TECHNICAL RESEARCH REPORT

The Berry-Hannay Phase of the Equal-Sided, Spring-Jointed, Four-Bar Mechanism: A Detailed Story

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The Berry-Hannay Phase of the Equal-Sided, Spring-Jointed, Four-Bar Mechanism: A Detailed Story

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Abstract

In this work we apply the moving systems approach developed by Marsden, Montgomery, and Ratiu to a free-floating, equal-sided, spring-jointed, four-bar mechanism that is being slowly rotated about its central axis and derive a formula for the induced geometric phase. We investigate the phase for a few specific systems using both analytic analysis and simulation.

1 Introduction

In this report we apply the moving systems approach developed by Marsden, Montgomery, and Ratiu [6] to an equal-sided, spring-jointed, four-bar mechanism, find a general formula for the Berry-Hannay phase in action-angle coordinates, and investigate the detailed form of the phase for particular spring potentials through analytic calculation and simulation. This work builds upon earlier work by Krishnaprasad and Yang on free-floating four-bar mechanisms [13].

It is hoped that this effort will prove useful in its own right and also serve as a stepping stone to the analysis of an equal-sided n-bar mechanism and from there to a new approach to the rotating,vibrating ring. The ring system, originally analyzed by G.H. Bryan in 1890 [3], has been used successfully in several gyroscope designs (e.g. [5] and [9]). It is to be expected that a deeper understanding will emerge by appealing to a non-linear, geometric approach directed at more accurate constitutive models. This work is a first step in that direction.

Here we consider an equal-sided, spring-jointed, four-bar mechanism. The mechanism is being adiabatically (slowly) rotated about its central axis at rate Ω. We assume the spring potentials are such that there exists a stable equilibrium shape and a stable periodic solution near that shape. The rest of this report is organized as follows. In Section 2 we present a brief description of the moving systems approach. In Section 3 we describe the four-bar mechanism and define the configuration and ambient spaces. In Section 4 the Lagrangian is derived and in Section 5 the Hamiltonian and the Berry-Hannay phase formula are calculated. In Section 6 we investigate the phase formula in detail and present some simulation results. We conclude in 7 with a few remarks about future work.

2 The Moving Systems Approach

The approach developed by Marsden, Montgomery, and Ratiu uses modern geometric tools to define the Berry-Hannay phase as the holonomy of the Cartan-Hannay-Berry connection. For background and details on the method please see [6]. Here we provide a brief description of the approach as given in [7].

Let S be a Riemannian manifold referred to as the ambient space. Let $Q \subset S$ be the configuration space for some system. Let M be the space of embeddings of Q into S. Consider a curve m_t in M. If a particle in Q follows a path $q(t)$ and Q moves along the curve m_t then the particle in S follows the path $m_t(q(t))$. The velocity in S is given by

$$T_{q(t)}m_t \cdot v + \mathcal{Z}_t(m_t(q(t)))$$

where $v \stackrel{\triangle}{=} \dot{q}$ and $\mathcal{Z}_t(m_t(q(t)))$ is the velocity vector $\frac{dm_t}{dt}$. Let $V(q(t))$ be a potential on Q and $U(m_t(q(t)))$ be a potential on S. The usual Lagrangian is given by

$$L(q,v) = \frac{1}{2}\|T_{q(t)}m_t \cdot v + \mathcal{Z}_t(m_t(q(t)))\|^2 - V(q(t)) - U(m_t(q(t))) \tag{1}$$

where the norm is given by the Riemannian metric on S. To compute the fiber derivative of L along $T_{q(t)}Q$ we take the derivative of L with respect to v in the direction w for $w \in T_{q(t)}Q$.

$$\frac{\partial L}{\partial v} \cdot w \stackrel{\triangle}{=} p \cdot w = < T_{q(t)}m_t \cdot v + \mathcal{Z}_t^T(m_t(q(t))), T_{q(t)}m_t \cdot w >_{m_t(q(t))} \tag{2}$$

where $p \cdot w$ is the natural pairing of the covector $p \in T^*_{q(t)}Q$ with w, $< \cdot, \cdot >_{m_t(q(t))}$ is the metric on S at the point $m_t(q(t))$ and ^{T} denotes the orthogonal projection onto $T_{q(t)}Q$ under the metric of S. Q inherits a metric from S under which m_t is an isometry. Using this metric we have

$$\begin{aligned} p \cdot w &= < v + \left(T_{q(t)}m_t\right)^{-1} \mathcal{Z}_t^T(m_t(q(t))), w >_{q(t)} && (3) \\ \Rightarrow p &= \left(v + \left(T_{q(t)}m_t\right)^{-1} \mathcal{Z}_t^T(m_t(q(t)))\right)^\flat && (4) \end{aligned}$$

where $\flat$ is the map

$$\begin{aligned} (\cdot)^\flat : T_qQ &\rightarrow T_q^*Q \\ z &\mapsto z^\flat \end{aligned}$$

defined by

$$z^\flat \cdot w = < z, w >_q \quad \forall w \in T_qQ \tag{5}$$

The Hamiltonian is given by

$$\begin{aligned} H(q,p) &= p \cdot v - L(q,v) \\ &= \frac{1}{2}\|p\|^2 - \mathcal{P}(Z_t) - \frac{1}{2}\|\mathcal{Z}^\perp\|^2 + V(q) + U(m_t(q)) && (6) \end{aligned}$$

where

$$\mathcal{P}(Z_t) \stackrel{\triangle}{=} p \cdot \left(T_{q(t)}m_t\right)^{-1} \left[\mathcal{Z}_t(m_t(q))\right]^T \tag{7}$$

Define the nominal Hamiltonian by setting $\mathcal{Z}$ and U to zero. Let G be a Lie group that acts on T^*Q and leaves the nominal Hamiltonian invariant. Assuming G has an invariant measure with respect to which we can average, we replace the Hamiltonian by its G-average

$$< H > (q,p) = \frac{1}{2}\|p\|^2 - < \mathcal{P}(Z_t) > - \frac{1}{2} < \|Z_t^{\perp}\|^2 > + V(q) + U(m_t(q)) \tag{8}$$

Under the assumption of a slowly moving system we discard $< \|Z_t^{\perp}\|^2 >$ as small. Let $X_{<\mathcal{P}(Z_t)>}$ be the Hamiltonian vector field associated to the extra term in the Hamiltonian coming from the imposed motion. Then $-X_{<\mathcal{P}(Z_t)>}$ can be interpreted as coming from the horizontal lift of Z_t relative to a connection on $T^*Q \times M$ known as the Cartan-Hannay-Berry connection. The holonomy of this connection, given by the integral of the lifted vector field around the closed loop in the base space, is known as the **Berry-Hannay** phase and is calculated from

$$\mathcal{H}_{m_t} = -\oint_{m_t} X_{<\mathcal{P}(Z_t)>} dt \tag{9}$$

3 The Equal-Sided Four-Bar Mechanism: Ambient and Configuration Spaces

In this section we describe the four-bar system and place it in the moving systems framework. We follow the approach of Yang and Krishnaprasad in [13]. The structure of an equal-sided four-bar mechanism is shown in figure 1. By a 'bar' we mean a planar rigid body on which the center of mass and pin joints are arbitrarily located. The bars are labeled sequentially from 0 to 3 and on each a body-fixed frame is defined such that its origin is at the body center of mass and the x-axis is parallel to the line connecting the pin joints. The positive direction of the x-axis of the i^{th} bar is defined to be towards the $(i+1)^{th}$ bar for $i = 0, 1, 2, 3$ (mod 4). We define the following:

