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Finite Gain l_p Stabilization Requires Analog Control

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Abstract

A causal feedback map, taking sequences of measurements and producing sequences of controls, is denoted as finite-set if its range is in a finite set. Bit-rate constrained or digital control are particular cases of finite-set feedback. In this paper, we show that the finite gain (FG) l_p stabilization, with $1 \leq p \leq \infty$, of a discrete-time, linear and time-invariant unstable plant is impossible by finite-set feedback. In addition, we show that, under finite-set feedback, weaker (local) versions of FG l_p stability are also impossible. These facts are not obvious, since recent results have shown that input to state stabilization (ISS) is viable by bit-rate constrained control. In view of such existing work, this paper leads to two conclusions: (1) in spite of ISS stability being attainable under finite-set feedback, small changes in the *amplitude* of the external excitation may cause, in relative terms, a large increase in the *amplitude* of the state (2) FG l_p stabilization requires logarithmic precision around zero. Since our conclusions hold with no assumption on the feedback structure, they cannot be derived from existing results. We adopt an information theoretic viewpoint, which also brings new insights into the problem of stabilization.

Key words: Digital Control, Limited Information, Bit Rate Constraints, Input to State Stability, Finite Gain Stability, Fundamental Limits

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1 Introduction

Consider the following feedback system:

$$X(k+1) = AX(k) + \mathcal{F}(X^k, k) + W(k),\ X(0) = 0,\ k \in \mathbb{N}_+ \tag{1}$$

where $W(k) \in \mathbb{R}^n$ represents the input, $X(k) \in \mathbb{R}^n$, $X^k = (X(0), \dots, X(k))$, $A \in \mathbb{R}^{n \times n}$ and $\mathcal{F}(\cdot, k) : \mathbb{R}^{n \times (k+1)} \to \mathbb{R}$ represents a feedback strategy.

Definition 1.1 ***(Finite-set feedback)*** *Let $\mathcal{F}^k$ be defined as:*

$$\mathcal{F}^k(X^k) = \left(\mathcal{F}(X(0), 0), \dots, \mathcal{F}(X^k, k)\right), X(k) \in \mathbb{R}^n, k \in \mathbb{N}_+$$

We say that (1) has ***finite-set feedback*** *if, for every $k \in \mathbb{N}_+$, the range of $\mathcal{F}^k$ lies in a finite set. The problem of stabilization under bit-rate constrained feedback is a sub-class of our framework, where, for a given $R \in \mathbb{R}_+$, the range of $\mathcal{F}^k$ has at most $2^{(k+1)R}$ elements.*

Stabilization under finite-set feedback involves, implicitly, quantization. The work in [3] has motivated the careful study of the effects of quantization in feedback, where it is shown that the naive quantization noise model is not appropriate. The formulation in [3] adopts a discrete-time, time-invariant, memory-less and finite valued quantization of the state, under which it is shown that asymptotic internal stabilization is impossible. The analysis in [2], gives a complete solution to the problem of finding a quadratic control Lyapunov function (QCLF) in the presence of memoryless quantization of either the state, or of the observation estimation error (output feedback). In [2], it is shown that, under the aforementioned framework, the existence of a QCLF requires a quantizer with an infinite number of levels, whose resolution increases logarithmically around zero. On the other hand, it is reported in [1] that, by allowing analog processing before and after quantization, global asymptotic stability can be achieved in the presence of bit-rate constraints. A meticulous analysis of internal stabilization for discrete-time linear systems, in the presence of memoryless piecewise non-linearities, is given in [4]. The stabilization of nonlinear systems is studied in [9].

Most external stability bounds, for bit-rate constrained feedback, assume that the amplitude of the external excitation is known [5], [17], [8], [14],[6]. Therefore, in all of the aforementioned publications, the notions of stability are not compatible with finite gain (FG) l_p , nor with the more general notion of input to state stability (ISS) [13]. Recently, the authors of [10] have addressed this issue, by devising a bit-rate constrained feedback scheme that guarantees stabilization in the ISS sense. In order to attain ISS, the controller must not depend on prior knowledge of the amplitude of the external excitation. In addition, ISS guarantees that the amplitude of the state decreases, as the amplitude of the external signals decreases. However, the sensitivity, in terms of how the state is amplified with respect to the external excitation, has to be characterized using *gain* notions such as FG l_p stability, where $1 \leq p \leq \infty$. These facts have motivated the investigation reported in this paper, i.e., the derivation of necessary conditions for FG l_p stabilization.

