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Distributional convergence of path durations in MANETs with dependent link excess lives

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Distributional Convergence of Path Durations in MANETs with Dependent Link Excess Lives

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***Abstract*— We investigate the issue of path selection in multi-hop wireless networks with the goal of identifying a scheme that can select a path with the largest expected duration. To this end we first study the distribution of path duration. We show that, under a set of mild conditions, when the hop count along a path is large, the distribution of path duration can be well approximated by an exponential distribution even when the distributions of link durations are dependent and heterogeneous. Secondly, we investigate the statistical relation between a path duration and the durations of the links along the path. We prove that the parameter of the exponential distribution, which determines the expected duration of the path, is related to the link durations only through their means and is given by the sum of the inverses of the expected link durations. Based on our analytical results we propose a scheme that can be implemented with existing routing protocols and select the paths with the largest expected durations.**

I. Introduction

Multi-hop wireless ad-hoc networks have been the focus of active research in recent years. Unlike a wireline network with a fixed infrastructure, a wireless ad-hoc network can be deployed with no infrastructure and mobile nodes can establish and maintain a network in an autonomous manner. Due to nodes' mobility, links are expected to be set up and torn down much more frequently than in a wireline network. As a result, a network topology varies with time as the connectivity between nodes changes dynamically. Frequent link failures and network topology changes in mobile ad-hoc networks (MANETs) render the routing protocols designed for wireline networks (e.g., the Internet) rather inefficient. A suite of new routing algorithms have been proposed for MANETs to deal with frequent network topology changes [8], [12], [14], [15]. A detailed discussion of available routing protocols is provided in the monographs [13], [19].

Due to nodes' mobility, links along a provided path may become unavailable in an unpredictable manner. When one or more links along a path in use become unavailable (which we call a path failure), the path is no longer valid and a path recovery procedure is triggered to find an alternate path. Detecting and recovering from a path failure can take a non-negligible amount of time (from applications' viewpoint), during which service to on-going traffic will be disrupted. Such a disruption in service can degrade the performance of time-critical applications. Furthermore, an initiation of path recovery incurs additional overhead. Therefore, from the perspective of providing reliable network service and minimizing control overhead, a good routing algorithm should take into consideration the *expected* duration as well as other requirements when selecting a path. The duration of a path refers to the amount of time for which the path remains available after its set-up until one of the links along the path fails for the first time.

Intuitively the duration of a path should depend on the durations of the links along the path and their dependence structure. Therefore, there is much interest in better understanding the statistical properties of link and path durations and their relation. Better understanding of their statistical properties will allow us to approximate the frequency of disruption in service and resulting overhead. Hence, it will help us evaluate the performance of on-demand routing protocols and the adverse effects of potentially frequent disruptions in service on the performance of upper layers (e.g., Transmission Control Protocol) without having to run time-consuming detailed simulations. A numerical example using the Random Waypoint (RWP) mobility model is given in [5, Section 8].

To the best of authors' knowledge there is very little known about the distribution of path durations and its relation with those of the links that provide them. Consequently, most of existing routing protocols select a path based on some heuristic argument; the Dynamic Source Routing (DSR) protocol selects the minimum hop path, whereas the Ad-hoc On-demand Distance Vector (AODV) routing protocol selects the first discovered path. Associativity Based Routing (ABR) protocol selects the path with maximum average age of the links. However, it is not clear how the hop count or the average age of the links along a path is related to its (expected) duration.

Along this line Sadagopan *et al.* [18] presented a simulation study of the distribution of multi-hop path durations under various mobility models. Their study shows that the distribution of path duration can be accurately approximated by an *exponential* distribution when the number of hops is larger than 3 or 4 for *all* mobility models considered. However, no clear explanation was offered for the emergence of an exponential distribution.

In order to correct the current state of affairs Han *et al.* [5] developed an approximate framework for studying the distributions of path and link durations. They showed that, under certain conditions, the distribution of path duration (under appropriate scaling) converges to an exponential distribution when the number of hops becomes large. This result is in line with the simulation results provided in [18], and is obtained as a simple application of Palm's Theorem [7, Thm. 5-14, p. 157]. In addition, they explored the connection between the expected duration of a path and the expected durations of the links along the path. To be more precise, they showed that when the number of hops is large, the inverse of the expected duration of a path is approximately given by the sum of the inverses of expected durations of the links along the path.

The results reported in [5] provide the first evidence that when the hop count is large, the distribution of path duration can indeed be approximated by an exponential distribution. The application of Palm's theorem in [5], however, requires that the link excess lives be mutually independent, which is in general *not* true. The excess life of a link in a path refers to the amount of time the link remains available until it is torn down for the first time after the path set-up. Two neighboring links along a path, for example, share a common node. Clearly, the excess lives of these two neighboring links depend on the mobility of the shared node, introducing some level of dependence in them. Moreover, such local dependence in link excess lives may be more evident under group mobility models where the mobility of a set of nodes may be correlated. Therefore, it is of much interest to see if the same distributional convergence to an exponential distribution holds *without* the independence assumption and how the parameter of the limit distribution (which decides the expected duration) is affected by the dependence.

In this paper we extend the results in [5] by relaxing the independence assumption on the link excess lives imposed in [5]. Instead we only require that the dependence of link excess lives go away asymptotically with increasing hop distance between the links. This assumption can be stated using what is known as a *mixing condition* (Section V-A). It allows the possibility of *strong* local dependence in link excess lives that can be exhibited, for example, by group mobility models. We demonstrate that, under some mild conditions (to be stated precisely), the same distributional convergence to an exponential distribution reported in [5] holds under this much weaker condition (Section V-B). Relaxation of the independence assumption necessitates a new set of tools for proving the distributional convergence and complicates the proof considerably; a suitable extension of Palm's theorem that deals with dependent processes is not available.

We also show that the parameter of the emerging exponential distribution is the same whether the link excess lives are mutually independent or not. In other words, the parameter of the exponential distribution is given by the sum of the inverses of the expected link durations. This suggests that for a sufficiently large hop count the dependence of link excess lives does not significantly affect the path duration distribution. Based on this observation, we outline a scheme that can be implemented in existing routing protocols to select the path with the largest expected duration with minimal communication overhead (Section VI).

The paper is organized as follows. A basic framework for modeling path durations is given in Section II. Section III introduces the set-up under which the asymptotic distribution of a path duration with increasing hop count is studied. In Section IV we study a simpler case in which link durations have the same distribution and their dependence is limited to a finite neighborhood. This is followed by a study of more general cases in which the link duration distributions may be heterogeneous and the dependence is not limited to a finite neighborhood in Section V. We outline how our results can be used to implement a scheme for selecting a path with the largest expected duration during a path discovery phase in Section VI.

A word on the notation and convention used throughout: We find it convenient to define all the random variables (rvs) of interest on some common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Two $\mathbb{R}$–valued rvs X and Y are said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. The independence between two rvs X and Y is denoted by $X \perp Y$. If G is a probability distribution on $\mathbb{R}_+$, let $\mathbf{m}(G)$ denote its first moment which is always assumed to be finite. Convergence in distribution (with n going to infinity) is denoted by $\Longrightarrow_n$. For any $\boldsymbol{x}$ in $\mathbb{R}^2$, with components (x_1, x_2), set $||\mathbf{x}|| = \sqrt{x_1^2 + x_2^2}$.

II. A Basic Framework

This section describes the same basic framework that we borrow from [5] for our analysis: Consider a MANET where a set of nodes creates and maintains network connectivity. We assume that an on-demand algorithm is used and a path between a source node and a destination node is set up only when a request is made.

Let $V = \{1, \ldots, N\}$ denote the set of N mobile communicating nodes. Each node moves across a domain $\mathbb{D}$ of $\mathbb{R}^2$ or $\mathbb{R}^3$ according to some mobility model. Due to nodes' mobility, links between nodes are set up and torn down dynamically. We assume that a link between two nodes is either up or down. Two nodes without a link between them establish such a link as soon as they become reachable, e.g., when they come within a transmission range of each other or when the signal to interference and noise ratio (SINR) at the receiver exceeds certain threshold, and packets from each other can be successfully decoded. The latter case captures the characteristics of the physical layer (e.g., path loss and channel fading) more accurately. Although this is not needed for the analysis, *communication* links are assumed bidirectional since such bidirectional communication is typically required between two nodes for reliable forwarding of packets, for instance, by means of acknowledgments for each transmission.

Establishing a path from a source node to a destination node requires *simultaneous* availability of a number of communication links that are up at the time of path request and collectively provide the desired connectivity between the source and the destination. The *duration* of a path provided by the underlying routing protocol is then defined as the amount of time that elapses until one of the links along the path goes down for the first time after the path set-up. A link may go down (which we call a link failure) due to either mobility or degradation in channel condition. For simplicity of analysis, path set-up delays are assumed negligible.

