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Bounding superposed on-off sources - Variability ordering and majorization to the rescue

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Bounding superposed on-off sources – Variability ordering and majorization to the rescue

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Abstract

We consider the problem of bounding the loss rate of the aggregation of independent on-off sources in a bufferless model by the loss rate resulting from the aggregation of i.i.d. on-off sources. This is done through a unified framework based on the interplay of well-known results from the theory of variability orderings with the concept of majorization ordering. In particular, we use a basic comparison result to readily derive a bound of Rasmussen et al. for heterogeneous sources and an upper bound of Mao and Habibi for homogeneous sources, and to discuss a second upper bound proposed by these authors. It is argued that this conjectured upperbound is too tight in general, and should be replaced by new and provably correct upper bound.

1 Introduction

Traffic burstiness has long been considered a key factor for provisioning link and buffer resources at ATM multiplexers. In a first step, these issues can be addressed with the help of a simple *bufferless* model fed by fluid-like input traffic. An information source is then characterized by its $\mathbb{R}_+$-valued rate process $\{R(t),\ t \geq 0\}$, so that the source bursts at time $t \geq 0$ with an instantaneous rate of $R(t)$ bps, say for sake of definiteness. For obvious practical reasons, it is customary to require the constraint $0 \leq R(t) \leq P$ ($t \geq 0$) where P is the peak rate of the source.

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1.1 Loss rates

In most situations of interest, the rate process $\{R(t),\ t \geq 0\}$ can be assumed ergodic (as we do from now on) in the sense that for all $x \geq 0$,

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathbf{1}\left[R(t) \leq x\right] dt = \mathbf{P}\left[R \leq x\right] \quad a.s. \tag{1}$$

for some $\mathbb{R}_+$-valued rv R. If the rate process is stationary and ergodic, then (1) holds with the steady-state rate variable R determined through the weak convergence $R(t) \Longrightarrow_t R$. Under (1) the source admits an average rate given by

$$m(R) := \lim_{T\to\infty} \frac{1}{T}\int_0^T R(t)dt = \mathbf{E}\left[R\right] \quad a.s. \tag{2}$$

If the traffic is offered for transmission over a a link operating at C bps, only $\min(R(t), C)$ bps can be accommodated, and in the absence of any buffer, the remaining $(R(t) - C)^+$ [1] bps represents the instantaneous loss rate over that link. Under (1) the (average) loss rate of the source $\{R(t),\ t \geq 0\}$ over the C bps link is well defined and given by

$$\begin{aligned} \mathrm{L}(R; C) &:= \lim_{T\to\infty} \frac{1}{T}\int_0^T (R(t) - C)^+ dt \\ &= \mathbf{E}\left[(R - C)^+\right] \quad a.s. \end{aligned} \tag{3}$$

1.2 Multiplexing sources

While the definition (3) for $\mathrm{L}(R; C)$ might appear too poor a marker of source behavior to be of any use, its evaluation is nevertheless helpful either for dimensioning link capacity or as the basis for a Call Admission Control (CAC) procedure [1, 2, 7]. In the latter instance, traffic carried on the link is typically obtained by multiplexing several independent information sources. If N sources $\{R_n(t),\ t \geq 0\}$ ($n = 1, \ldots, N$) are multiplexed on a link operating at C bps, the total instantaneous rate is then given by

$$R(t) = R_1(t) + \ldots + R_N(t), \quad t \geq 0.$$

Under appropriate ergodic assumptions, it follows that

$$\begin{aligned} \mathrm{L}(R; C) &= \mathrm{L}(R_1 + \ldots + R_N; C) \\ &= \mathbf{E}\left[(R_1 + \ldots + R_N - C)^+\right] \end{aligned} \tag{4}$$

[1] We write $x^+ = \max(x, 0)$ for any scalar x.

where the mutually independent rvs $R_1, \ldots, R_N$ are the steady-state rate variables for the component sources.

