ABSTRACT

Title of dissertation: LIFTING OF CHARACTERS ON P-ADIC ORTHOGONAL AND METAPLECTIC GROUPS

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Let F be $p-$adic field of characteristic zero. Consider a dual pair $(\widetilde{Sp}(2n), SO(2n+1)_+)$, where $\widetilde{Sp}(2n)$ is the metaplectic cover of the symplectic group $Sp(2n)$ and $SO(2n+1)_+$ is the split orthogonal group over F. We show that there is a matching of Cartan subgroups between $SO(2n+1)_+$ and $\widetilde{Sp}(2n)$ via stabilized orbit correspondence. We say two representations of $SO(2n+1)_+$ and $\widetilde{Sp}(2n)$ correspond, if their characters on matching Cartan subgroups differ by a transfer factor, which is essentially character of the difference of the two halves of the oscillator representation. We show that this correspondence is compatible with parabolic induction: if two representations of Levi factors correspond, then after parabolic induction the two resulting representations also correspond. These results were motivated by the paper *Lifting of characters on orthogonal and metaplectic groups* by J. Adams who considered the case $F = \mathbb{R}$.

LIFTING OF CHARACTERS ON P-ADIC ORTHOGONAL AND METAPLECTIC GROUPS

by

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TABLE OF CONTENTS

Chapter 1

Introduction

Investigating transfer and lifting of representations is a very important part of the theory of representations and automorphic forms. For example, the local Langlands conjecture states that representations of a *linear* algebraic group G are parameterized by data related to the "dual" group ${}^LG$. Assuming this, a map between dual groups $\phi : {}^LH \to {}^LG$ should be related to a "transfer" of representations between G and H. Conversely a transfer of representations between H and G should be explained in terms of such a homomorphism ϕ.

Another approach to this problem is the Howe Theta correspondence. This correspondence matches representations of G and G', for any dual pair of subgroups (G, G') of the metaplectic cover $\widetilde{Sp}(2N)$ of the symplectic group $Sp(2N)$. It is important, that in this case the groups G and G' need not be linear.

A particulary interesting example of a dual pair is $(SO(2n+1)_+, \widetilde{Sp}(2n)) \subset \widetilde{Sp}(2N)$, where $N = 2n(2n+1)$ and $SO(2n+1)_+$ denotes the split orthogonal group. The properties of such pairs in the real case were investigated by Adams and Barbasch in [A-B]. Their main result is that there is a natural bijection between genuine irreducible representations of the metaplectic group $\widetilde{Sp}(2n)$ and the irreducible representations of the groups $SO(p, q)$ where $p+q = 2n+1$. (A representation of $\widetilde{Sp}(2n)$ is called genuine if it does not factor to a representation of $Sp(2n)$.)

A natural question is if the dual pair correspondence can be interpreted on the level of characters. In the real case this problem was solved by Adams in [A1]. He defined a lifting of stable characters between orthogonal and metaplectic groups. A character of $\mathbf{G}(F)$ is stable, roughly speaking, if it is invariant by conjugation by $\mathbf{G}(\overline{F})$, where $\overline{F}$ denotes the algebraic closure of F. Stable characters arise naturally in the study of characters for linear groups, see Section 7.2 for a discussion of stability. For tempered representations the lifting of stable characters agrees with the stabilized dual pair correspondence. Moreover, this lifting has several other nice properties: it takes discrete series to discrete series and "small" representations to "small" representations (for example the trivial representation of $SO(n+1,n)$ lifts to the difference of the two halves of the oscillator representation of $\widetilde{Sp}(2n)$.

The purpose of this thesis is to study lifting of characters in the case of $p-adic$ fields of characteristic zero. The case when $n=1$ was solved by Schultz in [Sch]. He established a bijection between stable virtual characters of $\widetilde{SL}(2)$ and irreducible representations of $SO(3)_+$ via the character theory.

We are considering admissible representations of $SO(2n+1)_+$ and genuine admissible representations of $\widetilde{Sp}(2n)$. For a linear reductive group G over a local or real field it is well known (see for example the work of Harish-Chandra, [HC]) that a character of an admissible representation π defined as a distribution

$$f \mapsto tr \int f(x)\pi(x)dx, \quad f \in C_c^\infty(G)$$

is given by a locally integrable function. We assume the same holds for the metaplectic group $\widetilde{Sp}(2n)$[1].

First (Chapter 2) we investigate the orbit correspondence between strongly regular semisimple elements of $SO(2n+1)_+$ and $Sp(2n)$ and the resulting matching of Cartan subgroups. We also study the "stabilized" version of this correspondence. One can define the stable orbit correspondence in the following way: two strongly regular semisimple elements $g \in Sp(2n)$ and $g' \in SO(2n+1)_+$ stably correspond, if they have the same nontrivial eigenvalues. The result is that every strongly regular semisimple element of $SO(2n+1)_+$ stably corresponds to a strongly regular semisimple element of $Sp(2n)$. The converse (which is nontrivial because of the requirement on the orthogonal group to be split) is also true and it is proved in Chapter 3 by the methods of Galois cohomology. Alternatively, we obtain the result that every Cartan subgroup in the symplectic group can be embedded into the split orthogonal group, and vice versa.

Because of such correspondence we can define the matching of representations directly on the character level. We say that a character $\theta_{\rho'}$ of $SO(2n+1)_+$ lifts to a character θ_ρ of $\widetilde{Sp}(2n)$ if they satisfy the condition

$$\theta_\rho(\widetilde{g}) = \Phi(\widetilde{g})\theta_{\rho'}(g'), \tag{1.1}$$

for any pair of strongly regular semisimple elements $\widetilde{g}$ and g' such that $p(\widetilde{g})$ and g' stably correspond. Here $p : \widetilde{Sp}(2n) \longrightarrow Sp(2n)$ is the projection and Φ is a certain "transfer factor" which is necessary for technical reasons (i.e. to take care of Weyl

[1]Kazdhan and Patterson showed this fact for the metaplectic cover of $GL(n)$, see [K-P] Theorem I.5.1

denominators). It is also worth mentioning that characters matched by the above condition are necessarily stable (see Lemma 7.2.5). We discuss the details below.

We would like to show that if $\theta_{\rho'}$ is a stable virtual character of $SO(2n+1)_+$ then θ_ρ defined by (1.1) is a genuine stable virtual character of $\widetilde{Sp}(2n)$ and vice versa. (For the definition of stability for $\widetilde{Sp}(2n)$ see section 7.2.) In the real case, Adams used knowledge of discrete series characters to prove this first for discrete series. He then showed that the formula "commutes" with parabolic induction. In the p–adic case much less is known about characters of discrete series (in particular supercuspidal) representations.

The main result of this thesis is that lifting of characters commute with parabolic induction. It is stated in Theorem 8.0.8. We repeat it here.

Let $A \subset Sp(2n)$ and $A' \subset SO(2n+1)_+$ be isomorphic split tori. Let $M = Cent_{Sp(2n)}A, M' = Cent_{SO(2n+1)_+}A'$ and $\widetilde{M} = p^{-1}(M) \subset \widetilde{Sp}(2n)$.

Theorem 1.0.1 *Let ρ be a genuine admissible virtual representation of $\widetilde{M}$ and let ρ' be an admissible virtual representation of M'. Assume that their characters satisfy the condition*

$$\theta_\rho(\widetilde{x}) = \Phi_{\widetilde{M}}(\widetilde{x})\theta_{\rho'}(x'),$$

for any pair of strongly regular semisimple elements $\widetilde{x} \in \widetilde{M}$ and $x' \in M'$ such that $p(\widetilde{x})$ and x' stably correspond. Let $\pi = Ind_{\widetilde{P}}^{\widetilde{Sp}(2n)}\rho$ and $\pi' = Ind_{P'}^{SO(2n+1)_+}\rho'$ and denote by θ_π and $\theta_{\pi'}$ the characters of these representations. Then

$$\theta_\pi(\widetilde{x}) = \Phi_{\widetilde{Sp}(2n)}(\widetilde{x})\theta_{\pi'}(x'),$$

for any regular semisimple elements $\widetilde{x} \in \widetilde{Sp}(2n)$ and $x' \in SO(2n+1)_+$ such that

$p(\widetilde{x})$ and x' stably correspond.

This theorem reduces the problem of lifting characters to the supercuspidal case, meaning that the results of this thesis together with proving the result for supercuspidal representations would solve the general problem of lifting of characters in the case of $p-adic$ fields of characteristic zero.

Just as in the real case, the transfer factor is essentially the character of the *difference* of the two halves of the oscillator representation on $\widetilde{Sp}(2n)$. For the orthogonal group we use the formula for a character of an induced representation given by van Dijk in [D]:

$$\theta_{\pi'}(x') = \sum_{s_y \in W(A',T')} \theta_{\rho'}(y^{-1}x'y)\frac{|D_M(y^{-1}x'y)|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(x')|^{\frac{1}{2}}}, \quad x' \in T' \cap SO(2n+1)_{reg},$$

where $T' \subset M'$ is a Cartan subgroup with a split component $A_{T'}$ and $W(A',T')$ is the set of all injections $s : A' \to A_{T'}$ for which there exists $y \in SO(2n+1)_+$ such that $s(a) = yay^{-1}$ for all $a \in A'$. There is no known reference how to adapt this formula for $\widetilde{Sp}(2n)$. However, it looks like a very similar proof would hold in the metaplectic group. We will assume that the result holds and that the formula for the character of the representation π on $\widetilde{Sp}(2n)$ is

$$\theta_{\pi}(\widetilde{x}) = \sum_{s_y \in W(A,T)} \theta_{\rho}(y^{-1}\widetilde{x}y)\frac{|D_M(y^{-1}xy)|^{\frac{1}{2}}}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}}, \quad \widetilde{x} \in \widetilde{T} \cap \widetilde{Sp}(2n)_{reg}.$$

We compare the two formulas term by term, when A' and A are isomorphic split tori, T' and $p(\widetilde{T})$ are isomorphic Cartan subgroups and $x = p(\widetilde{x}) \in p(\widetilde{M})$ stably corresponds to $x' \in M'$. The main challenge is to show that the indexing sets $W(A,T)$ and $W(A',T')$ are in bijection and that the bijection has "nice" properties, i.e. preserves stable correspondence.

For example, in the case of minimal parabolic induction the transfer factor defined on $\widetilde{A} = p^{-1}(A)$ is given by the formula

$$\Phi((x,\epsilon)) = \gamma(\prod_{i=1,\dots,n} x_i, \eta)\epsilon,$$

where $x = diag(x_1, \dots, x_n, 1/x_1, \dots, 1/x_n) \in Sp(2n)$. Therefore if χ' is a character on a split Cartan subgroup in $SO(2n+1)_+$, its lift to a character on $\widetilde{A} \subset \widetilde{Sp}(2n)$ is defined by

$$\widetilde{\chi}((x,\epsilon)) = \chi(x')\gamma(\prod_{i=1,\dots,n} x_i, \eta)\epsilon,$$

where $x' = diag(x_1, \dots, x_n, 1/x_1, \dots, 1/x_n, 1) \in SO(2n+1)_+$. Also van Dijk's formula is much simpler in this case since the indexing sets are just isomorphic Weyl groups of type B_n and C_n. Hence if $\pi = Ind_P^{SO(2n+1)_+}\chi$ and $\widetilde{\pi} = Ind_{\widetilde{P}}^{\widetilde{Sp}(2n)}\widetilde{\chi}$ and θ_π and $\theta_{\widetilde{\pi}}$ denote their characters we have that

$$\theta_\pi(x') = \sum_{w \in W(B_n)} \frac{\chi(w \cdot x')}{|D_{SO(2n+1)_+}(x')|^{\frac{1}{2}}},$$

$$\theta_{\widetilde{\pi}}((x,\epsilon)) = \gamma(\prod_i x_i, \eta)\epsilon \sum_{w \in W(C_n)} \frac{\chi(w \cdot x')}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}}.$$

We also showed that the transfer factor Φ on $\widetilde{Sp}(2n)$ is precisely equal to

$$\gamma(\prod_i x_i, \eta)\epsilon \frac{|D_{SO(2n+1)_+}(x')|^{\frac{1}{2}}}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}} = \frac{\gamma(\prod_i x_i, \eta)\epsilon}{|\det(1+x)|^{\frac{1}{2}}}.$$

Therefore indeed

$$\theta_{\widetilde{\pi}}(x,\epsilon) = \Phi((x,\epsilon))\theta_\pi(x').$$

In the general case we had to describe the sets $W(A,T)$ and $W(A',T')$, in a different way, namely:

$$W(A,T) \cong \bigcup_{\{H \subset M : H \sim_{Sp(2n)} T\}/\sim_M} \frac{W(Sp(2n),H)}{W(M,H)}$$

and

$$W(A',T') \cong \bigcup_{\{H'\subset M': H'\sim_{SO(2n+1)_+} T'\}/\sim_{M'}} \frac{W(SO(2n+1)_+,H')}{W(M',H')}.$$

Then we show that one can choose the representatives of the sets $\{H \subset M : H \sim T\}/\sim_M$ and $\{H' \subset M' : H' \sim T'\}/\sim_{M'}$ to be isomorphic Cartan subgroups. Finally we replace the quotients $W(G,H)/W(M,H)$ with their stable versions, i.e. $W_{st}(G,H)/W_{st}(M,H)$. We define $W_{st}(G,H)$ to be the subgroup of $W(\mathbf{G},\mathbf{H})$ consisting of those elements that act on $H = \mathbf{H}(F)$. This replacement is valid, since we already proved that the characters θ_ρ and $\theta_{\rho'}$ are stable (see section 7.2). The advantage is that for any pair of isomorphic Cartan subgroups $H \subset Sp(2n)$ and $H' \subset SO(2n+1)_+$ the stable Weyl groups $W_{st}(Sp(2n),H)$ and $W_{st}(SO(2n+1)_+,H)$ are isomorphic (while in general $W(Sp(2n),H)$ and $W(SO(2n+1)_+,H)$ are not). Therefore, the bijection between the stable quotients of the Weyl groups descends from the group isomorphism, and we obtain all the required properties.

Chapter 2

Preliminaries

In this chapter we introduce some basic notions and constructions that we will use in this thesis. We start by defining the Hilbert symbol and we review some of its properties. We also recall the classification of nondegenerate quadratic forms in the case of $p - adic$ fields via rank, discriminant and Hasse invariant. We use it to classify orthogonal groups of odd rank. In section 3 we state the definition and properties of the Weil index. We end this chapter with introducing the notion of orbit correspondence together with its stabilized version. We summarize its basic properties which we will need later on.

Throughout the thesis we will keep the following notations. Let F a p-adic field of characteristic zero. We will denote its algebraic closure by $\overline{F}$ and by Γ the Galois group of $\overline{F}/F$. Let $\mathbf{G}$ be a connected split semisimple algebraic group defined over F. We will identify $\mathbf{G}$ with $\mathbf{G}(\overline{F})$, and we will also denote its F–points $\mathbf{G}(F)$ by G.

2.1 Hilbert symbol

For $a, b \in F^*$ we define the Hilbert symbol as follows (compare Serre, [Se1]):

$$(a,b) = \begin{cases} 1 & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a nonzero solution in } F^3, \\ -1 & \text{otherwise.} \end{cases}$$

Proposition 2.1.1 *The Hilbert symbol satisfies the formulas:*

(1) $(aa', b) = (a, b)(a', b)$,

(2) $(a, b^2) = 1$,

(3) $(a, b) = (b, a)$,

(4) $(a, -a) = 1 = (a, 1 - a)$,

(5) If $(a, b) = 1$ *for all* $a \in F^*$, *then* b *is a square.*

Proof. See [Se1], chapter III. □

2.2 Quadratic forms over F

In this section we classify odd orthogonal groups over F. A nice account of this material can be found in Serre, [Se1].

Let (V, Q) be a nondegenerate quadratic form of rank n over F. Choose an orthogonal basis for V and suppose that in this basis

$$Q(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2.$$

We define the following two invariants:

Definition 2.2.1 *The discriminant* $\delta(Q) = a_1 \cdots a_n \in F^*/F^{*2}$.

Definition 2.2.2 *The Hasse invariant* $\epsilon(Q) = \prod_{i<j}(a_i, a_j) = \pm 1$.

Proposition 2.2.3 *The number* $\epsilon(Q)$ *does not depend on the choice of an orthogonal basis.*

Proof. See [Se1], Thmeorem 5, chapter IV. □

Proposition 2.2.4 *Two nondegenerate quadratic forms over F of rank $n \geq 3$ are equivalent if and only if they have the same rank, discriminant and the same Hasse invariant.*

Proof. See [Se1], Theorem 7, chapter IV. □

We will study quadratic spaces of odd rank.

Proposition 2.2.5 *For a form Q of odd rank n to represent zero it is necessary and sufficient that:*

(1) $n = 3$ *and* $(-1, -\delta) = \epsilon$,

(2) $n \geq 5$.

Proof. See [Se1], Theorem 6, chapter IV. □

Corollary 2.2.6 *For p odd there are two orthogonal groups of rank $2n+1$, $n \geqslant 1$.*

Proof. There are eight classes of quadratic forms, one for each pair (δ, ϵ) (see also [Se1], Proposition 6, chapter IV). By Proposition 2.2.5 every nondegenerate form of rank $2n+1$ can be written as $(1, -1)^{n-1} \oplus (a, b, c)$. Scaling it to a new form $(x, -x)^{n-1} \oplus (xa, xb, xc)$ will change its discriminant by x and its Hasse invariant by $(x, -1)^n$, but it will not change the orthogonal group. Therefore there are at most two orthogonal groups. There are exactly two: the one preserving the form $(1, -1)^{n-1} \oplus (1, -1, 1)$ (this one contains the split Cartan subgroup F^{*n}) and the other one which does not contain F^{*n}, i.e. the one preserving the form $(1, -1)^{n-1} \oplus (1, -\Delta, x\Delta)$, where Δ is a non square in F and $(x, \Delta) = -1$. □

Notation 2.2.7 *We will denote by* $SO(2n+1)_+$ *the split orthogonal group and by* $SO(2n+1)_-$ *the nonsplit one.*

2.3 Weil index

Let η be a nontrivial additive character of F. If $a \in F^*$ then $a\eta$ denotes a character given by $a\eta(x) = \eta(ax)$. We define after Ranga Rao (see [R], Appendix):

$$\gamma(\eta) = \text{Weil index of } x \to \eta(x^2),$$

$$\gamma(a, \eta) = \gamma(a\eta)/\gamma(\eta), a \in F^\times.$$

We will write $\gamma_\eta(a) = \gamma(a\eta)$.

Proposition 2.3.1 *(1)* $\gamma(ab, \eta) = (a, b)\gamma(a, \eta)\gamma(b, \eta)$,

(2) $\gamma(ac^2, \eta) = \gamma(a, \eta)$,

(3) $\gamma(a, c\eta) = (a, c)\gamma(a, \eta)$,

(4) $\gamma(a, \eta)^2 = (-1, a)$,

(5) $\gamma(-1, \eta) = \gamma_\eta(1)^{-1}$.

Proof. See [R], appendix, Theorem A.4. and Corollary A.5. □

For the quadratic form $Q_a(x) = ax^2$ we define $\gamma_\eta(Q_a) = \gamma_\eta(a)$. If $Q = 0$, then $\gamma_\eta(Q) = 1$. If $Q = Q_1 \oplus Q_2$, then $\gamma_\eta(Q) = \gamma_\eta(Q_1)\gamma_\eta(Q_2)$.

Corollary 2.3.2 *(1)* $\gamma_\eta(ab)\gamma_\eta(1) = (a, b)\gamma_\eta(a)\gamma_\eta(b)$,

(2) $\gamma_\eta(a)\gamma_\eta(-a) = 1$,

(3) $\gamma_\eta(1)^4 = (-1,-1)$,

(4) $\gamma_\eta(Q) = \gamma_\eta(1)^{rkQ-1}\gamma_\eta(\delta(Q))\epsilon(Q)$, *where* δ *is the discriminant and* ϵ *is the Hasse invariant,*

(5) If Q *is hyperbolic, then* $\gamma_\eta(Q) = 1$.

2.4 Orbit correspondence and stability

In this section we define the orbit correspondence, the stabilized orbit correspondence and we investigate some of their properties. See Adams ([A1]) for more details.

