TECHNICAL RESEARCH REPORT

Control of Smart Actuators: A Viscosity Solutions Approach

by Xiaobo Tan, John S. Baras

CDCSS TR 2001-8
(ISR TR 2001-39)

CENTER FOR DYNAMICS AND CONTROL OF SMART STRUCTURES

The Center for Dynamics and Control of Smart Structures (CDCSS) is a joint Harvard University, Boston University, University of Maryland center, supported by the Army Research Office under the ODDR&E MURI97 Program Grant No. DAAG55-97-1-0114 (through Harvard University). This document is a technical report in the CDCSS series originating at the University of Maryland.

Web site http://www.isr.umd.edu/CDCSS/cdcss.html

Control of Smart Actuators: A Viscosity Solutions Approach

Xiaobo Tan John S. Baras

Department of Electrical and Computer Engineering and
Institute for Systems Research
University of Maryland, College Park, MD 20740

Abstract

Hysteresis in smart materials hinders their wider applicability in actuators. In this report we investigate control of smart actuators through the example of controlling a commercially available magnetostrictive actuator. At low frequencies, the magnetostriction can be related to the bulk magnetization through a square law, thus control of the magnetization amounts to control of the magnetostriction. The model we use is the low dimensional Jiles-Atherton model for ferromagnetic hysteresis, which is a hybrid system. For illustrative purpose, we consider an infinite horizon control problem. The approach we take features dynamic programming and Hamilton-Jacobi equations. In particular, we show that the value function of the control problem satisfies a Hamilton-Jacobi-Bellman equation (HJB) of some hybrid form in the viscosity sense. We further prove uniqueness of solutions to the (HJB), and provide a numerical scheme to approximate the solution together with a suboptimal controller synthesis method.

1 Introduction

Hysteresis in smart materials, e.g., magnetostrictives, piezoceramics, and shape memory alloys (SMAs), hinders the wider applicability of such materials in actuators. Hysteresis models can be classified into physics based models and phenomenological models. An example of physics based model is the Jiles-Atherton model for ferromagnetic hysteresis[9], where hysteresis is considered to arise from pinning of domain walls on defect sites. The most popular phenomenological hysteresis model used in control of smart actuators has been the Preisach model[1, 6, 7].

A fundamental idea in coping with hysteresis is inverse compensation[5, 11, 13]. Inverse compensation suffers from a couple of drawbacks, like no closed form and implementation difficulties. In this report, we will investigate the control of hysteretic actuator from a different perspective. We will study a special class of hysteretic systems which have low dimensional mathematical models.

To be specific, we will focus on control of a commercially available magnetostrictive actuator. At low frequencies, the magnetostriction can be related to the bulk magnetization through a square law, thus control of magnetostriction is equivalently control of bulk magnetization. We will employ the low dimensional bulk magnetization model[14] for the magnetization hysteresis. The model is a hybrid dynamical system, whose switching depends on both the state and the control. Conclusions and future work are provided in Section 6.

This report is organized as follows. In Section 2 we present the hysteresis model and explore its properties. In Section 3 we formulate an optimal control problem, and show the value function satisfies a Hamilton-Jacobi-Bellman equation (HJB) of some hybrid form in the viscosity sense. In Sectin 4, We prove that (HJB) admits a unique solution in the class of continuous functions to which the value function belongs. We describe some discrete approximation schemes to (HJB) in Section 5. This establishes the existence of a solution to (HJB) as well as provides a way for suboptimal control synthesis. Finally future work along the line of this report is provided in Section 6.

2 Mathematical Model of Hysteresis

2.1 The bulk ferromagnetic hysteresis model

Jiles and Atherton proposed a low dimensional model for ferromagnetic hysteresis, based upon the quantification of energy losses due to domain wall intersections with inclusions or pinning sites within the material[9].A modification of the Jiles-Atherton model was made by Venkataraman and Krishnaprasad with rigorous use of energy balancing principle[15, 14], and they called it the *bulk ferromagnetic hysteresis model*. Also based on the energy balancing principle, they derived a bulk magnetostrictive hysteresis model[16, 14]. At low frequencies, the magnetostriction can be related to the bulk magnetization through a square law[14], thus control of the bulk magnetization amounts to control of the magnetostriction. In this report, we will study optimal control of the bulk magnetization exclusively to highlight the methodology of hysteresis control. Extension to control of magnetostriction at high frequencies can be done following the ideas in [12].

We will use the bulk ferromagnetic hysteresis model [15, 14] in this report, which has a slightly different form from the Jiles-Atherton model. We now briefly outline the model. The bulk magnetization M is comprised of a reversible component M_{rev} and an irreversible component M_{irr}, and M_{rev} is related to M_{irr} and the anhysteretic magnetization M_{an} by:

$$M_{rev} = c(M_{an} - M_{irr}), \tag{1}$$

where c is called the reversibility coefficient, and M_{an} is given below.

For an input field H and a bulk magnetization M, we define $H_e = H + \alpha M$ to be the effective field, where α is a mean field parameter representing inter-domain coupling. Through thermodynamic considerations, the anhysteretic magnetization M_{an} can be expressed as

$$\begin{aligned} M_{an}(H_e) &= M_s(coth(\frac{H_e}{a}) - \frac{a}{H_e}) \\ &= M_s\mathcal{L}(z), \end{aligned} \tag{2}$$

where $\mathcal{L}(\cdot)$ is the Langevin function, $\mathcal{L}(z) = coth(z) - \frac{1}{z}$, with $z = \frac{H_e}{a}$, M_s is the saturation magnetization of the material and a is a parameter characterizing the shape of M_{an} curve. Energy balancing yields the expressions for M_{irr}:

$$\frac{dM_{irr}}{dH} = \tilde{\delta}\frac{\mu_0(M_{an} - M)}{k\delta(1-c)}\frac{dH_e}{dH}, \tag{3}$$

where μ_0 is the permeability of vacuum, k is a measure for the average energy required to break a pinning site, $\delta = sign(\dot{H})$, and

$$\tilde{\delta} = \begin{cases} 0, dH < 0 \text{ and } M < M_{an} \\ 0, dH > 0 \text{ and } M > M_{an} \\ 1, \text{ else} \end{cases}.$$

The function δ is defined to guarantee that pinning always opposes changes in magnetization, and $\tilde{\delta}$ is defined to guarantee that the incremental susceptibility is non-negative. Since by (1),

$$M = M_{irr} + M_{rev} = (1-c)M_{irr} + cM_{an}, \tag{4}$$

taking derivative with respect to H at both sides of (4), we have

$$\frac{dM}{dH} = (1-c)\frac{dM_{irr}}{dH} + c\frac{dM_{an}}{dH}.$$

From (2) and (3), we get after some manipulations,

$$\frac{dM}{dH} = \frac{\frac{ck\delta M_s}{\mu_0 a}\frac{\partial\mathcal{L}(z)}{\partial z} + \tilde{\delta}(M_{an} - M)}{\frac{k\delta}{\mu_0}(1 - \frac{\alpha c M_s}{a}\frac{\partial\mathcal{L}(z)}{\partial z}) - \tilde{\delta}\alpha(M_{an} - M)}, \tag{5}$$

where

$$\frac{\partial\mathcal{L}(z)}{\partial z} = \frac{1}{z^2} - csch^2(z). \tag{6}$$

Equation (5) describes a switched nonlinear system. In particular, letting

$$f_1(H,M) = c\frac{M_s\frac{\partial \mathcal{L}(z)}{\partial z}}{a-\alpha c M_s\frac{\partial \mathcal{L}(z)}{\partial z}},$$

$$f_2(H,M) = \frac{ckM_s\frac{\partial \mathcal{L}(z)}{\partial z} - \mu_0 a(M_{an}-M)}{k(a-\alpha c M_s\frac{\partial \mathcal{L}(z)}{\partial z}) + \mu_0\alpha a(M_{an}-M)},$$

$$f_3(H,M) = \frac{ckM_s\frac{\partial \mathcal{L}(z)}{\partial z} + \mu_0 a(M_{an}-M)}{k(a-\alpha c M_s\frac{\partial \mathcal{L}(z)}{\partial z}) - \mu_0\alpha a(M_{an}-M)},$$

and $u = \dot{H}$, we can rewrite (5) as

$$\begin{pmatrix} \dot{H} \\ \dot{M} \end{pmatrix} = \begin{pmatrix} 1 \\ f_i(H,M) \end{pmatrix} u, \tag{7}$$

with each f_i smooth in H and M, and the switching rule is:

$$i = \begin{cases} 1, u \le 0,\ M \le M_{an}(H_e) \text{ or } u \ge 0,\ M \ge M_{an}(H_e) \\ 2, u \le 0,\ M \ge M_{an}(H_e) \\ 3, u \ge 0,\ M \le M_{an}(H_e) \end{cases}.$$

