classification of a very special class of irreducible SFTs (the so-called *flower shifts*) up to weak equivalence. This begs the question of what would

be needed to classify arbitrary SFTs up to weak equivalence.

For an answer, first we investigate the structure of the most general class of SFTs—*reducible* SFTs. Reducible SFTs are classified up to flow equivalence by Huang in [23] and [24]; with Boyle, an alternative description appears in [11] and [14]. Kim and Roush show in [26] that a solution to the problem of classifying arbitrary SFTs up to topological conjugacy would follow from a solution in the irreducible case.

We contribute to the theory of reducible SFTs in section 3.4 by introducing the notion of a *phase matrix* for an SFT. We show that the set of phase matrices for an SFT is a computable conjugacy invariant and, with Theorem 3.4.5, we give necessary and sufficient conditions in terms of phase matrices for there to exist an extension of a given map $NW(S) \to NW(T)$ between nonwandering sets of SFTs S and T to a map $S \to T$. This enables us in section 3.5 to classify when there exists a map from one SFT to another. Then in section 3.6 we classify weak equivalence for SFTs, by showing that SFTs S and T are weak equivalent if and only if there exist maps $S \to T$ and $T \to S$.

The centerpiece of our work is an extension result (Theorem 3.4.5). As outlined in section 3.1, extension results are significant to symbolic dynamics. In section 3.6 we indicate that our work may be relevant to a difficult open problem on extensions (Problem 3.6.8). This problem is derived from the work of Maass [34], and is formulated in terms of *steady* maps. We mention a certain special class of SFT pairs (S, T) such that there exists at least one map $S \to T$ which fails to be steady.

A natural question related to Theorem 3.4.5 is, given a surjective map $NW(S) \to$

$NW(T)$ from the nonwandering set of and SFT S onto the nonwandering set of an SFT T, what are necessary and sufficient conditions for there to exist an extension of the given map to a surjective map $S \to T$? We hope to answer this question in the near future but, as of now, all we can say is that the answer to this question will certainly be nontrivial.

Also difficult and open is the question of when there exists a surjective map from one SFT onto another. An answer to the question posed in the preceding paragraph would provide a reduction of this question to the irreducible case. However the irreducible case has shown itself to be extraordinarily challenging, and will certainly involve deep algebraic invariants.

Chapter 2 was published as a paper in the journal Colloquium Mathematicum. Chapter 3 is a paper in preprint, soon to be submitted for publication.

Chapter 2

Right closing almost conjugacy for G-shifts of finite type

2.1 Introduction

For a finite group G, a *G-shift of finite type* (G-SFT) is a shift of finite type (X, σ) together with a continuous G action on X which commutes with the shift σ. For irreducible shifts of finite type, right closing almost conjugacy is classified in terms of entropy, period, and an algebraic invariant called ideal class [15]. Bill Parry [38] posed the following question: what additional invariants are necessary to classify right closing almost conjugacy for irreducible G-SFTs? With Theorem 2.4.1 we show that for mixing G-SFTs where the G action is free, there are no additional invariants. In section 2.5 we generalize Theorem 2.4.1 to mixing G-SFTs where the G action is no longer assumed to be free. In section 2.6 we generalize further to irreducible but periodic G-SFTs. As a corollary to our results we classify regular isomorphism for G-Markov chains with respect to measures of maximal entropy.

Without the right closing assumption, almost conjugacy for irreducible G-SFTs was classified by Roy Adler, Bruce Kitchens and Brian Marcus [1]. They were working in a more general setting, but by modifying the proofs given here we can arrive at the same classification of almost conjugacy for irreducible G-SFTs (as was also done in [37]).

I thank Mike Boyle for many helpful discussions about this problem.

2.2 Background and definitions

We assume some familiarity with shifts of finite type; [29] and [33] provide more complete backgrounds. All of the free G-SFTs we consider arise out of skew products, as in [16]. The study of skew products dates back to von Neumann, in the context of ergodic measure preserving transformations on a probability space. For an example of more recent work with skew products in ergodic theory, see [21]. Also see [39] and [41] (and their references) for recent results with skew products in Livsic theory.

2.2.1 Shifts of finite type

Let A be an $n \times n$ matrix over the nonnegative integers $\mathbb{Z}_+$. Then A is the adjacency matrix for a directed graph, $\mathcal{G}_A$, which has vertices $\{v_1, v_2, \ldots, v_n\}$, and the number of edges from v_I to v_J is A_{IJ}. Let $\mathcal{E}_A = \{\text{edges in } \mathcal{G}_A\}$, and put

$$\Sigma_A = \{x = (x_i)_{i \in \mathbb{Z}} \in (\mathcal{E}_A)^{\mathbb{Z}} \,|\, \text{each } x_i x_{i+1} \text{ is a path in } \mathcal{G}_A\}.$$

With the appropriate topology (the relative of the product of the discrete topology on $\mathcal{E}_A$), Σ_A is a compact metric space. The *shift* on Σ_A is the homeomorphism $\sigma : \Sigma_A \to \Sigma_A$ given by $(\sigma x)_i = x_{i+1}$. The pair (Σ_A, σ) is the *edge shift of finite type (SFT)* defined by A. Where σ is understood, we write just Σ_A to denote (Σ_A, σ).

A *map* between SFTs $\pi : \Sigma_A \to \Sigma_B$ is a continuous function such that $\pi \circ \sigma(x) = \sigma \circ \pi(x)$ for all $x \in \Sigma_A$. The map π is *one block* if it is induced by a function which sends each edge of $\mathcal{G}_A$ to an edge of $\mathcal{G}_B$. A *factor map* is a surjective map. An injective factor map is a *conjugacy*.

The matrix A is *irreducible* if for each entry A_{IJ} of A there is a natural number N such that $(A^N)_{IJ} > 0$. If A is irreducible we say also that the graph $\mathcal{G}_A$ and the edge SFT Σ_A are irreducible. The matrix A is *primitive* if there is a natural number N such that for each entry A_{IJ} of A, $(A^N)_{IJ} > 0$. If A is primitive we say also that the graph $\mathcal{G}_A$ is primitive; in this case the edge SFT Σ_A is *mixing*.

If λ is the Perron eigenvalue of A, then the *entropy* of Σ_A is $\log \lambda$. If $v = [v_1, v_2, \ldots, v_n]^T$ is a right Perron eigenvector with entries in the ring $\mathbb{Z}[1/\lambda]$, then the *ideal class* of Σ_A is the class of the $\mathbb{Z}[1/\lambda]$-ideal which is generated by the components $v_1, \ldots, v_n$ of v. A point $x \in \Sigma_A$ is *periodic* if there exists a natural number p such that $\sigma^p(x) = x$. In this case p is a *period* of x; the smallest period of x is called the *least period* of x. We define the period of the edge SFT Σ_A to be the greatest common divisor of the set of periods of periodic points in Σ_A. The period of the graph $\mathcal{G}_A$ is the period of Σ_A.

Any SFT (X, σ) is conjugate to some edge SFT (Σ_A, σ). Then, the terms *irreducible, mixing, entropy, ideal class* and *period* apply to X exactly as they apply to Σ_A. A point $x \in X$ is *doubly transitive* if both sets $\{\sigma^n(x) : n \geq 0\}$ and $\{\sigma^n : n \leq 0\}$ are dense in X. Two points $x = (x_n)_{n \in \mathbb{Z}}$ and $y = (y_n)_{n \in \mathbb{Z}}$ in X are *left asymptotic* if there is an integer n such that $x_k = y_k$ for all $k \leq n$. A map between SFTs $\pi \colon X \to Y$ is *1-1 a.e.* if it is injective on the set of doubly transitive points in X. The map π is *right closing* if, for each pair $x, y \in X$ of distinct left asymptotic points, $\pi(x) \neq \pi(y)$. We say the SFTs X and Y are *right closing almost conjugate as SFTs* if there is a third SFT Z which factors onto both X and Y by factor maps which are 1-1 *a.e.* and right closing.

2.2.2 G-shifts of finite type

Let G be a finite group. A G-*SFT* is an SFT (X, σ) together with a continuous right G action on X such that $\sigma(x \cdot g) = \sigma(x) \cdot g$ for all $x \in X$ and $g \in G$. We say the G-SFT X (or the G action on X) is *free* if, for each non-identity element g of G, $x \cdot g \neq x$ for all $x \in X$. We say X (or the G action on X) is *faithful* if, for each non-identity element g of G, there exists some $x \in X$ such that $x \cdot g \neq x$. If Y is another G-SFT, then a G-*map* $\pi \colon X \to Y$ is a map between SFTs such that $\pi(x \cdot g) = \pi(x) \cdot g$ for all $x \in X$ and $g \in G$. A G-*factor map* is a surjective G-map and a G-*conjugacy* is an injective G-factor map. Two G-SFTs X and Y are *right closing almost conjugate as G-SFTs* if there is a third G-SFT Z which factors onto both X and Y by 1-1 *a.e.* and right closing G-factor maps. We point out that right closing almost conjugate G-SFTs are in particular right closing almost conjugate SFTs. The terms we define above for SFTs, such as *irreducible, mixing, entropy, ideal class* and *period*, apply to a G-SFT X as they apply to X as an SFT.

2.2.3 Skew products and matrices over $\mathbb{Z}_+ G$

By $\mathbb{Z}G$ we mean the integral group ring of G. We write an element x of $\mathbb{Z}G$ as $x = \sum_{g \in G} n_g g$, where each $n_g \in \mathbb{Z}$. Then for each g in G we define $\pi_g(x) = n_g$. If $\pi_g(x) > 0$, then g is a *summand* of x. If $\pi_g(x) > 0$ for each g in G, then we say x is *very positive* and write $x \gg 0$. The *augmentation* of x is $|x| = \sum_{g \in G} \pi_g(x)$. If A is a matrix over $\mathbb{Z}G$, then $A \gg 0$ if $A_{IJ} \gg 0$ for each entry A_{IJ} of A. The augmentation $|A|$ is the matrix given by $|A|_{IJ} = |A_{IJ}|$ for each entry A_{IJ} of A. We

let $\mathbb{Z}_+G = \{x \in \mathbb{Z}G : \pi_g(x) \geq 0 \text{ for each } g \in G\}$.

