TECHNICAL RESEARCH REPORT

A Controllability Counterexample

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Abstract

Simulation by time discretizations can be qualitatively misleading; as counterexamples a class of non-controllable single-input two-dimensional bilinear control systems is presented whose Euler discretizations are controllable on the punctured plane.

Index Terms

bilinear systems, controllability, difference equations.

I. INTRODUCTION

BILINEAR control systems are good model systems on which to experiment with nonlinear control techniques. For computer simulation a discrete-time approximation to the system is needed, and sometimes one uses Euler's method. The Euler discretization of

$$\dot{x} = Ax + uBx \tag{1}$$

$$\text{is} \quad x(k+1) = x(k) + \tau(A + u(k)B)x(k) \tag{2}$$

where $k = 0, 1, \dots$ and τ is the step size.

Can controllability of (1) be concluded from the controllability of its computer model (2)? The Euler discretization of $\dot{x}_1 = ux_1$ is a counterexample in dimension one, and is the only one commonly mentioned. The two-dimensional examples below are more interesting and seem not to be widely known.

II. EXAMPLES

Start with a single-input symmetric BCS on $\mathbb{R}^2_*$ whose matrix B is the $\boldsymbol{\alpha}$ representation of the complex numbers $\mathbb{C}$.

$$\dot{x} = u(t)Bx, \text{ where } B := \boldsymbol{\alpha}(\lambda + \mu\sqrt{-1}) = \lambda I + \mu J = \begin{bmatrix} \lambda & \mu \\ -\mu & \lambda \end{bmatrix}, \ x(0) = \xi \tag{3}$$

$$\text{and} \quad \gamma_\xi := \left\{ x(t) = \exp\left(B \int_0^t u(s)ds \right) \xi, \ t \geq 0 \right\}$$

is the solution path. Assume that λ is negative. For each ξ, the path γ_ξ is a logarithmic spiral curve. This system is uncontrollable in the strongest sense. The discrete control system obtained from (3) by Euler's method is (after absorbing the time-step in the control v)

$$x^+ = (I + vB)x; \ x(0) = \xi; \ v(k) \in \mathbb{R}. \tag{4}$$

Assume that λ is negative. For control values $v(k) = \omega$ in the interval

$$U := \frac{-2\lambda}{\lambda^2 + \mu^2} < \omega < 0, \quad \|I + \omega B\|^2 = 1 + 2\omega\lambda + \omega^2(\lambda^2 + \mu^2) < 1.$$

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With $v(k) \in U$ the path of (4) approaches the origin, and for a constant control $v(k) = \omega \in U$ that path is a discrete subset of the continuous spiral

$$\eta_\xi := \{(1 + \omega B)^t \xi,\, t \in \mathbb{R}\}.$$

The conjecture that if $\lambda < 0$ then (4) is controllable was suggested by the observation that for any $\xi \in \mathbb{R}^2_*$ the union of the lines tangent to each spiral η_ξ covers $\mathbb{R}^2_*$ (as do the tangents to any asymptotically stable spiral path). It is not necessary to prove this conjecture here, since by using some symbolic calculation one can generate families of examples.

Proposition 1 *There exists an instance of B such that* (4) *is controllable.*

Proof: Let $\bar{v} = -2\lambda/(\lambda^2 + \mu^2)$; then the path $X(\bar{v}, k)\xi$, $k \in \mathbb{Z}$ lies on a circle of radius $\|\xi\|$. To show controllability we use a B for which there are nominal paths with constant control $v(k) \equiv \bar{v}$ that take ξ to itself at some time $k = N - 1$. New paths can be constructed by changing the value of $v(N-1)$ and adding a new control $v(N)$; in this way the paths can reach states[1] near ξ. If $v(N) = 0$ the path still will end at ξ. The convenient case $N = 6$ can be obtained when

$$\lambda := -1/2, \mu := \frac{-1}{2\sqrt{3}}. \text{ For } \bar{v} := 1,\ (I + \bar{v}B)^6 = I. \tag{5}$$

The two-step transition matrix for (4) is $f(s,t) := (I + tB)(I + sB)$. For $x \in \mathbb{R}^2_*$ the Jacobian determinant of the mapping $(s,t) \mapsto f(s,t)x$ is, for matrices $B = \lambda I + \mu J$,

$$\begin{aligned} \frac{\partial(f(s,t)x)}{\partial(s,t)} &= \mu(\lambda^2 + \mu^2)(t - s)\|x\|^2 \\ &= \frac{\sqrt{3}}{2}(t - s)\|x\|^2 \end{aligned}$$

for our chosen B.[2] Let $s = v(5) = \bar{v}$ and add the new step using the nomimal value $v(6) = t = 0$, to make sure that ξ is a point where this determinant is non-zero. The inverse of this mapping will be used to construct controls that start from ξ and reach any point in an open neighborhood of ξ; this neighborhood $U(\xi)$ will be constructed explicitly as follows.

