TECHNICAL RESEARCH REPORT

When Are On-Off Sources SIS? Conditions and Applications

by Sarut Vanichpun and Armand M. Makowski

CSHCN TR 2002-20
(ISR TR 2002-40)

WHEN ARE ON-OFF SOURCES SIS? CONDITIONS AND APPLICATIONS

Sarut Vanichpun and Armand M. Makowski

Department of Electrical and Computer Engineering

Institute for Systems Research

University of Maryland

College Park, MD 20742

E-mail: {sarut,armand}@eng.umd.edu

Phone: (301) 405-6844

Abstract

Recent advances from the theory of multivariate stochastic orderings can be used to formalize the "folk theorem" to the effect that positive correlations lead to larger buffer levels at a discrete-time infinite capacity multiplexer queue. For instance, it is known that if the input traffic is larger than its independent version in the supermodular (sm) ordering, then their corresponding buffer contents are similarly ordered in the increasing convex (icx) ordering.

A sufficient condition for the aforementioned sm comparison is the stochastic increasingness in sequence (SIS) property of the input traffic. In this paper, we provide conditions for the stationary on-off source to be SIS. We then use this result to find conditions under which the superposition of independent on-off sources and the $M|G|\infty$ input model are each sm greater than their respective independent version. Similar but weaker SIS conditions are also obtained for renewal on-off processes.

1 Introduction

Consider the following discrete-time queueing model that describes the operation of a multiplexer at a network node: A flow of packets arrive to a buffer with infinite capacity. Packets are transmitted out of the buffer in order of arrival over a communication link of constant rate. With time organized in contiguous timeslots of identical duration, let Q_t denote the number of packets still present in the system at the beginning of timeslot $[t, t+1)$ and let A_t denote the number of new packets arriving into the buffer during that timeslot. If the buffer output link can transmit c packets/slot, then the buffer content evolves according to the Lindley recursion

$$Q_{t+1} = [Q_t + A_t - c]^+, \quad t = 0, 1, \ldots \tag{1}$$

for some given initial condition Q_0.

For this model there is ample evidence on a number of fronts that positive correlations in the packet input process $\{A_t,\ t = 0, 1, \ldots\}$ lead to increased buffer occupancy and larger buffer levels over that associated with the corresponding independent version. This conclusion is already apparent in the simulation studies carried out by Livny et al. [9] (and references therein) with the help of the TES modeling tool. On the theoretical side, when considering an associated input stream [Definition 4.1], the effective bandwidth calculations [5] [6] lead naturally to an asymptotic version of this fact. Recently, this "folk theorem" has received a more formal grounding with the help of ideas from the theory of multivariate stochastic orderings established by Meester and Shanthikumar [11] and by Shaked and Shanthikumar [16] where input sequences to the Lindley recursion (1) are compared in the supermodular (sm) ordering [Definition 3.3] and the buffer contents in the increasing convex (icx) ordering [Definition 3.2]. The sm ordering is well suited to capture positive dependence in the components of a random vector, while the icx ordering formalizes comparability in terms of variability and size. A number of contributions along these lines can already be found in the monograph by Szekli [19].

Indeed, let $\{\hat{A}_t,\ t = 0, 1, \ldots\}$ denote the independent version [Definition 3.10] of the input process $\{A_t,\ t = 0, 1, \ldots\}$. According to general comparison results based on properties of the sm ordering [3] [22, Section VI], if the comparison

$$\{\hat{A}_t, t = 0, 1, \ldots\} \leq_{sm} \{A_t, t = 0, 1, \ldots\} \tag{2}$$

holds, then the corresponding buffer content processes are icx ordered with

$$\hat{Q}_t \leq_{icx} Q_t, \quad t = 0, 1, \ldots. \tag{3}$$

provided $\hat{Q}_0 = Q_0$. A steady state comparison is easily derived from (3) in a standard manner whenever appropriate [18, 21], but this issue will not be considered any further in this paper.

As we plan to make use of this framework, we need to identify the appropriate notion of positive dependence which ensures (2). Although the aforementioned notion of association might have been a natural candidate for capturing this positive dependence, it appears too weak to imply (2). The key insight was provided by Meester and Shanthikumar [11] through the notion of *stochastic increasingness in sequence* (SIS) [Definition 4.2] as the appropriate form of positive dependence. Not only does SIS imply association [2, Thm. 4.7, p. 146], but it also provides a sufficient condition for (2) to hold [11, Thm. 3.8, p. 350].

At this point it is natural to wonder whether the input process $\{A_t,\ t = 0, 1, \ldots\}$ obeys the SIS property under any of the standard traffic models, and more generally, whether the comparisons (2)-(3) hold. This issue was taken on by Vanichpun in his M.S. thesis [21] for three popular (discrete-time) traffic models, namely the Fractional Gaussian Noise traffic model the on-off source model and the $M|G|\infty$ traffic model. In the present paper we report on some of the results obtained for the discrete-time on-off source model, independent aggregation of independent on-off sources and the $M|G|\infty$ traffic model when interpreted as a limit of superposed on-off sources.

The main contributions of the paper can be summarized as follows:

(i) The statistics of an on-off source are fully determined by a pair of independent $\{1, 2, \ldots\}$-valued random variables (rvs) B and I describing the generic on-period and off-period durations, respectively. The main results for stationary on-off sources, presented in Propositions 6.1 and 6.2, provide simple and easily checkable sufficient conditions on the rvs B and I for the corresponding on-off source to be SIS;

(ii) Likhanov, Tsybakov and Georganas [8] have shown that the $M|G|\infty$ input model can be thought of the limit of a superposition of independent stationary on-off sources under an appropriate rescaling as the number of multiplexed sources becomes unboundedly large. With the help of this limiting process, we use the results for on-off sources to find sufficient conditions on the session duration rv so that (2) holds [Theorem 8.2];

(iii) A similar discussion can be carried out when using the *renewal* version (instead of the stationary version) of the component on-off processes. The main results along these lines are reported in Propositions 11.1 and 11.2.

Some of the proofs and details are omitted in the interest of brevity; they are available in [21]. A summary of some of the results is presented in the conference paper [22].

The paper is organized as follows: Some basic notation and definitions for integer-valued rvs are collected in Section 2, and stochastic orderings are introduced in Section 3. The key notion of stochastic increasingness in sequence is presented in Section 4. Stationary on-off sources are described in some details in Section 5, and the main results are presented in Section 6. The proofs of these results are given in Section 10 with some preliminary results derived in Section 9. The superposition of a finite number of on-off sources is considered in Section 7, and the $M|G|\infty$ model is discussed in Section 8. Non-stationary on-off sources are discussed in Section 11.

2 Notation and definitions

Equivalence in law or in distribution between rvs and sequences of rvs is denoted by $=_{st}$. In addition, weak convergence is denoted by $\Longrightarrow_K$ (with K going to infinity).

For any $\{1, 2, \ldots\}$-valued rv X, set

$$\mathcal{S}(X) := \{t = 1, 2, \ldots : \ \mathbf{P}[X \geq t] > 0\} \tag{4}$$

and let

$$T_X := \sup\{t = 1, 2, \ldots : \ \mathbf{P}[X \geq t] > 0\}. \tag{5}$$

Given that $t \to \mathbf{P}[X \geq t]$ is non-increasing, it is plain that $\mathcal{S}(X) = \{1, \ldots, T_X\}$ if T_X is finite and that $\mathcal{S}(X) = \{1, 2, \ldots\}$ if $T_X = \infty$. Finally, define the *hazard* function (also known as the *failure rate* function) of the rv X by

$$h_X(t) = \frac{\mathbf{P}[X = t]}{\mathbf{P}[X \geq t]}, \quad t \in \mathcal{S}(X). \tag{6}$$

Definition 2.1 *An $\{1, 2, \ldots\}$-valued rv X is decreasing failure rate (DFR) if the mapping $\mathcal{S}(X) \to \mathbb{R}_+ : t \to h_X(t)$ is decreasing.*

If the $\{1, 2, \ldots\}$-valued rv X has finite mean, we define its *forward recurrence time* $\hat{X}$ to be the $\{1, 2, \ldots\}$-valued rv with pmf given by

$$\mathbf{P}\left[\hat{X} = t\right] = \frac{\mathbf{P}\left[X \geq t\right]}{\mathbf{E}\left[X\right]}, \quad t = 1, 2, \ldots \tag{7}$$

Note that $\mathbf{P}\left[\hat{X} \geq t\right] = 0$ if and only if $\mathbf{P}\left[X \geq t\right] = 0$, whence $\mathcal{S}(\hat{X}) = \mathcal{S}(X)$.