If $\mathcal{G}$ is a directed graph and l is a labeling of the edges of $\mathcal{G}$ by elements of G, then we say $(\mathcal{G}, l)$ is a G-labeled graph. If A is a square matrix over $\mathbb{Z}_+G$, then $|A|$ is a square matrix over $\mathbb{Z}_+$ which, as before, is the adjacency matrix for a directed graph $\mathcal{G}_{|A|}$. The matrix A corresponds to a G-labeled graph $(\mathcal{G}_{|A|}, l_A)$, where l_A is defined as follows: for each pair I, J of vertices in $\mathcal{G}_{|A|}$, $A_{IJ} = \sum n_g g$ if and only if for each $g \in G$ exactly n_g of the edges from I to J are l_A-labeled g. The edge labeling l_A determines a function $\tau_A : \Sigma_{|A|} \to G$ by $\tau_A(x) = l_A(x_0)$ for each $x = (x_n)_{n \in \mathbb{Z}}$ in $\Sigma_{|A|}$. The function τ_A is locally constant: for each $x \in \Sigma_{|A|}$, τ_A is constant on a neighborhood of x (here τ_A is constant on $\{y \in \Sigma_{|A|} \mid y_0 = x_0\}$). The function τ_A is the skewing function defined by A. Given two locally constant functions $\tau_1, \tau_2 : \Sigma_{|A|} \to G$, we say τ_1 is cohomologous to τ_2 if there is another locally constant $h : \Sigma_{|A|} \to G$ such that $\tau_1(x) = [h(\sigma x)]^{-1} \cdot \tau_2(x) \cdot h(x)$ for each $x \in \Sigma_{|A|}$.

The $\mathbb{Z}_+G$ matrix A determines an automorphism $S_A : \Sigma_{|A|} \times G \to \Sigma_{|A|} \times G$ by $S_A(x, g) = (\sigma(x), \tau_A(x) \cdot g)$, where τ_A is the skewing function defined by A. We say the dynamical system $(\Sigma_{|A|} \times G, S_A)$ is the skew product defined by A. There is a free right G action on $(\Sigma_{|A|} \times G, S_A)$ which commutes with the automorphism S_A, given by $g : (x, h) \mapsto (x, h \cdot g)$. Often we write just S_A as an abbreviation for the skew product $(\Sigma_{|A|} \times G, S_A)$

We can present the skew product S_A as a free G-SFT (which we also denote by S_A) as follows. As an edge SFT S_A has graph $\mathcal{G}$, where the vertex set of $\mathcal{G}$ is the product of the vertex set of $\mathcal{G}_{|A|}$ with G, and for each edge e from I to J in $\mathcal{G}_{|A|}$, for each g in G, there is an edge from (I, g) to $(J, l_A(e) \cdot g)$ in $\mathcal{G}$. For each pair of

vertices v, v' of $\mathcal{G}$ we choose an ordering of the edges from v to v', and let g in G act by the one block map given by the unique automorphism of $\mathcal{G}$ which acts on the vertex set of $\mathcal{G}$ by $(J, h) \mapsto (J, h \cdot g)$, and which is order preserving.

In this way any skew product is a free G-SFT. Conversely, any free G-SFT is G-conjugate to a skew product S_A for some $\mathbb{Z}_+ G$ matrix A. We say a matrix A over $\mathbb{Z}_+ G$ is *very primitive* if there exists a natural number N such that $A^N \gg 0$. One easily checks that A is very primitive if and only if the G-SFT S_A is mixing.

Square matrices A and B over $\mathbb{Z}_+ G$ are *strong shift equivalent* (SSE) over $\mathbb{Z}_+ G$ if they are connected by a string of elementary moves of the following sort: there are R and S over $\mathbb{Z}_+ G$ such that $A = RS$ and $B = SR$. Parry has shown that A and B are SSE over $\mathbb{Z}_+ G$ if and only if the skew products S_A and S_B are G-conjugate [16, Prop. 2.7.1].

2.3   Some useful results

In this section we collect some results to be used later. We begin with the known classification of right closing almost conjugacy for irreducible SFTs, which is a corollary of [15, Theorem 7.1].

Theorem 2.3.1. *Irreducible SFTs are right closing almost conjugate as SFTs if and only if they have the same ideal class, entropy and period.*

Lemma 2.3.2 ($\mathbb{Z}_+ G$ Masking Lemma). *Let A and C be matrices over $\mathbb{Z}_+ G$ such that the skew product S_A is G-conjugate to a subsystem of the skew product S_C. Then there is a matrix B over $\mathbb{Z}_+ G$ such that A is a principal submatrix of B, and*

S_B and S_C are G-conjugate skew products.

Proof. If S_A is G-conjugate to a subsystem of S_C, then A is SSE over $\mathbb{Z}_+ G$ to a principal submatrix of C [16, Prop. 2.7.1]. Nasu's original Masking Lemma for matrices over $\mathbb{Z}$ [35, Lemma 3.18] also holds for matrices over an arbitrary semiring containing 0 and 1 [13, Appendix 1]; in particular it holds for matrices over $\mathbb{Z}_+ G$. This means there is a matrix B over $\mathbb{Z}_+ G$ such that A is a principal submatrix of B, and B is SSE over $\mathbb{Z}_+ G$ to C; S_B and S_C are G-conjugate skew products by [16, Prop. 2.7.1].

$\square$

Lemma 2.3.3. *Let A and B be matrices over $\mathbb{Z}_+ G$. A G-factor map $\pi \colon S_A \to S_B$ induces a factor map $\overline{\pi} \colon \Sigma_{|A|} \to \Sigma_{|B|}$ such that the skewing function τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Conversely, if $\overline{\pi} \colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a factor map such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$, then $\overline{\pi}$ induces a G-factor map $\pi \colon S_A \to S_B$. The G-map π is 1-1 a.e. and right closing if and only if the map $\overline{\pi}$ is 1-1 a.e. and right closing.*

Proof. Let $\pi \colon S_A \to S_B$ be a G-factor map. Write $\pi = \pi_1 \times \pi_2$, so that for an element $(x, g) \in \Sigma_{|A|} \times G$, $\pi(x, g) = (\pi_1(x, g), \pi_2(x, g))$. Let e denote the identity element of G. Then $\pi \colon (x, g) \mapsto (\pi_1(x, e), \pi_2(x, e) \cdot g)$, since π intertwines G actions. For $x \in \Sigma_{|A|}$, set $\overline{\pi}(x) = \pi_1(x, e)$ and $h(x) = \pi_2(x, e)$, so that $\pi(x, g) = (\overline{\pi}(x), h(x) \cdot g)$. Look componentwise at the equality $\pi \circ S_A = S_B \circ \pi$. The first component shows that $\overline{\pi} \colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a well-defined factor map. The second component shows that $\tau_A(x) = [h(\sigma x)]^{-1} \cdot (\tau_B \circ \overline{\pi})(x) \cdot h(x)$ for each $x \in \Sigma_{|A|}$. Hence τ_A is cohomologous to $\tau_B \circ \overline{\pi}$.

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Conversely, suppose $\overline{\pi}\colon \Sigma_{|A|} \to \Sigma_{|B|}$ is a factor map such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Then there is a locally constant map $h\colon \Sigma_{|A|} \to G$ such that for each $x \in \Sigma_{|A|}$, $\tau_A(x) = [h(\sigma x)]^{-1} \cdot (\tau_B \circ \overline{\pi})(x) \cdot h(x)$. Define $\pi\colon \Sigma_{|A|} \times G \to \Sigma_{|B|} \times G$ by $\pi(x, g) = (\overline{\pi}(x), h(x) \cdot g)$. Observe that π is a G-factor map.

For the last statement of the lemma, consider the following commutative diagram, where the maps $q_A\colon S_A \to \Sigma_A$ and $q_B\colon S_B \to \Sigma_B$ are each given by $(x, g) \mapsto x$.

$$
\begin{array}{ccc}
S_A & \xrightarrow{\ \pi\ } & S_B \\
{\scriptstyle q_A}\downarrow & & \downarrow{\scriptstyle q_B} \\
\Sigma_{|A|} & \xrightarrow{\ \overline{\pi}\ } & \Sigma_{|B|}
\end{array}
$$

Both maps q_A and q_B are $|G|$-to-1 everywhere. Therefore π is 1-1 *a.e.* if and only if $\overline{\pi}$ is 1-1 *a.e.* For the closing condition, note that if ϕ and ψ are maps between irreducible SFTs, then $\phi \circ \psi$ is right closing if and only if both ϕ and ψ are right closing [15, Props. 4.10 and 4.11]. Because the constant-to-one maps q_A and q_B are in particular right closing [29, Prop. 4.3.4], it follows that π is right closing if and only if $\overline{\pi}$ is right closing.

$\square$

If $(\mathcal{G}, l)$ is a G-labeled graph, then for a cycle $s = s_1 s_2 \ldots s_p$ in $\mathcal{G}$ we define the *weight* of s by $l(s) = l(s_1)l(s_2)\cdots l(s_p)$. The *ratio group* Δ_l is the subgroup of G given by

$$
\Delta_l = \{l(s) \cdot l(s')^{-1} \colon s, s' \text{ are cycles in } \mathcal{G} \text{ of the same length}\}
$$

Theorem 2.3.4 ($\mathbb{Z}G$ **Replacement Theorem**). *Let $(\mathcal{G}, l)$ and $(\mathcal{G}', l')$ be irreducible*

G-labeled graphs of the same period which define edge SFTs Σ and Σ' (respectively) and skewing functions $\tau \colon \Sigma \to G$ and $\tau' \colon \Sigma' \to G$ given by $\tau(x) = l(x_0)$ and $\tau'(x) = l'(x_0)$. Let $\pi \colon \Sigma \to \Sigma'$ be a factor map such that τ is cohomologous to $\tau' \circ \pi$. If $\Delta_l = \Delta_{l'}$, then there is a 1-1 a.e. factor map $\overline{\pi} \colon \Sigma \to \Sigma'$ such that τ is cohomologous to $\tau' \circ \overline{\pi}$. Moreover, if π is right closing, then $\overline{\pi}$ can be taken to be right closing as well.

In [5, Theorem 6.1], Ashley proves a version of his ($\mathbb{Z}$) Replacement Theorem for maps between irreducible Markov chains, which can be interpreted as follows. Let $\mathbb{R}^+$ denote the multiplicative group of positive real numbers. Let $(\mathcal{G}, l)$ and $(\mathcal{G}', l')$ be irreducible $\mathbb{R}^+$-labeled graphs of the same period, which define irreducible SFTs Σ and Σ' and locally constant functions $\tau \colon \Sigma \to \mathbb{R}^+$ and $\tau' \colon \Sigma' \to \mathbb{R}^+$ where $\tau(x) = l(x_0)$ and $\tau'(x) = l'(x_0)$. If the ratio groups Δ_l and $\Delta_{l'}$ are equal (as multiplicative subgroups of $\mathbb{R}^+$), and $\pi \colon \Sigma \to \Sigma'$ is a factor map such that τ is cohomologous to $\tau' \circ \pi$, then there is a 1-1 a.e. factor map $\overline{\pi} \colon \Sigma \to \Sigma'$ such that τ is cohomologous to $\tau' \circ \overline{\pi}$. Moreover, if π is right closing, then $\overline{\pi}$ can be taken to be right closing as well.