A. Reachability processes

We model the situation outlined above as follows: For a pair of distinct nodes i and j in V, we introduce a $\{0,1\}$-valued *reachability* process $\{\xi_{ij}(t),\ t \geq 0\}$ with the interpretation that $\xi_{ij}(t) = 1$ (resp. $\xi_{ij}(t) = 0$) if the unidirectional link from node i to node j, denoted by "link" (i,j), is up (resp. down) at time $t \geq 0$. Since the communication links are assumed bidirectional, we must have $\xi_{ij}(t) = \xi_{ji}(t)$. The process $\{\xi_{ij}(t),\ t \geq 0\}$ is simply an alternating on-off process, with successive up and down time durations given by the rvs $\{U_{ij}(k), k = 1,2,\ldots\}$ and $\{D_{ij}(k), k = 1,2,\ldots\}$, respectively.

The reachability processes can be defined in a number of ways. For example, for each i in V, let $\{\boldsymbol{X}_i(t),\ t \geq 0\}$ describe the trajectory of node i, i.e., $\boldsymbol{X}_i(t)$ denotes the position of node i at time $t \geq 0$. If we do not explicitly model channel fading between nodes, it is reasonable to assume that two nodes can communicate with each other reliably if the distance between them is smaller than some fixed transmission range $r_{min} > 0$. Hence, a link between two distinct nodes i and j in V exists at time $t \geq 0$ if and only if their distance is smaller than r_{min}, leading to the definition

$$\xi_{ij}(t) := \mathbf{1}\left[||\boldsymbol{X}_i(t) - \boldsymbol{X}_j(t)|| \leq r_{min}\right], \quad t \geq 0. \tag{1}$$

In the literature this model is known as the protocol model [4], [16].

Alternative models can take into account the physical layer characteristics of the channel. For instance, two nodes i and j in V can maintain a link between them at time $t \geq 0$ if and only if

$$\min\left(\frac{P_j \cdot F_{ji}(t)}{\Psi_i(t)}, \frac{P_i \cdot F_{ij}(t)}{\Psi_j(t)}\right) > \Gamma \tag{2}$$

for some threshold $\Gamma > 0$, where P_i is the maximum transmission power of node i, and $\boldsymbol{F}(t) = (F_{ij}(t))$ denotes the channel gain matrix (including fading) at time t with $F_{ji}(t) \geq 0$ and $F_{ii}(t) = 0$, $i,j = 1,\ldots N$. Different choices of $\Psi_i(t)$ in (2) lead to different physical layer models. In the simplest form, one can assume that a node i can decode the packets from node j if and only if the received signal power exceeds some threshold $\Gamma > 0$ [2], [17]. In this case the reachability process between nodes i and j is given by (2) with $\Psi_i(t) = 1$ as the numerators give the largest achievable received signal power at the nodes.

Similarly, if one assumes that packets can be successfully decoded if and only if the achieved SINR exceeds the threshold Γ [3], [4], then the reachability process between nodes i and j is again given by (2) with

$$\Psi_i(t) = W_i + \sum_{k \in TX(t)\setminus\{j\}} P_k(t) \cdot F_{ki}(t)\ , \tag{3}$$

where W_i is the noise variance at node i, $TX(t)$ is the set of transmitters at time t and $P_k(t)$ denotes the transmission power of node k. The right hand side of (3) represents the sum of noise power and interference at node i at time t. This implies that nodes i and j have connectivity if and only if the achieved SINR value using the maximum transmission power exceeds Γ in both directions.

B. Path duration

Next we endow V with a time-varying graph structure by introducing a time-varying set $E(t)$ of directed edges through the relation

$$E(t) := \{(i,j) \in V \times V :\ \xi_{ij}(t) = 1\}, \quad t \geq 0 \tag{4}$$

where by convention we set $\xi_{ii}(t) = 0$ for each i in V and all $t \geq 0$. Thus, a path can be established (in principle) between nodes s and d at time $t \geq 0$, if node d is reachable from node s by a path in the *undirected* graph derived from the directed graph $(V, E(t))$. Let $\mathcal{P}_{sd}(t) \subseteq 2^{E(t)}$ denote the set of paths from node s to node d providing this reachability. This set of paths is empty when the nodes s and d are not reachable from each other at time t. When non-empty, this set $\mathcal{P}_{sd}(t)$ may contain more than one path since multiple paths may exist between nodes s and d. In such a case, the routing protocol in use selects one of the paths in $\mathcal{P}_{sd}(t)$ and let $\mathcal{L}_{sd}(t)$ denote the set of links in the selected path.

For each link ℓ in $\mathcal{L}_{sd}(t)$, let $T_\ell(t)$ denote the time-to-live or excess life after time t, i.e., $T_\ell(t)$ is the amount of the time that elapses from time t onward until link ℓ is down. The time-to-live or duration $Z_{sd}(t)$ of the established path from node s to node d using the links in $\mathcal{L}_{sd}(t)$ is defined as the amount of time that elapses from time t until one of the links in $\mathcal{L}_{sd}(t)$ goes down, at which point a path recovery procedure is initiated. This quantity is simply given by

$$Z_{sd}(t) := \min\left(T_\ell(t) : \ell \in \mathcal{L}_{sd}(t)\right), \quad t \geq 0. \tag{5}$$

III. The Set-up and Modeling Assumptions

In this paper we are interested in studying the distribution of path duration as the number of hops becomes large. In the following subsection we first describe the set-up used to model this scenario. Then, we state the modeling assumptions under which the distributional convergence of path duration is established with increasing hop count.

A. The set-up

In order to study the distribution of path duration with a large hop count, we investigate the asymptotic distribution of path duration (under appropriate scaling of link excess lives) as the number of hop count increases. This is done by introducing a parametric scenario with a sequence of networks in which both the number of communicating nodes and the domain across which they travel increase:
For each $n = 1,2,\ldots$, let $V^{(n)} = \{1,\ldots,N^{(n)}\}$ and $\mathbb{D}^{(n)}$ denote the set of mobile nodes and the domain across which the nodes move, respectively. For each node i in $V^{(n)}$, the $\mathbb{D}^{(n)}$-valued process $\{\boldsymbol{X}_i^{(n)}(t),\ t \geq 0\}$ denotes the trajectory of node i in $\mathbb{D}^{(n)}$. The stochastic process that governs the arrival of path requests is assumed to be independent of these trajectory processes.

1. Scaling – We are interested in the situation where

$$N^{(n)} \sim nN^{(1)} \quad \text{and} \quad \text{Area}(\mathbb{D}^{(n)}) \sim n \cdot \text{Area}(\mathbb{D}^{(1)}) \tag{6}$$

as n goes to infinity;[1] it is customary to reparameterize so that $N^{(n)} = n$. When in force, the scaling (6) guarantees

$$\frac{N^{(n)}}{Area(\mathbb{D}^{(n)})} \sim \frac{N^{(1)}}{Area(\mathbb{D}^{(1)})},$$

so that the density of nodes, i.e., the number of nodes per unit area, is asymptotically constant.

2. Stationarity – As the system is expected to run for a long time, we can assume that steady state has been reached. This possibility is captured by taking the $N^{(n)}$ trajectory processes to be *jointly stationary*. Joint stationarity of the trajectory processes also implies that the $\frac{N^{(n)}\times(N^{(n)}-1)}{2}$ reachability processes are jointly stationary. For distinct $i < j$ in $V^{(n)}$, let the rvs $\{(U_{ij}^{(n)}(k), D_{ij}^{(n)}(k)),\ k = 2, 3, \ldots\}$ denote the sequence of up and down times for the reachability process $\{\xi_{ij}^{(n)}(t),\ t \geq 0\}$. Writing

$$\boldsymbol{W}^{(n)}(k) = \left((U_{ij}^{(n)}(k), D_{ij}^{(n)}(k)),\ i < j,\ i, j \in V^{(n)} \right) ,$$

$k = 1, 2, \ldots$, we require that the sequence of rvs $\{\boldsymbol{W}^{(n)}(k),\ k = 2, 3, \ldots\}$ be *strictly stationary*. In particular, for distinct $i < j$ in $V^{(n)}$, the sequence $\{(U_{ij}^{(n)}(k), D_{ij}^{(n)}(k)),\ k = 2, 3, \ldots\}$ constitutes a stationary sequence with generic marginals $(U_{ij}^{(n)}, D_{ij}^{(n)})$. We denote by $G_{ij}^{(n)}$ the cumulative distribution function (CDF) of $U_{ij}^{(n)}$. This model is general enough that link dynamics due to both mobility and channel fading can be captured by a suitable choice of the CDFs for $U_{ij}^{(n)}$.

