

TECHNICAL RESEARCH REPORT

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Very sharp transitions in one-dimensional MANETs

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Abstract— We investigate how quickly phase transitions can occur in one-dimensional geometric random graph models of MANETs. In the case of graph connectivity, we show that the transition width behaves like n^{-1} (when the number n of users is large), a significant improvement over general asymptotic bounds given recently by Goel et al. for monotone graph properties. We also discuss a similar result for the property that there exists no isolated user in the network. The asymptotic results are validated by numerical computations. Finally we outline how the approach used here could be applied in higher dimensions and for other graph properties.

Keywords: Geometric random graphs, MANETs, Threshold functions, Phase transition, Zero-one laws, Poisson paradigm, Connectivity, Node isolation.

I. INTRODUCTION

Over the past few years, geometric random graphs have provided a useful abstraction for studying large wireless networks [9, 12, 13, 15]. In that line of work, much attention has focused on the following basic model where n points are distributed uniformly and independently in the unit cube $[0, 1]^d$ (in \mathbb{R}^d) for some positive integer d . Given a fixed threshold $\tau > 0$, two points are said to be directly connected if their Euclidean distance is less than τ . This notion of connectivity gives rise to an undirected geometric random graph, denoted $\mathbb{G}_d(n; \tau)$.

As discussed in the monograph by Penrose [17, Chap. 1] (and references therein), these geometric random graph models (and variations thereof) are of wide applicability in statistical physics, cluster analysis and hypothesis testing. In the context of wireless networks, with $d \leq 3$, we interpret the n points as users equipped with a transmitter/receiver of transmission range τ . In first approximation, if we neglect details of channel behavior, it is reasonable to model two users as communicating with each other if their Euclidean distance is less than τ . This approach has been taken by a number of authors, e.g., [5, 6, 7, 9, 12, 13, 15].

Of particular interest for wireless networking is the property that the graph $\mathbb{G}_d(n; \tau)$ be (path) connected. Because no explicit expression is available for the probability that $\mathbb{G}_d(n; \tau)$ is connected, except in the one-dimensional case ($d = 1$), attention turns instead to the situation with n large as representing the regime of practical relevance – After all, designing and running wireless networks are more pressing tasks when the number of users is large in relation to available

resources. Interestingly enough, randomizing user locations makes it possible for many properties of $\mathbb{G}_d(n; \tau)$ (including connectivity) to reveal a *typical behavior* when n becomes large.

This manifests itself as follows: Consider a monotone increasing graph property A defined in the usual manner [1, 14],¹ graph connectivity being such a property. For each $n = 2, 3, \dots$, let $P_A(n; \tau)$ denote the probability that A occurs in $\mathbb{G}_d(n; \tau)$. The mapping $\tau \rightarrow P_A(n; \tau)$ is monotone increasing with $0 < P_A(n; \tau) < 1$ in some finite interval and $P_A(n; \tau) = 1$ outside it. As earlier simulation results already indicate for various properties of interest [5, 6, 7, 15], there exists a *phase transition* from $P_A(n; \tau) \simeq 0$ to $P_A(n; \tau) \simeq 1$ as τ varies across some critical range. A natural question therefore consists in estimating how quickly this transition is taking place.

To address this issue, for each $n = 2, 3, \dots$, we define

$$\tau_A(n; a) = \inf(\tau > 0 : P_A(n; \tau) \geq a), \quad a \in (0, 1) \quad (1)$$

and whenever a lies in the interval $(0, \frac{1}{2})$, we set

$$\delta_A(n; a) = \tau_A(n; 1 - a) - \tau_A(n; a).$$

The transition width $\delta_A(n; a)$ measures how quickly $P_A(n; \tau)$ climbs from level a to level $1 - a$, thereby giving an indication of the sharpness of the phase transition. Given the rather complex dependence of $\delta_A(n; a)$ on n and a , it is desirable to find asymptotic bounds (if nothing else) on its behavior for large n .

A similar line of inquiry has been carried out extensively for Bernoulli graphs (also known as Erdős-Renyi graphs) [3, 8]. Recently, Goel et al. [11] have derived such asymptotic bounds for any monotone graph property in $\mathbb{G}_d(n; \tau)$. For any such property A , their results imply that $\delta_A(n; a) = o(1)$, a fact captured by the terminology that the monotone property A has a *sharp* threshold. However, these general results do leave open the question as to whether these asymptotic bounds can be further sharpened for *specific* monotone graph properties, hopefully in cases of practical importance.