If instead of $\mathbb{R}^+$-labeled graphs we consider G-labeled graphs, then we have the statement of Theorem 2.3.4. To prove Theorem 2.3.4, one can easily check that Ashley's proof for $\mathbb{R}^+$-labeled graphs goes through for G-labeled graphs as well.

Theorem 2.3.5. *Let X and Y be mixing free G-SFTs. Let $\pi \colon X \to Y$ be a G-factor map which is right closing. Then there is a G-factor map $\pi' \colon X \to Y$ which is 1-1 a.e. and right closing.*

Proof. Since X and Y are mixing free G-SFTs, assume without loss of generality that $X = S_A$ and $Y = S_B$ for very primitive matrices A and B over $\mathbb{Z}_+ G$. By Lemma 2.3.3 the G-factor map π induces a map $\overline{\pi} \colon \Sigma_{|A|} \to \Sigma_{|B|}$ such that τ_A is cohomologous to $\tau_B \circ \overline{\pi}$. Since A and B are very primitive the periods of $\mathcal{G}_{|A|}$ and $\mathcal{G}_{|B|}$ are both 1, and furthermore $\Delta_{l_A} = \Delta_{l_B} = G$. So assume (by Theorem 2.3.4) that the map $\overline{\pi}$ is 1-1 *a.e.* and right closing. Again apply Lemma 2.3.3 to obtain a G-factor map $\pi' \colon S_A \to S_B$ which is 1-1 *a.e.* and right closing. $\qquad\square$

2.4 Right closing almost conjugacy for mixing free G-SFTs

For mixing SFTs, entropy and ideal class are a complete set of invariants of right closing almost conjugacy (Theorem 2.3.1). We show that there are no additional invariants of right closing almost conjugacy for mixing free G-SFTs.

Theorem 2.4.1. *Let X and Y be mixing free G-SFTs. Then the following are equivalent.*

1. *X and Y are right closing almost conjugate as G-SFTs.*

2. *X and Y are right closing almost conjugate as SFTs.*

3. *X and Y have the same entropy and ideal class.*

Moreover, assuming (2) or (3), the common extension of X and Y in (1) can be taken to be a free G-SFT.

Proof. (2) $\Leftrightarrow$ (3) follows from Theorem 2.3.1. Right closing almost conjugate G-SFTs are in particular right closing almost conjugate as SFTs, so (1) $\Rightarrow$ (2). It

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remains to show $(2) \Rightarrow (1)$.

Let X and Y be mixing free G-SFTs which are right closing almost conjugate as SFTs. Without loss of generality, assume that X and Y are skew products S_A and S_B for very primitive matrices A and B over $\mathbb{Z}_+G$. Let l_A, l_B, τ_A and τ_B denote the edge labelings and skewing functions defined by A and B, respectively (see section 2.2). Since S_A and S_B are right closing almost conjugate as SFTs, they have the same entropy and ideal class (Theorem 2.3.1). The factor maps $q_A \colon S_A \to \Sigma_{|A|}$ and $q_B \colon S_B \to \Sigma_{|B|}$ given by $(x, g) \mapsto x$ are $|G|$-to-1 everywhere. In particular they preserve entropy and ideal class, so $\Sigma_{|A|}$ and $\Sigma_{|B|}$ have the same entropy and ideal class. Hence $\Sigma_{|A|}$ and $\Sigma_{|B|}$ are right closing almost conjugate as SFTs (Theorem 2.3.1). Let $\Sigma_{|C|}$ be a common extension of $\Sigma_{|A|}$ and $\Sigma_{|B|}$ by 1-1 $a.e.$ right closing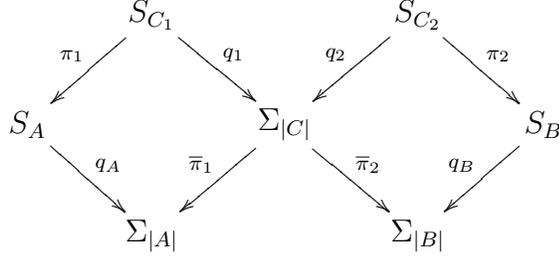 factor maps $\overline{\pi}_1 \colon \Sigma_{|C|} \to \Sigma_{|A|}$ and $\overline{\pi}_2 \colon \Sigma_{|C|} \to \Sigma_{|A|}$.

$$
\begin{array}{ccccccc}
S_A & & \Sigma_{|C|} & & S_B \\
& \searrow^{q_A} \quad \swarrow^{\overline{\pi}_1} & & \searrow^{\overline{\pi}_2} \quad \swarrow^{q_B} & \\
& \Sigma_{|A|} & & \Sigma_{|B|} &
\end{array}
$$
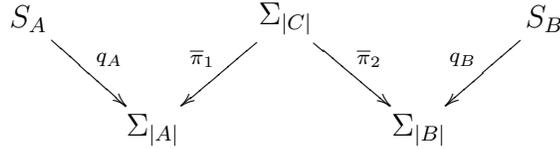
Without loss of generality, assume the factor maps $\overline{\pi}_1$ and $\overline{\pi}_2$ are one block. Define edge labelings l_1 and l_2 on $\mathcal{G}_{|C|}$ by $l_1 = l_A \circ \overline{\pi}_1$ and $l_2 = l_B \circ \overline{\pi}_2$. The labelings l_1 and l_2 correspond to matrices C_1 and C_2 (respectively) over $\mathbb{Z}_+G$ such that $|C_1| = |C_2| = |C|$. Define skewing functions $\tau_1 \colon \Sigma_{|C|} \to G$ and $\tau_2 \colon \Sigma_{|C|} \to G$ by $\tau_1(x) = l_1(x_0)$ and $\tau_2(x) = l_2(x_0)$. Define G-factor maps $\pi_1 \colon S_{C_1} \to S_A$ and $\pi_2 \colon S_{C_2} \to S_B$ by $\pi_1(x, g) = (\overline{\pi}_1(x), g)$ and $\pi_2(x, g) = (\overline{\pi}_2(x), g)$. Let $q_1 \colon S_{C_1} \to \Sigma_{|C|}$ and $q_2 \colon S_{C_2} \to \Sigma_{|C|}$ be the factor maps $(x, g) \mapsto x$. Then the following diagram commutes.

$$S_{C_1} \qquad\qquad S_{C_2}$$

$$\pi_1 \swarrow \qquad \searrow q_1 \qquad q_2 \swarrow \qquad \searrow \pi_2$$

$$S_A \qquad\qquad \Sigma_{|C|} \qquad\qquad S_B$$

$$\searrow q_A \qquad \overline{\pi}_1 \swarrow \qquad \overline{\pi}_2 \searrow \qquad q_B \swarrow$$

$$\Sigma_{|A|} \qquad\qquad \Sigma_{|B|}$$

The factor maps $\overline{\pi}_1$ and $\overline{\pi}_2$ are 1-1 $a.e.$ and right closing, so the factor maps π_1 and π_2 are as well (Lemma 2.3.3). In particular S_{C_1} and S_{C_2} are mixing free G-SFTs, so C_1 and C_2 are very primitive. Let l be the $(G \times G)$-labeling $l = l_1 \times l_2$. Then l corresponds to a $\mathbb{Z}_+(G \times G)$ matrix whose augmentation is $|C|$. Call this matrix C. Let $\tau \colon \Sigma_{|C|} \to G \times G$ denote the skewing function given by $\tau(x) = l(x_0)$.

Claim 2.4.2. *There is a vertex I in $\mathcal{G}_{|C|}$ and a natural number N such that there is a collection $\mathcal{U}$ of paths of length N from I to I with the following properties:*

1. *for each g in G there are at least $|G|$ paths $u \in \mathcal{U}$ with weights $l_1(u) = g$.*

2. *for each g in G there are at least $|G|$ paths $u \in \mathcal{U}$ with weights $l_2(u) = g$.*

3. *for each $u = u_1 u_2 \cdots u_N \in \mathcal{U}$ the point $x^u \in \Sigma_{|C|}$, defined by $x_i^u = u_j$ if $i \equiv j \bmod N$, has least period N.*

4. *if u and v are distinct paths in $\mathcal{U}$, then x^u and x^v are in different orbits under the shift.*

To prove the claim, let α be the element of $\mathbb{Z}_+ G$ given by $\alpha = \sum_{g \in G} g$. Fix a vertex I in $\mathcal{G}_{|C|}$. Let η be the number of cycles of length 1 in $\mathcal{G}_{|C|}$, and choose a positive integer k large enough so that $k - \eta \geq |G|$. Since C_1 and C_2 are very primitive matrices there is a positive integer $M = M(k)$ such that, for $i = 1, 2$ and

for all $m \geq M$, $k \cdot \alpha$ is a summand of $(C_i^m)_{II}$. Let $N \geq M$ be a prime number. Let $\mathcal{V}$ be the set of all N-paths from I to I. Each $v = v_1 v_2 \cdots v_N \in \mathcal{V}$ defines a point $x^v \in \Sigma_{|C|}$ by $x_i^v = v_j$ if $i \equiv j \mod N$. Since N is prime, each such x^v has least period either N or 1. Let $\mathcal{V}^1 = \{v \in V : x^v$ has least period 1$\}$ and $\mathcal{U} = \mathcal{V} - \mathcal{V}^1$.

It remains to verify that $\mathcal{U}$ satisfies the properties of the claim. Note that, for $i = 1, 2$, each monomial summand g of $(C_i^N)_{II}$ corresponds to a path $v \in \mathcal{V}$ with weight $l_i(v) = g$. Also, N was chosen so that $k \cdot \alpha$ is a summand of each $(C_i^N)_{II}$. So for $i = 1, 2$ and for each $g \in G$, there are at least k paths $v \in \mathcal{V}$ with weight $l_i(v) = g$. There are only η cycles of length 1 in $\mathcal{G}_{|C|}$, so in particular $|\mathcal{V}^1| \leq \eta$. But $k - \eta \geq |G|$. Hence, for $i = 1, 2$ and for each $g \in G$, there are at least k paths $u \in \mathcal{U}$ with weight $l_i(u) = g$, which verifies properties (1) and (2). Properties (3) and (4) are true by construction of $\mathcal{U}$. This proves the claim.

Now consider all points $x^u \in \Sigma_{|C|}$ such that $u \in \mathcal{U}$. Let $\overline{\Sigma}_{|C|}$ denote the smallest closed σ-invariant subset of $\Sigma_{|C|}$ containing all points of this form. Then $\overline{\Sigma}_{|C|} \times G$ is a closed S_C-invariant subset of $\Sigma_{|C|} \times G$, so it is a subsystem of the skew product S_C. Let $\overline{S}_C$ denote this subsystem of S_C.