Well-known results for renewal processes and independent on-off processes in equilibrium [7, Sections 5-6] can be generalized as follows: With $\ell = (i, j)$, in the notation introduced in Section II, we have

$$\mathbf{P}\left[T_\ell^{(n)}(0) \leq x \middle| \xi_{ij}^{(n)}(0) = 1 \right] = F_\ell^{(n)}(x), \quad x \in \mathbb{R} \tag{7}$$

where the conditional probability $F_\ell^{(n)}(x)$ is given by

$$F_\ell^{(n)} = \begin{cases} \frac{1}{\mathbf{m}(G_\ell^{(n)})} \int_0^x \left(1 - G_\ell^{(n)}(y) \right) dy & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{8}$$

for some link duration CDF $G_\ell^{(n)}$ with support in $\mathbb{R}_+$. In other words, $F_\ell^{(n)}$ is simply the distribution of the forward recurrence time associated with $U_\ell^{(n)}$. From (8) it is easy to see that the duration of a one-hop path has a non-increasing probability density function (PDF). If $X_\ell^{(n)}$ denotes any $\mathbb{R}_+$-valued rv distributed according to $F_\ell^{(n)}$, then the relation (7) simply states, with a little abuse of notation, that

$$\left[T_\ell^{(n)}(0) \leq x \middle| \xi_{ij}^{(n)}(0) = 1 \right] =_{st} X_\ell^{(n)}.$$

The rv (5) can now be viewed as the rv $Z^{(n)}$ defined by

$$Z^{(n)} := \min \left(X_\ell^{(n)} :\ \ell = 1, \ldots, H^{(n)} \right) \tag{9}$$

where $H^{(n)} = |\mathcal{L}_{sd}^{(n)}(0)|$. Due to the underlying stationarity assumptions, it clearly suffices to consider only the case $t = 0$ as we do from now on.

[1]From now on we omit this qualifier in all asymptotic equivalences.

B. Modeling assumptions

There are a few sources of difficulty in modeling and studying the distribution of path durations: First, the set $\mathcal{L}_{sd}(0)$ of links in the selected path is a *random* subset of $E(0)$, which depends on the reachability processes at $t = 0$. Second, the reachability processes are usually *not* mutually independent. This is clear from either formulation (1) or (2). In this subsection we explain how we handle these issues.

1. Asymptotics of the random set $\mathcal{L}_{sd}^{(n)}(0)$ – With increasing network size under scaling (6) the *average* number of hops in a path between two randomly selected nodes is expected to increase with n. For example, consider the protocol model (1) with a fixed domain.[2] We first select the locations of a source and a destination according to some stationary spatial distribution of the nodes. Then, for each $n = 3, 4, \ldots$, place the remaining $n - 2$ other nodes on the domain according to the same stationary distribution while decreasing the transmission range of the nodes as $1/\sqrt{n}$. If minimum hop routing is employed, the number of hops along the shortest path will increase approximately as $\sqrt{n}$. Thus, we assume that a pair of nodes s and d in $V^{(n)}$ can be selected such that $\lim_{n\to\infty} |\mathcal{L}_{sd}^{(n)}(0)| = \infty$, where for convenience the sequence $\{|\mathcal{L}_{sd}^{(n)}(0)|,\ n = 1, 2, \ldots\}$ is assumed to be deterministic.

2. Dependence of the reachability processes and link excess lives – As mentioned earlier, the link excess lives $\{X_\ell^{(n)},\ \ell = 1, \ldots, H^{(n)}\}$ in (9) are not mutually independent in general. The authors of [5] skirted this difficulty by assuming that the reachability processes $\{\xi_{ij}(t), t \geq 0\}$ are mutually independent so that the rvs $\{X_\ell^{(n)},\ \ell = 1, \ldots, H^{(n)}\}$ are mutually independent. They provided a simulation study (Section 9 in [5]) using the RWP mobility model without pause to justify this assumption; it shows that the correlation coefficient of link excess lives in (9) decays rapidly with increasing hop distance between the links. More specifically, it indicates that the correlation coefficient of link excess lives between two neighboring links is small and that of two links separated by intermediate link(s) is almost negligible.

This observation provides some evidence that the dependence of link excess lives may indeed decrease quickly with hop distance in some cases. However, the observed fast decrease of correlation in hop distance may be a consequence of the fact that the mobility of a node in the RWP model is independent of other nodes, and if the mobility of a set of nodes is strongly correlated (e.g., soldiers in a platoon partaking in a mission), this may no longer be true. In the following sections we relax the independence assumption of the reachability processes in [5] and replace it with what are commonly known as *mixing conditions*. These conditions impose a form of *asymptotic independence* as the hop distance between links increases, while allowing dependence in an (*unbounded*) neighborhood.

[2]Decreasing the transmission range while keeping the domain fixed has the same effect as increasing the domain size while keeping the transmission range fixed.

IV. Finite Dependence with Homogeneous Link Duration Distribution

In this section we consider a simpler case where link durations have the same CDF G with support in $\mathbb{R}_+$ and the dependence in link excess lives is limited to a *finite* local neighborhood. First, in order to model the link excess lives, we introduce a *stationary* sequence of rvs $\{X_i, i = 1, 2, \ldots\}$ whose CDF is given by

$$F(x) = \begin{cases} \frac{1}{\mathbf{m}(G)} \int_0^x (1 - G(y)) \ dy \ , & \text{if } x > 0 \\ 0 \ , & \text{if } x \leq 0 \end{cases} \ . \quad (10)$$

We let $X_\ell^{(n)} = X_\ell$ for all $n \in \mathbb{Z} := \{1, 2, \ldots\}$ such that $\ell \leq H(n)$, i.e., rv X_ℓ is used to model the excess life of the ℓ-th link in an $H(n)$-hop path. The path duration of an $H(n)$-hop path is modeled by rv $Z^{(n)} := \min(X_\ell^{(n)} : \ell = 1, \ldots, H(n))$. The rvs $X_i, i = 1, 2, \ldots$, are identically distributed from the stationarity assumption, but may not be mutually independent.

The aforementioned assumption of finite dependence of link excess lives is given by the following:

Assumption 1: (m-dependence [20]) The rvs X_i, $i = 1, 2, \ldots$, satisfy

$$X_\ell \perp X_{\ell'} \ \text{ if } \ |\ell - \ell'| > m \ ,$$

where m is a finite positive integer.

This assumption is consistent with the findings in [5, Fig. 9], where the dependence in link excess lives under the RWP mobility model appears to be limited to a very small neighborhood.

Assumption 2: For every $x \geq 0$ and any given $\epsilon > 0$, there exists an integer $n^\star = n^\star(x; \epsilon)$ such that

$$G\left(\frac{x}{n}\right) \leq \epsilon \ , \qquad n = n^\star, n^\star + 1, \ldots$$

Assumption 2 is equivalent to saying that a link duration is strictly positive with probability one, i.e., $\lim_{n\to\infty} G(x/n) = G(0) = 0$. It is obvious that this assumption holds trivially if the CDF G is continuous (i.e., link durations can be modeled as continuous rvs). Therefore, it is a reasonable assumption.

Theorem 1: Suppose that Assumptions 1 and 2 hold for the stationary sequence $\{X_i, i = 1, 2, \ldots\}$ and the CDF G. If the condition

$$\begin{aligned} &\lim_{c \downarrow 0} \frac{1}{\mathbf{P}[X_i < c]} \max_{|i-j| \leq m} \mathbf{P}[X_i < c, X_j < c] \\ &= \lim_{c \downarrow 0} \max_{|i-j| \leq m} \mathbf{P}[X_j < c \mid X_i < c] \\ &= 0 \end{aligned} \quad (11)$$

holds, then

$$\lim_{n\to\infty} \mathbf{P}\left[H(n) \cdot Z^{(n)} \leq x\right] = \begin{cases} 1 - e^{-\lambda x} \ , & \text{if } \ x > 0 \\ 0 \ , & \text{if } \ x \leq 0 \end{cases} \quad (12)$$

where $\lambda = (\mathbf{m}(G))^{-1}$.

Proof: A proof is provided in Appendix I. ∎

Theorem 1 tells us that as the number of hops $H(n)$ along a path increases the distribution of path duration can be well approximated by an exponential distribution with parameter $H(n) \cdot \lambda$ for all sufficiently large $H(n)$. Note that rv $Z^{(n)} = \min(X_\ell^{(n)} : \ell = 1, \ldots, H(n))$ tends to decrease with increasing $H(n)$. This is also obvious from the fact that $H(n) \cdot \lambda \to \infty$ as $n \to \infty$. Thus, in order to keep $Z^{(n)}$ from converging to a constant rv with value zero as $H(n)$ increases, rv $Z^{(n)}$ is scaled by the hop count $H(n)$ in (12).

