TECHNICAL RESEARCH REPORT

Continuous and Discrete Inverse Conductivity Problems

by Franklin Gavilanez, John Baras and Carlos Berenstein

TR 2004-8

CONTINUOUS AND DISCRETE INVERSE CONDUCTIVITY PROBLEMS[1]

FRANKLIN GAVILANEZ, JOHN BARAS and CARLOS BERENSTEIN

February, 2004

Abstract

Tomography using CT scans and MRI scans is now well-known as a medical diagnostic tool which allows for detection of tumors and other abnormalities in a noninvasive way, providing very detailed images of the inside of the body using low dosage X-rays and magnetic fields. They have both also been used for determination of material defects in moderate size objects. In medical and other applications they complement conventional tomography. There are many situations where one wants to monitor the electrical conductivity of different portions of an object, for instance, to find out whether a metal object, possibly large, has invisible cracks. This kind of tomography, usually called Electrical Impedance Tomography or EIT, has also medical applications like monitoring of blood flow. While CT and MRI are related to Euclidean geometry, EIT is closely related to hyperbolic geometry. A question that has arisen in the recent past is whether there is similar "tomographic" method to monitor the "health" of networks. Our objective is to explain how EIT ideas can in fact effectively be used in this context.

1 Introduction and preliminaries

Networks have become ubiquitous in present society and thus it has become important to avoid and detect disruptions. In particular, it is important to prevent malicious intruders from disrupting them. To achieve this sufficiently early, it is essential to count on a mathematical model that can allow early detection of attacks to the network. The mathematical tool that we consider to accomplish the early detection of disruptions is based on the use of tomographic ideas. One of the questions we are considering is how to find out whether an attack against the network by traffic overload is taking place by monitoring traffic only at the periphery of the network (input-output map), and hence, the use of a tomographic approach.

[1] This write-up is an extended version of the invited lecture given by the third author at PASI 2003. It represents research supported in part by grants NSF-DMS0070044 and ARO-DAAD19-01-1-0494.

We present first a general idea on how tomography can be used to do such monitoring and implemented as a diagnostic tool. Consider subsets A and Ω in $\mathbb{R}^2$ such that $A \subset \Omega$ and let $f \in C_o(\mathbb{R}^2)$ be such that $Supp f \subset \Omega$. Assume A represents an object from which we want to get some information. We could just ask for f to be rapidly decaying for $z \longrightarrow \infty$.

Let Φ be the collection of all the straight lines in Ω connecting any pair of points a and b where $a, b \in \partial\Omega$, $a \neq b$, i.e., $\Phi = \{\zeta \ line \ / a, b \in \zeta \cap \partial\Omega, \ a \neq b\}$, and $f(x)$ represents the distribution or density at the point $x \in \Omega$. Let the Radon transform

$$R(f) = \left\{ \int_{\zeta \in \Phi} f(x) dx \quad \right\} \tag{1.1}$$

be the set of all line integrals of the function f.

The concept of tomography then can be understood as the reconstruction of the function f from the set of values given by $R(f)$. To recover f is thus the same as finding the inverse of the operator R . Hence, we are able to know the value of f at any point x in Ω without having access to the interior of Ω.

A well-known example of tomography is transmission CT in diagnostic radiology, [31], where CT stands for computerized tomography. Essentially the setup consists of a detector and an X-rays beam source. A cross-section of the human body is scanned by a thin X-rays beam. Because the density of the tissue of the human body changes from its surface to its interior, there is a intensity loss which is recorded by the detector and processed by a computer to produce a two-dimensional image which in turn is displayed on a screen. Given that the X-rays go through the tissue, it is clear the X-rays absorption is related to the attenuation coefficient. Let $f(x)$ be the X-ray attenuation coefficient of the tissue at the point x. Taking a close look, X-rays traversing a small distance Δx at x suffer a relative intensity loss,

$$\frac{\Delta I}{I} = f(x)\Delta x. \tag{1.2}$$

If the X-rays are considered as straight lines, as indeed they essentially are, let ζ be the straight line representing the beam, I_o the initial intensity of the beam, and I_1 its intensity after having traversed the body. It follows from (1.1) that

$$\frac{I_1}{I_o} = \exp\{-\int_{\zeta} f(x) dx\} \tag{1.3}$$

thus the scanning process provides us with the line integral of the function f along each of the lines ζ. From the set knowledge of all of these integrals the problem is to reconstruct f. Equally well known by now is MRI, magnetic resonance tomography, where the underlying space is $\mathbb{R}^3$ and the integrals take place over the family of all planes in $\mathbb{R}^3$.

See [25] for other examples of imaging and detecting equipment based on tomographic principles and [34] as a recent overview of this kind of problems discussed here. We refer to [5] for more details on MRI.

Sections 2 and 3 are provide the background information on the Radon transform. In section 4, we also consider tomographic examples like the geodesic Radon transform in the hyperbolic plane, which appears naturally in relation to the inverse conductivity problem and, possibly, to internet tomography. We conclude in section 5 with some new results on this last subject. The key ingredient is the attempt to understand what happens in a network from "boundary measurements", that is, to determine whether all the nodes and routers are working or not and also measure congestion in the links between nodes by means of introducing test packets (ICMP packets) in the "external" nodes, that is, the routers. The question of finding out whether there are nodes that are in working order is a classical question in graph theory. For networks, it is also interesting to try to predict future problems due to congestion. (Note that nodes could fail to work for other reasons than congestion on the links starting at a given node.) This requires to monitor also traffic intensity, also known as load, congestion, etc., in different contexts. There is another analogy to mathematical tomography that arose independently and maybe closer to the consideration of this question in the context of electrical networks. Curtis and Morrow have done very interesting work in this context, both theoretical and in simulations, see, for instance, [20] and [19]. Another analogy in the same direction arises when we consider very large networks, as the internet, which could be considered as the discretization of an underlying continuous model. In this way, we can see the analogy with the well-known inverse conductivity problem and we could try to profit from the large body of mathematical research in this area. The analogy with this particular inverse problem indicates that if one were to pursue this "abstract" approach the "correct" geometry is closer to be hyperbolic than to be Euclidean [7]. On the other hand, as of this moment, we have found that those tomographic analogies are more useful for providing directions of research and methods to consider these problems than providing an exact correspondence between the two phenomena. It is in this context that [9] modelled "internet tomography" as an inverse Dirichlet-to-Neumann problem for a graph with weights. In this situation, one can prove that characteristics of the graph, namely, its connectivity and the traffic along links can be uniquely determined by boundary-value measurements as shown in [9] which is the natural analogue of the continuous inverse conductivity problem.

Among the questions that arise naturally using the inverse conductivity problem as a guiding model there are a number of questions that have been previously addressed using other points of view. Namely, the problems already addressed in [17] for internet tomography are:

1. Link-level inference, in other words link-level parameter estimation based on end-to-end path-level traffic measurements. Examples of this are unicast inference of link loss rates, unicast inference of link delay distributions, topology identification, loss rates by using multicast probing and so on.

2. Path-level inference (origin-destination tomography OD) in other words sender-receiver path-level traffic intensity estimation based on link-level traffic measurements. One example of this is time-varying OD traffic matrix estimation.

We would like to conclude by thanking the editors and the referee for his useful comments.