Here, we tackle this issue mainly for the probability of graph connectivity and for the probability of no isolated user. For ease of telling the story, we restrict the discussion to one-dimensional geometric random graph models for MANETs,

¹The case of monotone decreasing graph properties can be discussed *mutatis mutandis*.

i.e., $d = 1$; such models have been investigated in the references [5, 6, 7, 10] which contain some of the results we needed. Our main result takes the form of *exact* asymptotic expansions (in n) for the thresholds [Theorem 3.1 and Theorem 5.1]. This leads to transition widths of order n^{-1} with known preconstants, so that these graph properties are *very sharp* indeed! Such information can be leveraged in network design when network connectivity and node isolation are important concerns.

Some may construe the one-dimensional case ($d = 1$) as being perhaps too limited or not too relevant to practice. However, we stress that the main contribution of the paper lies in identifying an *approach of wide applicability* to establish sharp asymptotics: The key ingredients are the availability of a *Poisson paradigm* complementing the “zero-one” law usually occurring for many graph properties.

The paper is organized as follows: The model and preliminaries are given in Section II. The analytical results of two monotone network properties, namely *connectivity* and *nonexistence of isolated users*, are presented in Section III and Section V, respectively. In Section IV, we explain how the appropriate “zero-one” laws and companion Poisson convergence lead to the correct asymptotics for the threshold width. Some limited numerical validation of the asymptotics is provided in Section VI. In Section VII we briefly contrast our results against the results of Goel et al.; we also provide a rough roadmap to establish similar results in higher dimensions ($d \geq 2$) and for other graph properties. A proof of Theorem 3.1 is relegated in Appendix.

A word on the notation in use: The indicator function of an event E is simply $\mathbf{1}[E]$, and we use the notation \xrightarrow{P}_n (resp. \implies_n) to signify convergence in probability (resp. convergence in distribution) with n going to infinity.

II. MODEL AND PRELIMINARIES

The one-dimensional model has been considered by a number of authors [5, 6, 7, 10]. To define it, let $\{U_i, i = 1, 2, \dots\}$ denote a sequence of i.i.d. rvs distributed uniformly in the interval $[0, 1]$.

For each $n = 2, 3, \dots$, we think of U_1, \dots, U_n as the locations of n nodes (or users), labelled $1, \dots, n$, in the interval $[0, 1]$. Given a fixed distance $\tau > 0$, two nodes are said to be directly connected if their distance is at most τ , i.e., nodes i and j are connected if $|U_i - U_j| \leq \tau$, in which case an undirected edge is said to exist between these two users. This notion of connectivity gives rise to an undirected geometric random graph denoted $\mathbb{G}(n; \tau)$.

We introduce the rvs $X_{n,1}, \dots, X_{n,n}$ which denote the location of these n users when arranged in increasing order, i.e., $X_{n,1} \leq \dots \leq X_{n,n}$ with the convention $X_{n,0} = 0$ and $X_{n,n+1} = 1$. Also define

$$L_{n,k} := X_{n,k} - X_{n,k-1}, \quad k = 1, \dots, n+1.$$

Obviously $L_{n,1} + \dots + L_{n,n+1} = 1$. The rvs $X_{n,1}, \dots, X_{n,n}$ are the *order statistics* associated with the n i.i.d. rvs

U_1, \dots, U_n . It is well known [4, Eq. (6.4.3), p. 135] that for any fixed subset $I \subseteq \{1, \dots, n\}$, we have

$$\mathbf{P}[L_{n,k} > t_k, k \in I] = \left(1 - \sum_{k \in I} t_k\right)_+^n, \quad t_k \in [0, 1], k \in I.$$

with the notation $x_+^n = x^n$ if $x \geq 0$ and $x_+^n = 0$ if $x \leq 0$.

On several occasions, we shall find it convenient to consider the $\{0, 1\}$ -valued rvs $\chi_{n,1}(\tau), \dots, \chi_{n,n+1}(\tau)$ defined as the indicator functions

$$\chi_{n,k}(\tau) := \mathbf{1}[L_{n,k} > \tau], \quad k = 1, \dots, n+1.$$

III. CONNECTIVITY

Fix $\tau > 0$ and $n = 2, 3, \dots$. The geometric random graph $\mathbb{G}(n; \tau)$ is said to be (*path*) *connected* if every pair of users can be linked by at least one path over the edges of the graph, and we write

$$P_c(n; \tau) := \mathbf{P}[\mathbb{G}(n; \tau) \text{ is connected}].$$

Obviously, the graph $\mathbb{G}(n; \tau)$ is connected if and only if $L_{n,k} \leq \tau$ for all $k = 2, \dots, n$, so that

$$P_c(n; \tau) = \mathbf{P}[L_{n,k} \leq \tau, k = 2, \dots, n]. \quad (2)$$

The closed form expression

$$P_c(n; \tau) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (1 - k\tau)_+^n \quad (3)$$

was obtained by Desai and Manjunath [5] (as Eqn (8) with $z = 1$ and $r = \tau$). Related expressions were also derived earlier by Godehardt and Jaworski [10].