Let W be $2n+1$–dimensional vector space over F with a nondegenerate symmetric bilinear form $(\,,)$. Let V be a $2n$–dimensional vector space with a nondegenerate symplectic form $\langle\,,\rangle$. Let $G = SO(W)$ and $G' = Sp(V)$. Their Lie algebras will be denoted by $\mathfrak{g}$ and $\mathfrak{g}'$ respectively. Let's recall that $\mathfrak{g}$ consists of elements $X \in Hom(W,W)$ such that

$$(Xw,w') + (w,Xw') = 0, \quad w,w' \in W,$$

and $\mathfrak{g}'$ consists of elements $Y \in Hom(V,V)$ such that

$$\langle Yv,v'\rangle + \langle v,Yv'\rangle = 0 \quad v,v' \in V.$$

For $T \in Hom(W,V)$ define $T^* \in Hom(V,W)$ by:

$$\langle Tw,v\rangle = (w,T^*v), \quad w \in W, v \in V.$$

Let $\alpha : T \mapsto T^*T \in \mathfrak{g}$ and $\alpha' : T \mapsto TT^* \in \mathfrak{g}'$.

Definition 2.4.1 *For $X \in G$ (resp. G') or $\mathfrak{g}$ (resp. $\mathfrak{g}'$) such that $1+X$ is invertible we define the Cayley transform $C(X)$ as follows:*

$$C(X) = (1 - X)(1 + X)^{-1}.$$

The Cayley transform has the following properties:

Lemma 2.4.2 *(1) $C : G \to \mathfrak{g}$ and $C : \mathfrak{g} \to G$,*

(2) $C^2 = Id$,

(3) X is semisimple if and only if $C(X)$ is semisimple,

(4) C is equivariant for the adjoint action of G on $\mathfrak{g}$ and for the conjugation action on G.

The same statement holds for G' and $\mathfrak{g}'$.

Definition 2.4.3 *We say that a semisimple element g of G or G' is strongly regular if its centralizer is a Cartan subgroup. We say that g is regular if the identity component of its centralizer is a Cartan subgroup.*

Proposition 2.4.4 *The strongly regular elements form a dense open subset in G.*

Proof. See [St2], 2.15. □

Definition 2.4.5 *We say that $X \in \mathfrak{g}$ corresponds to $X' \in \mathfrak{g}'$ if there exists $T \in Hom(W, V)$ such that $\alpha(T) = X$ and $\alpha'(T) = X'$. We will write $X \overset{orbit}{\longleftrightarrow} X'$ This extends to a correspondence of orbits, i.e. if $O = G \cdot X$ and $O' = G' \cdot X'$ then $O \overset{orbit}{\longleftrightarrow} O'$. We say that a strongly regular element $g \in G$ corresponds to a strongly regular element $g' \in G'$ if $C(g) \overset{orbit}{\longleftrightarrow} C(g')$. We write $g \overset{orbit}{\longleftrightarrow} g'$.*

Proposition 2.4.6 *Fix discriminant $\delta \in F^*/F^{*2}$. Let W_+ be a $2n+1$–dimensional vector space over F with a nondegenerate split symmetric bilinear form with discriminant δ. Let W_- be a $2n+1$–dimensional vector space over F with a nondegenerate nonsplit symmetric bilinear form with discriminant δ.*

The orbit correspondence is a bijection between strongly regular semisimple adjoint orbits of $Sp(2n)$ and $SO(W_+) \cup SO(W_-)$.

Proof. The proof of this proposition is the same as the proof in [A1] (Proposition 2.5) of the analogous proposition in the case of real numbers. □

Example 2.4.7 *Elliptic elements in $SL(2)$ and $SO(3)_+$, $p \neq 2$.*

Let E/F be a field extension of degree 2, say $E = F(\sqrt{\Delta})$. Denote by E^1 the elements in E whose norm is 1. Choose the following embedding E^1 into $SL(2)$:

$$a + b\sqrt{\Delta} \mapsto \begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix}.$$

Fix a discriminant δ. Following the construction from [A1] (Proposition 2.5) we get

$$\begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix} \overset{orbit}{\longleftrightarrow} \begin{pmatrix} a & b\Delta & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where the latter preserves the quadratic form

$$\begin{pmatrix} \frac{b}{1+a} & 0 & 0 \\ 0 & -\frac{b\Delta}{(1+a)} & 0 \\ 0 & 0 & -\Delta\delta \end{pmatrix}.$$

Its discriminant is equal to δ and its Hasse invariant is equal to $(\delta, -1)(\Delta, \frac{b\delta}{1+a}) = (\delta, -1)(\Delta, 2b)$. (The equality follows from the fact that $(1+a)^2 - b^2\Delta = 2(1+a)$.) The form is split if and only if the Hasse invariant is equal to $(\delta, -1)$, i.e. if and only if $(\Delta, 2b) = 1$. This result coincides with Adams result in the case of the field of real numbers (see [A1], Example 2.13).

Definition 2.4.8 *Let $G = Sp(2n)$ or $SO(2n+1)_+$ and let $g, h \in G$ be strongly regular semisimple elements. We say that g and h are stably conjugate, if they are conjugate in $\boldsymbol{G}(\overline{F})$. We will write $g \sim_{st} h$.*

Lemma 2.4.9 *Let $G = Sp(2n)$ or $SO(2n+1)_+$ and let $g, h \in G$ be strongly regular semisimple elements. Then g and h are stably conjugate if and only if they have the same eigenvalues.*

Proof. Assume that g and h have the same eigenvalues. Choose elements $x, y \in \mathbf{G}(\overline{F})$ such that xgx^{-1} and yhy^{-1} are diagonal. Without loss of generality we can assume that $xgx^{-1} = yhy^{-1}$ (if $xgx^{-1} \neq yhy^{-1}$ then we can apply an action of an appropriate element of the Weyl group of the split Cartan subgroup). Therefore we have that $g = (x^{-1}y)h(x^{-1}y)^{-1}$. $\square$

Definition 2.4.10 *Let $g \in Sp(2n), g' \in SO(2n+1)_+$ be strongly regular semisimple elements. We say that g and g' stably correspond, if there exist elements $h \in Sp(2n)$ and $h' \in SO(2n+1)_+$ such that $g \sim_{st} h, g' \sim_{st} h'$ and $h \overset{orbit}{\longleftrightarrow} h'$. We will write $g \overset{stable}{\longleftrightarrow} g'$.*

Proposition 2.4.11 *Let $g \in Sp(2n), g' \in SO(2n+1)_+$ be strongly regular semisimple elements. Then g and g' stably correspond if and only if g and g' have the same nontrivial (i.e. $\neq 1$) eigenvalues.*

Proof. This follows directly from the definition of stable correspondence and from Lemma 2.4.9. □

We have the following proposition, which proof we will defer until the next chapter:

Proposition 2.4.12 *The stable correspondence is a bijection between strongly regular semisimple stable conjugacy classes of $Sp(2n)$ and $SO(2n+1)_+$.*

Example 2.4.13 *Stable correspondence between elliptic elements in $SO(3)_+$ and $SL(2)$ for $p \neq 2$.*

We keep the notations from the Example 2.4.7. Suppose that

$$\begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix} \overset{orbit}{\longleftrightarrow} \begin{pmatrix} a & b\Delta & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(3)_-,$$

where $SO(3)_-$ is the nonsplit orthogonal group. In such case the Hasse invariant of the corresponding quadratic form is equal to $(\delta, -1)$ (see Example 2.4.7). We pick any element $y \in F$ such that $(y, \Delta) = -1$. We have that

$$\begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix} \sim_{st} \begin{pmatrix} a & by(\Delta y^{-2}) \\ by & a \end{pmatrix} \overset{orbit}{\longleftrightarrow} \begin{pmatrix} a & \frac{b\Delta}{y} & 0 \\ by & a & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The latter preserves the quadratic form $diag(\frac{yb}{1+a}, -\frac{yb\Delta}{(1+a)}, -\Delta\delta)$. It's Hasse invariant is equal to $-(\delta,-1)(y,\Delta) = (\delta,-1)$. By Proposition 2.2.5 the form is split. Hence we showed that

$$\begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix} \overset{stable}{\longleftrightarrow} \begin{pmatrix} a & \frac{b\Delta}{y} & 0 \\ by & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2n+1)_+.$$

Now we will study Cartan isomorphic subgroups in the symplectic and orthogonal groups.

Lemma 2.4.14 *Any Cartan subgroup in an orthogonal group $SO(2n+1)_\pm$ can be embedded into $Sp(2n)$. Similarly, any Cartan subgroup in $Sp(2n)$ can be embedded into one of the orthogonal groups $SO(2n+1)_\pm$.*

Proof. Let $(\,,\,)$ be nondegenerate bilinear form on $2n+1$–dimensional vector space W. Let T be a Cartan subgroup in $SO(W)$ and $x \in T$ be a strongly regular element. We follow the construction of [A1] (Proposition 2.5) and we define a symplectic form on $W/ker(C(x))$ as follows:

$$\langle w, w' \rangle = (C(x)w, w').$$

This form is nondegenerate and is preserved by all $t \in T$:

$$\langle tw, tw' \rangle = (C(x)tw, tw') = (t^{-1}C(x)tw, w') = (C(x)w, w') = \langle w, w' \rangle.$$

Let $V = W/ker(C(x))$. Since x was strongly regular $dimV = n$, and by the above calculation $T \subset Sp(V)$.

The proof in the other direction is analogous. We start with a Cartan subgroup $T \subset Sp(V)$ and a strongly regular element $x \in T$. We define a bilinear form on V :

$$(v, v') = \langle C(x)v, v' \rangle.$$

This form is nondegenerate. We extend $(\, , \,)$ to a nondegenerate form on $W = V \oplus V_0$, where V_0 is a one dimensional space. We can choose a quadratic form on V_0 in such a way, that $(\, , \,)$ has a desired discriminant. However, there is no obvious way of modifying this form in order to obtain a split one. Finally note that, by similar argument as in the previous case, $T \subset SO(W)$. □

It is possible to embed every Cartan subgroup in $Sp(2n)$ into $SO(2n+1)_+$. Proving this requires more sophisticated tools, i.e. Galois cohomology and it is done in the next section.

Chapter 3

Galois Cohomology

In this chapter we use Galois cohomology to study stable conjugacy classes of Cartan subgroups in the symplectic and split orthogonal groups. As a corollary we obtain that $Sp(2n)$ and $SO(2n+1)_+$ have isomorphic Cartan subgroups and that the stable correspondence is a bijection between strongly regular semisimple conjugacy classes in those two groups.

3.1 Definitions

Here we review Galois cohomology; we refer the reader to Serre [Se2] for additional details. Let us recall that $\overline{F}$ denotes the algebraic closure of F and Γ is the Galois group of $\overline{F}/F$. Let $\mathbf{G}$ be a connected split semisimple algebraic group defined over F. We identify $\mathbf{G}$ with $\mathbf{G}(\overline{F})$ and when there is no confusion we will denote $\mathbf{G}(F)$ by G. The Galois group Γ acts on $\mathbf{G}$ and we define $H^0(\Gamma, \mathbf{G})$ to be the set of fixed elements $\mathbf{G}^\Gamma$. A $1-$cocycle of Γ in $\mathbf{G}$ is a continuous map $\sigma \mapsto a_\sigma(\sigma \in \Gamma, a_\sigma \in \mathbf{G})$, such that for all $\sigma, \tau \in \Gamma$:

$$a_{\sigma\tau} = a_\sigma \sigma(a_\tau).$$

The set of $1-$cocycles of Γ in $\mathbf{G}$ is denoted by $Z^1(\Gamma, \mathbf{G})$. Two cocycles a and a' are cohomologous if there exists $g \in \mathbf{G}$ such that $a'_\sigma = g^{-1} a_\sigma \sigma(g), \sigma \in \Gamma$. This is an equivalence relation and the set of equivalence classes is denoted by $H^1(\Gamma, \mathbf{G})$. In general $H^1(\Gamma, \mathbf{G})$ is a pointed set and its distinguished element is the equivalence class of the unit cocycle. It is a group provided that $\mathbf{G}$ is abelian. We say that a sequence of pointed sets is exact if the fiber of the distinguished point of each map is the image of its predecessor. Let $\mathbf{T}$ be a Cartan subgroup defined over F and let $N(\mathbf{T})$ be its normalizer in $\mathbf{G}$. If $n \in N(\mathbf{T})$, then we will denote its image in the Weyl group by $\overline{n}$. If $w \in W = W(\mathbf{G}, \mathbf{T})$ then $w \cdot t$ denotes the standard action of the Weyl group element w on the element $t \in \mathbf{T}$.

Proposition 3.1.1 *The short exact sequence* $1 \to \boldsymbol{T} \to N(\boldsymbol{T}) \to W \to 1$ *yields an exact sequence of pointed sets:*

$$1 \to \boldsymbol{T}^\Gamma \to N(\boldsymbol{T})^\Gamma \to W^\Gamma \xrightarrow{\delta} H^1(\Gamma, \boldsymbol{T}) \xrightarrow{\alpha} H^1(\Gamma, N(\boldsymbol{T})) \to H^1(\Gamma, W).$$

If $\overline{n} \in W^\Gamma$, *then* $\delta(\overline{n})$ *is the class of the cocycle* $\{n^{-1}\sigma(n)\}_{\sigma \in \Gamma}$.

Proof. See [Se2], I.5.4, Proposition 36. □

3.2 Stable conjugacy of Cartan subgroups

In this section G denotes $Sp(2n)$ or $SO(2n+1)_+$.

Definition 3.2.1 *Let* $\boldsymbol{T}$ *and* $\boldsymbol{T'}$ *be two Cartan subgroups in* $\boldsymbol{G}$ *that are defined over* F. *We say that* $\boldsymbol{T}(F)$ *and* $\boldsymbol{T'}(F)$ *are stably conjugate, if there exists* $g \in \boldsymbol{G}$ *such that* $g\boldsymbol{T}(F)g^{-1} = \boldsymbol{T'}(F)$.

Lemma 3.2.2 *If* $x, y \in G$ *are strongly regular semisimple elements that are stably conjugate, then the Cartan subgroups* $Cent_G(x)$ *and* $Cent_G(y)$ *are stably conjugate.*

Proof. Let $\mathbf{T} = Cent_{\mathbf{Sp(2n)}}(x)$ and $\mathbf{T}' = Cent_{\mathbf{Sp(2n)}}(y)$. Let $g \in \mathbf{G}$ be such that $gxg^{-1} = y$. Clearly $g\mathbf{T}g^{-1} = \mathbf{T}'$. We need to show that $g\mathbf{T}(F)g^{-1} = \mathbf{T}'(F)$. For $t \in \mathbf{T}(F)$ and $\sigma \in \Gamma$ we have

$$\sigma(gtg^{-1}) = \sigma(g)t\sigma(g)^{-1} = (\sigma(g)^{-1}g)\sigma(g)t\sigma(g)^{-1}(\sigma(g)^{-1}g)^{-1} = gtg^{-1}.$$

In the second equality we used the fact that $g\sigma(g)^{-1} \in \mathbf{T}'$ since

$$\sigma(g)^{-1}y\sigma(g) = \sigma(g^{-1}yg) = \sigma(x) = x = g^{-1}yg. \quad \square$$

Note that for a given Cartan subgroup T the group G acts on the set $\{T' \mid T' \text{ is a Cartan subgroup stably conjugate to } T\}$ by conjugation. We introduce the following notations:

$$\mathcal{C}_{st}(G,T) = \Big\{T' \mid T' \text{ is a Cartan subgroup stably conjugate to } T\Big\}\Big/ \; G,$$

$$\mathcal{C}_{st}(G) = \Big\{T \mid T \text{ is a Cartan subgroup in } G\Big\}\Big/ \text{ stable conjugacy.}$$

Lemma 3.2.3

$$\mathcal{C}_{st}(G,T) \overset{1-1}{\longleftrightarrow} \alpha(\ker[H^1(\Gamma, \boldsymbol{T}) \to H^1(\Gamma, \boldsymbol{G})]).$$

Proof. Let $\mathbf{T}$ and $\mathbf{T}'$ be two Cartan subgroups in $\mathbf{G}$ that are defined over F. If $g\mathbf{T}(F)g^{-1} = \mathbf{T}'(F)$ then for all $x \in \mathbf{T}(F)$ and all $\sigma \in \Gamma$ we have that $gxg^{-1} = \sigma(gxg^{-1}) = \sigma(g)x\sigma(g)^{-1}$. Therefore $g^{-1}\sigma(g)$ centralizes $\mathbf{T}(F)$, hence it belongs to $\mathbf{T}$. The map $\sigma \mapsto g^{-1}\sigma(g)$ is a $1-$cocycle whose cohomology class lies in the kernel of the map $H^1(\Gamma, \mathbf{T}) \to H^1(\Gamma, \mathbf{G})$. If $h \in \mathbf{G}$ is another element with the property $h\mathbf{T}(F)h^{-1} = \mathbf{T}'(F)$, then $g^{-1}h$ belongs to $N(\mathbf{T})$ and the cocycles $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ and $(h^{-1}\sigma(h))_{\sigma\in\Gamma}$ are cohomologous in $Z^1(\Gamma, N(\mathbf{T}))$.

Conversely, each element of $\ker[H^1(\Gamma, \mathbf{T}) \to H^1(\Gamma, \mathbf{G})]$ is the class of a cocycle of the form $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ for some $g \in \mathbf{G}$. Since $g^{-1}\sigma(g) \in \mathbf{T}$ we have that $g\mathbf{T}(F)g^{-1} \subset \mathbf{G}(F)$. Suppose now that the classes of the cocycles $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ and $(h^{-1}\sigma(h))_{\sigma\in\Gamma}$ are in the $\ker[H^1(\Gamma, \mathbf{T}) \to H^1(\Gamma, \mathbf{G})]$, and that they are cohomologous in $Z^1(\Gamma, N(\mathbf{T}))$. Choose $n \in N(\mathbf{T})$ such that $h^{-1}\sigma(h) = n^{-1}g^{-1}\sigma(g)\sigma(n)$. It follows that $x = gnh^{-1}$ belongs to $\mathbf{G}(F)$ and $g\mathbf{T}(F)g^{-1} = xhn^{-1}\mathbf{T}(F)nh^{-1}x^{-1} = xh\mathbf{T}(F)h^{-1}x^{-1}$, i.e. $g\mathbf{T}(F)g^{-1}$ and $h\mathbf{T}(F)h^{-1}$ are conjugate in $\mathbf{G}(F)$. Note that the cocycle $(n^{-1}\sigma(n))_{\sigma\in\Gamma}$ belongs to $Z^1(\Gamma, \mathbf{T})$, therefore $n^{-1}\mathbf{T}(F)n \subset \mathbf{G}(F)$. Therefore we proved the assertion. □

Proposition 3.2.4 *In the case of $\boldsymbol{G} = \boldsymbol{GL(n)}$, stable conjugacy coincides with conjugacy by the elements of $\boldsymbol{G}(F)$.*

Proof. This follows from the previous lemma, the fact that every torus in $\mathbf{GL}(\mathbf{n})$ is quasi split, and that for a quasi split tori $\mathbf{T}$ we have $H^1(\Gamma, \mathbf{T}) = 1$ (see [Pl-Rap], Chapter 2, Lemma 2.4). □

Proposition 3.2.5 *The map $H^1(\Gamma, N(\boldsymbol{T})) \to H^1(\Gamma, W)$ induced by the canonical map $N(\boldsymbol{T}) \to W$ is surjective. Moreover, given $a \in H^1(\Gamma, W)$ there exists a lift $\widetilde{a} \in H^1(\Gamma, N(\boldsymbol{T}))$ of a, which maps to the trivial class in $H^1(\Gamma, \boldsymbol{G})$.*

Proof. See [Rag], Theorem 1.1. □

Lemma 3.2.6 *Let* $\mathbf{T}$ *be a Cartan subgroup defined over F. Let $W = W(\mathbf{G}, \mathbf{T})$. Then*

$$\mathcal{C}_{st}(G) \xleftrightarrow{1-1} H^1(\Gamma, W).$$

Proof. Let $\mathbf{T}$ be a Cartan subgroup that is defined over F. If $\mathbf{T}'$ is another Cartan subgroup that is defined over F, then $g\mathbf{T}g^{-1} = \mathbf{T}'$ for some $g \in \mathbf{G}$. Since $\sigma(\mathbf{T}) = \mathbf{T}$ for all $\sigma \in \Gamma$, we have that $g^{-1}\sigma(g)$ normalizes $\mathbf{T}$, hence $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ is a cocycle in $Z^1(\Gamma, N(\mathbf{T}))$. If h is another element that conjugates $\mathbf{T}$ into $\mathbf{T}'$ then $h^{-1}g$ belongs to $N(\mathbf{T})$ and the cocycles $(g^{-1}\sigma(g))_{\sigma\in\Gamma}$ and $(h^{-1}\sigma(h))_{\sigma\in\Gamma}$ are cohomologous in $Z^1(\Gamma, N(\mathbf{T}))$ and thus also their images $(\overline{g^{-1}\sigma(g)})_{\sigma\in\Gamma}$ and $(\overline{h^{-1}\sigma(h)})_{\sigma\in\Gamma}$ are cohomologous in $Z^1(\Gamma, W)$. On the other hand, if the cocycles $(\overline{g^{-1}\sigma(g)})_{\sigma\in\Gamma}$ and $(\overline{h^{-1}\sigma(h)})_{\sigma\in\Gamma}$ are cohomologous in $Z^1(\Gamma, W)$ then $g\mathbf{T}g^{-1}$ and $h\mathbf{T}h^{-1}$ are stably conjugate (the proof of this fact is similar to the proof of analoguous statement in the proof of the previous lemma). Also, it follows from Proposition 3.2.5 that for each element of $H^1(\Gamma, W)$ there is a corresponding conjugacy class of $\mathbf{T}$. □

Proposition 3.2.7 *(1) There is a cannonical bijection between the sets* $\mathcal{C}_{st}(Sp(2n))$ *and* $\mathcal{C}_{st}(SO(2n+1)_+)$.

