TECHNICAL RESEARCH REPORT

Distribution of path durations in mobile ad-hoc networks - Palm's Theorem at work

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CSHCN TR 2003-20
(ISR TR 2003-43)

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Abstract

We study the distribution of a path duration in multi-hop wireless networks. We show that as the number of hops along a path increases, the path duration distribution can be accurately approximated by an exponential distribution under a set of mild conditions, even when the link duration distributions are not identical.

1 Introduction

Routing protocols for multi-hop wireless ad-hoc networks are classified as being either table-driven or on-demand. Table-driven routing protocols attempt to maintain a path between any two nodes at all times, whereas on-demand routing protocols establish a path between two nodes only upon request. Due to the mobility of nodes, links along provided paths may become unavailable in an unpredictable manner, thereby prompting path recovery. Thus, as the performance of various on-demand routing protocols and their overheads are likely to be shaped by the distribution of link and path durations, there is a need to better understand their characteristics. Indeed, accurate modeling of these link and path durations can help better evaluate the performance of current and new on-demand routing protocols without having to run time-consuming detailed simulations.

Clearly, these distributions are expected to depend on the mobility models used in the simulations as well as on the range of speed of nodes. Yet, Sadagopan *et al.* [5] have presented a recent numerical study of the distribution of multi-hop path durations, based on various mobility models. Their study shows that the distribution of path duration can be accurately approximated by an exponential distribution when the number of hops is larger than 3 or 4 for *all* mobility models considered. However, no explanation was offered for the emergence of the exponential distribution. In this paper, we develop an approximate framework for handling this issue and use it to *prove* that under a set of mild conditions, when the number of hops becomes large, the distribution of path duration can indeed be accurately approximated by an exponential distribution. Not surprisingly, these results are simply another incarnation of Palm's Theorem [3, Thm. 5-14, p. 157], the one-dimensional precursor of the celebrated Palm-Khintchine Theorem [3, Thm. 5-15, p. 160] to the effect that the superposition of a large number of independent equilibrium renewal processes, each with a small intensity, is asymptotically a Poisson process. We validate our results through numerical results.

The paper is organized as follows. We describe the model for studying a path duration in Section 2. Sections 3 and 4 present the convergence results of the path duration distribution, followed by a discussion on main results in Section 5. Numerical examples are presented in Section 6.

A word on the notation and convention used throughout: We find it convenient to define all the random variables (rvs) of interest on some common probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Two $\mathbb{R}$–valued rvs X and Y are said to be *equal in law* if they have the same distribution, a fact we denote by $X =_{st} Y$. For any $\alpha > 0$, we denote by E_α any rv that is exponentially distributed with parameter α, *i.e.*,

$$\mathbf{P}[E_\alpha \leq x] = \begin{cases} 1 - e^{-\alpha x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \tag{1}$$

All rvs will be $\mathbb{R}_+$-valued rvs. If H is a probability distribution on $\mathbb{R}_+$, let $m(H)$ denote its first moment which is always assumed to be finite.

2 A Framework and the Basic Model

Consider a mobile ad-hoc network, where a set of mobile nodes create and maintain a network. The routing algorithm is assumed to be an on-demand algorithm, *i.e.*, a path between a source (node) and a destination (node) is set up only when a request is made. Suppose that the path requests arrive at a node according to a Poisson process, and paths are established according to the given routing algorithm. These Poisson processes modeling request arrivals at different nodes are assumed mutually independent. The details on available on-demand routing protocols are outside the scope of this paper. We refer the interested reader to the monographs [4, 6] for additional information regarding these routing protocols.

Let $V = \{1, \ldots, \mathrm{I}\}$ denote the set of communicating nodes. Each node is mobile and moves across a domain D of $\mathbb{R}^2$ or $\mathbb{R}^3$ according to some mobility model. Since there is no fixed infrastructure and nodes are mobile, links between nodes are set up and torn down dynamically. We assume that two nodes i and j become neighbors and establish a link between them when the distance becomes smaller than some constant $r_{min} > 0$, and the link is torn down when this distance becomes larger than r_{min}. In the former case we say the link is up while in the latter case the link is termed to be down.