2 The Radon transform in $\mathbb{R}^2$

Let $\omega \in S^1$, then $\omega = (\cos\theta, \sin\theta)$, and take $p \in \mathbb{R}$. The locus of equation $x \cdot \omega = p$ represents the line l that is perpendicular to the line r passing through the origin and forming an angle θ with the real line $\mathbb{R}$. If B is the intersection of l and r, the euclidean distance d (signed) from $B = p\omega$ to the origin is equal to p.

Consider a nice function f or any reasonable function f , for instance $f$continuos and compactly supported, then consider the line integral with respect to Euclidean arc length ds,

$$Rf(\omega, p) := \int\limits_{x\cdot\omega=p} f(x)ds = \int\limits_{-\infty}^{\infty} f(x_o + t\omega^{\perp})dt \tag{2.1}$$

where x_o is a fixed point in l, i.e. it satisfies $x_o \cdot \omega = p$, and $\omega^{\perp} = (\cos\theta, -\sin\theta)$, the rotation of ω by $\pi/2$. When p and ω range over $\mathbb{R}$ and S^1respectively, we get all of the lines in $\mathbb{R}^2$. Usually x_o is taken as $p\omega$.

The map $f \to Rf$ is called the Radon transform and Rf is called the Radon Transform of f. We refer to [31], and others there in, for a detailed exposition of the Radon transform. Clearly Rf is a function defined on $S^1 \times R$, i.e. the family of all lines in $\mathbb{R}^2$ with the compatibility condition mentioned in [8]:

$$(Rf)(-\omega, -p) = Rf(\omega, p) \tag{2.2}$$

Given that l doesn't change when $\boldsymbol{\omega}$ and p are changed to $\lambda\boldsymbol{\omega}$ and λp, $\lambda \neq 0$, $\lambda \in \mathbb{R}$, then the Radon transform can be extended from $S^1 \times \mathbb{R}$ to $\mathbb{R}^2 \times \mathbb{R}$. The pair $(\lambda\omega, \lambda p)$ is identified with (ω, p), and the extension of the Radon transform satisfies

$$R\mu(\lambda\omega, \lambda p) = R\mu(\omega, p), \tag{2.3}$$

therefore, the Radon transform can be extended as an homogeneous function of degree zero, a very important property of this transform.

Rf can also be defined as

$$Rf(\omega, p) := \int_{\mathbb{R}^2} f(x)\delta(p - \omega \cdot x)dx \tag{2.4}$$

with δ the 1-dimensional delta, which allows us to obtain easily the properties below in the natural coordinate space of lines. In particular for $\omega = (\omega_1, \omega_2)$ with $|\omega| = 1$, then

$$R(\frac{\partial f(\omega, p)}{\partial x_i}) = \omega_i \ \frac{\partial Rf(\omega, p)}{\partial p} \tag{2.5}$$

and

$$\frac{\partial Rf(\omega, p)}{\partial \omega_i} = -\frac{\partial}{\partial p} R(x_i \ f)(\omega, p)$$

hence, it follows that if $P_m(x)$ is a homogeneous polynomial with constant coefficient of degree m and $|\omega| = 1$,

$$R(P_m(\partial x) \ f(\omega, p)) = P_m(\omega) \cdot \frac{\partial^m Rf(\omega, p)}{\partial p^m} \tag{2.6}$$

and

$$P_m(\partial_\omega)(Rf(\omega, p)) = (-1)^m \frac{\partial^m}{\partial p^m} R(P_m(x) \ f(\omega, p))$$

where

$$\partial_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}), \qquad \partial_\omega = (\frac{\partial}{\partial \omega_1}, \frac{\partial}{\partial \omega_2})$$

and obviously $x = (x_1, x_2)$.

One can similarly define the Radon transform in $\mathbb{R}^n$ and verify that the properties (2.5) and (2.6) extend to this case. In particular for the Laplacian Δ in $\mathbb{R}^n$,

$$R(\Delta f) = \frac{\partial^2 Rf(\omega, p)}{\partial p^2}, \tag{2.7}$$

where, for each direction $\omega \in S^{n-1}$ the right hand side is the Laplace operator in dimension 1. Note that in general

$$R(\Delta f)(\omega, p) = (\omega_1^2 + \cdots + \omega_n^2) \frac{\partial^2 Rf(\omega, p)}{\partial p^2}$$

As a consequence, if the function f depends also on time, and $\Box_n$ represents the wave operator in n dimensions we conclude that

$$R \, \Box_n f = \Box_1 Rf.$$

therefore, the Radon transform in n dimensions is localizable if and only if the wave equation is localizable. Fixing $\omega \in S^{n-1}$, one can express this identity by saying that the Radon transform interwines the wave operator $\Box_n = \Delta - \frac{\partial^2}{\partial t^2}$ in $n-$dimensions with the wave operator $\Box_1 = \frac{\partial^2}{\partial p^2} - \frac{\partial^2}{\partial t^2}$ in $1-$space dimension. It follows that the Radon transform can not be localized in even dimensions [10]. In spite of this observation one can obtain an *almost* localization of the Radon transform in $\mathbb{R}^2$. The key elements is the use of wavelets as it will be described in the next section. Meanwhile, for the sake of completeness we remind the reader of the standard inversion formula for the Radon transform in $\mathbb{R}^2$. It depends on the following identity, usually called the Fourier slice theorem. Namely, writing the Fourier transform $F_2(f)$ of a nice function f in $\mathbb{R}^2$ in polar coordinates (s, ω) we have

$$\int_{\mathbb{R}^2} f(x)\, e^{-is\omega\cdot x}\, dx = \int_{-\infty}^{\infty} Rf(\omega, p)\, e^{-isp}\, dp, \ \ x \in \mathbb{R}^2 \tag{2.8}$$

or, in a more concise form,

$$F_2(f) = F(Rf)$$

where, clearly, F_2 stands for the 2-dimensional Fourier transform and F stands for the 1-dimensional Fourier transform in the variable p which provides one standard inversion formula for the Radon transform

$$f = F_2^{-1} F(Rf) \tag{2.9}$$

There is another inversion formula that has a number of advantages for us, and we proceed to explain it now. To simplify we work in $X = \mathbb{S}(\mathbb{R}^2)$, the Schwartz space of functions f and $Y = \mathbb{S}(S^1 \times \mathbb{R})$ the Schwartz space of functions g. Let $f_{1,}\ f_2 \in X$ and $g_{1,}\, g_2 \in Y$, and $\langle f_{1,}\ f_2 \rangle_X$, $\langle g_{1,}\ g_2 \rangle_Y$ the inner products in X and Y respectively, then because of the linearity of the operator R, we write the equation that defines R^*, the adjoint operator of R

$$\langle Rf, g \rangle_Y = \langle f, R^* g \rangle_X \tag{2.10}$$

The explicit expression for $R^* g$ is given by

$$\int_{S^1} g(\omega, \omega \cdot x) d\omega = R^* g \tag{2.11}$$

R^* is called the *backprojection operator*.

The function $R^* g$ is such that for x fixed

$$R^* g(x) = \int_{S^1} g(\omega, \omega \cdot x) d\omega$$

is the integral of g over all lines passing through x.