Regarding the framework, the approaches in [2], [3] and [4] are significantly different from [1], [5], [17], [8] and [14]. The former addresses stabilization, under a given class of quantization schemes, while the latter is about control with bit-rate constraints. Each approach has its own motivation: specific quantization schemes are well suited for modeling measurement resolution, while bit-rate constraints describe an information-rate bottleneck in the feedback loop. It is important to make this distinction because necessary conditions for stabilization, derived for a given class of quantization schemes, cannot be used in deriving necessary conditions in terms of bit-rate constraints. For instance, [1] achieves global asymptotic stabilization by bit-rate constrained feedback, while in the scheme of [3] the state trajectory always converges to a chaotic orbit.

Our contribution is to show that FG l_p stabilization in not possible by finite-set feedback, and that includes bit-rate constrained control as a particular case. In addition, it follows from our analysis that finite-set feedback also rules out weaker (local) versions of FG l_p stabilization, and that, even though ISS is achievable [10], the amplitude of the state may increase arbitrarily with only a small change in the amplitude of the external excitation. The concept of logarithmic resolution was introduced in [2] for a class of quantization schemes. Our work comes in support of such fundamental notion, by proving that, regardless of the quantization scheme, the aforementioned weaker (local) versions of FG l_p stability also requires logarithmic resolution. Any quantization scheme requiring logarithmic resolution is not implementable in practice [1] and, for that reason, it introduces further limits to stability, even in the absence of bit-rate constraints. Our conclusions cannot be derived from existing results because they hold with no assumptions on the feedback structure. In particular, we allow arbitrary analog or digital pre-quantization processing (encoding) as well as post-quantization processing (decoding). In addition, we allow quantizers which may be time-varying and have infinite memory, or no quantizer at all. We use standard properties of information theory, which makes our proofs short and very general.

This paper has four sections. Section 2 discusses, without proofs, the necessary conditions for FG l_p stability and its implications on ISS, while section 3 contains the detailed proofs. Section 4 finalizes the paper with conclusions.

[1] For instance, logarithmic resolution can be achieved by non-linear gains before and after uniform quantization, without amplitude constraints. On the other hand, such non-linear gains will *explode* around zero.

We adopt the following notation: Complex (or real) variables are represented by small caps letters, while vectors use large caps letters, such as $Z \in \mathbb{C}^n$, where the element at the i-th coordinate is presented as Z_i. Exception to this rule is A, which is used to denote the dynamic matrix of the state space representation in (1). Sequences of complex (or real) variables are indicated as $z^k = (z(0), \ldots, z(k))$, $k \in \mathbb{N}_+ \bigcup \{\infty\}$. Similarly, a sequence of vectors is represented as $Z^k = (Z(0), \ldots, Z(k))$, $k \in \mathbb{N}_+ \bigcup \{\infty\}$. The absolute value is given by $|z| = \sqrt{Re\{z\}^2 + Im\{z\}^2}$. The p-norm of a vector $Z \in \mathbb{C}^n$ is defined as $\|Z\|_p = (\sum_{i=1}^n |Z_i|^p)^{\frac{1}{p}}$. Likewise, the ∞-norm of Z is computed as $\|Z\|_\infty = \max_{i \in \{1,\ldots,n\}} |Z_i|$. Infinite complex (real) sequences are indicated as $\vec{z} = (z(0), z(1), \ldots)$, while infinite vector sequences are represented as $\vec{Z} = (Z(0), Z(1), \ldots)$. The l_p norm of an infinite sequence is defined as $\|\vec{Z}\|_p = \left(\sum_{i=0}^\infty \|Z(i)\|_p^p\right)^{\frac{1}{p}}$ and the l_∞ norm is given by $\|\vec{Z}\|_\infty = \sup_{k \geq 0} \|Z(i)\|_\infty$. Complex (or real) random variables and vectors are represented by bold face letters, such as **z** and **Z**. With the exception of $\mathcal{K}$ (reserved), functions and maps are represented in calligraphic font, e.g., $\mathcal{Q}$. We denote $\mathbb{R}_+ \bigcup \{\infty\}$ as $\bar{\mathbb{R}}_+$. We also adopt the convention $0 \log_2 0 = 0$.