It is interesting to note that the parameter of the emerging exponential distribution is given by the same $\lambda = 1/\mathbf{m}(G)$ whether the rvs $\{X_i, i = 1, 2, \ldots\}$ are assumed to be locally dependent as here or mutually independent as assumed in [5].

The condition in (11) implies that as $c \downarrow 0$, the rare events $\{X_j < c\}$ do not occur in clusters in a local neighborhood of node i. One interpretation of this condition is as follows: Assume a very small c. Rare events of link excess lives being smaller than c are primarily caused by nodes being close to the edge of their transmission range and about to move out of the transmission range at the time of path set-up (rather than one or both of the nodes moving at an extremely high speed). Condition (11) implies that one pair of neighboring nodes being about to leave the transmission range of each other at the time of path set-up, does not mean the same is true for other pairs of neighboring nodes along a path, which is reasonable.

V. General Dependence with Heterogeneous Link Duration Distributions

In the previous section we considered the simpler case where the dependence in link excess lives is limited to a finite neighborhood. As mentioned earlier, this may be reasonable in some cases. However, we show that it can be relaxed considerably. To be precise, the same distributional convergence can be obtained even when the dependence of link excess lives goes away only *asymptotically* with increasing distance between links and the link duration distributions are heterogeneous.

In this section we first define the *mixing conditions* that describe the manner in which the dependence of link excess lives decays with the hop distance between the links. Then, we establish the distributional convergence of path duration in more general cases under the mixing conditions.

A. Mixing conditions

Suppose that $\mathbf{W} := \{W_i^{(n)}, n = 1, 2, \ldots; i = 1, \ldots, h(n)\}$ is an array of $\mathbb{R}$-valued rvs, where $\{h(n), n \geq 1\}$ is a sequence of positive integers with $\lim_{n\to\infty} h(n) = \infty$. Denote the joint CDF of rvs $\{W_{i_1}^{(n)}, W_{i_2}^{(n)}, \ldots, W_{i_n}^{(n)}\}$ by $\mathbf{J}_{i_1 \cdots i_n}^{(n)}$. For notational simplicity we write $\mathbf{J}_{i_1 \cdots i_n}^{(n)}(u)$ for $\mathbf{J}_{i_1 \cdots i_n}^{(n)}(u, \ldots, u)$.

Let $\{u_n, n \geq 1\}$ be a sequence of real numbers (which typically increases with n) and $\mathbf{A} := \{\alpha_{n,m}, n = 1, 2, \ldots; m = 1, \ldots, h(n)\}$ be an array of non-negative real numbers such that, for any integers

$$1 < i_1 < \cdots < i_p < j_1 < \cdots < j_q \leq h(n)$$

where $j_1 - i_p > m$, we have

$$\left| \mathbf{J}_{i_1 \ldots i_p j_1 \ldots j_q}^{(n)}(u_n) - \mathbf{J}_{i_1 \ldots i_p}^{(n)}(u_n) \mathbf{J}_{j_1 \ldots j_q}^{(n)}(u_n) \right| \leq \alpha_{n,m} \ . \quad (13)$$

Definition 1: ($D(u_n)$ condition [10], [11]) Suppose that we can find a sequence $\{m(n), n = 1, 2, \ldots\}$ and an array $\mathbf{A}$ of real numbers satisfying the condition in (13) such that (i) $\lim_{n\to\infty} m(n) = \infty$, (ii) $m(n) = o(h(n))$, i.e., $\lim_{n\to\infty} \frac{m(n)}{h(n)} = 0$, and (iii) $\lim_{n\to\infty} \alpha_{n,m(n)} = 0$. Then, we say that the array $\mathbf{W}$ satisfies the *condition* $D(u_n)$.

The condition $D(u_n)$ imposes a form of "dependence decay": As n increases, the dependence of two events $\{W_{i_1}^{(n)} \le u_n, \ldots, W_{i_p}^{(n)} \le u_n\}$ and $\{W_{j_1}^{(n)} \le u_n, \ldots, W_{j_q}^{(n)} \le u_n\}$ decreases as the distance $j_1 - i_p$ between the two sets of rvs increases. However, since $m(n) \to \infty$, it allows dependence in an *unbounded* neighborhood. One can easily verify that a sequence that satisfies the m-dependence condition in Assumption 1 satisfies the condition $D(u_n)$ with any sequence $\{u_n, n \ge 1\}$. The interpretation and role of this condition in our problem will be stated shortly.

In order to state the definition of the second mixing condition, we first need to introduce some notation. Let k be a *fixed* positive integer. We divide the interval $\{1, 2, \ldots, h(n)\}$ into $k+1$ disjoint subintervals[3]: The first k subintervals have a length $n' := \lfloor h(n)/k \rfloor$, where $\lfloor x \rfloor$ denotes the integer part of x, and the last interval has a length smaller than k. For $j = 1, 2, \ldots, k$, define

$$I_{k,j}^{(n)} = \{(j-1)\cdot n' + 1, \ldots, j \cdot n'\} ,$$

and

$$I_{k,k+1}^{(n)} = \{k \cdot n' + 1, \ldots, h(n)\} .$$

Note that $|I_{k,j}^{(n)}| = n'$ for $j = 1, \ldots, k$, and $0 \le |I_{k,k+1}^{(n)}| < k$, where $|I|$ denotes the cardinality of I.

Definition 2: The array $\mathbf{W}$ is said to satisfy the *condition* $D'(u_n)$ if

$$\lim_{n\to\infty} \left(\sum_{i,i' \in I_{k,j}^{(n)} : i < i'} \mathbf{P}\left[W_i^{(n)} > u_n, W_{i'}^{(n)} > u_n\right] \right) = o\left(\frac{1}{k}\right) \quad \text{for all } j = 1, \ldots, k . \tag{14}$$

A sufficient condition for the condition $D'(u_n)$ to hold is that

$$\lim_{n\to\infty} \left(\lfloor \frac{h(n)}{k} \rfloor^2 \cdot \sup_{i,i' \in I_{k,j}^{(n)} : i < i'} \mathbf{P}\left[W_i^{(n)} > u_n, W_{i'}^{(n)} > u_n\right] \right) = o\left(\frac{1}{k}\right) \qquad \text{for all } j = 1, \ldots, k . \tag{15}$$

The interpretation of the condition $D'(u_n)$ in the context of our problem will be given shortly.

B. Distributional convergence

Define $W_\ell^{(n)} = (X_\ell^{(n)})^{-1}, \ell = 1, \ldots, H(n)$. Let $\mathbf{W} := \{W_\ell^{(n)}, n = 1, 2, \ldots; \ell = 1, \ldots, H(n)\}$. We denote the CDF of rv $W_\ell^{(n)}$ by $\mathbf{J}_\ell^{(n)}$. We first make the following two assumptions. They are the same assumptions imposed in [5, Assumptions 1 and 2] for independent link excess lives cases.

Assumption 3: For every $x \ge 0$,[4]

$$\lim_{n\to\infty} \left(\max_{\ell=1,\ldots,H^{(n)}} G_\ell^{(n)}\left(\frac{x}{H(n)}\right) \right) = 0 .$$

A more concrete way to express Assumption 3 is as follows: For every $x \ge 0$ and any given $\epsilon > 0$, there exists an integer $n^\star = n^\star(x; \epsilon)$ such that

$$\max_{\ell=1,\ldots,H^{(n)}} G_\ell^{(n)}\left(\frac{x}{H(n)}\right) \le \epsilon, \quad n = n^\star, n^\star + 1, \ldots$$

It is clear that the interpretation of this assumption is the same as that of Assumption 2 and states that a link duration is strictly positive with probability one.

Assumption 4: (*scaling*) Let $\lambda_\ell^{(n)} = \left(\mathbf{m}(G_\ell^{(n)})\right)^{-1}$. There exists some positive constant λ such that

$$\lim_{n\to\infty} \frac{1}{H(n)} \sum_{\ell=1}^{H(n)} \lambda_\ell^{(n)} = \lambda . \tag{16}$$

Assumption 4 simply means that the link excess lives are scaled (by the average of the inverses of expected link durations divided by λ) so that we can define the parameter of the limit distribution. Under Assumption 3, one can show that Assumption 4 is equivalent to the following assumption.

Assumption 4A: There exists some positive constant λ such that, for any fixed $x \in (0, \infty)$, we have

$$\begin{aligned} \sum_{\ell=1}^{H(n)} \mathbf{P}\left[W_\ell^{(n)} > \frac{H(n)}{x}\right] &= \sum_{\ell=1}^{H(n)} \left(1 - \mathbf{J}_\ell^{(n)}\Big(\frac{H(n)}{x}\Big)\right) \\ &= \sum_{\ell=1}^{H(n)} \mathbf{P}\left[X_\ell^{(n)} < \frac{x}{H(n)}\right] \\ &\to \lambda \cdot x \qquad \text{as } n \to \infty . \end{aligned}$$

For the cases with dependent link excess lives, we introduce two additional assumptions.