The establishment of a path from a source node to a destination node requires the simultaneous availability of links that are up, one originating at the source node and another ending at the destination node, supporting connectivity between the source and the destination. The *path duration* is then defined as the length of time that elapses from the moment the path is established until that time when one of the links along the path goes down, as a result of mobility or interferences. For simplicity of analysis path setup delays are assumed to be negligible.

We model this situation as follows: Reflecting mobility, for distinct i and j in V, we introduce a $\{0,1\}$-valued *reachability* process $\{\xi_{ij}(t),\ t \geq 0\}$ with the interpretation that $\xi_{ij}(t) = 1$ (resp. $\xi_{ij}(t) = 0$) if the "link" (i, j) is up (resp. down) at time $t \geq 0$. The process $\{\xi_{ij}(t),\ t \geq 0\}$ is simply an alternating on-off process, with successive up and down time durations given by the rvs $\{U_{ij}(k), k = 1, 2, \ldots\}$ and $\{D_{ij}(k), k = 1, 2, \ldots\}$, respectively.

Next we can endow V with a time-varying graph structure by introducing a time-varying set of directed edges through the relation

$$E(t) := \{(i,j) \in V \times V : \xi_{ij}(t) = 1\}, \quad t \geq 0 \tag{2}$$

where by convention we have set $\xi_{ii}(t) = 0$ for each $i = 1, \ldots, I$ and all $t \geq 0$. Thus, a path can be established (in principle) between nodes i and j at time $t \geq 0$, if node j is reachable from node i in the *undirected* graph derived from the directed graph $(V, E(t))$. Let $L_{ij}(t)$ denote a set of links providing this reachability; this set of links is empty when the nodes i and j are not reachable from each other at time t. This set $L_{ij}(t)$ is not necessarily unique when non-empty, reflecting the fact that multiple paths may exist between nodes i and j; in that case, its determination is made by the routing protocol in use.

For each link ℓ in $L_{ij}(t)$, we denote the time-to-live or excess life after time t by $T_\ell(t)$, *i.e.*, $T_\ell(t)$ is the time that elapses from time t onward until the first moment that link ℓ is down. The time-to-live or duration of the path from node i to node j using the links in $L_{ij}(t)$ is then simply

$$Z_{ij}(t) := \min\left(T_\ell(t) : \ell \in L_{ij}(t)\right), \quad t \geq 0. \tag{3}$$

As discussed in Section 1, there is great interest in understanding the distributional properties of the rvs defined through (3). In order to make some progress, we shall make several simplifying assumptions:

1. First we assume that the reachability processes $\{\xi_{ij}(t),\ t \geq 0\}$ $(i \neq j \in V)$, are *mutually independent*; this is likely to be approximately the case in the limiting regime considered in this paper.[1]

2. Next, as the system is expected to run for a long time, we can expect steady state to be reached. We model this by taking each reachability process to be *stationary*, say with the rvs $\{(U_{ij}(k), D_{ij}(k)), k = 2, 3, , \ldots\}$ forming a strictly stationary sequence with generic marginals (U_{ij}, D_{ij}). We denote by G_{ij} the cumulative distribution function (CDF) of U_{ij}.

Well-known results for renewal processes and independent on-off processes in equilibrium [3, Section 5-6] can be generalized as follows: With $\ell = (i,j)$ we have

$$\mathbf{P}\left[T_\ell(0) \leq x | \xi_{ij}(0) = 1\right] = F_\ell(x) \tag{4}$$

where F_ℓ is the CDF given by

$$F_\ell(x) = \begin{cases} \frac{1}{m(G_\ell)} \int_0^x (1 - G_\ell(y))dy & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}, \tag{5}$$

and $m(G_\ell)$ is the first moment of distribution G_ℓ. In other words, F_ℓ is simply the distribution of the forward recurrence time associated with D_ℓ. Throughout we denote by X_ℓ a rv distributed according to F_ℓ. The relation (4) simply states, with a little abuse of notation, that

$$[T_\ell(0) \leq x | \xi_{ij}(0) = 1] =_{st} X_\ell.$$

[1]The approximation improves, for instance, as the number of nodes in the network increases.