$\mathbf{d}_+$	the vector from the body center of mass of the i^{th} bar to the pin joint with the $(i+1)^{th}$ bar
$\mathbf{d}_-$	the vector from the body center of mass of the i^{th} bar to the pin joint with the $(i-1)^{th}$ bar
l	the length of each bar, given by $\|\mathbf{d}_+ - \mathbf{d}_-\|$
$\mathbf{r}_i^c$	the vector from the system center of mass to the i^{th} body center of mass
$\mathbf{r}_c$	the vector from the inertial system to the system center of mass
θ_i	the angle between the i^{th} bar frame and the inertial frame
θ_{ij}	the angle between the i^{th} and j^{th} bars given by $\theta_i - \theta_j$
I	the moment of inertia of each bar
m	the mass of each bar

The loop closure constraints are

$$r_{i+1}^c = r_i^c + R(\theta_i)\mathbf{d}_+ - R(\theta_{i+1})\mathbf{d}_- \tag{10}$$

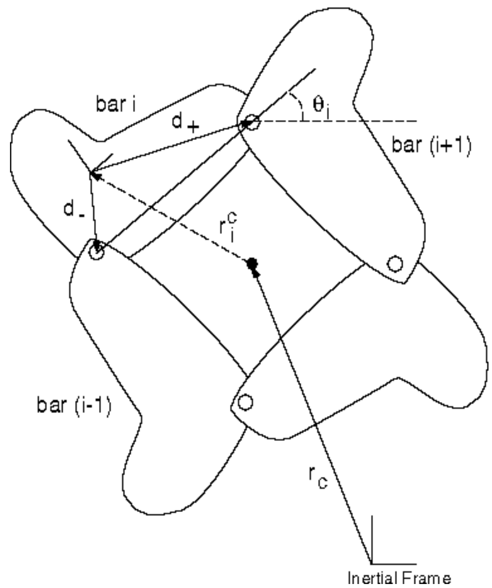

Figure 1: Equal-Sided Four-Bar Mechanism

We can manipulate these equations as follows

$$\begin{aligned} r_1^c &= r_0^c + R(\theta_0)\mathbf{d}_+ - R(\theta_1)\mathbf{d}_- \\ &= r_3^c + R(\theta_3)\mathbf{d}_+ - R(\theta_0)\mathbf{d}_- + R(\theta_0)\mathbf{d}_+ - R(\theta_1)\mathbf{d}_- \\ &= r_1^c + R(\theta_1)\mathbf{d}_+ - R(\theta_2)\mathbf{d}_- + R(\theta_3)\mathbf{d}_+ - R(\theta_0)\mathbf{d}_- + R(\theta_0)\mathbf{d}_+ - R(\theta_1)\mathbf{d}_- \\ \Rightarrow & \sum_{i=0}^{3} R(\theta_i)(\mathbf{d}_+ - \mathbf{d}_-) = 0 \end{aligned} \tag{11}$$

From [10] we know the configuration space for a free-floating four-link open chain is $R = \mathbb{R}^2 \times S^1 \times S^1 \times S^1 \times S^1$. The configuration space for a general four-bar mechanism is then $S = \{r \in R | F(r) = 0\}$ where $F(r)$ is the loop closure constraint, equation (11). In [13] it is shown that S is a manifold under certain conditions on the parameters of the mechanism. While those conditions are not met here, by explicitly requiring that the mechanism not pass through any singularities (joint angles of 0 or π) we can ensure S is a smooth submanifold of R. This is shown in the following lemma.

Lemma 3.1 *If $l_i = l$ $\forall i$ and the mechanism is restricted from achieving any singular configuration then S is a smooth submanifold of M.*

Proof Under the given assumptions we need only show that 0 is a regular value of F since then S is a smooth submanifold of M. This proof follows that of Theorem 3.2.1 in [12]. We have:

$$\frac{\partial F}{\partial m} = \left(\frac{\partial F}{\partial x} \quad \frac{\partial F}{\partial y} \quad \frac{\partial F}{\partial \theta_0} \quad \frac{\partial F}{\partial \theta_1} \quad \frac{\partial F}{\partial \theta_2} \quad \frac{\partial F}{\partial \theta_3} \right) \tag{12}$$

Since $\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}$ is the vector connecting the pin joints on the i^{th} bar we can write

$$R(\theta_i)(\mathbf{d}_{i,i+1} - \mathbf{d}_{i,i-1}) = l \begin{pmatrix} cos(\theta_i) \\ sin(\theta_i) \end{pmatrix}$$

Then

$$\frac{\partial F}{\partial m} = \begin{pmatrix} 0 & 0 & -lsin(\theta_0) & -lsin(\theta_1) & -lsin(\theta_2) & -lsin(\theta_3) \\ 0 & 0 & lcos(\theta_0) & lcos(\theta_1) & lcos(\theta_2) & lcos(\theta_3) \end{pmatrix} \tag{13}$$

The nontrivial 2×2 subdeterminants are:

$$\begin{aligned}
-l^2 sin(\theta_0)cos(\theta_1) + l^2 cos(\theta_0)sin(\theta_1) &= l^2 sin(\theta_1 - \theta_0) \stackrel{\triangle}{=} g_1(m) \\
-l^2 sin(\theta_0)cos(\theta_2) + l^2 cos(\theta_0)sin(\theta_2) &= l^2 sin(\theta_2 - \theta_0) \stackrel{\triangle}{=} g_2(m) \\
-l^2 sin(\theta_0)cos(\theta_3) + l^2 cos(\theta_0)sin(\theta_3) &= l^2 sin(\theta_3 - \theta_0) \stackrel{\triangle}{=} g_3(m) \\
-l^2 sin(\theta_1)cos(\theta_2) + l^2 cos(\theta_1)sin(\theta_2) &= l^2 sin(\theta_2 - \theta_1) \stackrel{\triangle}{=} g_4(m) \\
-l^2 sin(\theta_1)cos(\theta_3) + l^2 cos(\theta_1)sin(\theta_3) &= l^2 sin(\theta_3 - \theta_1) \stackrel{\triangle}{=} g_5(m) \\
-l^2 sin(\theta_2)cos(\theta_3) + l^2 cos(\theta_2)sin(\theta_3) &= l^2 sin(\theta_3 - \theta_2) \stackrel{\triangle}{=} g_6(m)
\end{aligned}$$

To ensure that 0 is a regular value of F we must simply ensure that for all possible values of θ_i at least one $g_i(m) \neq 0$. To have $g_i(m) = 0 \ \forall i$ we must have

$$\begin{aligned}
\theta_1 - \theta_0 &= 0 \text{ or } \pi \\
&\text{and} \\
\theta_2 - \theta_1 &= 0 \text{ or } \pi \\
&\text{and} \\
\theta_3 - \theta_2 &= 0 \text{ or } \pi \\
&\text{and} \\
\theta_0 - \theta_3 &= 0 \text{ or } \pi
\end{aligned}$$

However $\theta_{i+1} - \theta_i = 0$ or π is expressly forbidden by the restriction that the mechanism may not achieve any singular configuration. ■

While in the general four-bar mechanism the relations between the global angles can be quite complicated (see, for example, [12] or [2]) they have a particularly simple form for the equal-sided case.