Assumption 5: For any sequence $\{\hat{I}^{(n)}, n = 1, 2, \ldots\}$ of sets of consecutive positive integers, where $\hat{I}^{(n)} \subset \{1, \ldots, H(n)\}$,

$$\frac{1}{H(n)} \sum_{\ell \in \hat{I}^{(n)}} \lambda_\ell^{(n)} = O\left(\frac{|\hat{I}^{(n)}|}{H(n)}\right) .$$

A *sufficient* condition for Assumption 5 to hold is that there exists some arbitrarily small positive constant ε such that the expected link durations satisfy $\mathbf{m}(G_\ell^{(n)}) \ge \varepsilon$ for all $n = 1, 2, \ldots$ and $\ell = 1, \ldots, H(n)$. The interpretation of this assumption is that the *expected* link durations do *not* decrease to 0 with increasing network size. Since the link durations are likely to depend on the nodes' mobility and their transmission ranges but *not* directly on the network size, this is a reasonable

[3]We call a finite set of consecutive integers $\{i_1, \ldots, i_2\}$ an interval with length $i_2 - i_1 + 1$.

[4]In [5] the rvs $X_\ell^{(n)}$ are implicitly scaled by $H(n)$, while in this paper this scaling is carried out explicitly.

assumption.

Assumption 6: The array $\mathbf{W} = \{W_\ell^{(n)}, n = 1, 2, \ldots; \ell = 1, \ldots, H(n)\}$ satisfies the conditions $D(u_n)$ and $D'(u_n)$ with $u_n = \frac{H(n)}{x}$ for any $x \in (0, \infty)$.

The condition $D\left(u_n = \frac{H(n)}{x}\right)$ implies that, as n increases, the two events $\mathcal{E}_1 := \{X_{i_1}^{(n)} \geq x/H(n), \ldots, X_{i_p}^{(n)} \geq x/H(n)\}$ and $\mathcal{E}_2 := \{X_{j_1}^{(n)} \geq x/H(n), \ldots, X_{j_q}^{(n)} \geq x/H(n)\}$ become asymptotically independent, i.e., $\mathbf{P}[\mathcal{E}_1 \cap \mathcal{E}_2] - \mathbf{P}[\mathcal{E}_1] \cdot \mathbf{P}[\mathcal{E}_2] \to 0$, as the distance $j_1 - i_p$ between these two sets of link excess lives becomes larger. However, this condition holds trivially under Assumption 2 in our problem. The details are provided in Appendix II.

The condition $D'\left(u_n = \frac{H(n)}{x}\right)$ implies that the rare events $\left\{X_j^{(n)} \leq \frac{x}{H(n)}\right\}$ in a neighborhood are not strongly correlated as $n \to \infty$ (hence $\frac{x}{H(n)} \to 0$). The role and interpretation of this condition are similar to those of condition (11) in the m-dependence case (stated at the end of Section IV).

Theorem 2: Suppose that Assumptions 3 - 6 hold. Then, we have

$$\lim_{n\to\infty} \mathbf{P}\left[H(n) \cdot Z^{(n)} \leq x\right] = \begin{cases} 1 - e^{-\lambda x}, & \text{if } x > 0 \\ 0, & \text{if } x \leq 0 \end{cases} \quad (17)$$

Proof: The proof is given in Appendix III. ■

Theorem 2 states that the distribution of an h-hop path can be well approximated by an exponential distribution for all sufficiently large h. As a byproduct it also tells us that if the link duration distributions are given by $G_\ell, \ell = 1, \ldots, h$, the expected duration of the path can be approximated by $1/(\sum_{\ell=1,\ldots,h} (\mathbf{m}(G_\ell))^{-1})$. Somewhat surprisingly, the parameter of the emerging exponential distribution in (17) is the same as that of the exponential distribution with independent link excess lives [5, Theorem 2]. This holds with any *arbitrary* local dependence that may exist, and is consistent with the similar observation made in Section IV. This again suggests that the distribution of path duration is *not* significantly affected by the dependence of the reachability processes and link excess lives when the hop count is sufficiently large.

VI. An Outline of a Proposed Scheme

Detecting a link failure and finding an alternative path can take a non-negligible amount of time in practice. This is because link failures are often detected through a failure to receive/exchange a control message over a pre-determined period. When local recovery is unsuccessful after a link failure, packets queued at the originator of the failed link and additional packets on the way to the node which were to be routed using the link, will eventually be dropped by the node and must be retransmitted by their senders. These dropped packets lead to a waste of wireless resources. Moreover, losses of consecutive packets cause the transport layer protocol to back off, reducing its transmission rate. This may cause senders to rely on timeout to detect the packet losses, which can take more than a few seconds. Hence, frequent link failures along the paths in use will result in disruptions in service and degrade the performance of applications, especially that of time critical applications. For these reasons a routing algorithm should consider its expected duration in addition to other qualities (e.g., estimated available bandwidth or congestion level) when choosing a path .

In a large scale MANET the hop distance between a source and a destination is likely to be large [4]. Our results in the previous sections tell us that when hop counts are large, (i) the distribution of path duration can be well approximated by an exponential distribution and (ii) the inverse of the expected duration of a path is approximately given by the sum of the inverses of the expected durations of the links along the path. Thus, in order to approximate the expected duration of a path, a source needs to know only the *sum* of the inverses of the expected link durations.

Unfortunately, accurate estimation of the expected link durations along a path is difficult in practice. Instead, we approximate them using *average* link durations experienced by the nodes: Under our scheme each node maintains the *average* duration of the links that it establishes with other nodes. These average link durations are used as estimates to the expected link durations along a path and are provided to the source during a path discovery phase. Suppose that a node has routing information for a requested destination. Then, it generates a reply message and specifies the inverse of its estimate of the expected duration of the path to the destination in a field *inverse_path_duration_* (IPD) in the reply. A node that receives a reply message, first adds to the IPD value the inverse of its *average* of link durations, and then forwards it to the next upstream node. Finally, when the source receives the reply message, it adds the inverse of its average link duration to the IPD value. Then, the source chooses a path with the smallest IPD value, i.e., the largest estimated expected duration.

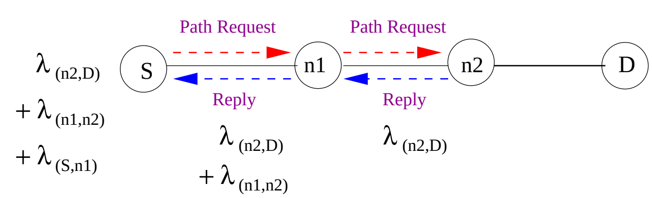

Fig. 1. An example of an estimation of expected path duration.

Let us explain this procedure using the example shown in Fig. 1. The source node S wants to find a path to destination node D and broadcasts a path request to its neighbors. Assume that node $n1$ does not have routing information for D and forwards the request to its neighbor, node $n2$. When node $n2$, a neighbor of D, receives the request, it generates a reply with the initial IPD value of $\lambda_{(n2,D)}$, which is the inverse of its average link duration. Here node $n2$'s average link duration is used as its estimate of the expected duration of the link with D. Then, it forwards the reply to node $n1$. Upon receiving the reply, node $n1$ adds the inverse $\lambda_{(n1,n2)}$ of its average link duration to the IPD value and forwards the reply to source node S. Again, node $n1$'s average link duration is used in place of the expected duration of the link with $n2$. When S receives the reply, it first adds $\lambda_{(S,n1)}$ to the IPD value of $\lambda_{(n2,D)} + \lambda_{(n1,n2)}$ in the reply. Then, it uses the inverse of the final IPD value as an estimate of the expected duration of the discovered path $\{(S, n1), (n1, n2), (n2, D)\}$. As only the *sum* of the inverses of average link durations is collected, this proposed modification can be easily implemented in available on-demand routing algorithms with minimal overhead.

It is also possible with our scheme to classify neighbors with whom nodes establish links and to maintain a *separate* average link duration for each type of neighbors. The reason for maintaining separate averages is as follows. A large scale MANET is likely to comprise many different types of nodes. For example, a Future Combat System (FCS) will include different types of vehicles (e.g., jeeps, tanks, etc.), soldiers, and possibly aerial vehicles. Clearly, the duration of a link between two nodes will depend on their mobility and capabilities. Thus, the durations of links a node sets up with its neighbors over time will be dependent on their types, i.e., their mobility and capabilities, as well as its own type.