Note the switching depends on both (sign of) u and the state variables H, M. We may represent model (7) in a more compact way. Let Γ=$\{(H,M) : M = M_{an}(H_e)\}$,

$$\Omega_1 = \{(H,M) : M < M_{an}(H_e)\},\quad \Omega_2 = \{(H,M) : M > M_{an}(H_e)\},$$

and denote $\overline{\Omega}_i = \Omega_i \cup \Gamma, i = 1,2$. Letting $x = (H,M)$, we can define

$$f_+(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix} & \text{if } x \in \overline{\Omega}_2 \\ \begin{pmatrix} 1 \\ f_3(x) \end{pmatrix} & \text{if } x \in \overline{\Omega}_1 \end{cases}, \text{and } f_-(x) = \begin{cases} \begin{pmatrix} 1 \\ f_1(x) \end{pmatrix} & \text{if } x \in \overline{\Omega}_1 \\ \begin{pmatrix} 1 \\ f_2(x) \end{pmatrix} & \text{if } x \in \overline{\Omega}_2 \end{cases}.$$

Since $f_i, 1 \le i \le 3$, coincide on Γ, f_+ and f_- are well defined and continuous. We then introduce the discrete control set $D = \{1,2\}$ and the continuous control sets

$$U_+ = \{u : u \le 0\},\quad U_- = \{u : u \ge 0\}.$$

A control action includes both the discrete mode control $d \in D$ and the continuous control u. Now the model (7) can be described as: at any $x \in \mathbb{R}^2$, if one chooses $d = 1$, then u must be picked from U_+, and the dynamics is governed by:

$$\dot{x} = f_+(x)u. \tag{8}$$

Similarly, if $d = 2$ is chosen, then u must be picked from U_-, and the dynamics is:

$$\dot{x} = f_-(x)u. \tag{9}$$

The state-dependent switching has now been incorporated into the definitions of f_+, f_-. Note the model(8),(9) is a *Duhem* hysteresis model [17]. The Duhem model characterizes a class of rate-independent hysteresis models with input $v(\cdot)$ and output $\omega(\cdot)$, with dynamics depending on the sign of $\dot{v}$. To be precise,

$$\dot{\omega} = g_1(v,\omega)(\dot{v})^+ - g_2(v,\omega)(\dot{v})^-, \tag{10}$$

where $(\dot{v})^+ = \max\{0, \dot{v}\}$ and $(\dot{v})^- = \max\{0, -\dot{v}\}$. Denoting $u = \dot{v}$, we can write (10) as

$$\begin{pmatrix} \dot{v} \\ \dot{\omega} \end{pmatrix} = \begin{pmatrix} 1 \\ g_i(v, \omega) \end{pmatrix} u, \tag{11}$$

with the switching rule:

$$i = \begin{cases} 1, u \leq 0, \\ 2, u \geq 0, \end{cases} .$$

Remark: Smith and Hom [10] proposed a model for ferroelectric hysteresis analogous to the Jiles-Atherton model for ferromagnetic materials. The model of Smith and Hom carries the same structure as that of (7), thus the approach presented in this report is fully applicable to control of actuators made of ferroelectric materials. This, in some sense, justifies the title of the report.

In the next two subsections, we will derive some properties of the model (7), which will be used in the analysis later.

2.2 Boundness of f_i

Lemma 1: $\mathcal{L}(z)$ *satisfies:*

$$0 < \frac{\partial \mathcal{L}(z)}{\partial z} \leq \frac{1}{3}, \tag{12}$$

$$|\mathcal{L}(z)| \leq 1. \tag{13}$$

The proof of Lemma 1 can be found in Appendix A.

Proposition 2:*If the parameters satisfy:*

$$T_1 := a - \frac{\alpha c M_s}{3} > 0, \tag{14}$$

$$T_2 := k(a - \frac{\alpha c M_s}{3}) - 2\mu_0 \alpha a M_s > 0, \tag{15}$$

then $0 < f_i \leq C_f, i = 1, 2, 3$ *for some constant* $C_f > 0$.

Proof. By (12) and (14)

$$0 < T_1 = a - \frac{\alpha c M_s}{3} \leq a - \alpha c M_s \frac{\partial \mathcal{L}(z)}{\partial z} < a.$$

We rewrite f_1 as

$$f_1 = -\frac{1}{\alpha} + \frac{a}{\alpha(a - \alpha c M_s \frac{\partial \mathcal{L}(z)}{\partial z})},$$

and note that it is a nondecreasing function of $\frac{\partial \mathcal{L}(z)}{\partial z}$. Since

$$\begin{aligned} f_1 &= 0 \text{ when } \frac{\partial \mathcal{L}(z)}{\partial z} = 0, \\ f_1 &= \frac{c M_s}{3a - \alpha c M_s} =: C_1 \text{ when } \frac{\partial \mathcal{L}(z)}{\partial z} = \frac{1}{3}, \end{aligned}$$

we get

$$0 < f_1 \leq C_1.$$

The function f_2 can be written as

$$f_2 = -\frac{1}{\alpha} + \frac{ka}{\alpha} \frac{1}{\underbrace{k(a - \alpha c M_s \frac{\partial \mathcal{L}(z)}{\partial z}) + \mu_0 \alpha a (M_{an} - M)}_{=:T}}$$

From the model (7), when f_2 is selected, $M_{an} - M \leq 0$. Since magnitudes of both M_{an} and M must be less than M_s, $M_{an} - M \geq -2M_s$. These facts together with (12) yield

$$0 < T_2 \leq T \leq ka.$$

Therefore

$$0 < f_2 \leq \frac{ka - T_2}{\alpha T_2} =: C_2.$$

Similarly we can show $0 < f_3 < C_2$. Picking $C_f = \max\{C_1, C_2\}$, we have $0 < f_i \leq C_f$ for $i = 1, 2, 3$. $\square$

Remark: Conditions (14) and (15) are satisfied for typical parameters. For example, if we take the identified parameters in [14], $\alpha = 1.9 \times 10^{-4}$, $a = 190$, $k = 48$ $Tesla$, $c = 0.3$, $M_s = 7.9 \times 10^5$ $Amp/meter$ and $\mu_0 = 4\pi \times 10^{-7}$ $Henry/meter$, we calculate $T_1 = 174.99$, $T_2 = 8.40 \times 10^3$.

2.3 Lipshitz continuity of the model

Proposition 3: *f_+ and f_- are Lipshitz continuous with Lipshitz constant L. Define $\tilde{f}_+(x, u) = f_+(x)u$, $\tilde{f}_-(x, u) = f_-(x)u$. If $U_+ = \{u : 0 \leq u \leq u_c\}$ and $U_- = \{u : -u_c \leq u \leq 0\}$ for some $u_c > 0$, then $\forall u \in U_+ (u \in U_-$, resp.), $\tilde{f}_+(x, u)$ ($\tilde{f}_-(x, u)$, resp.) is Lipshitz continuous with respect to x with Lipshitz constant $L_0 = Lu_c$.*

Remarks:

- The physical interpretation for $|u| \leq u_c$ is the operating bandwidth constraint on the magnetostrictive actuator.
- In the rest of the report, we will use $f_+(x)u$ ($f_-(x)u$, resp.) and $\tilde{f}_+(x, u)$ ($\tilde{f}_-(x, u)$, resp.) interchangeably.