Lemma 3.2 *$\theta_2 = \theta_0 + \pi$ and $\theta_3 = \theta_1 - \pi$*

Proof In figure 2 we show the four pin joints, the connecting lines defining the local x-axis directions, and the angles θ_i. The positive direction of each θ_i is defined to be in the counter-clockwise direction from the inertial frame x-axis. From the figure we immediately have

$$\begin{aligned} \theta_2 + (-\theta_0) = \pi \quad &\Rightarrow \quad \theta_2 = \pi + \theta_0 \\ \theta_1 + (-\theta_3) = \pi \quad &\Rightarrow \quad \theta_3 = \theta_1 - \pi \end{aligned}$$

■

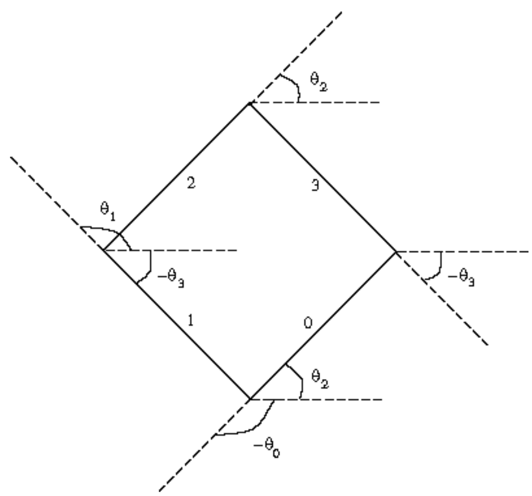

Figure 2: Relation between the θ_i

From this lemma we have the following set of equalities

$$\begin{array}{ccccc} \theta_{32} & = & \theta_{10} & & \\ \theta_{21} & = & \theta_{03} & = & \pi - \theta_{10} \\ \theta_{13} & = & \theta_{20} & = & \pi \end{array} \tag{14}$$

As in Yang and Krishnaprasad [13] we assume the inertial observer is placed at the system center of mass. (We can do this because the kinetic energy of the system is invariant under translations in inertial space and the configuration space can then be symplectically reduced by the translation group $\mathbb{R}^2$ as in Sreenath [10].) We take as the ambient space the manifold

$$S = S^1 \times S^1 \tag{15}$$

and choose as coordinates (θ_0, θ_{10}).

Again from [13] we know that this system is a simple mechanical system with symmetry and thus reduction to the shape space is possible. We take as the configuration space

$$Q = S^1 \tag{16}$$

and choose as a coordinate (θ_{10}).

The embedding is given by

$$m_t(\theta_{10}) = \begin{pmatrix} \Omega t + \theta_0(0) \\ \theta_{10}(t) \end{pmatrix} \tag{17}$$

where $\theta_0(0)$ is simply the initial angle.

4 The Lagrangian

4.1 Kinetic Energy

From classical mechanics the total kinetic energy of the system in the center of mass frame is given by

$$T = \frac{1}{2} I \sum_{i=0}^{3} \omega_i^2 + \frac{1}{2} m \sum_{i=0}^{3} \|\dot{r}_i^c\|^2 \tag{18}$$

Following [13] we can write this as

$$T = \frac{1}{2} < \widetilde{\omega}, \widetilde{M} \widetilde{\omega} > \tag{19}$$

where $\widetilde{\omega} = (\omega_0, \omega_1, \omega_2, \omega_3)$ and $\widetilde{M}$ is a 4x4 symmetric matrix whose elements for the equal-sided four-bar are given as

$$\begin{aligned} \widetilde{M}_{ii} &= I + \frac{3m}{8} \left(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2 \right) && (20) \\ \widetilde{M}_{i,i+1} &= \frac{m}{8} (< \mathbf{d}_-, R_{i+1,i} \mathbf{d}_+ > -3 < \mathbf{d}_+, R_{i+1,i} \mathbf{d}_- >) && (21) \\ \widetilde{M}_{i,i+2} &= -\frac{m}{8} (< \mathbf{d}_+, R_{i+2,i} \mathbf{d}_+ > + < \mathbf{d}_-, R_{i+2,i} \mathbf{d}_- >) && (22) \end{aligned}$$

For the equal-sided four-bar we have

$$\begin{pmatrix} \omega_2 \\ \omega_3 \end{pmatrix} = \mathbb{1} \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} \tag{23}$$

where $\mathbb{1}$ is the identity matrix. Defining

$$M = (\mathbb{1} \quad \mathbb{1}) \widetilde{M} \begin{pmatrix} \mathbb{1} \\ \mathbb{1} \end{pmatrix} \tag{24}$$

we have

$$T = \frac{1}{2} < \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix}, M \begin{pmatrix} \omega_0 \\ \omega_1 \end{pmatrix} > \tag{25}$$

M is symmetric and depends only on the joint angles. Given the relations (14), M depends only on θ_{10}. Now we want to write the kinetic energy in terms of (ω_0, ω_{10}) where $\omega_0 = \Omega$ from the embedding. Expanding equation (25) yields

$$\begin{aligned}
T &= \frac{1}{2}(\omega_0^2 M_{00} + 2\omega_0\omega_1 M_{10} + \omega_1^2 M_{11}) && (26)\\
&= \frac{1}{2}(\omega_0^2 M_{00} + 2\omega_0(\omega_1 - \omega_0 + \omega_0)M_{10} + (\omega_1 - \omega_0 + \omega_0)^2 M_{11}) && (27)\\
&= \frac{1}{2}(\omega_0^2 M_{00} + 2\omega_0(\omega_{10} + \omega_0)M_{10} + (\omega_{10} + \omega_0)^2 M_{11}) && (28)\\
&= \frac{1}{2}(\omega_0^2 M_{00} + 2(\omega_0\omega_{10} + \omega_0^2)M_{10} + (\omega_{10}^2 + 2\omega_0\omega_{10} + \omega_0^2)M_{11}) && (29)\\
&= \frac{1}{2} < \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix}, \begin{pmatrix} M_{00} + 2M_{10} + M_{11} & M_{11} + M_{10}) \\ M_{11} + M_{10} & M_{11} \end{pmatrix} \begin{pmatrix} \omega_0 \\ \omega_{10} \end{pmatrix} > && (30)\\
&\triangleq \frac{1}{2} < \widehat{\omega}, \widehat{M}(\theta_{10})\widehat{\omega} > && (31)
\end{aligned}$$

This defines a Riemannian metric K on S given by

$$K(m_t(q))(v_{m_t(q)}, w_{m_t(q)}) =< v_{m_t(q)}, \widehat{M}(q) w_{m_t(q)} > \qquad (32)$$

for $v_{m_t(q)}, w_{m_t(q)} \in T_{m_t(q)} m_t(Q)$.

4.2 Potential Energy

Each joint is equipped with an identical spring. Let the spring potential for each be given by $V_s(\theta_{i+1,i})$, $i = 0, 1, 2, 3 \text{ mod}(4)$ with V_s twice continuously differentiable. The total potential energy is then

$$\begin{aligned}
V(\theta_0, \theta_1, \theta_2, \theta_3) &= \sum_{i=0}^{3} V_s(\theta_{i+1,i}) && (33)\\
&= V_s(\theta_{10}) + V_s(\theta_{21}) + V_s(\theta_{32}) + V_s(\theta_{03}) && (34)\\
&= V_s(\theta_{10}) + V_s(\pi - \theta_{10}) + V_s(\theta_{10}) + V_s(\pi - \theta_{10}) && (35)\\
&= 2(V_s(\theta_{10}) + V_s(\pi - \theta_{10})) && (36)\\
&= V(\theta_{10}) && (37)
\end{aligned}$$

Then since $V_s \in C^2$ we have $V \in C^2$. We assume the potential energy is such that $\exists\ \alpha \in S^1$ such that

$$\begin{aligned}
\left.\frac{\partial V}{\partial \theta_{10}}\right|_{\alpha} &= 0 && (38)\\
\left.\frac{\partial^2 V}{\partial \theta_{10}^2}\right|_{\alpha} &> 0 && (39)
\end{aligned}$$

The Lagrangian is given by

$$
\begin{aligned}
L(\theta_{10}, \omega_{10}) &= \frac{1}{2} < \widehat{\omega}, \widehat{M}(\theta_{10})\widehat{\omega} > -V(\theta_{10}) && (40)\\
&= \frac{1}{2} < \begin{pmatrix} \Omega \\ \omega_{10} \end{pmatrix}, \widehat{M}(\theta_{10}) \begin{pmatrix} \Omega \\ \omega_{10} \end{pmatrix} > -V(\theta_{10}) && (41)
\end{aligned}
$$

where in the last step we have used the embedding, equation (17). We can now easily put this in the standard moving systems form

$$
L(\theta_{10}, \omega_{10}) = \frac{1}{2} < \left[\begin{pmatrix} 0 \\ \omega_{10} \end{pmatrix} + \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \right], \widehat{M}(\theta_{10}) \left[\begin{pmatrix} 0 \\ \omega_{10} \end{pmatrix} + \begin{pmatrix} \Omega \\ 0 \end{pmatrix} \right] > -V(\theta_{10}) \tag{42}
$$