The next lemma provides a simple characterization of the DFR property for the rv $\hat{X}$. Its proof is elementary and is therefore omitted.

Lemma 2.2 *For any $\{1, 2, \ldots\}$-valued rv X with finite mean, the corresponding $\{1, 2, \ldots\}$-valued rv $\hat{X}$ is DFR if and only if*

$$h_{\hat{X}}(t+1) \leq h_X(t) \quad \textit{whenever } t+1 \in \mathcal{S}(X).$$

3 Stochastic orderings

In this section, we summarize basic definitions concerning the stochastic orderings of random vectors. Additional information can be found in the monographs by Shaked and Shanthikumar [17], and by Stoyan [18].

Definition 3.1 *Let $\mathcal{F}$ be a class of Borel measurable functions $\varphi : \mathbb{R}^n \to \mathbb{R}$. We say that the two $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$ satisfy the relation $\mathbf{X} \leq_{\mathcal{F}} \mathbf{Y}$ if*

$$\mathbf{E}\left[\varphi(\mathbf{X})\right] \leq \mathbf{E}\left[\varphi(\mathbf{Y})\right] \tag{8}$$

for all functions φ in $\mathcal{F}$, whenever the expectations exist.

This generic definition has been specialized in the literature; here are two important examples which are used repeatedly in the sequel.

Definition 3.2 *The $\mathbb{R}^n$-valued rvs $\mathbf{X}$ and $\mathbf{Y}$ are ordered according to*

- *the usual stochastic ordering, written $\mathbf{X} \leq_{st} \mathbf{Y}$, if (8) holds for all increasing functions $\varphi : \mathbb{R}^n \to \mathbb{R}$;*

- *the increasing convex ordering, written* $\mathbf{X} \leq_{icx} \mathbf{Y}$*, if (8) holds for all increasing convex functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$.

The icx ordering is appropriate for comparing the variability of rvs. Several stochastic orderings have been found well suited for comparing the dependence structures of random vectors. Here we rely on the *supermodular* ordering which has recently been used in several queueing and reliability applications [3, 4, 16]. We begin by introducing the class of functions associated with this ordering.

Definition 3.3 *A function* $\varphi : \mathbb{R}^n \to \mathbb{R}$ *is supermodular (sm) if*

$$\varphi(\mathbf{x} \vee \mathbf{y}) + \varphi(\mathbf{x} \wedge \mathbf{y}) \geq \varphi(\mathbf{x}) + \varphi(\mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$$

where we set $\mathbf{x} \vee \mathbf{y} = (x_1 \vee y_1, \ldots, x_n \vee y_n)$ *and* $\mathbf{x} \wedge \mathbf{y} = (x_1 \wedge y_1, \ldots, x_n \wedge y_n)$.

We are now ready to define the supermodular ordering.

Definition 3.4 *The* $\mathbb{R}^n$*-valued rvs* $\mathbf{X}$ *and* $\mathbf{Y}$ *are ordered according to the supermodular ordering, written* $\mathbf{X} \leq_{sm} \mathbf{Y}$*, if (8) holds for all supermodular Borel measurable functions* $\varphi : \mathbb{R}^n \to \mathbb{R}$.

Additional information on the sm ordering can be found in [3, 4, 11, 12, 16, 20]. In Sections 7 and 8 we shall need the fact that the sm ordering is closed under convolution.

Lemma 3.5 *Let* $\mathbf{X}, \mathbf{Y}$ *and* $\mathbf{Z}$ *be independent* $\mathbb{R}^n$*-valued rvs. If* $\mathbf{X} \leq_{sm} \mathbf{Y}$*, then* $\mathbf{X} + \mathbf{Z} \leq_{sm} \mathbf{Y} + \mathbf{Z}$.

Iterating Lemma 3.5 readily leads to the following useful fact, but first, a definition:

Definition 3.6 *For* $\mathbb{R}^n$*-valued rvs* $\mathbf{X}$ *and* $\hat{\mathbf{X}}$*, we say that* $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$ *is an independent version of* $\mathbf{X} = (X_1, \ldots, X_n)$ *if the rvs* $\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n$ *are mutually independent with* $\hat{X}_k =_{st} X_k$, $k = 1, \ldots, n$.

Corollary 3.7 *Let* $\{\mathbf{X}_k,\ k = 1, 2, \ldots\}$ *denote a sequence of mutually independent* $\mathbb{R}^n$*-valued rvs. For each* $k = 1, 2, \ldots$*, let* $\hat{\mathbf{X}}_k = (\hat{X}_{k1}, \ldots, \hat{X}_{kn})$ *denote an independent version of* $\mathbf{X}_k$*. Assume the rvs* $\{\hat{\mathbf{X}}_k,\ k = 1, 2, \ldots\}$ *to be mutually independent. If* $\hat{\mathbf{X}}_k \leq_{sm} \mathbf{X}_k$ *for all* $k = 1, 2, \ldots$*, then for each* $K = 1, 2, \ldots$*, the rv* $\sum_{k=1}^K \hat{\mathbf{X}}_i$ *is an independent version of* $\sum_{k=1}^K \mathbf{X}_k$ *and*

$$\sum_{k=1}^{K} \hat{\mathbf{X}}_k \leq_{sm} \sum_{k=1}^{K} \mathbf{X}_k, \quad K = 1, 2, \ldots. \tag{9}$$

We also note [12, Thm. 3.1, p. 112]

Lemma 3.8 *Let* $\{\mathbf{X}_i,\ i = 1, 2, \ldots\}$ *and* $\{\mathbf{Y}_i,\ i = 1, 2, \ldots\}$ *denote two sequences of* $\mathbb{R}^n$*-valued rvs such that* $\mathbf{X}_n \Longrightarrow_n \mathbf{X}_\infty$ *and* $\mathbf{Y}_n \Longrightarrow_n \mathbf{Y}_\infty$*. If* $\mathbf{X}_n \leq_{sm} \mathbf{Y}_n$ *for each* $n = 1, 2, \ldots$*, then* $\mathbf{X}_\infty \leq_{sm} \mathbf{Y}_\infty$*.*

Finally, we find it useful to extend some of the earlier definitions to sequences of rvs.

Definition 3.9 *The two* $\mathbb{R}$*-valued sequences* $\mathbf{X} = \{X_n,\ n = 1, 2, \ldots\}$ *and* $\mathbf{Y} = \{Y_n,\ n = 1, 2, \ldots\}$ *satisfy the relation* $\mathbf{X} \leq_{sm} \mathbf{Y}$ *if* $(X_1, \ldots, X_n) \leq_{sm} (Y_1, \ldots, Y_n)$ *for all* $n = 1, 2, \ldots$*.*

Definition 3.10 *For sequences of* $\mathbb{R}$*-valued rvs* $\mathbf{X} = \{X_n,\ n = 1, 2, \ldots\}$ *and* $\hat{\mathbf{X}} = \{\hat{X}_n,\ n = 1, 2, \ldots\}$*, we say that* $\hat{\mathbf{X}}$ *is an independent version of* $\mathbf{X}$ *if for each* $n = 1, 2, \ldots$*, the* $\mathbb{R}^n$*-valued rv* $(\hat{X}_1, \ldots, \hat{X}_n)$ *is an independent version of the* $\mathbb{R}^n$*-valued rv* $(X_1, \ldots, X_n)$*.*

4 Positive dependence

Positive dependence in a collection of rvs can be captured in several ways. The association of rvs is one of the most useful such characterizations; it was introduced by Esary, Proschan and Walkup [7] and has proved useful in various settings [2].