As indicated already in [2, 3], evaluating (4) can be computationally prohibitive even in the simplest of cases due to the large number of sources that need to be multiplexed at any given time. This difficulty is further exacerbated when the component sources are statistically dissimilar (as is the case in practice) [2]. This state of affairs has prompted a search for *upper bounds* on loss rates which are *computationally efficient*, and yet sufficiently *tight* to provide good approximations.

1.3 On-off sources

Most of these efforts have been carried out for the class of on-off sources (e.g., [1, 2, 3, 7]). A source with rate process $\{R(t),\ t \geq 0\}$ is said to be a *(generalized) on-off* source if $R(t)$ alternates between two states, namely $R(t) = 0$ (resp. $R(t) = P$) when the source is silent (resp. active) at time $t \geq 0$. Under the ergodic assumption (1), such an on-off source admits a steady-state rate R with finite range $\{0, P\}$. In fact, it is easy to see that

$$\mathbf{P}[R = P] = 1 - \mathbf{P}[R = 0] = f(R)$$

where $f(R)$ is the *activity factor* of the source defined by

$$f(R) := \lim_{T \to \infty} \frac{1}{T} \int_0^T \mathbf{1}[R(t) > 0]\, dt \quad a.s. \tag{5}$$

For on-off sources, we have $m(R) = f(R)P$, so that such sources are fully (and equivalently) characterized by either of the pairs $(P, f(R))$ or $(P, m(R))$. We find it useful to represent the steady state rate R of the on-off source $\{R(t),\ t \geq 0\}$ with peak rate P and activity factor $f(R)$ as

$$R =_{st} PB(f(R))^2$$

where for p in $[0, 1]$, let $B(p)$ denote asn $\{0, 1\}$–valued (Bernoulli) random variable (rv) with $\mathbf{P}[B(p) = 1] = p$. We then refer to such an on-off source as the on-off source $(P, B(f(R)))$.

1.4 Earlier bounds and new results

Consider N independent on-off sources $(P_n, B(f_n))$ with peak rate P_n and activity factor f_n ($n = 1, \ldots, N$); the resulting steady state rate for the aggregate traffic is

$$\sum_{n=1}^{N} P_n B_n(f_n) \tag{6}$$

[2] For two $\mathbb{R}$-valued rvs X and Y with the same distribution, we write $X =_{st} Y$.

where $B_1(f_1), \ldots, B_N(f_N)$ are independent Bernoulli rvs. The upper bounds derived in the literature on loss rates (4) for the aggregate traffic (6) can be interpreted as loss rates for an aggregation of *fewer*, say $L \leq N$, *i.i.d.* on-off sources with common peak rate P_{new} and activity factor f_{new}. The resulting steady state rate for the aggregate traffic is now

$$\sum_{\ell=1}^{L} P_{\text{new}} B_\ell(f_{\text{new}}) \tag{7}$$

where $B_1(f_{\text{new}}), \ldots, B_L(f_{\text{new}})$ are i.i.d. Bernoulli rvs. As will become apparent in the forthcoming sections, the validity of the comparison

$$L(\sum_{n=1}^{N} P_n B_n(f_n) : C) \leq L(\sum_{\ell=1}^{L} P_{\text{new}} B_\ell(f_{\text{new}}); C), \quad C \geq 0 \tag{8}$$

entails tradeoffs in that a smaller value of L (desirable for obvious computational reasons) corresponds to a larger value for P_{new} (not desirable as it leads to looser bounds).

Rasmussen et al. [7, p. 353] conjectured that when the sources in (6) have identical peak rates, say P, but possibly different activity parameters, the aggregation (7) of N *homogeneous* on-off sources with identical peak rate $P_{\text{new}} = P$ and activity parameter $f_{\text{new}} = N^{-1}(f_1 + \ldots + f_N)$, provides an upper bound. This conjecture was recently established by Mao and Habibi [3, Thm. 1] from basic principles. These authors also establish another upper bound [3, Thm. 3], this time for N *homogeneous* on-off sources, by replacing them with a *reduced* number of homogeneous on-off sources. Finally, they conjecture the validity of an upper bound [3, Conjecture 1] which generalizes both the upper bound of Rasmussen et al. and their upper bound; a full discussion of this combined upper bound can be found in [4, Thm. 3, p. 127].