Thus

$$
\begin{aligned}
T_{q(t)} m_t \cdot v &= \begin{pmatrix} 0 \\ \omega_{10} \end{pmatrix} && (43)\\
\mathcal{Z}_t(m_t(q(t))) &= \begin{pmatrix} \Omega \\ 0 \end{pmatrix} && (44)
\end{aligned}
$$

5 The Hamiltonian and the Berry-Hannay Phase Formula

We first need to find $\mathcal{Z}_t(m_t(q(t)))^T$. Write

$$
\mathcal{Z}_t = \mathcal{Z}_t^T + \mathcal{Z}_t^{\perp} = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} v^{\perp 1} \\ v^{\perp 2} \end{pmatrix} \tag{45}
$$

for some $v \in T_{m_t(q)} m_t(Q)$ with $Z^{\perp}$ such that

$$
(0 \quad w)\widehat{M}\mathcal{Z}_t^{\perp} = 0 \quad \forall\, w \in T_{m_t(q)} m_t(Q) \tag{46}
$$

From equation (46) we have

$$
\begin{aligned}
& w[(M_{11} + M_{10})v^{\perp 1} + M_{11} v^{\perp 2}] = 0 \quad \forall\, w \in T_{m_t(q)} m_t(Q) && (47)\\
\Rightarrow \quad & v^{\perp 2} = -\left[\frac{M_{10} + M_{11}}{M_{11}} \right] v^{\perp 1} && (48)
\end{aligned}
$$

Now

$$
\begin{aligned}
& \mathcal{Z}_t = \begin{pmatrix} \Omega \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ v \end{pmatrix} + \begin{pmatrix} v^{\perp 1} \\ v^{\perp 2} \end{pmatrix} && (49)\\
\Rightarrow \quad & v^{\perp 1} = \Omega \qquad v = -v^{\perp 2} && (50)\\
\Rightarrow \quad & v = \left[\frac{M_{10} + M_{11}}{M_{11}} \right] \Omega && (51)
\end{aligned}
$$

Now consider the point $(m_t(q(t)), v) \in Tm_t(Q)$. We have

$$(Tm_t)^{-1}(m_t(q(t)), v) = (q(t), v) \tag{52}$$

Thus

$$(T_{q(t)}m_t)^{-1}\mathcal{Z}_t^T = \mathcal{Z}_t^T \tag{53}$$

From equations (7), (51), and (53) we have

$$\mathcal{P}(Z_t) = \left[\frac{M_{10} + M_{11}}{M_{11}}\right]\Omega p_{10} \tag{54}$$

Now from equations (45) and (48) we have

$$\mathcal{Z}^{\perp} = \begin{pmatrix} \Omega \\ -\frac{\widehat{M}_{10}}{\widehat{M}_{11}} \end{pmatrix} \tag{55}$$

Then

$$\begin{aligned}
\|\mathcal{Z}^{\perp}\|^2 &= \Omega^2 \begin{pmatrix} 1 & -\dfrac{\widehat{M}_{10}}{\widehat{M}_{11}} \end{pmatrix} \begin{pmatrix} \widehat{M}_{00} & \widehat{M}_{10} \\ \widehat{M}_{10} & \widehat{M}_{11} \end{pmatrix} \begin{pmatrix} 1 \\ -\frac{\widehat{M}_{10}}{\widehat{M}_{11}} \end{pmatrix} &(56)\\
&= \Omega^2 \left(\widehat{M}_{00} - \frac{\widehat{M}_{10}^2}{\widehat{M}_{11}}\right) &(57)\\
&= \Omega^2 \left(\frac{\widehat{M}_{00}\widehat{M}_{11} - \widehat{M}_{10}^2}{\widehat{M}_{11}}\right) &(58)\\
&= \Omega^2 \left(\frac{(M_{00} + 2M_{10} + M_{11})(M_{00}) - (M_{00} + M_{10})^2}{M_{11}}\right) &(59)\\
&= \Omega^2 \left(\frac{M_{00}M_{11} - M_{10}^2}{M_{11}}\right) &(60)
\end{aligned}$$

The restriction of the metric on S to Q is simply M_{11}. Thus from equations (6), (54), and (60) we have the Hamiltonian

$$H(\theta_{10}, p_{10}) = \frac{1}{2M_{11}}p_{10}^2 + V(\theta_{10}) - \left[\frac{M_{10} + M_{11}}{M_{11}}\right]\Omega p_{10} - \frac{\Omega^2}{2}\left(\frac{M_{00}M_{11} - M_{10}^2}{M_{11}}\right) \tag{61}$$

Due to the adiabatic assumption, we can drop the $\|\mathcal{Z}_t^{\perp}\|^2$ term to get

$$H(\theta_{10}, p_{10}) = \frac{1}{2M_{11}}p_{10}^2 + V(\theta_{10}) - \left[\frac{M_{10} + M_{11}}{M_{11}}\right]\Omega p_{10} \tag{62}$$

We see that the Hamiltonian of the nominal system is given by

$$H_{nom}(\theta_{10}, p_{10}) = \frac{1}{2M_{11}}p_{10}^2 + V(\theta_{10}) \tag{63}$$

5.1 The Nominal System

5.1.1 Periodic solutions

Recall that we assumed the nominal system had an equilibrium shape and a periodic solution around that equilibrium. Here we discuss under what conditions this assumption holds. The nominal Hamiltonian vector field is

$$X_{H_{nom}} = \left(\frac{p_{10}}{M_{11}} \frac{\partial}{\partial \theta_{10}}, \quad -\frac{\partial V}{\partial \theta_{10}} \frac{\partial}{\partial p_{10}} \right) \tag{64}$$

From equation (38) we know there is an equilibrium point at $(\theta_{10} = \alpha, p_{10} = 0)$. To ensure the existence of a periodic solution we appeal to the following theorem by Weinstein [11], paraphrased from [8].

Theorem 5.1 *(*A. Weinstein*) Consider $H : \mathbb{R}^{2n} \to \mathbb{R}$. If $H \in C^2$ near an equilibrium point z and the Hessian matrix at the equilibrium point is positive definite then for sufficiently small ϵ any energy surface $H(z) = H(0) + \epsilon^2$ contains at least n periodic orbits of the associated Hamiltonian system.*

The Hessian matrix of the Hamiltonian for the nominal system is

$$Hess = \begin{pmatrix} \frac{\partial^2 H}{\partial \theta_{10}^2} & 0 \\ 0 & \frac{\partial^2 H}{\partial p_{10}^2} \end{pmatrix} = \begin{pmatrix} \frac{\partial^2 V}{\partial \theta_{10}^2} & 0 \\ 0 & M_{11}^{-1} \end{pmatrix} \tag{65}$$

Since M is positive definite by its construction, $M_{11} > 0$. By assumption (equation (39)) the Hessian of V at α is positive definite. Therefore by theorem (5.1) there is a periodic solution around the equilibrium if the energy is sufficiently small.