Definition 4.1 *The* $\mathbb{R}$*-valued rvs* $\{X_1, \ldots, X_n\}$ *are associated if, with* $\mathbf{X} = (X_1, \ldots, X_n)$*, the inequality*

$$\mathbf{E}[f(\mathbf{X})g(\mathbf{X})] \geq \mathbf{E}[f(\mathbf{X})]\,\mathbf{E}[g(\mathbf{X})]$$

holds for all non-decreasing functions $f, g : \mathbb{R}^n \to \mathbb{R}$ *for which the expectations exist and are finite.*

Here, we focus on a stronger notion of positive dependence:

Definition 4.2 *The* $\mathbb{R}$*-valued rvs* $\{X_1, \ldots, X_n\}$ *are stochastic increasingness in sequence (SIS) if for each* $k = 1, 2, \ldots, n-1$*, the family of conditional distributions* $\{[X_{k+1}|X_1 = x_1, \ldots, X_k = x_k]\}$ *is stochastically increasing in* $\mathbf{x} = (x_1, \ldots, x_k)$*.*

More precisely, this definition states that for each $k = 1, 2, \ldots, n-1$, for $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^k$ with $\mathbf{x} \leq \mathbf{y}$ componentwise, it holds that

$$[X_{k+1}|(X_1, \ldots, X_k) = \mathbf{x}] \leq_{st} [X_{k+1}|(X_1, \ldots, X_k) = \mathbf{y}]$$

where $[X_{k+1}|(X_1, \ldots, X_k) = \mathbf{x}]$ denotes any rv distributed according to the conditional distribution of X_{k+1} given $(X_1, \ldots, X_k) = \mathbf{x}$ (with a similar interpretation for $[X_{k+1}|(X_1, \ldots, X_k) = \mathbf{y}]$).

These definitions can be extended to sequences in a natural way along the lines of Definition 3.9.

Definition 4.3 *The $\mathbb{R}$-valued sequence $\mathbf{X} = \{X_n,\ n = 1, 2, \ldots\}$ is SIS (resp. associated) if for each $n = 1, 2, \ldots$, the rvs $\{X_1, \ldots, X_n\}$ are SIS (resp. associated).*

If the $\mathbb{R}$-valued rvs $\{X_1, \ldots, X_n\}$ are SIS, then they are necessarily associated [2, Thm. 4.7, p. 146] but the converse may not be true. The next result was established by Meester and Shanthikumar [11], and relates the SIS property to the supermodular ordering. This fact will prove crucial for subsequent developments in this paper:

Theorem 4.4 *If the $\mathbb{R}_+$-valued rvs $\{X_1, \ldots, X_n\}$ are SIS, then*

$$(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n) \leq_{sm} (X_1, X_2, \ldots, X_n), \tag{10}$$

where $(\hat{X}_1, \hat{X}_2, \ldots, \hat{X}_n)$ is the independent version of $(X_1, X_2, \ldots, X_n)$.

5 Modeling on-off sources

A discrete-time on-off source with peak rate r is described by a succession of cycles, each such cycle comprising an off-period followed by an on-period. During the on-periods the source is active and produces "fluid" at constant rate r [1]; the source is silent during the off-periods: For each $n = 0, 1, \ldots$, let B_n and I_n denote the durations (in timeslots) of the on-period and off-period in the $(n+1)^{st}$ cycle, respectively. Thus, if the epochs $\{T_n,\ n = 0, 1, \ldots\}$ denote the beginning of successive cycles, with $T_0 := 0$ we have $T_{n+1} :=$

[1] For simplicity, we set this rate to be unity, say one packet/slot, i.e., $r = 1$.

$\sum_{\ell=0}^{n} I_\ell + B_\ell$ $(n = 0, 1, \ldots)$. The activity of the source is then described by the $\{0, 1\}$-valued process $\{A_t,\ t = 0, 1, \ldots\}$ given by

$$A_t := \sum_{n=0}^{\infty} \mathbf{1}\left[T_n + I_n \leq t < T_{n+1}\right] \tag{11}$$

for all $t = 0, 1, \ldots$, with the source active (resp. silent) during timeslot $[t, t+1)$ if $A_t = 1$ (resp. $A_t = 0$).

An *independent on-off* source is one for which (i) the $\{1, 2, \ldots\}$-valued rvs $\{I_n,\ n = 1, \ldots\}$ and $\{B_n,\ n = 1, \ldots\}$ are mutually independent rvs which are independent of the pair of rvs I_0 and B_0 associated with the initial cycle; and (ii) the rvs $\{I_n,\ n = 1, \ldots\}$ (resp. $\{B_n,\ n = 1, \ldots\}$) are i.i.d. rvs with generic off-period duration rv I (resp. on-period duration rv B). Throughout the generic rvs B and I are assumed to be independent $\{1, 2, \ldots\}$-valued rvs such that $0 < \mathbf{E}[B], \mathbf{E}[I] < \infty$, and we simply refer to the independent on-off process just defined as the on-off source (I, B).

In general, the activity process (11) is not stationary unless the $\mathbb{N}$-valued rvs I_0 and B_0 are selected appropriately. One possible way is to use the following variation on constructions given in [1, 15]: With

$$p := \frac{\mathbf{E}[B]}{\mathbf{E}[B] + \mathbf{E}[I]}, \tag{12}$$

we introduce the $\{0, 1\}$-valued rv U given by

$$\mathbf{P}[U = 1] = p = 1 - \mathbf{P}[U = 0]. \tag{13}$$

Let $\hat{B}$ and $\hat{I}$ denote two $\{1, 2, \ldots\}$-valued rvs distributed according to the forward recurrence time (7) associated with B and I, respectively. A stationary version of (11), still denoted $\{A_t,\ t = 0, 1, \ldots\}$, is now obtained by selecting (I_0, B_0) so that

$$(I_0, B_0) =_{st} (0, \hat{B})U + (\hat{I}, B)(1 - U) \tag{14}$$

with rvs U, B, $\hat{B}$ and $\hat{I}$ taken to be mutually independent and independent of the rvs $\{B_n, I_n,\ n = 1, \ldots\}$. With that selection, the rvs $\{A_t,\ t = 0, 1, \ldots\}$ form a stationary sequence with

$$\mathbf{P}[A_t = 1] = 1 - \mathbf{P}[A_t = 0] = p, \quad t = 0, 1, \ldots$$

where p is the average rate (12).

Note that the independent version of the stationary on-off process is a sequence $\{\hat{A}_t,\ t = 0, 1, \ldots\}$ of i.i.d. $\{0, 1\}$-valued rvs, with

$$\mathbf{P}\left[\hat{A}_t = 1\right] = 1 - \mathbf{P}\left[\hat{A}_t = 0\right] = p, \quad t = 0, 1, \ldots$$

where p is as above. It is easily seen that $\{\hat{A}_t,\ t = 0, 1, \ldots\}$ is also a stationary on-off process with geometric on-period and off-period, i.e., the corresponding on-period duration rv B (respectively, off-period duration rv I) is geometrically distributed with parameter p (respectively, $1 - p$) [2], i.e., $B =_{st} \mathcal{G}(p)$ and $I =_{st} \mathcal{G}(1 - p)$. In other words, $\{\hat{A}_t,\ t = 0, 1, \ldots\}$ can be interpreted as the discrete-time stationary on-off process $(\mathcal{G}(1 - p), \mathcal{G}(p))$.

6 The main results

As we have in mind to obtain the comparison (2) for the stationary on-off source, we seek the conditions for the stationary on-off source to satisfy the assumption of Theorem 4.4. The following proposition provides sufficient conditions on I and B for the discrete-time stationary on-off source (I, B) to have the SIS property.

Proposition 6.1 *Assume $1 < T_B$ and $1 < T_I$. The discrete-time stationary on-off source (I, B) satisfies the SIS property if the conditions (i)-(v) below hold, where*

(i) *The rvs I and B are DFR;*

(ii) $h_I(1) + h_B(2) \leq 1$ *and* $h_I(2) + h_B(1) \leq 1$;

(iii) *The rvs $\hat{I}$ and $\hat{B}$ are DFR;*

(iv) $\frac{1}{\mathbf{E}[I]} + \frac{1}{\mathbf{E}[B]} \leq 1$;

(v) $h_{\hat{I}}(2) + h_B(1) \leq 1$ *and* $h_{\hat{B}}(2) + h_I(1) \leq 1$.

[2]For α $(0 < \alpha < 1)$, an $\{1, 2 \ldots\}$-valued rv X is said to be a geometric rv with parameter α if it is distributed according to the pmf $\mathbf{P}[X = k] = \alpha^{k-1}(1 - \alpha)$ for all $k = 1, 2, \ldots$, in which case we write $X =_{st} \mathcal{G}(\alpha)$.