2 Necessary Conditions for FG l_p stability

In this section, we explain why FG l_p stabilization cannot be achieved with finite-set feedback. In addition, we define a weaker (local) version of FG stabilization, which we prove is also not possible by finite-set feedback. At a later point, in section 3, we prove a stronger result stating that logarithmic resolution is needed for such weak notion of stability. The implications of our results, in input to state stability (ISS), are discussed at the end of this section.

The following are reasons why feedback may be finite-set. (1) If the feedback loop comprises a uniform quantizer with amplitude constraints then the feedback is finite set. Notice that, without amplitude constraints, a uniform quantizer has infinite range. (2) Another case of finite set feedback is when the controller is implemented by a dynamical system operating on a finite alphabet, such as a digital computer. (3) Furthermore, control over a bit-rate constrained network is necessarily finite set. Notice that the feedback is finite-set, even if $\lim_{k\to\infty} r(k) = \infty$, where $r(k)$ is the maximum number of bits delivered at time instant k. This scenario is specially relevant to remote control applications, where information can be reliably transmitted only at a finite rate. Besides being finite, the rate of transmission might also be low due to security reasons, because of the communication medium (under-water missions) or in the presence of fading.

Our results hold for the following parameterized notion of stability:

Definition 2.1 *(**(ϵ, δ) FGI stability**) Let $\vec{X}$ be the solution of (1) and the constants $\epsilon, \delta \in \bar{\mathbb{R}}_+$ be given. The system represented by (1) is (ϵ, δ) FGI (finite gain internally) stable, if the following holds:*

$$\exists k_{min} > 0,\ \mathcal{G}(k_{min}, \epsilon, \delta) \overset{def}{=} \sup_{k > k_{min}} \left(\sup_{\vec{W} \in \mathbb{D}_{\epsilon,\delta}} \frac{\|X(k)\|_\infty}{\|W(0)\|_\infty} \right) < \infty \tag{2}$$

where $\mathbb{D}_{\epsilon,\delta} \overset{def}{=} \{\vec{W} \in \mathbb{R}^{n\times\infty} : 2^{-\epsilon} < \|W(0)\|_\infty < 2^\delta$ *and* $\forall k \geq 1, W(k) = 0\}$.

The following Theorem represents one of the main results of this paper.

Theorem 2.1 *Assume that the dynamical system represented by (1) has a non-Hurwitz (unstable) matrix A. In addition, consider the following conditions: **(C1)** there exists a real and positive δ such that (1) is (∞, δ) FGI stable; **(C2)** there exists a real and positive ϵ such that (1) is (ϵ, ∞) FGI stable; **(C3)** (1) is (∞, ∞) FGI stable . If at least one of these conditions holds, then there exists k_{min} such that the range of $\mathcal{F}^{k_{min}}$ is an infinite set.*

Proof: The proof follows from the results in section 3, more specifically, it is a direct consequence of corollaries 3.3, 3.4 and Remark 3.1. □

Definition 2.2 *The feedback system specified by (1) is FG l_p stable, if the following holds:*

$$\sup_{\vec{W} \in \mathbb{R}^{n \times \infty} - \{0\}} \frac{\|\vec{X}\|_p}{\|\vec{W}\|_p} = \beta_p < \infty \tag{3}$$

Notice that if there is at least one p, with $1 \leq p \leq \infty$, such that (1) is FG l_p stable then (1) is also (ϵ, δ) FGI stable for all $\epsilon, \delta \in \bar{\mathbb{R}}_+$. Therefore, Theorem 2.1 is sufficiently general to prove the following corollary.

Corollary 2.2 *Consider that A, the dynamic matrix of the dynamical system represented by (1), is non-Hurwitz (unstable). If there exists p, satisfying $1 \leq p \leq \infty$, such that (1) is FG l_p stable then there exists k_{min} such that the range of $\mathcal{F}^{k_{min}}$ is an infinite set.*

2.1 Comparative Analysis between ISS and (ϵ, δ) FGI Stability

We start by defining input to state stability (ISS) in discrete time [12], which is analogous to the continuous time version found in [13].