5.1.2 Action-angle coordinates

Since this is a one-dimensional system it is integrable and thus there exist action-angle coordinates (I, ϕ) [1]. Let $M(h)$ be the trajectory in phase space corresponding to the energy h. Then

$$I = \frac{1}{2\pi} \oint_{M(h)} p_{10} d\theta_{10} \tag{66}$$

The trajectory $M(h)$ and thus the action depends on the form of $V(\theta_{10})$. We can write in general

$$\begin{matrix} I = g_1(\theta_{10}, p_{10}) & & \theta_{10} = f_1(I, \phi) \\ & \Leftrightarrow & \\ \phi = g_2(\theta_{10}, p_{10}) & & p_{10} = f_2(I, \phi) \end{matrix} \tag{67}$$

The action is a constant of the motion in the nominal system and thus $H_{nom}(\theta_{10}, p_{10}) = H_{nom}(I)$. We are assuming the excursions are adiabatic and thus I remains conserved in the moving system. We can rewrite the Hamiltonian in terms of (I, ϕ)

$$H(I, \phi) = H_{nom}(I) - \left[\frac{M_{01}(f_1(I, \phi)) + M_{11}}{M_{11}} \right] \Omega f_2(I, \phi) \tag{68}$$

where we have explicitly shown the dependence on the coordinates. We take the average of the Hamiltonian over one cycle in ϕ to get

$$< H > (I,\phi) = H_{nom}(I) - < \mathcal{P}(Z_t) > \tag{69}$$

where

$$\begin{aligned} < \mathcal{P}(Z_t) > &= \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{M_{01}(f_1(I,\phi)) + M_{11}}{M_{11}}\right)\Omega f_2(I,\phi)d\phi && (70)\\ &= \frac{\Omega}{2\pi M_{11}}\int_0^{2\pi}(1 + M_{01}(f_1(I,\phi)))\, f_2(I,\phi)d\phi && (71)\\ &= \frac{\Omega}{2\pi M_{11}} f_3(I) && (72)\end{aligned}$$

for some $f_3(I)$. The horizontal lift of $\mathcal{Z}_t$ with respect to the Cartan-Hannay-Berry connection is then

$$-X_{<\mathcal{P}(Z_t)>} = -\frac{\Omega}{2\pi M_{11}}\frac{\partial f_3(I)}{\partial I}\frac{\partial}{\partial\phi} \tag{73}$$

and the Berry-Hannay phase is

$$\Delta\phi = -\frac{1}{2\pi M_{11}}\int_0^{2\pi}\frac{\partial f_3(I)}{\partial I}d\theta_0 = -\frac{1}{M_{11}}\frac{\partial f_3(I)}{\partial I} \tag{74}$$

6 Detailed Calculations of the Berry-Hannay Phase

From equations (24), (20),(22), and (14) we have

$$\begin{aligned} M_{00} &= \widetilde{M}_{00} + \widetilde{M}_{02} + \widetilde{M}_{20} + \widetilde{M}_{22} && (75)\\ &= 2(\widetilde{M}_{00} + \widetilde{M}_{02}) && (76)\\ &= 2(I + \frac{3m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) - \frac{m}{8}(< \mathbf{d}_+, R_{20}\mathbf{d}_+ > + < \mathbf{d}_-, R_{20}\mathbf{d}_- >)) && (77)\\ &= 2(I + \frac{3m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) - \frac{m}{8}(< \mathbf{d}_+, R_{\pi}\mathbf{d}_+ > + < \mathbf{d}_-, R_{\pi}\mathbf{d}_- >)) && (78)\\ &= 2(I + \frac{3m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) + \frac{m}{8}(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)) && (79)\\ &= 2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) && (80)\\ M_{11} &= \widetilde{M}_{11} + \widetilde{M}_{13} + \widetilde{M}_{31} + \widetilde{M}_{33} && (81)\\ &= 2(\widetilde{M}_{11} + \widetilde{M}_{13}) && (82)\\ &= 2(\widetilde{M}_{00} + \widetilde{M}_{02}) && (83)\\ &= M_{00} && (84)\end{aligned}$$

From equations (24), (21), and (14) we have

$$M_{01} = \widetilde{M}_{10} + \widetilde{M}_{12} + \widetilde{M}_{30} + \widetilde{M}_{32} \tag{85}$$

$$
\begin{aligned}
&= 2(\widetilde{M}_{10} + \widetilde{M}_{12}) && (86)\\
&= 2\frac{m}{8}\,(< \mathbf{d}_-, R_{10}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{10}\mathbf{d}_- > + < \mathbf{d}_-, R_{21}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{21}\mathbf{d}_- >) && (87)\\
&= \frac{m}{4}\,(< \mathbf{d}_-, R_{10}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{10}\mathbf{d}_- > + < \mathbf{d}_-, R_{\pi-\theta_{10}}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{\pi-\theta_{10}}\mathbf{d}_- >) && (88)
\end{aligned}
$$

Now

$$
\begin{aligned}
R_{\pi-\theta_{10}} &= \begin{pmatrix} cos(\pi-\theta_{10}) & -sin(\pi-\theta_{10}) \\ sin(\pi-\theta_{10}) & cos(\pi-\theta_{10}) \end{pmatrix} && (89)\\
&= \begin{pmatrix} cos(\pi)cos(\theta_{10}) + sin(\pi)sin(\theta_{10}) & -(sin(\pi)cos(\theta_{10}) - sin(\theta_{10})cos(\pi)) \\ sin(\pi)cos(\theta_{10}) - sin(\theta_{10})cos(\pi) & cos(\pi)cos(\theta_{10}) + sin(\pi)sin(\theta_{10}) \end{pmatrix} && (90)\\
&= \begin{pmatrix} -cos(\theta_{10}) & -sin(\theta_{10}) \\ sin(\theta_{10}) & -cos(\theta_{10}) \end{pmatrix} && (91)\\
&= \begin{pmatrix} -cos(\theta_{01}) & sin(\theta_{01}) \\ -sin(\theta_{01}) & -cos(\theta_{01}) \end{pmatrix} && (92)\\
&= -R_{01} && (93)
\end{aligned}
$$

Plugging this result into equation (88) yields

$$
\begin{aligned}
M_{01} &= \frac{m}{4}\,(< \mathbf{d}_-, R_{10}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{10}\mathbf{d}_- > - < \mathbf{d}_-, R_{01}\mathbf{d}_+ > +3 < \mathbf{d}_+, R_{01}\mathbf{d}_- >) && (94)\\
&= \frac{m}{4}\,(< \mathbf{d}_-, R_{10}\mathbf{d}_+ > -3 < \mathbf{d}_+, R_{10}\mathbf{d}_- > - < \mathbf{d}_+, R_{10}\mathbf{d}_- > +3 < \mathbf{d}_-, R_{10}\mathbf{d}_+ >) && (95)\\
&= m\,(< \mathbf{d}_-, R_{10}\mathbf{d}_+ > - < \mathbf{d}_+, R_{10}\mathbf{d}_- >) && (96)\\
&= m\,\big([(d_-^1 d_+^1 + d_-^2 d_+^2)cos(\theta_{10}) + (d_+^1 d_-^2 - d_+^2 d_-^1)sin(\theta_{10})] \\
&\qquad -[(d_-^1 d_+^1 + d_-^2 d_+^2)cos(\theta_{10}) + (d_-^1 d_+^2 - d_-^2 d_+^1)sin(\theta_{10})]\big) && (97)\\
&= 2m(d_+^1 d_-^2 - d_+^2 d_-^1)sin(\theta_{10}) && (98)
\end{aligned}
$$

Consider the diagram of a single bar in figure 3. From the figure we have

$$
\mathbf{d}_+ = \begin{pmatrix} \frac{l}{2} + \delta_x \\ \delta_y \end{pmatrix} \qquad \mathbf{d}_- = \begin{pmatrix} -(\frac{l}{2} - \delta_x) \\ \delta_y \end{pmatrix} \tag{99}
$$

Thus

$$
\begin{aligned}
d_+^1 d_-^2 - d_+^2 d_-^1 &= (\frac{l}{2} + \delta_x)(\delta_y) + (\delta_y)(\frac{l}{2} - \delta_x) && (100)\\
&= l\delta_y && (101)
\end{aligned}
$$

Plugging this into equation (98) gives

$$
M_{01} = 2ml\delta_y sin(\theta_{10}) \tag{102}
$$

By symmetry $M_{01} = M_{10}$. We now consider particular forms of the potential function.