Construct a $(G \times G)$-labeled graph $(\mathcal{H}, l_{\mathcal{H}})$ as follows. The vertex set of $\mathcal{H}$ consists of N vertices, $I_1, I_2, \ldots, I_N$. For $j = 1, 2, \ldots, N-1$, draw exactly one edge starting at I_j and ending at I_{j+1}, and give this edge the $l_{\mathcal{H}}$-label (e, e), where e is the identity element of G. From I_N to I_1 draw exactly $|\mathcal{U}|$ edges, call them $s_1, s_2, \ldots, s_{|\mathcal{U}|}$. Let $\mathcal{S} = \{s_1, s_2, \ldots, s_{|\mathcal{U}|}\}$, and fix a set bijection $\phi \colon \mathcal{S} \to \mathcal{U}$. For $s_i \in \mathcal{S}$, put $l_{\mathcal{H}}(s_i) = l(\phi(s_i)) = (l_1(\phi(s_i)), l_2(\phi(s_i)))$.

Let D be the $\mathbb{Z}_+(G \times G)$ adjacency matrix for the $(G \times G)$-labeled graph $(\mathcal{H}, l_{\mathcal{H}})$.
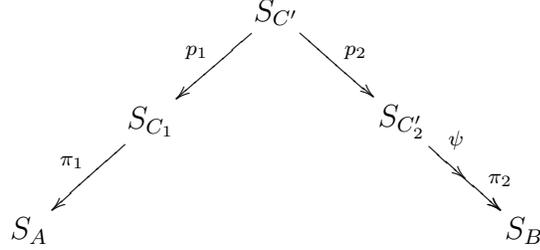
Observe that the set bijection $\phi\colon \mathcal{S} \to \mathcal{U}$ induces a $(G \times G)$-conjugacy between S_D and $\overline{S}_C$. Assume without loss of generality that D is a principal submatrix of C (Lemma 2.3.2), so that $(\mathcal{H}, l_\mathcal{H})$ is an induced sub-labeled graph of $(\mathcal{G}_{|C|}, l)$.

For each $g \in G$, at least $|G|$ of the edges $s_i \in \mathcal{S}$ have l-labels of the form $(g, \cdot)$, and at least $|G|$ of the $s_i \in \mathcal{S}$ have l-labels of the form $(\cdot, g)$ (by definition). Therefore there is a way to permute the second coordinates of the l-labelings of edges in $\mathcal{S}$ such that each $(g, h) \in G \times G$ labels at least one $s_i \in \mathcal{S}$. Equivalently, there exists a graph isomorphism $\overline{P}$ of $\mathcal{G}_{|C|}$ which fixes all edges except those in $\mathcal{S}$, and permutes the set $\mathcal{S}$ such that for any $(g, h) \in G \times G$, there is at least one edge $s_i \in \mathcal{S}$ with $(l_1(s_i), l_2 \circ \overline{P}(s_i)) = (g, h)$. Fix a graph isomorphism $\overline{P}$ with this property and set l' to be the $(G \times G)$-labeling of $\mathcal{G}_{|C|}$ given by $l' = l_1 \times (l_2 \circ \overline{P})$. Let P denote the automorphism of $\Sigma_{|C|}$ induced by $\overline{P}$. Let C_2' be the $\mathbb{Z}_+ G$ matrix defined by the edge labeling $l_2 \circ \overline{P}$ of $\mathcal{G}_{|C|}$. Note that the map $\psi\colon S_{C_2'} \to S_{C_2}$ given by $(x, g) \mapsto (P(x), g)$ is a G-conjugacy.

Let C' be the $\mathbb{Z}_+(G \times G)$ matrix defined by the edge labeling l' of $\mathcal{G}_{|C|}$, and let $\tau'\colon \Sigma_{|C|} \to G \times G$ be the skewing function given by $\tau'(x) = l'(x_0)$. Then $S_{C'}$ is the skew product $(\Sigma_{|C|} \times G \times G, S_{C'})$, where $S_{C'}(x, g, h) = (\sigma(x), \tau'(g, h)) = (\sigma(x), \tau_1(x) \cdot g, (\tau_2 \circ P)(x) \cdot h)$, and $G \times G$ acts by $(k, l)\colon (x, g, h) \mapsto (x, gk, hl)$. Note that C' is very primitive. (This is because, with $I = I_1$ and N as above, $(C''^N)_{II}$ has as a summand every element of $G \times G$.) Therefore $S_{C'}$ is a mixing free $(G \times G)$-SFT.

From now on, regard $S_{C'}$ as a mixing free G-SFT by restricting the $(G \times G)$-action to the diagonal: let an element $g \in G$ act by $(x, h, k) \mapsto (x, hg, kg)$. Let $p_1\colon S_{C'} \to S_{C_1}$ be the $|G|$-to-one factor map $(x, g, h) \mapsto (x, g)$, and let $p_2\colon S_{C'} \to S_{C_2'}$

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be the $|G|$-to-one factor map $(x, g, h) \mapsto (x, h)$. Note that p_1 and p_2 are G-factor maps; they are right closing because they are constant-to-one [29, Prop 4.3.4]. This gives a diagram of right closing G-factor maps:

$$
\begin{array}{ccccc}
 & & S_{C'} & & \\
 & {}^{p_1}\swarrow & & \searrow{}^{p_2} & \\
 & S_{C_1} & & S_{C_2'} & \\
 {}^{\pi_1}\swarrow & & & \;{}^{\psi}\searrow & \\
 & & & & \searrow{}^{\pi_2} \\
S_A & & & & S_B
\end{array}
$$

$S_{C'}$ is a mixing free G-SFT, so by Theorem 2.3.5, the right closing G-factor maps $\pi_1 \circ p_1$ and $\pi_2 \circ \psi \circ p_2$ can be replaced by 1-1 *a.e.* and right closing G-factor maps. This proves the theorem.

$\square$

2.5 General mixing G-SFTs

In this section we classify right closing almost conjugacy for mixing G-SFTs where the G action is no longer assumed to be free. We will need this generalization to classify the irreducible but periodic case in section 2.6. We begin with a result for faithful G-SFTs, which were defined in section 2.2.

Lemma 2.5.1. *Any irreducible faithful G-SFT is a 1-1 a.e. right closing G-factor of an irreducible free G-SFT.*
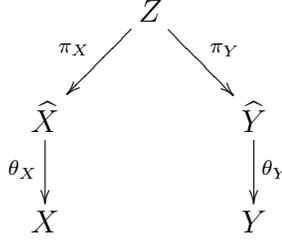
Lemma 2.5.1 is a corollary of [1, Theorem 3]. If X is a G-SFT, we let H^X denote the normal subgroup of G which acts by the identity map. Then X is a faithful (G/H^X)-SFT where, for all $g \in G$ and $x \in X$, $x \cdot (gH^X) = x \cdot g$.

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Theorem 2.5.2. *Let X and Y be mixing G-SFTs. Then the following are equivalent.*

1. *X and Y are right closing almost conjugate as G-SFTs*

2. *X and Y are right closing almost conjugate as SFTs, and $H^X = H^Y$.*

3. *X and Y have the same entropy and ideal class, and $H^X = H^Y$.*

Proof. $(2) \Leftrightarrow (3)$ follows from Theorem 2.3.1. If X and Y are right closing almost conjugate as G-SFTs, then in particular they are right closing almost conjugate as SFTs. Moreover, if Z is a common 1-1 *a.e.* right closing G-extension of X and Y, then $H^X = H^Z$ and $H^Y = H^Z$, because 1-1 *a.e.* G-factor maps preserve the subgroup H^Z. This proves $(1) \Rightarrow (2)$.

Conversely, suppose X and Y are right closing almost conjugate as SFTs, and $H = H^X = H^Y$. Then X and Y are faithful (G/H)-SFTs, where for all $x \in X$, $y \in Y$ and $g \in G$, $x \cdot (gH) = x \cdot g$ and $y \cdot (gH) = y \cdot g$. Hence there are free (G/H)-SFTs $\widehat{X}$ and $\widehat{Y}$, and 1-1 *a.e.* right closing (G/H)-factor maps $\theta_X \colon \widehat{X} \to X$ and $\theta_Y \colon \widehat{Y} \to Y$ (Lemma 2.5.1). Since X and Y are right closing almost conjugate as SFTs, they have the same entropy and ideal class. Since θ_X and θ_Y are right closing factor maps between irreducible SFTs, they preserve entropy and ideal classes. So $\widehat{X}$ and $\widehat{Y}$ have the same entropy and ideal class, and are therefore right closing almost conjugate as SFTs. Thus $\widehat{X}$ and $\widehat{Y}$ are right closing almost conjugate as (G/H)-SFTs, and the common extension can be taken to be a free (G/H)-SFT (Theorem 2.4.1). Let Z be a free (G/H)-SFT with 1-1 *a.e.* right closing (G/H)-factor maps $\pi_X \colon Z \to \widehat{X}$ and $\pi_Y \colon Z \to \widehat{Y}$.

$$
\begin{array}{ccc}
& Z & \\
{}^{\pi_X}\swarrow & & \searrow{}^{\pi_Y} \\
\widehat{X} & & \widehat{Y} \\
\theta_X\downarrow & & \downarrow\theta_Y \\
X & & Y
\end{array}
$$

For all $\widehat{x} \in \widehat{X}$, $\widehat{y} \in \widehat{Y}$ and $g \in G$, put $\widehat{x} \cdot g = \widehat{x} \cdot (gH)$ and $\widehat{y} \cdot g = \widehat{y} \cdot (gH)$. With these G actions, $\widehat{X}$ and $\widehat{Y}$ are G-SFTs, and θ_X and θ_Y are now G-maps. For all $z \in Z$ and $g \in G$, put $g \cdot z = z \cdot (gH)$. This G action makes Z a G-SFT as well, and π_X and π_Y are now G-maps. Thus Z together with the maps $\theta_X \circ \pi_X$ and $\theta_Y \circ \pi_Y$ give a right closing almost conjugacy between X and Y as G-SFTs.