Proof. We first prove f_- is Lipshitz continuous with Lipshitz constant L. We discuss three cases:

- **Case I:** Both $x_1, x_2 \in \overline{\Omega}_1$. In this case, mode 1 is active, and thus

$$\frac{\partial f_-(x)}{\partial x} = \begin{pmatrix} 0 & 0 \\ \frac{\partial f_1(H,M)}{\partial H} & \frac{\partial f_1(H,M)}{\partial M} \end{pmatrix}. \tag{16}$$

It can be shown, after some manipulations, that $|\frac{\partial f_1(H,M)}{\partial H}| \le C_1$, $|\frac{\partial f_1(H,M)}{\partial M}| \le \alpha C_1$ for some $C_1 > 0$. Therefore $|\frac{\partial f_-(x)}{\partial x}| \le L_1$ for some $L_1 > 0$, and the following holds:

$$|f_-(x_1) - f_-(x_2)| \le L_1 |x_1 - x_2|.$$

- **Case II:** Both $x_1, x_2 \in \overline{\Omega}_2$. In this case, mode 2 is active. Following similar steps as in Case I, we can show $|\frac{\partial f_-(x)}{\partial x}| \le C_2$ for some $L_2 > 0$ and therefore

$$|f_-(x_1) - f_-(x_2)| \le L_2 |x_1 - x_2|.$$

- **Case III:** $x_1 \in \overline{\Omega}_1, x_2 \in \overline{\Omega}_2$. Then there exist $x_0 \in \Gamma$, such that the line segment connecting x_1 and x_2 intersects Γ at x_0. We express $x_0 = \theta x_1 + (1-\theta)x_2$ with $0 \le \theta \le 1$. Thus

$$\begin{aligned} |f_-(x_1) - f_-(x_2)| &= |f_-(x_1) - f_-(x_0) + f_-(x_0) - f_-(x_2)| \\ &\le L_1|x_1 - x_0| + L_2|x_0 - x_2| \\ &= L_1(1-\theta)|x_1 - x_2| + L_2\theta|x_1 - x_2| \\ &\le L_-|x_1 - x_2|, \end{aligned}$$

with $L_- = \max\{L_1, L_2\}$.

Following exactly the same arguments, we can show ,

$$|f_+(x_1) - f_+(x_2)| \le L_+|x_1 - x_2|, \ \forall x_1, x_2.$$

We conclude the first part by taking $L = \max\{L_-, L_+\}$. The rest of the proposition follows trivially. □

3 Optimal Control: (HJB) and Viscosity Solutions

Dynamic Programming Principle (DPP) is one of the most important approaches in optimal control. When the value function is smooth, we can derive the Hamilton-Jacobi-Bellman equation (HJB), and in many cases, solving HJB amounts to solving the optimal control problem. The value function however, in general, is not smooth even for smooth systems, not to mention for a hybrid system, like that in our model. Crandall and Lions[4] introduced the notion of viscosity solutions to Hamilton-Jacobi equations. This turned out to be a very useful concept for optimal control since value functions of many optimal control problems do satisfy the HJB in the viscosity sense. And under mild assumptions, uniqueness and existence results for viscosity solutions hold.

We will explore this approach for control of smart actuators. This report is aimed at providing some flavors of this approach through the example of infinite-time horizon optimal control problem. We will study the properties of the value function, derive the Dynamic Programming Principle and show the value function indeed satisfies (HJB) of a special form.

3.1 Optimal control problem

For ease of presentation, we rewrite the model (8),(9) as

$$\dot{x} = \tilde{f}(x, u, d) = \begin{cases} \tilde{f}_+(x, u) \text{ if } d = 1, \ u \in U_+ \\ \tilde{f}_-(x, u) \text{ if } d = 2, \ u \in U_- \end{cases} . \tag{17}$$

We require $u(\cdot)$ to be measurable. This together with Proposition 3 guarantees that, for any initial condition x and any admissible control pair $\alpha(\cdot) := \{d(\cdot), u(\cdot)\}$, (17) has a unique solution $x(\cdot)$(the dependence on x and $\alpha(\cdot)$ is suppressed when no confusion arises).

Define the cost functional with initial condition x and control $\alpha(\cdot)$ as

$$J(x, \alpha(\cdot)) = \int_0^\infty l(x(t), u(t)) e^{-\lambda t} dt, \tag{18}$$

with $\lambda \geq 0$. The optimal control problem is: given initial condition x, find

$$V(x) = \inf_{\alpha(\cdot)} J(x, \alpha(\cdot)), \tag{19}$$

and if $V(x)$ is achievable, find the optimal control $\alpha^*(\cdot)$.

We make the following assumptions about $l(\cdot, \cdot)$:

- (A_1): $l(x, u)$ continuous in x and u, $l(x, u) \geq 0$, $\forall x, u$;
- (A_2): $l(0, 0) = 0$;
- (A_3): $|l(x_1, u) - l(x_2, u)| \leq C_l(1 + |x_1| + |x_2|)|x_1 - x_2|$, $\forall u$ for some $C_l > 0$.

Note (A_3) includes the case of quadratic cost.

3.2 Properties of the value function

Proposition 4 [Local Boundness]: *Under assumptions* $(A_1) - (A_3)$, $\forall \lambda > 0$, $V(\cdot)$ *is locally bounded, i.e.,* $\forall R \geq 0, \exists C_R \geq 0$, *such that* $|V(x)| \leq C_R \ \forall x \in \overline{B}(0, R) := \{x : |x| \leq R\}$.

Proof. First note since $l(\cdot, \cdot)$ is nonnegative, $V(x) \geq 0 \ \forall x$. Take $u(t) \equiv 0$,then $x(t) \equiv x$. Let $\alpha(t) = \{d(t), u(t)\}$ where $d(t) \equiv 1$. We have

$$\begin{aligned} V(x) \leq J(x, \alpha(\cdot)) &= \int_0^\infty l(x, 0) e^{-\lambda t} dt \\ &= \frac{l(x, 0)}{\lambda}. \end{aligned}$$

By (A_2) and (A_3), $l(x, 0) \leq C_l(1 + R)R$, and the proof is complete with $C_R := \frac{C_l(1+R)R}{\lambda}$. $\square$

Proposition 5 [Locally Lipshitz Continuity]: *Under assumptions* $(A_1)-(A_3)$*,* $\forall\lambda > 2L_0$ *with* L_0 *as defined in Proposition 3,* $V(\cdot)$ *is locally Lipshitz, i.e.,* $\forall R \geq 0, \exists L_R \geq 0$*, such that* $|V(x_1) - V(x_2)| \leq L_R|x_1 - x_2| \; \forall x_1, x_2 \in \overline{B}(0, R)$.

Before we prove Proposition 5, we first prove a lemma regarding the solution to (17).

Lemma 6: *Let* $x_1(\cdot), x_2(\cdot)$ *be solutions to (17) under some admissible control* $\alpha(\cdot) = \{d(\cdot), u(\cdot)\}$ *with initial condition* x_1, x_2 *respectively. Then*

1.

$$|x_1(t) - x_2(t)| \leq e^{L_0 t}|x_1 - x_2|;; \tag{20}$$

2.

$$|x_1(t)| \leq |x_1|e^{L_0 t} + \frac{C}{L_0}(e^{L_0 t} - 1), \tag{21}$$

where $C = \max_d |\tilde{f}(0, u_c, d)|$.

Proof. 1. Denote the sequence of mode switching times as $\{t_i, i = 0, 1, \cdots\}$ with $t_0 = 0$, and the mode during $[t_i, t_{i+1})$ as d_i. Then $\forall t \in [0, t_1)$,

$$\begin{aligned}\frac{d}{dt}|x_1(t) - x_2(t)|^2 &= 2(x_1(t) - x_2(t)) \cdot (\tilde{f}(x_1(t), u(t), d_0) - \tilde{f}(x_2(t), u(t), d_0) \\ &\leq 2L_0|x_1(t) - x_2(t)|^2,\end{aligned}$$

where the inequality comes from Proposition 3. Integrating both sides from 0 to t,

$$|x_1(t) - x_2(t)|^2 \leq |x_1 - x_2|^2 + 2L_0 \int_0^t |x_1(s) - x_2(s)|^2 ds,$$

and by Gronwall inequality,

$$|x_1(t) - x_2(t)|^2 \leq |x_1 - x_2|^2 e^{2L_0 t},$$

from which (20) follows. Now $\forall t \in [t_1, t_2)$, taking $x_1(t_1), x_2(t_1)$ as initial conditions, we follow the above procedures and get

$$\begin{aligned}|x_1(t) - x_2(t)| &\leq |x_1(t_1) - x_2(t_1)|e^{L_0(t-t_1)} \\ &\leq |x_1 - x_2|e^{L_0(t-t_1)}e^{L_0(t_1-0)} \\ &= |x_1 - x_2|e^{L_0 t}.\end{aligned}$$

Using the same argument successively, we can show that (20) holds $\forall t \geq 0$.