Definition 2.3 ***(ISS)*** *Let $\vec{x}$ be the solution of (1). We denote by $\mathcal{K}$ the set of positive, continuous, strictly increasing and unbounded functions $\mathcal{B}$ satisfying $\mathcal{B}(0) = 0$. We qualify the feedback loop (1) as input to state stable (ISS) (with zero initial conditions), if there exists $\mathcal{B} \in \mathcal{K}$ such that the following holds:*

$$\forall \vec{W} \in \mathbb{R}^{n \times \infty}, \|\vec{X}\|_\infty \leq \mathcal{B}\left(\|\vec{W}\|_\infty\right) \tag{4}$$

The following Remark follows readily from definitions 2.1 and 2.3, and it establishes a connection between ISS and (ϵ, δ) FGI stability.

Remark 2.1 *Consider that the system (1) is ISS and that $\mathcal{B} \in \mathcal{K}$ satisfies (4). For any arbitrary $\epsilon, \delta \in \bar{\mathbb{R}}_+$, the following holds:*

$$\sup_{\varrho \in (2^{-\epsilon}, 2^{\delta})} \frac{\mathcal{B}(\varrho)}{\varrho} \geq \sup_{\vec{W} \in \mathbb{D}_{\epsilon,\delta}} \frac{\|\vec{X}\|_\infty}{\|\vec{W}\|_\infty} \geq \sup_{k \geq 0} \mathcal{G}(k, \epsilon, \delta) \tag{5}$$

The following Corollary, shows that finite set feedback may impose fundamental constraints on the non-linear gain $\mathcal{B}$. Its proof follows from definition 2.1, Remark 2.1 and Theorem 2.1.

Corollary 2.3 *Consider that the matrix A, of the dynamical system represented by (1), is non-Hurwitz (unstable) and that the feedback loop is ISS, with $\mathcal{B} \in \mathcal{K}$ satisfying (4). If $\mathcal{F}$ implements a finite set feedback strategy, then the function $\mathcal{B}$ satisfies the following:*

$$\forall \delta > 0, \sup_{\varrho \in (0, 2^{\delta})} \frac{\mathcal{B}(\varrho)}{\varrho} = \infty$$

$$\forall \epsilon > 0, \sup_{\varrho > 2^{-\epsilon}} \frac{\mathcal{B}(\varrho)}{\varrho} = \infty$$

Since $\mathcal{B}$ is continuous and increasing, the unbounded growth-rate at zero creates a *cusp*-like shape (see Fig 1) which has been confirmed empirically by the authors [2] of [10].

The work in [17] addresses the problem of robustness in the presence of operator uncertainty, using induced norms, under the assumption that an upper bound on the amplitude of the external excitation is known. An alternative framework, in the absence of external excitation, can also be found in [7]. In the absence of a-priori bounds, Corollary 2.3 has further implications to robustness analysis. Since

[2] The author would like to thank Daniel Liberzon (UIUC) for sharing this information.

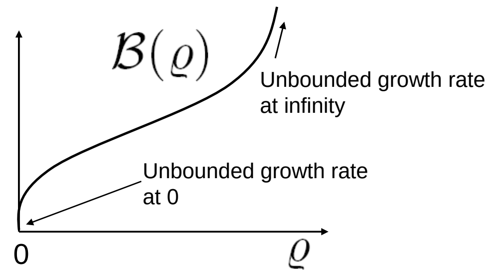

Fig. 1. Illustration of a function $\mathcal{B} \in \mathcal{K}$ which is not differentiable at zero and has unbounded sub-differential at infinity.

FG l_p stabilization is impossible under bit-rate constrained feedback, it follows that small gain arguments using l_p based induced norms are not viable. Thus, our results further support the use of ISS approaches to robustness, such as the work in [11].

3 Derivation of the necessary conditions

We start by listing the main definitions of Information Theory used throughout this section. All of the properties and definitions can be found either in [19] or [18].

Definition 3.1 *If* $\mathbf{E}$ *is a random variable, with alphabet* $\mathbb{E} = \mathbb{C}^q$*, then we define the differential entropy of* $\mathbf{E}$ *as* $h(\mathbf{E}) = -\int_{\mathbb{C}^q} p_E(\gamma) \log_2 p_E(\gamma) d(Re\{\gamma\} \times Im\{\gamma\})$.