The rest of the paper is now devoted to exploring the distributional properties of the rvs defined through (3). In view of (4) this amounts to considering the rvs Z_{ij} ($i, j \in V$ with $i \neq j$) defined by

$$Z_{ij} := \min\left(X_\ell : \ \ell \in L_{ij}(0)\right) \tag{6}$$

where the rvs $\{X_\ell, \ \ell \in L_{ij}(0)\}$ are mutually independent given $L_{ij}(0)$.

Typically, the set $L_{ij}(0)$ will itself be random, depending on the relative locations of nodes i and j in the network and the underlying routing protocol. For example, if nodes are uniformly distributed on the surface of a sphere or on a disk and nodes i and j are selected randomly, then $\mathbf{E}\left[\left|L_{ij}\right|\right] \propto \sqrt{\mathrm{I}}$, where I is the number of nodes in the network [2].[2] In this paper we are interested in studying the distributional properties of rv Z_{ij} as the number of links in L_{ij} increases. To this end, we take a sequence of sample paths provided by $L_{ij}^{(n)}, n = 1, 2, \ldots$, with $\left|L_{ij}^{(n)}\right| = n$. Such a sequence can be constructed, for instance, by increasing the number of nodes in the network and selecting a destination with increasing distance from the source in number of hops. For simplicity of notation, we assume that $L_{ij}^{(n)} = \{1, 2, \ldots, n\}$. With this in mind, we are now faced with the problem of exploring the distributional properties of the rvs defined through

$$Z_{ij}^{(n)} := \min\left(X_\ell^{(n)} : \ \ell = 1, \ldots, n\right) \tag{7}$$

where the rvs $X_1^{(n)}, \ldots, X_n^{(n)}$ are mutually independent rvs distributed according to the distribution of the forward recurrence time associated with $D_\ell^{(n)}$. We are now ready to discuss the asymptotic behavior of (7) for large n, and the emergence of the exponential distribution in the limit. We do so in two cases of increasing complexity.

3 I.I.D. Random Variables

The homogeneous case is discussed first, *i.e.*, all links have the same link duration distribution. When the number of hops n increases, the distribution of path duration, appropriately scaled, is shown to converge weakly to an exponential distribution: Let F denote the distribution of excess life of a link given in (5) associated with common link duration distribution G. For each $n = 1, 2, \ldots$, let $X_1^{(n)}, \ldots, X_n^{(n)}$ be a sequence of *i.i.d.* rvs with common distribution F, and we define the corresponding path duration $Z^{(n)}$ to be

$$Z^{(n)} := \min_{\ell = 1, \cdots, n} X_\ell^{(n)}.$$

Throughout we assume the non-degeneracy condition

Assumption 1 *The distribution G is non-degenerate on $\mathbb{R}_+$ with $G(0) < 1$.*

Let $\Longrightarrow_n$ denote convergence in distribution (with n going to infinity).

[2] Here, $|L|$ denotes the cardinality of the set L.

Theorem 3.1 *Under Assumption 1, with*

$$\lambda = \frac{1-G(0)}{m(G)},$$

it holds that $nZ_n \Longrightarrow_n E_\lambda$, *i.e.,*

$$\lim_{n\to\infty} \mathbf{P}[nZ_n \le x] = \begin{cases} 1-e^{-\lambda x} & \text{if } x>0 \\ 0 & \text{if } x \le 0 \end{cases} \tag{8}$$

In short, we write (8) as

$$Z_n \simeq_{st} \frac{1}{n} E_\lambda.$$

Proof: The proof is rather elementary: Fix $n = 1,2,\ldots$ and $x > 0$. We note that

$$\begin{aligned} \mathbf{P}[nZ_n > x] &= \mathbf{P}\left[\min_{\ell=1,\cdots,n} X_\ell^{(n)} > \frac{x}{n}\right] \\ &= \mathbf{P}\left[X_\ell^{(n)} > \frac{x}{n},\ \ell = 1,\ldots,n\right] \\ &= \left(1-F(\frac{x}{n})\right)^n \\ &= e^{n\log(1-F(\frac{x}{n}))} \end{aligned} \tag{9}$$

where, by standard facts from calculus, the Taylor series expansion of $z \to \log(1-z)$ at $z=0$ yields

$$\log\left(1-F\left(\frac{x}{n}\right)\right) = -F\left(\frac{x}{n}\right) + o\left(F\left(\frac{x}{n}\right)\right) \tag{10}$$

as n goes to infinity.