In order to get a formula for f from the Radon transform value, the next important property of the backprojection operator holds.

$$(R^*g) * f = R^*(g \circledast Rf) \tag{2.12}$$

where $\circledast$ stands for the convolution with respect to the second argument, and the Radon inversion formula is then given by.

$$\int_{\mathbb{R}^2} e^{i2\pi x\cdot\varsigma}\frac{|\varsigma|}{2}F_2(R^*Rf)(\varsigma)d\varsigma = f(x) \tag{2.13}$$

Introducing the square root Λ of the Laplacian operator Δ, we have

$$\Lambda(R^*Rf)(x) = f(x) \tag{2.14}$$

which is usually called *the backprojection inversion formula*.

3 Localization of the Radon transform

As explained above, we can not in general reconstruct the function f in a disk $D(a, r)$ of $\mathbb{R}^2$ using only lines l passing through $D(a, r)$. One can localize up to a baseline value of the function f, that is, one can recover for a disk $D(a, r)$ by using only the data $Rf(l)$ for passing through $D(a, r + \varepsilon)$, for arbitrary $\varepsilon > 0$, up to an additive constant [11] and [12] . The key element is the use of wavelets.

Let us recall the basic properties of the continuous wavelet transform (CWT) and the discrete wavelet transform (DWT).

Let $b \in \mathbb{R}$ be and f_b the translation of f by b, i.e. $f_b(x) = f(x - b)$, then

$$\widetilde{f}_b(\varsigma) = e^{-i2\pi b\varsigma}\widetilde{f}(\varsigma)$$

where $\widetilde{f}$ is the Fourier transform of f. Now let $D_a f$ be the dilation of f by the scaling factor $a \in \mathbb{R}$, $a > 0$ where $D_a f$ is defined as $D_a f(x) = \frac{1}{\sqrt{a}}f(\frac{x}{a})$ where the term $\frac{1}{\sqrt{a}}$ is chosen such that $\|f\|_2 = \|D_a f\|_2$, i.e., f and $D_a f$ have the same energy, then one has

$$\widetilde{(D_a f)}(\varsigma) = D_{1/a}\widetilde{f}(\varsigma) = \sqrt{a}\widetilde{f}(a\varsigma) \tag{3.1}$$

As pointed out in [27], equation (3.1) tells us that the Fourier transform $\widetilde{(D_a f)}(\varsigma)$ is dilated by $1/a$, then we lose in the $\varsigma-$domain (frequency) what we gained in $x-$domain (time). In other words, there is a trade-off between time and frequency localization if ς and x stand for frequency and time respectively.

3.1 Wavelets as a tool

In what follows for $f \in L_1(\mathbb{R})$ (or $f \in S(\mathbb{R})$) we denote

$$\tilde{f}(\varsigma) = \int_{-\infty}^{\infty} f(x)\, e^{-\, i2\pi\varsigma x} dx, \tag{3.2}$$

the usual Fourier transform of f. Let us recall from [27] the definition of the continuous wavelet transform (CWT) associated to a "mother" wavelet Ψ. Namely, following [8], given a "mother" wavelet $\Psi \in L_2(\mathbb{R}) \cap L_1(\mathbb{R})$ and $f \in L_2(\mathbb{R})$, we define the wavelet transform of f as

$$W_\Psi f(a,b) := \int_{-\infty}^{\infty} f(t)\, \overline{\Psi}(\frac{t-b}{a})\frac{dt}{\sqrt{a}} =< f, D_a\Psi_b(t) >_{L_2} \tag{3.3}$$

$b, a \in \mathbb{R}$, $a > 0$, where, for a function g and $b \in \mathbb{R}$ we let $g_b(t) = g(t-b)$.

One requires that the "mother" wavelet Ψ to be oscillatory, i.e. $\int_{-\infty}^{\infty} \Psi(x)dx = 0$. In fact, one assumes stronger condition

$$C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\overset{\sim}{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma < \infty, \tag{3.4}$$

called the *admissibility condition*. The admissibility condition is satisfied when Ψ has several vanishing moments,i.e., for $0 \leq k < s$

$$\int_{-\infty}^{\infty} x^k\ \Psi(x)dx = 0$$

The functions $D_a\Psi_b$ are called the wavelets

The function f can be reconstructed from its wavelet transform by means of the "*resolution identity*" formula

$$f = C_\Psi^{-1} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} < f, D_a\Psi_b(t) >_{L_2} D_a\Psi_b(t)dt$$

where $C_\Psi < \infty$ since $\Psi \in L_1(\mathbb{R})$.We refer to [27] for the general theory of wavelets.

Proposition 1 explains how to use wavelets to obtain (almost) localization.

Proposition 1 [10] *Let n be an even integer, and $h \in L_2(\mathbb{R})$ a function with compact support such that for some integer m ≥ 0 $\tilde{h}$ is n+m-1 times differentiable and satisfies*

1. $\gamma^j \tilde{h}^{(k)}(\gamma) \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ for 0$\leq j \leq m$, 0$\leq k \leq m+n-1$

2. $\int_{-\infty}^{\infty} t^{\,j} h(t)dt = 0$ *for* $0 \leq j < m+1$, i.e., h *has* $m+1$ *vanishing moments*
Then

$$I^{\,1-n}h(t) = o(|t|^{-n-m+1}) \textit{as } |t| \longmapsto \infty$$

and

$$t^{\,n+m-1} I^{\,1-n} h \in L_2(\mathbb{R})$$

The fact that $I^{\,1-n}h(t) = o(|t|^{-n-m+1})$ *as* $|t| \longmapsto \infty$ tells us that $I^{\,1-n}h$ decays as $|t|^{-(n+m-1)}$, and therefore, it does a good localization job.

For practical purposes, the continuos wavelet transform, CWT, is discretized and the discrete wavelet transform, DWT, is obtained. In order to discretize it, consider $m, n \in Z$ and the values a, b that appear in $W_\Psi f(a, b)$ are restricted to only discrete values $a = a_o^m$, $b = nb_o a_o^m$, $a_o > 1,\ b_o > 1$ fixed. (The fact that $a_o > 1,\ b_o > 1$ it really does not matter because m, n can be negative). The discrete wavelet transform DWT of f is defined as

$$W_{m,n}^{\Psi}(f) = a_o^{-m/2} \int_{-\infty}^{\infty} f(t)\overline{\Psi}(a_o^{-m}t - nb_o) \tag{3.5}$$

where, as before, it holds that $\int_{-\infty}^{\infty} \Psi(t) = 0$, and the wavelets are given by

$$\begin{aligned}\Psi_{m,n}(x) &= a_o^{-m/2}\Psi(a_o^{-m}x - nb_o) \\ &= a_o^{-m/2}\Psi(a_o^{-m}(x - nb_o a_o^m)\end{aligned}$$

hence $\Psi_{m,n}$ is localized around $nb_o a_o^m$ in time, (3.5) can be also expressed as $\langle f, \Psi_{m,n}\rangle$ which are called the wavelet coefficients.

It is important to point out that in the discrete case, in general, there does not exist a *resolution of the identity* formula to recover f, so the recovering of f must be done by using some other means , for instance numerical ones. The choice of the wavelet Ψ is essentially only restricted by the requirement that the admissibility condition holds, i.e., $C_\Psi = \int_{-\infty}^{\infty} \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma$ is finite. Following [21], the discretization is only restricted to positive values of a then the admissibility condition becomes

$$C_\Psi = \int_0^\infty \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma = \int_{-\infty}^0 \frac{\left|\widetilde{\Psi}(\varsigma)\right|^2}{|\varsigma|} d\varsigma < \infty$$

and since a, b will be discrete values only, then the dilation parameter is chosen as $a_o^{\,m}$, $m \in Z$ and $a_o \neq 1$ is fixed (usually $a_o > 1$). The value b_o is also fixed and it is chosen such that $\Psi(x - n\, b_o)$ covers the whole line.