$\square$

2.6 The irreducible but periodic case

Here we classify right closing almost conjugacy for irreducible but periodic G-SFTs. If (X, σ) is an irreducible G-SFT of period p, then we let $X^0, X^1, \ldots, X^{p-1}$ denote the cyclically moving subsets of X under σ. Then for $0 \leq n \leq p-1$, (X^n, σ^p) is a mixing SFT. The (X^n, σ^p) are pairwise conjugate SFTs and the action of G on (X, σ) permutes the (X^n, σ^p). If the entropy of (X, σ) is $\log(\lambda)$, then the entropy of each (X^n, σ^p) is $\log(\lambda^p)$. The ideal class (in $\mathbb{Z}[1/\lambda^p]$) of (X^n, σ^p) is determined by the ideal class (in $\mathbb{Z}[1/\lambda]$) of (X, σ). We let $\overline{X} = X^0$ and $\overline{\sigma} = \sigma^p|_{\overline{X}}$. Then as SFTs, X is conjugate to $\overline{X} \times \{0, \ldots, p-1\}$, where the shift for the latter is given by

$$\sigma(\overline{x}, n) = \begin{cases} (\overline{x}, n+1) & \text{if } 0 \le n \le p-2, \\ (\overline{\sigma}(\overline{x}), 0) & \text{if } n = p-1. \end{cases} \qquad (2.6.1)$$

We give to $\overline{X} \times \{0, \dots, p-1\}$ the G action which is the image under conjugacy of the G action on X, so that X is G-conjugate to $\overline{X} \times \{0, \dots, p-1\}$. Without loss of generality, we assume from now on that irreducible but periodic G-SFTs are of the form $(X, \sigma) = (\overline{X} \times \{0, \dots, p-1\}, \sigma)$, where the shift σ is given by 2.6.1.

By $\mathbb{Z}_p$ we mean the group of integers $\{0, 1, \dots, p-1\}$ with addition mod p. The G action on X determines a homomorphism $\phi_X \colon G \to \mathbb{Z}_p$, given by $\phi_X(g) = k$ if and only if $g \colon (\overline{X}, 0) \mapsto (\overline{X}, k)$. We refer to ϕ_X as the *action homomorphism* for the G-SFT (X, σ). Note that for $0 \le n \le p-1$ and for each $g \in G$, $g \colon (\overline{X}, n) \mapsto (\overline{X}, n + \phi_X(g)(\text{mod } p))$, where the action on the first coordinate is given by some automorphism U_g of $(\overline{X}, \overline{\sigma})$. The first coordinate automorphisms $\{U_g\}_{g \in G}$ define a G action on $(\overline{X}, \overline{\sigma})$, given by $g \colon \overline{x} \mapsto U_g(\overline{x})$. This G action on $\overline{X}$ is not necessarily free, even if the G action on X is free. We refer to the G-SFT $\overline{X}$ as the *base G-SFT* for X. We point out that base G-SFTs are mixing, so right closing almost conjugacy of base G-SFTs is classified by Theorem 2.5.2.

Theorem 2.6.2. *Let X and Y be irreducible G-SFTs. Then the following are equivalent.*

1. *X and Y are right closing almost conjugate as G-SFTs.*

2. *The base G-SFTs $\overline{X}$ and $\overline{Y}$ for X and Y are right closing almost conjugate as G-SFTs, and the action homomorphisms ϕ_X and ϕ_Y are the same.*

Proof. Suppose (X, σ) and (Y, σ) are right closing almost conjugate as G-SFTs. Then there is a G-SFT (Z, σ) and 1-1 *a.e.* right closing G-factor maps $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$. The maps π_X and π_Y preserve period, so Z must have period p, where p is the period of both X and Y. Furthermore Z must be irreducible because X and Y are irreducible. Without loss of generality, assume that $Z = \overline{Z} \times \{0, \ldots, p-1\}$ where $\overline{Z}$ is the base G-SFT for Z. Further assume $(\overline{X}, 0) = \pi_X(\overline{Z}, 0)$ and $(\overline{Y}, 0) = \pi_Y(\overline{Z}, 0)$, where $\overline{X}$ and $\overline{Y}$ are the base G-SFTs for X and Y respectively. Observe that for $0 \leq n \leq p-1$,

$$
\begin{aligned}
\pi_X(\overline{Z}, n) &= \pi_X \circ \sigma^n(\overline{Z}, 0) \\
&= \sigma^n \circ \pi_X(\overline{Z}, 0) \\
&= \sigma^n(\overline{X}, 0) \\
&= (\overline{X}, n).
\end{aligned}
$$

In particular $\phi_X = \phi_Z$ (since π_X intertwines G actions). Similarly $\phi_Y = \phi_Z$.

Let $P_Z \colon Z \to \overline{Z}$ be the G-factor map $(\overline{z}, n) \mapsto \overline{z}$ and let $P_X \colon X \to \overline{X}$ be the G-factor map $(\overline{x}, n) \mapsto \overline{x}$. Since $\pi_X(\overline{Z}, n) = (\overline{X}, n)$ for $0 \leq n \leq p-1$, there is a G-factor map $\overline{\pi}_X \colon \overline{Z} \to \overline{X}$ which makes the following diagram commute.

$$
\begin{array}{ccc}
Z & \xrightarrow{\pi_X} & X \\
{\scriptstyle P_Z}\downarrow & & \downarrow{\scriptstyle P_X} \\
\overline{Z} & \xrightarrow{\overline{\pi}_X} & \overline{X}
\end{array}
$$

$\overline{\pi}_X$ is 1-1 *a.e.* and right closing because π_X is. Similarly construct a 1-1 *a.e.* right closing G-factor map $\overline{\pi}_Y \colon \overline{Z} \to \overline{Y}$. Then $\overline{X}$ and $\overline{Y}$ are right closing almost conjugate as G-SFTs.

Conversely, suppose the base G-SFTs $(\overline{X}, \overline{\sigma})$ and $(\overline{Y}, \overline{\sigma})$ are right closing almost conjugate as G-SFTs, and $\phi = \phi_X = \phi_Y$. In particular, X and Y have the same period p. Let $(\overline{Z}, \overline{\sigma})$ be a G-SFT with 1-1 $a.e.$ right closing G-factor maps $\overline{\pi}_X \colon \overline{Z} \to \overline{X}$ and $\overline{\pi}_Y \colon \overline{Z} \to \overline{Y}$. Let $Z = \overline{Z} \times \{0, \ldots, p-1\}$ with the shift defined as in 2.6.1. Define a G action on Z by $g \colon (\overline{z}, n) \mapsto (\overline{z} \cdot g, n + \phi(g) (\bmod p))$. Define maps $\pi_X \colon Z \to X$ and $\pi_Y \colon Z \to Y$ by $\pi_X(\overline{z}, n) = (\overline{\pi}_X(\overline{z}), n)$ and $\pi_Y(\overline{z}, n) = (\overline{\pi}_Y(\overline{z}), n)$. Then π_X and π_Y are G-factor maps. They are 1-1 $a.e.$ and right closing because $\overline{\pi}_X$ and $\overline{\pi}_Y$ are.

$\square$

2.7 Regular isomorphism of G-Markov chains

Let (X, μ) and (Y, ν) be irreducible Markov chains with Markov measures μ and ν. Let α and β be the time zero partitions of X and Y, respectively. Consider the past σ-algebras $\alpha^- = \bigvee_{n=0}^{\infty} \sigma^n \alpha$ and $\beta^- = \bigvee_{n=0}^{\infty} \sigma^n \beta$. Then (X, μ) and (Y, ν) are *regularly isomorphic* if there is a measurable isomorphism $\phi \colon (X, \mu) \to (Y, \nu)$ such that

$$\phi^{-1}(\beta^-) \subset \sigma^{-N} \alpha^- = \alpha^- \vee \sigma^{-1} \alpha \vee \cdots \vee \sigma^{-N} \alpha \, ,$$

$$\phi(\alpha^-) \subset \sigma^{-N} \beta^- = \beta^- \vee \sigma^{-1} \beta \vee \cdots \vee \sigma^{-N} \beta$$

for some non-negative integer N. The idea of regular isomorphism was introduced and studied by Parry, first in [20] and also in [37]. For a regular isomorphism ϕ (in contrast to an arbitrary measurable isomorphism), to code the present $(\phi x)_0$, it

suffices to know the past and a bounded look into the future $x_{(-\infty,N]}$. Boyle and Tuncel [17] show this measurable coding relation has a more finite and continuous formulation, as follows.

Theorem 2.7.1. *Irreducible Markov chains (X,μ) and (Y,ν) are regularly isomorphic if and only if there exists an irreducible Markov chain (Z,η) and 1-1 a.e. right closing factor maps $\pi_X\colon (Z,\eta) \to (X,\mu)$ and $\pi_Y\colon (Z,\eta) \to (Y,\nu)$.*

A *G-Markov chain* is a Markov chain (X,μ) such that X is a G-SFT and μ is a G-invariant Markov measure on X. Say that irreducible G-Markov chains (X,μ) and (Y,ν) are *G-regularly isomorphic* if there is a regular isomorphism $\phi\colon (X,\mu) \to (Y,\nu)$ such that ϕ is G-equivariant. By Theorems 2.4.1 and 2.7.1 we have the following.

Corollary 2.7.2. *Mixing free G-Markov chains (X,μ_X) and (Y,μ_Y), with unique measures of maximal entropy μ_X and μ_Y, are G-regularly isomorphic if and only if (X,μ_X) and (Y,μ_Y) are regularly isomorphic as Markov chains.*

In the general irreducible case, G-regular isomorphism with respect to measures of maximal entropy can be classified in terms of the invariants of Theorem 2.6.2.

Chapter 3

Weak equivalence for shifts of finite type

3.1 Introduction

In [9], Beal and Perrin introduce an equivalence relation on subshifts called weak equivalence, and classify a special collection of irreducible shifts of finite type (SFTs) up to weak equivalence. We classify arbitrary SFTs up to weak equivalence. For this, first we prove an extension theorem: Given a homomorphism from the nonwandering set of an SFT S to the nonwandering set of an SFT T, we give necessary and sufficient conditions for the existence of an extension of the given homomorphism to a homomorphism from S to T. We use this extension result to give necessary and sufficient conditions for the existence of a homomorphism from one SFT to another.

Extension results are significant in symbolic dynamics. Boyle's extension result [10, Extension Lemma 2.4] is the key to characterizing when there exists an epimorphism from one mixing SFT onto another of lower entropy, and has had other applications as well. (For example, a refinement of that result, [17, Theorem 5.3], establishes that the Markovian property of a homomorphism is in a certain sense not local.) The extension theorem for inert automorphisms of Kim and Roush [25] is a central tool in the study of automorphisms of an SFT. (See [12], [27] and [28].) The study [7] provides a variety of surjective extension results. It follows from

the work of Maass [34] that an open problem on extensions, Problem 3.6.8 below, is very closely related to the problem of characterizing the limit sets of stable cellular automaton maps.

As a consequence of Lightwood's machinery for extending certain classes of homomorphisms of $\mathbb{Z}^2$ SFTs, we know that for a large class of examples the key issue for existence of an embedding between $\mathbb{Z}^2$ SFTs is the existence of a homomorphism between them [31, 32]. This provides some additional motivation for our existence result in the more tractable $d = 1$ case.

We thank Danrun Huang for calling our attention to the paper of Beal and Perrin, and we thank Mike Boyle for many helpful discussions about this problem.

3.2 Preliminaries

3.2.1 Shifts of finite type

We assume some familiarity with shifts of finite type. See [29] and [33] for more complete background.