If $\mathbf{S}$ is a random variable, with alphabet $\mathbb{S} = \mathbb{C}^{q'}$, then the conditional differential entropy of $\mathbf{E}$ given $\mathbf{S}$ is represented by $h(\mathbf{E}|\mathbf{S})$, and its computation is described in [19]. If $\mathbb{S}$ is countable then the computation of $h(\mathbf{E}|\mathbf{S})$ is described in [18].

Definition 3.2 *(Mutual Information) Let* $\mathbf{E}$ *and* $\mathbf{S}$ *be random variables with alphabets* $\mathbb{E} = \mathbb{C}^q$ *and* $\mathbb{S}$*, where* $\mathbb{S}$ *may be countable. The mutual information between* $\mathbf{E}$ *and* $\mathbf{S}$ *is given by* $I(\mathbf{E}, \mathbf{S}) = h(\mathbf{E}) - h(\mathbf{E}|\mathbf{S})$.

The following is a list of standard properties of h and I:

- **(P1)** $I(\phi(\mathbf{E}), \theta(\mathbf{S})) \leq I(\mathbf{E}, \mathbf{S})$, where equality holds if ϕ and θ are injective;
- **(P2)** $h(\mathbf{E}-\mathbf{S}|\mathbf{S}) = h(\mathbf{E}|\mathbf{S}) \leq h(\mathbf{E})$, or equivalently $h(\mathbf{E}|\mathbf{S}) = h(\mathbf{E}-\mathbf{S}|\mathbf{S}) \leq h(\mathbf{E}-\mathbf{S})$;
- **(P3)** $h(\mathbf{E}) \leq \frac{1}{2}\log_2(2\pi e Var(\mathbf{E}))$

In the subsequent analysis, we will refer to the above properties by (P1)-(P3). As it will be clear at a later stage, the following lemma is the critical basic result of this section.

Lemma 3.1 *Let* $\mathrm{s}_{\epsilon,\delta}$ *and* $\mathrm{e}_{\epsilon,\delta}$ *be parameterized families of positive real random variables, with real parameters* $\epsilon, \delta \in \mathbb{R}_+$. *Consider that* $\log_2 \mathbf{e}_{\epsilon,\delta}$ *is uniformly distributed in the interval* $(-\epsilon, \delta)$. *The following implication is true:*

$$\left(\exists \gamma \in (0,1),\ \forall (\epsilon,\delta) \in \mathbb{R}_+^2, |\mathbf{s}_{\epsilon,\delta} - \mathbf{e}_{\epsilon,\delta}| < \gamma \mathbf{e}_{\epsilon,\delta}\right) \implies \limsup_{(\epsilon+\delta)\to\infty} \frac{I(\mathbf{e}_{\epsilon,\delta}, \mathbf{s}_{\epsilon,\delta})}{\log_2(\epsilon+\delta)} \geq 1$$

Proof: We start by writing $I(\mathbf{e}_{\epsilon,\delta}, \mathbf{s}_{\epsilon,\delta}) \underset{(P1)}{=} I(\log_2 \mathbf{e}_{\epsilon,\delta}, \log_2 \mathbf{s}_{\epsilon,\delta})$, which can also be expressed, from definition 3.2, as $I(\mathbf{e}_{\epsilon,\delta}, \mathbf{s}_{\epsilon,\delta}) = h(\log_2 \mathbf{e}_{\epsilon,\delta}) - h(\log_2 \mathbf{e}_{\epsilon,\delta}| \log_2 \mathbf{s}_{\epsilon,\delta})$.