The proof of Proposition 6.1 is discussed in Section 10 with some preliminary results derived in Section 9. A somewhat more compact version of Proposition 6.1 but under stronger assumptions is given in

Proposition 6.2 *Assume* $1 < T_B$ *and* $1 < T_I$. *The conditions (i)-(v) in Proposition 6.1 are implied by the following conditions:*

(A.1) The rvs I *and* B *are DFR;*

(A.2) $\mathbf{P}[I = 1] + \mathbf{P}[B = 1] \leq 1$;

(A.3) The rvs $\hat{I}$ *and* $\hat{B}$ *are DFR;*

(A.4) $\frac{1}{\mathbf{E}[I]} + \frac{1}{\mathbf{E}[B]} \leq 1$.

Proof. Obviously, since (A.1), (A.3) and (A.4) coincide with (i), (iii) and (iv), respectively, we need only show that (ii) and (v) are implied by (A.1)–(A.4). To that end, we note that (A.2) is equivalent to

$$h_B(1) + h_I(1) \leq 1. \tag{15}$$

The fact that the rvs I and B are DFR implies $h_B(2) \leq h_B(1)$ and $h_I(2) \leq h_I(1)$, and (ii) follows from (15).

On the other hand, the rvs $\hat{I}$ and $\hat{B}$ being DFR, it follows from Lemma 2.2 that $h_{\hat{I}}(2) \leq h_I(1)$ and $h_{\hat{B}}(2) \leq h_B(1)$. Combining this observation with (15) we obtain (v). ■

7 Superposition of finitely many sources

Multiplexing is a major function in communication networks, with multiplexed traffic processes being created at routers and multiplexer buffers. Thus, with an on-off source representing a traffic stream, we construct multiplexed traffic by superposing a number of on-off sources. In this and the next sections we present results on the comparison (2) for the superposition of a finite and infinite number of independent on-off sources, respectively, the latter case giving rise to the so-called $M|G|\infty$ input model.

More formally, consider K independent, but not necessarily identically distributed, on-off sources. For each $k = 1, 2, \ldots, K$, let $\{A_t^k, t = 0, 1, \ldots\}$ denote the stationary on-off source (I^k, B^k). The superposition of these K on-off processes results in the stationary process $\{M_t^K, t = 0, 1, \ldots\}$ given by

$$M_t^K = \sum_{k=1}^{K} A_t^k, \quad t = 0, 1, \ldots. \tag{16}$$

Its traffic intensity

$$\sum_{k=1}^{K} p_k = \sum_{k=1}^{K} \frac{\mathbf{E}\left[B^k\right]}{\mathbf{E}\left[B^k\right] + \mathbf{E}\left[I^k\right]} \tag{17}$$

represents the average number of arrivals per slot generated by the superposition process.

Assume now that for each $k = 1, 2, \ldots, K$, the comparison

$$\{\hat{A}_t^k, t = 0, 1, \ldots\} \leq_{sm} \{A_t^k, t = 0, 1, \ldots\} \tag{18}$$

holds where $\{\hat{A}_t^k, t = 0, 1, \ldots\}$ is the independent version of $\{A_t^k, t = 0, 1, \ldots\}$. Proposition 6.1 provides sufficient conditions for (18) to hold.

By appealing to Corollary 3.7 it is a simple matter to conclude that

$$\{\sum_{k=1}^{K} \hat{A}_t^k, t = 0, 1, \ldots\} \leq_{sm} \{\sum_{k=1}^{K} A_t^k, t = 0, 1, \ldots\} \tag{19}$$

where the independent versions $\{\hat{A}_t^k, t = 0, 1, \ldots\}$, $k = 1, \ldots, K$, are taken to be mutually independent, and $\{\sum_{k=1}^{K} \hat{A}_t^k, t = 0, 1, \ldots\}$ is the independent version of $\{\sum_{k=1}^{K} A_t^k, t = 0, 1, \ldots\}$.

8 Superposition of infinitely many sources

We now consider the superposition of i.i.d. on-off sources as the number of sources grows unboundedly large. Some form of rescaling is needed in order to ensure a non-trivial limit. More precisely, for each $K = 1, 2, \ldots$, let the k^{th} on-off source $\{A_t^{(K,k)}, t = 0, 1, \ldots\}$ be a stationary on-off source with same on- and off-period duration rvs $(I^{(K)}, B)$. The resulting superposition process $\{M_t^K, t = 0, 1, \ldots\}$ given in (16) becomes

$$M_t^K = \sum_{k=1}^{K} A^{(K,k)}(t), \quad t = 0, 1, \ldots \tag{20}$$

with traffic intensity $\lambda_K \mathbf{E}[B]$ where

$$\lambda_K = \frac{K}{\mathbf{E}[B] + \mathbf{E}[I^{(K)}]}. \tag{21}$$

As K goes to infinity, Likhanov, Tsybakov and Georganas [8] have shown that the limiting process of the superposition (20) is a stationary $M|G|\infty$ input process, that is the sequence number of busy servers in the infinite server system fed by a discrete-time Poisson process with rate λ (customers per timeslot) and with generic service time B (expressed in timeslots).[3] We refer to this stationary $M|G|\infty$ process as the $M|G|\infty$ input process (λ, B); a more detailed treatment of $M|G|\infty$ input processes can be found in [10, 14]. This process is a versatile class of input traffic since both short-range and long-range dependent traffic can be generated by properly selecting the service distribution of B.

Theorem 8.1 *Let $\{M_t^K, t = 0, 1, \ldots\}$ be the superposition of K i.i.d. stationary on-off sources $(I^{(K)}, B)$. If $\lim_{K\to\infty} \lambda_K = \lambda$ for some $\lambda > 0$ and $\lim_{K\to\infty} \mathbf{P}\left[I^{(K)} \le r\right] = 0$ for each $r = 1, 2, \ldots$, then*

$$\{M_t^K, t = 0, 1, \ldots\} \Longrightarrow_K \{M_t, t = 0, 1, \ldots\}, \tag{22}$$

where $\{M_t, t = 0, 1, \ldots\}$ is the $M|G|\infty$ process (λ, B).

Theorem 8.1 is essentially a discrete-time version of the celebrated Palm-Khintchin Theorem with the session durations playing the role of marks. The on-period duration rv B in the on-off processes simply mutates into the session duration rv in the $M|G|\infty$ model. We note that this limiting process does *not* depend on the *fine* details of off-period duration distributions.

As shown below, this last fact provides a natural vehicle for establishing a form of the comparison result (2) for $M|G|\infty$ processes. Indeed, with Theorem 8.1 in mind, given a target $M|G|\infty$ model (λ, B), we can construct a sequence of superposition processes (20) that converges in distribution to the $M|G|\infty$ process (λ, B). However, as we have some latitude in selecting the rv $I^{(K)}$, we shall make a choice that guarantees the convergence (22) and yet ensures (19) for large K, say by satisfying the assumptions of Propositions 6.1 or 6.2 for the component processes. The desired conclusion would then follow from Lemma 3.8. This approach gives rise to the following comparison result:

[3] In this context it is helpful to think of B as modeling the duration of a session.

Theorem 8.2 *Let $\{M_t, t = 0, 1, \ldots\}$ be an $M|G|\infty$ input process (λ, B) such that B and $\hat{B}$ are DFR rvs. Its independent version $\{\hat{M}_t, t = 0, 1, \ldots\}$ is a sequence of i.i.d. Poisson rvs with mean $\lambda \mathbf{E}[B]$ and we have the comparison*

$$\{\hat{M}_t, t = 0, 1, \ldots\} \leq_{sm} \{M_t, t = 0, 1, \ldots\}. \tag{23}$$

Theorem 8.2 is not asserting the validity of the SIS property for $M|G|\infty$ input processes, but rather a consequence of it. We were not able to prove this SIS property, and in fact suspect that it does not hold true.

The proof of Theorem 8.2 given below relies on Theorem 8.1 and on the observations of Section 7. Another proof under no DFR assumption on the rvs B and $\hat{B}$ is provided in [22].