(2) This bijection matches isomorphic Cartan subgroups in $SO(2n+1)_+$ *and* $Sp(2n)$.

Proof. (1) Choose split Cartan subgroups $\mathbf{T}'_s(F) \subset SO(2n+1)_+$ and $\mathbf{T}_s(F) \subset Sp(2n)$. The Galois group Γ acts trivially on $W(\mathbf{Sp\ (2n)}, \mathbf{T}_s) \cong W(C_n)$ and also on $W(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s) \cong W(B_n)$, therefore the isomorphism between $W(C_n)$ and $W(B_n)$ induces an isomorphism between $H^1(\Gamma, W(C_n))$ and $H^1(\Gamma, W(B_n))$. By Lemma 3.2.6

$$\begin{array}{ccc} H^1(\Gamma, W(C_n)) & \xrightarrow{\sim} & H^1(\Gamma, W(B_n)) \\ {\scriptstyle 1-1}\downarrow & & {\scriptstyle 1-1}\downarrow \\ \mathcal{C}_{st}(Sp(2n)) & & \mathcal{C}_{st}(SO(2n+1)_+). \end{array}$$

Now we will show that the construction of the vertical bijection does not depend on the choice of the split Cartan subgroups $\mathbf{T}_s$ and $\mathbf{T}'_s$. It is enough to show this in the symplectic case. Assume then that $\mathbf{T}''_s \subset \mathbf{Sp(2n)}$ is another split Cartan subgroup defined over F. We have that $\mathbf{T}_s(F)$ and $\mathbf{T}''_s(F)$ are conjugate, say $\mathbf{T}''_s(F) = x\mathbf{T}_s(F)x^{-1}$ for some $x \in \mathbf{Sp(2n)}(F)$. (Note that this implies that $x = \sigma(x)$!) We identify $H^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}_s))$ with $H^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}''_s))$ via the isomorphism of the Weyl groups $W(\mathbf{Sp(2n)}, \mathbf{T}_s) \to W(\mathbf{Sp(2n)}, \mathbf{T}''_s)$ given by $\overline{n} \mapsto \overline{xnx^{-1}}, n \in N(\mathbf{Sp(2n)}, \mathbf{T}_s)$. Now, let $\mathbf{T}' \subset \mathbf{Sp(2n)}$ be an arbitrary Cartan subgroup defined over F. We pick $g \in \mathbf{Sp(2n)}$ such that $\mathbf{T}' = g\mathbf{T}_s g^{-1}$ and that gives us a cocycle $(\overline{g^{-1}\sigma(g)})_{\sigma\in\Gamma} \in Z^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}_s))$. We also have that $\mathbf{T}' = gx^{-1}\mathbf{T}''_s(gx^{-1})^{-1}$ and therefore we have another cocycle $(\overline{(gx^{-1})^{-1}\sigma(gx^{-1})})_{\sigma\in\Gamma} \in Z^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}''_s))$. However, these cocycles are identified via our isomorphism, since $\overline{g^{-1}\sigma(g)} \mapsto \overline{xg^{-1}\sigma(g)x^{-1}} = \overline{(gx^{-1})^{-1}\sigma(gx^{-1})}$.

Now we will prove (2). We want to show that the $F-$points of the Cartan subgroups associated to isomorphic cocyles in $H^1(\Gamma, W(B_n))$ and $H^1(\Gamma, W(C_n))$ are also isomorphic. First we choose an isomorphism $\psi_s : \mathbf{T}_s \to \mathbf{T}'_s$ that commutes with the Galois action, i.e.

$$\psi_s(\sigma(t)) = \sigma(\psi_s(t)) \quad \sigma \in \Gamma, t \in \mathbf{T}_s.$$

Let

$$\phi_s : W(\mathbf{Sp(2n)}, \mathbf{T}_s) \to W(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s)$$

be an isomorphism such that

$$\psi_s(w \cdot t) = \phi_s(w) \cdot \psi_s(t) \quad w \in W(\mathbf{Sp(2n)}, \mathbf{T}_s), t \in \mathbf{T}_s.$$

The isomorphism ϕ_s induces an isomorphism between $H^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}_s))$ and $H^1(\Gamma, W(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s))$. Let a be an arbitrary element of $H^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}_s))$. We will denote its image in $H^1(\Gamma, W(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s))$ by a'. By Propositon 3.2.5 there exist $g \in \mathbf{Sp(2n)}$ and $h \in \mathbf{SO(2n{+}1)}$ such that for all $\sigma \in \Gamma$ we have $g^{-1}\sigma(g) \in N(\mathbf{Sp(2n)}, \mathbf{T}_s)$, $h^{-1}\sigma(h) \in N(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s)$, $a_\sigma = \overline{g^{-1}\sigma(g)}$ and $a'_\sigma = \overline{h^{-1}\sigma(h)}$. Let $\mathbf{T} = g\mathbf{T}_s g^{-1}$ and $\mathbf{T}' = h\mathbf{T}'h^{-1}$. Denote by ψ the isomorphism between $\mathbf{T}$ and $\mathbf{T}'$ that makes the diagram below commute

$$\begin{array}{ccc} \mathbf{T}_s & \xrightarrow{\psi_s} & \mathbf{T}'_s \\ {\scriptstyle int(g^{-1})}\big\uparrow & & \big\downarrow{\scriptstyle int(h)} \\ \mathbf{T} & \xrightarrow{\psi} & \mathbf{T}'. \end{array}$$

We will show that $\psi(\mathbf{T}(F)) = \mathbf{T}'(F)$. Let $t \in \mathbf{T}(F)$ and $\sigma \in \Gamma$. By the construction of ψ we have

$$\psi(t) = h(\psi_s(g^{-1}tg))h^{-1}.$$

Therefore

$$\sigma(\psi(t)) = \sigma(h(\psi_s(g^{-1}tg))h^{-1}) = \sigma(h)\psi_s(\sigma(g^{-1})t\sigma(g))\sigma(h^{-1}).$$

Hence $\sigma(\psi(t)) = \psi(t)$ if and only if

$$\psi_s(g^{-1}hg) = (h^{-1}\sigma(h))\psi_s(\sigma(g^{-1})t\sigma(g))(h^{-1}\sigma(h))^{-1}.$$

The above equality is true, since $\overline{h^{-1}\sigma(h)} = \phi_s(\overline{g^{-1}\sigma(g)}) \in W(\mathbf{SO(2n{+}1)}, \mathbf{T}'_s)$ and by the choice of the isomorphisms ψ_s and ϕ_s we have that

$$\phi_s(\overline{g^{-1}\sigma(g)}) \cdot \psi_s(\sigma(g^{-1})t\sigma(g)) = \psi_s(\overline{g^{-1}\sigma(g)} \cdot \sigma(g^{-1})t\sigma(g)) = \psi_s(g^{-1}tg). \quad \square$$

Corollary 3.2.8 *Any Cartan subgroup in $Sp(2n)$ can be embedded in $SO(2n{+}1)_+$.*

Proof. This follows directly from Proposition 3.2.7. For an alternate proof, see [Pl-Rap], p. 340. □

Corollary 3.2.9 *The stable correspondence is a bijection between strongly regular, semisimple stable conjugacy classes of $Sp(2n)$ and $SO(2n+1)_+$.*

Proof. The statement follows from the Corollaries 2.4.14 and 3.2.8, and the fact that every stable strongly regular semisimple conjugacy class is determined by the set of the eigenvalues of its elements (see Lemma 2.4.9). □

Corollary 3.2.10 *Let $x \in Sp(2n)$ and $x' \in SO(2n+1)_+$ be a pair of strongly regular semisimple elements such that $x \overset{stably}{\longleftrightarrow} x'$. Let $T = Cent_{Sp(2n)}x$ and $T' = Cent_{SO(2n+1)_+}(x')$. There exists an isomorphism $\psi : T \longrightarrow T'$ such that $\psi(x) = x'$.*

Proof. The proof is basically reversing the steps of the proof of Proposition 3.2.7. The one difference is that we will have an additional condition on the isomorphism ψ_s between the split Cartan subgroups $\mathbf{T}_s$ and $\mathbf{T}'_s$, namely

$$t \overset{stably}{\longleftrightarrow} \psi_s(t)$$

for all strongly regular semisimple elements $t \in \mathbf{T}_s$. Here are the remaining details:

Let $g \in \mathbf{Sp}\ \mathbf{(2n)}$ be such that $\mathbf{T} = g\mathbf{T}_s g^{-1}$. As before we find an element $h \in \mathbf{SO(2n+1)}$ such that the classes of cocycles $(\overline{g^{-1}\sigma(g)})_{\sigma\in\Gamma} \in H^1(\Gamma, W(\mathbf{Sp(2n)}, \mathbf{T}_s))$ and $(\overline{h^{-1}\sigma(h)})_{\sigma\in\Gamma} \in H^1(\Gamma, W(\mathbf{SO(2n+1)}, \mathbf{T}'_s))$ are isomorphic. Then we show (as in the proof of Proposition 3.2.7) that ψ_s induces an isomorphism $\psi' : T \to h\mathbf{T}'_s h^{-1}(F)$. The Cartan subgroup T' is stably conjugate to $h\mathbf{T}'_s h^{-1}(F)$, so we compose the map ψ' with conjugation by an appropriate element to get the isomorphism $\psi'' : T \to T'$.

We also have that $x \xleftrightarrow{stable} \psi''(x) \sim_{st} x'$, hence we conjugate one more time to obtain the required condition $\psi(x) = x'$.

Chapter 4

Metaplectic group

In this chapter we define the metaplectic double cover of the symplectic group $Sp(2n)$. We introduce the formula for the character of the difference of the two halves of the oscillator representation, which is the main ingredient for defining the transfer factor. We also classify all genuine Weyl group action invariant characters of a split Cartan subgroup in the metaplectic group. This result we will use in the discussion about the uniqueness of the transfer factor later on.

4.1 Construction of the metaplectic group

The basic references for this section are Maktouf ([Mat]) and Lion and Vergne (see [L-V]).

We fix an additive character η. Let V be a $2n$–dimensional space equipped with a symplectic form $\langle\,,\rangle$. An orientation of V is a nonzero element e of $\bigwedge^{2n} V$. We will write V^e for an oriented vector space. For any two lagrangian subspaces l_1, l_2 we define a map $g_{l_1,l_2} : l_1 \to (l_2)^*$

$$g_{l_1,l_2}(v)(w) = \langle v, w\rangle, \quad v \in l_1, w \in l_2.$$

The map g_{l_1,l_2} induces an isomorphism between $l_1/l_1 \cap l_2$ and $(l_2/l_1 \cap l_2)^*$. If l_1 and l_2 are oriented we will denote this isomorphism by $g_{l_1^{e_1},l_2^{e_2}}$. We choose an orientation on $l_1 \cap l_2$ and we can consider $\det g_{l_1^{e_1},l_2^{e_2}} \mod F^{*2}$. It does not depend on the choice of orientation on $l_1 \cap l_2$. We define:

$$m(l_1^{e_1}, l_2^{e_2}) = \gamma_\eta(1)^{2(1-(n-dim(l_1\cap l_2))}\gamma_\eta(\det g_{l_1^{e_1}, l_2^{e_2}})^{-2}.$$

Example 4.1.1 *Split Cartan subgroup*

Let V be a $2n$–dimensional vector space equipped with a symplectic form

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

Let $x = \mathrm{diag}(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n) \in Sp(V)$. Let l^e be the lagrangian spanned by $e_1, \ldots, e_n$ with orientation $e = e_1 \wedge \cdots \wedge e_n$. Then xl^e is also spanned by $e_1, \ldots, e_n$, but has an orientation $x_1e_1 \wedge \cdots \wedge x_ne_n = (x_1 \cdots x_n)e$. We also choose the orientation e on $l \cap xl$, so the induced orientations on 0–dimensional spaces $l/(l \cap xl)$ and $(xl/(l \cap xl))^*$ are 1 and $x_1 \cdots x_n$ respectively. It follows that $\det(g_{l^e, xl^e}) = x_1 \cdots x_n$ and $m(l^e, xl^e) = \gamma_\eta(1)^2\gamma_\eta(x_1 \cdots x_n)^{-2}$.

For any triple of lagrangians l_1, l_2, l_3 denote by Q_{123} a quadratic form defined on $l_1 \times l_2 \times l_3$ as follows:

$$Q(x, y, z) = \langle x, y\rangle + \langle y, z\rangle + \langle z, x\rangle.$$

Let $c(l_1, l_2, l_3) = \gamma_\eta(Q_{123})$. The following lemma will be useful:

Lemma 4.1.2 *Let l_1 and l_2 be two lagrangian subspaces. Suppose that $l = (l \cap l_1) + (l \cap l_2)$. Then $c(l_1, l, l_2) = 1$.*

Proof. See [L-V], Lemma 1.5.11., page 44. □

We define $\widetilde{Sp}(2n)$ to be the set of pairs (x,ψ), where $x \in Sp(2n)$ and ψ is a function on the set of lagrangians of the space V satisfying conditions:

$$\psi(l') = \psi(l)c(l', l, xl)c(l', l, x^{-1}l')^{-1},$$

$$\psi^2(l) = m(l^e, xl^e)^{-1}.$$

The value of $m(l^e, xl^e)$ is independent of the choice of an orientation e. The multiplication is defined as follows:

$$(x,\psi)(x',\psi') = (xx', \psi''),$$

$$\psi''(l) = \psi(l)\psi'(l)c(l, xl, xx'l).$$

Remark. Note that this definition of the metaplectic group $\widetilde{Sp}(2n)$ depends on the fixed additive character η.

4.2 The character of the oscillator representation

Let $\omega(\eta) = \omega_+(\eta) + \omega_-(\eta)$ be the oscillator representation of $\widetilde{Sp}(2n)$ attached to the fixed additive character η. Since we work with a fixed η, for simplicity we will drop η from the notation. In this section we introduce explicit formulas for the characters of ω_+ and ω_-. See Maktouf ([Mak]) for the details.

Let $(x,\psi) \in \widetilde{Sp}(2n)$ where x is a regular semisimple element. We decompose V into a direct sum of subspaces W_1 and W_2 such that W_1 has an x invariant lagrangian l_1, and W_2 has a lagrangian l_2 such that $l_2 \cap (xl_2) = 0$. On $(1-x^{-1})^{-1}l_2$ we define the quadratic form Q_{x,l_2} :

$$Q_{x,l_2}(v) = \langle (x^{-1} - 1)v, v\rangle.$$

Proposition 4.2.1 *Let $\Theta_{\omega_\pm}$ denote the character of $\omega_\pm$. Let $(x,\psi) \in \widetilde{Sp}(2n)$ be such that x is regular semisimple element. Then*

$$\Theta_{\omega_\pm}(x,\psi) = \frac{1}{2}\psi(l_1+l_2)(\overline{\gamma_\eta(Q_{x,l_2})}|\det(1-x)|^{-\frac{1}{2}} \pm \overline{\gamma_\eta(Q_{-x,l_2})}|\det(1+x)|^{-\frac{1}{2}}).$$

Proof. See [Mak], section 31, page 296. □

Corollary 4.2.2 *Let $(x,\psi) \in \widetilde{Sp}(2n)$ and assume that x is a regular semisimple element. The character of the sum (difference) of the two halves of the oscillator representation is given by the formula:*

$$\Theta_{\omega_+ \pm \omega_-}((x,\psi)) = \psi(l_1+l_2)\overline{\gamma_\eta(Q_{\pm x,l_2})}|\det(1\mp x)|^{-1/2}.$$

Corollary 4.2.3 *Let (x,ψ) and $(-I,\psi') \in \widetilde{Sp}(2n)$, where x is a regular semisimple element. There exists a constant λ (depending only on the choice of ψ') such that*

$$\Theta_{\omega_+ + \omega_-}((x,\psi)(-I,\psi')) = \lambda\, \Theta_{\omega_+ - \omega_-}((x,\psi)).$$

Proof. First note that $(x,\psi)(-I,\psi') = (-x,\psi\psi')$. Indeed, for any lagrangian l, the value of the cocycle $c(l,xl,-xl)$ is 1, by Lemma 4.1.2. Therefore

$$\Theta_{\omega_+ + \omega_-}((-x,\psi\psi')) = \psi'(l_1+l_2)\psi(l_1+l_2)\overline{\gamma_\eta(Q_{-x,l_2})}|\det(1+x)|^{-1/2}$$

$$= \psi'(l_1+l_2)\Theta_{\omega_+ - \omega_-}((x,\psi)).$$

Now we claim that the function ψ' is constant. Recall that for any pair of lagrangians l and l' the following identity holds:

$$\psi(l') = \psi(l)c(l',l,-l)c(l',l,-l')^{-1}.$$

By Lemma 4.1.2 we get that $c(l', l, -l) = 1$. Consider now the form $Q_{l',l,-l'}$ (see section 4.1 for the description). Note that $Q_{l',l,-l'} = -Q_{l,l',-l'}$ and therefore $c(l', l, -l') = \gamma_\eta(Q_{l',l,-l'}) = \gamma_\eta(-Q_{l,l',-l'}) = \gamma_\eta(Q_{l,l',-l'})^{-1} = c(l, l', -l')^{-1} = 1$ (the middle equality follows from Corollary 2.3.2 and the last one follows again from Lemma 4.1.2). Now we define $\lambda = \psi(l') = \psi(l)$. $\square$

Example 4.2.4 *Split Cartan subgroup*

We use the notations of Example 4.1.1. Let $x = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n)$. Assume further that all $x_i \neq 1$. In this case $V = W_1, W_2 = 0, l_2 = 0, Q_{x,l_2} = 0$ and one can take l_1 to be the lagrangian l spanned by $e_1, \ldots, e_n$ with orientation $e = e_1 \wedge \cdots \wedge e_n$. Since $\psi^2(l) = m(l^e, xl^e)^{-1} = \gamma_\eta(x_1 \cdots x_n)^2 \gamma_\eta(1)^{-2}$ we have two choices for the value of the function ψ on l :

$$\psi_\epsilon(l) = \epsilon\gamma_\eta(x_1 \cdots x_n)\gamma_\eta(1)^{-1} = \epsilon\gamma(x_1 \cdots x_n, \eta),$$

where $\epsilon = \pm 1$. Therefore

$$\Theta_{\omega_+ \pm \omega_-}((x, \psi_\epsilon)) = \frac{\epsilon\gamma(x_1 \cdots x_n, \eta)}{|\det(1 \mp x)|^{1/2}}.$$

4.3 Characters of a split Cartan subgroup in $\widetilde{Sp}(2n)$

Here we introduce an alternate definition of $\widetilde{Sp}(2n)$ due to Steinberg [St] and Matsumoto [Mat]. A nice summary of these results is also given by Savin in [Sa].

In this section we assume $p \neq 2$. Let R be the set of roots of type C_n. Choose set of simple roots $\Delta = \{\alpha_1, \ldots, \alpha_n\}$. Let $\kappa(\ ,\ )$ be a Killing form normalized so that

$\kappa(\alpha,\alpha)=2$ for $\alpha\in R$. We define a bilinear form on the root lattice:

$$B(\alpha_i,\alpha_j)=\begin{cases}0 & \text{if } i<j,\\ 1/2\kappa(\alpha_i,\alpha_i) & \text{if } i=j,\\ \kappa(\alpha_i,\alpha_j) & \text{if } i>j.\end{cases}$$

The following theorem is due to Steinberg, [St], see also Savin, [Sa], Theorem 2.1.