Fix a in the interval $(0, 1)$. For each $n = 2, 3, \dots$, the mapping $\tau \rightarrow P_c(n; \tau)$ can be shown to be strictly monotone. This property guarantees the existence and uniqueness of solutions to the equation

$$P_c(n; \tau) = a, \quad \tau \in (0, 1). \quad (4)$$

Let $\tau_c(n; a)$ denote this unique solution, and whenever a lies in the interval $(0, \frac{1}{2})$, we set

$$\delta_c(n; a) := \tau_c(n; 1 - a) - \tau_c(n; a).$$

The main result concerning the behavior of $\tau_c(n; a)$ for large n is given first.

Theorem 3.1: *For every a in the interval $(0, 1)$, it holds that*

$$\tau_c(n; a) = \frac{\log n}{n} - \frac{1}{n} \log \left(\log \left(\frac{1}{a} \right) \right) + o(n^{-1}). \quad (5)$$

Theorem 3.1 is established in Appendix. The desired result on the width of the transition interval flows as an easy corollary.

Corollary 3.2: *For every a in the interval $(0, \frac{1}{2})$, we have*

$$\delta_c(n; a) = \frac{C(a)}{n} + o(n^{-1}) \quad (6)$$

with constant $C(a)$ given by

$$C(a) = \log \left(\frac{\log a}{\log(1-a)} \right). \quad (7)$$

It is a simple matter to check that $a \rightarrow C(a)$ is decreasing on the interval $(0, \frac{1}{2})$ with $\lim_{a \downarrow 0} C(a) = \infty$ and $\lim_{a \uparrow \frac{1}{2}} C(a) = 0$. These qualitative features are in line with one's intuition.

IV. HOW TO GUESS THE RESULT

We now present a plausibility argument which allows us to guess the validity of Theorem 3.1, and which eventually paves the way to its proof: Our point of departure is the "zero-one" law available for the property of graph connectivity under the asymptotic regime created by having n become large when the threshold parameter is scaled appropriately with n . Here, we need take a $[0, 1]$ -valued sequence $\{\tau(n), n = 2, 3, \dots\}$ of the form

$$\tau(n) = \frac{1}{n} (\log n + \alpha(n)), \quad n = 2, 3, \dots$$

such that $\lim_{n \rightarrow \infty} \tau(n) = 0$; this amounts to $\alpha_n = o(n)$. Under these conditions, it is known that

$$\lim_{n \rightarrow \infty} P_c(n; \tau(n)) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \alpha(n) = -\infty \\ 1 & \text{if } \lim_{n \rightarrow \infty} \alpha(n) = +\infty \end{cases} \quad (8)$$

This result follows from Theorem 1 in [2, p. 352], but can also be derived by direct arguments based on first and second moments [14, Eqn. (3.10), p. 55] [14, Remark 3.1, p. 55].

The convergence (8) identifies the critical scaling

$$\tau_c^*(n) = \frac{\log n}{n}, \quad n = 2, 3, \dots$$

as the *threshold* function which defines a *boundary* in the space of scalings. Intuition suggests that mild fluctuations about this boundary, say of the order n^{-1} , are likely to hold the key to the form of $\tau(n; a)$ for large n . To explore this idea further, for each x in \mathbb{R} , define the $[0, 1]$ -valued sequence $\{\sigma(n; x), n = 1, 2, \dots\}$ by

$$\sigma(n; x) = \min \left(1, \left(\frac{\log n + x}{n} \right)_+ \right), \quad n = 1, 2, \dots \quad (9)$$

so that

$$\sigma(n; x) = \frac{\log n + x}{n} \quad (10)$$

for n large enough. The next result complements the "zero-one" law (8) and can be easily extracted from Theorem 12 in [10, p. 157].