Given $n > 0$, let $\Sigma_n = \{0, \dots, n-1\}^{\mathbb{Z}}$ denote the set of doubly-infinite sequences $(x_i)_{i \in \mathbb{Z}}$ where each $x_i \in \{0, \dots, n-1\}$. Give to Σ_n the product of the discrete topology on $\{0, \dots, n-1\}$. Then Σ_n is compact and metrizable. The *shift* is the homeomorphism $\sigma : \Sigma_n \to \Sigma_n$ given by $\sigma(x)_i = x_{i+1}$. The pair (Σ_n, σ) (or just Σ_n for short) is the *full n-shift*. A *full shift* is a full n-shift for some n. If S is a closed, σ-invariant subset of a full shift, then the pair (S, σ) (or just S for short) is a *subshift*.

A *word* on $\{0, \ldots, n-1\}$ is a finite concatenation $w_1 \ldots w_k$, where each $w_i \in \{0, \ldots, n-1\}$. Given a subshift S, a point $x \in S$, and coordinates $i < j$, often we write $x[i, j]$ to denote the word $x_i \cdots x_j$.

If a subshift S may be written

$$S = \Sigma_n \setminus \{x \in \Sigma_n \; : \; \text{no word in } \mathcal{F} \text{ appears in } x\}$$

for a finite set of words $\mathcal{F}$, then S is a *shift of finite type (SFT)*.

A *homomorphism* $f : S \to T$ from one SFT to another is a σ-equivariant function such that, for each $x \in S$, $f(x)_0$ is determined by $x_{-i} \cdots x_0 \cdots x_j$ for some $i, j \geq 0$. Then f is r-*block* where $r = i + j + 1$. If f is surjective it is an *epimorphism*. An injective epimorphism is an *isomorphism*. Any homomorphism $f : S \to T$ is isomorphic to a one-block homomorphism, in the sense that there exists an SFT S' and an isomorphism $\varphi : S' \to S$ such that $f \circ \varphi$ is one-block.

3.2.2 Edge shifts

Let A be an $n \times n$ matrix over the non-negative integers which has no 0-row or 0-column. Then A is the adjacency matrix for a directed graph $\mathcal{G}(A)$: $\mathcal{G}(A)$ has vertex set $\{v_1, \ldots, v_n\}$ where the number of edges from v_i to v_j is $A(i, j)$. Let $\mathcal{E}(A) = \{\text{edges in } \mathcal{G}(A)\}$ and $\mathcal{V}(A) = \{\text{vertices in } \mathcal{G}(A)\}$. Put

$$\Sigma_A = \{x = (x_i)_{i \in \mathbb{Z}} \in \mathcal{E}(A)^{\mathbb{Z}} \mid \text{each } x_i x_{i+1} \text{ is a path in } \mathcal{G}(A)\}.$$

Give to Σ_A the relative of the product of the discrete topology on $\mathcal{E}(A)$. Then, by thinking of edges in $\mathcal{E}(A)$ as numbers in $\{0, \ldots, |\mathcal{E}(A)| - 1\}$, Σ_A is a special type of

subshift, called an *edge shift.* Any edge shift is an SFT. Also, any SFT is isomorphic to an edge shift.

3.2.3 Irreducible shifts of finite type

Let A be as in section 3.2.2. Then A, $\mathcal{G}(A)$, and Σ_A are *irreducible* if, for each $1 \leq i, j \leq n$, there exists $L = L(i, j)$ such that $A^L(i, j) > 0$. If there exists a uniform $L > 0$ such that $A^L(i, j) > 0$ for all $1 \leq i, j \leq n$, then A and $\mathcal{G}(A)$ are *primitive*. If A is primitive, then Σ_A is *mixing.*

A point $x \in \Sigma_A$ is *periodic* if there exists $p > 0$ such that $\sigma^p(x) = x$. In this case p is a *period* of x. When A is irreducible, we define the period of A, $\mathcal{G}(A)$ and Σ_A to be the greatest common divisor of the set of periods of the periodic points in Σ_A. Note that for primitive A, the period of A is 1.

Given a path $w = w_0 \cdots w_{l-1}$ of edges in $\mathcal{G}(A)$, the *length* of w is $|w| = l$. Denote by $\mathrm{in}(w)$ and $\mathrm{ter}(w)$ the initial and terminal vertices of w, respectively. If A is irreducible, say that vertices v_i and v_j in $\mathcal{V}(A)$ are *period equivalent* if there is a path w in $\mathcal{G}(A)$ with $\mathrm{in}(w) = v_i$ and $\mathrm{ter}(w) = v_j$ such that $|w|$ is divisible by $p = $ the period of A. This defines an equivalence relation on $\mathcal{V}(A)$, and induces a partition of $\mathcal{V}(A)$ into a disjoint union of p sets

$$\mathcal{V}(A) = \mathcal{V}^0(A) \sqcup \cdots \sqcup \mathcal{V}^{p-1}(A).$$

The $\mathcal{V}^i(A)$ are called the *cyclically moving vertex sets* of $\mathcal{V}(A)$. This partition of $\mathcal{V}(A)$ induces a partition of Σ_A into p sets

$$\Sigma_A = \Sigma_A^0 \sqcup \cdots \sqcup \Sigma_A^{p-1},$$

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where $\Sigma_A^i = \{x \in \Sigma_A : \text{in}(x_0) \in \mathcal{V}^i(A)\}$. The Σ_A^i are called the *cyclically moving subsets* of Σ_A. By re-labelling if necessary, we may assume that $\sigma(\Sigma_A^i) = \Sigma_A^{(i+1) \mod p}$ for $0 \le i \le p - 1$. Each Σ_A^i is both invariant and mixing under σ^p.

3.2.4 Reducible shifts of finite type

Let A be as in section 3.2.2. The *nonwandering* set of Σ_A, denoted $NW(A)$, is the closure of the set of periodic points in Σ_A. The nonwandering set $NW(A)$ is an SFT, and is a disjoint union

$$NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K,$$

where each $C(A)_i$ is an irreducible SFT. The $C(A)_i$ are called the *irreducible components* of Σ_A. If $K > 1$, then Σ_A is *reducible*.

For $1 \le i \le K$, let p_i denote the period of $C(A)_i$. Then $C(A)_i$ is partitioned into cyclically moving subsets $C(A)_i^0, \ldots, C(A)_i^{p_i-1}$, each of which is both invariant and mixing under σ^{p_i}, as in Section 3.2.3. Therefore we can find $l_i \ge 1$ such that if q and q' are any two vertices in the same cyclically moving subset of the graph $\mathcal{G}(C(A)_i)$, then there exists a path w in $\mathcal{G}(C(A)_i)$ of length $|w| = p_i \cdot l_i$ such that $\text{in}(w) = q$ and $\text{ter}(w) = q'$. Such l_i is called a *cyclic transition length* for $C(A)_i$.

3.3 Homomorphisms between irreducible SFTs

Let S and T be subshifts. Write $S \xrightarrow{PER} T$ if, for each periodic point $x \in S$ of period p, there is a periodic point $y \in T$ of period q, such that q divides p.

Theorem 3.3.1. *Given irreducible SFTs S and T, there exists a homomorphism $f : S \to T$ if and only if $S \xrightarrow{\text{PER}} T$. Moreover, assuming $S \xrightarrow{\text{PER}} T$, then given any pair of cyclically moving subsets S^0 and T^0 in S and T respectively, $f : S \to T$ may be chosen so that $f(S^0) \subset T^0$.*

Proof. If $f : S \to T$ is a block code and $x \in S$ is a periodic point of period p, then $f(x) \in T$ must be a periodic point of some period dividing p, by the σ-equivariance of f.

Conversely, first suppose $S \xrightarrow{\text{PER}} T$ and T is mixing. In this case, let $\overline{S}$ be the empty subshift, and let $\overline{f} : \overline{S} \to T$ be the empty homomorphism. By [10, Extension Lemma 2.4], $\overline{f}$ extends to a homomorphism $f : S \to T$.

Now suppose $S \xrightarrow{\text{PER}} T$ where T is no longer assumed to be mixing. Decompose S and T into disjoint unions of cyclically moving subsets, each of which is invariant and mixing under the pth power of the shift, where p is the period of S (the period of T divides p by assumption). Fix cyclically moving subsets S^0 of S and T^0 of T, and let $X = (S^0, \sigma^p)$ and $Y = (T^0, \sigma^p)$. Then $X \xrightarrow{\text{PER}} Y$ and Y is mixing so, by the previous paragraph, there exists a homomorphism $f_0 : X \to Y$. Every other cyclically moving subset of S is equal to $\sigma^i(S^0)$ for some unique $1 \le i < p$. So define a homomorphism f_i on $\sigma^i(S^0)$ by $f_i = \sigma^i \circ f_0 \circ \sigma^{-i}$, and define $f : S \to T$ by $f(x) = f_i(x)$ where $x \in \sigma^i(S^0)$.

$\square$

3.4 Extending homomorphisms defined on a nonwandering set

Let Σ_A be a reducible edge shift. Write the nonwandering set $NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K$ as a disjoint union of irreducible components, and let p_i denote the period of $C(A)_i$. Fix in each $C(A)_i$ one cyclically moving subset $C(A)_i^0$, and enumerate the remaining cyclically moving subsets in $C(A)_i$ by $C(A)_i^1, \ldots, C(A)_i^{p_i-1}$, where $C(A)_i^k = \sigma^k(C(A)_i^0)$.

Given this enumeration of the cyclically moving subsets, for each i, j, define the set of *connection paths* from $C(A)_i$ to $C(A)_j$, denoted $CP_A(i, j)$, to be the set of paths in $\mathcal{G}(A)$ of the form $x_0 \cdots x_{t-1}$ which have initial vertex in some $C(A)_i^s$, and terminal vertex in some $C(A)_j^r$. The number $s + t - r$, taken mod $\gcd(p_i, p_j)$, is the *phase change* of the path $x_0 \cdots x_{t-1}$.

Now define a $K \times K$ matrix P_A as follows. The entries of P_A are sets. Specifically, $P_A(i, j) \subset \mathbb{Z}_{\gcd(p_i, p_j)}$ is the set of phase changes of paths in $CP(A_i, A_j)$. Let $\mathcal{M}(A)$ denote the set of phase matrices for A. Note that the finitely many possible enumerations of the cyclically moving subsets determine the finitely many matrices in $\mathcal{M}(A)$. By *a phase matrix* for A we mean a matrix P_A determined by some enumeration of cyclically moving subsets.

Given x and x' in Σ_A, say that x and x' are *backwardly asymptotic* if there exists $k \in \mathbb{Z}$ such that $x_i = x'_i$ for all $i \leq k$, and *forwardly asymptotic* if there exists $k \in \mathbb{Z}$ such that $x_i = x'_i$ for all $i \geq k$.