Proposition 4.3.1 *Let $n>1$. The universal two fold central extension $\widetilde{Sp}(2n)$ of the symplectic group $Sp(2n)$ has a presentation with formal set of generators $\{e_\alpha(t),\alpha\in R,t\in F\}$ subject to relations:*

$$e_\alpha(t+s)=e_\alpha(t)e_\alpha(s),$$

$$[e_\alpha(x),e_\alpha(y)]=\begin{cases}e_{\alpha+\beta}((-1)^{B(\alpha,\beta)}xy) & \textit{if } \alpha+\beta \textit{ is a root,}\\ 1 & \textit{if not and } \alpha+\beta\neq 0.\end{cases}$$

We define now certain elements which will play an important role in this section:

$$w_\alpha(t)=e_\alpha(t)e_{-\alpha}(-t^{-1})e_\alpha(t),$$

$$h_\alpha(t)=w_\alpha(t)w_\alpha(-1),\quad t\in F^*.$$

In the case of $n=1$ we have the following:

Proposition 4.3.2 *The two fold cover $\widetilde{SL}(2)$ of the group $SL(2)$ is generated by the set $\{e_\alpha(t),\alpha\in R,t\in F\}$ subject to relations:*

$$e_\alpha(t)e_\alpha(s)=e_\alpha(t+s),\quad t,s\in F,\alpha\in R$$

$$w_\alpha(t)e_\alpha(u)w_\alpha(t)^{-1}=e_{-\alpha}(-t^{-2}u),\quad t\in F^*,u\in F,\alpha\in R$$

$$h_\alpha(s)h_\alpha(t)=(s,t)h_\alpha(st),\quad t,s\in F^*,\alpha\in R.$$

Proof. [St], Theorem 3.3 □

Let

$$c_\alpha(s,t) = \begin{cases} 1 & \text{if } \alpha \text{ is a short root,} \\ (s,t) & \text{otherwise.} \end{cases}$$

(see [Mat], p. 30). The split Cartan subgroup $\widetilde{T} \subset \widetilde{Sp}(2n)$ is generated by $h_\alpha(t)$ for all simple roots α and $t \in F^*$. The generators satisfy following relations (see [Mat], p. 38):

$$h_\alpha(t)h_\beta(s) = h_\beta(s)h_\alpha(t),$$

$$h_\alpha(s)h_\alpha(t) = c_\alpha(s,t)h_\alpha(st).$$

The Weyl group $\widetilde{W}$ of $\widetilde{T}$ is generated by the set $\{w_\alpha(-1), \alpha \in \Delta\}$. The Weyl group W of T is a quotient of $\widetilde{W}$ by a subgroup generated by $\{w_\alpha(-1)^2, \alpha \in \Delta\}$ (compare [Sa], p. 116).

Corollary 4.3.3 *The action of $\widetilde{W}$ coincides with the action of W.*

Proof. Note that $w_\alpha(-1)^2 = h_\alpha(-1)$. Since $\widetilde{T}$ is abelian, $h_\alpha(-1)$ acts trivially on $\widetilde{T}$.
□

Define now a character $\chi_\eta : \widetilde{T} \to \{\pm 1, \pm i\}$ as follows: Let

$$\chi_\eta(-1) = -1,$$

$$\chi_\eta(h_\alpha(t)) = \begin{cases} 1 & \text{if } \alpha \text{ is a short root,} \\ \gamma(t,\eta) & \text{otherwise.} \end{cases}$$

Hence χ_η is defined on the generators of $\widetilde{T}$; the following proposition shows that χ_η extends (uniquely) to a character of $\widetilde{T}$.

Proposition 4.3.4 *The map χ_η extends uniquely to a genuine, $W-$ action invariant character of the Cartan subgroup $\widetilde{T}$.*

Proof. First we need to check that χ_η is well defined. The only interesting case is when α is long. We have: $\chi_\eta(h_\alpha(s)h_\alpha(t)) = \chi_\eta((s,t)h_\alpha(st)) = (s,t)\gamma(st,\eta) = \gamma(s,\eta)\gamma(t,\eta) = \chi_\eta(h_\alpha(s))\chi_\eta(h_\alpha(t))$. To check that χ_η is $W-$ invariant it is enough to show that for all $t \in F^*$ and all simple roots α, β :

$$\chi_\eta(w_\alpha(1)h_\beta(t)w_\alpha(-1)) = \chi_\eta(h_\beta(t)).$$

By [Mat] $w_\alpha(1)h_\beta(t)w_\alpha(-1) = h_\beta(t)h_\alpha(t^{-d})$, where $d = \langle\alpha, \beta^\vee\rangle$, therefore we need to check that $\chi_\eta(h_\alpha(t^{-d})) = 1$. This is obviously true when α is short or $\alpha = 2e_n$ and $\beta \in \{e_1 - e_2, \ldots, e_{n-2} - e_{n-1}\}$. If $\alpha = 2e_n$ and $\beta = e_{n-1} - e_n$ or $\beta = 2e_n$ then $d = \mp 2$ and $\chi_\eta(h_\alpha(t^{\pm 2})) = \gamma(t^{\pm 2}, \eta) = 1$. $\square$

As a corollary we get that for every coset $xF^{*2} \in F^*/F^{*2}$ there exists a different genuine $W-$ invariant characters of $\widetilde{T}$, namely $\chi_{x\eta}$.

Proposition 4.3.5 *Every genuine $W-$ invariant character of $\widetilde{T}$ is of the form $\chi_{x\eta}$, where xF^{*2} is a coset in F^*/F^{*2}.*

*Proof.*Let χ be a genuine $W-$ invariant character of $\widetilde{T}$. By the definition of genuine character we have that $\chi(-1) = -1$. $W-$ invariance implies

$$\forall_{\alpha,\beta\in R'} \quad \chi(h_\alpha(t^{-\langle\alpha,\beta^\vee\rangle})) = 1, \quad t \in F^*.$$

For a short root α let β be such that $\langle\alpha, \beta^\vee\rangle = -1$ to get $\chi(h_\alpha(t)) = 1$. For a long root α let β be such that $\langle\alpha, \beta^\vee\rangle = -2$, then $\chi(h_\alpha(t^2)) = 1$. In any case

$$\chi(h_\alpha(t))^2 = \chi((t,-1)h_\alpha(t^2)) = (t,-1) = \pm 1,$$

and therefore χ has values in the set $\{\pm 1, \pm i\}$. Now let χ and χ' be two genuine $W-$ invariant characters of $\widetilde{T}$. Their product $\mu = \chi\chi'$ factors to T. Since μ is also $W-$ invariant, it is uniquely determined by its values on $h_\alpha(t)$ for a fixed long root α and $t \in F^*$. Therefore without lost of generality we can treat it as a character of F^*. We have $\mu(t^2) = \mu(t)^2 = \chi(t)^2\chi'(t)^2 = 1$, hence μ factors to a character of the quotient F^*/F^{*2} with values ± 1. Therefore the number of genuine $W-$ invariant characters of $\widetilde{T}$ is equal to the number of such characters, which is the cardinality of F^*/F^{*2}.

□

Chapter 5

Transfer factor

In this chapter we define a transfer factor and we study some of its properties. We will use the transfer factor later on, to define a lifting of characters between $SO(2n+1)_+$ and $\widetilde{Sp}(2n)$.

Let $G = Sp(2n)$ or $SO(2n+1)_+$. For $x \in G$ we define $D_G(x)$ by

$$\det(t+1-Ad(x)) = D_G(x)t^k + \cdots + \text{ terms of higher degree,}$$

where k is the rank of G.

Lemma 5.0.6 *Let T be a Cartan subgroup in G and let $x \in T$. Then $D_G(x) = \prod_{\alpha \in R}(1-\alpha(x))$, where R is the set of roots of T in G.*

We denote the set of regular elements of G (i.e. elements $x \in G$ such that $D_G(x) \neq 0$) by G_{reg}.

Lemma 5.0.7 *Let $x \in Sp(2n)$ be a regular semisimple element and let $x \overset{stable}{\longleftrightarrow} x' \in SO(2n+1)_+$. Then*

$$\frac{D_{Sp(2n)}(x)}{D_{SO(2n+1)_+}(x')} = \det(1+x).$$

Proof. Assume that x has eigenvalues $x_1, \ldots, x_{2n}$. Therefore x' has the same eigenvalues together with 1. By the previous lemma we have that

$$\frac{D_{Sp(2n)}(x)}{D_{SO(2n+1)_+}(x')} = \frac{\prod(1-x_i^2)\prod(1-x_i/x_j)}{\prod(1-x_i)\prod(1-x_i/x_j)} = \prod(1+x_i) = \det(1+x). \quad \square$$

Definition 5.0.8 *The transfer factor on $\widetilde{Sp}(2n)$ is equal to the difference of the two halves of the oscillator representation, i.e.*

$$\Phi_{\widetilde{Sp}(2n)} = \Theta_{\omega_+ - \omega_-}.$$

Now we will define the transfer factor on Levi subgroups of the metaplectic group. Let A be a split torus in $Sp(2n)$ and M be its centralizer in $Sp(2n)$. Assume that M is of the form $GL(n_1) \times \ldots \times GL(n_k) \times Sp(2m)$. Let $\widetilde{A} = p^{-1}(A)$, where $p : \widetilde{Sp}(2n) \to Sp(2n)$. Let further $A' = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SO(2n+1)_+$ and $M' = Cent_{SO(2n+1)_+} A' \cong GL(n_1) \times \ldots \times GL(n_k) \times SO(2m+1)_+$.

Definition 5.0.9 *Let $g = (g_1, \ldots g_k, g_{k+1}) \in M$ and $g' = (g'_1, \ldots g'_k, g'_{k+1}) \in M'$ be strongly regular semisimple elements. We say that g Levi-stably corresponds to g' if $g_i \in GL(n_i)$ is conjugate to $g'_i \in GL(n_i)$ for $i \in \{1, \ldots, k\}$ and if g_{k+1} stably corresponds to g'_{k+1} in the sense of definition 2.4.10. We will write $g \overset{L-stable}{\longleftrightarrow} g'$.*

Definition 5.0.10 *We define the transfer factor on $\widetilde{M} = p^{-1}(M)$ as follows:*

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_M(g)|^{\frac{1}{2}}} \frac{|D_{M'}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}),$$

where $p(\widetilde{g}) = g \overset{L-stable}{\longleftrightarrow} g'$.

Note that this definition agrees with the earlier definition of the transfer factor defined on $\widetilde{Sp}(2n)$. Indeed, when $M = Sp(2n)$ and $M' = SO(2n+1)_+$ we get

$$\Phi_{\widetilde{Sp}(2n)} = \Theta_{\omega_+ - \omega_-}.$$

In particular we have that

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_M(g)|^{\frac{1}{2}}} \frac{|D_{M'}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} \Phi_{\widetilde{Sp}(2n)}(\widetilde{g}). \tag{5.1}$$

Note also that in the special case when $M = A \cong F^{*n}$ we have that

$$\Phi_{\widetilde{A}}(\widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{g}) |\det(1+g)|^{\frac{1}{2}},$$

since by Lemma 5.0.7

$$\frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} = |\det(1+g)|^{\frac{1}{2}}.$$

Lemma 5.0.11 *(1) $\Phi_{\widetilde{M}}$ restricted to $\widetilde{Sp}(2m)$ is equal to the character of the difference of the two halves of the oscillator representation of $\widetilde{Sp}(2m)$ (i.e. it is equal to $\Phi_{\widetilde{Sp}(2m)}$),*

(2) $\Phi_{\widetilde{M}}$ restricted to each $\widetilde{GL}(n_i)$ is equal to a character χ, where $\chi(g, \epsilon) = \gamma(\det(g), \eta)\epsilon$.

Proof. Let V be a $2n$–dimensional vector space with a symplectic form $\langle\, , \rangle$. Choose a symplectic basis $e_1, \dots e_n, f_1, \dots f_n$. Let $N = n_1 + \cdots + n_k$. Embed $GL(N)$ and $Sp(2m)$ into $Sp(2n)$ as follows:

$$x \longmapsto \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & I_m & 0 & 0 \\ 0 & 0 & {}^t x^{-1} & \\ 0 & 0 & 0 & I_m \end{pmatrix},$$

$$y = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto \begin{pmatrix} I_N & 0 & 0 & 0 \\ 0 & A & 0 & B \\ 0 & 0 & I_N & 0 \\ 0 & C & 0 & D \end{pmatrix}.$$

We will identify x and y with their images in $Sp(2n)$. Now decompose V into direct sum of subspaces V_1 and V_2, where V_1 is spanned by $\{e_1, \dots e_N, f_1, \dots f_N\}$ and V_2 is spanned by $\{e_{N+1}, \dots e_n, f_{N+1}, \dots, f_n\}$. Note that $x \in GL(N)$ acts on V_1 and fixes V_2, similarly $y \in Sp(2m)$ fixes V_1 and acts on V_2. Consider now $(g, \psi_g) = (x, \psi_x)(y, \psi_y) \in \widetilde{Sp}(2n)$, where x and y are as above and $g = xy$ is strongly regular. First we decompose y further as follows

$$y = \begin{pmatrix} y' & 0 & 0 & 0 \\ 0 & A' & 0 & B' \\ 0 & 0 & {}^t y'^{-1} & 0 \\ 0 & C' & 0 & D' \end{pmatrix},$$

where $y' \in GL(s)$ for some integer $0 \le s \le m$ and

$$\begin{pmatrix} A' & B' \\ C' & D' \end{pmatrix}$$

belongs to the part of the Cartan subgroup $Cent_{Sp(2m)}(y)$ that is a product of norm one tori. We proceed now as in section 4.2 and we choose l_1 be the subspace generated by $e_1, \dots, e_N, \dots, e_{N+s}$ and l_2 be the subspace generated by $e_{N+s+1}, \dots, e_n$. Their sum $l_1 + l_2$ is a lagrangian and we will denote it by l. It follows that $xl = l$, hence by Lemma 4.1.2

$$\psi_{xy}(l) = \psi_x(l)\psi_y(l)c(l, xl, xyl) = \psi_x(l)\psi_y(l).$$

If $(x, \psi_x) = (x_1, \psi_1) \dots (x_{n_k}, \psi_{n_k})$ where each $x_{n_i} \in GL(n_i)$ then (again by Lemma 4.1.2) $\psi_x(l) = \psi_{x_{n_1}}(l) \dots \psi_{x_{n_k}}(l)$. We have that $\psi_{x_{n_i}}(l) = \pm\gamma(\det(x_{n_i}), \eta)$ is a character on $\widetilde{GL}(n_i)$.

Note now that $(1+g^{-1})^{-1}l_2 = (1+y^{-1})^{-1}l_2 \subset V_2$ and for $v \in (1+y^{-1})^{-1}l_2$ we have

$$Q_{-g,l_2}(v) = \langle(-g^{-1}-1)v, v\rangle = \langle(-y^{-1}-1)v, v\rangle = Q_{-y,l_2}(v).$$

Therefore $\gamma_\eta(Q_{-g,l_2}) = \gamma_\eta(Q_{-y,l_2})$. Combining all of this together we get that

$$\Theta_{\omega_+-\omega_-}((g,\psi_g)) = \psi_{x_1}(l)\cdots\psi_{x_{n_k}}(l)\psi_y(l)\overline{\gamma_\eta(Q_{-y,l_2})}|\det(1+g)|^{-1/2}.$$

Note also that by Lemma 5.0.7

$$\frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_{Sp(2m)}(y)|^{\frac{1}{2}}}\frac{|D_{SO(2m+1)_+}(y')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} = \frac{|\det(1+g)|^{\frac{1}{2}}}{|\det(1+y)|^{\frac{1}{2}}},$$

hence we can rewrite the formula for the transfer factor as follows:

$$\Phi_{\widetilde{M}}((g,\psi_g)) = \psi_{x_1}(l)\cdots\psi_{x_{n_k}}(l)\psi_y(l)\overline{\gamma_\eta(Q_{-y,l_2})}|\det(1+y)|^{-1/2}$$

$$= \psi_{x_1}(l)\cdots\psi_{x_{n_k}}(l)\Theta_{\omega_+-\omega_-}((y,\psi_y)). \quad \square$$

Chapter 6

Inducing

Here we recall van Dijk's formula for the character of an induced representation. Furthermore we show in the case of a minimal parabolic induction that if two representations correspond then after parabolic induction the two resulting representations still correspond. We also discuss the uniqueness of the transfer factor in that case. At the end we study some quotients of the Weyl groups that appear in van Dijk's formula.

6.1 Induced character formula

We state here van Dijk's result concerning the formula for the character of an induced representation. See [D] for more details.

Let $G = Sp(2n)$ or $SO(2n+1)_+$. Let A be a split torus in G and M its centralizer in G. For any Cartan subgroup T of G we denote by A_T the split component of T and by $W(A,T)$ the set of all injections $s : A \to A_T$ for which there exists $y \in G$ such that $s(a) = yay^{-1}$ for all $a \in A$.

For $s = s_y \in W(A,T)$ and any representation ρ of M, let $s\rho$ be the representation of $M^s = yMy^{-1}$ defined by $s\rho(m) = \rho(y^{-1}my)$. The following theorem is due to Van Dijk ([D], Theorem 3, page 238):

Proposition 6.1.1 *Consider the parabolic subgroup $P = MN$. Let ρ be any admis-*

sible representation of M with a character θ_ρ. We extend ρ to a representation of P by putting $\rho(mn) = \rho(m), m \in M, n \in N$. Let T be any Cartan subgroup of G that is conjugate to a Cartan subgroup in M. The character θ_π of the representation $\pi = Ind_P^G(\rho)$ has the formula:

$$\theta_\pi(x) = \sum_{s \in W(A,T)} \theta_{s\rho}(x) \frac{|D_{M^s}(x)|^{\frac{1}{2}}}{|D_G(x)|^{\frac{1}{2}}}, \quad x \in T \cap G_{reg}.$$

If T is a Cartan subgroup that is not conjugate to a Cartan subgroup in M, then θ_π vanishes on $T \cap G_{reg}$.

Van Dijk's theorem does not apply directly to $\widetilde{Sp}(2n)$. It seems that an argument similar to van Dijk's proof would hold for the metaplectic group. However, there is no known reference for this fact. Writing down the details of such proof is one more possible research project after completing this thesis. For the purpose of this work, we will assume that van Dijk's result holds. Let ρ be an admissible representation of a parabolic subgroup $\widetilde{P} = p^{-1}(P)$ where $P = MN$ and $M = Cent_{Sp(2n)}(A)$. Denote its character by θ_ρ and let $\widetilde{T} = p^{-1}(T)$ be any Cartan subgroup of $\widetilde{Sp}(2n)$ that is conjugate to a Cartan subgroup in $\widetilde{M} = p^{-1}(M)$. We will assume that the character θ_π of the representation $\pi = Ind_{\widetilde{P}}^{\widetilde{Sp}(2n)}(\rho)$ is given by the formula:

$$\theta_\pi(\widetilde{x}) = \sum_{s \in W(\widetilde{A},\widetilde{T})} \theta_{s\rho}(\widetilde{x}) \frac{|D_{M^s}(x)|^{\frac{1}{2}}}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}}, \quad \widetilde{x} \in \widetilde{T} \cap \widetilde{Sp}(2n)_{reg}.$$

If $\widetilde{T}$ is not conjugate to a Cartan subgroup in $\widetilde{M} = p^{-1}(M)$, then $\theta(\widetilde{x}) = 0$ for $\widetilde{x} \in \widetilde{T} \cap \widetilde{Sp}(2n)_{reg}$. The set $W(\widetilde{A}, \widetilde{T})$ is defined similarly as for the linear case, i.e. if $\widetilde{A} = p^{-1}(A)$ and $A_{\widetilde{T}}$ is the split component of the Cartan subgroup $\widetilde{T}$, then we denote by $W(\widetilde{A}, \widetilde{T})$ the set of all injections $s : \widetilde{A} \to A_{\widetilde{T}}$ for which there exists $\widetilde{y} \in \widetilde{Sp}(2n)$ such that $s(\widetilde{a}) = \widetilde{y}\widetilde{a}\widetilde{y}^{-1}$ for all $\widetilde{a} \in \widetilde{A}$.