2. $\forall x_1(t) \neq 0$, we can write

$$\begin{aligned}|x_1(t)|\frac{d}{dt}|x_1(t)| &= \frac{1}{2}\frac{d}{dt}|x_1(t)|^2 = x_1(t) \cdot \tilde{f}(x_1(t), u(t), d(t)) \\ &= x_1(t) \cdot (\tilde{f}(0, u(t), d(t)) + \tilde{f}(x_1(t), u(t), d(t)) - \tilde{f}(0, u(t), d(t))) \\ &\leq C|x_1(t)| + L_0|x_1(t)|^2,\end{aligned}$$

from which we obtain

$$\frac{d}{dt}|x_1(t)| \leq C + L_0|x_1(t)|.$$

Integrating it from 0 to t and then apply Gronwall inequality, we have (21). $\square$

Proof of Proposition 5. For $\epsilon > 0$, let $\alpha^\epsilon(\cdot) = \{d^\epsilon(\cdot), u^\epsilon(\cdot)\}$ be $\epsilon-$optimal for x_2, i.e.,

$$V(x_2) \geq J(x_2, \alpha^\epsilon(\cdot)) - \epsilon.$$

Since $V(x_1) \leq J(x_1, \alpha^\epsilon(\cdot))$, we have

$$\begin{aligned}
V(x_1) - V(x_2) &\leq J(x_1, \alpha^\epsilon(\cdot)) - J(x_2, \alpha^\epsilon(\cdot)) \\
&\leq \int_0^\infty e^{-\lambda t}|l(x_1(t), u^\epsilon(t)) - l(x_2(t), u^\epsilon(t))|dt + \epsilon \\
&\leq \int_0^\infty e^{-\lambda t}C_l(1 + |x_1(t)| + |x_2(t)|)|x_1(t) - x_2(t)|dt + \epsilon,
\end{aligned}$$

where the last inequality is from (A_3). Using Lemma 6,

$$\begin{aligned}
V(x_1) - V(x_2) &\leq C_l|x_1 - x_2|\int_0^\infty ((\frac{2C}{L_0} + |x_1| + |x_2|)e^{(2L_0-\lambda)t} \\
&\quad +(1 - \frac{2C}{L_0})e^{(L_0-\lambda)t})dt + \epsilon \\
&= (C_0 + \frac{(|x_1| + |x_2|)}{\lambda - 2L_0})C_l|x_1 - x_2| + \epsilon \\
&\leq L_R|x_1 - x_2| + \epsilon,
\end{aligned}$$

where C_0 is a constant and $L_R := (C_0 + \frac{2R}{\lambda - 2L_0})C_l$. Since ϵ is arbitrary, we have

$$V(x_1) - V(x_2) \leq L_R|x_1 - x_2|.$$

But x_1 and x_2 are symmetric, we must also have

$$V(x_1) - V(x_2) \leq L_R|x_1 - x_2|.$$

Therefore

$$|V(x_1) - V(x_2)| \leq L_R|x_1 - x_2|.$$

$\square$

Remark: Proposition 2 can be exploited to yield sharper estimates for $|x_1(t) - x_2(t)|$ and $|x_1(t)|$, as shown in the next lemma. This might be used to weaken the assumptions in Proposition 5.

Lemma 7: *Let $x_1(\cdot), x_2(\cdot)$ be solutions to (17) under some admissible control $\alpha(\cdot) = \{d(\cdot), u(\cdot)\}$ with initial condition x_1, x_2 respectively. Then*

1.

$$|x_1(t) - x_2(t)| \leq |x_1 - x_2| + 2L_0 t; \tag{22}$$

2.
$$|x_1(t) - x_1| \leq C_0 t, \tag{23}$$

where
$$C_0 = u_c \left| \begin{array}{c} 1 \\ C_f \end{array} \right|.$$

*Proof.*1. For $x_1(t) \neq x_2(t)$,

$$\begin{aligned}
& |x_1(t) - x_2(t)| \frac{d}{dt} |x_1(t) - x_2(t)| \\
= & \frac{1}{2} \frac{d}{dt} |x_1(t) - x_2(t)|^2 \\
= & (x_1(t) - x_2(t)) \cdot (\tilde{f}(x_1(t), u(t), d(t)) - \tilde{f}(x_2(t), u(t), d(t))) \\
\leq & 2L_0 |x_1(t) - x_2(t)|,
\end{aligned}$$

where the last inequality is from Proposition 3. Therefore we have

$$\frac{d}{dt} |x_1(t) - x_2(t)| \leq 2L_0,$$

from which (22) follows.

2. (23) is a straightforward consequence of Proposition 2. □

3.3 Dynamic programming principle

Proposition 8 [Dynamic Programming Principle]: *Assume $(A_1) - (A_3)$, $\lambda < 2L_0$. Denote any admissible control pair $\{d(\cdot), u(\cdot)\}$ as $\alpha(\cdot)$. The following is true:*

$$V(x) = \inf_{\alpha(\cdot)} \{ \int_0^t e^{-\lambda s} l(x(s), u(s)) ds + e^{-\lambda t} V(x(t)) \}, \ \forall t \geq 0. \tag{24}$$

Proof. Denote the right hand side of (24) as $W(x)$. Note under the assumptions, $\forall t \geq 0$, $V(x)$ and $W(x)$ are locally bounded, i.e., for any bounded x, $V(x) < \infty, W(x) < \infty$. We will first show $V(x) \geq W(x)$ and then the converse.

1. For $\epsilon > 0$, let $\alpha^\epsilon(\cdot)$ be $\epsilon-$optimal, i.e.,

$$J(x, \alpha^\epsilon(\cdot)) \leq V(x) + \epsilon.$$

Now

$$\begin{aligned}
J(x, \alpha^\epsilon(\cdot)) &= \int_0^t e^{-\lambda s} l(x(s), u^\epsilon(s)) ds + \int_t^\infty e^{-\lambda s} l(x(s), u^\epsilon(s)) ds \\
&= \int_0^t e^{-\lambda s} l(x(s), u^\epsilon(s)) ds + e^{-\lambda t} \int_0^\infty e^{-\lambda s} l(x(s+t), u^\epsilon(s+t)) ds.
\end{aligned}$$

Let $\overline{x}(s) = x(s+t), \overline{\alpha}(s) = \alpha^\epsilon(s+t)$, we have

$$\begin{aligned}\int_0^\infty e^{-\lambda s} l(x(s+t), u^\epsilon(s+t))ds &= \int_0^\infty e^{-\lambda s} l(\overline{x}(s), \overline{u}(s))ds \\ &= J(x(t), \overline{u}(\cdot)) \geq V(x(t)).\end{aligned}$$

Therefore

$$V(x) + \epsilon \geq W(x),$$

and since ϵ is arbitrary,$V(x) \geq W(x)$.

2. For $\epsilon > 0$, pick $\alpha_0(\cdot) = \{d_0(\cdot), u_0(\cdot)\}$ such that

$$\int_0^t e^{-\lambda s} l(x(s), u_0(s))ds + e^{-\lambda t} V(x(t)) \leq W(x) + \epsilon.$$

Then pick $\alpha_1(\cdot) = \{d_1(\cdot), u_1(\cdot)\}$ such that

$$J(x(t), \alpha_1(\cdot)) \leq V(x(t)) + \epsilon.$$

Now define $\overline{\alpha}(\cdot) = \{\overline{d}(\cdot), \overline{u}(\cdot)\}$ by

$$\overline{\alpha}(s) = \begin{cases} \alpha_0(s), & s \leq t \\ \alpha_1(s-t), & s > t \end{cases}.$$

We then have

$$\begin{aligned}W(x) + \epsilon &\geq \int_0^t e^{-\lambda s} l(x(s), u_0(s))ds + e^{-\lambda t}\left(\int_t^\infty e^{-\lambda(s)} l(x(s), u_1(s))ds - \epsilon\right) \\ &\geq \int_0^\infty e^{-\lambda s} l(x(s), \overline{u}(s))ds - \epsilon \\ &= J(x, \overline{\alpha}(\cdot)) - \epsilon \geq V(x) - \epsilon,\end{aligned}$$

which implies

$$W(x) \geq V(x) - 2\epsilon.$$

Since ϵ is arbitrary, we have $W(x) \geq V(x)$. $\square$

3.4 Hamilton-Jacobi-Bellman equation

In this subsection, we will show that the value function $V(\cdot)$ satisfies the HJB equation in the viscosity sense. Viscosity solutions to Hamilton-Jacobi equations were first introduced by Crandall and Lions[4]. Here we use one of the three equivalent definitions of viscosity solutions[3]:

Definition[Viscosity Solution]: Let W be a continuous function from an open set $O \in \mathbb{R}^n$ into $\mathbb{R}$ and let DW denote the gradient of W (when W is differentiable). We call W a *viscosity solution* to a nonlinear first order partial differential equation

$$F(x, W(x), DW(x)) = 0, \tag{25}$$

provided $\forall \phi \in C^1(O)$,

- (*Viscosity Subsolution*) if $W - \phi$ attains a local maximum at $x_0 \in O$, then

$$F(x_0, W(x_0), D\phi(x_0)) \leq 0.$$

- (*Viscosity Supersolution*) if $W - \phi$ attains a local minimum at $x_0 \in O$, then

$$F(x_0, W(x_0), D\phi(x_0)) \geq 0.$$

Viscosity solutions have a couple of nice properties[3, 4]. We mention one elementary property here(*consistency with the notion of classical solution)*, that is: 1) any classical solution to (25) is a viscosity solution; 2) the viscosity solution satisfies (25) in the classical sense at any point where it is differentiable.