Theorem 4.1: *For each x in \mathbb{R} , it holds that*

$$\lim_{n \rightarrow \infty} P_c(n; \sigma(n; x)) = p(x) \quad (11)$$

with

$$p(x) = e^{-e^{-x}}. \quad (12)$$

To see in what sense the convergence (11) underpins Theorem 3.1, consider the following heuristic arguments: For each x in \mathbb{R} , the convergence (11) yields the approximation

$$P_c(n; \sigma(n; x)) \simeq p(x)$$

for large enough n . The mapping $p : \mathbb{R} \rightarrow \mathbb{R}_+ : x \rightarrow p(x)$ is strictly monotone and continuous with $\lim_{x \rightarrow -\infty} p(x) = 0$ and $\lim_{x \rightarrow \infty} p(x) = 1$. Therefore, for each a in the interval $(0, 1)$, there exists a unique value, denoted x_a , such that $p(x_a) = a$. In fact,

$$x_a = -\log(-\log a). \quad (13)$$

Given a in the interval $(0, 1)$, we find that

$$P_c(n; \sigma(n; x_a)) \simeq a$$

for large n . This suggests (but not quite yet proves) that $\sigma(n; x_a)$ and $\tau(n; a)$ behave in tandem asymptotically, thereby laying the grounds for the validity of (5) – Just insert (13) into (10) and (12). These ideas form the basis for the proof of Theorem 3.1 found in Appendix.

To gain some perspective on (11)–(12), we need to introduce the notion of breakpoint user. For each $i = 1, \dots, n$, user i is said to be a *breakpoint* user in the random graph $\mathbb{G}(n; \tau)$ whenever (i) it is not the leftmost user in $[0, 1]$ and (ii) there are no users in the random interval $[U_i - \tau, U_i]$. The number $B_n(\tau)$ of breakpoint nodes in $\mathbb{G}(n; \tau)$ is given by

$$B_n(\tau) = \sum_{k=2}^n \chi_{n,k}(\tau),$$

so that

$$P_c(n; \tau) = \mathbf{P}[B_n(\tau) = 0].$$

Theorem 4.1 is a mere byproduct of a stronger result on *Poisson convergence* [10, Thm. 12, p. 157], namely that $B_n(\sigma(n; x)) \Rightarrow_n \Pi(e^{-x})$ where $\Pi(\mu)$ denotes a Poisson rv with parameter μ . See also a similar result for the corresponding model on the unit circle [16, Thm. 8, p. 172].

V. ISOLATED NODES

Similar arguments can be made for graph properties other than the property of graph connectivity just discussed. Here is another example: Fix $\tau > 0$ and $n = 2, 3, \dots$. For each $i = 1, \dots, n$, node i is said to be *isolated* in the random graph $\mathbb{G}(n; \tau)$ whenever $|U_i - U_j| > \tau$ for all $j \neq i, j = 1, \dots, n$. In terms of the order statistics introduced earlier, we see that the user at location $X_{n,k}$ is isolated (i) if $L_{n,2} > \tau$ for $k = 1$; (ii) if $L_{n,k} > \tau$ and $L_{n,k+1} > \tau$ whenever $k = 2, \dots, n-1$; and (iii) if $L_{n,n} > \tau$ for $k = n$. As a result, the total number $I_n(\tau)$ of isolated nodes in $\mathbb{G}(n; \tau)$ is given by

$$I_n(\tau) = \chi_{n,2}(\tau) + \sum_{k=2}^{n-1} \chi_{n,k}(\tau) \chi_{n,k+1}(\tau) + \chi_{n,n}(\tau).$$

The probability $P_i(n; \tau)$ that there is no isolated node in $\mathbb{G}(n; \tau)$ is simply given by

$$P_i(n; \tau) = \mathbf{P}[I_n(\tau) = 0].$$

The distribution of $I_n(\tau)$ is computed in [10, Thm. 4, p. 148]. With $r = 0$ in these expressions we find

$$\begin{aligned} P_i(n; \tau) &= \sum_{k=0}^{k(\tau)} (-1)^k \sum_{j=j(\tau)}^k \binom{n-k-1}{k-j} \binom{k+1}{j} (1 - (2k-j)\tau)^n \end{aligned} \quad (14)$$

where $k(\tau) = \min(n-1, \lfloor \frac{1}{\tau} \rfloor)$ and $j(\tau) = \max(0, 2k - \lfloor \frac{1}{\tau} \rfloor, 2k - n + 1)$.

Fix a in the interval $(0, 1)$. For each $n = 2, 3, \dots$, the mapping $\tau \rightarrow P_i(n; \tau)$ being strictly monotone, the equation

$$P_i(n; \tau) = a, \quad \tau \in (0, 1). \quad (15)$$

admits a unique solution, denoted $\tau_i(n; a)$. The main result concerning the behavior of $\tau_i(n; a)$ for large n parallels Theorem 3.1.

Theorem 5.1: *For every a in the interval $(0, 1)$, it holds that*

$$\tau_i(n; a) = \frac{\log n}{2n} - \frac{1}{2n} \log \left(\log \left(\frac{1}{a} \right) \right) + o(n^{-1}). \quad (16)$$

For every a in the interval $(0, \frac{1}{2})$, this leads readily to

$$\begin{aligned} \delta_i(n; a) &:= \tau_i(n; 1-a) - \tau_i(n; a) \\ &= \frac{C(a)}{2n} + o(n^{-1}) \end{aligned} \quad (17)$$

with constant $C(a)$ given by (7).