Here, we revisit these upper bounds by establishing a general comparison result for weighted sums of independent Bernoulli rvs [Proposition 4]. We show how this general result provides a *unified* vehicle for readily deriving the bound of Rasmussen et al. for heterogeneous sources and the upper bound of Mao and Habibi for homogeneous sources, and for discussing their second upper bound. It is argued that this conjectured upperbound is too tight in general. We then use Proposition 4 to generate a new upper bound which is provably correct.

The proper framework for addressing these issues (and similar comparisons more generally) is one that combines stochastic orderings [9] with the notion of majorization [6]: The variability orderings we use are tailor-made for comparing loss rates as in (8), while majorization is useful for formally comparing degrees of heterogeneity. The relevant definitions and facts are given in Section 2. This is followed in Section 3 by a discussion of three simple operations that reduce variability; this material readily yields the general comparison result in Section 4. Applications of the general result are presented in Section 5.

2 Stochastic orderings and majorization

The basic tools are introduced in this section.

2.1 Variability orderings

For $\mathbb{R}$–valued rvs X and Y, we say that X is smaller than Y in the convex (resp. increasing convex) ordering if

$$\mathbf{E}[\varphi(X)] \leq \mathbf{E}[\varphi(Y)] \tag{9}$$

for all mappings $\varphi : \mathbb{R} \to \mathbb{R}$ which are convex (resp. increasing and convex) provided the expectations in (9) exist; we write $X \leq_{cx} Y$ (resp. $X \leq_{icx} Y$). We refer to these orderings as the *variability* orderings. Additional material on these orderings can be found in the monographs [8] and [9].

2.2 Key facts

We now present well-known facts that help shape the approach taken here. First, an equivalent definition of the convex increasing ordering [9, Thm. 1.3.1, p. 9].

Proposition 1 *For $\mathbb{R}$-valued rvs X and Y with finite expectations, we have $X \leq_{icx} Y$ if and only if*

$$\mathbf{E}\left[(X-a)^+\right] \leq \mathbf{E}\left[(Y-a)^+\right], \quad a \in \mathbb{R}.$$

Proposition 1 makes it clear why the variability orderings are likely vehicles for carrying out the comparisons discussed earlier. Put simply, establishing the comparison $\mathrm{L}(R_1; C) \leq \mathrm{L}(R_2; C)$ for *all* values of C between the loss rates of two information sources with steady-state rates R_1 and R_2 is *equivalent* to the comparison $R_1 \leq_{icx} R_2$.

Next, we explore the impact of the constraint $\mathbf{E}[X] = \mathbf{E}[Y]$ [9, Thm. 1.3.1, p. 9].

Proposition 2 *For $\mathbb{R}$-valued rvs X and Y with finite expectations, we have $X \leq_{cx} Y$ if and only if $X \leq_{icx} Y$ and $\mathbf{E}[X] = \mathbf{E}[Y]$.*

Finally, the convex ordering is closed under independent addition [9, p. 9].

Proposition 3 *Consider two sets of mutually independent* $\mathbb{R}$*-valued rvs* $X_1, \ldots, X_N$ *and* $Y_1, \ldots, Y_N$*. If* $X_n \leq_{cx} Y_n$ *for each* $n = 1, \ldots, N$*, then*

$$X_1 + \ldots + X_N \leq_{cx} Y_1 + \ldots + Y_N.$$

2.3 Majorization

Let K denote some given positive integer. For any vector $\boldsymbol{x} = (x_1, \ldots, x_K)$ in $\mathbb{R}^K$, let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(K)}$ denote the components of $\boldsymbol{x}$ arranged in increasing order. For vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\mathbb{R}^K$, we say that $\boldsymbol{x}$ is *majorized* by $\boldsymbol{y}$, and write $\boldsymbol{x} \prec \boldsymbol{y}$, whenever the conditions

$$\sum_{i=1}^{k} x_{(i)} \geq \sum_{i=1}^{k} y_{(i)}, \quad k = 1, 2, \ldots, K \tag{10}$$