Lemma 6.1.2 $W(\widetilde{A},\widetilde{T}) \cong W(A,T)$.

Proof. Consider $s = s_y \in W(A,T)$ and choose any $\widetilde{y} \in \widetilde{Sp}(2n)$ such that $p(\widetilde{y}) = y$. Then for all $\widetilde{a} \in \widetilde{A}$ we have that $\widetilde{y}\widetilde{a}\widetilde{y}^{-1} \subset A_{\widetilde{T}}$. Hence $s_{\widetilde{y}} \in W(\widetilde{A},\widetilde{T})$. This is independent of the choice of the element $\widetilde{y}$. Indeed, let $\widetilde{y}' \in \widetilde{Sp}(2n)$ be another element that maps to y. Since they differ by a central element in $\widetilde{Sp}(2n)$ we have that $\widetilde{y}\widetilde{a}\widetilde{y}^{-1} = \widetilde{y}'\widetilde{a}\widetilde{y}'^{-1}$ for all $\widetilde{a} \in \widetilde{A}$. Therefore $s_{\widetilde{y}} = s_{\widetilde{y}'}$.

On the other hand, if $s = s_{\widetilde{y}} \in W(\widetilde{A},\widetilde{T})$, then clearly $s = s_y$, where $y = p(\widetilde{y})$ belongs to $W(A,T)$. $\square$

Example 6.1.3 *Minimal parabolic induction for $G = SO(2n+1)_+$.*

Let $A \cong F^{*n} \subset SO(2n+1)_+$. Then $M = Cent_{SO(2n+1)_+}A = A$. Let χ be a character of A. We extend it to a character of P by $\chi(mn) = \chi(m)$, and then we induce it to a representation π on $SO(2n+1)_+$. We consider its character $\theta_{\pi.}$ If T is any Cartan subgroup with the split part $A_T \cong F^{*k}$ where $k < n$, then $W(A,T) = \emptyset$, hence $\theta_\pi \equiv 0$ on T_{reg}. If $T \cong A$, then $W(A,A) = W(B_n)$ is the Weyl group of type B_n. Let $x \in T$. We have that $D_M(x) = 1$. We conclude:

$$\theta_\pi(x) = \sum_{w \in W(B_n)} \frac{\chi(w \cdot x)}{|D_{SO(2n+1)_+}(x)|^{\frac{1}{2}}}.$$

Example 6.1.4 *Minimal parabolic induction for $G = \widetilde{Sp}(2n)$.*

Let $A \cong F^{*n} \subset Sp(2n), \widetilde{A} = p^{-1}(A)$, where $p : \widetilde{Sp}(2n) \to Sp(2n)$ is the projection map. Define a character $\widetilde{\chi}$ on $\widetilde{A}$ as follows:

$$\widetilde{\chi}((x,\epsilon)) = \chi(x)\gamma(\prod_{i=1,\ldots,n} x_i, \eta)\epsilon,$$

where $x = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n) \in Sp(2n)$, and χ is a character of F^{*n}. We extend it to a character on $\widetilde{P}$ and we consider the character $\theta_{\widetilde{\pi}}$ of the induced representation $\widetilde{\pi}$. As before it is enough to find its value on the split Cartan subgroup $\widetilde{A}$. In this case $\widetilde{M} = Cent_{\widetilde{Sp}(2n)}\widetilde{A} = \widetilde{A}$, $D_M(x) = 1$, and $W(\widetilde{A}, \widetilde{A}) \cong W(A, A) = W(C_n)$ is the Weyl group of type C_n. Note also, that the character $(x, \epsilon) \mapsto \gamma(\prod_{i=1,\ldots,n} x_i, \eta)\epsilon$ is $W-$ action invariant (see Proposition 4.3.4 for the proof), therefore

$$(w\widetilde{\chi})((x, \epsilon)) = \chi(w \cdot x)\gamma(\prod_i x_i, \eta)\epsilon, \quad w \in W(C_n), (x, \epsilon) \in \widetilde{A}.$$

Combining all of these together we get that

$$\theta_{\widetilde{\pi}}((x, \epsilon)) = \sum_{w \in W(C_n)} \frac{\chi(w \cdot x)\gamma(\prod_i x_i, \eta)\epsilon}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}}.$$

6.2 Minimal parabolic induction

Here we state and prove the main theorem of this thesis in the special case of the minimal parabolic induction. This section is intended as an example and it is not needed for the purpose of the proof in the general case.

Consider split Cartan subgroups $A \cong F^{*n} \subset Sp(2n)$, $\widetilde{A} = p^{-1}(A) \subset \widetilde{Sp}(2n)$ and $A' \cong F^{*n} \subset SO(2n+1)_+$. Recall that the transfer factor defined on $\widetilde{A}$ is equal to:

$$\Phi_{\widetilde{A}}(\widetilde{x}) = \Theta_{\omega_+ - \omega_-}(\widetilde{x})|\det(1 + p(\widetilde{x}))|^{\frac{1}{2}}. \tag{6.1}$$

Let χ be a character on A' and $\widetilde{\chi}$ be a character of $\widetilde{A}'$. Assume that they satisfy the matching condition:

$$\widetilde{\chi}(\widetilde{x}) = \Phi_{\widetilde{A}}(\widetilde{x})\chi(x'), \tag{6.2}$$

for all elements $\widetilde{x}$ and x' such that $p(\widetilde{x}) = x = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n) \in A$, and $x' = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n, 1) \in A'$ are strongly regular. We extend χ and $\widetilde{\chi}$ to characters defined on P and $\widetilde{P}$ respectively. We will show the following:

Theorem 6.2.1 *Let the characters $\chi, \widetilde{\chi}$ be as above. Denote the characters of the induced representations $\pi = Ind_P^{SO(2n+1)_+}\chi$ and $\widetilde{\pi} = Ind_{\widetilde{P}}^{\widetilde{Sp}(2n)}\widetilde{\chi}$ by θ_π and $\theta_{\widetilde{\pi}}$ respectively. Then*

$$\theta_{\widetilde{\pi}}(\widetilde{g}) = \Phi_{\widetilde{Sp}(2n)}(\widetilde{g})\theta_\pi(g'),$$

whenever $p(\widetilde{g}) \overset{stable}{\longleftrightarrow} g'$.

Remark 1. Note that we work with a fixed additive character η. The construction of the metaplectic group $\widetilde{Sp}(2n)$, and the oscillator representation $\omega_+ + \omega_-$ both depend on η. Therefore also the transfer factor and the matching of the characters χ and $\widetilde{\chi}$ depend on η.

Proof. It is enough to show the equality of the characters on the diagonal elements $\widetilde{x}$ and x', where $p(\widetilde{x}) \overset{stable}{\longleftrightarrow} x'$. By the definition of stable correspondence, they have the same nontrivial eigenvalues. Since we will average over the Weyl group, we can assume without loss of generality that $x' = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n, 1) \in SO(2n+1)_+$ and $\widetilde{x} = (x, \epsilon) \in \widetilde{Sp}(2n)$, where $x = diag(x_1, \ldots, x_n, 1/x_1, \ldots, 1/x_n) \in Sp(2n)$. Recall that in Example 4.2.4 we evaluated the character $\Theta_{\omega_+ - \omega_-}$ on an element (x, ϵ) :

$$\Theta_{\omega_+ - \omega_-}((x, \epsilon)) = \frac{\gamma(\prod_{i=1,\ldots,n} x_i, \eta)\epsilon}{|\det(1 + x)|^{1/2}}.$$

Combining this with the equation (6.1) we get the formula for the transfer factor

$$\Phi_{\widetilde{A}}((x,\epsilon)) = \gamma(\prod_{i=1,\dots,n} x_i, \eta)\epsilon.$$

Therefore, given a character χ, the matching character $\widetilde{\chi}$ must satisfy

$$\widetilde{\chi}((x,\epsilon)) = \chi(x')\gamma(\prod_{i=1,\dots,n} x_i, \eta)\epsilon.$$

In Examples 6.1.3 and 6.1.4 we calculated the formulas of the characters of the induced representations π and $\widetilde{\pi}$:

$$\theta_\pi(x') = \sum_{w\in W(B_n)} \frac{\chi(w\cdot x')}{|D_{SO(2n+1)_+}(x')|^{\frac{1}{2}}},$$

$$\theta_{\widetilde{\pi}}((x,\epsilon)) = \gamma(\prod_i x_i, \eta)\epsilon \sum_{w\in W(B_n)} \frac{\chi(w\cdot x')}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}}.$$

Since (compare Lemma 5.0.7)

$$\frac{|D_{SO(2n+1)_+}(x')|^{\frac{1}{2}}}{|D_{Sp(2n)}(x)|^{\frac{1}{2}}} = \frac{1}{|\det(1+x)|^{\frac{1}{2}}}$$

we get

$$\theta_{\widetilde{\pi}}(x,\epsilon) = \frac{\gamma(\prod_i x_i,\eta)\epsilon}{|\det(1+x)|^{\frac{1}{2}}}\theta_\pi(x') = \Phi_{\widetilde{Sp}(2n)}((x,\epsilon))\theta_\pi(x'). \quad \square$$

Remark 2. (Uniqueness of the transfer factor) Take χ to be a trivial character on $A' \subset SO(2n+1)_+$. Then the matching character $\widetilde{\chi}$ on $\widetilde{A}$ is of the form

$$\widetilde{\chi}((x,\epsilon)) = \Phi_{\widetilde{A}}((x,\epsilon)) = \gamma(\prod_{i=1,\dots,n} x_i, \eta)\epsilon.$$

By Proposition 4.3.4 $\widetilde{\chi}$ is a genuine W–action invariant character of the split Cartan subgroup $\widetilde{A}$. However, by Proposition 4.3.5 every genuine W–action invariant character $\widetilde{\chi}'$ of $\widetilde{A}$ is of the form

$$\widetilde{\chi}'((x,\epsilon)) = \gamma(\prod_{i=1,\dots,n} x_i, \eta')\epsilon$$

for some additive character η'.

6.3 Weyl groups

We keep the notations from the previous sections. The goal of this section is to show that there exists a bijection between the sets $W(A,T)$ and $W(A',T')$, where $A \subset Sp(2n)$ and $A' \subset SO(2n+1)_+$ are isomorphic split tori and $T \subset Sp(2n)$ and $T' \subset SO(2n+1)_+$ are isomorphic Cartan subgroups. Recall also that $M = Cent_{Sp(2n)}(A)$ and $M' = Cent_{SO(2n+1)_+}(A')$.

First we show that in any case, the set $W(A,T)$ can be identified with a subset of the Weyl group of the split Cartan subgroup. This is a generalization of the case of minimal parabolic induction, when $W(A,T)$ was isomorphic to $W(C_n)$. Analogous statement holds for $W(A',T')$. We denote by A_T the split component of T and by A_s a split Cartan subgroup of $Sp(2n)$. We assume without loss of generality that $A, A_T \subset A_s$.

Lemma 6.3.1 *Let $s \in W(A,T)$. There exists $w_s \in W(Sp(2n), A_s)$ such that $s(a) = w_s \cdot a$ for all $a \in A$.*

Proof. Let $s = int(g) : A \to A_T$. Consider $int(g)(A_s) = gA_sg^{-1}$. We have that $gAg^{-1} \subset A_T \subset A_s$ and $gAg^{-1} \subset gA_sg^{-1}$, therefore $A_s, gA_sg^{-1} \subset Cent_{Sp(2n)}(gAg^{-1})$. Since A_s and gA_sg^{-1} are split Cartan subgroups in $Cent_{Sp(2n)}(gAg^{-1})$ they must be conjugate by $h \in Cent_{Sp(2n)}(gAg^{-1})$. Then $int(hg)(A_s) = A_s$ and $int(hg)(A) = int(g)(A)$. We take w_s to be the image of hg in the Weyl group of A_s.

$$\begin{array}{ccccc}
A_s & \xrightarrow{int(g)} & gA_sg^{-1} & \xrightarrow{int(h)} & A_s \\
\uparrow & & \uparrow & & \uparrow \\
A & \xrightarrow{int(g)} & gAg^{-1} & = & gAg^{-1}
\end{array}$$

□

Lemma 6.3.2 *(1)* $W(A,T) \cong \bigcup_{\{H\subset M: H\sim_{Sp(2n)} T\}/\sim_M} W(Sp(2n),H)/W(M,H)$,

(2) $W(A',T') \cong \bigcup_{\{H'\subset M': H'\sim_{SO(2n+1)_+} T'\}/\sim_{M'}} W(SO(2n+1)_+,H')/W(M',H')$.

Proof. It is enough to prove the symplectic case only, since the prove of (2) is analogous. Let $T_1,\dots,T_k$ be the representatives of the set $\{H \subset M : H \sim_{Sp(2n)} T\}/\sim_M$. For each $i=1,\dots,k$ choose $t_i \in Sp(2n)$ such that $T = t_i T_i t_i^{-1}$. Note that we have $A_T = t_i A_{T_i} t_i^{-1}$ and also note that $A \subset A_{T_i}$ (since every T_i is a Cartan subgroup in $M = Cent_{Sp(2n)}(A)$). Consider now $int(g) \in W(A,T)$. We have $A \mapsto gAg^{-1} \subset A_T \subset T$. It follows that $A \subset g^{-1}Tg$. The Cartan subgroup $g^{-1}Tg$ is contained in M :

$$(g^{-1}tg)a(g^{-1}tg)^{-1} = a \quad \forall a\in A, t\in T \Longleftrightarrow t(gag^{-1})t^{-1} = gag^{-1} \quad \forall a\in A, t\in T,$$

and that is true since $t \in T$ and $gag^{-1} \in A_T \subset A$. Therefore $g^{-1}Tg = m_g T_i m_g^{-1}$, for some $m_g \in M$ and $i=1,\dots,k$. Since $gm_gT_i(gm_g)^{-1} = T = t_iT_it_i^{-1}$ we get that $t_i^{-1}gm_g \in N(Sp(2n),T_i)$. Now define a map

$$\Psi : W(A,T) \longrightarrow \bigcup_{T_i} W(Sp(2n),T_i)/W(M,T_i),$$

$$\Psi(int(g)) = [t_i^{-1}gm_g] \in W(Sp(2n),T_i)/W(M,T_i).$$

It is well defined. Indeed, suppose $int(g)|_A = int(h)|_A$, i.e. $gag^{-1} = hah^{-1}$ for all $a \in A$. It follows that $g^{-1}h \in M$. If $h^{-1}Th = m_hT_jm_h^{-1}$, then $hm_hT_j(hm_h)^{-1} = T = gm_gT_i(gm_g)^{-1}$, hence T_i and T_j are conjugate in M, therefore $i=j$ and $m_g^{-1}g^{-1}hm_h \in N(M,T_i)$. Therefore $\Psi(int(h)) = [t_i^{-1}hm_h] = [t_i^{-1}gm_g] = \Psi(int(g))$.

Let now $\overline{g} \in W(Sp(2n),T_i)/W(M,T_i)$, where $g \in N(Sp(2n),T_i)$. Since $A \subset$

A_{T_i} we get that $A \stackrel{int(g)}{\longrightarrow} A_{T_i} \stackrel{int(t_i)}{\longrightarrow} A_T$. Define now a map

$$\Phi : \bigcup_{T_i} W(Sp(2n), T_i)/W(M, T_i) \longrightarrow W(A, T),$$

$$\Phi(\overline{g}) = int(t_i g).$$

It is well defined since M (and hence T_i) act trivially on A. Note finally that $\Psi(\Phi(\overline{g})) = \Psi(int(t_i g)) = [gm_{t_i g}] = \overline{g}$ and $\Phi(\Psi(intg)) = \Phi([t_i^{-1} gm_g]) = int(gm_g) = int(g)$. □

Consider now a parabolic subgroup $P = MN \subset Sp(2n)$, $M = Cent_{Sp(2n)} A$. Let T be a Cartan subgroup in M.

Proposition 6.3.3 *Any Cartan subgroup in $Sp(2n)$ can be decomposed into a direct product $(F^*)^a \times K_1^* \times \cdots \times K_l^* \times E_1^1 \times \cdots \times E_k^1$, where each K_i is some nontrivial field extension of F and each E_j^1 is the group of norm units of E_j over L_j for some tower of field extensions $E_j \stackrel{2}{-} L_j - F$.*

Proof. See [Ho], Lemma p. 296. □

Write

$$T \cong T_F \times T_K \times T_N \subset Sp(2a) \times Sp(2b) \times Sp(2c), \tag{6.3}$$

where $a + b + c = n$, and T_F is the split torus, T_K is a product of nontrivial field extensions, and T_N is a product of norm one tori.

Now we want to show that there exists a a bijection between the quotients $W(Sp(2n), H)/W(M, H)$ and $W(SO(2n+1)_+, H')/W(M', H')$ for any pair of isomorphic Cartan subgroups $H \subset M \subset Sp(2n)$ and $H' \subset M' \subset SO(2n+1)_+$. First we will describe the group $W(Sp(2b), T_K)$. Note that we do not need to calculate the Weyl group $W(Sp(2c), T_N)$, since it will not appear in the quotient $W(Sp(2n), T)/W(M, T)$ (see Lemma 6.3.10).

Assume first that T_K is a Cartan subgroup in $Sp(2b)$ coming from a single field extension, i.e. $T_K = K^*$ for some field extension $K \overset{b}{-} F$ of degree b. We choose a symplectic basis and we consider the following embedding of K^* into $Sp(2b)$:

$$\iota(z) = \begin{pmatrix} M_z & 0 \\ 0 & {}^tM_{z^{-1}} \end{pmatrix},$$

where $M_z \in GL(b)$ denotes the matrix of multiplication by $z \in K^*$. Let $Z = Z_K \in GL(b)$ be such that $ZM_zZ^{-1} = {}^tM_z$, for all $z \in K^*$. Such Z exists, since the Cartan subgroups $\{M_z, z \in K^*\} \subset GL(b)$ and $\{{}^tM_z, z \in K^*\} \subset GL(b)$ are stably conjugate in $GL(b)$, and therefore they are conjugate in $GL(b)$ (see Proposition 3.2.4). Let

$$\delta_K = \begin{pmatrix} 0 & {}^tZ^{-1} \\ -Z & 0 \end{pmatrix} \in Sp(2b).$$

The element δ_K acts on the Cartan subgroup $K^* \subset Sp(2b)$ as follows

$$\delta_K \begin{pmatrix} M_z & 0 \\ 0 & {}^tM_{z^{-1}} \end{pmatrix} \delta_K^{-1} = \begin{pmatrix} M_{z^{-1}} & 0 \\ 0 & {}^tM_z \end{pmatrix}.$$

Lemma 6.3.4

$$W(Sp(2b), K^*) \cong W(GL(b), K^*) \times \mathbb{Z}_2.$$

Proof. Let $\mathbf{T}_K$ be a Cartan subgroup defined over the algebraic closure of F, such that its $F-$points $\mathbf{T}_K(F) = K^*$. Then $W(Sp(2b), K^*)$ is a subgroup of the Weyl group of $\mathbf{T}_K$ defined over $\overline{F}$. The latter is of type C_b and consists of permutations and "sign changes". From these the only operations that preserve $\mathbf{T}_K(F) \subset Sp(2b)$

are those coming from the action of $GL(b)$ on K^* and the action of the element δ_K, i.e. the simultaneous sign change of all the eigenvalues. Finally, note that δ_K commutes with the subgroup $W(GL(b), K^*)$. $\square$

We generalize the above lemma. We consider

$$T_K \cong K_1^* \times \ldots \times K_l^*,$$

where each K_i is a nontrivial field extensions of F.

Lemma 6.3.5

$$W(Sp(2b), T_K) \cong W(GL(b), T_K) \ltimes (\mathbb{Z}_2)^l.$$

Proof. The subgroup $(\mathbb{Z}_2)^l$ is generated by the elements $\{\delta_{K_i}, i = 1, \ldots, l\}$. Each δ_{K_i} acts on K_i^* as described earlier, i.e. it replaces $\iota(z)$ with $\iota(z^{-1})$ for $z \in K_i^*$ and leaves all the other K_j^* fixed. The subgroup $(\mathbb{Z}_2)^l$ is normal in $W(Sp(2b), T_K)$. The rest of the proof is analogous to the proof of the previous lemma. $\square$

Lemma 6.3.6

$$N_{Sp(2n)}(T) \cong N_{Sp(2a)}(T_F) \times N_{Sp(2b)}(T_K) \times N_{Sp(2c)}(T_N).$$

Proof. If $n \in N_{Sp(2n)}(T)$ then n has to also normalize every individual component of T, i.e T_F, T_K and T_N. That is because the eigenvalues of elements of each of these components are of different nature: every element of T_F is diagonalizable over F, while the eigenvalues of an element in T_K belong to some nontrivial field extension $K_i - F$. Furthermore, the eigenvalues of an element in T_N are norm one elements in some nontrivial field extension $E_i - L_i - F$.