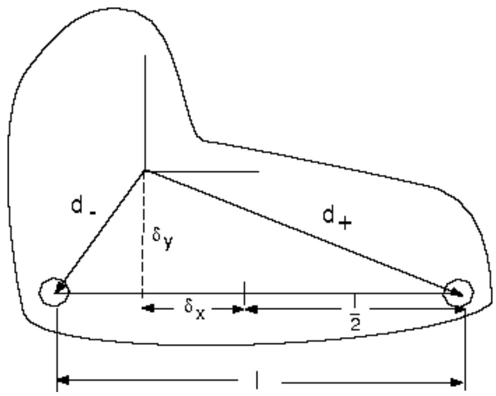

Figure 3: Single bar diagram

6.1 Small oscillations (or quadratic potential)

Let us expand $V(\theta_{10})$ about the equilibrium point α.

$$V(\theta_{10}) = \bar{V}(\alpha) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \theta_{10}^2}\right|_\alpha (\theta_{10} - \alpha)^2 + h.o.t. \tag{103}$$

Perform a change of coordinates $\psi_{10} = \theta_{10} - \alpha$ to get

$$V(\psi_{10}) = \bar{V}(\alpha) + \frac{1}{2}\left.\frac{\partial^2 V}{\partial \psi_{10}^2}\right|_0 \psi_{10}^2 + h.o.t. \tag{104}$$

Assume the oscillations are small enough such that we can neglect all higher order terms (or that the potential is quadratic to begin with). The nominal Hamiltonian is then

$$H_{nom} = \frac{p_{10}^2}{2M_{11}} + \frac{k}{2}\psi_{10}^2 \tag{105}$$

where we have taken advantage of the fact that we can shift the energy by a constant to drop the term $V(0)$ and have made the obvious definition for k. This is the Hamiltonian for a harmonic oscillator. The phase portrait for a given energy is shown in figure 4. From [1] we know that

$$I = \frac{h}{\omega} \tag{106}$$

where ω is the frequency of oscillation and h is the energy corresponding to the initial conditions. To find the frequency ω consider the Hamiltonian vector field of the nominal system

$$\dot{\psi}_{10} = \frac{p_{10}}{M_{11}} \qquad \dot{p}_{10} = -k\psi_{10} \tag{107}$$

$$\Rightarrow \qquad \ddot{\psi}_{10} = -\frac{k}{M_{11}}\psi_{10} \tag{108}$$

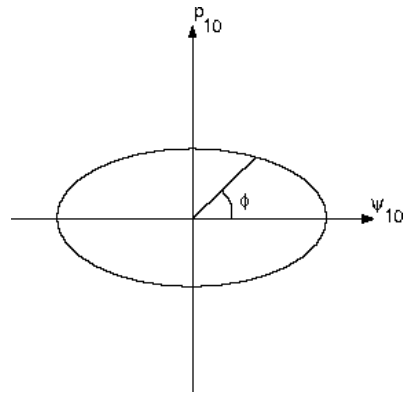

Figure 4: Harmonic oscillator phase portrait

Let $\psi_{10} = Acos(\omega t + \eta)$. Then $\ddot{\psi}_{10} = -\omega^2 \psi_{10}$ and $\omega = \sqrt{\frac{k}{M_{11}}}$. Putting this into equation (106) gives

$$I = \frac{\frac{p_{10}^2}{2M_{11}} + \frac{k}{2}\psi_{10}^2}{\sqrt{\frac{k}{M_{11}}}} \tag{109}$$

$$= \frac{p_{10}^2 + kM_{11}\psi_{10}^2}{2\sqrt{kM_{11}}} \tag{110}$$

The angle variable is the phase of the oscillation. We thus have

$$\psi_{10} = Acos(\phi) \tag{111}$$

where

$$A = \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \tag{112}$$

Similarly

$$p_{10} = M_{11}\dot{\psi}_{10} \tag{113}$$

$$= -M_{11}Asin(\phi)\dot{\phi} \tag{114}$$

$$= -M_{11}Asin(\phi)\sqrt{\frac{k}{M_{11}}} \tag{115}$$

From these two we get

$$tan\phi = -\frac{p_{10}}{\psi_{10}\sqrt{kM_{11}}} \tag{116}$$

From this we can get the functions $f_1(I,\phi), f_2(I,\phi)$

$$\theta_{10} = f_1(I,\phi) = \alpha + \psi_{10} \tag{117}$$

$$= \alpha + \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \cos\phi \tag{118}$$

$$p_{10} = f_2(I,\phi) \tag{119}$$

$$= -\left[2I\sqrt{kM_{11}}\right]^{\frac{1}{2}} \sin\phi \tag{120}$$

Then from equations (54), (102), (118), and (120) we have

$$\mathcal{P}(Z_t) = -\Omega\left[1 + \frac{2ml\delta_y}{M_{11}} sin\left(\alpha + \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}} \cos\phi\right)\right]\left[2I\sqrt{kM_{11}}\right]^{\frac{1}{2}} \sin\phi \tag{121}$$

The average of $\mathcal{P}(Z_t)$ over one period of ϕ is then

$$< \mathcal{P}(Z_t) > = -\frac{\Omega\left[2I\sqrt{kM_{11}}\right]^{\frac{1}{2}}}{2\pi}\int_0^{2\pi}\left[1 + \frac{2ml\delta_y}{M_{11}}\sin\left(\alpha + \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}}\cos\phi\right)\right]\sin\phi d\phi \tag{122}$$

$$= -\frac{\Omega\left[2I\sqrt{kM_{11}}\right]^{\frac{1}{2}}}{2\pi}\left[\frac{2ml\delta_y}{M_{11}}\right]\int_0^{2\pi}\sin\left(\alpha + \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}}\cos\phi\right)\sin\phi d\phi \tag{123}$$

$$= \frac{\Omega\left[2I\sqrt{kM_{11}}\right]^{\frac{1}{2}}}{2\pi}\left[\frac{2ml\delta_y}{M_{11}}\right]\cos\left(\alpha + \left[\frac{2I}{\sqrt{kM_{11}}}\right]^{\frac{1}{2}}\cos\phi\right)\Bigg|_0^{2\pi} \tag{124}$$

$$= 0 \tag{125}$$

Thus in the small-angle approximation (or with linear springs) we have that the Berry-Hannay phase is 0.

6.2 Simulations

Since the calculation of action-angle coordinates involves solving the dynamics of the nominal system they cannot be found for generic spring potentials. We turn then to simulation for insight into nonlinear springs. In this section we derive the full equations of motion for (θ_{10}, p_{10}), including the centrifugal terms, and then show the results of simulations experiments using both quadratic spring potentials to validate the above results and quartic spring potentials to investigate nonlinear effects. All simulations are done using the generic joint shown in Figure 5 with parameters $l_1 = 1, \beta = \frac{\pi}{4}, \lambda = 1$. With these parameters we have

$$m = 2 \tag{126}$$

$$\delta_y = -0.4619 \tag{127}$$

$$l = 0.3827 \tag{128}$$

$$\|\mathbf{d}_-\|^2 = \|\mathbf{d}_+\|^2 = 0.25 \tag{129}$$

$$I = 0.2399 \tag{130}$$

From these we can calculate the nominal angular frequency for the nominal solution to be $\omega = 1.6441\frac{rad}{s}$ and thus the frequency is $f = 0.2617$ Hz and the period is $T = 3.82$ s.