hold with

$$\sum_{i=1}^{K} x_i = \sum_{i=1}^{K} y_i. \tag{11}$$

Additional information regarding majorization can be found in the monograph [6]. Note that for any $\boldsymbol{x}$ in $\mathbb{R}^K$, we have $x_{\rm av}\boldsymbol{e} \prec \boldsymbol{x}$ with $\boldsymbol{e} = (1, \ldots, 1)$ in $\mathbb{R}^K$, and

$$x_{\rm av} = \frac{1}{K}(x_1 + \ldots + x_K).$$

3 Reducing variability

Below we identify three operations that reduce variability, thus leading to comparisons in the ordering $\leq_{cx}$.

3.1 Normalized Bernoulli rvs

We begin with a comparison result for renormalized Bernoulli rvs. Recall that for p in $[0, 1]$, let $B(p)$ denote an $\{0, 1\}$–valued rv with $\mathbf{P}[B(p) = 1] = p$.

Lemma 1 *The collection of rvs* $\{p^{-1}B(p),\ p \in (0, 1]\}$ *is monotone decreasing in the convex ordering, i.e.,*

$$q^{-1}B(q) \leq_{cx} p^{-1}B(p), \quad p < q.$$

In other words, increasing p makes $p^{-1}B(p)$ less variable.

Proof. We need to show that

$$\mathbf{E}\left[\varphi(q^{-1}B(q))\right] \leq \mathbf{E}\left[\varphi(p^{-1}B(p))\right], \quad p < q \tag{12}$$

for any convex mapping $\varphi : \mathbb{R} \to \mathbb{R}$, where

$$\mathbf{E}\left[\varphi(p^{-1}B(p))\right] = p(\varphi(p^{-1}) - \varphi(0)) + \varphi(0), \quad p \in (0,1].$$

Hence, it suffices to establish (12) for convex mappings $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\varphi(0) = 0$. However, under this constraint, it is well known that $x \to x^{-1}\varphi(x)$ is non-decreasing on $(0, \infty)$ and the conclusion follows. ■

3.2 Heterogeneity decreases variability

For $\boldsymbol{p}$ in $[0,1]^K$, we define the rv $S_K(\boldsymbol{p})$ as the sum

$$S_K(\boldsymbol{p}) \equiv \sum_{k=1}^{K} B_k(p_k)$$

where the Bernoulli rvs $B_1(p_1), \ldots, B_K(p_K)$ are assumed mutually independent.

Lemma 2 *For vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ in $[0,1]^K$, it holds that $S_K(\boldsymbol{q}) \leq_{cx} S_K(\boldsymbol{p})$ whenever $\boldsymbol{p} \prec \boldsymbol{q}$.*

Proof. For any integer–convex mapping $\varphi : \mathbb{N} \to \mathbb{R}$, we define the mapping $\Phi_K : [0,1]^K \to \mathbb{R}$ by

$$\Phi_K(\boldsymbol{p}) \equiv \mathbf{E}\left[\varphi(S_K(\boldsymbol{p}))\right], \quad \boldsymbol{p} \in [0,1]^K. \tag{13}$$

It is well known [6, F.1, p. 360] that the mapping Φ_K is Schur–concave in that the condition $\boldsymbol{p} \prec \boldsymbol{q}$ implies $\Phi_K(\boldsymbol{q}) \leq \Phi_K(\boldsymbol{p})$, and the conclusion $S_K(\boldsymbol{q}) \leq_{cx} S_K(\boldsymbol{p})$ follows from the definition of the convex ordering $\leq_{cx}$. ■

The next result, originally due to Hoeffding [6, p. 359], is an immediate consequence of Lemma 2.

Lemma 3 *For any vector $\boldsymbol{p}$ in $[0,1]^K$, it holds that $S_K(\boldsymbol{p}) \leq_{cx} S_K(p_{\text{av}}\boldsymbol{e})$ where $p_{\text{av}} = \frac{1}{K}(p_1 + \ldots + p_K)$.*

3.3 Linear combinations

Let $\{X_n,\ n = 1, 2, \ldots\}$ denote a sequence of i.i.d. $\mathbb{R}$-valued rvs. The following result is an easy consequence of Proposition B.2 in [6, p. 287]; see also B.2.b in [6, p. 288].