Proof. Fix $K = 1, 2, \ldots$ such that $K > \lambda(1 + \mathbf{E}[B])$: For each $t = 0, 1, \ldots$, we have the superposition process (20) where for each $k = 1, \ldots, K$, the component process $\{A_t^{(K,k)}, t = 0, 1, \ldots\}$ is the on-off source $(I^{(K)}, B)$. First, we select the rv $I^{(K)}$ so that $\lambda_K = \lambda$, with (21) yielding the relation

$$\mathbf{E}\left[I^{(K)}\right] = \frac{K - \lambda \mathbf{E}[B]}{\lambda}. \tag{24}$$

Next, if we take $I^{(K)} =_{st} \mathcal{G}(1-\alpha(K))$ for some $0 < \alpha(K) < 1$, then $\mathbf{E}\left[I^{(K)}\right] = \alpha(K)^{-1}$ and the mean value condition (24) implies

$$\alpha(K) = \frac{\lambda}{K - \lambda \mathbf{E}[B]}.$$

It is plain that $\lim_{K\to\infty} \alpha(K) = 0$ and $\lim_{K\to\infty} \mathbf{P}\left[I^{(K)} \leq r\right] = 0$ for each $r = 1, 2, \ldots$. Consequently, the sequence of rvs $\{I^{(K)},\ K = 1, 2, \ldots\}$ satisfies the requirements of Theorem 8.1 so that (22) holds with $\{M_t, t = 0, 1, \ldots\}$ the $M|G|\infty$ input process (λ, B).

Next we turn to the the SIS conditions of Proposition 6.2 for the component on-off processes defined above. For each $K = 1, 2, \ldots$, it is easy to check that $I^{(K)} =_{st} \hat{I}^{(K)}$ and that these rvs are DFR since $h_{I^{(K)}}(t) = \alpha(K)$ for all $t = 1, 2, \ldots$. Thus, by taking the rvs B and $\hat{B}$ to be DFR, Conditions (A.1) and (A.3) are satisfied. Conditions (A.2) and (A.4) require that

$$\mathbf{P}\left[I^{(K)} = 1\right] + \mathbf{P}[B = 1] \leq 1$$

and

$$\frac{1}{\mathbf{E}\left[I^{(K)}\right]}+\frac{1}{\mathbf{E}\left[B\right]}\leq 1,$$

respectively. But $\lim_{K\to\infty}\mathbf{P}\left[I^{(K)}=1\right]=0$ and $\lim_{K\to\infty}\mathbf{E}\left[I^{(K)}\right]=\infty$, whence Conditions (A.2) and (A.4) are indeed satisfied if K is large enough, say $K>K^{\star}$ for some $K^{\star}>0$ – Indeed recall that $\mathbf{E}\left[B\right]\geq 1$ since B is $\{1,2,\ldots\}$-valued.

Thus, whenever $K>K^{\star}$, the rvs $I^{(K)}$ and B satisfy conditions (A.1)-(A.4), whence for each $k=1,\ldots,K$, the component process $\{A_t^{(K,k)},t=0,1,\ldots\}$ is SIS, and by Theorem 4.4, we get

$$\{\hat{A}_t^{(K,k)},t=0,1,\ldots\}\leq_{sm}\{A_t^{(K,k)},t=0,1,\ldots\}\tag{25}$$

where $\{\hat{A}_t^{(K,k)},t=0,1,\ldots\}$ is the independent version of $\{A_t^{(K,k)},t=0,1,\ldots\}$.

As in Section 7, upon combining (25) and Corollary 3.7, we obtain

$$\{\hat{M}_t^{K},t=0,1,\ldots\}\leq_{sm}\{M_t^{K},t=0,1,\ldots\}.\tag{26}$$

with $\{\hat{M}_t^{K},t=0,1,\ldots\}$ denoting the independent version of $\{M_t^{K},t=0,1,\ldots\}$. We have

$$\hat{M}_t^{K}=\sum_{k=1}^{K}\hat{A}_t^{(K,k)}\quad t=0,1,\ldots$$

where the independent versions $\{\hat{A}_t^{(K,k)},t=0,1,\ldots\}$, $k=1,\ldots,K$, are taken to be mutually independent. The independent version $\{\hat{A}_t^{(K,k)},t=0,1,\ldots\}$ is a sequence of i.i.d. $\{0,1\}$-valued rvs with $\mathbf{P}\left[\hat{A}_t^{k}=1\right]=p(K)$ for all $t=0,1,\ldots$ where

$$p(K)=\frac{\mathbf{E}\left[B\right]}{\mathbf{E}\left[B\right]+\mathbf{E}\left[I^{(K)}\right]}=\frac{\mathbf{E}\left[B\right]}{\mathbf{E}\left[B\right]+\alpha(K)^{-1}}=\frac{1}{K}\lambda\mathbf{E}\left[B\right].$$

Consequently, for each $t=0,1,\ldots$, the rv $\hat{M}_t^{K}$ is a Binomial rv with parameters $(K,p(K))$, whence the independent version $\{\hat{M}_t^{K},t=0,1,\ldots\}$ is a sequence of i.i.d. Binomial rvs with parameters $(K,\frac{\lambda\mathbf{E}[B]}{K})$.

We are now ready to let K go to infinity in (26): By Lemma 3.8 we obtain the desired conclusion (23) if we show that

$$\{\hat{M}_t^{K},t=0,1,\ldots\}\Longrightarrow_K\{\hat{M}_t,t=0,1,\ldots\}\tag{27}$$

for some collection $\{\hat{M}_t, t = 0, 1, \ldots\}$ of i.i.d. Poisson rvs with parameter $\lambda\mathbf{E}[B]$. Indeed, it is well known [14, Prop. 2, p. 277] that the stationary $M|G|\infty$ input process (λ, B) has Poisson marginals with parameter $\lambda\mathbf{E}[B]$, whence such a limit $\{\hat{M}_t, t = 0, 1, \ldots\}$ would indeed be the independent version of the $M|G|\infty$ input process $\{M_t, t = 0, 1, \ldots\}$. However, by Poisson's Convergence Theorem, for each $t = 0, 1, \ldots$, the Binomial rv $\hat{M}_t^K$ with parameters $(K, \frac{\lambda\mathbf{E}[B]}{K})$ converges in distribution to a Poisson rv with mean $\lambda\mathbf{E}[B]$, and the convergence (27) of the sequences of i.i.d. rvs readily follows. ■

9 Expressions for stationary on-off sources

We begin the proof of Proposition 6.1 by developing some needed expressions: Indeed, for the stationary on-off source (I, B), the SIS condition takes a much simpler form which we now present: For each $t = 0, 1, \ldots$, with the notation $\mathbf{A}^t = (A_0, \ldots, A_t)$, we need to establish the inequalities

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] \leq \mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{y}^t\right] \tag{28}$$

for any pair $\mathbf{x}^t = (x_0, \ldots, x_t)$ and $\mathbf{y}^t = (y_0, \ldots, y_t)$ in $\{0,1\}^{t+1}$ such that $\mathbf{x}^t \leq \mathbf{y}^t$ componentwise in $\{0,1\}^{t+1}$ with

$$\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0 \quad \text{and} \quad \mathbf{P}\left[\mathbf{A}^t = \mathbf{y}^t\right] > 0. \tag{29}$$

As we proceed to evaluate the relevant conditional probabilities, in all cases we rely on the basic observation that

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \frac{\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t; A_{t+1} = 1\right]}{\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right]} \tag{30}$$

for every $\mathbf{x}^t$ in $\{0,1\}^{t+1}$ for which $\mathbf{P}[\mathbf{A}^t = \mathbf{x}^t] > 0$. We first find the expression (30) of the stationary on-off source for the case $t = 0$.