The proof of Theorem 5.1 is omitted due to space limitations. However, that this proof follows a pattern similar to that of Theorem 3.1 should come as no surprise in view of the following observations: Here it is appropriate to consider a $[0, 1]$ -valued sequence $\{\tau(n), n = 2, 3, \dots\}$ of the form

$$\tau(n) = \frac{1}{2n} (\log n + \alpha(n)), \quad n = 2, 3, \dots$$

such that $\lim_{n \rightarrow \infty} \tau(n) = 0$ (whence $\alpha_n = o(n)$). Under these conditions, it is known that

$$\lim_{n \rightarrow \infty} P_i(n; \tau(n)) = \begin{cases} 0 & \text{if } \lim_{n \rightarrow \infty} \alpha(n) = -\infty \\ 1 & \text{if } \lim_{n \rightarrow \infty} \alpha(n) = +\infty \end{cases} \quad (18)$$

This follows from Theorem 2 in [2, p. 353], but can also be derived by lengthy arguments using the methods of first and second moments. This time the critical scaling is given by

$$\tau_i^*(n) = \frac{\log n}{2n} = \frac{1}{2} \tau_c^*(n), \quad n = 2, 3, \dots$$

and the complement to the “zero-one” law (18) takes the form

$$\lim_{n \rightarrow \infty} P_i(n; \frac{1}{2} \sigma(n; x)) = p(x), \quad x \in \mathbb{R} \quad (19)$$

with $p(x)$ given by (12).

Again (19) flows from a Poisson convergence result, namely $I_n(\frac{1}{2} \sigma(n; x)) \Rightarrow_n \Pi(e^{-x})$. Although we do not provide a detailed proof of this fact due to space limitations, we offer some pointers as to its validity: First, for small enough τ in the interval $(0, 1)$, we note that

$$\begin{aligned} \mathbf{E}[I_n(\tau)] &= 2\mathbf{E}[\chi_{n,1}(\tau)] + (n-2)\mathbf{E}[\chi_{n,1}(\tau)\chi_{n,2}(\tau)] \\ &= 2(1-\tau)^n + (n-2)(1-2\tau)^n \end{aligned}$$

for all $n = 2, 3, \dots$, whence

$$\lim_{n \rightarrow \infty} \mathbf{E} \left[I_n \left(\frac{1}{2} \sigma(n; x) \right) \right] = e^{-x} = \mathbf{E}[\Pi(e^{-x})].$$

Next, the announced Poisson convergence holds if and only if $J_n(\frac{1}{2} \sigma(n; x)) \Rightarrow_n \Pi(e^{-x})$ with

$$J_n(\tau) := \sum_{k=2}^{n-1} \chi_{n,k}(\tau) \chi_{n,k+1}(\tau), \quad n = 3, 4, \dots$$

for all $\tau > 0$. This is so because $\chi_{n,2}(\frac{1}{2} \sigma(n; x)) \xrightarrow{P} n0$ and $\chi_{n,n}(\frac{1}{2} \sigma(n; x)) \xrightarrow{P} n0$. The Poisson convergence paradigm is now in place once we recognize that the rv $J_n(\frac{1}{2} \sigma(n; x))$ is the sum of $(n-2)$ identically distributed indicator functions which become vanishingly small and increasingly decorrelated with n large. See also a similar result for the corresponding model on the unit circle [16, Thm. 6, p. 169].

VI. NUMERICAL VALIDATION

Below we present some limited numerical results validating the asymptotic results obtained here.

A. Evaluation

We consider n users which are uniformly and independently distributed in the interval $[0, 1]$, with n ranging from $n = 1000$ to $n = 9000$ in increments of 1000.

Given a in $(0, 1)$, the threshold $\tau_c(n; a)$ is calculated by solving the equation (3) which now takes the simpler form

$$a = P_c(n; \tau) = \sum_{k=0}^{k(\tau)} (-1)^k \binom{n-1}{k} (1-k\tau)^n \quad (20)$$

with $k(\tau) = \min(n-1, \lfloor \frac{1}{\tau} \rfloor)$. In these calculations, some care needs to be exercised owing to possible buffer overflow associated with the evaluation of combinatorial coefficients. To avoid computing directly the coefficients $\binom{n-1}{k}$, $k = 0, 1, \dots, k(\tau)$, we focus instead on evaluating the quantities $b_k = \binom{n-1}{k} (1-k\tau)^n$, $k = 0, 1, \dots, k(\tau)$. Note that we can calculate $b_0, b_1, \dots, b_{k(\tau)}$ sequentially through the simple recursion

$$b_{k+1} = \frac{n-k-1}{k+1} \left(1 - \frac{\tau}{1-k\tau} \right)^n \cdot b_k$$

for all $k = 0, 1, \dots, k(\tau) - 1$ with $b_0 = 1$.