Now, for reasonable Ψ and suitable a_o, b_o, there exist $\Psi_{m,n}$ so that the discrete wavelet coefficients $\langle\, f, \Psi_{m,n}\rangle$ characterize completely f which is given by

$$f = \sum_{m,n} \langle\, f, \Psi_{m,n}\rangle\, \Psi_{m,n}$$

then any function in $L_2(\mathbb{R})$ can be written as a superposition of the wavelets $\Psi_{m,n}$.

3.2 Wavelets and the Radon transform

Now we want to state some results that relate wavelets and the Radon transform, which is of interest for tomography, [11], [12].

Proposition 2 *Let $\rho \in L_2(\mathbb{R})$ real valued, even, and satisfying*

$$\int_{-\infty}^{\infty} \frac{\left|\widetilde{\rho}(r)\right|^2}{r^3} dr < \infty \tag{3.6}$$

where $\widetilde{\rho}$ stands for the 1-dimensional Fourier transform of ρ. define the radial function Ψ in $\mathbb{R}^2$ by

$$F_2\Psi(\varsigma) = \frac{\widetilde{\rho}(|\varsigma|)}{|\varsigma|}$$

where as before, F_2 is the 2-dimensional Fourier transform, then Ψ is a wavelet for n=2 and the wavelet transform of f is such that

$$W_\Psi f(a,b) = a^{-1/2} \int_{S^1} (W_\rho R_\omega\ f)(a, b\omega) d\omega$$

where $R_\omega\ f$ is such that $R_\omega\ f(p) = Rf(w,p)$.

Proposition 3 *Let Ψ be a separable 2-dimensional wavelet, i.e.,*

$$\Psi(x) = \Psi^1(x_1)\Psi^2(x_2), \quad x \equiv (x_1, x_2)$$

where for i=1,2 $\left|\widetilde{\Psi}(\gamma)\right| \leq C_1(1+|\gamma|)^{-1}$ for all $\gamma \in \mathbb{R}$. Defining the family of the one-dimensional functions $\{\rho_\omega\}_{\omega\in S^1}$ by

$$\widetilde{\rho_\omega}\,(\gamma) = \frac{1}{2}\,|\gamma|\,\widetilde{\Psi}^1(\gamma\omega_1)\widetilde{\Psi}^2(\gamma\omega_2), \quad \omega = (\omega_1, \omega_2)$$

i.e. $\rho_\omega = F_1^{-1}(\widetilde{\rho_\omega})$. *Then for every* $f \in L_1(R^2) \cap L_2(R^2)$,

$$(W_\Psi f)(a,x) = a^{-1/2} \int_{S^1} \left(W_{\rho_\omega} \, R_\omega f \right)(a, x \cdot \omega) d\omega$$

The proposition shows that the wavelet transform of a function $f(x)$ given any mother wavelet and at any scale can be obtained by backprojecting the wavelets transform of the Radon transform of f using wavelets that vary with each angle, the argument of $\omega,$ but which are admissible for each angle, i.e. $C_\Psi < \infty$.

4 The hyperbolic Radon transform and EIT

In this section we discuss the Radon transform on the hyperbolic plane, state some formulae analogous to the ones that were given in section 2 to invert the Radon transform. The backprojection inversion formula is one of them, and later we will see how the hyperbolic Radon transform is related to electric impedance tomography (EIT).

In [6] and [7] it is shown that the hyperbolic Radon transform is involved in the problem of reconstructing the conductivity distribution on a plate by using electrical impedance tomography EIT.

4.1 The hyperbolic Radon transform

Let D be the unit disk of the complex plane, i.e. $D = \{z \in \mathbf{C} / \ |z| < 1\}$. In D, a Riemannian structure is defined through the hyperbolic metric of arc-length ds given by

$$ds = \frac{2\,|dz|}{(1-|z|^2)} \tag{4.1}$$

with dz the Euclidean distance in $\mathbb{R}^2$, and the hyperbolic distance between two points $z, w \in D$ is given by

$$d(z,w) = \operatorname{arcsin} h \left(\frac{|z-w|}{(1-|z|^2)^{1/2}(1-|w|^2)^{1/2}} \right)$$

The set of lines that are diameters of D, and the set of intersections between the Euclidean circles and D such that the resultant lines (intersections) are perpendicular to the boundary ∂D of D are the geodesics or h-lines for the metric (4.1).

If $z \in D$ is expressed in polar coordinates by (w, r) where $w = z/\mid z\mid, r = d(z, 0)$, then the metric (4.1) becomes

$$ds^2 = dr^2 + \sinh^2 r \; dw^2$$

where dw^2 is the usual metric on ∂D. This indicates that the area in hyperbolic geometry is exponential on the radius r. Let us recall that if E is a set contained in the hyperbolic disk D, then the hyperbolic area of E, $h - area(E)$, is given by

$$h - area(E) = \iint_E dxdy \frac{4}{[(1-|\ z|^2)]^2} dxdy, \quad z = (x, y)$$

and the hyperbolic length of any curve γ in D, $h - length(\gamma)$, is given by

$$h - length(\gamma) = \int_\gamma \frac{2|dz|}{1-|\ z|^2}$$

In terms of the Euclidean Laplacian Δ, the Laplace-Beltrami operator Δ_H in polar coordinates on D can be expressed as

$$\Delta_H = \frac{\left(1 - |z|^2\right)^2}{4} \Delta = \frac{\partial^2}{\partial r^2} + \coth r \frac{\partial}{\partial r} + \sinh^{-1} r \frac{\partial^2}{\partial w^2} \tag{4.2}$$

and in the Euclidean coordinates, $z = (x, y)$, (4.2) becomes

$$\Delta_H = \frac{(1 - x^2 - y^2)^2}{4} (\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}).$$

Following [8], we denote the Radon transform of a function f within the hyperbolic plane by $R_H f$ which is a function on the family of geodesics in D. It is defined as follows,

$$Rf(\gamma) = R_H f(\gamma) = \int_\gamma f(z) ds(z), \quad \gamma\ geodesic\ in\ D, \tag{4.3}$$

where f is a function such that (4.3) makes sense. For instance, if f is a function with compact support, i.e., $f \in C_o(D)$ or $f \in \mathbb{S}(D)$, the Schwartz space, which is the space of functions rapidly decaying as $|z| \to 1$. In fact, f has to decay a bit faster than e^{-r} because the element of length $ds(z)$ grows as e^r. (As $ds^2 = dr^2 + \sinh^2 r\ dw^2$, ds grows as e^r). If Γ is the space of all the geodesics in D, then the dual (adjoint) transform R^* (backprojection operator) is given by

$$R^* \Phi(z) = \int_{\Gamma_z} \Phi(\gamma) d\mu_z(\gamma)$$

where Γ_z is the set of all the geodesics passing through z and $d\mu_z$ is the normalized measure of Γ_z .