Lemma 9.1 *For the stationary on-off source (I, B), we have*

$$\mathbf{P}[A_1 = 1 | A_0 = 0] = h_{\hat{I}}(1) \tag{31}$$

and

$$\mathbf{P}[A_1 = 1 | A_0 = 1] = 1 - h_{\hat{B}}(1). \tag{32}$$

Proof. The conclusions (31) and (32) are easy consequences of the facts

$$\begin{aligned}
\mathbf{P}[A_1 = 1|A_0 = 0] &= \frac{\mathbf{P}[A_0 = 0, A_1 = 1]}{\mathbf{P}[A_0 = 0]} \\
&= \frac{\mathbf{P}[I_0 = 1, B_0 \geq 1]}{\mathbf{P}[I_0 \geq 1]} \\
&= \frac{\mathbf{P}[I_0 = 1, B_0 \geq 1 | I_0 > 0]}{\mathbf{P}[I_0 \geq 1 | I_0 > 0]} \\
&= \frac{\mathbf{P}\left[\hat{I} = 1\right]}{\mathbf{P}\left[\hat{I} \geq 1\right]} \mathbf{P}[B \geq 1]
\end{aligned}$$

since $\mathbf{P}[B \geq 1] = 1$, and

$$\begin{aligned}
\mathbf{P}[A_1 = 1|A_0 = 1] &= \frac{\mathbf{P}[A_0 = 1, A_1 = 1]}{\mathbf{P}[A_0 = 1]} \\
&= \frac{\mathbf{P}[I_0 = 0, B_0 \geq 2]}{\mathbf{P}[I_0 = 0, B_0 \geq 1]} \\
&= \frac{\mathbf{P}[B_0 \geq 2 | I_0 = 0]}{\mathbf{P}[B_0 \geq 1 | I_0 = 0]} \\
&= \frac{\mathbf{P}\left[\hat{B} \geq 2\right]}{\mathbf{P}\left[\hat{B} \geq 1\right]}.
\end{aligned}$$

■

To describe the results when $t = 1, 2, \ldots$, we associate with any $\mathbf{x}^t$ in $\{0, 1\}^{t+1}$ the index $\ell(\mathbf{x}^t)$ of "last change" given by

$$\ell(\mathbf{x}^t) := \min\{r = 0, 1, \ldots, t : \ x_r = \ldots = x_t\}.$$

If $\ell(\mathbf{x}^t) > 0$, then $x_{\ell(\mathbf{x}^t)-1} \neq x_{\ell(\mathbf{x}^t)} = \ldots = x_t$, while if $\ell(\mathbf{x}^t) = 0$, then $x_0 = x_1 = \ldots = x_t$. Fix $t = 1, 2, \ldots$ throughout.

Proposition 9.2 *For the stationary on-off source (I, B), for each $\mathbf{x}^t$ in $\{0, 1\}^{t+1}$ with $x_t = 1$, we have*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \begin{cases} 1 - h_B(t - \ell(\mathbf{x}^t) + 1) & \text{if } \ell(\mathbf{x}^t) > 0 \\ 1 - h_{\hat{B}}(t+1) & \text{if } \ell(\mathbf{x}^t) = 0 \end{cases} \quad (33)$$

provided $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$.

Proof. With $x_t = 1$, we already note the relations

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_{t+1} = 1\right] \end{aligned}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 1\right] \end{aligned}$$

If $\ell(\mathbf{x}^t) > 0$, then with some rv B independent of $\{A_s, 0 \le s < \ell(\mathbf{x}^t)\}$, we conclude that

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), B \ge t - \ell(\mathbf{x}^t) + 2\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[B \ge t - \ell(\mathbf{x}^t) + 2\right] \end{aligned} \tag{34}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t), B \ge t - \ell(\mathbf{x}^t) + 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \le s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[B \ge t - \ell(\mathbf{x}^t) + 1\right]. \end{aligned} \tag{35}$$

The first half of (33) follows readily by combining (34) and (35) through (30).

On the other hand, if $\ell(\mathbf{x}^t) = 0$, then $\mathbf{x}^t = (1, \ldots, 1)$ and it holds that

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = A_{t+1} = 1\right] \\ &= \mathbf{P}\left[\hat{B} \ge t + 2\right] \end{aligned} \tag{36}$$

and

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 1\right] \\ &= \mathbf{P}\left[\hat{B} \ge t + 1\right]. \end{aligned} \tag{37}$$

The second half of (33) is obtained by combining (36) and (37) via (30). ■

Proposition 9.3 *For the stationary on-off source* (I, B)*, for each* $\mathbf{x}^t$ *in* $\{0,1\}^{t+1}$ *with* $x_t = 0$*, we have*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = \begin{cases} h_I(t - \ell(\mathbf{x}^t) + 1) & \text{if } \ell(\mathbf{x}^t) > 0 \\ \\ h_{\hat{I}}(t+1) & \text{if } \ell(\mathbf{x}^t) = 0 \end{cases} \tag{38}$$

provided $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$.

Proof. The proof follows a pattern similar to that of Proposition 9.2. With $x_t = 0$, we obtain the relations

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 0,\ A_{t+1} = 1\right] \end{aligned}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t),\ A_{\ell(\mathbf{x}^t)} = \ldots = A_t = 0\right]. \end{aligned}$$

If $\ell(\mathbf{x}^t) > 0$, then with some pair of independent rvs I and B which are independent of $\{A_s, 0 \leq s < \ell(\mathbf{x}^t)\}$, we conclude that

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t), I = t - \ell(\mathbf{x}^t) + 1,\ B \geq 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[I = t - \ell(\mathbf{x}^t) + 1\right] \mathbf{P}\left[B \geq 1\right] \end{aligned} \tag{39}$$

and

$$\begin{aligned} & \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t), I \geq t - \ell(\mathbf{x}^t) + 1\right] \\ = \ & \mathbf{P}\left[A_s = x_s, 0 \leq s < \ell(\mathbf{x}^t)\right] \mathbf{P}\left[I \geq t - \ell(\mathbf{x}^t) + 1\right]. \end{aligned} \tag{40}$$

Combining (39) and (40) through (30) we get the first half of (38).

On the other hand, if $\ell(\mathbf{x}^t) = 0$, then $\mathbf{x}^t = (0, \ldots, 0)$ and it holds that

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0,\ A_{t+1} = 1\right] \\ &= \mathbf{P}\left[I_0 = t+1, B_0 \geq 1\right] \\ &= \mathbf{P}\left[I_0 > 0\right] \mathbf{P}\left[\hat{I} = t+1\right] \mathbf{P}\left[B \geq 1\right] \end{aligned} \tag{41}$$

and

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0\right] \\ &= \mathbf{P}\left[I_0 \geq t+1\right] \\ &= \mathbf{P}\left[I_0 > 0\right] \mathbf{P}\left[\hat{I} \geq t+1\right]. \end{aligned} \tag{42}$$

We conclude to the second half of (38) by combining (41) and (42) via (30). ■

As a byproduct of the proofs of Propositions 9.2 and 9.3, we note the following necessary conditions for $\mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] > 0$ to hold: With $x_t = 1$, it follows from (35) and (37) that we need $t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(B)$ if $\ell(\mathbf{x}^t) > 0$ and $t+1$ in $\mathcal{S}(B)$ if $\ell(\mathbf{x}^t) = 0$. Similarly, with $x_t = 0$, we conclude from (40) and (42) that we need $t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(I)$ if $\ell(\mathbf{x}^t) > 0$ and $t+1$ in $\mathcal{S}(I)$ if $\ell(\mathbf{x}^t) = 0$.

10 A proof of Proposition 6.1

For each $t = 0, 1, \ldots$, we need to show that (28) holds for any pair of distinct elements $\mathbf{x}^t$ and $\mathbf{y}^t$ in $\{0,1\}^{t+1}$ with $\mathbf{x}^t \leq \mathbf{y}^t$ such that (29) holds.

For $t = 0$, (29) automatically holds here since $\mathbf{P}\left[A_0 = 1\right] = 1 - \mathbf{P}\left[A_0 = 0\right] = p$ with $0 < p < 1$. By Lemma 9.1 we see that (28) reduces to

$$h_{\hat{I}}(1) \leq 1 - h_{\hat{B}}(1) \tag{43}$$

which is equivalent to (iv).