Using the fact that $h(\log_2 \mathbf{e}_{\epsilon,\delta}| \log_2 \mathbf{s}_{\epsilon,\delta}) \underset{(P2),(P3)}{\leq} \frac{1}{2}\log_2(2\pi e Var(\log_2 \mathbf{e}_{\epsilon,\delta} - \log_2 \mathbf{s}_{\epsilon,\delta}))$, we arrive at the inequality below:

$$I(\mathbf{e}_{\epsilon,\delta}, \mathbf{s}_{\epsilon,\delta}) \geq h(\log_2 \mathbf{e}_{\epsilon,\delta}) - \frac{1}{2}\log_2(2\pi e Var(\log_2 \mathbf{s}_{\epsilon,\delta} - \log_2 \mathbf{e}_{\epsilon,\delta})) \qquad (6)$$

From the assumptions of the Lemma, we have $\log_2(1-\gamma) \leq \log_2 \frac{\mathbf{s}_{\epsilon,\delta}}{\mathbf{e}_{\epsilon,\delta}} \leq \log_2(1+\gamma)$, which implies that $\sup_{\epsilon,\delta\in\mathbb{R}_+} Var(\log_2 \mathbf{s}_{\epsilon,\delta} - \log_2 \mathbf{e}_{\epsilon,\delta}) < \infty$. As such, we conclude the proof from (6), as well as by using definition 3.1 to compute $h(\log_2 \mathbf{e}_{\epsilon,\delta}) = \log_2(\epsilon+\delta)$. $\square$

3.1 Necessary condition for (∞, δ) FGI stabilization

In the subsequent analysis, we consider the following first-order system:

$$x(k+1) = ax(k) + \mathcal{F}(x^k, k) + w(k), x(0) = 0 \tag{7}$$

where $x(k) \in \mathbb{C}$, $a \in \mathbb{C}$, $|a| > 1$, $w(k) \in \mathbb{C}$ and $\mathcal{F}(\cdot, k) : \mathbb{C}^k \to \mathbb{C}$ is a causal map implementing a certain feedback strategy.

Remark 3.1 ***(Validity of using (7) in deriving necessary conditions of stability for the multi-state feedback system represented by (1).*** *) Assume that the matrix A in (1) is not Hurwitz. Choose a mode of A corresponding to the lower-right most element of a non-Hurwitz Jordan block. Let $a \in \mathbb{C}$, with $|a| > 1$, describe the eigenvalue of such a mode. It is clear that the minimum cardinality which the range of a feedback map must have to stabilize (7), in the (ϵ, δ) FGI sense, is a lower bound to the cardinality required for stabilizing (1) in the same sense.*

Definition 3.3 *Let $\vec{x}$ be the solution of (7). Given $\epsilon, \delta \in \bar{\mathbb{R}}_+$, k_{min} and a gain $\alpha > 0$, we define the following set of (ϵ, δ) FGI stabilizing feedback strategies:*

$$\mathbb{T}_{\epsilon,\delta}(a, \alpha, k_{min}) \stackrel{def}{=} \{\mathcal{F} : \forall k > k_{min}, \forall \vec{w} \in \mathbb{D}_{\epsilon,\delta},\ |x(k)| \leq \alpha |w(0)|\}$$

Theorem 3.2 *Let δ be a given positive real constant and $\vec{x}$ be the solution of (7). Consider that $\mathcal{F}$ stabilizes (7) in the (∞, δ) FGI sense ,i.e., assume that there exists a real and positive gain α and an integer k_{min} such that the following holds:*

$$\forall k \geq k_{min}, \forall \vec{w} \in \mathbb{D}_{\infty,\delta}, |x(k)| \leq \alpha |w(0)|$$

For every real and positive ϵ, select a map $\mathcal{F}_{\epsilon,\delta}$ in the set $\mathbb{T}_{\epsilon,\delta}(a, \alpha, k_{min})$ and denote by $\mathbb{F}$ the resulting ϵ-parameterized family of maps $\{\mathcal{F}_{\epsilon,\delta}\}_{\epsilon \in \mathbb{R}_+}$. For any given $\mathbb{F}$, *there exists an integer k_∞ for which the following holds:*

$$\limsup_{\epsilon\to\infty} \frac{\sharp Range\left(\mathcal{F}^{k_\infty}_{\epsilon,\delta}\right)}{\epsilon} \geq 1$$

where $\sharp Range\left(\mathcal{F}^{k_\infty}_{\epsilon,\delta}\right)$ is the cardinality of the map defined by:

$$\mathcal{F}^{k_\infty}_{\epsilon,\delta}(x^{k_\infty}) = \left(\mathcal{F}_{\epsilon,\delta}(x(0),0),\dots,\mathcal{F}_{\epsilon,\delta}(x^{k_\infty},k_\infty)\right)$$