We now present the first main result of this report: the value function $V(\cdot)$ satisfies a Hamilton-Jacobi-Bellman equation of a special form in the viscosity sense.

Theorem 9 [Hamilton-Jacobi-Bellman Equation] *Assuming* $(A_1) - (A_3), \lambda > 2L_0$, *the value function* $V(x)$ *is a viscosity solution to the following equation:*

$$(HJB) \;\; \lambda W(x) + \max\{\sup_{u \in U_+}\{-uf_+(x) \cdot DW(x) - l(x,u)\}, \sup_{u \in U_-}\{-uf_-(x) \cdot DW(x) - l(x,u)\}\} = 0. \tag{26}$$

Remark: we may replace sup in (26) by max since U_- and U_+ are compact.

Proof. 1. We first show $V(\cdot)$ is a viscosity subsolution. For any $u \in U_-$, take $\alpha(\cdot) = \{d(\cdot), u(\cdot)\}$ with $d(t) = 2,\; u(t) = u \;\forall t$. From (24), for any $t \geq 0$

$$V(x) \leq \int_0^t l(x(s), u)e^{-\lambda s} ds + e^{-\lambda t} V(x(t)),$$

which we rewrite as

$$V(x(t)) - V(x) + \int_0^t l(x(s), u)e^{-\lambda s} ds + V(x(t))(e^{-\lambda t} - 1) \geq 0. \tag{27}$$

Now suppose that $V - \phi$ with $\phi \in C^1(\mathbb{R}^2)$ has a local maximum at x, then since V is continuous,

$$V(x(t)) - \phi(x(t)) \leq V(x) - \phi(x),$$

for t sufficiently small. This together with (27) implies

$$\phi(x(t)) - \phi(x) + \int_0^t l(x(s), u)e^{-\lambda s} ds + V(x(t))(e^{-\lambda t} - 1) \geq 0. \tag{28}$$

Divide (28) by t and let $t \to 0$, we obtan

$$uf_-(x) \cdot D\phi(x) + l(x,u) - \lambda V(x) \geq 0, \; \forall u \in U_-,$$

i.e.,

$$\lambda V(x) + \sup_{u \in U_-}\{-uf_-(x) \cdot D\phi(x) - l(x,u)\} \leq 0.$$

Similarly, we have

$$\lambda V(x) + \sup_{u \in U_+} \{-uf_+(x) \cdot D\phi(x) - l(x,u)\} \leq 0.$$

Therefore

$$\begin{aligned}\lambda V(x) + \max\{ &\sup_{u \in U_-} \{-uf_-(x) \cdot D\phi(x) - l(x,u)\}, \\ &\sup_{u \in U_+} \{-uf_+(x) \cdot D\phi(x) - l(x,u)\}\} \leq 0. \end{aligned} \tag{29}$$

2. The proof of supersolution is much more technically involved. We mainly follow the approach in Bardi and Capuzzo-Dolcetta[2].

For $t > 0, \epsilon > 0$, we can find $\alpha(\cdot) = \{d(\cdot), u(\cdot)\}$, such that

$$V(x) \geq \int_0^t l(x(s), u(s)) e^{-\lambda s} ds + e^{-\lambda t} V(x(t)) - t\epsilon. \tag{30}$$

Since by (A_3),

$$|l(x(s), u(s)) - l(x, u(s))| \leq C_l(1 + |x(s)| + |x|)|x(s) - x|,$$

and by Lemma 7,

$$|x(s) - x| \leq C_0 s,$$

we then have

$$\int_0^t l(x(s), u(s)) e^{-\lambda s} ds = \int_0^t l(x, u(s)) e^{-\lambda s} ds + o(t).$$

And by (30),

$$V(x) \geq \int_0^t l(x, u(s)) e^{-\lambda s} ds + e^{-\lambda t} V(x(t)) - t\epsilon + o(t). \tag{31}$$

Now suppose that $V - \phi$ with $\phi \in C^1(\mathbb{R}^2)$ has a local minimum at x, then since V is continuous,

$$V(x(t)) - \phi(x(t)) \geq V(x) - \phi(x),$$

for t sufficiently small. This and (31) imply

$$\phi(x) - \phi(x(t)) \geq \int_0^t l(x, u(s)) e^{-\lambda s} ds - (1 - e^{-\lambda t}) V(x(t)) - t\epsilon + o(t). \tag{32}$$

We also have

$$\begin{aligned}\phi(x) - \phi(x(t)) &= -\int_0^t D\phi(x(s)) \cdot \tilde{f}(x(s), u(s), d(s)) ds \\ &= -\int_0^t D\phi(x) \cdot \tilde{f}(x, u(s), d(s)) ds + o(t), \end{aligned} \tag{33}$$

since

$$
\begin{aligned}
|\tilde{f}(x(s),u(s),d(s)) - \tilde{f}(x,u(s),d(s))| &\leq L_0|x(s)-x| \\
&\leq C_0L_0s,
\end{aligned}
$$

and

$$
|D\phi(x(s)) - D\phi(x)| \leq \omega_{D\phi}(|x(s)-x|),
$$

where $\omega_{D\phi}(\cdot)$ is the modulus of continuity of $D\phi$.

Combining (32) and (33) gives rise to

$$
\begin{aligned}
\int_0^t (-D\phi(x)\cdot\tilde{f}(x,u(s),d(s)) - l(x,u(s)))ds + \int_0^t (1-e^{-\lambda s})l(x,u(s))ds \\
+(1-e^{-\lambda t})V(x(t)) \geq -t\epsilon + o(t). \qquad (34)
\end{aligned}
$$

Since $U_- \cup U_+$ is compact, $l(x,u(s)) \leq M_x$, $\forall s \geq 0$, for some constant M_x depending on x. Thus

$$
\int_0^t (1-e^{-\lambda s})l(x,u(s))ds = o(t).
$$

Since

$$
\begin{aligned}
-D\phi(x)\cdot\tilde{f}(x,u(s),d(s)) - l(x,u(s)) \leq \max\{\sup_{u\in U_-}\{-uf_-(x)\cdot D\phi(x) - l(x,u)\}, \\
\sup_{u\in U_+}\{-uf_+(x)\cdot D\phi(x) - l(x,u)\}\} =: Q(x,D\phi(x)),
\end{aligned}
$$

we obtain from (34)

$$
\int_0^t Q(x,D\phi(x))ds + (1-e^{-\lambda t})V(x(t)) \geq -t\epsilon + o(t). \qquad (35)
$$

By dividing (35) by t and letting $t \to 0$,

$$
\lambda V(x) + Q(x,D\phi(x)) \geq -\epsilon.
$$

Since ϵ is arbitrary, we get

$$
\begin{aligned}
\lambda V(x) + \max\{\sup_{u\in U_-}\{-uf_-(x)\cdot D\phi(x) - l(x,u)\}, \\
\sup_{u\in U_+}\{-uf_+(x)\cdot D\phi(x) - l(x,u)\}\} \geq 0. \qquad (36)
\end{aligned}
$$

Combining (29) and (36) yields the desired result.□

4 Uniqueness of (HJB)

We would like to characterize the value function $V(\cdot)$ as a unique solution to (HJB). The uniqueness result basically comes from Theorem 1.5 in [8]. In [8], the author gave only a sketch of proof. Here for completeness, we will provide the full proof.