For $t = 1, 2, \ldots$, three cases present themselves, depending on whether (a) $x_t = y_t = 1$; (b) $x_t = y_t = 0$; and (c) $x_t = 0 < y_t = 1$. Recall that we are only interested in the situations where (29) is satisfied. We consider each one of three cases in turn:

Case (a) – With $x_t = y_t = 1$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{y}^t) \leq \ell(\mathbf{x}^t)$. If $\ell(\mathbf{y}^t) > 0$, then $\ell(\mathbf{x}^t) > 0$ as well. By Proposition 9.2, the inequality (28) reduces to

$$h_B(t - \ell(\mathbf{y}^t) + 1) \leq h_B(t - \ell(\mathbf{x}^t) + 1) \tag{44}$$

with $t - \ell(\mathbf{x}^t) + 1 \leq t - \ell(\mathbf{y}^t) + 1$ in $\mathcal{S}(B)$. The inequality (44) does hold when B is DFR. If $\ell(\mathbf{y}^t) = 0$, then $\ell(\mathbf{x}^t) > 0$ (for otherwise $\mathbf{x}^t = \mathbf{y}^t$) and Proposition 9.2 this time shows that (28) is equivalent to

$$h_{\hat{B}}(t+1) \leq h_B(t - \ell(\mathbf{x}^t) + 1)$$

with both $t+1$ and $t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(B)$. The validity of this last inequality is an easy consequence of Lemma 2.2 and of the fact that both B and $\hat{B}$ are DFR rvs since on the considered range, we have

$$h_{\hat{B}}(t+1) \leq h_B(t) \leq h_B(t - \ell(\mathbf{x}^t) + 1).$$

Case (b) – With $x_t = y_t = 0$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ now implies $\ell(\mathbf{x}^t) \leq \ell(\mathbf{y}^t)$. If $\ell(\mathbf{x}^t) > 0$, then $\ell(\mathbf{y}^t) > 0$ and by Proposition 9.3, the inequality (28) reduces to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq h_I(t - \ell(\mathbf{y}^t) + 1) \tag{45}$$

with $t - \ell(\mathbf{y}^t) + 1 \leq t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(I)$. The inequality (45) is implied by the fact that the rv I is DFR under (i). If $\ell(\mathbf{x}^t) = 0$, then $\ell(\mathbf{y}^t) > 0$ (for otherwise $\mathbf{x}^t = \mathbf{y}^t$) and Proposition 9.3 shows that (28) is equivalent to

$$h_{\hat{I}}(t+1) \leq h_I(t - \ell(\mathbf{y}^t) + 1)$$

with both $t+1$ and $t - \ell(\mathbf{y}^t) + 1$ in $\mathcal{S}(I)$. As before, this is the case on the specified range by virtue of Lemma 2.2 which yields

$$h_{\hat{I}}(t+1) \leq h_B(t) \leq h_I(t - \ell(\mathbf{x}^t) + 1)$$

since I and $\hat{I}$ are DFR rvs.

Case (c) – With $x_t = 0 < y_t = 1$, four possible scenarios need to be considered when invoking Propositions 9.2 and 9.3 to rewrite the inequality (28) in reduced form: First, if $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t) = 0$, then (28) can be rewritten as

$$h_{\hat{I}}(t+1) \leq 1 - h_{\hat{B}}(t+1) \tag{46}$$

with $t+1$ in both $\mathcal{S}(B)$ and $\mathcal{S}(I)$, and this inequality does hold by virtue of (iii) and (iv) (or equivalently, (43)).

Next, if $\ell(\mathbf{x}^t) = 0$ and $\ell(\mathbf{y}^t) > 0$, then (28) becomes

$$h_{\hat{I}}(t+1) \le 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \tag{47}$$

with $t - \ell(\mathbf{y}^t) + 1 \le t$ in $\mathcal{S}(B)$ and $t+1$ in $\mathcal{S}(I)$. Because $\hat{I}$ and B are DFR, we get

$$h_{\hat{I}}(t+1) + h_B(t - \ell(\mathbf{y}^t) + 1) \le h_{\hat{I}}(2) + h_B(1) \tag{48}$$

and (47) indeed holds by virtue of (v). Symmetrically, if $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) = 0$, then (28) reads

$$h_I(t - \ell(\mathbf{x}^t) + 1) \le 1 - h_{\hat{B}}(t+1) \tag{49}$$

with $t - \ell(\mathbf{x}^t) + 1 \le t$ in $\mathcal{S}(I)$ and $t+1$ in $\mathcal{S}(B)$. Now, the fact that $\hat{B}$ and I are DFR leads to

$$h_{\hat{B}}(t+1) + h_I(t - \ell(\mathbf{x}^t) + 1) \le h_{\hat{B}}(2) + h_B(1) \tag{50}$$

and (49) is now immediate from (v).

Lastly, if $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) > 0$, then (28) is equivalent to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \le 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \tag{51}$$

with $t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(I)$ and $t - \ell(\mathbf{y}^t) + 1$ in $\mathcal{S}(B)$. If $\ell(\mathbf{x}^t) \neq \ell(\mathbf{y}^t)$, say $\ell(\mathbf{x}^t) < \ell(\mathbf{y}^t)$, then $1 \le t - \ell(\mathbf{y}^t) + 1$ while $2 \le t - \ell(\mathbf{x}^t) + 1$, and (51) is now a simple consequence of (ii) since the rvs I and B are DFR. A symmetric argument holds *mutatis mutandis* if $\ell(\mathbf{y}^t) < \ell(\mathbf{x}^t)$. We complete the proof by noting that the case $\ell(\mathbf{x}^t) = \ell(\mathbf{y}^t)$ is not possible under the constraints $x_t = 0$, $y_t = 1$ and $\mathbf{x}^t \le \mathbf{y}^t$ componentwise.

11 Renewal on-off sources

In the definition of on-off processes, the initial pair (I_0, B_0) could have been selected so that

$$(I_0, B_0) =_{st} (I, B) \tag{52}$$

with I and B independent. The resulting on-off source is said to be the renewal on-off source (I, B). Under these assumptions, the independent version $\{\hat{A}_t,\ t = 0, 1, \ldots\}$ is simply a collection of mutually independent $\{0,1\}$-valued rvs with

$$\mathbf{P}\left[\hat{A}_t = 1\right] = \mathbf{P}\left[A_t = 1\right], \quad t = 0, 1, \ldots$$

and therefore cannot be interpreted anymore as an on-off process, stationary or renewal.

We now present conditions for the renewal on-off source (I, B) to be SIS. The analog of Proposition 6.1 is given next.

Proposition 11.1 *The renewal on-off source (I, B) satisfies the SIS property whenever conditions (i)-(ii) of Proposition 6.1 hold.*

Here, the analog of Proposition 6.2 takes the form

Proposition 11.2 *The conditions (i)-(ii) in Proposition 6.1 are implied by conditions (A.1)-(A.2) of Proposition 6.2.*

Proposition 11.2 is established in a manner similar to that of Proposition 6.2, and we therefore omit the proof. As we now turn to the proof of Proposition 11.1, we first obtain the expression (30) for the renewal on-off source (I, B) and then derive the corresponding SIS conditions. Here, we have $I_0 =_{st} I$ so that $\mathbf{P}[A_0 = 0] = 1$. This observation leads to the following analog of Lemma 9.1.

Lemma 11.3 *For the renewal on-off source (I, B), we have*

$$\mathbf{P}[A_1 = 1 | A_0 = 0] = h_I(1). \tag{53}$$

Proof. As in the proof of Lemma 9.1, the conclusion (53) is an easy consequence of the facts

$$\mathbf{P}[A_1 = 1 | A_0 = 0] = \frac{\mathbf{P}[I_0 = 1, B_0 \geq 1]}{\mathbf{P}[I_0 \geq 1]} = \frac{\mathbf{P}[I_0 = 1]}{\mathbf{P}[I_0 \geq 1]} = \mathbf{P}[I_0 = 1]$$

by the independence of the rvs B_0 and I_0, and the fact that $\mathbf{P}[B_0 \geq 1] = 1$ ■

In the renewal case, the analogs of Propositions 9.2 and 9.3 can be expressed more compactly as the next proposition shows:

Proposition 11.4 *Fix $t = 1, 2, \ldots$. For the renewal on-off source (I, B), for each $\mathbf{x}^t$ in $\{0,1\}^{t+1}$ with $\mathbf{P}[\mathbf{A}^t = \mathbf{x}^t] > 0$, we have the following: If $x_t = 1$, then*

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = 1 - h_B(t - \ell(\mathbf{x}^t) + 1), \quad \ell(\mathbf{x}^t) > 0 \tag{54}$$

and if $x_t = 0$, then

$$\mathbf{P}\left[A_{t+1} = 1 | \mathbf{A}^t = \mathbf{x}^t\right] = h_I(t - \ell(\mathbf{x}^t) + 1), \quad \ell(\mathbf{x}^t) \geq 0. \tag{55}$$

Proof. Note that with $\ell(\mathbf{x}^t) = 0$ and $x_t = 1$, we have $\mathbf{x}^t = (1, \ldots, 1)$ and the event cannot occur since $I_0 =_{st} I$ implies $A_0 =_{st} 0$.