We claim that $n \in Sp(2a) \times Sp(2b) \times Sp(2c)$. Indeed, there exist elements $n_F \in N_{Sp(2a)}(T_F)$ and $n_K \in N_{Sp(2b)}(T_K)$ whose actions on T_F and T_K coincide with the action of n. That is because n normalizes T_F and T_K and the fact that all operations allowed on T_F and T_K are realized by $W(Sp(2a), T_F)$ and $W(Sp(2b), T_K)$ (see the proof of Lemma 6.3.5). Therefore the element $nn_F^{-1}n_K^{-1}$ fixes $T_F \times T_K$, hence it belongs to the centralizer of $T_F \times T_K$; in particular it is contained in $Sp(2a) \times Sp(2b) \times Sp(2c)$. $\square$

Lemma 6.3.7

$$W(Sp(2n), T) \cong W(Sp(2a), T_F) \times W(Sp(2b), T_K) \times W(Sp(2c), T_N).$$

Now we will study the Weyl group of T in M, where

$$M \cong Sp(2m) \times GL(n_1) \times \ldots \times GL(n_k)$$

for some $m + n_1 + \cdots + n_k = n$. Let us recall that

$$T \cong T_F \times T_K \times T_N \subset Sp(2a) \times Sp(2b) \times Sp(2c).$$

Accordingly we write

$$a = a_0 + a_1 + \cdots + a_k,$$

$$b = b_0 + b_1 + \cdots + b_k,$$

$$c = c_0 + 0 + \cdots + 0.$$

We decompose the Cartan subgroup T as follows:

$$T \cong (F^{*a_0} \times T_{K_0} \times T_N) \times (F^{*a_1} \times T_{K_1}) \times \cdots \times (F^{*a_k} \times T_{K_k}),$$

where $F^{*a_0} \times T_{K_0} \times T_N$ is a Cartan subgroup in $Sp(2m)$, $F^{*a_i} \times T_{K_i}$ is a Cartan subgroup in $GL(n_i)$ for $i = 1, \ldots, k$, and each Cartan subgroup $T_{K_i} \subset GL(b_i)$ for $i = 0, \ldots, k$ is a product of nontrivial field extensions.

Lemma 6.3.8

$$W(M,T) \cong W(Sp(2a_0), F^{*a_0}) \times W(Sp(2b_0), T_{K_0}) \times W(Sp(2c), T_N)$$
$$\times \prod_{i=1}^{k} (W(GL(a_i), F^{*a_i}) \times W(GL(b_i), T_{K_i}).$$

Proof. The statement about the factor of T that is contained in $Sp(2m)$ follows from Lemma 6.3.7; the proof about factors contained in $GL(n_i)$'s is analogous to the proof of Lemma 6.3.6. □

Lemma 6.3.9 *The quotient* $W(Sp(2n), T)/W(M, T)$ *is isomorphic to*

$$\frac{W(C_a)}{W(C_{a_0}) \times \prod_{i=1}^{k} S_{a_i}} \times \frac{W(Sp(2b), T_K)}{W(Sp(2b_0), T_{K_0}) \times \prod_{i=1}^{k} W(GL(b_i), T_{K_i})}.$$

We simplify the above formula further. Assume that T_{K_0} is a product of l_0 nontrivial field extensions. By Lemma 6.3.5 we have that $W(Sp(2b_0), T_{K_0}) \cong W(GL(b_0), T_K) \ltimes (\mathbb{Z}_2)^{l_0}$ and $W(Sp(2b), T_K) \cong W(GL(b), T_K) \ltimes (\mathbb{Z}_2)^l$. Hence we have the following lemma:

Lemma 6.3.10

$$\frac{W(Sp(2n), T)}{W(M, T)} \cong \frac{W(C_a)}{W(C_{a_0}) \times \prod_{i=1}^{k} S_{a_i}} \times \frac{W(GL(b), T_K) \ltimes (\mathbb{Z}_2)^l}{(\prod_{i=0}^{k} W(GL(b_i), T_{K_i})) \ltimes (\mathbb{Z}_2)^{l_0}}.$$

Consider now $A' = \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \subset SO(2n+1)_+$. Let M' be the centralizer of A' in $SO(2n+1)_+$. Let $T'_N \in SO(2m+1)_+$ be a torus that is isomorphic to the torus $T_N \in Sp(2m)$ described earlier in the section. Let

$$T' \cong T_F \times T_K \times T'_N \subset SO(2a) \times SO(2b) \times SO(2c+1). \qquad (6.4)$$

As before, we calculate the Weyl group of T' in $SO(2n+1)_+$ and in M'.

Lemma 6.3.11

$$W(SO(2n+1)_+, T') \cong W(B_a) \times (W(GL(b), T_K) \ltimes (\mathbb{Z}_2)^l) \times W(SO(2c+1)_+, T_N).$$

Proof. Assume that $SO(2n+1)_+$ preserves the bilinear form $\begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Consider first $T_F \times I \times I$. Note that we have an action of the Weyl group of type D_a on the torus T_F. Let e_1 be an element of $O(2a)$ that acts on $diag(x_1, \ldots, x_a, 1/x_1, \ldots, 1/x_a)$ by interchanging x_1 and x_1^{-1}. Let $\bar{e}_1 = e_1 \times -I \times -I \in SO(2n+1)_+$. Now, $\det(\bar{e}_1) = 1$, $\bar{e}_1$ commutes with T_K and T_N and together with $W(D_a)$ it generates $W(B_a)$.

The proof of the statement concerning the torus $T_K \cong K_1^* \times \ldots \times K_l^*$ is similar to the proof of Lemma 6.3.5. The only difference is that we replace the elements δ_{K_i} with δ'_{K_i}, where

$$\delta'_{K_i} = \begin{pmatrix} 0 & {}^tZ_i^{-1} & 0 \\ Z_i & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and the matrices Z_i are the same ones that were used to construct $\delta'_{K_i}s$. □

In order to calculate the Weyl group of T' in M' we need to refine the decomposition of T'. We have

$$M' \cong SO(2m+1)_+ \times GL(n_1) \times \ldots \times GL(n_k),$$

$$T' \cong (F^{*a_0} \times T_{K_0} \times T'_N) \times (F^{*a_1} \times T_{K_1}) \times \cdots \times (F^{*a_k} \times T_{K_k}),$$

where F^{*a_i} and T_{K_i} are as before, i.e. $F^{*a_0} \times T_{K_0} \times T'_N$ is a Cartan subgroup in $SO(2m+1)$, $F^{*a_i} \times T_{K_i}$ is a Cartan subgroup in $GL(n_i)$ for $i = 1, \ldots, k$, and each Cartan subgroup $T_{K_i} \subset GL(b_i)$ for $i = 0, \ldots, k$ is a product of nontrivial field extensions.

Lemma 6.3.12 *$W(M', T')$ is isomorphic to*

$$W(SO(2a_0+1), F^{*a_0}) \times (W(GL(b_0), T_{K_0}) \ltimes (\mathbb{Z}_2)^{l_0}) \times W(SO(2c+1), T'_N)$$
$$\times \prod_{i=1}^{k} (W(GL(a_i), F^{*a_i}) \times W(GL(b_i), T_{K_i}).$$

Proof. The proof is analogous to the proof of the previous proposition. □

Lemma 6.3.13

$$\frac{W(SO(2n+1)_+, T')}{W(M', T')} \cong \frac{W(B_a)}{W(B_{a_0}) \times \prod_{i=1}^{k} S_{a_i}} \times \frac{W(GL(b), T_K) \ltimes (\mathbb{Z}_2)^l}{(\prod_{i=0}^{k} W(GL(b_i), T_{K_i})) \ltimes (\mathbb{Z}_2)^{l_0}}.$$

Lemma 6.3.14

$$W(Sp(2n), T)/W(M, T) \cong W(SO(2n+1)_+, T')/W(M', T').$$

To show the final result, we will need two more lemmas.

Lemma 6.3.15 *Suppose that $T \cong T_1 \times T_N$ is a Cartan subgroup in $GL(a+b) \times Sp(2c) \subset Sp(2n)$, where T_1 is a Cartan subgroup in $GL(a+b)$ and T_N is a product of norm one tori of various field extensions $E \overset{2}{-} L - F$ (see Lemma 6.3.3). Suppose also that T is conjugate in $Sp(2n)$ to a Cartan subgroup $H \subset GL(a+b) \times Sp(2c)$. Then:*

(1) $H \cong H_1 \times H_N$ where H_1 is conjugate in $GL(a+b)$ to T_1 and H_N is stably conjugate to T_N in $Sp(2c)$,

(2) If $x \in Sp(2n)$ is such that $xTx^{-1} = H$, then $x \in Sp(2(a+b)) \times Sp(2c)$,

(3) H is conjugate to $H_1 \times T_N$ in $I \times Sp(2c) \subset M$.

Proof. (1) follows from the fact that conjugate elements of T_N and H_N (and hence of T_1 and H_1) have the same eigenvalues.

To show (2) consider a $2n-$dimensional vector space V with a nondegenerate symplectic form $\langle \, , \rangle$ that is preserved by $Sp(2n)$. Let $g = (d_1, e_1) \in T_1 \times T_N$ be a regular element. Decompose V into a direct product of subspaces W_1 and W_2, where $d_1 \in Sp(W_1)$ and $e_1 \in Sp(W_2)$. By (1) $x(d_1, 1)x^{-1} = (d_2, 1)$. Let $w_2 \in W_2$. Since $(d_1, 1)w_2 = w_2$ we have that $(x(d_1, 1)x^{-1})xw_2 = xw_2$. Since d_2 is regular (i.e. has no trivial eigenvalues) and since $(d_2, 1)$ fixes W_2 we get that $xw_2 \in W_2$. Note also that x takes W_1 into W_1, since $0 = \langle W_1, W_2 \rangle = \langle xW_1, xW_2 \rangle = \langle xW_1, W_2 \rangle$ and preserves $\langle \, , \rangle|_{W_i}$, hence the assertion follows.

To show (3) write $x = (x_1, x_2) \in Sp(2(a+b)) \times Sp(2c)$. Then $(1, x_2^{-1})H(1, x_2) = H_1 \times T_N$, and $(1, x_2^{-1}) \in I \times Sp(2c)$. □

Below is the "$SO(2n+1)$" version:

Lemma 6.3.16 *Suppose that $T' \cong T_1 \times T'_N$ is a Cartan subgroup in $GL(a+b) \times SO(2c+1)_+ \subset SO(2n+1)_+$, where T_1 is a Cartan subgroup in $GL(a+b)$ and T'_N is the Cartan subgroup in $SO(2c+1)_+$ coming from the groups of norm units of various field extensions $E \overset{2}{-} L - F$. Suppose also that T is conjugate in $SO(2n+1)_+$ to a Cartan subgroup $H \subset GL(a+b) \times SO(2c+1)_+$. Then:*

(1) $H \cong H_1 \times H'_N$ *where* H_1 *is conjugate in* $GL(a+b)$ *to* T_1 *and* H'_N *is stably conjugate to* T'_N *in* $SO(2c+1)_+$*,*

(2) If $x \in SO(2n+1)$ *is such that* $xTx^{-1} = H$*, then* $x \in O(2(a+b)) \times O(2c+1)$*,*

(3) H *is conjugate to* $H_1 \times T'_N$ *in* $I \times SO(2c+1)_+ \subset M'$*.*

Proof. The proof of (1) and (2) is similar to the proof of the previous lemma. The only significant difference might happen in (3). Namely, if $x = (x_1, x_2) \in O(2(a+b)) \times O(2c+1)$, and $detx_2 = -1$ then we replace $(1, x_2)$ with $(1, -x_2) \in SO(2c+1)_+$, and the rest follows. □

Lemma 6.3.17 $W(A, T) \cong W(A', T')$.

Proof. Recall that $T \cong T_1 \times T_N$ and $T' \cong T_1 \times T'_N$, where $T_1 \cong T_F \times T_K$ is a Cartan subgroup in $GL(a+b)$ and $T_N \subset Sp(2c)$ and $T'_N \subset SO(2c+1)_+$ are isomorphic Cartan subgroups. By Lemma 6.3.2 and Lemma 6.3.14 it is enough to show that we can choose the representatives of the indexing sets in the formula of Lemma 6.3.2 to be isomorphic Cartan subgroups. Let $T_1, \ldots, T_k$ be the representatives of the set $\{H \subset M : H \sim_G T\}/\sim_M$ in $Sp(2n)$. By Lemma 6.3.15 we can decompose each T_i into a product $T_{i_1} \times T_{i_N}$, where T_{i_1} is a Cartan subgroup in $GL(a+b)$ that is conjugate to T_1 in $GL(a+b)$ and T_{i_N} is a Cartan subgroup in $Sp(2c)$, that is stably conjugate in $Sp(2c)$ to T_N. Also by Lemma 6.3.15, each T_i is conjugate in M to $T_{i_1} \times T_N$. Without loss of generality we will assume then that $T_{i_N} = T_N$ for all i. Therefore $T_{1_1} \times T_N, \ldots, T_{k_1} \times T_N$ is a complete list of representatives of the indexing set on the "$Sp(2n)$ side".

We form a list $T'_1 \dots, T'_k$ on the "$SO(2n+1)$ side" as follows. Define $T'_i \cong T_{i_1} \times T'_N \subset SO(2n+1)_+$. We need to show the following: (1) $T'_i \sim_{SO(2n+1)_+} T'$, (2) $T'_i \nsim_{M'} T'_j$, (3) the list is complete.

Assertion (1) follows from the fact that T_{i_1} and T_1 are conjugate in $GL(a+b)$, hence also $T_{i_1} \times T'_N$ and $T' \cong T_1 \times T'_N$ are conjugate in $SO(2n+1)_+$. Part (2) is true because $T_i \nsim_M T_i$ for $i \neq j$..

Finally, note that $T'_1, \dots, T'_k$ exhaust the list of the representatives of the quotient $\{H' \subset M' : H' \sim_{SO(2n+1)_+} T'\} / \sim_{M'}$ in $SO(2n+1)_+$. If $H' \sim_{SO(2n+1)_+} T' \cong T'_1 \times T'_N$, then $H' \cong H_1 \times H''_N$, where H_1 is conjugate to T'_1 in $GL(a+b)$ and H''_N is stably conjugate in $SO(2c+1)_+$ to T'_N. Without loss of generality we can therefore assume that $H_1 = T_{j_1}$, for some j. Hence $H' \cong T_{j_1} \times H''_N \sim_{SO(2n+1)_+} T_{j_i} \times T'_N$. By part (3) of Lemma 6.3.16 we get that $H' \sim_{M'} T_{j_1} \times T'_N$. That completes the proof. $\square$

Chapter 7

Stability

In this chapter we again study quotients of the Weyl groups that appear in van Dijk's formula and we show that we can replace them with their stable versions. We define stable conjugacy for the metaplectic group $\widetilde{Sp}(2n)$. We also define a matching correspondence between the representations of $\widetilde{Sp}(2n)$ and $SO(2n+1)_+$ as follows: two representations correspond if their characters are identified by the stable orbit correspondence, up to a transfer factor. Further we show that corresponding representations are necessarily stable and that the representation obtained from inducing a stable representation is also stable.

7.1 Stabilized Weyl groups

The bijection between the sets $W(A,T)$ and $W(A',T')$ obtained in the previous chapter does not have sufficiently nice properties. The reason is, that it is just a bijection between sets, it does not come from a Weyl group isomorphism (in general the groups $W(Sp(2n),T)$ and $W(SO(2n+1)_+,T')$ are not isomorphic, see the example below). In order to obtain the required properties, we need to consider the stable Weyl group, i.e. $W_{st}(G,T) = \{w \in W(\mathbf{G}\ ,\mathbf{T}) \mid w \text{ acts on } \mathbf{T}(F)\}$. The same technique was also used by Adams in [A1].

Example 7.1.1 *Weyl group of the elliptic Cartan subgroup in $SL(2)$ and $SO(3)_+$,* *for p odd.*

Let Δ be a non-square in F^*. We consider the elliptic Cartan subgroup of norm one elements in the field extension $F(\sqrt{\Delta})/F$:

$$a + b\Delta \mapsto \begin{pmatrix} a & b\Delta & 0 \\ b & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \in SO(2n+1)_+.$$

The Weyl group of this Cartan subgroup in $SO(2n+1)_+$ is $\mathbb{Z}_2$; the nontrivial conjugation that takes $a+b\Delta$ to $a-b\Delta$ is realized by a diagonal matrix $diag(1,-1,-1) \in SO(2n+1)_+$. In this case $W_{st}(SO(3)_+, T'_N) = W(SO(3)_+, T'_N)$.

However, the Weyl group of the same Cartan subgroup in $SL(2)$, i.e.

$$a + b\Delta \mapsto \begin{pmatrix} a & b\Delta \\ b & a \end{pmatrix},$$

depends on the sign of $(\Delta, -1)$. (Note that the matrix $diag(1, -1)$ does *not* belong to $SL(2)$). For example, if -1 is a square in F, then then the conjugation operation can be realized by $diag(\sqrt{-1}, -\sqrt{-1})$. More generally, it can be shown by a direct computation that the most general form of the matrix that conjugates $a + b\Delta$ to $a - b\Delta$ is

$$\begin{pmatrix} x & -y\Delta \\ y & -x \end{pmatrix}, \quad x, y \in F, \quad x^2 - \Delta y^2 \neq 0.$$

Its determinant is equal to $-x^2 + \Delta y$, hence this matrix can be realized in $SL(2)$ if and only if we can find x and y such that $x^2 - \Delta y = -1$, i.e. if and only if -1 is a norm, i.e. if and only if $(\Delta, -1) = 1$. Therefore $W(SL(2), T_N) \cong \mathbb{Z}_2$, if $(\Delta, -1) = 1$, and $W(SL(2), T_N) \cong \{1\}$ otherwise.

Note now, that the element $diag(\sqrt{-1}, -\sqrt{-1}) \in W(\mathbf{SL(2)}, \mathbf{T}_N)$ acts on $T_N = \mathbf{T}_N(F)$. Therefore the stabilized Weyl group $W_{st}(SL(2), T_N) \cong \mathbb{Z}_2$ and it is isomorphic to $W_{st}(SO(2n+1)_+, T'_N)$.

Let $T = \mathbf{T}(F) \subset Sp(2n)$ and $T' = \mathbf{T}'(F) \subset SO(2n+1)_+$ be isomorphic Cartan subgroups. Let's recall that in Chapter 3 we constructed a commutative diagram

$$\begin{array}{ccc} \mathbf{T}_s = g^{-1}\mathbf{T}g & \xrightarrow{\psi_s} & \mathbf{T}'_s = h^{-1}\mathbf{T}'h \\ {\scriptstyle int(g^{-1})}\uparrow & & \downarrow{\scriptstyle int(h)} \\ \mathbf{T} & \xrightarrow{\psi} & \mathbf{T}' \end{array}$$

and an isomorphism between the Weyl groups

$$\phi_s : W(\mathbf{Sp(2n)}, \mathbf{T}_s) \longrightarrow W(\mathbf{SO(2n+1)}, \mathbf{T}'_s),$$

with the following properties:

- $\psi(\mathbf{T}(F)) = \mathbf{T}'(F)$,
- $x \overset{stably}{\longleftrightarrow} \psi(x)$ for all strongly regular semisimple elements $x \in T$,
- $\psi_s(w \cdot t) = \phi_s(w) \cdot \psi_s(t),$ for all $t \in \mathbf{T}_s \quad w \in W(\mathbf{Sp(2n)}, \mathbf{T}_s)$.

Denote by ϕ the isomorphism between the Weyl groups $W(\mathbf{Sp(2n)}, \mathbf{T})$ and $W(\mathbf{SO(2n+1)}, \mathbf{T}'_s)$ induced by the commutative diagram:

$$\begin{array}{ccc} W(\mathbf{Sp(2n)}, \mathbf{T}_s) & \xrightarrow{\phi_s} & W(\mathbf{SO(2n+1)}, \mathbf{T}'_s) \\ {\scriptstyle int(g^{-1})}\uparrow & & \downarrow{\scriptstyle int(h)} \\ W(\mathbf{Sp(2n)}, \mathbf{T}) & \xrightarrow{\phi} & W(\mathbf{SO(2n+1)}, \mathbf{T}'). \end{array}$$

Lemma 7.1.2 *The isomorphism* $\phi : W(\boldsymbol{Sp(2n)}, \boldsymbol{T}) \longrightarrow W(\boldsymbol{SO(2n{+}1)}, \boldsymbol{T'})$ *satisfies the property* $\psi(w \cdot t) = \phi(w) \cdot \psi(t)$, *for all* $t \in \boldsymbol{T}$ *and all* $w \in W(\boldsymbol{Sp(2n)}, \boldsymbol{T})$.