Remark 3.4.1. Let P_A be a phase matrix for A. Then $c \in P_A(i, j)$ if and only if there exist x, y and z in Σ_A such that

- z is backwardly asymptotic to x and forwardly asymptotic to y,

- $x \in C(A)_i^s$ and $y \in C(A)_j^r$, and

- $s - r \equiv c \mod \gcd(p_i, p_j)$.

It follows that, if $\phi : \Sigma_A \to \Sigma_C$ is an isomorphism which respects the numberings of the irreducible components and cyclically moving subsets, then the associated phase matrices P_A and P_C are equal. In particular, the set $\mathcal{M}(A)$ of possible phase matrices for A is an isomorphism invariant.

Example 3.4.2. Let A be the adjacency matrix for the following graph.
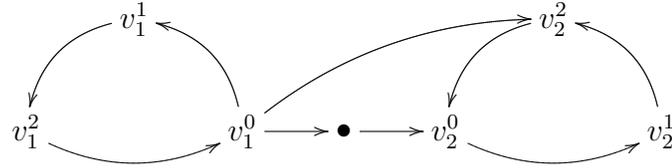
Let $C(A)_1$ correspond to the cycle on the left and let $C(A)_2$ correspond to the cycle on the right. Choose the cyclically moving subset $C(A)_1^0$ to be those points $x \in C(A)_1$ such that the initial vertex of x_0 is v_1^0, and $C(A)_2^0$ to be those points $x \in C(A)_2$ such that the initial vertex of x_0 is v_2^0. Then the phase matrix for this choice is

$$P_A = \begin{pmatrix} \{0\} & \{2\} \\ \emptyset & \{0\} \end{pmatrix}.$$

Remark 3.4.3. Let $\mathcal{D}(A)$ denote the set of $K \times K$ diagonal matrices D, where each $D(i,i)$ is an element of the group $\mathbb{Z}_{p_i}$. Note that, if M and M' are any two phase

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matrices in $\mathcal{M}(A)$, then there is a matrix $D \in \mathcal{D}(A)$ such that $M' = DMD^{-1}$, by which we mean that, for each i, j,

$$M'(i, j) = \{D(i, i) + s - D(j, j) \; : \; s \in M(i, j)\}. \qquad (3.4.4)$$

Conversely, given $M \in \mathcal{M}(A)$ and $D \in \mathcal{D}(A)$, the matrix DMD^{-1} is in $\mathcal{M}(A)$. Thus the set $\mathcal{M}(A)$ is computable via the following finite process:

1. Arbitrarily choose an enumeration of the cyclically moving subsets for A.

2. Construct the phase matrix P_A for this choice, as defined above, and include this P_A in $\mathcal{M}(A)$.

3. For each $D \in \mathcal{D}(A)$, add to $\mathcal{M}(A)$ the matrix $DP_A D^{-1}$.

Now let Σ_B be another reducible edge shift with nonwandering set $NW(B) = C(B)_1 \sqcup \cdots \sqcup C(B)_L$, and let q_i denote the period of the irreducible component $C(B)_i$. Suppose there exists a homomorphism $f_0 : NW(A) \to NW(B)$. Then f_0 induces a set function $g : \{1, \ldots, K\} \to \{1, \ldots, L\}$ by the following rule: $C(B)_{g(i)}$ is the irreducible component of Σ_B which contains $f_0(C(A)_i)$.

In each irreducible component $C(B)_i$, arbitrarily choose one cyclically moving subset $C(B)_i^0$, and enumerate the remaining cyclically moving subsets in $C(B)_i$ by $C(B)_i^k = \sigma^k(C(B)_i^0)$. Let P_B be the phase matrix determined by this choice. Say that a phase matrix P_A for A is *compatible* with (P_B, f_0) if P_A is the phase matrix determined by a choice of the cyclically moving subsets $C(A)_i^0$ in $C(A)_i$ such that each $f_0(C(A)_i^0) \subset C(B)_{g(i)}^0$. For such P_A let $\overline{P}_A$ be the matrix such that each $\overline{P}_A(i, j)$ is the set of elements in $P_A(i, j)$ taken $\mod \gcd(q_{g(i)}, q_{g(j)})$.

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Theorem 3.4.5. *Let Σ_A and Σ_B be reducible edge shifts, as above, and let f_0 : $NW(A) \to NW(B)$ be a homomorphism. Let $g : \{1,\ldots,K\} \to \{1,\ldots,L\}$ be the set function induced by f_0. Choose a phase matrix P_B for B, and let P_A be a phase matrix for A compatible with (P_B, f_0).*

Then f_0 extends to a homomorphism $f : \Sigma_A \to \Sigma_B$ if and only if each $\overline{P}_A(i,j)$ is contained in $P_B(g(i), g(j))$.

Proof. Suppose f_0 extends to $f : \Sigma_A \to \Sigma_B$. Let $\overline{c} \in \overline{P}_A(i,j)$ and let $c \in P_A(i,j)$ satisfy $c \equiv \overline{c} \mod \gcd(q_{g(i)}, q_{g(j)})$. By Remark 3.4.1 there exist x, y and z in Σ_A such that

- z is backwardly asymptotic to x and forwardly asymptotic to y,

- $x \in C(A)_i^s$ and $y \in C(A)_j^r$, and

- $s - r \equiv c \mod \gcd(p_i, p_j)$.

Let $\overline{s}$ ($\overline{r}$, respectively) denote s (r, resp.) taken modulo $q_{g(i)}$ ($q_{g(j)}$, resp.). Then

- $f(z)$ is backwardly asymptotic to $f(x)$ and forwardly asymptotic to $f(y)$,

- $f(x) \in C(B)_{g(i)}^{\overline{s}}$ and $f(y) \in C(B)_{g(j)}^{\overline{r}}$, and

- $\overline{s} - \overline{r} \equiv \overline{c} \mod \gcd(q_{g(i)}, q_{g(j)})$.

Hence $\overline{c} \in P_B(g(i), g(j))$.

For the converse first note that we may assume, without loss of generality, that f_0 is one-block. For, if not, then we can choose an SFT Σ_C and an isomorphism $\varphi : \Sigma_C \to \Sigma_A$ such that $f_0 \circ \varphi|_{NW(C)}$ is one-block. The isomorphism φ can be taken to

respect the numberings of the irreducible components and cyclically moving subsets, which implies $P_C = P_A$, by Remark 3.4.1. It follows that f_0 extends to all of Σ_A if and only if $f_0 \circ \varphi|_{NW(C)}$ extends to all of Σ_C.

For the converse, we will use the following.

Claim 3.4.6. *Suppose each $\overline{P}_A(i,j) \subset P_B(g(i), g(j))$. Then there exists $M > 0$ such that, for each i, j and for each path $Q \in CP_A(i,j)$ of length $|Q| \geq M$, there exists a path $Q' \in CP_B(g(i), g(j))$ such that $|Q'| = |Q|$, $\mathrm{in}(Q') = f_0(\mathrm{in}(Q))$, and $\mathrm{ter}(Q') = f_0(\mathrm{ter}(Q))$.*

To prove Claim 3.4.6, first recall that each $C(A)_i$ has a cyclic transition length l_i. WLOG assume that l_i is a cyclic transition length for $C(B)_{g(i)}$ as well. (If not, just make l_i larger.) Let

$$l = \max_i \{l_i\},$$

and let

$$p = \prod_i p_i.$$

Note that if v and v' are any two vertices in the same cyclically moving subset of some $C(B)_{g(i)}$, then there is a path in $C(B)_{g(i)}$ of length lp from v to v'.

Next, pick T large enough so that, for each i, j and for each $\overline{c} \in \overline{P}_A(i,j)$, there exists a path $Q \in CP_B(g(i), g(j))$ of length at most T with phase change $\overline{c}$.

Then, for each i, j, pick $R_{i,j}$ large enough so that all integers r greater than $R_{i,j}$ and divisible by $\gcd(q_{g(i)}, q_{g(j)})$ are contained in the $\mathbb{N}$-ideal

$$< q_{g(i)}, q_{g(j)} >_{\mathbb{N}} := \{ m q_{g(i)} + n q_{g(j)} \ : \ m \in \mathbb{N}, \ n \in \mathbb{N} \}.$$

Let $R = \max_{i,j} R_{i,j}$, and set

$$M = T + R + 2lp + 2\max_i q_{g(i)}.$$

To see that M satisfies the requirements of Claim 3.4.6, let $Q \in CP_A(i,j)$ have length $|Q| \geq M$ and phase change c. Assume that $\text{in}(Q)$ is in some $C(A)_i^s$ and $\text{ter}(Q)$ is in some $C(A)_j^r$, so that $c \equiv s + |Q| - r \mod \gcd(p_i, p_j)$. Let $Q_1 \in CP_B(g(i), g(j))$ have phase change $\bar{c} \equiv c \mod \gcd(q_{g(i)}, q_{g(j)})$ and $|Q_1| \leq T$.

Extend Q_1 to the left in $\mathcal{G}(C(B)_{g(i)})$ by at most $q_{g(i)}$ and to the right in $\mathcal{G}(C(B)_{g(j)})$ by at most $q_{g(j)}$ to a path $Q_2 \in CP_B(g(i), g(j))$ with $\text{in}(Q_2) \in C(B)_{g(i)}^{\bar{s}}$ and $\text{ter}(Q_2) \in C(B)_{g(j)}^{\bar{r}}$, where $\bar{s} \equiv s \mod q_{g(i)}$ and $\bar{r} \equiv r \mod q_{g(j)}$. Note that Q_2 has phase change $\bar{c}$. Also, $|Q| - 2lp - |Q_2|$ is divisible by $\gcd(q_{g(i)}, q_{g(j)})$ and greater than R, so

$$|Q| - 2lp - |Q_2| \ \in\ < q_{g(i)}, q_{g(j)} >_{\mathbb{N}} .$$

Hence

$$|Q| - 2lp = |Q_2| + a q_{g(i)} + b q_{g(j)}$$

for some $a, b \geq 0$. So we may extend Q_2 to the left in $\mathcal{G}(C(B)_{g(i)})$ by $a q_{g(i)}$ and to the right in $\mathcal{G}(C(B)_{g(j)})$ by $b q_{g(j)}$ to a path $Q_3 \in CP_B(g(i), g(j))$ with

- $\text{in}(Q_3) \in C(B)_{g(i)}^{\bar{s}}$,

- $\text{ter}(Q_3) \in C(B)_{g(j)}^{\bar{r}}$, and

- $|Q| = |Q_3| + 2lp$.

Finally, by choice of l, we may extend Q_3 to the left in $\mathcal{G}(C(B)_{g(i)})$ by lp and to the right in $\mathcal{G}(C(B)_{g(j)})$ by lp to a path $Q' \in CP_B(g(i), g(j))$ with

- $\mathrm{in}(Q') = f_0(\mathrm{in}(Q))$,

- $\mathrm{ter}(Q') = f_0(\mathrm{ter}(Q))$, and

- $|Q| = |Q'|$.