A careful inspection of the proofs of Propositions 9.2 and 9.3 shows that both (54) and (55) hold when $\ell(\mathbf{x}^t) > 0$. Hence, only the case $\ell(\mathbf{x}^t) = 0$ with $x_t = 0$ needs to be considered in order to complete the proof of (55). In that case, we have $\mathbf{x}^t = (0, \ldots, 0)$, whence (41) and (42) now become

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t,\ A_{t+1} = 1\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0,\ A_{t+1} = 1\right] \\ &= \mathbf{P}\left[I_0 = t+1, B_0 \geq 1\right] \\ &= \mathbf{P}\left[I = t+1\right] \mathbf{P}\left[B \geq 1\right] \end{aligned} \tag{56}$$

and

$$\begin{aligned} \mathbf{P}\left[\mathbf{A}^t = \mathbf{x}^t\right] &= \mathbf{P}\left[A_0 = \ldots = A_t = 0\right] \\ &= \mathbf{P}\left[I_0 \geq t+1\right] = \mathbf{P}\left[I \geq t+1\right]. \end{aligned} \tag{57}$$

We conclude to the desired result by combining (56) and (57) via (30). ■

A proof of Proposition 11.1 can now be given: For each $t = 0, 1, \ldots$, we need to show that (28) holds for distinct elements $\mathbf{x}^t$ and $\mathbf{y}^t$ in $\{0,1\}^{t+1}$ satisfying (29) and such that $\mathbf{x}^t \leq \mathbf{y}^t$.

For $t = 0$, $x_0 = 0$ and $y_0 = 1$, so that there is no need for comparison here since $\mathbf{P}[A_0 = y_0] = 0$.

Fix $t = 1, 2, \ldots$. As in the proof of Proposition 6.1, three cases present themselves, depending on whether (a) $x_t = y_t = 1$; (b) $x_t = y_t = 0$; and (c) $x_t = 0 < y_t = 1$.

Case (a) – With $x_t = y_t = 1$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{y}^t) \leq \ell(\mathbf{x}^t)$. By Proposition 11.4, the inequality (28) can be rewritten as

$$h_B(t - \ell(\mathbf{y}^t) + 1) \leq h_B(t - \ell(\mathbf{x}^t) + 1) \tag{58}$$

with $t - \ell(\mathbf{x}^t) + 1 \leq t - \ell(\mathbf{y}^t) + 1$ in $\mathcal{S}(B)$. It is plain that (58) holds because B is assumed DFR.

Case (b) – With $x_t = y_t = 0$, the condition $\mathbf{x}^t \leq \mathbf{y}^t$ implies $\ell(\mathbf{x}^t) \leq \ell(\mathbf{y}^t)$. By Proposition 11.4, the inequality (28) reduces to

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq h_I(t - \ell(\mathbf{y}^t) + 1) \tag{59}$$

with $t - \ell(\mathbf{y}^t) + 1 \leq t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(I)$, and the validity of (59) is implied by the fact that the rv I is DFR.

Case (c) – With $x_t = 0 < y_t = 1$, invoking Proposition 11.4 we can rewrite (28) in reduced form as

$$h_I(t - \ell(\mathbf{x}^t) + 1) \leq 1 - h_B(t - \ell(\mathbf{y}^t) + 1) \tag{60}$$

with $t - \ell(\mathbf{x}^t) + 1$ in $\mathcal{S}(I)$ and $t - \ell(\mathbf{y}^t) + 1$ in $\mathcal{S}(B)$. As in the proof of Proposition 6.1, we must have $\ell(\mathbf{x}^t) \neq \ell(\mathbf{y}^t)$ if both $\ell(\mathbf{x}^t) > 0$ and $\ell(\mathbf{y}^t) > 0$. Hence, it is a simple matter to check in all feasible situations that either $2 \leq t - \ell(\mathbf{x}^t) + 1$ or $2 \leq t - \ell(\mathbf{y}^t) + 1$. The validity of (60) is then guaranteed under (ii) since the rvs I and B are DFR rvs.

Acknowledgement

This material is based upon work supported by the Space and Naval Warfare Systems Center – San Diego under Contract No. N66001-00-C-8063. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the Space and Naval Warfare Systems Center – San Diego.

References

[1] Agrawal, R., Makowski, A.M. and Nain, Ph. (1999). On a reduced load equivalence for a multi–fluid buffer under heavy tail assumptions. *Queueing Systems – Theory & Applications* 33: 5–41.

[2] Barlow, R.E. and Proschan, F. (1975). *Statistical Theory of Reliability and Life Testing*. New York (NY): International Series in Decision Processes, Holt, Rinehart and Winston.

[3] Bäuerle, N. (1997). Monotonicity results for $MR|GI|1$ queues. *Journal of Applied Probability* 34: 514–524.

[4] Bäuerle, N. and Rolski, T. (1998). A monotonicity result for the workload in Markov-modulated queues. *Journal of Applied Probability* 35: 741–747.

[5] Duffield, N. G. and O'Connell, N. (1995), "Large deviations and overflow probabilities for the general single server queue, with applications," *Proceedings of the Cambridge Philosophical Society* 118, 363–374.

[6] Glynn, P.W. and Whitt, W. (1994). "Logarithmic asymptotics for steady–state tail probabilities in a single–server queue," *Journal of Applied Probability* 31: 131–159.

[7] Esary, J.D., Proschan, F. and Walkup, D.W. (1967). Association of random variables, with applications. *Annals of Mathematical Statistics* 38: 1466–1474.

[8] Likhanov, N. and Tsybakov, B. and Georganas, N.D. (1995). Analysis of an ATM buffer with self-similar ("fractal") input traffic. In *Proceedings of IEEE INFOCOM 1995*, Boston, April 1995, pp. 985–992.

[9] Livny, M., Melamed, B. and Tsiolis, A.K. (1993). The impact of autocorrelation on queuing systems. *Management Science* 39: 322–339.

[10] Makowski, A.M. and Parulekar, M. (2000). Buffer Asymptotics for $M|G|\infty$ input processes. In *Self–Similar Network Traffic and Performance Evaluation*, Eds. K. Park and W. Willinger. New York (NY): John Wiley & Sons, pp. 215–248.

[11] Meester, L.E. and Shanthikumar, J.G. (1993) Regularity of stochastic processes: A theory of directional convexity. *Probability in the Engineering and Informational Sciences* 7: 343–360.

[12] Müller, A. and Scarsini, M. (2000). Some remarks on the supermodular order. *Journal of Multivariate Analysis* 73: 107–119.

[13] Park, K. and Willinger, W.(Eds.) (2000). *Self–Similar Network Traffic and Performance Evaluation*. New York (NY): John Wiley & Sons.

[14] Parulekar, M. and Makowski, A.M. (1997). Tail probabilities for a multiplexer driven by $M|G|\infty$ input processes (I): Preliminary Asymptotics. *Queueing Systems – Theory & Applications* 27: 271–296.

[15] Resnick, S. and Samorodnitsky, G. (2000). Fluid queues, on/off processes, and teletraffic modeling with highly variable and correlated inputs. In *Self–Similar Network Traffic and Performance Evaluation*, Eds. K. Park and W. Willinger. New York (NY): John Wiley & Sons, pp. 171–192.

[16] Shaked, M. and Shanthikumar, J.G. (1997). Supermodular stochastic orders and positive dependence of random vectors. *Journal of Multivarate Analysis* 61: 86–101.

[17] Shaked, M. and Shanthikumar, J.G. (1994). *Stochastic Orders and Their Applications*. New York (NY): Academic Press.

[18] Stoyan, D. (1984). *Comparison Methods for Queues and Other Stochastic Models*. English Translation (D.J. Daley, Editor). New York (NY): J. Wiley & Sons.

[19] Szekli, R. (1995). *Stochastic Ordering and Dependence in Applied Probability*, Lecture Notes in Statistics **97**. New York (NY): Springer-Verlag.

[20] Toyoizumi, H. and Shanthikumar, J.G. and Wolff, R.W. (1997). Two extremal autocorrelated arrival processes. *Probability in the Engineering and Informational Sciences* 11: 441–450.

[21] Vanichpun, S. (2001). *The Effects of Positive Correlations On Buffer Occupancy: Comparisons and Lower-Bounds Via Supermodular Ordering*. M.S. Thesis, Department of Electrical and Computer Engineering, University of Maryland, College Park (MD), December 2001.

[22] Vanichpun, S. and Makowski, A.M. (2002). The effects of positive correlations on buffer occupancy: Lower bounds via supermodular ordering. In *Proceeding of IEEE INFOCOM 2002*, New York (NY), June 2002.