Lemma 4 *For each positive integer K, it holds that*

$$\sum_{k=1}^{K} a_k X_k \leq_{cx} \sum_{k=1}^{K} b_k X_k \tag{14}$$

whenever $\boldsymbol{a} \prec \boldsymbol{b}$ in $\mathbb{R}^K$.

An immediate corollary to Lemma 4 is obtained by taking positive integers $L < K$, and observing that $\boldsymbol{a} \prec \boldsymbol{b}$ in $[0,1]^K$ with

$$\boldsymbol{a} = K^{-1}(1, \ldots, 1) \quad \text{and} \quad \boldsymbol{b} = L^{-1}(1, \ldots, 1, 0, \ldots, 0).$$

Lemma 5 *For positive integers $L < K$, it holds that*

$$\frac{1}{K}\sum_{k=1}^{K} X_k \leq_{cx} \frac{1}{L}\sum_{\ell=1}^{L} X_\ell. \tag{15}$$

This last result was first derived by Marshall and Proschan [6, B.2.c, p. 288], and formalizes the notion that averaging decreases variability.

4 The main result

Consider N *independent* on-off sources as described in Section 1.3, where for each $n = 1, \ldots, N$, the n^{th} source $(P_n, B_n(f_n))$ has peak rate P_n and activity factor f_n so that its average rate m_n is given by

$$m_n = P_n f_n.$$

As these N sources are multiplexed, the resulting total average rate is simply

$$m_{\text{total}} = m_1 + \ldots + m_N. \tag{16}$$

Proposition 4 *With $P^\star$ selected so that*

$$\max_{n=1,\ldots,N} P_n := P_{\max} \leq P^\star, \tag{17}$$

set

$$f^\star := \frac{m_{\text{total}}}{NP^\star}. \tag{18}$$

For any positive integer $L \le N$, it holds that

$$\sum_{n=1}^{N} P_n B_n(f_n) \le_{cx} \frac{NP^\star}{L} \sum_{\ell=1}^{L} B_\ell(f^\star) \tag{19}$$

where the rvs $B_1(f^\star), \ldots, B_L(f^\star)$ are i.i.d. Bernoulli rvs.

Thus, the aggregation of heterogeneous independent on-off sources can be upper bounded in the sense of the convex ordering by an aggregation of fewer related i.i.d on-off sources.

Proof. For each $n = 1, \ldots, N$, define

$$f_n^\star := \frac{P_n}{P^\star} f_n = \frac{m_n}{P^\star}$$

and note that

$$P_n f_n = P^\star f_n^\star = m_n,$$

so that $f_n^\star$ lies in $(0, 1]$ since $f_n^\star \le f_n$. From this last equality we conclude by Lemma 1 that

$$f_n^{-1} B_n(f_n) \le_{cx} f_n^{\star -1} B_n(f_n^\star). \tag{20}$$

where we take the Bernoulli rvs $B_1(f_1^\star), \ldots, B_N(f_N^\star)$ to be mutually independent rvs.

With this in mind, we now get

$$\begin{aligned} \sum_{n=1}^{N} P_n B_n(f_n) &= \sum_{n=1}^{N} P_n f_n \left(f_n^{-1} B_n(f_n) \right) \\ &= \sum_{n=1}^{N} P^\star f_n^\star \left(f_n^{-1} B_n(f_n) \right) \\ &\le_{cx} \sum_{n=1}^{N} P^\star f_n^\star \left(f_n^{\star -1} B_n(f_n^\star) \right) \\ &= P^\star \sum_{n=1}^{N} B_n(f_n^\star) \end{aligned} \tag{21}$$

where the inequality follows from (20) via Lemma 3.