VII. Conclusion

We studied the issue of designing a scheme for selecting paths with the largest expected durations with the aim of providing reliable network services in MANETs. To this end we first investigated the distributional properties of path duration in multi-hop wireless networks. We extended the results in [5] and proved that, under certain conditions, the distribution of path duration (appropriately scaled) converges to an exponential distribution as the number of hops increases even when link excess lives are not mutually independent. Moreover, we showed that under the given conditions, the parameter of the emerging exponential distribution is not affected by the dependence of the link excess lives. Based on these results we proposed a new scheme that can be easily incorporated into existing routing protocols. The required information under our scheme can be be piggybacked in reply messages, introducing only minimal communication overhead.

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Appendix I
Proof of Theorem 1

The proof of the theorem follows directly from the theorem in [20]: Define $W_\ell = (X_\ell)^{-1}, \ell = 1, 2, \ldots$. Then, $\{W_\ell, \ell = 1, 2, \ldots\}$ is a sequence of rvs unbounded above that satisfies the m-dependence assumption and

$$\lim_{c \uparrow \infty} \frac{1}{\mathbf{P}\left[W_i > c\right]} \max_{|i-j| \leq m} \mathbf{P}\left[W_i > c, W_j > c\right] = 0$$

from (11).

Fix $x > 0$ and let $c_n(x) = \frac{n}{x}, n = 1, 2, \ldots$. First, note that

$$\begin{aligned}
&\lim_{n \to \infty} n \cdot \mathbf{P}\left[W_1 > \frac{n}{x \cdot \mathbf{m}(G)}\right] \\
&= \lim_{n \to \infty} n \cdot \mathbf{P}\left[X_1 < \frac{x \cdot \mathbf{m}(G)}{n}\right] \\
&= \lim_{n \to \infty} n \cdot \frac{1}{\mathbf{m}(G)} \int_0^{\frac{x \cdot \mathbf{m}(G)}{n}} (1 - G(y))\ dy \qquad \text{(from (10))} \\
&= \lim_{n \to \infty} \frac{n}{\mathbf{m}(G)} \left(\frac{x \cdot \mathbf{m}(G)}{n} + o\left(\frac{1}{n}\right)\right) \\
&= x\ .
\end{aligned}$$

Since the conditions in the theorem in [20, pp. 798] are satisfied by $\{W_i, i = 1, 2, \ldots\}$,

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\left[W_i \le \frac{n}{x}; i = 1, \ldots, n\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[X_i \ge \frac{x}{n}; i = 1, \ldots, n\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[\min(X_i : i = 1, \ldots, n) \ge \frac{x}{n}\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[n \cdot \min(X_i; i = 1, \ldots, n) \ge n \cdot \frac{x}{n}\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[n \cdot \min(X_i; i = 1, \ldots, n) \ge x\right] \\
&= \exp(-\frac{x}{\mathbf{m}(G)}) \qquad (18) \\
&= \exp(-\lambda x) \ .
\end{aligned}$$

where (18) follows from the theorem in [20].

Appendix II
Condition $D(u_n)$

First, recall the definition of the events.

$$\begin{aligned}
\mathcal{E}_1 &= \{X_{i_1} \ge x/H(n), \ldots, X_{i_p} \ge x/H(n)\} \\
\mathcal{E}_2 &= \{X_{j_1} \ge x/H(n), \ldots, X_{j_q} \ge x/H(n)\} \ ,
\end{aligned}$$

From the well known bounds, we have

$$\begin{aligned}
\mathbf{P}[\mathcal{E}_1] &\ge 1 - \sum_{k=1}^{p} \mathbf{P}\left[X_{i_k}^{(n)} < x/H(n)\right] \\
\mathbf{P}[\mathcal{E}_2] &\ge 1 - \sum_{l=1}^{q} \mathbf{P}\left[X_{j_l}^{(n)} < x/H(n)\right] \qquad (19) \\
\mathbf{P}[\mathcal{E}_1 \cap \mathcal{E}_2] &\ge 1 - \sum_{k=1}^{p} \mathbf{P}\left[X_{i_k}^{(n)} < x/H(n)\right] \\
&\quad - \sum_{l=1}^{q} \mathbf{P}\left[X_{j_l}^{(n)} < x/H(n)\right] \ .
\end{aligned}$$

Note that both $\sum_{k=1}^{p} \mathbf{P}\left[X_{i_k}^{(n)} < x/H(n)\right]$ and $\sum_{l=1}^{q} \mathbf{P}\left[X_{j_l}^{(n)} < x/H(n)\right]$ go to 0 as $n \to \infty$ because $x/H(n) \to 0$. Therefore, the lower bounds in (19) converge to 1. Therefore, it is clear that $|\mathbf{P}[\mathcal{E}_1 \cap \mathcal{E}_2] - \mathbf{P}[\mathcal{E}_1]\mathbf{P}[\mathcal{E}_2]| \to 0$ as $n \to \infty$.

Appendix III
Proof of Theorem 2

In order to prove the theorem, we show that, for any fixed $x \in (0, \infty)$,

$$\lim_{n\to\infty} \mathbf{P}\left[H(n) \cdot Z^{(n)} > x\right] = \exp(-\lambda x) \ . \qquad (20)$$

To prove (20) we show the following equivalent statement.

$$\lim_{n\to\infty} \mathbf{P}\left[\max_{\ell=1,\ldots,H(n)} W_\ell^{(n)} < \frac{H(n)}{x}\right] = \exp(-\lambda x) \qquad (21)$$

from the equality

$$\begin{aligned}
&\lim_{n\to\infty} \mathbf{P}\left[H(n) \cdot Z^{(n)} > x\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[\min_{\ell=1,\ldots,H(n)} X_\ell^{(n)} > \frac{x}{H(n)}\right] \\
&= \lim_{n\to\infty} \mathbf{P}\left[\max_{\ell=1,\ldots,H(n)} W_\ell^{(n)} < \frac{H(n)}{x}\right] \ .
\end{aligned}$$

Before doing so, we first need to introduce some notation used in the proof. Let E be a set of positive integers. We define $M^{(n)}(E) := \max(W_j^{(n)} : j \in E)$. If $E = \{j_1, \ldots, j_2\}$ and $E' = \{j_1', \ldots, j_2'\}$ are two intervals with $j_1' > j_2$, we say that E and E' are separated by $j_1' - j_2$.

Let k be a fixed positive integer. For each $n = 1, 2, \ldots$, we first divide the interval $\{1, \ldots, H(n)\}$ into $k+1$ consecutive disjoint subintervals as done Section V-A. Then, we further divide each of the first k subintervals into two disjoint subintervals: Let $n' := \lfloor H(n)/k \rfloor$. For $j = 1, \ldots, k$, define

$$I_{k,j}^{(n)} = \{(j-1) \cdot n' + 1, \ldots, j \cdot n'\} \ ,$$

and

$$I_{k,k+1}^{(n)} = \{k \cdot n' + 1, \ldots, H(n)\} \ .$$

Let $\{m(n), n = 1, 2, \ldots\}$ be a sequence of integers such that, for all sufficiently large n, $k < m(n) < n'$,

$$\lim_{n\to\infty} m(n) = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{m(n)}{H(n)} = 0 \ .$$

For $j = 1, \ldots, k$, we divide the subinterval $I_{k,j}^{(n)}$ into the following two disjoint subintervals.

$$\begin{aligned}
\underline{I}_{k,j}^{(n)} &= \{(j-1) \cdot n' + 1, \ldots, j \cdot n' - m(n)\} \\
\text{and} \quad \bar{I}_{k,j}^{(n)} &= \{j \cdot n' - m(n) + 1, \ldots, j \cdot n'\} \ .
\end{aligned}$$

It is clear that $|\underline{I}_{k,j}^{(n)}| = n' - m(n)$ and $|\bar{I}_{k,j}^{(n)}| = m(n)$.