Before stating the theorem, we need first identify structural properties of our (HJB). We rewrite (26) as:

$$\lambda W(x) + H(x, DW(x)) = 0, \tag{37}$$

where

$$H(x,p) = \max\{\sup_{u\in U_-}\{-p\cdot \tilde{f}_-(x,u) - l(x,u)\}, \sup_{u\in U_+}\{-p\cdot \tilde{f}_+(x,u) - l(x,u)\}\}.$$

Proposition 10: *Assume (A_3), $H(x,p)$ satisfies the following:*

$$\begin{aligned} |H(x_1,p) - H(x_2,p)| &\leq C_R(1+|p|)|x_1 - x_2|,\ \forall x_1, x_2 \in \overline{B}(0,R),\ \forall p & (38)\\ |H(x,p_1) - H(x,p_2)| &\leq C_0|p_1 - p_2|, \forall x,\ \forall p_1, p_2 & (39)\end{aligned}$$

for some $C_R > 0, C_0 > 0$, with C_R dependent on R.

Proof. 1. Without loss of generality, suppose $u_1 \in U_-$ attains the maximum in $H(x_1,p)$. Since $H(x_2,p) \geq -p\cdot \tilde{f}_-(x_2,u_1) - l(x_2,u_1)$,

$$\begin{aligned} H(x_1,p) - H(x_2,p) &\leq -p\cdot\tilde{f}_-(x_1,u_1) - l(x_1,u_1) + p\cdot\tilde{f}_-(x_2,u_1) + l(x_2,u_1)\\ &\leq |p|L_0|x_1-x_2| + C_l(1+|x_1|+|x_2|)|x_1-x_2|\\ &\leq C_R(1+|p|)|x_1-x_2|,\end{aligned}$$

where C_R is a constant dependent on R. By symmetry, we have $H(x_2,p) - H(x_1,p) \leq C_R(1+|p|)|x_1-x_2|$.

2. Without loss of generality, suppose $u_1 \in U_-$ attains the maximum in $H(x,p_1)$. Since $H(x,p_2) \geq -p_2\cdot\tilde{f}_-(x,u_1) - l(x,u_1)$,

$$\begin{aligned} H(x,p_1) - H(x,p_2) &\leq -p_1\cdot\tilde{f}_-(x,u_1) - l(x,u_1) + p_2\cdot\tilde{f}_-(x,u_1) + l(x,u_1)\\ &= -\tilde{f}_-(x,u_1)\cdot(p_1-p_2)\\ &\leq C_0|p_1-p_2|,\end{aligned}$$

where the last inequality is from boundness of $\tilde{f}_-$. Again by symmetry, we have the other half of the inequality. □

Remark: As we have seen above, despite the hybrid structure of our physical model, $H(x,p)$ enjoys nice structural properties, which enables us to prove the uniqueness result.

From Proposition 5, we know that the value function $V(\cdot)$ of our optimal control problem belongs to the following class

$$\mathcal{P}(\mathbb{R}^2) = \{W(\cdot) : |W(x_1) - W(x_2)| \leq C(1+R)|x_1 - x_2|, \forall x_1, x_2 \in \overline{B}(0,R)\}.$$

Note $W(\cdot) \in \mathcal{P}(\mathbb{R}^2)$ implies

$$W(\cdot) \in \mathcal{Q}(\mathbb{R}^2) = \{W(\cdot) \in C(\mathbb{R}^2) : \sup_{\mathbb{R}^2} \frac{|W(x)|}{1+|W(x)|^2} < \infty\}.$$

The following theorem is adapted from Theorem 1.5 in [8].

Theorem 11: *Assuming $(A_1) - (A_3), \lambda > 2L_0$, if (37) has a viscosity solution in $\mathcal{P}(\mathbb{R}^2)$, it is unique.*

Proof. Without loss of generality, we take $\lambda = 1$. Let $W(\cdot), V(\cdot) \in \mathcal{P}(\mathbb{R}^2)$ be viscosity solutions to (37). For $\epsilon > 0, \alpha > 0, m > 2$, define

$$\Phi(x,y) = W(x) - V(y) - \frac{1}{\epsilon}|x-y|^2 - \alpha(< x >^m + < y >^m)$$

where

$$< x >:= (1+|x|^2)^{\frac{1}{2}}.$$

Since $W(\cdot), V(\cdot) \in \mathcal{Q}(\mathbb{R}^2)$,

$$\lim_{|x|+|y| \to \infty} \Phi(x,y) = -\infty.$$

And since $\Phi(\cdot,\cdot)$ is continuous, there exists (x_0, y_0) such that Φ attains the global maximum. First we need to get an estimate for $|x_0|, |y_0|$ and $|x_0 - y_0|$.

From $\Phi(0,0) \leq \Phi(x_0, y_0)$,

$$W(0) - V(0) - 2\alpha \leq W(x_0) - V(y_0) - \frac{1}{\epsilon}|x_0 - y_0|^2 - \alpha(< x_0 >^m + < y_0 >^m).$$

$W(\cdot), V(\cdot) \in \mathcal{Q}(\mathbb{R}^2)$ leads to

$$< x_0 >^m + < y_0 >^m \leq C_\alpha(1 + < x_0 >^2 + < y_0 >^2),$$

where C_α is a constant independent of ϵ (but dependent on α). Since $m > 2$, there exists $R_\alpha > 0$ (independent of ϵ), such that

$$|x_0| \leq R_\alpha, \ |y_0| \leq R_\alpha.$$

From $\Phi(x_0, x_0) + \Phi(y_0, y_0) \leq 2\Phi(x_0, y_0)$,

$$\begin{aligned} \frac{1}{\epsilon}|x_0 - y_0|^2 &\leq W(x_0) - W(y_0) + V(x_0) - V(y_0) \\ &\leq C(1+R_\alpha)|x_0 - y_0|, \end{aligned}$$

where the last inequality comes from $W(\cdot), V(\cdot) \in \mathcal{P}(\mathbb{R}^2)$. Therefore we get

$$|x_0 - y_0| \leq \epsilon C'_\alpha, \tag{40}$$

with C'_α depending on α only.

Now define

$$\phi(x) = V(y_0) + \frac{1}{\epsilon}|x - y_0|^2 + \alpha(< x >^m + < y_0 >^m),$$
$$\psi(y) = W(x_0) - \frac{1}{\epsilon}|x_0 - y|^2 - \alpha(< x >^m + < y_0 >^m).$$

Since $(W - \phi)(\cdot)$ achieves maximum at x_0,

$$\lambda W(x_0) + H(x_0, D\phi(x_0)) \leq 0,$$

i.e.,

$$W(x_0) \leq -H(x_0, \frac{2}{\epsilon}(x_0 - y_0) + \alpha m < x_0 >^{m-2} x_0). \tag{41}$$

Similarly, since $(V - \phi)(\cdot)$ achieves minimum at y_0, we obtain

$$V(y_0) \geq -H(y_0, \frac{2}{\epsilon}(x_0 - y_0) - \alpha m < y_0 >^{m-2} y_0). \tag{42}$$

Subtracting (42) from (41), we have

$$W(x_0) - V(y_0) \leq H(y_0, \frac{2}{\epsilon}(x_0 - y_0) - \alpha m < y_0 >^{m-2} y_0) - H(x_0, \frac{2}{\epsilon}(x_0 - y_0) + \alpha m < x_0 >^{m-2} x_0).$$

From (39)

$$H(y_0, \frac{2}{\epsilon}(x_0 - y_0) - \alpha m < y_0 >^{m-2} y_0) \leq H(y_0, \frac{2}{\epsilon}(x_0 - y_0)) + \alpha C_0 m < y_0 >^{m-1},$$
$$-H(x_0, \frac{2}{\epsilon}(x_0 - y_0) + \alpha m < x_0 >^{m-2} x_0) \leq -H(x_0, \frac{2}{\epsilon}(x_0 - y_0)) + \alpha C_0 m < x_0 >^{m-1},$$

and therefore

$$\begin{aligned} W(x_0) - V(y_0) &\leq H(y_0, \frac{2}{\epsilon}(x_0 - y_0)) - H(x_0, \frac{2}{\epsilon}(x_0 - y_0)) + \alpha C_0 m(< x_0 >^{m-1} + < y_0 >^{m-1}) \\ &\leq C_{R_\alpha}(1 + \frac{2}{\epsilon}|x_0 - y_0|)|x_0 - y_0| + \alpha C_0 m(< x_0 >^{m-1} + < y_0 >^{m-1}) \end{aligned}$$

where the last inequality follows from (38).