Next, we observe that

$$\frac{1}{N} \sum_{n=1}^{N} f_n^\star = \frac{1}{N} \sum_{n=1}^{N} \frac{m_n}{P^\star} = f^\star.$$

Invoking Lemma 3, with i.i.d. Bernoulli rvs $B_1(f^\star), \ldots, B_N(f^\star)$, we find that

$$\begin{aligned}
P^\star \sum_{n=1}^N B_n(f_n^\star) &\leq_{cx} P^\star \sum_{n=1}^N B_n(f^\star) \\
&= NP^\star \left(\frac{1}{N} \sum_{n=1}^N B_n(f^\star) \right) \\
&\leq_{cx} NP^\star \left(\frac{1}{L} \sum_{\ell=1}^L B_\ell(f^\star) \right) \qquad (22)
\end{aligned}$$

where the second comparison follows from Lemma 5. Combining (21) and (22) readily leads to (19). ■

5 Proposition 4 in action

Proposition 4 will now be used to discuss the bounds of Rasmussen et al. [7] and of Mao and Habibi [3, 4, 5]. Given N independent on-off sources $(P_n, B_n(f_n))$ $(n = 1, \ldots, N)$, all these results express bounds of the form

$$\sum_{n=1}^N P_n B_n(f_n) \leq_{cx} P_{\text{new}} \sum_{\ell=1}^L B_\ell(f_{\text{new}}) \qquad (23)$$

with i.i.d. on-off sources $(P_{\text{new}}, B_\ell(f_{\text{new}}))$ $(\ell = 1, \ldots, L)$ for appropriate constants $P_{\text{new}} \geq P_{\max}$ and f_{new} in $(0, 1]$, and positive integer $L \leq N$.

This upper bound (23) will flow from Proposition 4 if $P^\star$ and $f^\star$ are selected according to (17)-(18), so as to ensure the identification

$$P_{\text{new}} = \frac{NP^\star}{L} \qquad (24)$$

and

$$f_{\text{new}} = f^\star = \frac{m_{\text{total}}}{NP^\star} = \frac{m_{\text{total}}}{LP_{\text{new}}} \qquad (25)$$

once $L \leq N$ is chosen.

5.1 The bound by Rasmussen et al. [7]

Rasmussen et al. assume $P_1 = \ldots = P_N =: P_{\text{c}}$, and arbitrary activity factors $f_1, \ldots, f_N$, so that $m_{\text{total}} = P_c \sum_{n=1}^N f_n$. Applying Proposition 4 with $P^\star = P_{\text{c}} = P_{\max}$ and $L = N$, we find that

$$f^\star = \frac{m_{\text{total}}}{NP^\star} = \frac{1}{N} \sum_{n=1} f_n$$

and the bound of Rasmussen et al. is obtained in the form (23) with $L = N$, $f_{\text{new}} = f^\star$ and $P_{\text{new}} = P_c$. As should be clear from Lemma 3, this bound is simply a well-known stochastic comparison result for sums of Bernoulli rvs due to Hoeffding [6, p. 359].

5.2 The first bound by Mao and Habibi [3, Thm. 3]

We are in the homogeneous case with $P_1 = \ldots = P_N =: P_{\text{c}}$ and $m_1 = \ldots = m_N =: m_{\text{c}}$. Consequently, we have $P_{\max} = P_{\text{c}}$, $m_{\text{total}} = Nm_{\text{c}}$ and

$$f_n = \frac{m_n}{P_n} = \frac{m_{\text{c}}}{P_{\text{c}}}, \quad n = 1, \ldots, N.$$

Whenever $P^\star$ and L are selected so that $P^\star \geq P_{\text{c}}$ and $L = \lceil \frac{N}{U} \rceil$ for some positive integer U, it is plain that $L \leq N$ while (18) yields

$$f^\star = \frac{m_{\text{total}}}{NP^\star} = \frac{m_{\text{c}}}{P^\star}.$$

Now, select $P^\star \geq P_{\text{c}}$ so that $\frac{NP^\star}{L} = UP_{\text{c}}$; this is always feasible by taking $P^\star = \frac{U}{N}\lceil \frac{N}{U} \rceil P_{\text{c}}$. Applying Proposition 4 under these conditions, we get the upper bound of Theorem 3 in [3] in the form (23) with $L = \lceil \frac{N}{U} \rceil$, $f_{\text{new}} = f^\star$ and $P_{\text{new}} = UP_c$.