We denote $M^{(n)}(I_{k,j}^{(n)})$, $j = 1, \ldots, k$, by $M_{k,j}^{(n)}$ for notational convenience. To prove (21) we will first show

$$\lim_{n\to\infty} \prod_{j=1}^{k(n)} \mathbf{P}\left[M_{k(n),j}^{(n)} \le u_n\right] = \exp(-\lambda x) \ ,$$

where $u_n = H(n)/x$ from Section V-B. Then, we will prove

$$\begin{aligned}
&\left|\mathbf{P}\left[\max_{\ell=1,\ldots,H(n)} W_\ell^{(n)} \le u_n\right] - \prod_{j=1}^{k(n)} \mathbf{P}\left[M_{k(n),j}^{(n)} \le u_n\right]\right| \\
&\quad \to 0 \qquad \text{as } n \to \infty \ .
\end{aligned}$$

From the definition,

$$\{M_{k,j}^{(n)} > u_n\} = \bigcup_{i \in I_{k,j}^{(n)}} \{W_i^{(n)} > u_n\} \qquad \text{for } j = 1, \ldots, k \ .$$

Hence, we have the following lower and upper bounds.

$$\begin{aligned}
&\sum_{i \in I_{k,j}^{(n)}} \mathbf{P}\left[W_i^{(n)} > u_n\right] \\
&\qquad - \sum_{i,i' \in I_{k,j}^{(n)}, i \ne i'} \mathbf{P}\left[W_i^{(n)} > u_n, W_{i'}^{(n)} > u_n\right] \\
&\le \ \mathbf{P}\left[M_{k,j}^{(n)} > u_n\right] \qquad (22) \\
&\le \ \sum_{i \in I_{k,j}^{(n)}} \mathbf{P}\left[W_i^{(n)} > u_n\right]
\end{aligned}$$

From these bounds in (22) we obtain

$$\prod_{j=1}^{k}\left(1-\sum_{i\in I_{k,j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\right)$$
$$\leq\prod_{j=1}^{k}\left(1-\mathbf{P}\left[M_{k,j}^{(n)}>u_n\right]\right)=\prod_{j=1}^{k}\mathbf{P}\left[M_{k,j}^{(n)}\leq u_n\right]$$
$$\leq\prod_{j=1}^{k}\left(1-\sum_{i\in I_{k,j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\right. \quad (23)$$
$$\left.+\sum_{i,i'\in I_{k,j}^{(n)},i\neq i'}\mathbf{P}\left[W_i^{(n)}>u_n,W_{i'}^{(n)}>u_n\right]\right)\ .$$

Now take a sequence $\{k(n),n=1,2,\ldots\}$ of positive integers such that (i) $\lim_{n\to\infty}k(n)=\infty$, (ii) $\lim_{n\to\infty}\frac{k(n)}{m(n)}=0$, (iii) $\lim_{n\to\infty}k(n)\cdot\alpha_{n,m(n)}=0$, and (iv) $\lim_{n\to\infty}\frac{m(n)\cdot k(n)}{H(n)}=0$. The existence of such a sequence is guaranteed when the condition $D(u_n=\frac{H(n)}{x})$ holds. We can show that

$$\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\to 0 \qquad \text{from Assumption 5} \quad (24)$$

and

$$\sum_{j=1}^{k(n)}\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\to\lambda\cdot x \quad (25)$$

from Assumptions 4A and 5. The first claim in (24) can be proved as follows.

$$\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]=\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[X_i^{(n)}<\frac{x}{H(n)}\right]$$
$$\leq\sum_{i\in I_{k(n),j}^{(n)}}\lambda_i^{(n)}\cdot\frac{x}{H(n)} \qquad \text{from (8)}$$
$$=x\cdot O\left(\frac{|I_{k(n),j}^{(n)}|}{H(n)}\right) \qquad \text{from Assumption 5}$$
$$=x\cdot O\left(\frac{\lfloor H(n)/k(n)\rfloor}{H(n)}\right)$$
$$=x\cdot O\left(\frac{1}{k(n)}\right)$$

Since $k(n)\to\infty$ as $n\to\infty$, the claim (24) follows.

We first state a well known convergence result without a proof. We will make use of it shortly.

Lemma 1: Consider an array $\{c_{n,i},n=1,2,\ldots;i=1,2,\ldots,k(n)\}$ of non-negative real numbers, where $c_{n,i}<1$ and $\lim_{n\to\infty}k(n)=\infty$. Suppose that $\max_{i=1,\ldots,k(n)}c_{n,i}\to 0$ and $\sum_{i=1}^{k(n)}c_{n,i}\to\lambda$ as $n\to\infty$. Then, the following holds.

$$\lim_{n\to\infty}\prod_{i=1}^{k(n)}(1-c_{n,i})=\exp(-\lambda)$$

Eqs. (24) - (25) and Lemma 1 imply

$$\lim_{n\to\infty}\prod_{j=1}^{k(n)}\left(1-\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\right)=\exp(-\lambda x)\ . \quad (26)$$

By the same argument, we also have

$$\lim_{n\to\infty}\prod_{j=1}^{k(n)}\left(1-\sum_{i\in I_{k(n),j}^{(n)}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\right.$$
$$\left.+\sum_{i,i'\in I_{k(n),j}^{(n)},i\neq i'}\mathbf{P}\left[W_i^{(n)}>u_n,W_{i'}^{(n)}>u_n\right]\right)$$
$$=\exp(-\lambda x) \quad (27)$$

because Assumption 6 (more specifically condition $D'(u_n=\frac{H(n)}{x})$) tells us

$$\sum_{i,i'\in I_{k(n),j}^{(n)},i\neq i'}\mathbf{P}\left[W_i^{(n)}>u_n,W_{i'}^{(n)}>u_n\right]\right)=o\left(\frac{1}{k(n)}\right)\ .$$

Since both the lower and upper bounds in (23) converge to $\exp(-\lambda x)$ from (26) and (27), we obtain

$$\lim_{n\to\infty}\prod_{j=1}^{k(n)}\left(\mathbf{P}\left[M_{k(n),j}^{(n)}\leq u_n\right]\right)=\exp(-\lambda x)\ . \quad (28)$$

We introduce a lemma used to complete the proof of the theorem. The proof of the lemma is provided in Appendix IV.

Lemma 2: For the sequence $m(n),n\geq 1$, satisfying the condition $D(u_n=\frac{H(n)}{x})$ and $k(n)$ satisfying the aforementioned conditions, we have

$$\lim_{n\to\infty}\left|\mathbf{P}\left[M_n\leq u_n\right]-\prod_{j=1}^{k(n)}\mathbf{P}\left[M_{k(n),j}^{(n)}\leq u_n\right]\right|=0\ ,$$

where $M_n:=\max(W_1^{(n)},\ldots,W_{H(n)}^{(n)})$.

Eq. (28) and Lemma 2 now tell us

$$\lim_{n\to\infty}\mathbf{P}\left[H(n)\cdot Z^{(n)}>x\right]$$
$$=\lim_{n\to\infty}\mathbf{P}\left[\max_{\ell=1,\ldots,H(n)}W_\ell^{(n)}\leq\frac{H(n)}{x}\right]$$
$$=\lim_{n\to\infty}\mathbf{P}\left[M_n\leq u_n\right]$$
$$=\lim_{n\to\infty}\prod_{j=1}^{k(n)}\mathbf{P}\left[M_{k(n),j}^{(n)}\leq u_n\right]$$
$$=\exp(-\lambda x)\ ,$$

and the theorem follows.

Appendix IV
Proofs of Lemma 2

We first introduce some auxiliary results used to prove the lemma.

Lemma 3: Suppose that $\mathbf{A}=\{\alpha_{n,m},n=1,2,\ldots;m=1,\ldots,H(n)\}$ is an array of non-negative real numbers that satisfies condition (13). Let n, r, and m be fixed positive integers and $E_1,\ldots,E_r$ subintervals of $\{1,\ldots,H(n)\}$ such that any two subintervals E_i and E_j, $i\neq j$, are separated by at least m. Then, we have

$$\left|\mathbf{P}\left[\bigcap_{j=1}^{r}\{M^{(n)}(E_j)\le u_n\}\right]-\prod_{j=1}^{r}\mathbf{P}\left[M^{(n)}(E_j)\le u_n\right]\right| \le (r-1)\cdot\alpha_{n,m}\ .$$

Proof: For notational convenience, we write $A_j^{(n)}=\{M^{(n)}(E_j)\le u_n\}$. Let $E_j=\{k_j,\ldots,l_j\}$, where $k_1\le l_1<k_2\le\ldots\le l_r$. Then, since $k_2-l_1\ge m$, we get

$$\begin{aligned}&\left|\mathbf{P}\left[A_1^{(n)}\cap A_2^{(n)}\right]-\mathbf{P}\left[A_1^{(n)}\right]\mathbf{P}\left[A_2^{(n)}\right]\right|\\&=\left|\mathbf{J}^{(n)}_{k_1\ldots l_1,k_2\ldots l_2}(u_n)-\mathbf{J}^{(n)}_{k_1\ldots l_1}(u_n)\mathbf{J}^{(n)}_{k_2\ldots l_2}(u_n)\right|\\&\le\alpha_{n,m}\ .\end{aligned}$$