The asymptotics (5) and (6) suggest approximating $\tau_c(n; a)$ and $\delta_c(n; a)$ through the two quantities

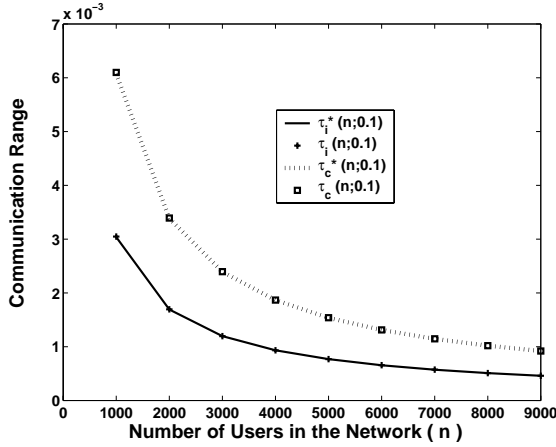
$$\tau_c^*(n; a) := \frac{\log n}{n} - \frac{1}{n} \log \left(\log \left(\frac{1}{a} \right) \right) \quad \text{and} \quad \delta_c^*(n; a) := \frac{C(a)}{n}.$$

Their accuracy is measured by the error variables $\xi_c(n; a) := |\tau_c(n; a) - \tau_c^*(n; a)|$ and $\varepsilon_c(n; a) := |\delta_c(n; a) - \delta_c^*(n; a)|$.

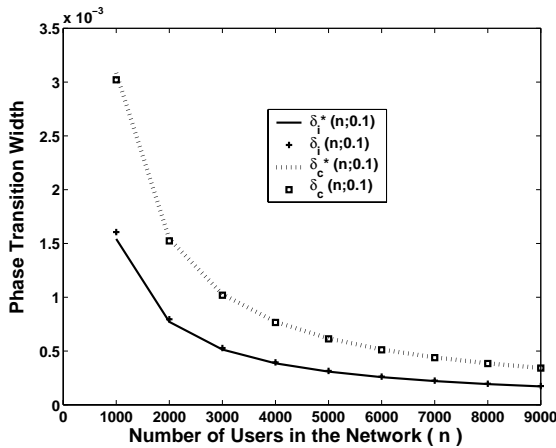
We evaluate $\tau_i(n; a)$ through (14). Overflow issues are circumvented by considering the quantities

$$c_j = \binom{n-k-1}{k-j} \binom{k+1}{j} ((1-(2k-j)\tau)^n,$$

which again can be computed iteratively but in decreasing order. The quantities $\tau_i(n; a)$ and $\delta_i(n; a)$ are approximated by $\tau_i^*(n; a) := \tau_c^*(n; a)/2$ and $\delta_i^*(n; a) := \delta_c^*(n; a)/2$, respectively. The accuracy of these approximations is quantified by the error variables $\xi_i(n; a) = |\tau_i(n; a) - \tau_i^*(n; a)|$ and $\varepsilon_i(n; a) = |\delta_i(n; a) - \delta_i^*(n; a)|$.



(a)



(b)

Fig. 1. Communication range and phase transition width when $a = 0.1$

B. Results

Below we display results for $a = 0.1$. The quantities $\tau_c(n; a)$, $\tau_c^*(n; a)$, $\tau_i(n; a)$ and $\tau_i^*(n; a)$ are plotted in Fig.1(a). The results for $\delta_c(n; a)$, $\delta_c^*(n; a)$, $\delta_i(n; a)$ and $\delta_i^*(n; a)$ are displayed in Fig.1(b). The symbols represent the numerical results (as per computations explained above) and the lines represent the approximations. It is plain that the approximations provide highly accurate results. In addition, for a given n , $\tau_i(n; a)$ and $\delta_i(n; a)$ are about half of $\tau_c(n; a)$ and $\delta_c(n; a)$, respectively, as expected from the asymptotic results.

We further investigate how accurate these approximations are. By virtue of Theorem 3.1, Corollary 3.2, and Theorem 5.1, the approximation errors, namely $\xi_c(n; a)$, $\varepsilon_c(n; a)$, $\xi_i(n; a)$ and $\varepsilon_i(n; a)$ should be of order $o(n^{-1})$. This is indeed reflected by Table.I and Fig.2 upon noting that $n\xi_c(n; a)$, $n\varepsilon_c(n; a)$, $n\xi_i(n; a)$ and $n\varepsilon_i(n; a)$ all go to zero as n grows large.