Proof. This follows from the fact that $\psi_s(w \cdot t) = \phi_s(w) \cdot \psi_s(t)$, for all $t \in \mathbf{T}_s$ and all $w \in W(\mathbf{Sp(2n)}, \mathbf{T}_s)$ and from the form of vertical isomorphisms (intertwining action of the same elements g^{-1} and h) in the diagrams above. □

Recall that $\overline{F}$ denotes the algebraic closure of F and $\Gamma = \Gamma(\overline{F}/F)$ is the Galois group $\overline{F}$ over F. Let $\mathbf{G} = \mathbf{Sp(2n)}$ or $\mathbf{SO(2n{+}1)}$ and let $\mathbf{H} \subset \mathbf{G}$ denote a Cartan subgroup that is defined over F. We denote by $W(\mathbf{G}, \mathbf{H})^{\Gamma}$ the subgroup of $W(\mathbf{G}, \mathbf{H})$ that is fixed by the action of Γ.

Lemma 7.1.3 $W(\boldsymbol{G}, \boldsymbol{H})^{\Gamma} = \{w \in W(\boldsymbol{G}\ , \boldsymbol{H}) \mid w \textit{ acts on } \boldsymbol{H}(F)\}$

Proof. First note that $W(\mathbf{G}, \mathbf{H})^{\Gamma}$ acts on $\mathbf{H}(F)$. Indeed, if $w \in W(\mathbf{G}, \mathbf{H})^{\Gamma}$ and $t \in \mathbf{H}(F)$ then $\sigma(w \cdot t) = \sigma(w) \cdot \sigma(t) = w \cdot t$, for all $\sigma \in \Gamma$. On the other hand, assume that $w \in W(\mathbf{G}, \mathbf{H})$ acts on $\mathbf{H}(F)$. Let $t \in \mathbf{H}(F)$ be a regular element. We have that $\sigma(w \cdot t) = w \cdot t$, i.e. $w^{-1}\sigma(w)$ commutes with t. Therefore $w^{-1}\sigma(w) \in \mathbf{H}$, hence $\sigma(w) = w$. □

Definition 7.1.4 *Let* $G = Sp(2n)$ *or* $SO(2n + 1)_+$. *Let* $H = \boldsymbol{H}(F)$ *be a Cartan subgroup in* G. *We define the stable Weyl group* $W_{st}(G, H)$ *to be* $W(\boldsymbol{G}, \boldsymbol{H})^{\Gamma}$.

Lemma 7.1.5 $\phi(W_{st}(Sp(2n), T)) = W_{st}(SO(2n + 1)_+, T')$.

Proof. Let $w \in W_{st}(Sp(2n), T)$ and $t' \in T'$. Note that $\psi(T) = T'$, hence $t' = \psi(t)$, for some $t \in T$. Since $w \cdot t \in T$ we get that $\phi(w) \cdot t' = \phi(w) \cdot \psi(t) = \psi(w \cdot t) \in T'$. By the above lemma, $\phi(w) \in W_{st}(SO(2n + 1)_+, T')$. □

Let us recall the decompositions of tori T and T' :

$$T \cong T_F \times T_K \times T_N \subset Sp(2a) \times Sp(2b) \times Sp(2c) \quad \text{(see (6.3))},$$

$$T' \cong T_F \times T_K \times T'_N \subset SO(2a) \times SO(2b) \times SO(2c+1)_+ \quad \text{(see (6.4))}.$$

In Lemmas 6.3.7, 6.3.5 and 6.3.11 we calculated the Weyl groups of these tori:

$$W(Sp(2n), T) \cong W(C_a) \times (W(GL(b), T_K) \ltimes \mathbb{Z}_2^l) \times W(Sp(2c), T_N),$$

$$W(SO(2n+1)_+, T') \cong W(B_a) \times (W(GL(b), T_K) \ltimes \mathbb{Z}_2^l) \times W(SO(2c+1)_+, T'_N).$$

Note that $W_{st}(SO(2a+1)_+, T_F) = W(SO(2a+1)_+, T_F)$ and $W_{st}(Sp(2a), T_F) = W(Sp(2a), T_F)$. Also it follows from the proof of Lemmas 6.3.5 and 6.3.11 that $W_{st}(Sp(2b), T_K) = W(Sp(2b), T_K) = W(GL(b), T_K) \ltimes \mathbb{Z}_2^l$ and $W_{st}(SO(2b+1)_+, T_K) = W(GL(b), T_K) \ltimes \mathbb{Z}_2^l$. Therefore

$$W_{st}(Sp(2n), T) \cong W(C_a) \times (W(GL(b), T_K) \ltimes \mathbb{Z}_2^l) \times W_{st}(Sp(2c), T_N),$$

$$W_{st}(SO(2n+1)_+, T') \cong W(B_a) \times (W(GL(b), T_K) \ltimes \mathbb{Z}_2^l) \times W_{st}(SO(2c+1)_+, T'_N).$$

Analogous statements hold for the stable Weyl groups $W_{st}(M, T)$ and $W_{st}(M', T')$ (compare Lemma 6.3.8 and Lemma 6.3.12 for the description of $W(M, T)$ and $W(M', T')$):

$$W_{st}(M, T) \cong W(Sp(2a_0), F^{*a_0}) \times (W(GL(b_0), T_{K_0}) \ltimes (\mathbb{Z}_2)^{l_0})$$
$$\times \prod_{i=1}^{k} S_{a_i} \times \prod_{i=1}^{k} W(GL(b_i), T_{K_i}) \times W_{st}(Sp(2c), T_N),$$

$$W_{st}(M', T') \cong W(SO(2a_0+1), F^{*a_0}) \times (W(GL(b_0), T_{K_0}) \ltimes (\mathbb{Z}_2)^{l_0})$$

$$\times \prod_{i=1}^{k} S_{a_i} \times \prod_{i=1}^{k} W(GL(b_i), T_{K_i}) \times W_{st}(SO(2c+1)_+, T'_N).$$

Lemma 7.1.6 *(1)* $W_{st}(Sp(2n), T) \cong W_{st}(SO(2n+1)_+, T')$,

(2) $W_{st}(M, T) \cong W_{st}(M', T')$,

(3) $\frac{W_{st}(Sp(2n),T)}{W_{st}(M,T)} \cong \frac{W(Sp(2n),T)}{W(M,T)} \cong \frac{W(SO(2n+1)_+,T')}{W(M',T')} \cong \frac{W_{st}(SO(2n+1)_+,T')}{W_{st}(M',T')}$.

Proof. The statements follow from Lemma 7.1.5 applied to $W_{st}(Sp(2c), T_N)$ and $W_{st}(SO(2n+1)_+, T'_N)$ and from the preceding remarks. □

We summarize this section in the following lemma:

Lemma 7.1.7 *Let $T \subset Sp(2n)$ and $T' \subset SO(2n+1)_+$ be isomorphic Cartan subgroups. Then there exist an isomorphism $\psi : T \to T'$ and an isomorphism $\phi : W_{st}(Sp(2n), T) \to W_{st}(SO(2n+1)_+, T')$ such that*

(1) $\psi(t) \overset{stable}{\longleftrightarrow} t$, *for all strongly regular semisimple elements* $t \in T$,

(2) $\phi : W_{st}(Sp(2n), T_N) \to W_{st}(SO(2c+1)_+, T'_N)$,

(3) $\psi(w \cdot g) = \phi(w) \cdot \psi(g), \quad w \in W_{st}(Sp(2n), T),\ g \in T.$

7.2 Stability in $\widetilde{Sp}(2n)$

We start with the following lemma which is crucial in defining stability for the metaplectic group.

Lemma 7.2.1 *Let $\widetilde{g}, \widetilde{h} \in \widetilde{Sp}(2n)$ be such that $p(\widetilde{g})$ is stably conjugate to $p(\widetilde{h})$ in $Sp(2n)$. Then $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \pm\Theta_{\omega_+ - \omega_-}(\widetilde{h})$.*

Proof. Let $\widetilde{g}$ and $\widetilde{h}$ be as in the statement of the lemma. The character of the oscillator representation is given by

$$\Theta_{\omega_+ + \omega_-}(\widetilde{g}) = \pm \frac{\gamma(1)^{2n-1}\gamma(\det(g-1))}{|\det(g-1)|^{1/2}}$$

(see [T], Theorem 1C). Therefore the lemma holds with $\Theta_{\omega_+ + \omega_-}$ in place of $\Theta_{\omega_+ - \omega_+}$, i.e.

$$\Theta_{\omega_+ + \omega_-}(\widetilde{g}) = \pm \Theta_{\omega_+ + \omega_-}(\widetilde{h}).$$

Let $x \in \widetilde{Sp}(2n)$ be such that $p(x) = -I \in Sp(2n)$. In Corollary 4.2.3 we showed that $\Theta_{\omega_+ + \omega_-}(x\widetilde{g}) = \lambda\ \Theta_{\omega_+ - \omega_+}(\widetilde{g})$, where λ is a constant independent of $\widetilde{g}$. Now we have that

$$\Theta_{\omega_+ - \omega_+}(\widetilde{g}) = \lambda^{-1}\ \Theta_{\omega_+ + \omega_-}(x\widetilde{g}) = \pm\lambda^{-1}\ \Theta_{\omega_+ + \omega_-}(x\widetilde{h}) = \pm\Theta_{\omega_+ - \omega_+}(\widetilde{h}). \quad \square$$

Definition 7.2.2 *Let $\widetilde{g}, \widetilde{h} \in \widetilde{Sp}(2n)$. We say that $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ if $p(\widetilde{g})$ is stably conjugate to $p(\widetilde{h})$ in $Sp(2n)$ and $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{h})$.*

This definition is nonstandard and one could for example use the character of the *sum* of the two halves of the oscillator representation instead of the difference. However, this definition matches our choice of the transfer factor and hence makes the results hold. It also agrees with the definition in the real case (see Adams [A1] work for details).

Now we will define stably conjugacy in the Levi factor $\widetilde{M} = p^{-1}(M)$. Let $M \cong Sp(2m) \times GL(n_1) \times \ldots \times GL(n_k)$. We consider the following cover of $\widetilde{M}$:

$$\begin{array}{c} \widetilde{\widetilde{M}} \cong \widetilde{Sp}(2m) \times \widetilde{GL}(n_1) \times \ldots \times \widetilde{GL}(n_k) \\ {\scriptstyle p'} \big\downarrow \\ \widetilde{M} \cong p^{-1}(Sp(2m) \times GL(n_1) \times \ldots \times GL(n_k)). \end{array}$$

Here the cocycle on each $\widetilde{GL}(n_i)$ is given by the Hilbert symbol of the appropriate determinants, i.e. $c(x,y) = (\det(x), \det(y))$.

Definition 7.2.3 *Let $\widetilde{g}, \widetilde{h} \in \widetilde{M}$. We say that $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ in $\widetilde{M}$ if there exist elements $(\widetilde{g}_0, \widetilde{g}_1, \dots, \widetilde{g}_k), (\widetilde{h}_0, \widetilde{h}_1, \dots, \widetilde{h}_k) \in \widetilde{Sp}(2m) \times \widetilde{GL}(n_1) \times \dots \times \widetilde{GL}(n_k)$ such that*

- $p'((\widetilde{g}_0, \widetilde{g}_1, \dots, \widetilde{g}_k)) = \widetilde{g}$,
- $p'((\widetilde{h}_0, \widetilde{h}_1, \dots, \widetilde{h}_k)) = \widetilde{h}$,
- *$\widetilde{g}_i$ is conjugate to $\widetilde{h}_i$ in $\widetilde{GL}(n_i)$ for $i = 1, \dots, k$,*
- *$\widetilde{g}_0$ is stably conjugate to $\widetilde{h}_0$ in $\widetilde{Sp}(2m)$.*

Lemma 7.2.4 *Let $\widetilde{g}, \widetilde{h} \in \widetilde{M}$. The following conditions are equivalent:*

(1) $\widetilde{g}$ is stably conjugate to $\widetilde{h}$ in $\widetilde{M}$,

(2) $p(\widetilde{g})$ is stably conjugate to $p(\widetilde{h})$ in M and $\Phi_{\widetilde{M}}(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{h})$,

(3) $p(\widetilde{g})$ is stably conjugate to $p(\widetilde{h})$ in M and $\Theta_{\omega_+ - \omega_-}(\widetilde{g}) = \Theta_{\omega_+ - \omega_-}(\widetilde{h})$.

Proof. (1) $\Longrightarrow$ (2) Clearly $p(\widetilde{g})$ is stably conjugate to $p(\widetilde{h})$ in M. Choose elements $(\widetilde{g}_0, \widetilde{g}_1, \dots, \widetilde{g}_k)$ and $(\widetilde{h}_0, \widetilde{h}_1, \dots, \widetilde{h}_k) \in \widetilde{\widetilde{M}}$ that satisfy the conditions of the definition above. By the definition of stable conjugacy in $\widetilde{Sp}(2m)$ we have that $\Theta_{\omega_+ - \omega_-}(\widetilde{g}_0) = \Theta_{\omega_+ - \omega_-}(\widetilde{h}_0)$. Since each $\widetilde{g}_i \in \widetilde{GL}(n_i)$ is conjugate to $\widetilde{h}_i \in \widetilde{GL}(n_i)$ we also have that $\chi_i(\widetilde{g}_i) = \chi_i(\widetilde{h}_i)$, where χ_i is a character of $\widetilde{GL}(n_i)$ described in Lemma 5.0.11. Again by Lemma 5.0.11 we get that the values of the transfer factor on the elements $\widetilde{g}$ and

$\widetilde{h}$ are equal:

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \Theta_{\omega_+-\omega_-}(\widetilde{g}_0)\prod_{i=1}^{k}\chi_i(\widetilde{g}_i) = \Theta_{\omega_+-\omega_-}(\widetilde{h}_0)\prod_{i=1}^{k}\chi_i(\widetilde{h}_i) = \Phi_{\widetilde{M}}(\widetilde{h}).$$

(2) $\Longleftrightarrow$ (3) Recall the definition of the transfer factor on $\widetilde{M}$:

$$\Phi_{\widetilde{M}}(\widetilde{g}) = \frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_M(g)|^{\frac{1}{2}}}\frac{|D_{M'}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}}\Theta_{\omega_+-\omega_-}(\widetilde{g}),$$

where $g = p(\widetilde{g})$ and $g' \in M'$ is chosen in such a way that $g' \overset{L-stable}{\longleftrightarrow} g$. Since g and $h = p(\widetilde{h})$ are stably conjugate in M, we have $g' \overset{L-stable}{\longleftrightarrow} h$, $D_M(g) = D_M(h)$ and $D_{Sp(2n)}(g) = D_{Sp(2n)}(h)$. Therefore

$$\Phi_{\widetilde{M}}(\widetilde{h}) = \frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_M(g)|^{\frac{1}{2}}}\frac{|D_{M'}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}}\Theta_{\omega_+-\omega_-}(\widetilde{h}),$$

and the assertion follows.

(3) $\Longrightarrow$ (1) Let $p(\widetilde{g}) = (g_0, \ldots, g_k) \in M$ and $p(\widetilde{h}) = (h_0, \ldots, h_k) \in M$. For $i = 1, \ldots, k$ we fix ϵ_i and we let $\widetilde{g}_i = (g_i, \epsilon_i), \widetilde{h}_i = (h_i, \epsilon_i) \in \widetilde{GL}(n_i)$. Then $\widetilde{g}_i$ is conjugate to $\widetilde{h}_i$ in $\widetilde{GL}(n_i)$. Indeed, since g_i and h_i are conjugate in $GL(n_i)$, there exists $y_i \in GL(n_i)$ such that $y_i g_i y_i^{-1} = h_i$. We lift y_i to an element $\widetilde{y}_i = (y_i, \epsilon_i) \in \widetilde{GL}(n_i)$ and we get that

$$\widetilde{y}_i\widetilde{g}_i\widetilde{y}_i^{-1} = (y_i, \epsilon_i)(g_i, \epsilon_i)(y_i^{-1}, \epsilon_i) = (y_i g_i y_i^{-1}, \epsilon_i) = (h_i, \epsilon_i) = \widetilde{h}_i.$$

Now we choose lifts of the elements g_0 and $h_0 \in Sp(2m)$ to elements $\widetilde{g}_0 = (g_0, \epsilon)$ and $\widetilde{h}_0 = (h_0, \epsilon') \in \widetilde{Sp}(2m)$ in such a way that $\Theta_{\omega_+-\omega_-}(\widetilde{g}_0) = \Theta_{\omega_+-\omega_-}(\widetilde{h}_0)$ (it is possible by Lemma 7.2.1). By our construction we get that $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k))$ is stably conjugate to $p'((\widetilde{h}_0, \widetilde{h}_1, \ldots, \widetilde{h}_k))$ in $\widetilde{M}$. We still need to show that $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)) = \widetilde{g}$ and $p'((\widetilde{h}_0, \widetilde{h}_1, \ldots, \widetilde{h}_k)) = \widetilde{h}$. Note that $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)) = \widetilde{g}$ if and only if

$\Theta_{\omega_+-\omega_-}(p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k))) = \Theta_{\omega_+-\omega_-}(\widetilde{g})$ and $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)) \neq \widetilde{g}$ if and only if $\Theta_{\omega_+-\omega_-}(p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k))) = -\Theta_{\omega_+-\omega_-}(\widetilde{g})$. By our assumption $\Theta_{\omega_+-\omega_-}(\widetilde{g}) = \Theta_{\omega_+-\omega_-}(\widetilde{h})$. Therefore we have only two possibilities: either $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)) = \widetilde{g}$ and $p'((\widetilde{h}_0, \widetilde{h}_1, \ldots, \widetilde{h}_k)) = \widetilde{h}$ (in which case we are done) or $p'((\widetilde{g}_0, \widetilde{g}_1, \ldots, \widetilde{g}_k)) \neq \widetilde{g}$ and $p'((\widetilde{h}_0, \widetilde{h}_1, \ldots, \widetilde{h}_k)) \neq \widetilde{h}$. In the second case we simultaneously change the choice of ϵ to $-\epsilon$ and ϵ' to $-\epsilon'$. $\square$

Let $\widetilde{T}$ be a Cartan subgroup in $\widetilde{Sp}(2n)$. Let $T = p(\widetilde{T})$ and let $W_{st}(Sp(2n), T)$ be the stable Weyl group of T in $Sp(2n)$. We define the action of $w \in W_{st}(Sp(2n), T)$ on $\widetilde{t} \in \widetilde{T}$ as follows:

$$w \cdot \widetilde{t} = \widetilde{h}, \text{ where } p(\widetilde{t}) = w \cdot p(\widetilde{h}) \text{ and } \widetilde{t} \sim_{st} \widetilde{h},$$

or equivalently

$$w \cdot \widetilde{t} = \widetilde{h}, \text{ where } p(\widetilde{t}) = w \cdot p(\widetilde{h}) \text{ and } \Theta_{\omega_+-\omega_-}(\widetilde{t}) = \Theta_{\omega_+-\omega_-}(\widetilde{h}).$$

To check that this is a group action, we need to show that for any elements $w_1, w_2 \in W_{st}(Sp(2n), T)$ and any element $\widetilde{x} \in \widetilde{Sp}(2n)$ we have that

$$\Theta_{\omega_+-\omega_-}((w_1 w_2) \cdot \widetilde{x}) = \Theta_{\omega_+-\omega_-}(w_1 \cdot (w_2 \cdot \widetilde{x})).$$

But that is true, since by the definition of the action of the stable Weyl group and by the definition of stable conjugacy $\Theta_{\omega_+-\omega_-}(\widetilde{x}) = \Theta_{\omega_+-\omega_-}((w_1 w_2) \cdot \widetilde{x})$ and $\Theta_{\omega_+-\omega_-}(\widetilde{x}) = \Theta_{\omega_+-\omega_-}(w_2 \cdot \widetilde{x}) = \Theta_{\omega_+-\omega_-}(w_1 \cdot (w_2 \cdot \widetilde{x}))$.