Next, we observe from (5) that

$$\begin{aligned} nF\left(\frac{x}{n}\right) &= \frac{x}{m(G)} \cdot \frac{\int_0^{n^{-1}x}(1-G(t))\,dt}{n^{-1}x} \\ &= \frac{x}{m(G)} \cdot \int_0^1 (1-G(n^{-1}x\tau))d\tau, \end{aligned} \tag{11}$$

and therefore

$$\lim_{n\to\infty} nF\left(\frac{x}{n}\right) = \frac{x}{m(G)} \lim_{n\to\infty} \int_0^1 (1-G(n^{-1}x\tau))d\tau = \frac{1-G(0)}{m(G)}x \tag{12}$$

by the Bounded Convergence Theorem (and the right-continuity of G at $t=0$). Letting n go to infinity in (10) and using (12), we conclude that

$$\lim_{n\to\infty} n\log(1-F(\frac{x}{n})) = -\frac{1-G(0)}{m(G)}x,$$

and (8) readily follows from (9). ∎

The result could also be obtained from more general results on the asymptotics of extremes by applying Theorem 2.1.5 in [1, p. 56].

4 Independent Random Variables

We extend the results in Section 3 to the inhomogeneous case where the excess life rvs are still independent, but not necessarily identically distributed. Thus, for each $n = 1, 2, \ldots$, let $X_1^{(n)}, \ldots, X_n^{(n)}$, be a collection of independent rvs. For each $\ell = 1, \ldots, n$, the rv $X_\ell^{(n)}$ is distributed according to the distribution $F_\ell^{(n)}$ associated through by (5) with some link duration distribution $G_\ell^{(n)}$. Here, the distributions $\{G_\ell^{(n)},\ \ell = 1, \ldots, n;\ n = 1, 2, \ldots\}$ are no longer assumed identical.

To state the requisite assumptions, we introduce the quantities

$$\lambda_\ell^{(n)} = \frac{1}{m(G_\ell^{(n)})}, \quad \ell = 1, \ldots, n,\ n = 1, 2, \ldots$$

We assume that the following conditions hold:

Assumption 2 *There exists a sequence* $\{d_n,\ n = 1, 2, \ldots\}$ *of positive scalars such that*

$$d_n \sum_{\ell=1}^{n} \lambda_\ell^{(n)} = \lambda < \infty \, .$$

Assumption 3 *Given* $\varepsilon > 0$, *for each* $x > 0$ *and n sufficiently large, it holds that*

$$G_\ell^{(n)}(d_n \cdot x) \leq \varepsilon \, , \quad \ell = 1, \ldots, n \, .$$

Here, $\{d_n,\ n = 1, 2, \ldots\}$ are the appropriate scaling constants.

Theorem 4.1 *Under Assumptions 2 and 3, it holds that*

$$\lim_{n \to \infty} \mathbf{P}\left[Z^{(n)} \leq d_n \cdot x\right] = \begin{cases} 1 - e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} ,$$

where $Z^{(n)} := \min_{\ell=1,\ldots,n} X_\ell^{(n)}$.

Proof: The proof of the theorem is a simple application of Palm's Theorem [3, Thm. 5-14, p. 157]: Fix $n = 1, 2, \ldots$, and consider n independent renewal processes $\{N_\ell^{(n)}(t), t \geq 0\}$, $\ell = 1, \ldots, n$, in *equilibrium* which are determined as follows. For each $\ell = 1, \ldots, n$, the time between the first and second renewal epochs of the renewal processes $\{N_\ell^{(n)}(t), t \geq 0\}$ is distributed according to $G_\ell^{(n)}$. Moreover, let $Y_\ell^{(n)}$ denote the epoch of the first renewal of the renewal processes $\{N_\ell^{(n)}(t), t \geq 0\}$. The requirement that the process be in equilibrium implies that $Y_\ell^{(n)}$ is distributed according to the excess life distribution in (5) associated with $G_\ell^{(n)}$.