Any geodesic passing through the point $z \in D$ depends only on one of the end points. (The other end point determines the same geodesic through z.) Therefore, Γ_z is completely determined by $\partial D = S^1$ hence Γ_z can be parametrized by $w \in S^1$ and $d\mu_z$ is naturally associated to $\frac{1}{2\pi}dw$.

Having done this parametrization, the purpose now is to invert the operator R_H. In order to invert R_H one can proceed in the spirit of Radon's inversion formula. See [24]. Following [10], one try to find a filtered backprojection type formula like 2.13. Recall that if $k \in L^1_{loc}([0,\infty))$ we can associate a radial kernel such that for $f \in C_o(D)$, the convolution operator with respect to this radial kernel k is defined as

$$k * f(z) = k *_H f(z) = \int_D f(w)k(d(z,w))dm(w) \tag{4.4}$$

where $dm(w)$ is the measure for the hyperbolic area which in polar coordinates is given by

$$dm = \sinh r dr dw$$

Recall the formula for R^*R can be written as

$$R^*Rf = \frac{2}{|x|} * f.$$

The analogous result for the hyperbolic Radon transform R_H is given by

$$R^*_H R_H f = k * f, \ where\ k(t) = \frac{1}{\pi \sinh t} \tag{4.5}$$

In [7] and [8] it is shown that by letting f $S(t) = \coth t - 1$, we obtain

$$\frac{1}{4\pi}\Delta_H \ S *_H R^*_H R_H = I, \tag{4.6}$$

the analogue to the backprojection inversion formula given before.

The Fourier transform in the hyperbolic disk D for a radial function k is given by

$$\widetilde{k}(\lambda) = 2\pi \int_0^\infty k(t) P_{i\lambda-1/2}(\cosh t)\sinh t dt, \quad for\ \lambda \in \mathbb{R}$$

where $P_\nu(r)$ is the *Legendre* function if index ν. If m is another radial function then

$$\widetilde{(k * m)}(\lambda) = \widetilde{k}(\lambda)\widetilde{m}(\lambda)$$

as we know it, [8]. It follows that as $\widetilde{k}(\lambda) \neq 0 \ \forall \lambda \in \mathbb{R}$ then the operator R_H, which takes f to $k *_H f$, is injective.

4.2 Electrical impedance tomography (EIT)

EIT has a number of applications to medicine and non-destructive evaluation. For instance, to determine the existence and lengths of internal cracks in the wings of an airplane. These applications are related to the inverse problem which is formulated now.

Let D the unit disk in $\mathbb{R}^2$ and β an strictly positive function defined on $\overline{D}$ which is unknown and represents the conductivity distribution inside the disk. When currents are introduced at the boundary ∂D, let Ψ be a given integrable function representing such currents and such that the average of the values of Ψ on ∂D is zero

$$\int_{\partial D} \Psi ds = 0$$

and consider the boundary problem with Neumann conditions

$$\begin{cases} div(\beta grad\, u) = 0, & in\ \ D \\ \beta \frac{\partial u}{\partial n} = \Psi, & on\ \ \partial D \end{cases} \tag{4.7}$$

where Ψ is given and n is the outer unit normal vector on ∂D. This problem has a unique solution u where the uniqueness of u is up to an additive constant. The function u is the potential distribution on D so $grad\ u$ is the electrical field. The variation of u on ∂D has to correspond to the known values of Ψ on ∂D, then, if s represents the tangent vector to ∂D, it follows that the tangential derivative of u, $\frac{\partial u}{\partial s}$, depends linearly on Ψ. So, for Ψ given and β, the unknown conductivity, there exists a solution u. This defines a mapping

$$\beta, \Psi \longrightarrow \frac{\partial u}{\partial s}$$

where β is the only remaining function to be found.

Let Λ_β be

$$\Lambda_\beta : \Psi \longrightarrow \frac{\partial u}{\partial s}$$

Λ_β is a linear operator from the Sobolev space $H^{\alpha}(\partial D)$ into $H^{\alpha-1}(\partial D)$, and β determines Λ_β.Given that β is to be found, then we consider the nonlinear mapping

$$\beta \longrightarrow \Lambda_\beta \tag{4.8}$$

Now the problem (Calderón's problem) consists in determining β once Λ_β is given. In other words, the problem is to find the inverse of the mapping (4.8), and this problem is called the *inverse conductivity problem*.

Some questions arise here. Is the mapping (4.8) injective?. If so, how can the inverse of Λ_β be found?. The injectivity of it in two dimensions was proven by Nachman [29] and for dimensions higher than two by Sylvester and Uhlmann [34] and [33]. See [4] and references therein for some uniqueness results. For the linearized problem, the injectivity was proven by Calderon [13]. What we explain next is how to try to find an approximate inverse.

4.3 The approximate solution to the EIT problem

As it was stated before, β is called the conductivity distribution (of some object, for instance, lungs tissue), and $1/\beta$ is called impedance, hence the name of EIT. The value of β corresponding to different constituents like human lungs tissue, blood and so on are already known, then one only looks for a profile of the areas occupied by them. EIT can measure the rate of pumping of the heart. In fact, there is already a patented device based on EIT that measures that rate.

In the case of the determination of cracks, if $\beta > 0, \beta$ can be assumed as known, then the curves and the existence of them are to be determined.

We want to emphasize that β is generally not a constant but in the case we consider now β is close to being a constant positive value β_o. Assume that β_o is initially known, and what we want to know is how much it deviates from β_o where β_o. To simplify we set $\beta_o = 1$ so the deviation of β is governed by

$$\beta = 1 + \delta\beta$$

where $|\delta\beta| << 1$, and $\delta\beta$ is a function depending on the position. If $\delta\beta = 0$ at some point w in the object being studied D, then there is no any "abnormal" situation at w. It is also assumed that there is no any deviation on ∂D, i.e. $\delta\beta = 0$. If U is the solution of (4.7) for $\beta = 1$, i.e.,

$$\begin{cases} div(grad\, U) = 0, \ \ in \ \ D \\ \frac{\partial U}{\partial n} = \Psi, \ \ on \ \ \partial D \end{cases}$$

and since $div(grad\, U) = \Delta U$ (the Laplacian of U), it follows that

$$\begin{cases} \Delta U = 0, \ \ in \ \ D \\ \frac{\partial U}{\partial n} = \Psi, \ \ on \ \ \partial D \end{cases} \tag{4.9}$$

now let u be the corresponding solution of (4.7) for the perturbed conductivity $\beta = 1 + \delta\beta$, then there is a perturbation δU, hence $u = U + \delta U$. The perturbation δU satisfies

$$\begin{cases} \Delta(\delta U) = -\langle grad\, \delta\beta, grad\, U \rangle, \ \ in \ \ D \\ \frac{\partial U}{\partial n} = -(\delta\beta)\Psi, \ \ on \ \ \partial D \end{cases}$$

and since Ψ represents the input of the currents and they can be arbitrarily chosen with the only constraint

$$\int_{\partial D} \Psi ds = 0$$

then the input Ψ, can be well approximated by linear combination of dipoles where a dipole at a point $w \in \partial D$ is given by $-\pi \frac{\partial}{\partial s}\delta_w$, δ_w the Dirac delta at w. It follows that the problem (4.9) for the dipole (input) $-\pi \frac{\partial}{\partial s}\delta_w$ at w becomes

$$\begin{cases} \Delta U_w = 0, \;\; in \;\; D \\ \frac{\partial U_w}{\partial n} = -\pi \frac{\partial}{\partial s}\delta_w, \;\; on \;\; \partial D \end{cases} \tag{4.10}$$

and the solution U_w of (4.10) has level curves which are arcs of circles that pass through w and are perpendicular to ∂D. Therefore, the level curves of U_w are exactly the geodesics given by the hyperbolic metric. At this point, the hyperbolic Radon transform is involved in the problem and can be used to solve it.