We will be considering representations ρ' of M' and ρ of $\widetilde{M}$ that satisfy the following matching condition:

$$\theta_\rho(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{g}) \theta_{\rho'}(g'), \tag{7.1}$$

for all strongly regular semisimple elements $\widetilde{g} \in \widetilde{M}, g' \in M'$ such that $p(\widetilde{g}) \overset{L-stable}{\longleftrightarrow} g'$.

Lemma 7.2.5 *Let ρ be a representation on $\widetilde{M}$ and ρ' be a representation on M'. Assume that their characters satisfy (7.1). Then the characters θ_ρ and $\theta_{\rho'}$ are stable on $\widetilde{M}$ and M' respectively.*

Proof. Let $g', h' \in M'$ be strongly regular semisimple elements that are stably conjugate in M'. Choose any element $\widetilde{g} \in \widetilde{Sp}(2n)$ such that $p(\widetilde{g}) \overset{L-stable}{\longleftrightarrow} g'$. We also have that $p(\widetilde{g}) \overset{L-stable}{\longleftrightarrow} h'$, hence we can apply (7.1) twice to get

$$\Phi_{\widetilde{M}}(\widetilde{g})\theta_{\rho'}(g') = \theta_\rho(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{g})\theta_{\rho'}(h').$$

That implies $\theta_{\rho'}(g') = \theta_{\rho'}(h')$.

Let now $\widetilde{g}$ and $\widetilde{h}$ be stably conjugate in $\widetilde{M}$. By Lemma 7.2.4 that this implies that the values of the transfer factor at $\widetilde{g}$ and $\widetilde{h}$ are equal. Let $g' \in SO(2n+1)_+$ be any element such that $p(\widetilde{g}) \overset{L-stable}{\longleftrightarrow} g'$. We also have that $p(\widetilde{h}) \overset{L-stable}{\longleftrightarrow} g'$. Hence

$$\theta_\rho(\widetilde{g}) = \Phi_{\widetilde{M}}(\widetilde{g})\theta_{\rho'}(g') = \Phi_{\widetilde{M}}(\widetilde{h})\theta_{\rho'}(g') = \theta_\rho(\widetilde{h}). \quad \square$$

Remark. Let $G = Sp(2n)$ or $SO(2n+1)_+$. Recall van Dijk's formula (Proposition 6.1.1) for the character of an induced representation:

$$\theta_\pi(x) = \sum_{w \in W(G,T)/W(M,T)} \theta_\rho(w^{-1} \cdot x)\frac{|D_M(w^{-1} \cdot x)|^{\frac{1}{2}}}{|D_G(x)|^{\frac{1}{2}}}, \quad x \in T \cap G_{reg}.$$

In Lemma 7.1.6 we showed that the Weyl group quotients that appear in this formula are isomorphic to the quotients of the stable Weyl groups. Since the character θ_ρ that we are considering is stable (see Lemma 7.2.5), we have that

$$\theta_\rho(w^{-1} \cdot x) = \theta_\rho(w_{st}^{-1} \cdot x),$$

where $w \in \frac{W(G,T)}{W(M,T)}$ and $w_{st} \in \frac{W_{st}(G,T)}{W_{st}(M,T)}$ are matched by the isomorphism. Therefore we can replace those quotients in van Dijk's formula without further referring to this isomorphism.

The following fact is well known for the linear case. However, since the proofs for linear and non-linear case are similar, we present them both.

Proposition 7.2.6 *If the character of a representation ρ' is stable on M', then the character of the induced representation $\pi' = Ind_P^G \rho'$ is stable on $SO(2n+1)_+$. Analogous statement holds for $Sp(2n)$ and $\widetilde{Sp}(2n)$.*

Proof. Let x and x' be two strongly regular semisimple elements that are stably conjugate in $SO(2n+1)_+$. Assume that $x, x' \in M'$. First we will show, that there exists $g \in SO(2n+1)_+$ such that gxg^{-1} is stably conjugate in M' to x'. We decompose $x = (x_1, x_N)$, where x_1 belongs to $GL(k)$, for some integer k and x_N belongs to T_N which is a product of norm one tori. Similarly, we decompose $x' = (x'_1, x'_N)$, with $x'_1 \in GL(k)$ and $x'_N \in T'_N$. Since x and x' are stably conjugate, we have that x_1 and x'_1 are stably conjugate in $GL(k)$, hence they are conjugate by some $g_1 \in GL(k)$. Let $g = g_1 \times I$. Since $x_N \in T_N \subset M'$ and $x'_N \in T'_N \subset M'$ are stably conjugate and we have that $gxg^{-1} = (x'_1, x_N)$ is stably conjugate in M to $x' = (x'_1, x'_N)$.

Now we use van Dijk's formula (see Proposition 6.1.1) to evaluate the character θ_π at x and x'. Let $T = Cent_{SO(2n+1)_+}(x'_1, x_N)$ and $T' = Cent_{SO(2n+1)_+}(x'_1, x'_N)$. We have that

$$\theta_\pi(x) = \theta_\pi(gxg^{-1}) = \sum_{s \in W(A,T)} \theta_{s\rho'}(x'_1, x_N) \frac{|D_{M'^s}(x'_1, x_N)|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(x'_1, x_N)|^{\frac{1}{2}}},$$

$$\theta_\pi(x') = \sum_{s\in W(A,T')} \theta_{s\rho'}(x_1', x_N') \frac{|D_{M'^s}(x_1', x_N')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(x_1', x_N')|^{\frac{1}{2}}}.$$

Note that the Cartan subgroups T and T' have the same split parts, therefore $W(A,T) = W(A,T')$. Since (x_1', x_N) is stably conjugate in M' to (x_1', x_N'), we have that the values of $D_{M's}$ and $D_{SO(2n+1)_+}$ at those two points are equal. Finally θ_ρ is stable on M, hence $\theta_{s\rho'}(x_1', x_N) = \theta_{s\rho'}(x_1', x_N')$. That completes the proof in the linear case.

The proof of the $\widetilde{Sp}(2n)$ case is similar. Let $\widetilde{x}$ and $\widetilde{x}'$ be two elements that are stably conjugate in $\widetilde{Sp}(2n)$, i.e. $p(\widetilde{x})$ is stably conjugate to $p(\widetilde{x}')$ in $Sp(2n)$ and $\Theta_{\omega_+-\omega_-}(\widetilde{x}) = \Theta_{\omega_+-\omega_-}(\widetilde{x}')$. We find an element $g \in Sp(2n)$ such that $gp(\widetilde{x})g^{-1}$ is stably conjugate in M to $p(\widetilde{x}')$ (such g exists by similar argument as in the $SO(2n+1)_+$ case). We choose an element $\widetilde{g} \in \widetilde{Sp}(2n)$ such that $p(\widetilde{g}) = g$. We have that $\Theta_{\omega_+-\omega_-}(\widetilde{g}\widetilde{x}\widetilde{g}^{-1}) = \Theta_{\omega_+-\omega_-}(\widetilde{x}) = \Theta_{\omega_+-\omega_-}(\widetilde{x}')$, hence by Lemma 7.2.4 the elements $\widetilde{g}\widetilde{x}\widetilde{g}^{-1}$ and $\widetilde{x}'$ are stably conjugate in $\widetilde{M}$. The rest of the proof is similar to the linear case. □

Chapter 8

Parabolic induction

This chapter contains the proof of the main results of the thesis, which is that the lifting of representations is compatible with parabolic induction. This result reduces the problem of lifting of representations to the supercuspidal case.

We keep the notations from the previous chapters. Recall that $M' \cong SO(2m+1)_+ \times GL(n_1) \times \ldots \times GL(n_k), M \cong Sp(2m) \times GL(n_1) \times \ldots \times GL(n_k)$ and $\widetilde{M} = p^{-1}(M)$. We also consider the following cover of $\widetilde{M}$:

$$\begin{array}{c} \widetilde{\widetilde{M}} \cong \widetilde{Sp}(2m) \times \widetilde{GL}(n_1) \times \ldots \times \widetilde{GL}(n_k) \\ \big\downarrow \\ \widetilde{M} \cong p^{-1}(Sp(2m) \times GL(n_1) \times \ldots \times GL(n_k)). \end{array}$$

Here the cocycle on each $\widetilde{GL}(n_i)$ is given by the Hilbert symbol of the appropriate determinants, i.e. $c(x,y) = (\det(x), \det(y))$. If ρ is a representation on $\widetilde{M}$, we will denote its lift to $\widetilde{\widetilde{M}}$ by $\widetilde{\rho}$.

Theorem 8.0.7 *(Reduction to the $Sp(2m)$ and $SO(2m+1)_+$ factors) Let ρ be a genuine representation on $\widetilde{M}$ and let ρ' be a representation on M'. Assume that their characters satisfy the matching condition (7.1):*

$$\theta_\rho(\widetilde{x}) = \Phi_{\widetilde{M}}(\widetilde{x})\theta_{\rho'}(x'), \quad p(\widetilde{x}) \overset{L-stable}{\longleftrightarrow} x'.$$

Decompose $\widetilde{\rho} = \widetilde{\rho}_0 \otimes \ldots \otimes \widetilde{\rho}_k$ and $\rho' = \rho'_0 \otimes \ldots \otimes \rho'_k$ accordingly to the decomposition of $\widetilde{\widetilde{M}}$ and M'. Then

(1) $\theta_{\widetilde{\rho}_0}(\widetilde{x}) = \Theta_{\omega_+ - \omega_-}(\widetilde{x})\theta_{\rho_0'}(x')$, *where* $\Theta_{\omega_+ - \omega_-}$ *is the character of the difference of the two halves of the oscillator representation on* $\widetilde{Sp}(2m)$, $\widetilde{x} \in \widetilde{Sp}(2m), x' \in SO(2m+1)_+$ *and* $p(\widetilde{x}) \overset{stable}{\longleftrightarrow} x'$,

(2) For $i = 1, \ldots, k$ *we have that* $\theta_{\widetilde{\rho}_i}(\widetilde{x}) = \chi_\eta(\widetilde{x})\theta_{\rho_i'}(x')$, *where* χ_η *is a character of* $\widetilde{GL}(n_i)$ *defined by* $\chi_\eta(g, \epsilon) = \gamma(\det(g), \eta)\epsilon$.

Proof. This is an immediate consequence of Lemma 5.0.11, where we showed that the transfer factor restricted to $\widetilde{Sp}(2m)$ is equal to the character of the difference of two halves of the oscillator representation on $\widetilde{Sp}(2m)$ and that the transfer factor restricted to each $\widetilde{GL}(n_i)$ is equal to a character χ_η, where $\chi_\eta(g, \epsilon) = \gamma(\det(g), \eta)\epsilon$.

□

The following theorem is the main result of this thesis. It reduces the problem of lifting characters to the supercuspidal case.

Theorem 8.0.8 *Let* ρ *be a genuine admissible virtual representation of* $\widetilde{M}$ *and* ρ' *be an admissible virtual representation of* M'. *Assume that their characters satisfy the matching condition (7.1), i.e.*

$$\theta_\rho(\widetilde{x}) = \Phi_{\widetilde{M}}(\widetilde{x})\theta_{\rho'}(x'), \quad p(\widetilde{x}) \overset{L-stable}{\longleftrightarrow} x'.$$

(Recall that this implies that the characters θ_ρ *and* $\theta_{\rho'}$ *are stable). Let* $\pi = Ind_{\widetilde{M}}^{\widetilde{Sp}(2n)}\rho$ *and* $\pi' = Ind_{M'}^{SO(2n+1)_+}\rho'$ *and denote by* θ_π *and* $\theta_{\pi'}$ *the characters of these representations. Then*

$$\theta_\pi(\widetilde{g}) = \Phi_{\widetilde{Sp}(2n)}(\widetilde{g})\theta_{\pi'}(g') \quad for \quad p(\widetilde{g}) \overset{stable}{\longleftrightarrow} g'.$$

Remark 1. (Connections to the case of the minimal parabolic induction) Assume that $A \cong F^{*n}$ is a split Cartan subgroup. We have $A \cong M$ hence by the definition of the transfer factor and by Lemma 5.0.7 we have that

$$\Phi_{\widetilde{A}}(\widetilde{g}) = \frac{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}) = |\det(1+g)|^{\frac{1}{2}} \Theta_{\omega_+ - \omega_-}(\widetilde{g}),$$

for $p(\widetilde{g}) = g \overset{L-stable}{\longleftrightarrow} g'$. This coincides with the transfer factor Φ defined by the equation (6.1) in the case of the minimal parabolic induction. Therefore the pair of characters χ and $\widetilde{\chi}$ described in Theorem 6.2.1 satisfies the matching condition (7.1). Hence indeed Theorem 6.2.1 is a special case of the theorem above.

Remark 2. As in the case of the minimal parabolic induction, note that the matching correspondence depends on the fixed additive character η. See the remark after Theorem 6.2.1 for an explanation.

Proof. Assume that the characters of the representations ρ and ρ' satisfy (7.1), i.e.

$$\theta_\rho(\widetilde{x}) = \Phi_{\widetilde{M}}(\widetilde{x})\theta_{\rho'}(x'), \quad p(\widetilde{x}) \overset{L-stable}{\longleftrightarrow} x'.$$

Let $\widetilde{g} \in \widetilde{Sp}(2n)$ and $g' \in SO(2n+1)_+$ be such that $g = p(\widetilde{g}) \in Sp(2n)$ and g' are strongly regular semisimple and $g \overset{stable}{\longleftrightarrow} g'$. Let $\widetilde{T}$ be the centralizer of $\widetilde{g}$ in $\widetilde{Sp}(2n)$. Let T' be the centralizer of g' in $SO(2n+1)_+$. If $\widetilde{T}$ (resp. T') is not conjugate to a Cartan subgroup in $\widetilde{M}$ (resp. M'), then the value of the character θ_π (resp. $\theta_{\pi'}$) is zero. Therefore without loss of generality we can assume that $\widetilde{T}$ is a Cartan subgroup in $\widetilde{M}$ and T' is a Cartan subgroup in M'. Then T' is isomorphic to the Cartan subgroup $p(\widetilde{T})$ in $Sp(2n)$. By Proposition 6.1.1 we have:

$$\theta_\pi(\widetilde{g}) = \sum_{s \in W(A,T)} \theta_{s\rho}(\widetilde{g}) \frac{|D_{M^s}(g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}} \ ,$$

and

$$\theta_{\pi'}(g') = \sum_{s' \in W(A',T')} \theta_{s'\rho'}(g') \frac{|D_{M'^{s'}}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}} \ .$$

We use Lemma 6.3.2 to rewrite the indexing sets as follows

$$W(A,T) \cong \bigcup_{\{H \subset M : H \sim_G T\}/\sim_M} W(Sp(2n),H)/W(M,H),$$

$$W(A',T') \cong \bigcup_{\{H' \subset M' : H' \sim_G T'\}/\sim_{M'}} W(SO(2n+1)_+,H')/W(M',H').$$

Now we use the methods of the proof of Lemma 6.3.17 to choose the representatives $T_1,\dots,T_k$ of the set $\{H \subset M : H \sim_G T\}/\sim_M$ and the representatives $T'_1,\dots,T'_k$ of the set $\{H' \subset M : H' \sim_G T'\}/\sim_{M'}$ to be isomorphic Cartan subgroups. Therefore we have

$$W(A,T) \cong \bigcup_{i=1,\dots,k} W(Sp(2n),T_i)/W(M,T_i),$$

$$W(A',T') \cong \bigcup_{i=1,\dots,k} W(SO(2n+1)_+,T'_i)/W(M',T'_i),$$

where T_i and T'_i are isomorphic Cartan subgroups for $i = 1,\dots,k$. By Lemma 7.2.5, the characters θ_ρ and $\theta_{\rho'}$ are stable, therefore we can use Lemma 7.1.6 to replace the Weyl group quotients with their stabilized versions:

$$W(A,T) \cong \bigcup_{i=1,\dots,k} W_{st}(Sp(2n),T_i)/W_{st}(M,T_i),$$

$$W(A',T') \cong \bigcup_{i=1,\dots,k} W_{st}(SO(2n+1)_+,T'_i)/W_{st}(M',T'_i).$$

For each i choose isomorphisms $\psi_i : T_i \to T'_i$ and

$$\phi_i : W_{st}(Sp(2n),T_i)/W_{st}(M,T_i) \to W_{st}(SO(2n+1)_+,T'_i)/W_{st}(M',T'_i)$$

such that $\psi_i(w \cdot t) = \phi_i(w) \cdot \psi_i(t)$, for $w \in W_{st}(Sp(2n),T_i)/W_{st}(M,T_i)$ and $t \in T_i$ (see Lemma 7.1.7). If $w \in W_{st}(Sp(2n),T_i)/W_{st}(M,T_i)$ then we will denote its image via the isomorphism ϕ_i by w'. We rewrite the character formula as follows:

$$\theta_\pi(\widetilde{g}) = \sum_i \sum_w \theta_{w\rho}(\widetilde{g}) \frac{|D_{M^w}(g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}},$$

$$\theta_{\pi'}(g') = \sum_i \sum_{w'} \theta_{w'\rho'}(g') \frac{|D_{M'^{w'}}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}}.$$

We will compare these formulas term by term. We fix i, and for simplicity of the rest of the proof we assume that $T_i = T$, $T'_i = T', \psi_i = \psi$, etc. Recall that $g = p(\widetilde{g}) \overset{stable}{\longleftrightarrow} g'$. We can find $w'' \in W_{st}(SO(2n+1)_+, T')$ such that $g \overset{L-stable}{\longleftrightarrow} w'' \cdot g'$. Since we average over the quotient $W_{st}(SO(2n+1)_+, T')/W_{st}(M', T')$ and since the action of $W_{st}(M', T')$ does not affect the $L-stable$ conjugacy classes, we can assume without loss of generality that $g \overset{L-stable}{\longleftrightarrow} g'$. Note that

$$\theta_{w\rho}(\widetilde{g}) \frac{|D_{M^w}(g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}} = \theta_\rho(w^{-1} \cdot \widetilde{g}) \frac{|D_M(w^{-1} \cdot g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}}$$

and that

$$p(w^{-1} \cdot \widetilde{g}) \overset{L-stable}{\longleftrightarrow} w'^{-1} \cdot g'.$$

Now we can apply our assumption to get

$$\Phi_{\widetilde{M}}(w^{-1} \cdot \widetilde{g}) \theta_{\rho'}(w'^{-1} \cdot g') \frac{|D_M(w^{-1} \cdot g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}},$$

which can be rewritten as

$$\Phi_{\widetilde{M}}(w^{-1} \cdot \widetilde{g}) \frac{|D_M(w^{-1} \cdot g)|^{\frac{1}{2}}}{|D_{Sp(2n)}(g)|^{\frac{1}{2}}} \frac{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}}{|D_{M'}(w'^{-1} \cdot g')|^{\frac{1}{2}}} \theta_{\rho'}(w'^{-1} \cdot g') \frac{|D_{M'}(w'^{-1} \cdot g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}}.$$

By the equation (5.1) it can be shortened to

$$\Phi_{\widetilde{Sp}(2n)}(w^{-1} \cdot \widetilde{g}) \theta_{\rho'}(w'^{-1} \cdot g') \frac{|D_{M'}(w'^{-1} \cdot g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}},$$

which is equal to

$$\Phi_{\widetilde{Sp}(2n)}(\widetilde{g})\theta_{w'\rho'}(g')\frac{|D_{M'^{w'}}(g')|^{\frac{1}{2}}}{|D_{SO(2n+1)_+}(g')|^{\frac{1}{2}}},$$

since

$$\Phi_{\widetilde{Sp}(2n)}(w^{-1}\cdot\widetilde{g}) = \Theta_{\omega_+-\omega_-}(w^{-1}\cdot\widetilde{g}) = \Theta_{\omega_+-\omega_-}(\widetilde{g}) = \Phi_{\widetilde{Sp}(2n)}(\widetilde{g}).$$

That proves the assertion. □

We reduced the problem of lifting of representations to the supercuspidal case. It remains to prove this for supercuspidal representations. The first step will be to try to prove it for depth zero supercuspidal representations. Depth is an invariant of an admissible representation and it is a nonnegative rational number. It does not change after parabolic induction and the depths of two irreducible admissible representations that are paired by the local theta correspondence are equal. Moreover, depth zero characters are used to construct other supercuspidal representations of positive depths. It may be possible to make some progress on this using L-packets of stable depth zero supercuspidal representations constructed by DeBacker and Reeder in [DB-R].

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