Next, we define

$$N^{(n)}(t) = \sum_{\ell=1}^{n} N_\ell^{(n)}(t), \quad t \geq 0.$$

Then, we readily see that the earliest renewal takes place at a time given by

$$\inf\{t \geq 0 \mid N^{(n)}(t) \geq 1\} = \min_{\ell=1,\ldots,n} Y_\ell^{(n)}.$$

Palm's Theorem [3] tells us that $\min_{\ell=1,\ldots,n} \frac{Y_\ell^{(n)}}{d_n}$ converges in distribution to an exponential rv with parameter λ. Since $Y_\ell^{(n)} =_{st} X_\ell^{(n)}$, from equilibrium renewal theory, one can see that

$$\lim_{n\to\infty} \mathbf{P}\left[\min_{\ell=1,\ldots,n} X_\ell^{(n)} \leq d_n \cdot x\right] = \begin{cases} 1-e^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} .$$

This completes the proof. ■

5 Discussion

The results in Sections 3 and 4 indicate that when the number of hops is large, the distribution of path duration can be accurately approximated by an exponential rv under a set of mild conditions. In fact, in the i.i.d. case the only assumption required for convergence is that the link duration be non-degenerate. In addition, (5) tells us that the probability density function (PDF) of the duration of an one-hop path is a non-increasing function. This observation contrasts with the numerical results (Fig. 6) in [5], where the authors suggest, based on limited simulation results, that the one-hop path duration may not have a non-increasing PDF. We suspect that this is due to the limited number of statistics they were able to collect from the simulation as a result of low mobility. Note that the PDF plots become much smoother with increasing mobility or speed of nodes in [5] because of a larger number of collected samples (*e.g.,* Fig. 6 and 7 vs. Fig. 8 - 10).

6 Numerical Examples

We illustrate the analytical results of Section 3 through numerical examples. When the excess lives $X_\ell^{(n)}, \ell = 1,\ldots,n$, of the links are i.i.d. rvs, the CDF of the rv $Z^{(n)} = \min_{\ell=1,\ldots,n} X_\ell^{(n)}$ is given by

$$P(Z^{(n)} \leq x) = 1 - (1-F(x))^n, \quad x \geq 0$$

and the corresponding PDF of $Z^{(n)}$ is simply

$$f_{Z^{(n)}}(x) = \frac{n}{m(G)}(1-F(x))^{n-1}(1-G(x)), \quad x \geq 0.$$

Here, we use two different link duration distributions, namely the uniform distribution over [0, 10] and the gamma distribution. Note that the link duration distributions in [5] resemble a gamma distribution under several mobility models.

Fig. 1 plots the PDF of $Z^{(n)}$ for the values $n = 1, 2, 4$, with the selection

$$G(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{x}{10} & \text{if } 0 \leq x \leq 10 \\ 1 & \text{if } x > 10. \end{cases}$$

With increasing n the probability mass gets concentrated ever closer to the origin, as expected because $Z^{(n)}$ tends to become smaller. However, it is clear that the PDF of $Z^{(n)}$ begins to resemble that of an exponential rv with increasing n.

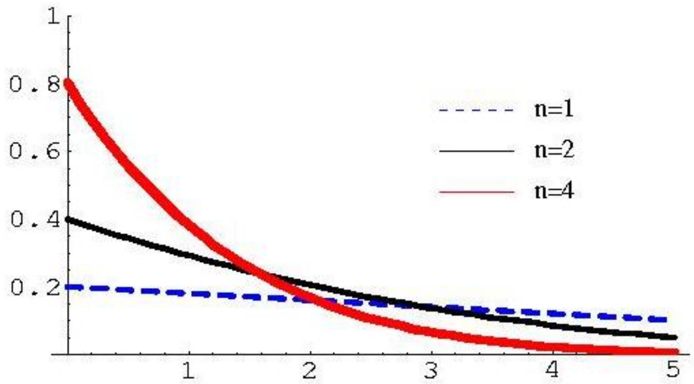

Figure 1: The PDF of a path duration with uniform link duration distribution.