Now fix α, construct a sequence $\{\epsilon_k\}$ with $\lim_{k\to\infty} \epsilon_k = 0$. We denote the corresponding maximizers of Φ as (x_{0k}, y_{0k}). Since $\forall k, (x_{0k}, y_{0k}) \in \overline{B}(0, R_\alpha)$, by extracting a subsequence if necessary, we get

$$\lim_{k\to\infty} (x_{0k}, y_{0k}) \to (x_\alpha, y_\alpha) \in \overline{B}(0, R_\alpha). \tag{43}$$

Also from (40), we have $x_\alpha = y_\alpha$. For each ϵ_k, from $\Phi(x, x) \leq \Phi(x_0, y_0)$,

$$\begin{aligned} W(x) - V(x) - 2\alpha < x >^m &\leq W(x_0) - V(y_0) - \frac{1}{\epsilon_k}|x_0 - y_0|^2 - \alpha(< x_0 >^m + < y_0 >^m) \\ &\leq W(x_0) - V(y_0) - \alpha(< x_0 >^m + < y_0 >^m) \\ &\leq C_{R_\alpha}(1 + \frac{2}{\epsilon}|x_0 - y_0|)|x_0 - y_0| + \alpha C_0 m(< x_0 >^{m-1} + < y_0 >^{m-1}) \\ &\quad -\alpha(< x_0 >^m + < y_0 >^m) \end{aligned}$$

and letting $k \to \infty$,using (40)

$$W(x) - V(x) \le 2\alpha(C_0 m < x_\alpha >^{m-1} - < x_\alpha >^m) + 2\alpha < x >^m .$$

Since $Cm < x_\alpha >^{m-1} - < x_\alpha >^m \le C''$ for some $C'' > 0$,

$$W(x) - V(x) \le 2\alpha(C'' + < x >^m).$$

Letting $\alpha \to 0$, we get

$$W(x) - V(x) \le 0, \ \forall x,$$

Since W and V are symmetric, we also have $V(x) - W(x) \le 0 \ \forall x$. Thus we get $W(x) = V(x), \ \forall x$. $\square$

5 Discrete Approximation Schemes

The approximation will be accomplished in two steps. First we approximate the continuous time optimal control problem by a discrete time problem, derive the discrete Bellman equation (DBE), and show the value function of the disrete problem converges to that of the continuous problem locally uniformly. Following [2], we call this step "semi-discrete" approximation. Then we indicate how to further discretize (DBE) in the spatial variable, which is called "fully-discrete" approximation. The approaches we take here follows closely those in [2](Chapter VI and Appendix A).

5.1 Semi-discrete approximation

Consider a discrete time optimal control problem obtained by discretizing the original continuous time one with time step $h \in (0, \frac{1}{\lambda})$. The dynamics is given by

$$x[n] = x[n-1] + h\tilde{f}(x[n-1], u[n-1], d[n-1]), \ x[0] = x, \tag{44}$$

and the cost is given by

$$J_h(x, \alpha[\cdot]) = \sum_{n=0}^{\infty} hl(x[n], u[n])(1 - \lambda h)^n, \tag{45}$$

where $\alpha[\cdot] = \{d[\cdot], u[\cdot]\}$ is the control. The value function is defined to be

$$V_h(x) = \inf_{\alpha[\cdot]} J_h(x, \alpha[\cdot]). \tag{46}$$

The following lemma is analogous to Lemma 6.

Lemma 12: *Let $x_1[\cdot], x_2[\cdot]$ be solutions to (44) under some admissible control $\alpha[\cdot] = \{d[\cdot], u[\cdot]\}$ with initial condition x_1, x_2 respectively. Then*

1.

$$|x_1[n] - x_2[n]| \le (1 + hL_0)^n |x_1 - x_2|; \tag{47}$$

2.
$$|x_1[n]| \leq (1+hL_0)^n |x_1| + \frac{C((1+hL_0)^n - 1))}{L_0}, \tag{48}$$

where $C = \max_d |\tilde{f}(0, u_0, d)|$.

It's not hard to show $V_h(\cdot)$ has the following property:

Lemma 13: $V_h(\cdot) \in \mathcal{P}(\mathbb{R}^2)$*, and the coefficient* C *in defining* $\mathcal{P}(\mathbb{R}^2)$ *can be made independent of* h*.*

Remark: Lemma 13 implies that $V_h(\cdot)$ is locally bounded and locally Lipshitz continuous.

Following standard arguments, one can show $V_h(\cdot)$ satisfies the discrete Bellman equation (DBE):

Theorem 14: $V_h(\cdot)$ *satisfies:*

$$(DBE) \quad V_h(x) = \min\{\inf_{u \in U_-} \{(1-\lambda h)V_h(x+huf_-(x))+hl(x,u)\}, \inf_{u \in U_+} \{(1-\lambda h)V_h(x+huf_+(x))+hl(x,u)\}\}. \tag{49}$$

It's of interest to know whether (49) characterizes the value function $V_h(\cdot)$. Unlike in [2](Chapter VI), where a bounded value function was considered, we have $V_h(\cdot)$ unbounded. But it turns out that with a little bit additional assumption, (49) has a unique solution.

Proposition 15: *Assume* (A_1) *and* (A_3)*. If* $\frac{2(1-\lambda h)}{\sqrt{C_0^2+4}-C_0} < 1$*, then there exists unique solution in* $\mathcal{P}(\mathbb{R}^2)$ *to (49), where*
$$C_0 = |u_c| \left| \begin{array}{c} 1 \\ C_f \end{array} \right|.$$

Proof. Let $\tilde{V}_h(x) = V_h(x) < x >^{-m}, m > 2$, where $< x >= (1+|x|^2)^{1/2}$. Since $V_h \in \mathcal{P}(\mathbb{R}^2)$, $\tilde{V}_h$ is bounded. In terms of $\tilde{V}_h$, (49) is rewritten as

$$\begin{aligned} \tilde{V}_h(x) \quad &= \quad \min\{\inf_{u \in U_-} \{(1-\lambda h)\tilde{V}_h(x+h\tilde{f}_-(x,u))(\frac{< x+h\tilde{f}_-(x,u) >}{< x >})^m + hl(x,u) < x >^{-m}\}, \\ &\quad \inf_{u \in U_+} \{(1-\lambda h))\tilde{V}_h(x+h\tilde{f}_+(x,u))(\frac{< x+h\tilde{f}_+(x,u) >}{< x >})^m + hl(x,u) < x >^{-m}\}\} \quad (50) \\ =: &\quad (G(\tilde{V}_h))(x). \end{aligned}$$

It suffices to show (50) has a unique solution. It's clear that the operator $G(\cdot)$ maps any $\tilde{W} \in BC(\mathbb{R}^2)$ into $BC(\mathbb{R}^2)$, where $BC(\mathbb{R}^2)$ denotes the set of bounded continuous functions. Now take $\tilde{V}, \tilde{W} \in BC(\mathbb{R}^2)$. For any x, without loss of generality, assume $u_1 \in U_-$ achieves the minimum in

$G(\tilde{V})(x)$. Then

$$\begin{aligned} G(\tilde{W})(x) - G(\tilde{V})(x) &\le (1-\lambda h)\left(\frac{< x + h\tilde{f}_-(x,u_1) >}{< x >}\right)^m \tilde{W}(x + h\tilde{f}_-(x,u)) \\ &\quad -(1-\lambda h)\left(\frac{< x + h\tilde{f}_-(x,u_1) >}{< x >}\right)^m \tilde{V}(x + h\tilde{f}_-(x,u)) \\ &\le (1-\lambda h)\left(\frac{< x + h\tilde{f}_-(x,u_1) >}{< x >}\right)^m \sup_{x\in\mathbb{R}^2} (\tilde{W} - \tilde{V})(x). \end{aligned}$$

Using the boundness of $\tilde{f}_-$, one can show that

$$\left(\frac{< x + h\tilde{f}_-(x,u_1) >}{< x >}\right)^m \le \left(\frac{2}{\sqrt{C_0^2+4} - C_0}\right)^{\frac{m}{2}}.$$

Since we may choose m arbitrarily close to 2,

$$G(\tilde{W})(x) - G(\tilde{V})(x) \le \rho \sup_{x\in\mathbb{R}^2} |(\tilde{W} - \tilde{V})(x)|,$$

where $\rho < 1$. By symmetry, we have

$$G(\tilde{V})(x) - G(\tilde{W})(x) \le \rho \sup_{x\in\mathbb{R}^2} |(\tilde{W} - \tilde{V})(x)|.$$

Thus $\| G(W) - G(V) \|_\infty \le \rho \| W - V \|_\infty$. Since $BC(\mathbb{R}^2)$ is a Banach space, by Contraction Mapping Principle, there exists a unique solution to (50) in $BC(\mathbb{R}^2)$. $\Box$

The following theorem asserts that $V_h(\cdot)$ converges to $V(\cdot)$ as $h \to 0$. The proof can be found in [2](Chapter VI)(with minor modification).