By the same argument

$$\begin{aligned}&\left|\mathbf{P}\left[A_1^{(n)}\cap A_2^{(n)}\cap A_3^{(n)}\right]-\mathbf{P}\left[A_1^{(n)}\right]\mathbf{P}\left[A_2^{(n)}\right]\mathbf{P}\left[A_3^{(n)}\right]\right|\\&\le\left|\mathbf{P}\left[A_1^{(n)}\cap A_2^{(n)}\cap A_3^{(n)}\right]-\mathbf{P}\left[A_1^{(n)}\cap A_2^{(n)}\right]\mathbf{P}\left[A_3^{(n)}\right]\right|\\&\quad+\left|\mathbf{P}\left[A_1^{(n)}\cap A_2^{(n)}\right]-\mathbf{P}\left[A_1^{(n)}\right]\mathbf{P}\left[A_2\right]\right|\cdot\mathbf{P}\left[A_3\right]\\&\le 2\ \alpha_{n,m}\end{aligned}$$

since $E_1\cup E_2\subseteq\{k_1,\ldots,l_2\}$ and $k_3-l_2\ge m$. By applying the same argument repeatedly, the lemma follows. ∎

Lemma 4: Suppose that the condition $D(u_n)$ holds. For any fixed k, the following statements hold. A proof is provided in Appendix V:

(i)

$$\begin{aligned}0&\le\mathbf{P}\left[\bigcap_{j=1}^{k}\{M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n\}\right]-\mathbf{P}\left[M_n\le u_n\right]\\&\le\sum_{j=1}^{k}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k,j})\right]\\&\quad+\mathbf{P}\left[u_n<M^{(n)}(I^{(n)}_{k,k+1})\right]\ ,\end{aligned}\tag{29}$$

(ii)

$$\left|\mathbf{P}\left[\bigcap_{j=1}^{k}\left\{M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n\right\}\right]-\prod_{j=1}^{k}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n\right]\right| \le(k-1)\cdot\alpha_{n,m(n)}\ ,$$

(iii)

$$\begin{aligned}&\prod_{j=1}^{k}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n\right]-\prod_{j=1}^{k}\mathbf{P}\left[M^{(n)}_{k,j}\le u_n\right]\\&\le\prod_{j=1}^{k}\left(1+\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k,j})\right]\right)-1\ .\end{aligned}$$

Lemma 5: Suppose that condition $D(u_n=\frac{H(n)}{x})$ holds. Let $\{k(n),n=1,2,\ldots\}$ be a sequence that satisfies the conditions in Appendix III. Then, for every $j=1,2,\ldots$,

$$\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right]=o\left(\frac{1}{k(n)}\right)\ ,$$

for all sufficiently large n.

Proof: Under Assumption 5,

$$\begin{aligned}&\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right]\\&=\mathbf{P}\left[\Big(\bigcap_{i\in\underline{I}^{(n)}_{k(n),j}}\{W_i^{(n)}\le u_n\}\Big)\bigcap\Big(\bigcup_{i\in\bar{I}^{(n)}_{k(n),j}}\{W_i^{(n)}>u_n\}\Big)\right]\\&\le\sum_{i\in\bar{I}^{(n)}_{k(n),j}}\mathbf{P}\left[W_i^{(n)}>u_n\right]\\&=O\left(\frac{|\bar{I}^{(n)}_{k(n),j}|}{H(n)}\right)\qquad\text{from Assumption 5}\\&=O\left(\frac{m(n)}{H(n)}\right)\\&=\frac{1}{k(n)}\cdot O\left(\frac{m(n)\cdot k(n)}{H(n)}\right)\\&=o\left(\frac{1}{k(n)}\right)\end{aligned}$$

where the last equality follows from the assumption $m(n)\cdot k(n)=o(H(n))$. ∎

We now proceed with the proof of Lemma 2. First, by rewriting the difference, the following bound holds.

$$\begin{aligned}&\left|\mathbf{P}\left[M_n\le u_n\right]-\prod_{j=1}^{k(n)}\mathbf{P}\left[M^{(n)}_{k(n),j}\le u_n\right]\right|\\&\le\left|\mathbf{P}\left[M_n\le u_n\right]-\mathbf{P}\left[\bigcap_{j=1}^{k(n)}\left\{M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n\right\}\right]\right|\\&\quad+\left|\mathbf{P}\left[\bigcap_{j=1}^{k(n)}\left\{M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n\right\}\right]\right.\\&\qquad\left.-\prod_{j=1}^{k(n)}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n\right]\right|\\&\quad+\left|\prod_{j=1}^{k(n)}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n\right]\right.\\&\qquad\left.-\prod_{j=1}^{k(n)}\mathbf{P}\left[M^{(n)}_{k(n),j}\le u_n\right]\right|\ .\end{aligned}\tag{30}$$

We now upper bound each term in (30) using the bounds derived in Lemma 4.

$$\begin{aligned}(30)&\le\sum_{j=1}^{k(n)}\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right]\\&\quad+\mathbf{P}\left[u_n<M^{(n)}(I^{(n)}_{k(n),k(n)+1})\right]\\&\quad+(k(n)-1)\cdot\alpha_{n,m(n)}\\&\quad+\prod_{j=1}^{k(n)}\left(1+\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j})\le u_n<M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right]\right)\\&\quad-1\ .\end{aligned}\tag{31}$$

In order to complete the proof, it suffices to show that (31) converges to 0: First, note that

$$\begin{aligned}
&\mathbf{P}\left[u_n < M^{(n)}(I^{(n)}_{k(n),k(n)+1})\right] \\
&= \mathbf{P}\left[\bigcup_{i\in I^{(n)}_{k(n),k(n)+1}} \left\{W_i^{(n)} > u_n\right\}\right] \\
&\leq \sum_{i\in I^{(n)}_{k(n),k(n)+1}} \mathbf{P}\left[W_i^{(n)} > u_n\right] \qquad (32)\\
&\to 0
\end{aligned}$$

from Assumption 5 because

$$\limsup_{n\to\infty} \frac{|I^{(n)}_{k(n),k(n)+1}|}{H(n)} \leq \limsup_{n\to\infty} \frac{k(n)-1}{H(n)} = 0 \ .$$

Second, Lemma 5 tells us

$$\begin{aligned}
&\lim_{n\to\infty} \sum_{j=1}^{k(n)} \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right] \\
&= \lim_{n\to\infty} \sum_{j=1}^{k(n)} o\Big(\frac{1}{k(n)}\Big) \qquad (33)\\
&= 0 \ .
\end{aligned}$$

Similarly, Lemmas 1 and 5 and (33) imply

$$\begin{aligned}
&\lim_{n\to\infty} \prod_{j=1}^{k(n)} \Big(1 + \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k(n),j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k(n),j})\right]\Big) \\
&= \lim_{n\to\infty} \prod_{j=1}^{k(n)} \Big(1 + o\Big(\frac{1}{k(n)}\Big)\Big) \qquad (34)\\
&= 1 \ .
\end{aligned}$$

From (32) - (34) with the assumption $\lim_{n\to\infty} k(n) \cdot \alpha_{n,m(n)} = 0$, the right hand side of (31) goes to 0 as $n \to \infty$. This completes the proof of Lemma 2.

Appendix V
Proof of Lemma 4

Claim (i) of Lemma 4 follows from the observation that

$$\{M_n \leq u_n\} \subset \bigcap_{j=1}^{k} \{M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n\}$$

and their difference is given by the event

$$\begin{aligned}
&\Big(\bigcup_{j=1}^{k} \Big\{M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k,j})\Big\}\Big) \\
&\qquad \bigcup \{u_n < M^{(n)}(I^{(n)}_{k,k+1})\} \ . \qquad (35)
\end{aligned}$$

The probability of the event in (35) can be bounded using the union bound in (29).

Claim (ii) follows directly from Lemma 3 by replacing E_j with $\underline{I}^{(n)}_{k,j}$.

In order to prove claim (iii), we first note that, for $j = 1, \ldots, k$,

$$\begin{aligned}
&\mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n\right] - \mathbf{P}\left[M^{(n)}_{k,j} \leq u_n\right] \\
&= \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k,j})\right] \ . \qquad (36)
\end{aligned}$$

Now the claim follows from

$$\begin{aligned}
&\prod_{j=1}^{k} \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n\right] - \prod_{j=1}^{k} \mathbf{P}\left[M^{(n)}_{k,j} \leq u_n\right] \\
&= \prod_{j=1}^{k} \Big(\mathbf{P}\left[M^{(n)}_{k,j} \leq u_n\right] \\
&\qquad + \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k,j})\right]\Big) \\
&\quad - \prod_{j=1}^{k} \mathbf{P}\left[M^{(n)}_{k,j} \leq u_n\right] \\
&\leq \prod_{j=1}^{k} \Big(1 + \mathbf{P}\left[M^{(n)}(\underline{I}^{(n)}_{k,j}) \leq u_n < M^{(n)}(\bar{I}^{(n)}_{k,j})\right]\Big) - 1 \ ,
\end{aligned}$$

where the first equality follows from (36) and the inequality holds because $\mathbf{P}\left[M^{(n)}_{k,j} \leq u_n\right] \leq 1$.