This completes the proof of Claim 3.4.6

Now define $f : \Sigma_A \to \Sigma_B$ as follows. Let $M > 0$ be as in Claim 3.4.6, and let $Q \to Q'$ be a chosen associated map on connection paths Q of length at least M. Consider the set of paths VWX in $\mathcal{G}(A)$ which satisfy

1. V is in a component $C(A)_i$ and $|V| = M$,

2. X is in a component $C(A)_j$ and $|X| = M$,

3. the only $C(A)_i$-vertex of W is its initial vertex, and the only $C(A)_j$-vertex of W is its terminal vertex, and

4. W does not contain any path of length M from $NW(A)$.

Let $x \in \Sigma_A$. For each $k \in \mathbb{Z}$ such that $x[k, k + |VWX| - 1]$ is a path satisfying $(1) - (4)$ above, set $f(x)[k, k + |VW| - 1] = (VW)'$. Elsewhere in x, define $f(x)_i = f_0(x)_i$. As there is an upper bound on the lengths of paths VWX which satisfy $(1) - (4)$ above, f is a homomorphism. By construction, f_0 is the restriction of f to $NW(A)$.

$\square$

3.5 Homomorphisms between reducible SFTs

Our main theorem below characterizes when there exists a homomorphism between arbitrary SFTs.

Theorem 3.5.1. *Let Σ_A and Σ_B be reducible edge shifts with nonwandering sets $NW(A) = C(A)_1 \sqcup \cdots \sqcup C(A)_K$ and $NW(B) = C(B)_1 \sqcup \cdots \sqcup C(B)_L$. Let P_B be a phase matrix for B. Then there exists a homomorphism $f : \Sigma_A \to \Sigma_B$ if and only if we may choose a set function $g : \{1, \ldots, K\} \to \{1, \ldots, L\}$ and a phase matrix P_A for A such that, for $1 \le i, j \le K$,*

1. $C(A)_i \xrightarrow{PER} C(B)_{g(i)}$, and

2. $\overline{P}_A(i, j) \subset P_B(g(i), g(j))$.

Proof. Given a homomorphism $f : \Sigma_A \to \Sigma_B$, let $C(B)_{g(i)}$ be the irreducible component of Σ_B containing $f(C(A)_i)$. This defines a set function $g : \{1, \ldots, K\} \to \{1, \ldots, L\}$. Then, for $1 \le i \le K$, $f|_{C(A)_i} : C(A)_i \to C(B)_{g(i)}$ is a homomorphism, so $C(A)_i \xrightarrow{PER} C(B)_{g(i)}$, by Theorem 3.3.1.

For $1 \le j \le L$ and $i \in g^{-1}(j)$, choose $C(A)_i^0$ to be a cyclically moving subset of $C(A)_i$ such that $f(C(A)_i^0) \subset C(B)_j^0$. Let P_A be the phase matrix for A induced by this choice of the cyclically moving subsets $C(A)_i^0$ in $C(A)_i$. Then P_A is compatible with (P_B, f_0), where $f_0 = f|_{NW(A)}$. By Theorem 3.4.5 each $\overline{P}_A(i, j) \subset P_B(g(i), g(j))$.

Conversely suppose we may choose a set function $g : \{1, \ldots, K\} \to \{1, \ldots, L\}$ and a phase matrix P_A for A such that conditions (1) and (2) of Theorem 3.5.1 are satisfied for $1 \le i, j \le K$. By condition (1) and Theorem 3.3.1, there exists a

homomorphism $f_0 : NW(A) \to NW(B)$ such that $C(B)_{g(i)}$ is the irreducible component of Σ_B containing $f_0(C(A)_i)$. Moreover, by Theorem 3.3.1, f_0 may be chosen so that each $C(B)^0_{g(i)}$ contains $f_0(C(A)^0_i)$ (i.e. P_A is compatible with (P_B, f_0)). It then follows from Theorem 3.4.5 that f_0 extends to a homomorphism $f : \Sigma_A \to \Sigma_B$.

$\square$

3.6   Weak equivalence of SFTs

Definition 3.6.1. Let S and T be subshifts. If $\varphi : \Sigma_n \to \Sigma_m$ is a homomorphism between full shifts Σ_n and Σ_m which contain S and T respectively, such that $\varphi^{-1}(T) = S$, then write $\varphi : S \xrightarrow{\sim} T$. If such a φ exists, write $S \xrightarrow{\sim} T$

Definition 3.6.2 (Beal and Perrin). Subshifts S and T are *weak equivalent* if $S \xrightarrow{\sim} T$ and $T \xrightarrow{\sim} S$.

A *flower shift* is an SFT presented by a directed graph made up of loops that all begin and end at a single vertex. If the lengths of the loops of a flower shift S are $s := (s_1, \ldots, s_n)$, then we define $< s >_{\mathbb{N}}$ to be the $\mathbb{N}$-ideal $s_1\mathbb{N} + \cdots + s_n\mathbb{N}$. Given flower shifts S and T, which define ideals $< s >_{\mathbb{N}}$ and $< t >_{\mathbb{N}}$, Beal and Perrin [9] prove that S and T are weak equivalent if and only if $< s >_{\mathbb{N}} = < t >_{\mathbb{N}}$.

Proposition 3.6.3. *The following are equivalent for subshifts S and T.*

1. *$S \xrightarrow{\sim} T$.*

2. *There exists an SFT S' containing S, a subshift T' containing T, and a homomorphism $f : S' \to T'$ such that $f^{-1}(T) = S$.*

Proof. (1) $\Rightarrow$ (2): Let (f, S', T') be $(\varphi, \Sigma_n, \Sigma_m)$ from Definition 3.6.1.

(2) $\Rightarrow$ (1): Choose $r \geq 2$ so that f is r-block. We may assume WLOG that S' is a 1-*step* SFT, which means that a point x is in S' if and only if $x_i x_{i+1}$ is an allowed word of length 2 in S' for each $i \in \mathbb{Z}$. Let Σ_{n-1} and Σ_{m-1} be full shifts containing S' and T' respectively. Define an r-block homomorphism $\varphi : \Sigma_n \to \Sigma_m$ by:

$$\varphi(x_0 \cdots x_{r-1}) = \begin{cases} f(x_0 \cdots x_{r-1}) & \text{if } x_0 \cdots x_{r-1} \text{ is allowed in } S' \\ \\ m-1 & \text{if } x_0 \cdots x_{r-1} \text{ is not allowed in } S' \end{cases}$$

Then $\varphi : \Sigma_n \to \Sigma_m$ is well-defined, and $\varphi^{-1}(T) = S$.

$\square$

Theorem 3.6.4. *Let S be an SFT and let T be a subshift. Then $S \xrightarrow{\sim} T$ if and only if there exists a homomorphism $f : S \to T$.*

Proof. If $\varphi : S \xrightarrow{\sim} T$, then $f = \varphi|_S$ is a homomorphism. Conversely, if $f : S \to T$ is a homomorphism, then letting $S' = S$ and $T' = T$, condition (2) of Proposition 3.6.3 is satisfied. Hence $S \xrightarrow{\sim} T$.

$\square$

The requirement that S be an SFT can not be removed from Theorem 3.6.4. To see this note that, if S is a non-SFT subshift and T is an SFT, then it can not be the case that $S \xrightarrow{\sim} T$. However there are plenty of examples of homomorphisms from non-SFT subshifts to SFT subshifts.

Corollary 3.6.5. *Let S and T be SFTs. Then S and T are weak equivalent if and only if there exist homomorphisms $f : S \to T$ and $g : T \to S$.*

Corollary 3.6.5 follows directly from Theorem 3.6.4. Corollary 3.6.5 combined with Theorem 3.5.1 provide a classification of weak equivalence for SFTs.

The next proposition is an easy observation which we record to further emphasize the local nature of the weak equivalence relation. For a subshift S, let $W_k(S)$ denote the set of words of length k occurring in points in S. Given $n \in \mathbb{N}$, let S_n denote the largest SFT such that $W_n(S_n) = W_n(S)$. If $f : S \to T$ is an R-block homomorphism between subshifts then, for $r \geq R$, the R-block rule defining f also defines a homomorphism $f_r : S_r \to T_{r-R}$.

Proposition 3.6.6. *Suppose $\varphi : S \xrightarrow{\sim} T$, and let $f = \varphi|_S$. Then there exists a positive integer R such that, for all $r \geq R$, f_r is well-defined and $f_r^{-1}(T) = S$.*

Proof. We have $\varphi : \Sigma_n \to \Sigma_m$ for some n and m. Choose R so that φ is R-block. Observe that $S \subset S_R \subset \Sigma_n$ because $S_R \subset S_1$ and S_1 is the smallest full shift containing S. Also $\varphi|_{S_R} = f_R$ because φ and f are defined by the same R-block rule. Therefore $\varphi|_{S_r} = f_r$ for all $r \geq R$. We know $\varphi^{-1}(T) = S$, so $(\varphi|_{S_r})^{-1}(T) = S$ and hence $f_r^{-1}(T) = S$ for all $r \geq R$. $\square$

Say that a homomorphism f defined on a subshift S is *steady* if there is an SFT S' containing S such that f is well-defined on S', and f applied to S' has the same image as does f. A subshift S is *sofic* if there exists an SFT R and an epimorphism $R \to S$.

Remark 3.6.7. If $S \xrightarrow{\sim} T$ where S is a non-SFT subshift, then there exists a homomorphism $f : S \to T$ which fails to be steady. This is true in particular if S is strictly sofic. To see this, suppose $\varphi : S \xrightarrow{\sim} T$ and let $f = \varphi|_S$. If f were steady,

then there would exist an SFT S' containing S such that f is well-defined on S' and $f(S') = T$. Then, for all r sufficiently large, f_r is well-defined and $S_r \subset S'$, so $f_r^{-1}(T) = S_r$. But $S_r \neq S$ because S is non-SFT, contradicting Proposition 3.6.6.

Remark 3.6.7 points to a seeming resemblance between Proposition 3.6.6 and the following difficult open problem.

Problem 3.6.8. *Give necessary and sufficient conditions for a homomorphism defined on a sofic shift to be steady.*

One motivation for Problem 3.6.8 is to characterize the limit sets of stable cellular automaton maps. These limit sets are the subshifts T for which there exists a cellular automaton map f and a positive integer r such that $T = \text{Image}(f^r) = \text{Image}(f^{r+1})$. In [34], Maass studies such limit sets, and shows that they are precisely the mixing sofic shifts S which contain a receptive fixed point and which admit a steady epimorphism $f : S \to S$.

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