In [7] is shown that the linearized problem can in fact be described explicitly in the context of hyperbolic geometry using R_H and a radial convolution operator with kernel k. Let k be given by

$$k(t) = \frac{\cosh^{-2}(t) - 3\cosh^{-4}(t)}{8\pi}$$

then, as the boundary data function $\mu = \frac{\partial(\partial U)}{\partial s}$ defined on the space of the geodesics in D, the relation between $\delta\beta$ and μ can be shown to be

$$R_H(k *_H \delta\beta) = \mu$$

and because of the backprojection operator, one obtains

$$R_H^* R_H(k *_H \delta\beta) = R_H^*\mu$$

hence

$$\frac{1}{4\pi}\Delta_H(S *_H (R_H^*\mu)) = k *_H \delta\beta \tag{4.11}$$

Computing the hyperbolic Fourier transform of k, $\widetilde{k}$, which can be done exactly, it can be seen that $\widetilde{k}(\lambda) \neq 0, \; \forall\lambda \in \mathbb{R}$, and consequently, the convolution operator with kernel or symbol k, $k*_H$ is invertible. Formula (4.11) requires to invert the convolution operator of symbol k to compute $\delta\beta$. Barber and Brown [2] proposed an approximate inversion and Santosa and Vogelius [32] shows that the inversion formula suggested by [2] is a generalized radon transform.

To numerically implement the reconstruction of $\delta\beta$ it is necessary to invert the geodesic Radon transform and perform a deconvolution. The difficulty of numerically implementing (4.11) lies in the fact that it is complicated to numerically implement a two-dimensional non-Euclidean convolution on the hyperbolic space. In [26], Lissianoi and Ponomarev focus on the problem of numerically inverting the geodesic Radon transform by developing an algorithm, and the problem regarding the deconvolution is also considered there. For this purpose, they consider the inversion formula (4.6) and use it to derive an inversion formula for the geodesic Radon transform that it is more suitable for computations. The interesting open problem here is to be able to define a class of “discrete hyperbolic wavelets” that provides the localization described in section 3 for the Euclidean Radon transform and has computation properties similar to those of the Euclidean wavelets. For examples of discrete hyperbolic wavelets, we refer to [22]and[23].

5 A similar problem on networks

To conclude this report, we would like to mention a question raised by the second author in his PASI 2003 lecture and the new results obtained since then. The question we refer to is what is now being called Internet Tomography [17] and [1]. The problem is to be able to find out whether a network, usually a communications network, is suffering some sort of breakdown. By that we mean that traffic along the network either can not reach every node in the network, or when we add a measure of traffic around nodes, the traffic is so large in some parts of the network that it would take very long to go from one node to another. When the network is large, the information is naturally gathered at the “periphery” of the network and hence the name of internet tomography. The similarity to usual tomography becomes closer when one uses as a way to measure the traffic “packets” sent from the “boundary of the network” and measures whether they arrive to the other boundary points and, more often, how long it takes to get there.

Computer scientists have done “experimental” work on this subject and have suggested that the natural model of internet tomography is a graph situated in a portion of the 2-sphere or, what is essentially the same thing, in the hyperbolic plane [28]**.** Before we proceed further, let us note that we have alluded to two natural types of “disruptions” of the network. First, when thought as a planar graph, if a node or collection of nodes have ceased to exist because of an “intrusion”, the “topology” of the network has changed. There has been very significant work on this direction by experts on graph theory. The important work of Fan Chung and her collaborators offers crucial insights into this question. (See, for instance [14], [15] and [16].) Another situation, that resembles more what “conventional” tomography is supposed to help with, arises when traffic among certain nodes starts to increase to levels where the graph structure remains intact but there is significant slow down due to this large amount of traffic. Communication networks and, regretfully, road networks are a well known example of this second phenomenon. In either case, the desire is to be able to detect this problem when it is incipient to try to devise a solution to it. It is the latter problem that is of interest to us. One can see that Munzner’s suggestion leads to a question closely resembling EIT, and it is natural to consider it a problem in hyperbolic tomography of the kind described earlier. On the other hand, we have just obtained a significant result on the inversion of the Neumann-Dirichlet problem by studying it directly on "weighted" graphs [9].

Let us explain now a bit more in detail what these recent results are and what new questions they open up. To understand the ideas better let us consider a very simple example of a planar network, the square network G [20]. This network is constructed as follows. The nodes of G are the integer lattice points $p = (i, j)$ with $0 \leq i \leq n+1$ and $0 \leq j \leq n{+}1$ and exclude the points $(0, 0)$, $(n{+}1, 0)$, $(0, n{+}1)$, and $(n{+}1, n{+}1)$. Let V be the set of nodes, and $int\ V$ the *interior* of V consisting of the nodes $p = (i, j)$ with $0 \leq i \leq n$ and $0 \leq j \leq n$. The boundary of G is denoted $\partial\Omega$ and it is equal to $V \setminus intV$. Let p be a node then it has four neighboring nodes which are the nodes at unit distance from p. Call the set of these neighboring nodes as $N(p)$. If p is an interior node then $N(p)$ is in V, and if p is on $\partial\Omega$ then it has only one neighboring node which is the interior node that has unit distance from p. If a line segment l connects a pair of neighboring nodes p and q in $intV$ or if it connects a boundary node p to its neighboring interior node q is called *edge* or *conductor* and denoted pq. In the case in which p is on the boundary, the edge is called a boundary edge. The set of edges is denoted by E, and usually the graph G is denoted by $G(V, E)$.

Let ω a non-negative real-valued function on E, the value $\omega(pq)$ is called the *conductance* of pq and $1/\omega(pq)$, the *resistance* of pq, and ω is the *conductivity* (ω is also called a *weight*). A function $u : V \rightarrow \mathbb{R}$ gives a current across each conductor pq by *Ohm's law*, $I = \omega(pq)(u(p) - u(q))$ (I the current). The function u is called ω-$harmonic$ if for each interior node p,

$$\sum_{q \in N(p)} \omega(pq)(u(q) - u(p)) = 0$$

then the sum of the currents flowing out of each interior node is zero, and this is *Kirkhoff's law.* Let Φ a function defined at the boundary nodes, the network will acquire a unique ω-$harmonic$ function u with $u(p) = \Phi(p)$ for each $p \in \partial G$ in other words, Φ induces u and u is called the *potential* induced by Φ. Considering a conductor pq then the potential drop across this conductor is $\Delta u(pq) = u(p) - u(q)$. The potential function u determines a current $I_\Phi(p)$ through each boundary node p, by $I_\Phi(p) = \omega(pq)(u(p) - u(q)$, q being the interior neighbor of p. As in the continuous case, for each conductivity ω on E, the linear map Λ_ω from boundary functions to boundary functions is defined by $\Lambda_\omega \Phi = I_\Phi$ where the boundary function Φ is called Dirichlet data, the boundary current I_Φ is called Neumann data, and the map Λ_ω is called the Dirichlet-to-Neumann map.