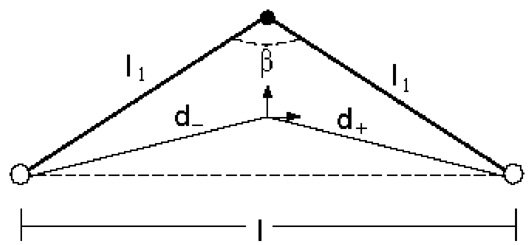

Figure 5: Generic bar

6.2.1 Equations of motion

The Hamiltonian from equation (62) is

$$H(\theta_{10}, p_{10}) = \frac{1}{2M_{11}} p_{10}^2 + V(\theta_{10}) - \left[\frac{M_{10} + M_{11}}{M_{11}}\right] \Omega p_{10} - \frac{\Omega^2}{2}\left(\frac{M_{00}M_{11} - M_{10}^2}{M_{11}}\right) \tag{131}$$

$$= \frac{1}{2M_{11}} p_{10}^2 + V(\theta_{10}) - \left[\frac{M_{10} + M_{11}}{M_{11}}\right] \Omega p_{10} - \frac{\Omega^2}{2}\left(\frac{M_{00}^2 - M_{10}^2}{M_{11}}\right) \tag{132}$$

Then from Hamilton's equations we have

$$\dot{\theta}_{10} = \frac{\partial H}{\partial p_{10}} = \frac{p_{10}}{M_{11}} - \left[\frac{M_{10} + M_{11}}{M_{11}}\right]\Omega \tag{133}$$

$$\dot{p}_{10} = -\frac{\partial H}{\partial \theta_{10}} = \frac{\Omega p_{10}}{M_{11}}\frac{\partial M_{10}}{\partial \theta_{10}} - \frac{\Omega^2}{2M_{11}}\left(2M_{10}\frac{\partial M_{10}}{\partial \theta_{10}}\right) - \frac{\partial V(\theta_{10})}{\partial \theta_{10}} \tag{134}$$

Recall that

$$M_{00} = 2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) \tag{135}$$

$$M_{10} = 2ml\delta_y sin(\theta_{10}) \tag{136}$$

Thus

$$\frac{\partial M_{10}}{\partial \theta_{10}} = 2ml\delta_y cos(\theta_{10}) \tag{137}$$

Put this all together to get

$$\dot{\theta}_{10} = \frac{p_{10} - [2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) + 2ml\delta_y sin(\theta_{10})]\Omega}{2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)} \tag{138}$$

$$\dot{p}_{10} = \left[\frac{2ml\delta_y \Omega cos(\theta_{10})}{2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)}\right][p_{10} - 2\Omega ml\delta_y sin(\theta_{10})] - \frac{\partial V}{\partial \theta_{10}} \tag{139}$$

6.2.2 Quadratic potential

Let

$$V_s(\psi) = \frac{k_1}{2}(\psi - \alpha)^2 \tag{140}$$

From equation (37) we have

$$V(\theta_{10}) = k_1 \left[(\theta_{10} - \alpha)^2 + (\pi - \theta_{10} - \alpha)^2\right] \tag{141}$$

Then

$$\frac{\partial V}{\partial \theta_{10}} = 2k_1(2\theta_{10} - \pi) \tag{142}$$

and the equations of motion are

$$\dot{\theta}_{10} = \frac{p_{10} - [2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2) + 2ml\delta_y sin(\theta_{10})]\Omega}{2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)} \tag{143}$$

$$\dot{p}_{10} = \left[\frac{2ml\delta_y \Omega cos(\theta_{10})}{2I + m(\|\mathbf{d}_+\|^2 + \|\mathbf{d}_-\|^2)}\right][p_{10} - 2\Omega ml\delta_y sin(\theta_{10})] - 2k_1(2\theta_{10} - \pi) \tag{144}$$

Consider first the simulation experiments shown in figure 6. In these experiments the initial conditions were taken to be at $\theta_{10}(0) = \frac{3\pi}{8}$, $p_{10}(0) = 0$. The first plot, with $\Omega = 0$, is the nominal system. In the second plot we have $\Omega = 0.001\frac{rad}{s}$. For such a small value of Ω the system is very close to the nominal. As we continue to increase Ω the system varies significantly from the nominal. As this effect is dependent on the rate of rotation we know it is not a geometric phase. It instead arises from the effect of the centrifugal terms on the dynamics. These simulations, then, agree with the theoretical calculations above. Notice that the centrifugal terms will also affect a system starting at the nominal equilibrium point ($\theta_{10}(0) = pi/2, p_{10} = 0$). In the next simulation the system was started from that equilibrium and run for various values of Ω. In figure 7 we see the results for $\Omega = 0$, $\Omega = 0.001$, and $\Omega = 0.1$. The first plot verifies that the initial condition is an equilibrium point for the system. The effect on the system at $\Omega = 0.001$ is quite small (notice the scale on the figures) while it has increased significantly at $\Omega = 0.1$. These simulations show that the equilibrium point of the nominal system becomes unstable and bifurcates to periodic solutions (a Hopf bifurcation) as Ω varies from 0.

We now turn to the quartic potential.

6.2.3 Quartic potential

Let

$$V_s(\psi) = \frac{k_1}{2}(\psi - \alpha)^2 + \frac{k_2}{4}(\psi - \alpha)^4 \tag{145}$$

From equation (37) we have

$$V(\theta_{10}) = k_1 \left[(\theta_{10} - \alpha)^2 + (\pi - \theta_{10} - \alpha)^2\right] + \frac{k_2}{2}\left[(\theta_{10} - \alpha)^4 + (\pi - \theta_{10} - \alpha)^4\right] \tag{146}$$

and then

$$\frac{\partial V}{\partial \theta_{10}} = 2k_1(2\theta_{10} - \pi) + 2k_2\left[(\theta_{10} - \alpha)^3 - (\pi - \theta_{10} - \alpha)^3)\right] \tag{147}$$

leading to the equations of motion

$$\dot{\theta}_{10} = \frac{p_{10} - [2I + m(\|d_+\|^2 + \|d_-\|^2) + 2ml\delta_y sin(\theta_{10})]\Omega}{2I + m(\|d_+\|^2 + \|d_-\|^2)} \tag{148}$$

$$\dot{p}_{10} = \left[\frac{2ml\delta_y \Omega cos(\theta_{10})}{2I + m(\|d_+\|^2 + \|d_-\|^2)}\right][p_{10} - 2\Omega ml\delta_y sin(\theta_{10})] \\ -2k_1(2\theta_{10} - \pi) - 2k_2\left[(\theta_{10} - \alpha)^3 - (\pi - \theta_{10} - \alpha)^3)\right] \tag{149}$$

Simulations were performed for the same initial conditions and values of Ω as in the quadratic potential case. Here the spring constants were taken to be $k_1 = 1, k_2 = 10$. The plots in figure 8 show the simulations runs for the initial conditions $\theta_{10}(0) = \frac{3\pi}{8}, p_{10} = 0$. The first plot in this figure shows the nominal system. Comparing this to the first plot in figure 6 shows the effect of the nonlinear springs on the system. As in the quadratic potential we see that there is no geometric phase effect on the system. The centrifugal terms again affect the dynamics with significant effect when Ω is no longer small with respect to the angular frequency in the nominal solution.

7 Conclusion

In this report we have derived a formula for the Berry-Hannay phase of the equal-sided, spring-jointed, four-bar mechanism and shown that in the small angle approximation this geometric phase is zero.

In future work we hope to apply the moving systems method to an equal-sided, spring-jointed, n-bar mechanism. Since it is well known that the vibrating ring exhibits nodal precession even for very small vibration amplitudes we hope, by analogy, to find a non-trivial Berry-Hannay phase in the n-bar mechanism and expect this phase to manifest itself as a rotating wave.