Proof: For every $\epsilon \in \mathbb{R}_+$, select $\vec{\mathbf{w}}_{\epsilon,\delta}$ as a singleton, where $\log_2|\mathbf{w}_{\epsilon,\delta}(0)|$ is uniformly distributed in $(-\epsilon,\delta)$ and $\mathbf{w}_{\epsilon,\delta}(k)=0$, for $k\geq 1$. Using α and k_{min} from the assumptions of the Theorem, we know that $\forall k > k_{min}, \forall\epsilon\in\bar{\mathbb{R}}_+,\ |\mathbf{x}_{\epsilon,\delta}(k)| \leq \alpha|\mathbf{w}_{\epsilon,\delta}(0)|$ is satisfied, where $\vec{\mathbf{x}}_{\epsilon,\delta}$ is the solution of:

$$\mathbf{x}_{\epsilon,\delta}(k+1) = a\mathbf{x}_{\epsilon,\delta}(k) + \mathcal{F}_{\epsilon,\delta}(\mathbf{x}^k_{\epsilon,\delta},k) + \mathbf{w}_{\epsilon,\delta}(k), \mathbf{x}_{\epsilon,\delta}(0)=0 \tag{8}$$

For simplicity, we adopt $\mathbf{u}_{\epsilon,\delta}(k) = \mathcal{F}_{\epsilon,\delta}(\mathbf{x}^k_{\epsilon,\delta},k)$, which leads to:

$$\forall\epsilon\in\bar{\mathbb{R}}_+, |\mathbf{x}_{\epsilon,\delta}(k)| \geq |a|^{k-1}|\mathcal{L}\left(\mathbf{u}^k_{\epsilon,\delta},k\right) - |\mathbf{w}_{\epsilon,\delta}(0)|| \tag{9}$$

where $\mathcal{L}\left(\mathbf{u}^k_{\epsilon,\delta},k\right) = |a|^{1-k}|\sum_{i=0}^{k-1} a^{k-i}\mathbf{u}_{\epsilon,\delta}(i)|$. Take k_∞ satisfying $k_\infty > k_{min}$ and $|a|^{-k_\infty}\alpha < 1$. Using (9), we arrive at:

$$\forall k > k_\infty, \forall\epsilon\in\bar{\mathbb{R}}_+,\ |\mathcal{L}\left(\mathbf{u}^k_{\epsilon,\delta},k\right) - |\mathbf{w}_{\epsilon,\delta}(0)|| \leq \gamma|\mathbf{w}_{\epsilon,\delta}(0)| \tag{10}$$

where our choice of k_∞ implies that $\gamma = |a|^{-k_\infty}\alpha < 1$. From (10) and Lemma 3.1, we have:

$$\forall k > k_\infty, \limsup_{\epsilon\to\infty} \frac{I\left(\mathcal{L}\left(\mathbf{u}^k_{\epsilon,\delta},k\right), |\mathbf{w}_{\epsilon,\delta}(0)|\right)}{\log_2\epsilon} \geq 1 \tag{11}$$

$$\forall k > k_\infty, \limsup_{\epsilon\to\infty} \frac{I\left(\mathbf{u}^k_{\epsilon,\delta}, \mathbf{w}_{\epsilon,\delta}(0)\right)}{\log_2\epsilon} \geq 1 \tag{12}$$

where (12) follows from (11) and (P1). In view of (12), the proof is concluded by the following remark:

Remark 3.2 *Let* $\mathbf{E}$ *and* $\mathbf{S}$ *be vectors of complex random variables. If* $\mathbf{S}$ *has a countable alphabet* $\mathbb{S}$*, then the following holds [18,19]:*

$$I(\mathbf{E},\mathbf{S}) \leq H(\mathbf{S}) \stackrel{def}{=} \sum_{\gamma \in \mathbb{S}} p_S(\gamma) \log_2 p_S(\gamma) \tag{13}$$