The problem to consider is to recover the conductivity ω from Λ_ω, which is analogous to the the inverse problem in the continuous case. The two basic problems are the connectivity and conductivity of the network. Note that the connectivity of the network or the situation where the network remains connected but some edges disappear is a topological problem, the *configuration* of the graph has changed. For detailed theory about electrical networks, planar graphs, recovering of a graph and harmonic functions, we refer to [18] and the work of Curtis and Morrow [19].

The discrete or finite nature of graphs makes working on graphs basically easier than investigating these problems in the continuous case, although it gives rise to several disadvantages. For example, solutions of the Laplace equation for graphs have neither the local uniqueness property nor is their uniqueness guaranteed by the Cauchy data, contrary to the continuous case where they are the most important mathematical tools used to study the inverse conductivity problem and related problems [9]. The inverse problem that we study is to identify the connectivity of the nodes and the conductivity on the edges between each adjacent pair of nodes.

Given a network with a pattern of traffic measured as the "usual" load between adjacent nodes (e.g., number of messages) one can associate to it a Laplace operator denoted Δ_ω, where the weight ω is a sequence of values representing the usual loads between every pair of adjacent nodes in the network.

We define the degree $d_\omega x$ of a vertex in the weighted graph G with weight ω by

$$d_\omega x = \sum_{y \in V} \omega(x, y)$$

and the weighted ω-$Laplacian$ $\Delta_\omega f$ by

$$\Delta_\omega f := \sum_{y \in V} [f(y) - f(x)] \cdot \frac{\omega(x, y)}{d_\omega x}, \; x \in V$$

A graph $S = S(V', E')$ is said to be a *subgraph* of $G(E, V)$ if $V\prime \subset V$ and $E\prime \subset E$. In this case, we call G a *host graph* of S. The integration of a function $f : G \to \mathbb{R}$ on a graph $G = G(V, E)$ is defined by

$$\int_G f := \sum_{x \in V} f(x) d_\omega x \text{ or simply } \int_G f d_\omega$$

For a subgraph S of a graph $G = G(V, E)$ the (vertex) *boundary* ∂S of S is defined to be the set of all vertices $z \in V$ not in S but adjacent to some vertex in S, i.e.,

$$\partial S := \{z \in V \mid z \sim y \text{ for some } y \in S\}$$

and we define the *inner boundary* $\mathring{\partial} S$ by

$$\mathring{\partial} S := \{z \in S \mid y \sim z \text{ for some } y \in \partial S\}$$

Also, by $\overline{S}$ we denote a graph whose vertices and edges are in $S \cup \partial S$.The (outward) normal derivative $\frac{\partial f}{\partial_\omega n}(z)$ at $z \in \partial S$ is defined to be

$$\frac{\partial f}{\partial_\omega n}(z) := \sum_{y \in S} [f(z) - f(y)] \cdot \frac{\omega(z, y)}{d'_\omega z},$$

where $d'_\omega z = \sum_{y \in S} \omega(z, y)$

An attack by saturation corresponds to a new weight ω' where the load on each edge has either remained the same or increased (substantially in some parts of the network).

Associated to the weight ω there is a Laplace operator in the network we consider the response to diagnostic "proves" applied to the outside boundary. The boundary observations (outputs) correspond to the Neumann-to-Dirichlet map for the Laplacian Δ_ω.

The theorem below shows that the Neumann-to-Dirichlet map for $\Delta_{\omega'}$ is different to that for Δ_ω.

Theorem 4 [9] *Let ω_1 and ω_2 be weights with $\omega_1 \leq \omega_2$ on $\overline{S} \times \overline{S}$, G a graph and f_1, $f_2 : \overline{S} \to \mathbb{R}$ be functions satisfying that for j=1, 2,*

$$\begin{cases} \Delta_{\omega_j} f_j(x) = 0, \ x \in S \\ \frac{\partial f}{\partial_{\omega_j} n}(z) = \Phi(z), \ z \in \partial S \\ \int_S f_j d_{\omega_j} = K \end{cases}$$

for a given function $\Phi : \partial S \to \mathbb{R}$ with $\int_{\partial S} \Phi = 0$, and for a suitably chosen number $K > 0$.

If we assume that

(i) $\omega_1(z, y) = \omega_2(z, y)$ on $\partial S \times \overset{\circ}{\partial S}$

(ii) $f_1|_{\partial S} = f_2|_{\partial S}$, then we have

$$f_1 = f_2 \ on \ \overline{S}$$

and

$$\omega_1 = \omega_2 \ on \ \overline{S} \times \overline{S}$$

whenever $f_1(x) \neq f_1(y)$ and $f_2(x) \neq f_2(y)$.

Note that

$$\begin{cases} \Delta_\omega f(x) = 0, \ x \in S \\ \frac{\partial f}{\partial_\omega n}(z) = \Phi(z), \ z \in \partial S \end{cases}$$

is known as the Neumann boundary value problem *NBVP*. In [9] it is shown that the NBVP has a unique solution up to an additive constant.

The second conclusion of the theorem shows not only whether or not each pair of nodes is connected by a link, but also how nice the link is. Moreover, the proof gives an algorithm to detect if the weights change on the edges.

The conditions $\omega_1 \leq \omega_2$ (monotonicity condition) and $\int_G f_j d_{\omega_j} = K$ (the normalization condition) are essential for the uniqueness of the result as the following example from [9] shows.

We know that the NBVP has a unique solution up to an additive constant. Therefore, the Dirichlet data $f\,|_{\partial S}$, $z \in \partial S$ is well-defined up to an additive constant. Here we discuss the inverse conductivity problem on the network (graph) S with nonempty boundary, which consists in recovering the conductivity (connectivity or weight) ω of the graph by using the Dirichlet-to-Neumann map with one boundary measurement. In order to deal with this inverse problem, we need at least to know or be given the boundary data such as $f(x)$, $\frac{\partial f}{\partial_\omega n}(z)$ for $z \in \partial S$ and ω near the boundary. So it is natural to assume that $f\,|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S \times \mathring{\partial S}}$ are known (given or measured). But even though we are given all these data on the boundary, we are not guaranteed, in general, to be able to identify the conductivity ω uniquely. To illustrate this we consider a graph S whose vertices are {1,2,3} and $\partial S = \{0, 4\}$ with the weight

$$\omega(0,1) = 1,\ \omega(0,k) = 0 \ \ (k = 2,3,4),$$

and

$$\omega(3,4) = 1,\ \omega(k,4) = 0 \ \ (k = 0,1,2).$$

Let $f : \overline{S} \to \mathbb{R}$ be satisfying $\Delta_\omega f(k) = 0$, $k = 1,2,3$. Assume that $f(0) = 0$, $f(1) = 1$, $f(3) = 3$, $f(4) = 4$, $f(2) =$ unknown. Thus, since $\mathring{\partial S} = \{1, 3\}$, the boundary data $f\,|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ a and $\omega|_{\partial S \times \mathring{\partial S}}$ are known .