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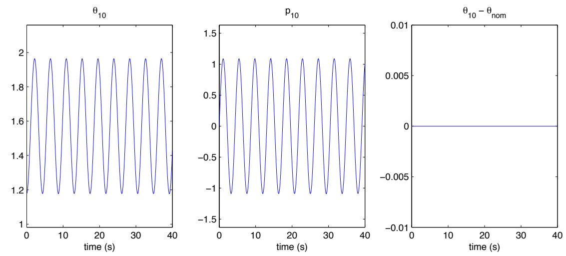
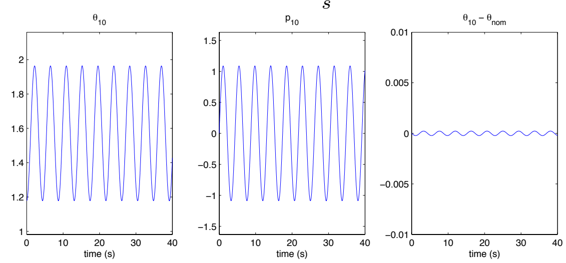
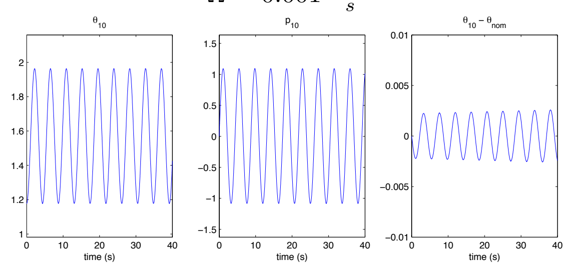


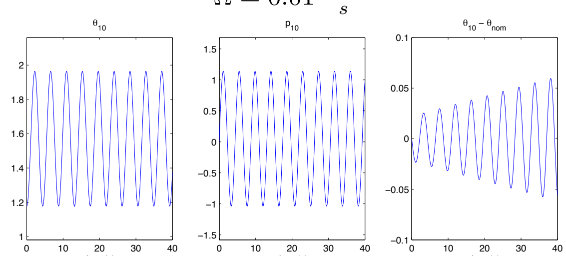
$\Omega = 0 \ \frac{rad}{s}$



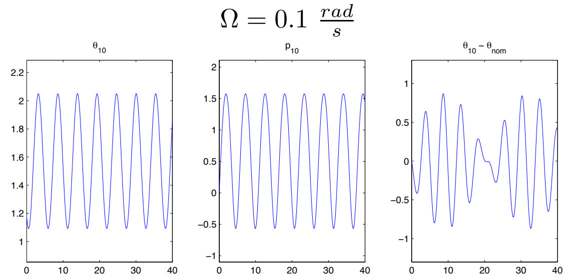
$\Omega = 0.001 \ \frac{rad}{s}$



$\Omega = 0.01 \ \frac{rad}{s}$

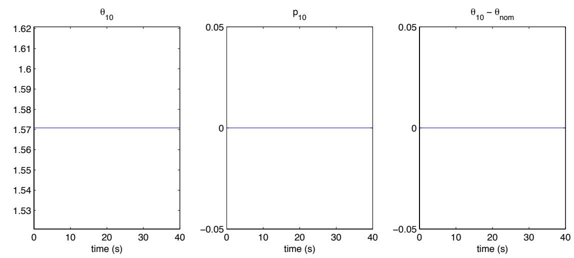


$\Omega = 0.1 \ \frac{rad}{s}$

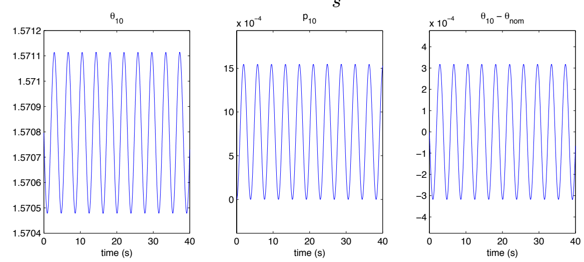


$\Omega = 1.0 \ \frac{rad}{s}$

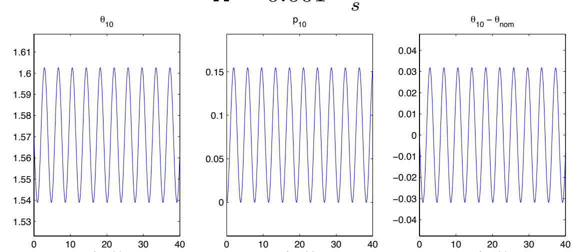
Figure 6: Simulation run: $\theta_{10}(0) = \frac{3\pi}{8}$, $p_{10}(0) = 0$



$\Omega = 0 \ \frac{rad}{s}$

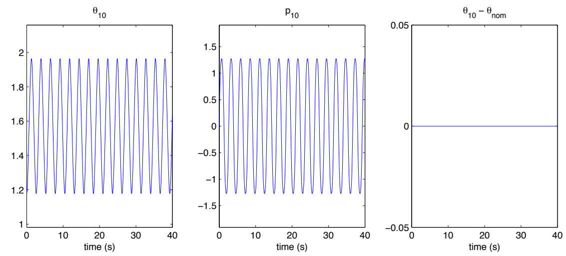


$\Omega = 0.001 \ \frac{rad}{s}$

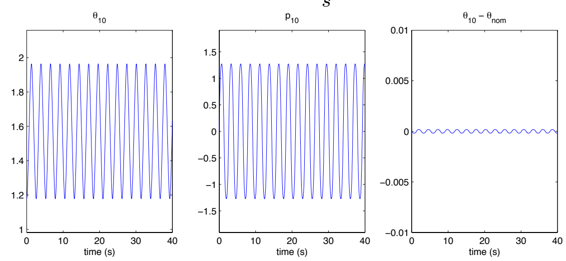


$\Omega = 0.1 \ \frac{rad}{s}$

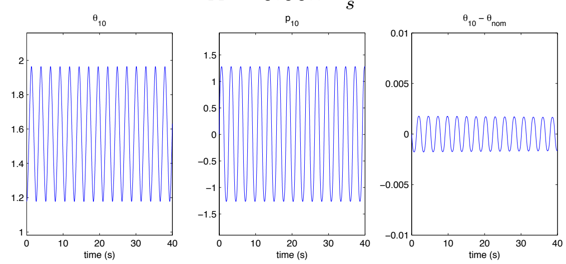
Figure 7: Simulation run: $\theta_{10}(0) = \frac{\pi}{2}$, $p_{10}(0) = 0$



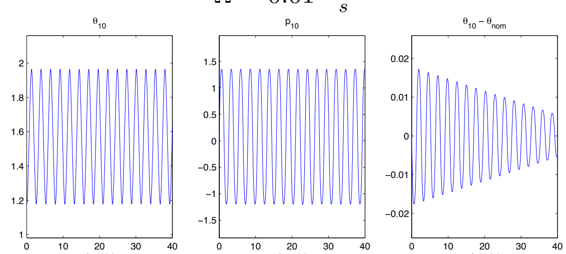
$\Omega = 0\ \frac{rad}{s}$



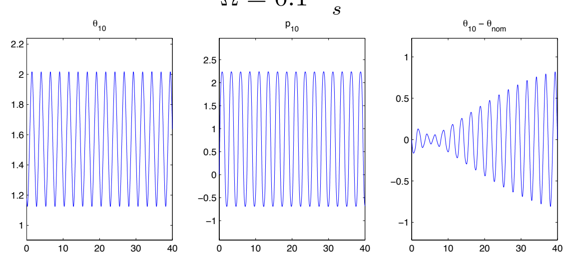
$\Omega = 0.001\ \frac{rad}{s}$



$\Omega = 0.01\ \frac{rad}{s}$



$\Omega = 0.1\ \frac{rad}{s}$



$\Omega = 1.0\ \frac{rad}{s}$

Figure 8: Simulation run (nonlinear springs): $\theta_{10}(0) = \frac{3\pi}{8}$, $p_{10}(0) = 0$