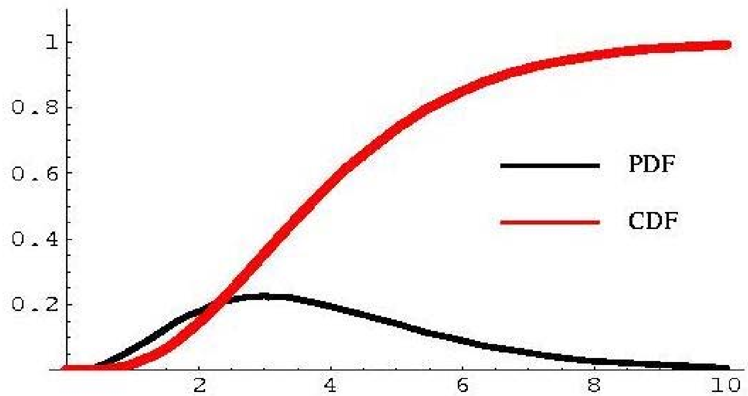

Figure 2: PDF and CDF of gamma distribution.

In the second example, the link duration is distributed according to the gamma distribution

$$G(x) = \begin{cases} 1-(1+x+\frac{x^2}{2})e^{-x} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases} . \tag{13}$$

The corresponding PDF and CDF are plotted in Fig. 2. Fig. 3 plots the PDF of the rv $Z^{(n)}$ for $n = 1, 2, 4$, when G is given by (13). As in the uniform case, the distribution of path duration is seen to converge to that of an exponential rv with increasing n.

In these examples it takes only a few hops in a path before the path duration distribution begins to resemble an exponential distribution, in line with the numerical results in [5].

7 Conclusion

We have studied the distributional properties of path duration in multi-hop wireless networks. We have shown that, under a set of mild conditions, the distribution of path duration converges to an exponential distribution with appropriate scaling as the number of hops increases.

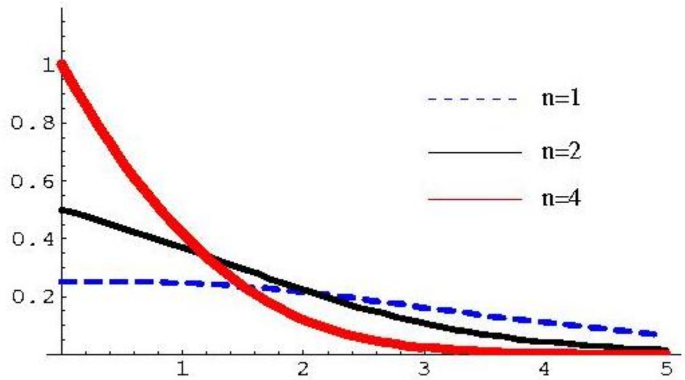

Figure 3: Path duration pdf with gamma link duration distribution.

For our analysis we have assumed that the link durations are mutually independent. Although this may not be strictly true in general, one would expect the correlation between links to go away as their spatial distance in the network increases. We are currently investigating the correlation structure of the link durations to understand the implications of dependence of link durations on the on-demand routing protocols and distributional properties of path duration.

References

[1] J. Galambos, *The Asymptotic Theory of Extreme Order Statistics*, John Wiley & Sons, New York, New York, 1978.

[2] P. Gupta and P. R. Kumar, “The capacity of wireless networks”. *IEEE Trans. Information Theory*, Vol. 46(2), pp. 388-404, March 2000.

[3] D. P. Heyman and M. J. Sobel, *Stochastic Models in Operations Research, Volume I*, McGraw-Hill, New York, New York, 1982.

[4] C. E. Perkins, *Ad-hoc Networking,* Addison-Wesley Longman, Incorporated, 2000.

[5] N. Sadagopan, F. Bai, B. Krishnamachari, and A. Helmy. “PATHS: Analysis of PATH Duration Statistics and the Impact on Reactive MANET Routing Protocols”. *ACM MobiHoc*, Annapolis, MD, June, 2003.

[6] C.-K. Toh, *Ad-hoc Mobile Wireless Networks: Protocols and Systems*, Prentice Hall PTR, 2001.