5.3 The second bound by Mao and Habibi [4, Thm. 3]

In [3, Conjecture 1], Mao and Habibi propose an upper bound that combines their earlier bound with that of Rasmussen et al.. This second bound is discussed as Theorem 3 in both [4, p. 127] and [5], and deals with N *arbitrary* independent on-off sources $(P_n, B_n(f_n))$ $(n = 1, \ldots, N)$.

Pick an arbitrary target value $P_{\text{new}} \geq P_{\max}$. The conjectured bound is of the form (23) with $L = L_{\text{MH}}$ where

$$L_{MH} = \lceil \frac{P_{\text{total}}}{P_{\text{new}}} \rceil \quad \text{with} \quad P_{\text{total}} = P_1 + \ldots + P_N \tag{26}$$

and

$$f_{\text{new}} = \frac{m_{\text{total}}}{L_{\text{MH}} P_{\text{new}}}. \tag{27}$$

At first, we try to check whether this second upper bound does indeed flow from Proposition 4 by appropriately selecting $P^\star$ and $f^\star$ according to (17)-(18) when $L = L_{\text{MH}}$: Obviously $L_{\text{MH}} \leq N$ since $P_{\text{total}} \leq NP_{\max}$ and (27) is just (25). Next, the value of $P^\star$ which meets the requirement (24) obviously has to be given by $P^\star = \frac{L_{\rm MH}}{N} P_{\rm new}$. Unfortunaly, the constraint $P^\star \geq P_{\max}$ is usually not satisfied as can be seen upon rewriting it in equivalent form as

$$\frac{NP_{\max}}{P_{\rm new}} \leq \lceil \frac{P_{\rm total}}{P_{\rm new}} \rceil. \tag{28}$$

In fact, if (28) is to hold at all, it can only hold as an equality, in which case we must have $P^\star = P_{\max}$. Thus, in general, the conjectured bound of Mao and Habibi is *not* a consequence of Proposition 4 unless

$$\frac{NP_{\max}}{P_{\rm new}} = \lceil \frac{P_{\rm total}}{P_{\rm new}} \rceil. \tag{29}$$

5.4 A new provably correct upper bound

The state of affairs just uncovered leads us to suspect that the conjectured upper bound might be in error, and indeed the end of Part 2b of the proof given in [4, p. 137] appears to be in error. As will become apparent shortly, the bound proposed earlier is "too tight" with too few terms.

To see this, we recall that the comparison (19) will lead to (23) with a prescribed *target* value $P_{\rm new} \geq P_{\max}$ upon selecting $P^\star$ and $f^\star$ according to (17)-(18), and a positive integer $L \leq N$, so that (24) and (25) hold. Thus, once $P_{\rm new}$ is prescribed, the integer $L \leq N$ and the *auxiliary* variable $P^\star$ determine each other via the relation

$$L = \frac{NP^\star}{P_{\rm new}} \tag{30}$$

under the constraints $P_{\max} \leq P^\star \leq P_{\rm new}$.

Reductions in computations are achieved by selecting the *smallest* admissible value of L, say $L_{\min}$, so that $P^\star$ given via (24) satisfies (17). These constraints yield

$$L_{\min} := \min\{L = 1, \ldots, N : \ \frac{L}{N} P_{\rm new} \geq P_{\max}\} \tag{31}$$

whence $L_{\min}$ and the corresponding $P^\star$ are now given by

$$L_{\min} = \lceil N \frac{P_{\max}}{P_{\rm new}} \rceil \quad \text{and} \quad P^\star = \frac{L_{\min}}{N} P_{\rm new}, \tag{32}$$

so that

$$f^\star = \frac{m_{\rm total}}{NP^\star} = \frac{m_{\rm total}}{L_{\min} P_{\rm new}} = f_{\rm new} \tag{33}$$

The new upper bound is therefore of the form (23) with $L_{\min}$ terms. We note that $L_{\rm MH} \leq L_{\min}$.

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