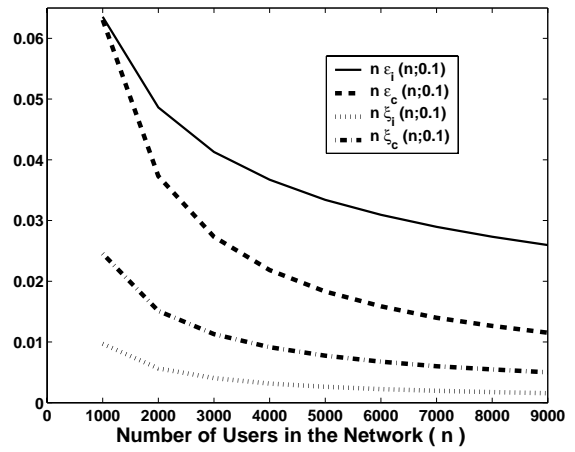


Fig. 2. The asymptotic behavior of error variables

TABLE I

THE ASYMPTOTIC BEHAVIOR OF ERROR VARIABLES

n	100	1000	10000	100000
$n\xi_c(n; 0.1)$	0.1022	0.0245	0.0046	0.0007
$n\varepsilon_c(n; 0.1)$	0.3213	0.0630	0.0106	0.0016
$n\xi_i(n; 0.1)$	0.0446	0.0097	0.0013	0.0001
$n\varepsilon_i(n; 0.1)$	0.1364	0.0636	0.0247	0.0090

VII. DISCUSSION

For $d = 1$, the model considered by Goel et al. [11] coincides with the one-dimensional situation considered here. They show [11, Thm. 1.1] that for every monotone graph property A , the corresponding transition width for property A satisfies

$$\delta_A(n; a) = O\left(\sqrt{\frac{-\log a}{n}}\right). \quad (21)$$

The results obtained here for the property of graph connectivity and for the property of no isolated nodes markedly improve on (21) in that *exact* asymptotics were provided and the rate of decay (namely n^{-1}) is found to be a lot faster than the rough asymptotic bound given at (21).

These authors also show [11, Thm. 1.2] that there *exists* some monotone graph property, say B , such that

$$\delta_B(n; a) = \Omega\left(\sqrt{\frac{-\log a}{n}}\right), \quad (22)$$

in which case from (21) there exist positive constants C_- and C_+ such that

$$C_- \sqrt{\frac{-\log a}{n}} \leq \delta_B(n; a) \leq C_+ \sqrt{\frac{-\log a}{n}}$$

for n sufficiently large. Obviously, graph connectivity and node isolation cannot be such properties!

The discussion of Sections IV and V provides a roadmap to deriving corresponding results in higher dimensions ($d \geq 2$) and for other graph properties: For a given graph property A , we first need to identify the critical threshold associated with the “zero-one” law it satisfies. The effect of “small” perturbations (of the property-specific appropriate order) from

the critical threshold can then be explored with the help of the Poisson convergence paradigm. The resulting Poisson convergence has its roots in the fact that many graph properties can be captured through counting sums of many indicator functions which become vanishingly small and increasingly decorrelated with n large under the appropriate (perturbed) scaling.

Poisson convergence is a common occurrence. A typical example is the existence in $\mathbb{G}_d(n, \tau)$ of at least one copy of a given graph G ; see [14, Chap. 3] for a discussion in the case of Bernoulli graphs. For $d \geq 2$, the properties considered here, connectivity and node isolation, have similar critical thresholds [2, 18]. For $d = 2$, with points distributed uniformly over a disk of unit radius (rather than over a square), it is known [12, 13, 18] that the critical threshold is such that

$$\tau^*(n)^2 = \frac{\log n}{n}, \quad n = 2, 3, \dots$$

Moreover, Venkatesh [18] has recently shown that the number of isolated users indeed converges to a Poisson rv $\Pi(e^{-x})$ when this critical scaling is perturbed to

$$\sigma(n; x) = \min \left(1, \sqrt{\left(\frac{\log n + x}{n} \right)_+} \right), \quad n = 1, 2, \dots$$

It is reasonable to expect that Poisson convergence does hold in arbitrary dimensions, although the authors are not aware of its proof at this time. These issues will be pursued elsewhere.