On the other hand, $H(\mathbf{S}) \leq \log_2 \sharp\mathbb{S}$ *is a standard fact, where* $0 \leq \sharp\mathbb{S} \leq \infty$ *is the cardinality of the alphabet of* $\mathbf{S}$*. Given a vector of complex random variables* $\mathbf{Z}$ *and a countable range function* $\mathcal{P}$*, we can use (13) to arrive at* $2^{I(\mathbf{E},\mathcal{P}(\mathbf{Z}))} \leq \sharp Range(\mathcal{P})$*, where* $0 \leq \sharp Range(\mathcal{P}) \leq \infty$ *is the cardinality of the range of* $\mathcal{P}$*. Notice that if the range of* $\mathcal{P}$ *is uncountable, then it is immediately infinite and* $2^{I(\mathbf{E},\mathcal{P}(\mathbf{Z}))} \leq \sharp Range(\mathcal{P})$ *is satisfied.*$\square$

Corollary 3.3 *Let* δ *be a given positive real constant and* $\vec{x}$ *be the solution of (7). If* $\mathcal{F}$ *stabilizes (7) in the* (∞, δ) *FGI sense then there exists* k_∞*, such that the range of* $\mathcal{F}^{k_\infty}$ *is an infinite set.*

Proof: For every ϵ positive select $\mathcal{F}_{\epsilon,\delta} = \mathcal{F}$. Since we assume that $\mathcal{F}$ is (∞, δ) FGI stabilizing, we conclude that the aforementioned family satisfies the assumptions of Theorem 3.2, which concludes the proof.$\square$

The proof of the following Corollary is analogous to the proof of Corollary 3.3.

Corollary 3.4 *Let* ϵ *be a given positive real constant and* $\bar{x}$ *be the solution of (7). If* $\mathcal{F}$ *stabilizes (7) in the* (ϵ, ∞) *FGI sense, then there exists* k_∞ *such that the range of* $\mathcal{F}^{k_\infty}$ *is an infinite set.*

3.2 (∞, δ) FGI stabilization requires a logarithmically increasing resolution

In this subsection, we show that any (∞, δ) FGI stabilizing feedback is infinite-set, with a *resolution* increasing logarithmically as the infinity norm of $\vec{w}$ decreases.

Definition 3.4 *Let $\mathcal{F}$ be the causal feedback map in (7). Clearly $\mathcal{F}$ is ultimately a function of $\vec{w}$ and we can define $\mathcal{F}|_{\mathbb{D}_{\epsilon,\delta}}$ by restricting the domain of $\mathcal{F}$ to $\vec{w} \in \mathbb{D}_{\epsilon,\delta}$.*

Corollary 3.5 *Let δ be a given and $\vec{x}$ be the solution of (7). Consider that $\mathcal{F}$ stabilizes (7) in the (∞, δ) ISS sense, for some k_{min} and gain $\alpha > 0$. The following holds:*

$$\exists k_\infty > 0, \limsup_{\epsilon \to \infty} \frac{\sharp Range\left(\mathcal{F}|^{k_\infty}_{\mathbb{D}_{\epsilon,\delta}}\right)}{\epsilon} \geq 1 \tag{14}$$

where $\mathcal{F}|_{\mathbb{D}_{\epsilon,\delta}}$ is the restricted $\mathcal{F}$ of definition 3.4 and $\sharp Range(\mathcal{F}|^{k_\infty}_{\mathbb{D}_{\epsilon,\delta}})$ is the cardinality of the range of $\mathcal{F}|^{k_\infty}_{\mathbb{D}_{\epsilon,\delta}}$.

Proof: For every positive and real ϵ, choose $\mathcal{F}_{\epsilon,\delta} = \mathcal{F}|_{\mathbb{D}_{\epsilon,\delta}}$. Since $\mathcal{F}$ is (∞, δ) FGI stabilizing, the family $\mathcal{F}_{\epsilon,\delta} = \mathcal{F}|_{\mathbb{D}_{\epsilon,\delta}}$ satisfies the conditions of Theorem 3.2 and the proof follows immediately. $\square$

4 Conclusions

We have established that FG l_p stabilization is not possible by finite-set feedback, and that the resolution of the controller must increase logarithmically as the amplitude of the external excitation decreases. The main implication of this result is the following: ISS stabilization is achievable with finite-set feedback, but the sensitivity with respect to external signals becomes arbitrarily large for small and large disturbances. This fact is a fundamental limitation and cannot be avoided, in particular, FG l_p stabilization can only be accomplished by analog control. The absence of FG l_p stability also precludes the use of standard, *induced norm based*, small gain theorem approaches for robustness analysis.

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