In fact,

$$\frac{\partial f}{\partial_\omega n}(0) = f(0) - f(1) = -1$$
$$\frac{\partial f}{\partial_\omega n}(4) = f(4) - f(3) = 1$$

the problem is to determine

$$\omega(1,2)=x,\ \omega(2,3)=y,\ \omega(1,3)=z,\ \text{and } f(2).$$

From $\Delta_\omega f(k)=0$, $k=1,2,3$, we have

$$\begin{array}{l} f(1)=\frac{f(0)+xf(2)+3z}{1+x+z}=1, \\ f(2)=\frac{xf(1)+yf(3)}{x+y}, \\ f(3)=\frac{zf(1)+yf(2)+f(4)}{z+y+1}=3 \end{array}$$

This system is equivalent to

$$(1)\quad \begin{cases} x(y-1)+y(x-1)+2z(x+y)=0, \\ f(2)=\frac{x+3y}{x+y}. \end{cases}$$

This systems has infinitely many solutions. For instance, assume that $z=0$, that is, the two vertices 1 and 3 are not adjacent. Then (1) is reduced to

$$(2)\quad \begin{cases} \frac{1}{x}+\frac{1}{y}=2 \\ f(2)=\frac{x+3y}{x+y}. \end{cases}$$

It is easy to see that there are infinitely many pairs (x,y) of nonnegative numbers satisfying the first equation in (2), so that $f(2)$ is undetermined as a result. In view of the above example, in order to determine the weight ω uniquely we need some more information than just $f|_{\partial S}$, $\frac{\partial f}{\partial_\omega n}|_{\partial S}$ and $\omega|_{\partial S\times\partial \overset{\circ}{S}}$. If we impose in this example the additional constraints that

$$x\geq 1,\ y\geq 1 \text{ and } z\geq 0$$

in (1), then the equation (1) yields a unique triple of solution $x=1$, $y=1$, $z=0$ and $f(2)=2$.

There are many problems to be answered, for instance what happens if the number of nodes is not finite? What is the hyperbolic version of the discrete case?. If we allow to consider also $\omega=0$ then the presence of zero weights tells us that the conductivity on the edge (a particular one) is either down or the nodes connected to that edge “disappear” in the sense that the edge length becomes infinite and this is because uniqueness is not true. We still need to get stronger results to determine the configuration of a network (connectivity). Let us add that very recently Bensoussan and Menaldi [3] have given a slightly different proof of theorem 4 relying on the fact that Δ_ω is a positive operator.

[1] J. Baras, C. A. Berenstein and F. Gavilánez, Local monitoring of the internet network. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-7/TR_2003-7.phtml

[2] D. C. Barber and B. H. Brown, Inverse problems in partial differential equations (Arcata, CA, 1989), 151-164, SIAM, Philadelphia, PA, 1990.

[3] A. Bensoussan and J. L. Menaldi, Difference Equations on weighted graphs, 2003

[4] R. M. Brown, and R. Torres, Uniqueness in the inverse conductivity problem for conductivities with $\frac{3}{2}$-derivatives in L^p, $p > 2n$, JFAA, 9 (2003), 563-574.

[5] R. W. Brown, E. Haacke, M. Thompson, R. Venkatesan, Magnetic Resonance Imaging: Physical Principles and Sequence Design. Wiley, May 1999.

[6] C. A. Berenstein and E. Casadio Tarabusi, The inverse conductivity problem and the hyperbolic Radon transform, "75 years of Radon Transform", S. Gindikin and P. Michor, editors. International Press, 1994.

[7] C. A. Berenstein and E. Casadio Tarabusi, Integral geometry in hyperbolic spaces and electrical impedance tomography, SIAM J. Appl. Math. 56 (1996), 755-764.

[8] C. A. Berenstein, Radon transforms, wavelets, an applications, Lecture notes in Mathematics. Berlin: Springer, Vol. 1684, 1998, pp.1-33.

[9] C. A. Berenstein and S-Y. Chung, $\omega-$Harmonic functions and inverse conductivity problems on networks. Available at http://techreports.isr.umd.edu/TechReports/ISR/2003/TR_2003-16/TR_2003-16.phtml.

[10] C. A. Berenstein, Local tomography and related problems, AMS Contemporary Mathematics, Vol. 278, 2001.

[11] C. A. Berenstein, and D. Walnut, Local inversion of the Radon transform in even dimensions using wavelets, "75 years of Radon transform", S. Gindikin and P. Michor, editors, International Press, 1994, 45-69.

[12] C. A. Berenstein, and D. Walnut, Wavelets and local tomography, "Wavelets in Medicine and Biology", A. Aldroubi and M. Unser, editors, CRC Press, 1996, 231-261.

[13] A. P. Calderon, On an inverse boundary value problem, Seminar on numerical analysis and its applications to continuum physics, Soc. Brz. Math. 1980, 65-73.

[14] F. Chung, and R. Ellis, A chip-firing game and Dirichlet eigenvalues, U. of California.

[15] F. Chung, Spectral Graph Theory,AMS, 1997.

[16] F. Chung, and K. Oden, Weighted Graph Laplacians and Isoperimetric Inequalities, U. of California-Harvard University.1999.

[17] M. Coates, A. Hero III, R. Nowak, and B. Yu, Internet tomography, IEEE Signal processing magazine, may 2002.

[18] E. B. Curtis, T. Ingerman, and J. A. Morrow, Circular planar graphs and resistors networks, Linear algebra and its applications, 283 (1998), 115-150.

[19] E. B. Curtis, and J. A. Morrow, Inverse problems for electrical networks, Series on Applied Mathematics, Vol. 13, 2000.

[20] E. B. Curtis, and J. A. Morrow, The Dirichlet to Neumann problem for a resistor network, AMS, 1990

[21] T. Daubechies, Ten lectures in wavelets. SIAM, 1992.

[22] Evans T. and D. Walnut, Wavelets in hyperbolic geometry. MS dissertation, Department of Mathematics, George Mason University, Fairfax: 1999, pp.1-51.

[23] F. Gavilánez, Multiresolution Analysis on the Hyperbolic plane, Master's thesis, University of Maryland, College Park, 2002.

[24] S. Helgason, The Radon transform. Boston: Birkhauser, 1999.

[25] A. I. Katsevich and A. G. Ramm, The Radon transform and local tomography. Boca Raton: CRC Press, 1996.

[26] S. Lissianoi S. and I. Ponomarev, On the inversion of the geodesic Radon transform on the hyperbolic plane. Journal of Inverse problems, vol. 13, IOP Publishing Ltd. 1997, pp. 1053-1062.

[27] S. Mallat, A wavelet tour of signal processing. London: Academic press, 2001.

[28] T. Munzner, Interactive Visualization of Large Graphs and Networks, Ph.D. Dissertation, Stanford University, June 2000.

[29] A.I. Nachman, Global uniqueness for a two-dimensional inverse boundary problem, Annals Math. 143 (1996), 71-96.

[30] F. Natterer, and F. Wubbeling, Mathematical methods in image reconstruction. SIAM 2001.

[31] F. Natterer, The mathematics of computerized tomography. SIAM 2001.

[32] F. Santosa, and M. Vogelius, A backprojection algorithm for electrical impedance imaging. SIAM J. Appl. Math. 50 (1990), no. 1, 216-243.

[33] J. Sylvester, and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, Ann. of Math. (2) 125 (1987), no. 1, 153-169.

[34] G. Uhlmann, Developments in inverse problems since Calderon's foundational paper, in Harmonic analysis and partial differential equations, U. Chicago Press, 1999, 295-345.