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APPENDIX

A PROOF OF THEOREM 3.1

Fix x in \mathbb{R} . We restate (11) by noting that for each $\varepsilon > 0$, there exists a finite integer $n^*(\varepsilon, x)$ such that

$$p(x) - \varepsilon < P_c(n; \sigma(n; x)) < p(x) + \varepsilon, \quad n \geq n^*(\varepsilon, x). \quad (23)$$

Now fix a in the interval $(0, 1)$, and pick ε sufficiently small such that $0 < 2\varepsilon < a$ and $a + 2\varepsilon < 1$. Repeatedly applying (23) with $x = x_{a+\varepsilon}$ and $x = x_{a-\varepsilon}$, we get

$$p(x_{a+\varepsilon}) - \varepsilon < P_c(n; \sigma(n; x_{a+\varepsilon})) < p(x_{a+\varepsilon}) + \varepsilon \quad (24)$$

whenever $n \geq n^*(\varepsilon, x_{a+\varepsilon})$, and

$$p(x_{a-\varepsilon}) - \varepsilon < P_c(n; \sigma(n; x_{a-\varepsilon})) < p(x_{a-\varepsilon}) + \varepsilon \quad (25)$$

whenever $n \geq n^*(\varepsilon, x_{a-\varepsilon})$. In the remainder of this proof, all inequalities are now understood to hold for $n \geq n^*(a; \varepsilon)$ where we have set

$$n^*(a; \varepsilon) = \max(n^*(x_a), n^*(\varepsilon, x_{a+\varepsilon}), n^*(\varepsilon, x_{a-\varepsilon})).$$

where $n^*(x)$ denotes the finite integer beyond which the representation (10) holds.

Since $p(x_{a\pm\varepsilon}) = a \pm \varepsilon$, the two chains of inequalities (24) and (25) can be rewritten as

$$a < P_c(n; \sigma(n; x_{a+\varepsilon})) < a + 2\varepsilon$$

and

$$a - 2\varepsilon < P_c(n; \sigma(n; x_{a-\varepsilon})) < a.$$

Thus,

$$P_c(n; \tau_c(n; a)) < P_c(n; \sigma(n; x_{a+\varepsilon})) < P_c(n; \tau_c(n; a + 2\varepsilon))$$

and

$$P_c(n; \tau_c(n; a - 2\varepsilon)) < P_c(n; \sigma(n; x_{a-\varepsilon})) < P_c(n; \tau_c(n; a)),$$

and the strict monotonicity of $\tau \rightarrow P_c(n; \tau)$ yields

$$\tau_c(n; a) < \sigma(n; x_{a+\varepsilon}) < \tau_c(n; a + 2\varepsilon)$$

and

$$\tau_c(n; a - 2\varepsilon) < \sigma(n; x_{a-\varepsilon}) < \tau_c(n; a).$$

Combining these last two inequalities, we conclude that

$$\sigma(n; x_{a-\varepsilon}) < \tau_c(n; a) < \sigma(n; x_{a+\varepsilon}). \quad (26)$$

Upon writing

$$\xi(n; a) = \tau_c(n; a) - \sigma(n; x_a), \quad n = 2, 3, \dots \quad (27)$$

we obtain from (26) that

$$\sigma(n; x_{a-\varepsilon}) - \sigma(n; x_a) < \xi(n; a) < \sigma(n; x_{a+\varepsilon}) - \sigma(n; x_a)$$

with

$$\sigma(n; x_{a-\varepsilon}) - \sigma(n; x_a) = \frac{x_{a-\varepsilon} - x_a}{n}$$

and

$$\sigma(n; x_{a+\varepsilon}) - \sigma(n; x_a) = \frac{x_{a+\varepsilon} - x_a}{n}.$$

As a result, $x_{a-\varepsilon} - x_a \leq \liminf_{n \rightarrow \infty} (n\xi(n; a))$ and $\limsup_{n \rightarrow \infty} (n\xi(n; a)) \leq x_{a+\varepsilon} - x_a$. Given that ε can be taken to arbitrary small, it follows that

$$\liminf_{n \rightarrow \infty} (n\xi(n; a)) = \limsup_{n \rightarrow \infty} (n\xi(n; a)) = 0$$

since

$$\lim_{\varepsilon \downarrow 0} (x_{a-\varepsilon} - x_a) = \lim_{\varepsilon \downarrow 0} (x_{a+\varepsilon} - x_a) = 0.$$

Thus, $\lim_{n \rightarrow \infty} (n\xi(n; a)) = 0$, whence $\xi(n; a) = o\left(\frac{1}{n}\right)$.

Reporting into (27) leads to

$$\tau_c(n; a) = \sigma(n; x_a) + o(n^{-1}), \quad n = 2, 3, \dots$$

and the desired result readily follows from (9) and (13).