Let $v(k) := \bar{v}$ for the five values $k = 0, \ldots, 4$. For target states $\xi + x$ in $U(\xi)$, a changed value $v(5) := s$ and a new value $v(6) := t$ are needed; finding them is simplified, by the use of (5), to the solution of two quadratic equations in (s,t)

$$(I + tB)(I + sB)\xi = (I + \bar{v}B)(\xi + x). \text{ If } \xi = (1, 0)$$
$$\text{then in either order } (s,t) = \frac{1}{6}\left(3 + 2\sqrt{3}x_2 \pm \sqrt{3}\sqrt{3 + 12x_1 + 4x_2^2}\right),$$

which are real in the region $U(\xi) = \{\xi + x \mid (3 + 12x_1 + 4x_2^2) > 0\}$ that was to be constructed. If $P \in \boldsymbol{\alpha}(\mathbb{C}_*)$ then

$$PB = BP \text{ so } Pf(x,t)\xi = f(s,t)P\xi = Px$$

generalizes our construction to any ξ because $\mathbb{C}_*$ is transitive on $\mathbb{R}^2_*$.

Apply the Lemma in the Appendix to conclude controllability.

□

The purpose of this study was to show actual controllability for the discretization of a control system whose paths are one-dimensional. If all that is wanted is to show that the accessibility property (open reachable sets) can hold for the Euler discretizations of bilinear systems whose attainable sets have no interior, there are many more examples. For our two-dimensional example (3) suppose the spiral

[1] Note that similar examples can be constructed for which the trajectories $(I + \bar{v}B)^k\xi$ are dense in the circle.

[2] Mathematica was used for these calculations.

degenerates ($B = J$) to a circle through ξ; the region exterior to that circle (but not the circle itself) is easily seen to be attainable with (4) using the lines $\{(I + tJ)x \mid t \in \mathbb{R}\}$, so strong accessibility can be concluded, but not controllability.

For BCS on $\mathbb{R}^n_*$ with skew-symmetric matrices A and B, if the attainable sets are the concentric spheres $\|x\|^2 = \|\xi\|^2$ then the Euler discretization has the exteriors of those spheres attainable. Replace $\|x\|^2$ with a Lyapunov function $V(x) \gg 0$ to obtain a similar accessibility property for Euler discretizations of nonlinear control systems $\dot{x} = f(x, u)$ for which V is the only invariant function.

As to other numerical methods that must be watched with care, see [1] for strange behavior of variable-step Runge-Kutta-Felberg methods analogous to bursting phenomena in adaptive control system. About twenty years ago it was pointed out by several authors that numerical methods could introduce chaos.

III. Conclusion

It has been shown here that Euler discretizations of an uncontrollable bilinear system can have attainable sets which are of larger dimension than those of the continuous-time system and can be controllable, as shown by a numerical example. The simple lemma in the Appendix may be useful in other proofs of controllability.

Appendix

Lemma 1 *The system $x(k+1) = f(x(k), u(k))$ is controllable on a connected submanifold $S \subset \mathbb{R}^n$ if and only if there exist controls such that for every initial state ξ a neighborhood $N(\xi) \subset S$ of ξ is attainable.*

Proof: The “only if” part follows from the definition of controllability.

For the “if” part, given any two states $\xi, \zeta \in S$ we need to construct a finite set of points $p_0 = \xi, \ldots, p_k = \zeta$ such that there exists a control u for which the path from p_{j-1} terminates, in a finite time, at p_j.

First construct any continuous curve $\gamma(\tau)$ in S of finite length in the metric induced on $S \subset \mathbb{R}^n$ such that $\gamma(0) = \xi$, $\gamma(1) = \zeta$. For each point $p = \gamma(\tau)$ there exists an attainable neighborhood $N(p) \subset S$. The collection of neighborhoods $\cup_x N(x)$ covers γ, which is compact, so we can find a finite subcover $N_1, \ldots, N_k$ where $p_1 = N(\xi)$ and the k neighborhoods are ordered along γ. From ξ choose a control that reaches $p_1 \in N_1 \cap N_2$, and so forth; $\zeta \in N_k$ can be reached from p_{k-1}. $\square$

This seems to be a folk-lemma of no known provenance.

References

[1] Skufca, J. D., “Analysis matters: A surprising instance of failure of Runge–Kutta–Felberg ODE solvers,” *SIAM Review*, vol.46, no. 4, pp. 729-737, 2004.