Theorem 16[2]: *Under assumptions which guarantee uniqueness of (HJB) and (DBE),*

$$\sup_{x\in\mathcal{K}} |V_h(x) - V(x)| \to 0 \ \ as \ \ h \to 0, \tag{51}$$

for every compact $\mathcal{K} \subset \mathbb{R}^2$, where $V_h(\cdot)$ and $V(\cdot)$ are the unique solutions to (DBE) and (HJB) respectively.

Remarks:

- Theorem 16 also serves as a proof of existence of a solution to (HJB).
- In solving (49), one obtains the optimal control $\alpha_h^*[\cdot]$ for the discrete time problem. A sub-optimal control for the continuous time problem is $\alpha_h(\cdot)$ defined by $\alpha_h(t) \equiv \alpha_h^*[k]$, $\forall t \in [kh,(k+1)h), k \in \mathbb{N}$. As $h \to 0$, one can show $J(x, \alpha_h(\cdot)) \to V(x)$.

5.2 Fully-discrete approximation

Theoretically the solution to (49) can be obtained by first computing $\tilde{V}(\cdot)$ via successive approximation and then transforming back to $V(\cdot)$. An approximation scheme for solving (DBE) is described in [2](Appendix A). It was shown there that when space discretization gets finer and finer, the solution obtained via solving a finite system of equations converges to $V_h(\cdot)$.

6 Conclusions and Future Work

In this report, we have studied control of a magnetostrictive actuator, taking the infinite horizon optimal control problem as an example. We characterized the value function as the (unique) solution of a Hamilton-Jacobi-Bellman equation of a hybrid form. And we pointed out how to solve the (HJB) and obtain a suboptimal control by discrete time approximation.

Future work includes extension of this approach to other control problems of practical interests, which are listed below:

- **Finite Horizon Control Problems.** Such problems arise, for instance, in tracking control of the actuator.
- **Problems with State-Space Constraints.** The state-space constraints come from physics, as well as limitation on the operating range of the input current.
- **Time-Optimal Control Problem.** An important example of this is fast positioning.
- H_∞ **Control Problem.** To account for exogenous disturbances and unmodeled dynamics, we can introduce a noise term into the model and investigate robust control of the actuator using H_∞ control theory.

For each of the problems discussed above, we need to study the exact form of the Hamilton-Jacobi equation (together with its initial/boundary conditions) satisfied by the value function, the numerical solution of the equation and the controller synthesis method.

A Proof of Lemma 1

Proof.

$$\begin{aligned}
\frac{\partial \mathcal{L}(z)}{\partial z} &= \frac{1}{z^2} - csch^2(z) \\
&= \frac{1}{z^2} - \frac{1}{(\frac{e^z - e^{-z}}{2})^2} \\
&= \frac{1}{z^2} - \frac{1}{(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots)^2}.
\end{aligned} \tag{52}$$

Therefore

$$\frac{\partial \mathcal{L}(z)}{\partial z} > 0, \ \forall z \neq 0. \tag{53}$$

Further manipulation on (52) yields

$$\frac{\partial \mathcal{L}(z)}{\partial z} = \frac{(2 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)(\frac{1}{3!} + \frac{z^2}{5!} + \cdots)}{(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)^2},$$

from which we obtain

$$\frac{\partial \mathcal{L}}{\partial z}(0) = \frac{1}{3}. \tag{55}$$

$$\tag{56}$$

Combining (53) and (55) we have

$$\frac{\partial \mathcal{L}(z)}{\partial z} > 0. \tag{57}$$

Since in addition,

$$\lim_{z \to \infty} \mathcal{L}(\mathrm{z}) = 1, \quad \lim_{z \to -\infty} \mathcal{L}(\mathrm{z}) = -1,$$

we have (13).

To prove $\frac{\partial \mathcal{L}(z)}{\partial z} \leq \frac{1}{3}$, it suffices to show

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(z)}{\partial z^2} &> 0 \ \forall z < 0, \\
\frac{\partial^2 \mathcal{L}(z)}{\partial z^2} &< 0 \ \forall z > 0.
\end{aligned}$$

But

$$\begin{aligned}
\frac{\partial^2 \mathcal{L}(z)}{\partial z^2} &= \frac{8(e^z + e^{-z})}{(e^z - e^{-z})^3} - \frac{2}{z^3} \\
&= 2\frac{(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots) - (1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)^3}{z^3(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots)^3},
\end{aligned} \tag{59}$$

so we need only to show that the numerator of (59) is always less than 0 $\forall z \neq 0$. We first note that the coefficient of $z^{2k}, k > 1$ in the second term is

$$\begin{aligned}
3(\frac{1}{(2k+1)!} + \frac{1}{(2k-1)!3!} + \cdots) &> 3\frac{1}{(2k-1)!3!} \\
&> \frac{1}{(2k)!}(\frac{3(2k)}{3!}) \\
&> \frac{1}{(2k)!},
\end{aligned}$$

while $\frac{1}{(2k)!}$ is the coefficient of z^{2k} in the first term. For $k = 0, 1$, the coefficients of both terms cancel out. The proof is now complete. $\square$

References

[1] A.A. Adly, I.D. Mayeygoyz, and A. Bergqvist. Preisach modeling of magnetostrictive hysteresis. *J. Appl. Phys.*, 69(8):5777–5779, 1991.

[2] M. Bardi and I. Capuzzo-Dolcetta. *Optimal Control and Viscosity Solutions of Hamilton-Jacobi-Bellman Equations*. Birkhäuser, Boston, 1997.

[3] M.G. Crandall, L.C. Evans, and P.L. Lions. Some properties of viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 282(2):487–502, 1984.

[4] M.G. Crandall and P.L. Lions. Viscosity solutions of Hamilton-Jacobi equations. *Transactions of the American Mathematical Society*, 277(1):1–42, 1983.

[5] W.S. Galinaitis and R.C. Rogers. Control of a hysteretic actuator using inverse hysteresis compensation. In V.V. Varadan, editor, *Mathematics and Control in Smart Structures*, volume 3323 of *SPIE*, pages 267–277, 1998.

[6] R.B. Gorbet, D.W.L. Wang, and K.A. Morris. Preisach model identification of a two-wire SMA actuator. In *Proceedings of IEEE International Conference on Robotics and Automation*, pages 2161–2167, 1998.

[7] D. Hughes and J.T. Wen. Preisach modeling and compensation for smart material hysteresis. In *Active Materials and Smart Structures*, volume 2427 of *SPIE*, pages 50–64, 1994.

[8] H. Ishii. Uniqueness of unbounded viscosity solution of Hamilton-Jacobi equations. *Indiana University Mathematics Journal*, 33(5):721–748, 1984.

[9] D.C. Jiles and D.L. Atherton. Theory of ferromagnetic hysteresis. *J. Magn. Magn. Mater.*, 61:48–60, 1986.

[10] R. C. Smith and C. L. Hom. A domain wall theory for ferroelectric hysteresis. Technical Report CRSC-TR99-01, CRSC, North Carolina State University, 1999.

[11] R.C. Smith. Inverse compensation for hysteresis in magnetostrictive transducers. Technical Report CRSC-TR98-36, CRSC, North Carolina State University, 1998.

[12] X. Tan. Control of smart actuators. Ph.D. Dissertation Research Proposal, Department of Electrical and Computer Engineering, University of Maryland, College Park, 2001.

[13] G. Tao and P.V. Kokotović. Adaptive control of plants with unknown hysteresis. *IEEE. Trans. Automat. Contr.*, 40(2):200–212, 1995.

[14] R. Venkataraman. *Modeling and Adaptive Control of Magnetostrictive Actuators*. PhD thesis, University of Maryland, College Park, 1998.

[15] R. Venkataraman and P. S. Krishnaprasad. Qualitative analysis of a bulk ferromagnetic hysteresis model. In *Proceedings of the 37th IEEE Conference on Decision and Control, Tampa, FL*, pages 2443–2448, December 1998.

[16] R. Venkataraman and P.S. Krishnaprasad. A model for a thin magnetostrictive actuator. In *Proceedings of the 32nd Conference on Information Sciences and Systems, Princeton, NJ*, Princeton, March 1998.

[17] A. Visintin. *Differential Models of Hysteresis*. Springer, 1994.